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Analysis on weighted homogeneous trees

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Chapter 1

Introduction

The object of this thesis is the analysis on a metric measure space $(\mathcal{V}, d, \mu)$ built on the nodes $\mathcal{V}$ of an infinite homogeneous tree of order $q+1$ endowed with the usual distance d and a weighted counting measure μ that takes into account the special levelled structure of the tree and is defined by the formula

$$\int_{\mathcal{V}} f \, d\mu = \sum_{x \in \mathcal{V}} f(x) q^{\ell(x)} \,,$$

where ℓ is a suitable level function on the tree.

The space $(\mathcal{V}, d, \mu)$ exhibits a number of interesting properties. For example, it has exponential growth at infinity, namely

$$\mu\big(B(x,r)\big) \sim q^{\ell(x)} \, q^r \qquad \forall x \in \mathcal{V}, r > 0 \,,$$

where $B(x,r)$ denotes the ball centred at a vertex x of radius r. Thus the doubling property is not satisfied, namely

$$\lim_{r \to +\infty} \frac{\mu\big(B(x,2r)\big)}{\mu\big(B(x,r)\big)} = +\infty \qquad \forall x \in \mathcal{V} \,.$$

Due to the lack of the doubling property, the classical Calderón–Zygmund theory developed on spaces of homogeneous type in the second half of the 20th century [Ste93] does not apply to this setting. It is then significant to construct a new Calderón–Zygmund theory adapted to this space. To this extent we discuss the abstract Calderón–Zygmund theory introduced by Hebisch and Steger in [HS03] and we show that it can be applied in the setting described above. In particular, such theory can be used to study boundedness properties of singular integrals related to a Laplacian operator $\mathcal{L}$ which acts on a function $f : \mathcal{V} \to \mathbb{C}$ as follows:

$$\mathcal{L}f(x) = f(x) - \frac{1}{2\sqrt{q}} \sum_{y\in\mathcal{V}:d(x,y)=1} q^{\frac{\ell(y)-\ell(x)}{2}} f(y) \qquad \forall x \in \mathcal{V}.$$

The Laplacian $\mathcal{L}$ turns out to be bounded on $L^p(\mathcal{V},\mu)$, for $p \in [1,\infty]$, self-adjoint on $L^2(\mathcal{V},\mu)$ and its L^2-spectrum is $[0,2]$.

We are able to define suitable linear operators X_j, $j = 0,\ldots,q$, which play the role of first derivatives of functions in this setting, in such a way that

$$\mathcal{L} = \frac{1}{2(q+1)} \sum_{j=0}^{q} X_j^* X_j.$$

Following the sketch of the proof given in [HS03] we show that the singular integral operators $X_j\mathcal{L}^{-1/2}$, $j = 0,\ldots,q$, are of weak-type $(1,1)$ and bounded on $L^p(\mathcal{V},\mu)$ for $1 < p \leq 2$. This are the analogue of the classical Riesz transforms in this setting. The L^p-boundedness of such operators for $p > 2$ is an open problem, on which we are still working.

As a future developement, we are also interested in the definition of suitable Hardy and BMO spaces adapted to this setting: these are function spaces which turn out to be good substitute for L^1 and L^∞ in the study of boundedness of singular integral operators. Due again to the lack of the doubling property, the classical theory of Hardy spaces [Ste93] cannot be used here. We also show that $(\mathcal{V},d,\mu)$ does not satisfy the isoperimetric property (which we define basing on [HLW06]), i.e. it does not exist a positive constant C such that for every bounded set A

$$\mu(A_1) \geq C\mu(A)\,,$$

where $A_1 = \{x \in A : d(x,A^c) \leq 1\}$. The isoperimetric property was a key ingredient for the recent theory of Hardy spaces developed in [CMM09] for metric spaces, possibly of exponential volume growth. So also that theory cannot be applied here. We shall try to construct a new Hardy-BMO theory using the sets which appear in the Calderón–Zygmund theory in the spirit of [Val09].

Chapter 2

Preliminaries

In this chapter we present some preliminaries that will be useful in the following. First we introduce L^p-weak spaces and review their main properties. In this way we build the setting for the main topic of the section, interpolation theory, that we meet in the form of the Marcinkiewicz interpolation theorem. We end the chapter with an introduction to functional calculus, focusing on continuous functional calculus.

2.1 L^p-weak spaces and interpolation

L^p-weak spaces are spaces of function larger than Lebesgue spaces L^p that can often be used as substitutes to L^p when studying the boundedness of operators. In this way, thanks to interpolation theory techniques, the problem is often simplified. For these reasons the subject is fundamental for our work and we will broadly exploit the results presented in this section. For this part the exposition is based on Chapter 1 of the book by L. Grafakos [Gra08].

2.1.1 The distribution function

Let X be a measurable space and μ a positive, not necessarily finite, measure on X.

Definition 2.1.1. The function spaces $L^p(X,\mu)$ are defined as follows:

$$\begin{aligned} 0<p<\infty, \quad L^p(X,\mu) = \{ f : X \to \mathbb{C} \ \ \mu - \textit{measurable s.t.} \\ \int_X |f|^p d\mu < +\infty \}, \end{aligned}$$

$$\begin{aligned} p=\infty, \quad L^\infty(X,\mu) = \{ f : X \to \mathbb{C} \ \ \mu - \textit{measurable s.t.} \\ \exists B>0 : \mu\left(\{x : |f(x)|>B\}\right)=0 \}. \end{aligned}$$

Two functions f, g are considered equal if they are equal μ-almost everywhere, i.e.:

$$\mu\left(\{x \in X : f(x) \neq g(x)\}\right) = 0.$$

Definition 2.1.2. The L^p quasinorm is defined as follows:

$$\begin{aligned} 0<p<\infty, \quad & \|f\|_{L^p(X,\mu)} = \left(\int_X |f(x)|^p d\mu \right)^{\frac{1}{p}}, \\ p=\infty, \quad & \|f\|_{L^\infty(X,\mu)} = \inf\{B>0 : \mu\left(\{x : |f(x)|>B\}\right)=0\}. \end{aligned}$$

Proposition 2.1.1. If $1 \leq p \leq \infty$, then the Minkowski's inequality holds, i.e. we have:

$$\|f+g\|_{L^p(X,\mu)} \leq \|f\|_{L^p(X,\mu)} + \|g\|_{L^p(X,\mu)}.$$

It can be shown that $L^p(X,\mu)$ are normed Banach spaces.

If $0<p<1$, Minkowski's inequality does not hold but the following inequality holds:

$$\|f+g\|_{L^p(X,\mu)} \leq 2^{\frac{1-p}{p}} \left(\|f\|_{L^p(X,\mu)} + \|g\|_{L^p(X,\mu)} \right).$$

It can be shown that $L^p(X,\mu)$ are quasinormed Banach spaces.

Definition 2.1.3. Let f be a measurable function on X. The distribution function of f is the function $d_f : [0,\infty) \to [0,\infty)$ defined as:

$$d_f(\alpha) = \mu\left(\{x \in X : |f(x)| > \alpha\}\right) \qquad \forall \alpha > 0\,.$$

The distribution function d_f provides information about the size of f but not about its local behavior. Note that d_f is a decreasing function of α (not necessarily strictly decreasing).

Proposition 2.1.2. Let f and g be measurable functions on (X,μ). Then for all $\alpha, \beta > 0$ we have:

1. $|g| \leq |f| \, \mu - a.e. \implies d_g \leq d_f$;
2. $d_{cf}(\alpha) = d_f(\frac{\alpha}{|c|}) \quad \forall c \in \mathbb{C} \setminus \{0\}$;
3. $d_{f+g}(\alpha+\beta) \leq d_f(\alpha) + d_g(\beta)$;
4. $d_{fg}(\alpha\beta) \leq d_f(\alpha) + d_g(\beta)$.

It is possible to express the L^p norm of f in terms of its distribution function d_f, as stated in the following proposition.

Proposition 2.1.3. Let f be in $L^p(X,\mu)$ for $0 < p < \infty$. Then we have the following characterization of the L^p norm of f:

$$\|f\|^p_{L^p(X,\mu)} = p \int_0^\infty \alpha^{p-1} d_f(\alpha) d\alpha.$$

Proof. We proceed by direct computation:

$$\begin{aligned} p \int_0^\infty \alpha^{p-1} d_f(\alpha) d\alpha &= p \int_0^\infty \alpha^{p-1} \int_X \chi_{\{x:|f(x)|>\alpha\}} d\mu(x) \, d\alpha \\ &= \int_X \int_0^{|f(x)|} p\alpha^{p-1} d\alpha \, d\mu(x) \quad \text{by Fubini theorem} \\ &= \int_X |f(x)|^p d\mu(x) \\ &= \|f\|^p_{L^p(X,\mu)}. \end{aligned}$$

□

2.1.2 L^p-weak spaces

Definition 2.1.4. If $0 < p < \infty$, then the weak L^p-space is the set of all μ-measurable functions f such that the quantity:

$$\begin{aligned} \|f\|_{L^{p,\infty}} &= \inf \left\{ c > 0 : d_f(\alpha) \leq \frac{c^p}{\alpha^p} \quad \forall \alpha > 0 \right\} \\ &= \sup \left\{ \gamma d_f(\gamma)^{\frac{1}{p}} : \gamma > 0 \right\} \end{aligned}$$

is finite.

If $p = \infty$, then the weak L^∞-space is by definition $L^\infty(X,\mu)$.

The weak L^p spaces are denoted by $L^{p,\infty}(X,\mu)$. Two functions in $L^{p,\infty}(X,\mu)$ are considered equal if they are equal μ-a.e.

Proposition 2.1.4. Properties of $\|\cdot\|_{L^{p,\infty}}$.

i) $\forall k \in \mathbb{C}\setminus\{0\}: \quad \|kf\|_{L^{p,\infty}} = |k|\|f\|_{L^{p,\infty}}$,

ii) $\|f+g\|_{L^{p,\infty}} \le c_p\,(\|f\|_{L^{p,\infty}} + \|g\|_{L^{p,\infty}})$ where $c_p = \max(2, 2^{\frac{1}{p}})$,

iii) $\|f\|_{L^{p,\infty}} = 0 \implies f = 0 \quad \mu$-a.e.

In view of i),ii),iii) $L^{p,\infty}$ is a quasinormed linear space for $0 < p < \infty$.

Weak L^p spaces are larger than usual L^p spaces, as illustrated in the following proposition.

Proposition 2.1.5. For $0 < p < \infty$, it holds:

$$\forall f \in L^p(X,\mu): \quad \|f\|_{L^{p,\infty}} \le \|f\|_{L^p}\,,$$

hence

$$L^p(X,\mu) \subset L^{p,\infty}(X,\mu).$$

Proof. Let $f \in L^p(X,\mu)$. For every $\alpha > 0$ we have:

$$\begin{aligned} \alpha^p d_f(\alpha) &\le \int_{\{x:|f(x)|>\alpha\}} |f(x)|^p d\mu \\ &\le \int_X |f(x)|^p d\mu \\ &= \|f\|_{L^p}^p\,, \end{aligned}$$

so that

$$\forall \alpha > 0, \quad \alpha d_f(\alpha)^{\frac{1}{p}} \le \|f\|_{L^p}\,.$$

This implies that

$$\|f\|_{L^{p,\infty}} = \sup\left\{\alpha d_f(\alpha)^{\frac{1}{p}} : \alpha > 0\right\} \le \|f\|_{L^p}.$$

□

We observe that the inclusion $L^p(X,\mu) \subset L^{p,\infty}(X,\mu)$ is strict. For example: if $X = \mathbb{R}^n$ and $\mu = |\cdot|$ is the Lebesgue measure, then the function $h(x) = |x|^{-\frac{n}{p}}$ satisfies:

$h \notin L^p(\mathbb{R}^n,\mu)$ but $h \in L^{p,\infty}(\mathbb{R}^n,\mu)$ with $\|h\|_{L^{p,\infty}(\mathbb{R}^n,\mu)} = \nu_n = |B_{\mathbb{R}^n}(0,1)|$.

The following proposition provides a first glimpse at interpolation.

Proposition 2.1.6. Let $0 < p < q \leq \infty$, $f \in L^{p,\infty}(X,\mu) \cap L^{q,\infty}(X,\mu)$. Then:

$$f \in L^r(X,\mu) \quad \forall r \text{ s.t. } p < r < q$$

and

$$\|f\|_{L^r} \leq \left(\frac{r}{r-p} + \frac{r}{q-r}\right)^{\frac{1}{r}} \|f\|_{L^{p,\infty}}^{\frac{\frac{1}{r}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}}} \|f\|_{L^{q,\infty}}^{\frac{\frac{1}{p}-\frac{1}{r}}{\frac{1}{p}-\frac{1}{q}}}.$$

Proof. We first consider the case $q < \infty$.

We know that

$$d_f(\alpha) \leq \min\left(\frac{\|f\|_{L^{p,\infty}}^p}{\alpha^p}, \frac{\|f\|_{L^{q,\infty}}^q}{\alpha^q}\right) \qquad \forall \alpha > 0\,.$$

Observe that

$$\frac{\|f\|_{L^{p,\infty}}^p}{\alpha^p} \leq \frac{\|f\|_{L^{q,\infty}}^q}{\alpha^q} \iff \alpha^{q-p} \leq \frac{\|f\|_{L^{q,\infty}}^q}{\|f\|_{L^{p,\infty}}^p} \iff \alpha \leq \left(\frac{\|f\|_{L^{q,\infty}}^q}{\|f\|_{L^{p,\infty}}^p}\right)^{\frac{1}{q-p}}.$$

We set

$$B = \left(\frac{\|f\|_{L^{q,\infty}}^q}{\|f\|_{L^{p,\infty}}^p}\right)^{\frac{1}{q-p}}.$$

Now we can evaluate the L^r norm of f for $p < r < q$ by means of Proposition 2.1.3:

$$\begin{aligned}
\|f\|_{L^r(X,\mu)}^r &= r\int_0^\infty \alpha^{r-1} d_f(\alpha)d\alpha \\
&\leq r\int_0^\infty \alpha^{r-1} \min\left(\frac{\|f\|_{L^{p,\infty}}^p}{\alpha^p}, \frac{\|f\|_{L^{q,\infty}}^q}{\alpha^q}\right) d\alpha \\
&= r\int_0^B \alpha^{r-1-p}\|f\|_{L^{p,\infty}}^p d\alpha + r\int_B^\infty \alpha^{r-1-q}\|f\|_{L^{q,\infty}}^q d\alpha.
\end{aligned}$$

Note that the first integral converges since $r-p > 0$ while the second integral converges since $r-q<0$. So we can proceed:

$$\begin{aligned}
\|f\|^r_{L^r(X,\mu)} &\le \frac{r}{r-p}\|f\|^p_{L^{p,\infty}}B^{r-p} + \frac{r}{q-r}\|f\|^q_{L^{q,\infty}}B^{r-q} \\
&= \frac{r}{r-p}\|f\|^p_{L^{p,\infty}}\left(\frac{\|f\|^q_{L^{q,\infty}}}{\|f\|^p_{L^{p,\infty}}}\right)^{\frac{r-p}{q-p}} + \frac{r}{q-r}\|f\|^q_{L^{q,\infty}}\left(\frac{\|f\|^q_{L^{q,\infty}}}{\|f\|^p_{L^{p,\infty}}}\right)^{\frac{r-q}{q-p}} \\
&= \frac{r}{r-p}\left(\|f\|^p_{L^{p,\infty}}\right)^{1-\frac{r-p}{q-p}}\left(\|f\|^q_{L^{q,\infty}}\right)^{\frac{r-p}{q-p}} + \frac{r}{q-r}\left(\|f\|^q_{L^{q,\infty}}\right)^{1+\frac{r-q}{q-p}}\left(\|f\|^p_{L^{p,\infty}}\right)^{\frac{q-r}{q-p}} \\
&= \frac{r}{r-p}\left(\|f\|^p_{L^{p,\infty}}\right)^{\frac{q-r}{q-p}}\left(\|f\|^q_{L^{q,\infty}}\right)^{\frac{r-p}{q-p}} + \frac{r}{q-r}\left(\|f\|^q_{L^{q,\infty}}\right)^{\frac{r-p}{q-p}}\left(\|f\|^p_{L^{p,\infty}}\right)^{\frac{q-r}{q-p}} \\
&= \left(\frac{r}{r-p}+\frac{r}{q-r}\right)\left(\|f\|^q_{L^{q,\infty}}\right)^{\frac{r-p}{q-p}}\left(\|f\|^p_{L^{p,\infty}}\right)^{\frac{q-r}{q-p}}.
\end{aligned}$$

Now we pass to the case $q=\infty$.

We know that $d_f(\alpha)=0$ for $\alpha > \|f\|_{L^\infty(X,\mu)}$, thus:

$$\begin{aligned}
\|f\|^r_{L^r(X,\mu)} &= r\int_0^\infty \alpha^{r-1}d_f(\alpha)d\alpha \\
&= r\int_0^{\|f\|_{L^\infty}} \alpha^{r-1}d_f(\alpha)d\alpha \\
&\le r\int_0^{\|f\|_{L^\infty}} \alpha^{r-1}\alpha^{-p}\|f\|^p_{L^{p,\infty}}d\alpha \\
&= \frac{r}{r-p}\|f\|^p_{L^{p,\infty}}\|f\|^{r-p}_{L^\infty}.
\end{aligned}$$

□

2.1.3 Interpolation

Definition 2.1.5. Consider two measure spaces (X,μ), (Y,ν).

- Bounded operators T that map $L^p(X,\mu)\to L^q(Y,\nu)$ are called of strong type (p,q).
- Bounded operators T that map $L^p(X,\mu)\to L^{q,\infty}(Y,\nu)$ are called of weak type (p,q).

We can make the following useful classification:

Definition 2.1.6. Let (X,μ), (Y,ν) be two measure spaces. Consider the linear space U and the set V:

$$U = \{f : (X,\mu) \to \mathbb{C}\ , f \text{ measurable}\}$$
$$V = \{f : (Y,\nu) \to \mathbb{C}\ , f \text{ measurable and finite a.e.}\}\ .$$

$T : U \to V$ is called:

- linear, if $\forall f, g \in U$, $\forall \lambda \in \mathbb{C}$

$$T(f+g) = T(f) + T(g) \quad \text{and} \quad T(\lambda f) = \lambda T(f),$$

- sublinear, if $\forall f, g \in U$, $\forall \lambda \in \mathbb{C}$

$$|T(f+g)| \leq |T(f)| + |T(g)| \quad \text{and} \quad |T(\lambda f)| = |\lambda||T(f)|,$$

- quasilinear, if $\forall f, g \in U$, $\forall \lambda \in \mathbb{C}$

$$|T(f+g)| \leq K\left(|T(f)| + |T(g)|\right) \quad \text{and} \quad |T(\lambda f)| = |\lambda||T(f)|$$

  for some constant $K > 0$.

Theorem 2.1.7 (Marcinkiewic interpolation theorem)**.** *Let (X,μ), (Y,ν) be two measure spaces and $0 < p_0 < p_1 \leq \infty$.*
Let T be a sublinear operator such that

$$T : L^{p_0}(X) + L^{p_1}(X) \to \{\textit{measurable functions on } Y\}.$$

Assume that $\exists A_0, A_1 > 0$ such that:

$$\|T(f)\|_{L^{p_0,\infty}(Y)} \leq A_0 \|f\|_{L^{p_0}(X)} \quad \forall f \in L^{p_0}(X),$$
$$\|T(f)\|_{L^{p_1,\infty}(Y)} \leq A_1 \|f\|_{L^{p_1}(X)} \quad \forall f \in L^{p_1}(X).$$

Then $\forall p$ such that $p_0 < p < p_1$ and $\forall f \in L^p(X)$ we have:

$$\|T(f)\|_{L^p(Y)} \leq A\|f\|_{L^p(X)}$$

where $A = 2\left(\frac{p}{p-p_0} + \frac{p}{p_1-p}\right)^{\frac{1}{p}} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}}$.

Proof. We first consider the case $p_1 < \infty$.

Fix $f \in L^p(X)$ and $\alpha > 0$.

We split $f = f_0^\alpha + f_1^\alpha$, where $f_0^\alpha \in L^{p_0}(X)$ and $f_1^\alpha \in L^{p_1}(X)$. The splitting is obtained by cutting $|f|$ at height $\delta\alpha$ for some $\delta\alpha$ to be determined later.

Set:

$$f_0^\alpha(x) = \begin{cases} f(x) & \text{for } |f(x)| > \delta\alpha \\ 0 & \text{for } |f(x)| \leq \delta\alpha \end{cases},$$

$$f_1^\alpha(x) = \begin{cases} f(x) & \text{for } |f(x)| \leq \delta\alpha \\ 0 & \text{for } |f(x)| > \delta\alpha \end{cases}.$$

Observe that:

- f_0^α (the unbounded part of f) is in $L^{p_0}(X)$, indeed:

$$\begin{aligned} \|f_0^\alpha\|_{L^{p_0}}^{p_0} &= \int_{|f|>\delta\alpha} |f|^{p_0} d\mu \\ &= \int_{|f|>\delta\alpha} |f|^p |f|^{p_0-p} d\mu \\ &\leq (\delta\alpha)^{p_0-p} \|f\|_{L^p}^p \end{aligned}$$

  since $p_0 < p$ and then $|f|^{p_0-p} \leq (\delta\alpha)^{p_0-p}$.

- f_1^α (the bounded part of f) is in $L^{p_1}(X)$, indeed:

$$\begin{aligned} \|f_1^\alpha\|_{L^{p_1}}^{p_1} &= \int_{|f|\leq\delta\alpha} |f|^{p_1} d\mu \\ &= \int_{|f|\leq\delta\alpha} |f|^p |f|^{p_1-p} d\mu \\ &\leq (\delta\alpha)^{p_1-p} \|f\|_{L^p}^p \end{aligned}$$

  since $p < p_1$ and then $|f|^{p_1-p} \leq (\delta\alpha)^{p_1-p}$.

By the sublinearity property of T:

$$|T(f)| \leq |T(f_0^\alpha)| + |T(f_1^\alpha)|$$

which implies that for every $\alpha > 0$

$$\{x : |T(f)(x)| > \alpha\} \subseteq \left\{x : |T(f_0^\alpha)(x)| > \tfrac{\alpha}{2}\right\} \cup \left\{x : |T(f_1^\alpha)(x)| > \tfrac{\alpha}{2}\right\}$$

because $\forall x$ such that $\alpha < |T(f)(x)| \le |T(f_0^\alpha)(x)| + |T(f_1^\alpha)(x)|$ one of the following occurs:

$$|T(f_0^\alpha)(x)| > \tfrac{\alpha}{2} \text{ and } |T(f_1^\alpha)(x)| > \alpha - |T(f_0^\alpha)(x)| \quad \text{or}$$
$$|T(f_0^\alpha)(x)| \le \tfrac{\alpha}{2} \text{ and } |T(f_1^\alpha)(x)| > \alpha - |T(f_0^\alpha)(x)| \ge \tfrac{\alpha}{2}\,.$$

Therefore

$$d_{T(f)}(\alpha) \le d_{T(f_0^\alpha)}(\tfrac{\alpha}{2}) + d_{T(f_1^\alpha)}(\tfrac{\alpha}{2}).$$

Since f_0^α is in L^{p_0}, $T(f_0^\alpha)$ is in $L^{p_0,\infty}$ and

$$\|T(f_0^\alpha)\|_{L^{p_0,\infty}} = \sup\{\gamma d_{T(f_0^\alpha)}(\gamma)^{\frac{1}{p_0}}, \gamma > 0\} \le A_0 \|f_0^\alpha\|_{L^{p_0}}\,,$$

so that $d_{T(f_0^\alpha)}(\frac{\alpha}{2}) \le \frac{\|T(f_0^\alpha)\|^{p_0}_{L^{p_0,\infty}}}{\left(\frac{\alpha}{2}\right)^{p_0}}$.

Similarly, since f_1^α is in L^{p_1}, $T(f_1^\alpha)$ is in $L^{p_1,\infty}$ and:

$$\|T(f_1^\alpha)\|_{L^{p_1,\infty}} = \sup\{\gamma d_{T(f_1^\alpha)}(\gamma)^{\frac{1}{p_1}}, \gamma > 0\}$$

so that $d_{T(f_1^\alpha)}(\frac{\alpha}{2}) \le \frac{\|T(f_1^\alpha)\|^{p_1}_{L^{p_1,\infty}}}{\left(\frac{\alpha}{2}\right)^{p_1}}$.

Then:

$$\begin{aligned} d_{T(f)}(\alpha) &\le \frac{\|T(f_0^\alpha)\|^{p_0}_{L^{p_0,\infty}}}{\left(\frac{\alpha}{2}\right)^{p_0}} + \frac{\|T(f_1^\alpha)\|^{p_1}_{L^{p_1,\infty}}}{\left(\frac{\alpha}{2}\right)^{p_1}} \\ &\le \frac{A_0^{p_0}}{\left(\frac{\alpha}{2}\right)^{p_0}} \|f_0^\alpha\|^{p_0}_{L^{p_0}} + \frac{A_1^{p_1}}{\left(\frac{\alpha}{2}\right)^{p_1}} \|f_1^\alpha\|^{p_1}_{L^{p_1}} \\ &= \frac{(2A_0)^{p_0}}{\alpha^{p_0}} \int_{|f|>\delta\alpha} |f(x)|^{p_0} d\mu + \frac{(2A_1)^{p_1}}{\alpha^{p_1}} \int_{|f|\le\delta\alpha} |f(x)|^{p_1} d\mu. \end{aligned}$$

In this way we obtain:

$$\begin{aligned}
\|T(f)\|_{L^p}^p &= p\int_0^\infty \alpha^{p-1} d_{T(f)}(\alpha)d\alpha \\
&\le p(2A_0)^{p_0}\int_0^\infty \alpha^{p-1}\alpha^{-p_0}\int_{|f|>\delta\alpha}|f(x)|^{p_0}d\mu\, d\alpha \\
&\quad + p(2A_1)^{p_1}\int_0^\infty \alpha^{p-1}\alpha^{-p_1}\int_{|f|\le\delta\alpha}|f(x)|^{p_1}d\mu\, d\alpha \\
&= p(2A_0)^{p_0}\int_X |f(x)|^{p_0}\int_0^{\frac{1}{\delta}|f(x)|}\alpha^{p-1-p_0}d\alpha\, d\mu \\
&\quad + p(2A_1)^{p_1}\int_X |f(x)|^{p_1}\int_{\frac{1}{\delta}|f(x)|}^\infty \alpha^{p-1-p_1}d\alpha\, d\mu.
\end{aligned}$$

Notice that the first integral converges since $p > p_0 \implies -1+(p-p_0) > -1$ while the second integral converges since $p < p_1 \implies -1+(p-p_1) < -1$. Then:

$$\begin{aligned}
\|T(f)\|_{L^p}^p &\le \frac{p(2A_0)^{p_0}}{p-p_0}\frac{1}{\delta^{p-p_0}}\int_X |f(x)|^{p_0}|f(x)|^{p-p_0}d\mu \\
&\quad + \frac{p(2A_1)^{p_1}}{p_1-p}\frac{1}{\delta^{p-p_1}}\int_X |f(x)|^{p_1}|f(x)|^{p-p_1}d\mu \\
&= p\left(\frac{p(2A_0)^{p_0}}{p-p_0}\frac{1}{\delta^{p-p_0}} + \frac{p(2A_1)^{p_1}}{p_1-p}\frac{1}{\delta^{p-p_1}}\right)\|f\|_{L^p}^p.
\end{aligned}$$

We pick $\delta > 0$ such that:

$$(2A_0)^{p_0}\frac{1}{\delta^{p-p_0}} = (2A_1)^{p_1}\delta^{p_1-p}$$

and obtain the thesis.

We consider now the case $p_1 = \infty$.

We write as before:

$$f_0^\alpha(x) = \begin{cases} f(x) & \text{for } |f(x)| > \gamma\alpha \\ 0 & \text{for } |f(x)| \le \gamma\alpha\,, \end{cases}$$

$$f_1^\alpha(x) = \begin{cases} f(x) & \text{for } |f(x)| \le \gamma\alpha \\ 0 & \text{for } |f(x)| > \gamma\alpha\,. \end{cases}$$

As in the previous case, one can verify that:

- f_0^α (the unbounded part of f) is in $L^{p_0}(X)$.
- f_1^α (the bounded part of f) is in $L^\infty(X)$ by definition and $\|f_1^\alpha\|_{L^\infty} \leq \gamma\alpha$.

By assumption T is bounded from $L^\infty(X)$ to $L^\infty(Y)$ (since $L^{\infty,\infty} = L^\infty$) and:

$$\|T(f_1^\alpha)\|_{L^\infty} \leq A_1\|f_1^\alpha\|_{L^\infty} \leq A_1\gamma\alpha = \frac{\alpha}{2}$$

provided we choose $\gamma = (2A_1)^{-1}$.

It follows that:

$$\mu\left(\{x : |T(f_1^\alpha)(x)| > \tfrac{\alpha}{2}\}\right) = 0.$$

Therefore (exploiting the sublinearity property of T as in the previous case):

$$d_{T(f)}(\alpha) \leq d_{T(f_0^\alpha)}(\tfrac{\alpha}{2}) + d_{T(f_1^\alpha)}(\tfrac{\alpha}{2}) = d_{T(f_0^\alpha)}(\tfrac{\alpha}{2}).$$

By assumption T maps $L^{p_0}(X)$ to $L^{p_0,\infty}(Y)$ and we know that:

$$d_{T(f_0^\alpha)}(\alpha) \leq \frac{\|T(f_0^\alpha)\|^{p_0}_{L^{p_0,\infty}}}{\alpha^{p_0}} \quad \text{and} \quad \|T(f_0^\alpha)\|_{L^{p_0,\infty}} \leq A_0\|f_0^\alpha\|_{L^{p_0}}.$$

It follows that

$$\begin{aligned}
d_{T(f_0^\alpha)}(\tfrac{\alpha}{2}) &\leq \frac{\|T(f_0^\alpha)\|^{p_0}_{L^{p_0,\infty}}}{\left(\frac{\alpha}{2}\right)^{p_0}} \\
&\leq A_0^{p_0}\frac{\|f_0^\alpha\|^{p_0}_{L^{p_0}}}{(\alpha)^{p_0}}2^{p_0} \\
&= \frac{(2A_0)^{p_0}}{\alpha^{p_0}}\int_{|f|>\gamma\alpha} |f(x)|^{p_0}d\mu
\end{aligned}$$

with $\gamma = (2A_1)^{-1}$. Finally we obtain:

$$\begin{aligned}
\|T(f)\|^p_{L^p} &= p\int_0^\infty \alpha^{p-1}d_{T(f)}(\alpha)d\alpha \\
&\leq p\int_0^\infty \alpha^{p-1}d_{T(f_0^\alpha)}(\tfrac{\alpha}{2})d\alpha \\
&\leq p\int_0^\infty \alpha^{p-1}\frac{(2A_0)^{p_0}}{\alpha^{p_0}}\int_{|f|>\frac{\alpha}{2A_1}} |f(x)|^{p_0}d\mu d\alpha \\
&= p(2A_0)^{p_0}\int_X |f(x)|^{p_0}\int_0^{2A_1|f(x)|} \alpha^{p-p_0-1}d\alpha d\mu \\
&= \frac{p(2A_1)^{p-p_0}(2A_0)^{p_0}}{p-p_0}\int_X |f(x)|^p d\mu \\
&= \frac{p(2A_1)^{p-p_0}(2A_0)^{p_0}}{p-p_0}\|f\|^p_{L^p}.
\end{aligned}$$

This proves the theorem with constant

$$A = 2\left(\frac{p}{p-p_0}\right)^{\frac{1}{p}} A_1^{1-\frac{p_0}{p}} A_0^{\frac{p_0}{p}}.$$

□

2.2 Continuous functional calculus

Continuous functional calculus aims to make sense of the expression $f(A)$ for each self-adjoint operator A on a Hilbert space H and each continuous function $f \in C(\sigma(A))$, where $\sigma(A)$ denotes the spectrum of A. For this section we refer the reader to the book by Reed and Simon [RS80], which we follow in our description.

First, let P be a polynomial in $\mathbb{C}[z]$ and A a self-adjoint operator on a Hilbert space H. Suppose

$$P(z) = \sum_{n=0}^{N} a_n z^n.$$

We define $P(A) \in B(H)$ (where $B(H)$ is the space of bounded linear operators on H) as:

$$P(A) = \sum_{n=0}^{N} a_n A^n$$

where $A^0 = I$ and $A^n = A \cdot A^{n-1} \quad \forall n \geq 1$.

To extend this definition to all continuous functions we must first prove two lemmas.

Lemma 2.2.1. *Let* $P(z) = \sum_{n=0}^{N} a_n z^n \in \mathbb{C}[z]$ *and* $P(A) = \sum_{n=0}^{N} a_n A^n \in B(H)$.

Then:

$$\sigma(P(A)) = \{P(\lambda) : \lambda \in \sigma(A)\}.$$

Proof.

- First we prove that $\lambda \in \sigma(A) \implies P(\lambda) \in \sigma(P(A))$.

  Let $\lambda \in \sigma(A)$.

  $z = \lambda$ is a root of $P(z) - P(\lambda)$ so we can write:

$$\tilde{P}(z) = P(z) - P(\lambda) = (z-\lambda)Q(z)$$

where λ is not a root for $Q(z)$. Then:

$$\tilde{P}(A) = P(A) - P(\lambda)I = (A - \lambda I)Q(A).$$

By hypothesis $\lambda \in \sigma(A)$ which means that $(A - \lambda I)$ is not invertible. So also $P(A) - P(\lambda)I$ is not invertible, that is $P(\lambda) \in \sigma(P(A))$.

- Now we prove that $\mu \in \sigma(P(A)) \implies \mu = P(\lambda)$ with $\lambda \in \sigma(A)$.
  Let $\mu \in \sigma(P(A))$.
  Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of $\hat{P}(z) = P(z) - \mu$, that is:

  $$\hat{P}(z) = P(z) - \mu = a(z - \lambda_1)(z - \lambda_2)\ldots(z - \lambda_n)$$

  Then

  $$\hat{P}(A) = P(A) - \mu I = a(A - \lambda_1 I)(A - \lambda_2 I)\ldots(A - \lambda_n I).$$

  If $\lambda_i \notin \sigma(A)$, $\forall i : 1 < i < n$ then $P(A) - \mu$ is invertible as:

  $$(P(A) - \mu I)^{-1} = a^{-1}(A - \lambda_1)^{-1}(A - \lambda_2)^{-1}\ldots(A - \lambda_n)^{-1}$$

  while by hypothesis $\mu \in \sigma(P(A))$, that is $P(A) - \mu I$ is not invertible.
  So we conclude that for some i it holds $\lambda_i \in \sigma(A)$, i.e. $\mu = P(\lambda_i)$ with $\lambda_i \in \sigma(A)$.

□

Lemma 2.2.2. *Let A be a bounded self-adjoint operator on a Hilbert space H.*

Then:

$$\|P(A)\|_{B(H)} = \sup_{\lambda \in \sigma(A)} |P(\lambda)|.$$

Proof.

$$\begin{aligned}
\|P(A)\|^2_{B(H)} &= \|P(A)^* P(A)\| \\
&= \|(\bar{P}P)(A)\| \quad \text{since } P(A)^* = (\bar{P})(A) \\
&= \sup_{\lambda \in \sigma((\bar{P}P)(A))} |\lambda| \\
&= \sup_{\lambda \in \sigma(A)} |(\bar{P}P)(\lambda)| \\
&= \left(\sup_{\lambda \in \sigma(A)} |P(\lambda)| \right)^2.
\end{aligned}$$

The third equality follows from the fact that $\|T\| = r(T) = \sup_{\lambda \in \sigma(T)} |\lambda| \quad \forall T$ self-adjoint where $r(T)$ denotes the spectral ray of T and because $(\bar{P}P)$ has real coefficients, so $(\bar{P}P(A))$ is self-adjoint. $\square$

Lemma 2.2.2 shows that, given A self-adjoint on H, the function $\tilde{\phi} : \mathbb{P}(\sigma(A))_{\|\cdot\|_{C(\sigma(A))}} \to B(H)$ defined by $\tilde{\phi}(P) = P(A)$ is linear and bounded.

Polynomials on $\sigma(A)$ are dense in $C(\sigma(A))$ and $B(H)$ is a Banach space. Then $\tilde{\phi}$ admits an unique bounded linear extension $\phi : C(\sigma(A)) \to B(H)$.

We write $f(A) = \phi(f)$ to emphasize the dependence on A. Now we have all the necessary ingredients to prove the following theorem.

Theorem 2.2.3. *Let A be a self-adjoint operator on a Hilbert space H.*

Then there is a unique map $\phi : C(\sigma(A)) \to B(H)$ with the following properties:

a) ϕ is an algebraic $*$-homomorphism, that is:

- $\phi(fg) = \phi(f)\phi(g)$,
- $\phi(\lambda f) = \lambda\phi(f)$,
- $\phi(1) = I$,
- $\phi(\bar{f}) = \phi(f)^*$.

b) ϕ is continuous, that is, $\|\phi(f)\|_{B(H)} \leq C\|f\|_\infty$.

c) Let f be the function $f(x) = x$. Then $\phi(f) = A$.

Moreover ϕ has the additional properties:

d) If $A\psi = \lambda\psi$, then $\phi(f)\psi = f(\lambda)\psi$.

e) $\sigma(\phi(f)) = \{f(\lambda) : \lambda \in \sigma(A)\}$.

f) If $f \geq 0$, then $\phi(f) \geq 0$.

g) $\|\phi(f)\|_{B(H)} = \|f\|_\infty$.

Proof. The uniqueness of ϕ follows from the previous considerations observing that *a*) and *c*) imply that $\phi(P) = P(A)$ for all polynomials P.

So the only candidate for ϕ is the extension of $\tilde{\phi}$ described before. Indeed, ϕ coincides with $\tilde{\phi}$ on polynomials so by continuity it agrees with its unique extension on all $C(\sigma(A))$.

For the existence of ϕ we must prove that the extension of $\tilde{\phi}$ (which we denote with ϕ) satisfies properties $a) - g)$.

First we observe that ϕ satisfies $a) - g)$ for all polynomials, because $\tilde{\phi}$ does.

This can be shown by direct computation. Take p, q polynomials and $\lambda \in \mathbb{C}$. Then:

$a)$

$$\begin{aligned}
\tilde{\phi}(p \cdot q) &= (p \cdot q)(A) \\
&= \left(\sum_{n=0}^{N} a_n z^n \cdot \sum_{n=0}^{M} b_n z^n \right)(A) \\
&= \left(\sum_{n=0}^{N+M} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n \right)(A) \\
&= \sum_{n=0}^{N+M} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) A^n \\
&= \sum_{n=0}^{N} a_n A^n \cdot \sum_{n=0}^{M} b_n A^n \\
&= p(A) q(A) \\
&= \tilde{\phi}(p) \tilde{\phi}(q).
\end{aligned}$$

$$\begin{aligned}
\tilde{\phi}(\lambda p) &= (\lambda p)(A) \\
&= \lambda p(A) \\
&= \lambda \tilde{\phi}(p).
\end{aligned}$$

$$\begin{aligned}
\tilde{\phi}(1) &= A^0 \\
&= I \quad \text{by definition.}
\end{aligned}$$

$$\begin{aligned}\tilde{\phi}(\bar{p}) &= \bar{p}(A) \\ &= \sum_{n=0}^{N} \bar{a}_n A^n \\ &= \left(\sum_{n=0}^{N} a_n A^n\right)^* \\ &= p(A)^* \\ &= \tilde{\phi}(p)^*.\end{aligned}$$

b) and g) follow from Lemma 2.2.2.

c) is true by definition.

d)

$$\begin{aligned}\tilde{\phi}(p)\psi &= p(A)\psi \\ &= \left(\sum_{n=0}^{N} a_n A^n\right)\psi \\ &= \sum_{n=0}^{N} a_n (A^n \psi) \\ &= \sum_{n=0}^{N} a_n \lambda^n \psi \\ &= p(\lambda)\psi.\end{aligned}$$

e) follows from lemma 2.2.1.

f) can be shown directly for all $f \in C(\sigma(A))$. Let $f \in C(\sigma(A))$, $f \geq 0$.

$$f \geq 0 \implies f = g^2 \quad \text{with } g \text{ real, } g \in C(\sigma(A)).$$

Thus

$$\phi(f) = \phi(g^2) = \phi(g \cdot \bar{g}) = \phi(g)\phi(g)^*$$

$$\begin{aligned}(x, \phi(f)x) &= (x, \phi(g)\phi(g)^* x) \\ &= (\phi(g)x, \phi(g)x) \\ &= \|\phi(g)x\|^2 \geq 0 \quad \forall x \in H.\end{aligned}$$

Then ϕ satisfies $a) - g)$ for all continuous functions on $\sigma(A)$ by continuity. As an example we check d) and g).

Let $f \in C(\sigma(A))$, $f = \lim_{n\to\infty} p_n$ for some sequence of polynomials $\{p_n\}_n$.

d)

$$\begin{aligned}
\phi(f) &= \phi(\lim_{n\to\infty} p_n) \\
&= \lim_{n\to\infty} \phi(p_n) \\
&= \lim_{n\to\infty} \tilde{\phi}(p_n) \\
&= \lim_{n\to\infty} p_n(A).
\end{aligned}$$

Then

$$\begin{aligned}
\phi(f)\psi &= \left(\lim_{n\to\infty} p_n(A)\right)\psi \\
&= \lim_{n\to\infty} (p_n(A)\psi) \\
&= \lim_{n\to\infty} (p_n(\lambda)\psi) \\
&= \left(\lim_{n\to\infty} p_n(\lambda)\right)\psi \\
&= f(\lambda)\psi.
\end{aligned}$$

g)

$$\begin{aligned}
\|\phi(f)\|_{B(H)} &= \|\phi(\lim_{n\to\infty} p_n)\|_{B(H)} \\
&= \|\lim_{n\to\infty} \phi(p_n)\|_{B(H)} \\
&= \lim_{n\to\infty} \|\tilde{\phi}(p_n)\|_{B(H)} \\
&= \lim_{n\to\infty} \|p_n\|_\infty \\
&= \|\lim_{n\to\infty} p_n\|_\infty \\
&= \|f\|_\infty.
\end{aligned}$$

□

Chapter 3

Abstract Calderón–Zygmund theory

Calderón–Zygmund theory was first developed in the middle of the 20th century: its initial setting was the euclidean space $\mathbb{R}^n$ endowed with the euclidean distance and the Lebesgue measure. Since then a lot of effort has been made in order to extend the theory to the setting of abstract metric measure spaces, the key aspect being to identify the essential properties which provide the foundation of the theory. One of the main results consisted in discovering that a sufficient condition to have the Calderón–Zygmund property is the doubling property of the measure. While this result provided a unified solution to the quest for extension for a wide class of spaces, it also revealed a route for further exploration on spaces lacking the doubling property. This is the case for the infinite homogeneous tree which is the subject of the analysis of our work, as it does not exhibit nor the doubling nor the isoperimetric property. It is then significant to investigate the Calderón–Zygmund property for this space and to this extent in this chapter we study the abstract theory proposed by Hebisch and Steger in [HS03].

Troughout we adopt the usual convention that C stands for a positive constant, whose precise value varies from occurrence to occurrence.

3.1 Abstract Calderon–Zygmund property

Definition 3.1.1. We say that a metric measure space (M, d, μ) has the Calderón–Zygmund property if there exists a positive constant C such that:

$$\forall f \in L^1(M,\mu),\ \forall \lambda > C\frac{\|f\|_{L^1(M,\mu)}}{\mu(M)} \quad (\lambda > 0 \text{ if } \mu(M) = \infty)$$

f admits a decomposition of the form

$$f = \sum_i f_i + g \tag{3.1}$$

such that there exist sets Q_i, numbers r_i and points x_i satisfying the following properties:

1. $f_i = 0$ outside Q_i,
2. $\int_M f_i d\mu = 0 \quad \forall i$,
3. $Q_i \subset B(x_i, Cr_i) \quad \forall i$,
4. $\sum_i \mu(Q_i^*) \leq C\frac{\|f\|_{L^1(M,\mu)}}{\lambda}$ where $Q_i^* = \{x \in M : d(x, Q_i) < r_i\}$,
5. $\sum_i \|f_i\|_{L^1(M,\mu)} \leq C\|f\|_{L^1(M,\mu)}$,
6. $|g| \leq C\lambda$.

The sets Q_i are called Calderón–Zygmund sets and the decomposition (3.1) is called the Calderón–Zygmund decomposition of f at level λ.

Since $g = f - \sum_i f_i$ we have that:

$$\begin{aligned}
\|g\|_{L^1(M,\mu)} &= \left\| f - \sum_i f_i \right\|_{L^1(M,\mu)} \\
&\leq \|f\|_{L^1(M,\mu)} + \left\| \sum_i f_i \right\|_{L^1(M,\mu)} \\
&\leq \|f\|_{L^1(M,\mu)} + \sum_i \|f_i\|_{L^1(M,\mu)} \\
&\leq \|f\|_{L^1(M,\mu)} + C\|f\|_{L^1(M,\mu)} \quad \text{by property 5)} \\
&\leq C'\|f\|_{L^1(M,\mu)}
\end{aligned}$$

that is

$$\|g\|_{L^1(M,\mu)} \leq C' \|f\|_{L^1(M,\mu)} . \tag{3.2}$$

Moreover, from Property 6. it follows that

$$g \in L^\infty(M,\mu) \quad \text{and} \quad \|g\|_{L^\infty(M,\mu)} \leq C\lambda. \tag{3.3}$$

Then from the Marcinkiewic interpolation Theorem 2.1.7 it holds that

$$\begin{cases} g \in L^1(M,\mu) \\ g \in L^\infty(M,\mu) \end{cases} \implies$$

$$g \in L^r(M,\mu) \quad \forall r \text{ s.t. } 1 < r < \infty \quad \text{and} \quad \|g\|^r_{L^r(M,\mu)} \leq \frac{r}{r-1} \|g\|_{L^1(M,\mu)} \|g\|^{r-1}_{L^\infty(M,\mu)} .$$

In particular, for $r = 2$ we have:

$$\begin{aligned} \|g\|^2_{L^2(M,\mu)} &\leq 2 \|g\|_{L^1(M,\mu)} \|g\|_{L^\infty(M,\mu)} \\ &\leq 2C' \|f\|_{L^1(M,\mu)} C\lambda \quad \text{for 3.2 and 3.3} \\ &= C''\lambda \|f\|_{L^1(M,\mu)} \end{aligned}$$

that is

$$\|g\|^2_{L^2(M,\mu)} \leq C''\lambda \|f\|_{L^1(M,\mu)} . \tag{3.4}$$

3.2 Abstract Calderón–Zygmund theorem

Given a measurable function $k(x,y)$ locally integrable on $M \times M$, K denotes the integral operator with kernel k defined $\forall f \in C_c(M)$ as:

$$(Kf)(x) = \int_M k(x,y)f(y)d\mu(y) , \qquad \text{for } x \notin \operatorname{supp} f.$$

This integral is well defined at least for compactly supported functions f.

Theorem 3.2.1. *Consider (M,d,μ) with the Calderón–Zygmund property. Suppose that T is a linear operator which is bounded on $L^2(M,\mu)$ and admits a locally integrable kernel $k(x,y)$ that satisfies the condition:*

$$\sup_{Q_i} \sup_{y,z\in Q_i} \int_{(Q_i^*)^c} |k(x,y) - k(x,z)| d\mu(x) < \infty \tag{3.5}$$

where the supremum is taken over all Calderón–Zygmund sets Q_i. Then T extends from $L^1(M,\mu) \cap L^2(M,\mu)$ to a bounded operator from $L^1(M,\mu)$ to $L^{1,\infty}(M,\mu)$ and on $L^p(M,\mu)$, for all $p \in (1,2]$.

Remark If T is a linear operator bounded on $L^2(M,\mu)$ such that

$$T : L^2(M,\mu) \to L^2(M,\mu)$$
$$T = \sum_{n\in\mathbb{Z}} K_n \quad \text{with } K_n \text{ integral operator with kernel } k_n$$

such that for appropriate constants $C > 0$, $0 < c < 1$, $a > 0$, $b > 0$ the following conditions are satisfied:

$$\text{i)} \int_M |k_n(x,y)|(1 + c^n d(x,y))^a d\mu(x) \le C \quad \forall y \in M\,,$$
$$\text{ii)} \int_M |k_n(x,y) - k_n(x,z)| d\mu(x) \le C(c^n d(y,z))^b \quad \forall y,z \in M\,,$$

then T satisfies the hypothesis 3.5 of Theorem 3.2.1. These conditions are formulated by Hebisch and Steger and they are more convenient to verify, even if less intuitive. For this reason we prefer the following formulation of Theorem 3.2.1.

Theorem 3.2.2. *Consider (M,d,μ) with the Calderón–Zygmund property.*

Suppose that T is a linear bounded operator on $L^2(M,\mu)$ such that $T = \sum_{n\in\mathbb{Z}} K_n$ with K_n integral operator with kernel k_n such that for appropriate constants $C > 0$, $0 < c < 1$, $a > 0$, $b > 0$ the following conditions are satisfied:

$$\text{i)} \int_M |k_n(x,y)|(1 + c^n d(x,y))^a d\mu(x) \le C \quad \forall y \in M\,, \tag{3.6}$$
$$\text{ii)} \int_M |k_n(x,y) - k_n(x,z)| d\mu(x) \le C(c^n d(y,z))^b \quad \forall y,z \in M\,. \tag{3.7}$$

Then T is of weak type $(1,1)$ and bounded on $L^p(M,\mu)$ for every $1 < p \le 2$, i.e.:

$$T : L^1(M,\mu) \to L^{1,\infty}(M,\mu) \quad \text{is bounded}$$
$$T : L^p(M,\mu) \to L^p(M,\mu) \quad \text{is bounded for } 1 < p \le 2\,.$$

Before proving Theorem 3.2.2 we state and prove two lemmas.

Lemma 3.2.3. *Let f_i, Q_i, r_i, x_i, Q_i^* as in Definition 3.1.1.*

Then there exists a positive constant C such that for every i:

$$\sum_{n\in\mathbb{Z}: c^n r_i \ge 1} \int_{(Q_i^*)^c} |K_n f_i(x)| d\mu(x) \le C \, \|f_i\|_{L^1(M,\mu)}\,.$$

Proof. We first estimate

$$\begin{aligned}
\int_{(Q_i^*)^c} |K_n f_i(x)| d\mu(x) &= \int_{(Q_i^*)^c} \left| \int_M k_n(x,y) f_i(y) d\mu(y) \right| d\mu(x) \\
&\leq \int_{(Q_i^*)^c} \int_M |k_n(x,y)||f_i(y)| d\mu(y) d\mu(x) \\
&= \int_{(Q_i^*)^c} \int_{Q_i} |k_n(x,y)||f_i(y)| d\mu(y) d\mu(x) \quad \text{since supp } f_i \subseteq Q_i \text{ for 1)} \\
&= \int_{Q_i} \int_{(Q_i^*)^c} |k_n(x,y)||f_i(y)| d\mu(x) d\mu(y) \\
&= \int_{Q_i} |f_i(y)| \left(\int_{(Q_i^*)^c} |k_n(x,y)| d\mu(x) \right) d\mu(y) \\
&= \|f_i\|_{L^1(M,\mu)} \sup_{y \in Q_i} \int_{(Q_i^*)^c} |k_n(x,y)| d\mu(x).
\end{aligned}$$

By definition points in $(Q_i^*)^c$ have distance $\geq r_i$ from each point in Q_i. Then the points x with $d(x,y) \geq r_i$ from a fixed $y \in Q_i$ are a superset of $(Q_i^*)^c$. So we can proceed with the inequalities:

$$\begin{aligned}
\int_{(Q_i^*)^c} |K_n f_i(x)| d\mu(x) &\leq \|f_i\|_{L^1(M,\mu)} \sup_{y \in Q_i} \int_{(Q_i^*)^c} |k_n(x,y)| d\mu(x) \\
&\leq \|f_i\|_{L^1(M,\mu)} \sup_{y \in Q_i} \int_{x: d(x,y) \geq r_i} |k_n(x,y)| d\mu(x) \\
&\leq \|f_i\|_{L^1(M,\mu)} \sup_{y \in M} \int_{x: d(x,y) \geq r_i} |k_n(x,y)| d\mu(x) \\
&\leq (c^n r_i)^{-a} \|f_i\|_{L^1(M,\mu)} \sup_{y \in M} \int_{x: d(x,y) \geq r_i} |k_n(x,y)| (1 + c^n d(x,y))^a d\mu(x) \\
&\leq C (c^n r_i)^{-a} \|f_i\|_{L^1(M,\mu)} \quad \text{for 3.6.}
\end{aligned}$$

where we have used the fact that $\left(\frac{1+c^n d(x,y)}{c^n r_i} \right)^a > 1 \quad \forall x : d(x,y) \geq r_i$. By summing over all indices n such that $c^n r_i \geq 1$ we get:

$$\begin{aligned}
\sum_{n \in \mathbb{Z}: c^n r_i \geq 1} \int_{(Q_i^*)^c} |K_n f_i|(x) d\mu(x) &\leq C \|f_i\|_{L^1(M,\mu)} \sum_{n \in \mathbb{Z}: c^n r_i \geq 1} (c^n r_i)^{-a} \\
&= C \|f_i\|_{L^1(M,\mu)} r_i^{-a} \sum_{n \in \mathbb{Z}: c^n r_i \geq 1} (c^a)^{-n}.
\end{aligned}$$

We have that:

$$
\begin{aligned}
r_i^{-a} \sum_{n\in\mathbb{Z}:c^n r_i\geq 1} (c^a)^{-n} &= r_i^{-a} \sum_{m\in\mathbb{Z}:c^{-m} r_i\geq 1} (c^a)^{m} \\
&= r_i^{-a} \sum_{m=\lceil \frac{log(r_i)}{log(c)}\rceil}^{+\infty} (c^a)^{m}.
\end{aligned}
$$

Indeed, $c^{-m} r_i \geq 1 \;\;\Leftrightarrow -mlog(c) + log(r_i) \geq 0$ where $log(c) < 0$ because $c \in (0,1)$. So the sum is over integer numbers $m \geq \frac{log(r_i)}{log(c)}$. We can express such numbers as $m = k + \lceil \frac{log(r_i)}{log(c)}\rceil$ with $k \in \mathbb{N}$, so we obtain:

$$
\begin{aligned}
r_i^{-a} \sum_{n\in\mathbb{Z}:c^n r_i\geq 1} (c^a)^{-n} &= r_i^{-a} \sum_{m=\lceil \frac{log(r_i)}{log(c)}\rceil}^{+\infty} (c^a)^{m} \\
&= r_i^{-a} \sum_{k=0}^{+\infty} (c^a)^{k+\lceil \frac{log(r_i)}{log(c)}\rceil} \\
&= r_i^{-a} (c^a)^{\lceil \frac{log(r_i)}{log(c)}\rceil} \sum_{k=0}^{+\infty} (c^a)^{k} \\
&\leq r_i^{-a} (c^a)^{\frac{log(r_i)}{log(c)}} \frac{1}{1-c^a} \quad \text{since } c^a < 1 \\
&= r_i^{-a} e^{log(c) a \frac{log(r_i)}{log(c)}} \frac{1}{1-c^a} \\
&= r_i^{-a} e^{a log(r_i)} \frac{1}{1-c^a} \\
&= r_i^{-a} r_i^{a} \frac{1}{1-c^a} \\
&= \frac{1}{1-c^a}.
\end{aligned}
$$

In conclusion:

$$
\begin{aligned}
\sum_{n\in\mathbb{Z}:c^n r_i\geq 1} \int_{(Q_i^*)^c} |K_n f_i|(x) d\mu(x) &\leq C \left\| f_i \right\|_{L^1(M,\mu)} \frac{1}{1-c^a} \\
&\leq C' \left\| f_i \right\|_{L^1(M,\mu)}.
\end{aligned}
$$

□

Lemma 3.2.4. *Let f_i, Q_i, r_i, x_i, Q_i^* as in Definition 3.1.1.*

Then there exists a positive constant C such that for every i:

$$\sum_{n\in\mathbb{Z}:c^n r_i<1}\int_M |K_n f_i(x)|d\mu(x) \le C\,\|f_i\|_{L^1(M,\mu)}\,.$$

Proof. We first estimate:

$$\int_M |K_n f_i(x)|d\mu(x) = \int_M \left|\int_{Q_i} k_n(x,y)f_i(y)d\mu(y)\right| d\mu(x).$$

We have that $\int_{Q_i} f_i(y)d\mu(y) = 0$ by Property 2. so $k_n(x,x_i)\int_{Q_i} f_i(y)d\mu(y) = \int_{Q_i} k_n(x,x_i)f_i(y)d\mu(y) = 0$ and we can proceed with the inequalities:

$$\begin{aligned}
\int_M \left|\int_{Q_i} k_n(x,y)f_i(y)d\mu(y)\right| d\mu(x) &= \int_M \left|\int_{Q_i} (k_n(x,y)-k_n(x,x_i))\, f_i(y)d\mu(y)\right| d\mu(x)\\
&\le \int_M\int_{Q_i} |(k_n(x,y)-k_n(x,x_i))|\,|f_i(y)|d\mu(y)d\mu(x)\\
&= \int_{Q_i}\int_M |(k_n(x,y)-k_n(x,x_i))|\,|f_i(y)|d\mu(x)d\mu(y)\\
&= \int_{Q_i} |f_i(y)|\left(\int_M |k_n(x,y)-k_n(x,x_i)|\,d\mu(x)\right)d\mu(y)\\
&\le \int_{Q_i} |f_i(y)|\sup_{y\in Q_i}\left(\int_M |k_n(x,y)-k_n(x,x_i)|\,d\mu(x)\right)d\mu(y)\\
&\le \|f_i\|_{L^1(M,\mu)}\sup_{y\in Q_i} C\,(c^n d(y,x_i))^b \quad \text{for 3.7}\\
&\le \|f_i\|_{L^1(M,\mu)}\, C'(c^n r_i)^b.
\end{aligned}$$

Where the last inequality follows from the fact that $x_i \in Q_i$, $y \in Q_i$ and from property 3) $Q_i \subset B(x_i, Cr_i)$ so it holds $d(y,x_i) \le Cr_i$. Now we can compute

$$\begin{aligned}
\sum_{n\in\mathbb{Z}:c^n r_i<1}\int_M |K_n f_i|(x)d\mu(x) &\le \sum_{n\in\mathbb{Z}:c^n r_i<1}\|f_i\|_{L^1(M,\mu)}\, C'(c^n r_i)^b\\
&= \|f_i\|_{L^1(M,\mu)}\, C' r_i^b \sum_{n\in\mathbb{Z}:c^n r_i<1}(c^b)^n.
\end{aligned}$$

We have that:

$$r_i^b\sum_{n\in\mathbb{Z}:c^n r_i<1}(c^b)^n = r_i^b\sum_{n=\lfloor-\frac{log(r_i)}{log(c)}\rfloor+1}^{+\infty}(c^b)^n.$$

Indeed, $c^n r_i < 1 \Leftrightarrow n log(c) + log(r_i) < 0$ where $log(c) < 0$ because $c \in (0,1)$. So the sum is over integer numbers $n > -\frac{log(r_i)}{log(c)}$. We can express such numbers as $n = \lfloor -\frac{log(r_i)}{log(c)} \rfloor + 1 + k$ with $k \in \mathbb{N}$, so we obtain:

$$\begin{aligned} r_i^b \sum_{n=\lfloor -\frac{log(r_i)}{log(c)} \rfloor + 1}^{+\infty} (c^b)^n &= r_i^b \frac{(c^b)^{\lfloor -\frac{log(r_i)}{log(c)} \rfloor + 1}}{1-c^b} \\ &\leq \frac{r_i^b}{1-c^b} e^{b log(c)\left(-\frac{log(r_i)}{log c}\right)} \\ &= \frac{r_i^b}{1-c^b} r_i^{-b} \\ &= \frac{1}{1-c^b}. \end{aligned}$$

In conclusion:

$$\begin{aligned} \sum_{n \in \mathbb{Z}: c^n r_i < 1} \int_M |K_n f_i|(x) d\mu(x) &\leq \|f_i\|_{L^1(M,\mu)} C' \frac{1}{1-c^b} \\ &\leq C'' \|f_i\|_{L^1(M,\mu)} . \end{aligned}$$

□

Proof of Theorem 3.2.2. By hypothesis $T : L^2(M,\mu) \to L^2(M,\mu)$ is linear bounded, i.e. T is strong-type $(2,2)$ and then it is also weak-type $(2,2)$.

Thanks to the Marcinkiewic interpolation theorem it is sufficient to show that T is weak-type $(1,1)$ to conclude the proof.

Let us take $f \in L^1(M,\mu)$.

We fix $\lambda > 0$. Then either $\lambda \leq C\frac{\|f\|_{L^1(M,\mu)}}{\mu(M)}$ or $\lambda > C\frac{\|f\|_{L^1(M,\mu)}}{\mu(M)}$, where C is the constant appearing in the Calderón–Zygmund property 3.1.1.

We consider first the case $\lambda > 0$ and $\lambda \leq C\frac{\|f\|_{L^1(M,\mu)}}{\mu(M)}$.

$$\begin{aligned} d_{Tf}(\lambda) &= \mu(\{x \in M : Tf(x) > \lambda\}) \\ &\leq \mu(M) \\ &\leq \frac{C\|f\|_{L^1(M,\mu)}}{\lambda} \end{aligned}$$

and so

$$\lambda d_{Tf}(\lambda) \leq C\|f\|_{L^1(M,\mu)} \quad \forall \lambda : \ 0 < \lambda \leq \frac{C\|f\|_{L^1(M,\mu)}}{\mu(M)}.$$

Now we consider the case $\lambda > C\frac{\|f\|_{L^1(M,\mu)}}{\mu(M)}$.

Since (M, d, μ) has the Calderón–Zygmund property, for such λ and $f \in L^1(M,\mu)$ we have a decomposition $f = \sum_i f_i + g$ which satisfies Properties 1–6 in Definition 3.1.1.

We define the sets:

$$E = \{x \in M : \sum_{n,i} |K_n f_i(x)| > \tfrac{\lambda}{2}\},$$
$$E_1 = \bigcup_i Q_i^*,$$

where Q_i^* are the sets associated to the Calderón–Zygmund decomposition of f at level λ.

We have that:

$$\begin{aligned}
\frac{\lambda}{2}\mu(E \setminus E_1) &\leq \int_{E\setminus E_1} \sum_{n,i} |K_n f_i(x)|\, d\mu(x) \\
&\leq \int_{(E_1)^c} \sum_{n,i} |K_n f_i(x)|\, d\mu(x) \\
&= \sum_i \int_{(E_1)^c} \sum_n |K_n f_i(x)|\, d\mu(x) \\
&= \sum_i \int_{\bigcap_s (Q_s^*)^c} \sum_n |K_n f_i(x)|\, d\mu(x) \quad \text{since } E_1^c = \bigcap_s (Q_s^*)^c \\
&\leq \sum_i \int_{(Q_i^*)^c} \sum_n |K_n f_i(x)|\, d\mu(x)
\end{aligned}$$

since $\bigcap_s (Q_s^*)^c \subset (Q_j^*)^c \; \forall j$ so we can enlarge the domain by integrating each $\sum_n |K_n f_i|$ over the respective $(Q_i^*)^c \supset \bigcap_s (Q_s^*)^c$. So

$$\begin{aligned}
\frac{\lambda}{2}\mu(E \setminus E_1) &\leq \sum_i \left(\sum_n \int_{(Q_i^*)^c} |K_n f_i(x)|\, d\mu(x) \right) \\
&\leq \sum_i C \|f_i\|_{L^1(M,\mu)} \quad \text{for Lemmas 3.2.3 and 3.2.4} \\
&\leq C \sum_i \|f_i\|_{L^1(M,\mu)} \\
&\leq C\tilde{C} \|f\|_{L^1(M,\mu)} \quad \text{from Property 5) of Definition 3.1.1} \\
&\leq C' \|f\|_{L^1(M,\mu)} .
\end{aligned}$$

So:

$$\mu(E \setminus E_1) \leq \frac{2C' \|f\|_{L^1(M,\mu)}}{\lambda}. \tag{3.8}$$

Now we claim that

$$\mu\left(\{x \in M : |Tf(x)| > \lambda\}\right) \leq \mu\left(\{x \in M : |Tg(x)| > \tfrac{\lambda}{2}\}\right) + \mu(E). \tag{3.9}$$

Indeed, consider $x \in M : |Tf(x)| > \lambda$. Then, since $f = \sum_i f_i + g$ and T is linear:

$$\begin{aligned}\lambda < |Tf(x)| &= \left|\sum_i Tf_i(x) + Tg(x)\right| \\ &\leq |Tg(x)| + \left|\sum_{i,n} K_n f_i(x)\right| \\ &\leq |Tg(x)| + \sum_{i,n} |K_n f_i(x)|.\end{aligned}$$

Then one of the following occurs:

$$\begin{aligned}&|Tg(x)| > \tfrac{\lambda}{2} \quad \text{and} \quad \sum_{i,n} |K_n f_i(x)| > \lambda - |Tg(x)| \quad \text{or} \\ &|Tg(x)| \leq \tfrac{\lambda}{2} \quad \text{and} \quad \sum_{i,n} |K_n f_i(x)| > \lambda - |Tg(x)| \geq \tfrac{\lambda}{2}.\end{aligned}$$

So we have shown that

$$\{x \in M : |Tf(x)| > \lambda\} \subset \{x \in M : |Tg(x)| > \tfrac{\lambda}{2}\} \cup \{x \in M : \sum_{i,n} |K_n f_i(x)| > \tfrac{\lambda}{2}\},$$

which proves Claim 3.9. Then we can proceed with the estimate of the distribution function of Tf.

$$\begin{aligned}\mu\left(\{x \in M : |Tf(x)| > \lambda\}\right) &\leq \mu\left(\{x \in M : |Tg(x)| > \tfrac{\lambda}{2}\}\right) + \mu(E) \\ &\leq \frac{4\bar{C}\|g\|^2_{L^2(M,\mu)}}{\lambda^2} + \mu(E_1) + \mu(E \setminus E_1).\end{aligned}$$

The last inequality is justified by the following: we have shown that $g \in L^2(M,\mu)$ (see 3.4) and by hypothesis $T : L^2(M,\mu) \to L^2(M,\mu) \subset L^{2,\infty}(M,\mu)$ is bounded. So $Tg \in L^{2,\infty}(M,\mu)$ and $\|Tg\|_{L^{2,\infty}(M,\mu)} \leq \|Tg\|_{L^2(M,\mu)}$. Then:

$$\|Tg\|^2_{L^{2,\infty}(M,\mu)} = \sup\{\gamma^2 d_{Tg}(\gamma) : \gamma > 0\}$$

$$\begin{aligned}
\implies d_{Tg}(\tfrac{\lambda}{2}) &\leq \frac{\|Tg\|^2_{L^{2,\infty}(M,\mu)}}{\left(\frac{\lambda}{2}\right)^2} \\
&\leq \frac{4\|Tg\|^2_{L^2(M,\mu)}}{\lambda^2} \\
&\leq \frac{4\bar{C}\|g\|^2_{L^2(M,\mu)}}{\lambda^2}.
\end{aligned} \tag{3.10}$$

Moreover,

$$E \subset (E \cup E_1) = E_1 \cup (E \setminus E_1) \implies \mu(E) \leq \mu(E_1) + \mu(E \setminus E_1).$$

We can now proceed with the estimate above applying 3.4 e 3.8:

$$\begin{aligned}
\mu\left(\{x \in M : |Tf(x)| > \lambda\}\right) &\leq \frac{4\bar{C}\,\|g\|^2_{L^2(M,\mu)}}{\lambda^2} + \mu(E_1) + \mu(E \setminus E_1) \\
&\leq \frac{4\bar{C}\,\|g\|^2_{L^2(M,\mu)}}{\lambda^2} + \mu\left(\bigcup_i Q_i^*\right) + \frac{2C'\,\|f\|_{L^1(M,\mu)}}{\lambda} \\
&\leq \frac{C''\lambda\,\|f\|_{L^1(M,\mu)}}{\lambda^2} + \sum_i \mu(Q_i^*) + \frac{2C'\,\|f\|_{L^1(M,\mu)}}{\lambda} \\
&\leq \frac{C''\,\|f\|_{L^1(M,\mu)}}{\lambda} + \frac{C\,\|f\|_{L^1(M,\mu)}}{\lambda} + \frac{2C'\,\|f\|_{L^1(M,\mu)}}{\lambda} \\
&\leq \frac{C'''\,\|f\|_{L^1(M,\mu)}}{\lambda}.
\end{aligned}$$

In this way we have shown that

$$\lambda d_{Tf}(\lambda) \leq C'''\,\|f\|_{L^1(M,\mu)} \quad \forall \lambda > \frac{C\,\|f\|_{L^1(M,\mu)}}{\mu(M)}.$$

We can now conclude the proof:

$$\|Tf\|_{L^{1,\infty}(M,\mu)} = \sup\{\lambda d_{Tf}(\lambda) : \lambda > 0\} \leq \tilde{C}\|f\|_{L^1(M,\mu)}\,,$$

thus T is of weak type $(1,1)$. □

Chapter 4

Weighted homogeneous trees

In this chapter we introduce the infinite homogeneous tree and we describe how it can be equipped with suitable distance and measure. We study the properties of the corresponding metric measure space $(\mathcal{V}, d, \mu)$, such as the lack of the doubling property. These properties motivate the need for the abstract Calderón–Zygmund theory presented in Chapter 3, which will be applied to the tree in the next chapter. We also introduce various operators over $(\mathcal{V}, d, \mu)$, including laplacian operators, which will be studied in Chapter 6.

4.1 The infinite homogeneous tree T

Definition 4.1.1. An infinite homogeneous tree of order $q + 1$ is a graph $T = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes the set of vertices and $\mathcal{E}$ denotes the set of edges, with the following properties:

- T is connected and acyclic;
- each vertex has exactly $q + 1$ neighbours.

On $\mathcal{V}$ we can define the distance $d(x, y)$ between two vertices x and y as the length of the shortest path between x and y. We also fix an infinite geodesic g in T, that is a connected subset $g \subset \mathcal{V}$ such that:

- for each element $v \in g$ there are exactly two neighbours of v in g;
- for every couple (u, v) of elements in g, the shortest path joining u and v is contained in g.

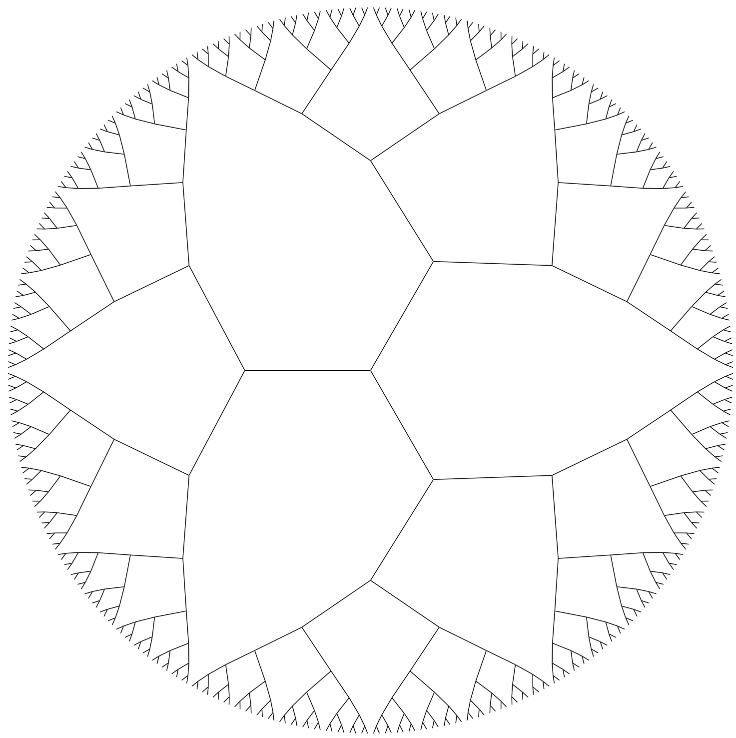


Figure 4.1: A representation of the infinite homogeneous tree of order 3.

We define a mapping $N : g \to \mathbb{Z}$ such that:

$$|N(x) - N(y)| = d(x, y) \qquad \forall x, y \in g\,. \tag{4.1}$$

This corresponds to the choice of an origin $o \in g$ (the only vertex for which $N(o) = 0$) and an orientation for g; in this way we obtain a numeration of the vertices in g.

We define the level function $l : \mathcal{V} \to \mathbb{Z}$ as:

$$l(x) = N(x') - d(x, x')$$

where x' is the only vertex in g such that $d(x, x') = \min\{d(x, z) : z \in g\}$.

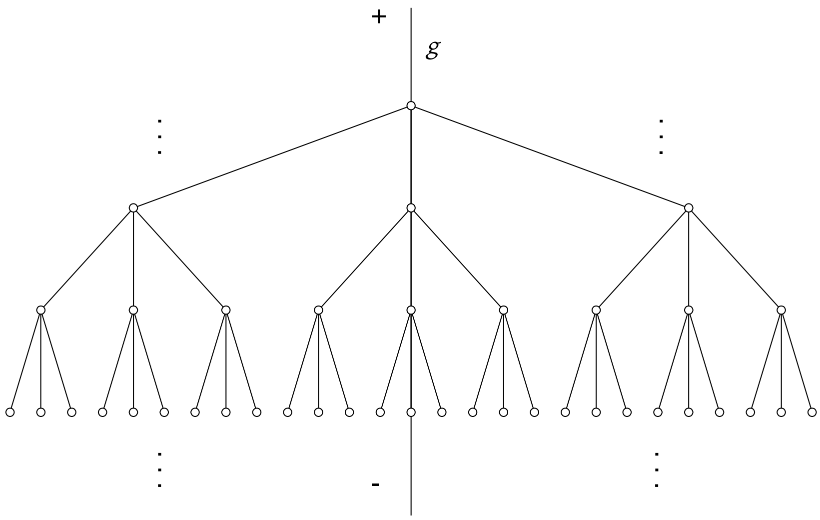


Figure 4.2: Representation of a portion of T with $q = 3$

Let μ be the measure on T defined by the formula:

$$\int_{\mathcal{V}} f \, d\mu = \sum_{x \in \mathcal{V}} f(x) q^{l(x)}. \tag{4.2}$$

Then μ is a weighted counting measure:

- the weight of a vertex depends only on its level;
- the weight associated to a certain level is given by q times the weight of the level immediately underneath (note that for each vertex in the higher level there are q vertices on the lower level).

4.2 Laplacian operators on T

We denote by $C(\mathcal{V})$ the space of complex valued functions defined on the nodes of the infinite homogeneous tree, that is:

$$C(\mathcal{V}) = \{f : \mathcal{V} \to \mathbb{C}\} .$$

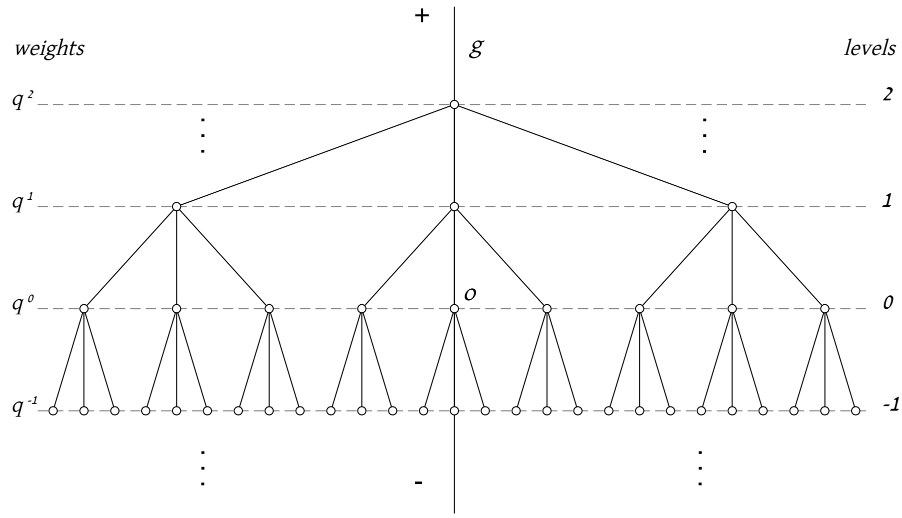


Figure 4.3: Representation of the measure μ ($q = 3$)

Consider a function $f \in C(\mathcal{V})$.

Definition 4.2.1. We define the operator A by the formula:

$$(Af)(x) = \frac{1}{2\sqrt{q}} \sum_{y \in \mathcal{V}: d(x,y)=1} q^{\frac{l(y)-l(x)}{2}} f(y) \qquad \forall x \in \mathcal{V}. \tag{4.3}$$

We observe that the difference between levels of vertices involved in the sum defining $(Af)(x)$ can be either $+1$ or -1. In particular:

$$l(y) - l(x) = \begin{cases} +1 & \text{for just one neighbour of } x \\ -1 & \text{for } q \text{ neighbours of } x \end{cases}$$

This implies that $\forall x \in \mathcal{V}$

$$\frac{q^{\frac{l(y)-l(x)}{2}}}{\sqrt{q}} = \begin{cases} 1 & \text{for just one neighbour of } x \\ \frac{1}{q} & \text{for } q \text{ neighbours of } x \end{cases}$$

More precisely, we define the sets of vertices:

$$V^{+}(x) = \{y \in \mathcal{V} : d(x,y) = 1 \ , \ l(y) = l(x) + 1\} \, ,$$

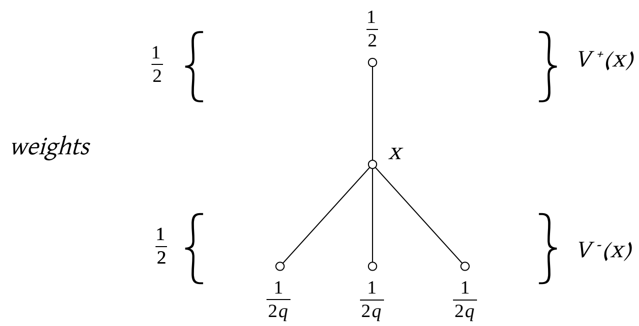
$$V^-(x) = \{y \in \mathcal{V} : d(x,y) = 1 \ , \ l(y) = l(x) - 1\}\,.$$

We call $V^+(x)$ the set of parents of x (there is only one parent for every x) and we call $V^-(x)$ the set of children of x (every vertex has q children).

Then we can express $(Af)(x)$ as follows:

$$(Af)(x) = \frac{1}{2} \sum_{y \in V^+(x)} f(y) + \frac{1}{2} \left(\frac{1}{q} \sum_{y \in V^-(x)} f(y) \right) \qquad \forall x \in \mathcal{V}.$$

In this way we see that $(Af)(x)$ is a weighted average of the values $f(y)$ on the neighborhood of x: the value of f on the neighbor on the upper level has weight $\frac{1}{2}$ while the values of f on each one of the q neighbors on the lower level have weight $\frac{1}{2q}$, so that each level contributes for a weight of $\frac{1}{2}$.



Representation of the action of A ($q = 3$)

4.2.1 Properties of the operator A

Proposition 4.2.1. The operator A satisfies the following properties:

i) $A : L^1(\mathcal{V}, \mu) \to L^1(\mathcal{V}, \mu)$ is a bounded linear operator of norm 1;

ii) $A : L^\infty(\mathcal{V},\mu) \to L^\infty(\mathcal{V},\mu)$ is a bounded linear operator and $\|A\|_{B(L^\infty(\mathcal{V},\mu))} \leq 1$;

iii) $A : L^p(\mathcal{V},\mu) \to L^p(\mathcal{V},\mu)$ is a bounded linear operator for $1 < p < \infty$ and $\|A\|_{B(L^p(\mathcal{V},\mu))} \leq 1$;

iv) $A \in B(L^2(\mathcal{V},\mu))$ is self-adjoint on $L^2(\mathcal{V},\mu)$.

Proof. We first prove i). Take $f \in L^1(\mathcal{V},\mu)$. Then:

$$
\begin{aligned}
\|Af\|_{L^1(\mathcal{V},\mu)} &= \sum_{x\in\mathcal{V}} |Af(x)|\, q^{l(x)} = \sum_{x\in\mathcal{V}} \left| \frac{1}{2} \sum_{y\in V^+(x)} f(y) + \frac{1}{2}\left(\frac{1}{q} \sum_{y\in V^-(x)} f(y) \right) \right| q^{l(x)} \\
&\leq \sum_{x\in\mathcal{V}} \left(\frac{1}{2} \sum_{y\in V^+(x)} |f(y)| + \frac{1}{2q} \sum_{y\in V^-(x)} |f(y)| \right) q^{l(x)} \\
&= \frac{1}{2} \sum_{x\in\mathcal{V}} \sum_{y\in V^+(x)} |f(y)|\, q^{l(x)} + \frac{1}{2q} \sum_{x\in\mathcal{V}} \sum_{y\in V^-(x)} |f(y)|\, q^{l(x)} \\
&= \frac{1}{2} \sum_{y\in\mathcal{V}} \sum_{x\in V^-(y)} |f(y)|\, q^{l(x)} + \frac{1}{2q} \sum_{y\in\mathcal{V}} \sum_{x\in V^+(y)} |f(y)|\, q^{l(x)} \\
&= \frac{1}{2} \sum_{y\in\mathcal{V}} |f(y)| \sum_{x\in V^-(y)} q^{l(x)} + \frac{1}{2q} \sum_{y\in\mathcal{V}} |f(y)| \sum_{x\in V^+(y)} q^{l(x)} \\
&= \sum_{y\in\mathcal{V}} |f(y)| \left(\frac{1}{2} \sum_{x\in V^-(y)} q^{l(x)} + \frac{1}{2q} \sum_{x\in V^+(y)} q^{l(x)} \right) \\
&= \sum_{y\in\mathcal{V}} |f(y)| \left(\frac{1}{2} \sum_{x\in V^-(y)} q^{l(y)-1} + \frac{1}{2q} \sum_{x\in V^+(y)} q^{l(y)+1} \right) \\
&= \sum_{y\in\mathcal{V}} |f(y)|\, q^{l(y)} \left(\frac{1}{2} \sum_{x\in V^-(y)} \frac{1}{q} + \frac{1}{2q} \sum_{x\in V^+(y)} q \right) \\
&= \sum_{y\in\mathcal{V}} |f(y)|\, q^{l(y)} \left(\frac{1}{2q} q + \frac{1}{2} 1 \right) \\
&= \sum_{y\in\mathcal{V}} |f(y)|\, q^{l(y)} \\
&= \|f\|_{L^1(\mathcal{V},\mu)} .
\end{aligned}
$$

Note that $\|A\|_{L^1(\mathcal{V},\mu)\to L^1(\mathcal{V},\mu)} = 1$, indeed if we repeat the steps with f positive the inequality becomes an equality.

We now prove (ii): Let us consider $f \in L^\infty(\mathcal{V},\mu)$, that is $\sup_{x\in\mathcal{V}} |f(x)| \le M$ for some $M > 0$. Then:

$$\begin{aligned}
|Af(x)| &= \left| \frac{1}{2} \sum_{y\in V^+(x)} f(y) + \frac{1}{2}\left(\frac{1}{q} \sum_{y\in V^-(x)} f(y) \right) \right| \\
&= \frac{1}{2} \sum_{y\in V^+(x)} |f(y)| + \frac{1}{2q} \sum_{y\in V^-(x)} |f(y)| \\
&= \frac{1}{2} \sum_{y\in V^+(x)} M + \frac{1}{2q} \sum_{y\in V^-(x)} M \\
&= \frac{1}{2} 1M + \frac{1}{2q} qM = M\,.
\end{aligned}$$

This proves (ii). Property (iii) follows by the Marcinkiewicz interpolation theorem 2.1.7.

To prove (iv), let $f, g \in L^2(\mathcal{V},\mu)$:

$$\begin{aligned}
\langle Af, g\rangle_{L^2(\mathcal{V},\mu)} &= \sum_{x\in\mathcal{V}} (Af)(x)\overline{g(x)} q^{l(x)} \\
&= \sum_{x\in\mathcal{V}} \left(\frac{1}{2} \sum_{y\in V^+(x)} f(y) + \frac{1}{2}\left(\frac{1}{q} \sum_{y\in V^-(x)} f(y) \right)\right) \overline{g(x)} q^{l(x)} \\
&= \frac{1}{2} \sum_{x\in\mathcal{V}} \sum_{y\in V^+(x)} f(y)\overline{g(x)} q^{l(x)} + \frac{1}{2q} \sum_{x\in\mathcal{V}} \sum_{y\in V^-(x)} f(y)\overline{g(x)} q^{l(x)} \\
&= \frac{1}{2} \sum_{y\in\mathcal{V}} \sum_{x\in V^-(y)} f(y)\overline{g(x)} q^{l(y)-1} + \frac{1}{2q} \sum_{y\in\mathcal{V}} \sum_{x\in V^+(y)} f(y)\overline{g(x)} q^{l(y)+1} \\
&= \sum_{y\in\mathcal{V}} f(y) q^{l(y)} \left(\frac{1}{2q} \sum_{x\in V^-(y)} \overline{g(x)} \right) + \sum_{y\in\mathcal{V}} f(y) q^{l(y)} \left(\frac{1}{2} \sum_{x\in V^+(y)} \overline{g(x)} \right) \\
&= \sum_{y\in\mathcal{V}} f(y) q^{l(y)} \overline{\left(\frac{1}{2} \sum_{x\in V^+(y)} g(x) + \frac{1}{2q} \sum_{x\in V^-(y)} g(x) \right)} \\
&= \sum_{y\in\mathcal{V}} f(y) \overline{(Ag)(y)} q^{l(y)} \\
&= \langle f, Ag\rangle_{L^2(\mathcal{V},\mu)}\,.
\end{aligned}$$

□

4.2.2 The spectrum of the operator A

It follows from the previous results that the $L^2(\mathcal{V},\mu)$ spectrum of A is real and lies in $[-1,1]$. In this section we prove a more precise statement.

Definition 4.2.2. For every $f \in C(\mathcal{V})$ we define the operator L by:

$$(Lf)(x) = \sum_{y:d(x,y)=1} f(y)$$

The following relationship between the operators A and L holds.

Proposition 4.2.2. For A and L as defined before it holds the following:

$$2\sqrt{q}\, q^{\frac{l}{2}} A = L q^{\frac{l}{2}} \tag{4.4}$$

where $q^{\frac{l}{2}}$ denotes the multiplication operator defined for every $f \in C(\mathcal{V})$ by:

$$(q^{\frac{l}{2}} f)(x) = q^{\frac{l(x)}{2}} f(x).$$

Proof.

$$\begin{aligned}
(Lq^{\frac{l}{2}})(f)(x) &= \sum_{y:d(x,y)=1} (q^{\frac{l}{2}} f)(y) \\
&= \sum_{y:d(x,y)=1} q^{\frac{l(y)}{2}} f(y) \\
&= \sum_{y\in V^+(x)} q^{\frac{l(x)+1}{2}} f(y) + \sum_{y\in V^-(x)} q^{\frac{l(x)-1}{2}} f(y) \\
&= q^{\frac{l(x)}{2}} \left(\sqrt{q} \sum_{y\in V^+(x)} f(y) + \frac{1}{\sqrt{q}} \sum_{y\in V^-(x)} f(y) \right) \\
&= 2\sqrt{q}\, q^{\frac{l(x)}{2}} \left(\frac{1}{2} \sum_{y\in V^+(x)} f(y) + \frac{1}{2q} \sum_{y\in V^-(x)} f(y) \right) \\
&= 2\sqrt{q} q^{\frac{l(x)}{2}} Af(x) \\
&= 2\sqrt{q} \left(q^{\frac{l}{2}} A \right) (f)(x).
\end{aligned}$$

□

We observe the following interesting fact.

Proposition 4.2.3.

$$q^{\frac{l}{2}} : L^2(\mathcal{V},\mu) \longrightarrow L^2(\mathcal{V},\#)$$

is a surjective linear isometry, where # denotes the counting measure. Then $L^2(\mathcal{V},\mu)$ and $L^2(\mathcal{V},\#)$ are isometrically isomorphic.

Proof.

Let $f \in L^2(\mathcal{V},\mu)$.

Then $q^{\frac{l}{2}} f \in L^2(\mathcal{V},\#)$ since

$$\begin{aligned} \|q^{\frac{l}{2}} f\|^2_{L^2(\mathcal{V},\#)} &= \sum_{x\in\mathcal{V}} \left| q^{\frac{l(x)}{2}} f(x) \right|^2 \\ &= \sum_{x\in\mathcal{V}} |f(x)|^2 q^{l(x)} \\ &= \|f\|^2_{L^2(\mathcal{V},\mu)}. \end{aligned}$$

This shows that $q^{\frac{l}{2}}$ is a linear isometry (and so it is injective).

Moreover, $q^{\frac{l}{2}}$ is surjective. To prove it we define the operator

$$q^{-\frac{l}{2}} : f(x) \longrightarrow (q^{-\frac{l}{2}} f)(x) = q^{-\frac{l(x)}{2}} f(x)$$

for every $f \in C(\mathcal{V})$ and show that $q^{-\frac{l}{2}} : L^2(\mathcal{V},\#) \longrightarrow L^2(\mathcal{V},\mu)$ is again a linear isometry.

Let $f \in L^2(\mathcal{V},\#)$. Then $q^{-\frac{l}{2}} f \in L^2(\mathcal{V},\mu)$, indeed

$$\begin{aligned} \|q^{-\frac{l}{2}} f\|^2_{L^2(\mathcal{V},\mu)} &= \sum_{x\in\mathcal{V}} \left| q^{-\frac{l(x)}{2}} f(x) \right|^2 q^{l(x)} \\ &= \sum_{x\in\mathcal{V}} |f(x)|^2 \\ &= \|f\|^2_{L^2(\mathcal{V},\#)}. \end{aligned}$$

Finally, given $f \in L^2(\mathcal{V},\#)$, f is the image of $q^{-\frac{l}{2}} f \in L^2(\mathcal{V},\mu)$ through $q^{\frac{l}{2}}$:

$$q^{\frac{l}{2}} \left(q^{-\frac{l}{2}} f \right) = f$$

so $q^{\frac{l}{2}}$ is surjective and $q^{-\frac{l}{2}}$ is its inverse operator. □

Going back to the spectrum of A, we can prove the following.

Proposition 4.2.4. The $L^2(\mathcal{V},\mu)$ spectrum of A is precisely:

$$\sigma(A) = [-1, 1]. \tag{4.5}$$

Proof. We can exploit the relationship (4.4) between A and L to compute $\sigma(A)$ given $\sigma(L)$.

We write (4.4) in the form:

$$A = \frac{1}{2\sqrt{q}}\left(q^{-\frac{l}{2}} L q^{+\frac{l}{2}}\right)$$

and observe that

$$\begin{aligned}
\lambda \in \sigma(L) \Leftrightarrow & \ L - \lambda I \quad \text{is not invertible} \\
\Leftrightarrow & \ \frac{1}{2\sqrt{q}} q^{-\frac{l}{2}} (L - \lambda I)\, q^{+\frac{l}{2}} \quad \text{is not invertible} \\
\Leftrightarrow & \ \frac{1}{2\sqrt{q}}\left(q^{-\frac{l}{2}} L\, q^{+\frac{l}{2}} - \lambda I\right) \quad \text{is not invertible} \\
\Leftrightarrow & \ A - \frac{\lambda}{2\sqrt{q}} I \quad \text{is not invertible} \\
\Leftrightarrow & \ \frac{\lambda}{2\sqrt{q}} \in \sigma(A).
\end{aligned}$$

The L^p spectrum of the laplacian $\mathbb{L} = I - \frac{1}{q+1}L$ is known in literature (we refer to [FTP83]). In particular for the L^2 spectrum we have that:

$$\sigma(\mathbb{L}) = [1 - \gamma(0), 1 + \gamma(0)]$$

where γ is the function defined by the formula:

$$\gamma(z) = \frac{q^{\frac{1}{2}}}{q+1}\left(q^{iz} + q^{-iz}\right).$$

So:

$$\begin{aligned}
& \sigma(\mathbb{L}) = \left[1 - \frac{2\sqrt{q}}{q+1}, 1 + \frac{2\sqrt{q}}{q+1}\right] \\
\Rightarrow & \ \sigma(L) = [-2\sqrt{q}, 2\sqrt{q}] \\
\Rightarrow & \ \sigma(A) = [-1, 1]\ .
\end{aligned}$$

□

4.2.3 The Laplacian $\mathcal{L}$

Definition 4.2.3. On the infinite homogeneous tree $T = (\mathcal{V}, \mathcal{E})$ with measure μ and distance d we define the laplacian $\mathcal{L}$ as

$$\mathcal{L} = I - A.$$

For each vertex $x \in \mathcal{V}$ we can fix a labeling of the neighbors of x in such a way that the father of x is labelled with 0 and the children of x are labeled with j for $j = 1, \dots, q$.

In the following we denote by $v_j(x),\ j = 0, \dots, q$ the j^{th} neighbor of x.

Definition 4.2.4. We define the linear operators $X_j,\ j = 0, \dots, q$ which take a function $f \in C(\mathcal{V})$ and give the functions $X_j f$ defined by:

$$(X_j f)(x) = f(v_j(x)) - f(x)\,. \tag{4.6}$$

$X_j f$ can be thought as a "first derivative" of the function f.

Definition 4.2.5. For every function $f \in C(\mathcal{V})$ we also define the gradient ∇f by the formula:

$$(\nabla f)(x) = \sum_{y \in \mathcal{V}: d(x,y)=1} |f(y) - f(x)| \qquad \forall x \in \mathcal{V}\,. \tag{4.7}$$

We notice that:

$$\nabla f(x) = \sum_{j=0}^{q} |X_j f(x)|\,.$$

We now show that there is an interesting link between laplacian and gradient on the tree, since they can both be expressed in terms of the operators X_j.

First of all, we compute the adjoint operators of $X_j,\ j = 0, \dots, q$.

$$\begin{aligned}\langle X_j f, g\rangle &= \sum_{x \in \mathcal{V}} (X_j f)(x)\overline{g(x)} q^{l(x)} \\ &= \sum_{x \in \mathcal{V}} \left(f(v_j(x)) - f(x)\right) \overline{g(x)} q^{l(x)}.\end{aligned}$$

We must distinguish the cases:

- $j = 0$, i.e. $v_j(x)$ is the father of x.

$$\begin{aligned}
\langle X_0 f, g\rangle &= \sum_{x\in\mathcal{V}} \left(f(v_0(x)) - f(x)\right)\overline{g(x)}q^{l(x)} \\
&= \sum_{x\in\mathcal{V}} f(v_0(x))\overline{g(x)}q^{l(x)} - \sum_{x\in\mathcal{V}} f(x)\overline{g(x)}q^{l(x)} \\
&= \sum_{x\in\mathcal{V}} \sum_{y\in V^+(x)} f(y)\overline{g(x)}q^{l(x)} - \sum_{x\in\mathcal{V}} f(x)\overline{g(x)}q^{l(x)} \\
&= \sum_{y\in\mathcal{V}} \sum_{x\in V^-(y)} f(y)\overline{g(x)}q^{l(x)} - \sum_{x\in\mathcal{V}} f(x)\overline{g(x)}q^{l(x)} \\
&= \sum_{y\in\mathcal{V}} \sum_{x\in V^-(y)} f(y)\overline{g(x)}q^{l(y)-1} - \sum_{x\in\mathcal{V}} f(x)\overline{g(x)}q^{l(x)} \\
&= \sum_{y\in\mathcal{V}} \sum_{j=1}^{q} f(y)\overline{g(v_j(y))}q^{l(y)-1} - \sum_{y\in\mathcal{V}} f(y)\overline{g(y)}q^{l(y)} \\
&= \sum_{y\in\mathcal{V}} f(y)\left(\frac{1}{q}\sum_{j=1}^{q}\overline{g(v_j(y))} - \overline{g(y)}\right)q^{l(y)} \\
&= \sum_{y\in\mathcal{V}} f(y)\overline{\left(\frac{1}{q}\sum_{j=1}^{q} g(v_j(y)) - g(y)\right)}q^{l(y)} \\
&= \langle f, X_0^* g\rangle .
\end{aligned}$$

This shows that:

$$\begin{aligned}
(X_0^* g)(x) &= \frac{1}{q}\sum_{j=1}^{q} g(v_j(x)) - g(x) \\
&= \frac{1}{q}\sum_{j=1}^{q} \left(X_j g\right)(x) \\
&= \left(\frac{1}{q}\sum_{j=1}^{q} X_j g\right)(x).
\end{aligned}$$

That is:

$$X_0^* = \frac{1}{q}\sum_{j=1}^{q} X_j. \tag{4.8}$$

- $j \neq 0$, i.e. $v_j(x)$ is a child of x. To treat this case it is useful to define the set of vertices y which are the j^{th} child of their father:

$$F_j = \{y \in \mathcal{V} : y = v_j(v_0(y))\}\,.$$

$$\begin{aligned}
\langle X_j f, g\rangle &= \sum_{x\in\mathcal{V}} \left(f(v_j(x)) - f(x)\right) \overline{g(x)} q^{l(x)} \\
&= \sum_{x\in\mathcal{V}} f(v_j(x))\overline{g(x)} q^{l(x)} - \sum_{x\in\mathcal{V}} f(x)\overline{g(x)} q^{l(x)} \\
&= \sum_{y\in F_j} f(y)\overline{g(v_0(y))} q^{l(y)+1} - \sum_{y\in\mathcal{V}} f(y)\overline{g(y)} q^{l(y)} \\
&= \sum_{y\in\mathcal{V}} f(y) \left(q\,\overline{g(v_0(y))}\chi_{F_j}(y) - \overline{g(y)}\right) q^{l(y)} \\
&= \sum_{y\in\mathcal{V}} f(y) \overline{\left(q\,g(v_0(y))\chi_{F_j}(y) - g(y)\right)} q^{l(y)} \\
&= \left\langle f, X_j^* g\right\rangle
\end{aligned}$$

where χ_{F_j} is the characteristic function of the set F_j.

This shows that:

$$(X_j^* g)(x) = q\,g(v_0(x))\chi_{F_j}(x) - g(x) \quad \text{for } j \neq 0. \tag{4.9}$$

We are now in the position to show the relationship between the Laplacian $\mathcal{L}$ and the operators X_j which we introduced before.

Proposition 4.2.5. Let A and X_j be as previously defined. We have that:

$$\mathcal{L} = I - A = \frac{1}{2(q+1)} \sum_{j=0}^{q} X_j^* X_j. \tag{4.10}$$

Proof. First we compute $X_j^* X_j$ for $j = 0, \ldots, q$. As before we distinguish two cases.

- $j = 0$:

$$X_0^* X_0 = \left(\frac{1}{q}\sum_{j=1}^{q} X_j\right) X_0 = \frac{1}{q}\sum_{j=1}^{q} X_j X_0$$

so we have to compute the composition X_jX_0 for all $j \neq 0$.

$$\begin{aligned}((X_jX_0)f)(x) &= (X_0f)(v_j(x)) - (X_0f)(x)\\ &= f(v_0(v_j(x))) - f(v_j(x)) - f(v_0(x)) + f(x)\\ &= f(x) - f(v_j(x)) - f(v_0(x)) + f(x).\end{aligned}$$

Then we have:

$$\begin{aligned}((X_0^*X_0)f)(x) &= \frac{1}{q}\sum_{j=1}^{q}(f(x) - f(v_j(x)) - f(v_0(x)) + f(x))\\ &= 2f(x) - f(v_0(x)) - \frac{1}{q}\sum_{j=1}^{q} f(v_j(x))\\ &= 2\left(f(x) - \frac{1}{2}f(v_0(x)) - \frac{1}{2q}\sum_{j=1}^{q} f(v_j(x))\right)\\ &= 2\,((I-A)f)(x).\end{aligned}$$

- $j \neq 0$.

$$\begin{aligned}\big((X_j^*X_j)f\big)(x) &= q(X_jf)(v_0(x))\chi_{F_j}(x) - (X_jf)(x)\\ &= q\,(f(v_j(v_0(x)) - f(v_0(x)))\,\chi_{F_j}(x) - f(v_j(x)) + f(x)\\ &= q\,f(x)\chi_{F_j}(x) - q\,f(v_0(x))\chi_{F_j}(x) - f(v_j(x)) + f(x).\end{aligned}$$

Then we have:

$$\begin{aligned}\sum_{j=1}^{q}\big((X_j^*X_j)f\big)(x) &= \sum_{j=1}^{q}\big(q\,f(x)\chi_{F_j}(x) - q\,f(v_0(x))\chi_{F_j}(x) - f(v_j(x)) + f(x)\big)\\ &= 2qf(x) - qf(v_0(x)) - \sum_{j=1}^{q} f(v_j(x))\\ &= 2qf(x) - 2q\left(\frac{1}{2}f(v_0(x)) + \frac{1}{2q}\sum_{j=1}^{q} f(v_j(x))\right)\\ &= 2q\,((I-A)f)(x).\end{aligned}$$

Summing up, we have:

$$\begin{aligned}\sum_{j=0}^{q}\big((X_j^*X_j)f\big)(x) &= 2\,((I-A)f)(x) + 2q\,((I-A)f)(x)\\ &= 2(q+1)\,((I-A)f)(x).\end{aligned}$$

□

4.3 Properties of the metric measure space $(\mathcal{V}, d, \mu)$

4.3.1 The measure of spheres and balls

In this section we shall compute the μ-measure of the sphere $S_r(x_0)$ of radius r centered in $x_0 \in \mathcal{V}$ defined by:

$$S_r(x_0) = \{x \in \mathcal{V} : d(x, x_0) = r\}$$

and of the ball $B_r(x_0)$ of radius r centered in $x_0 \in \mathcal{V}$ defined by:

$$B_r(x_0) = \{x \in \mathcal{V} : d(x, x_0) \leq r\}\,.$$

Let $x_0 \in \mathcal{V}$ with $l(x_0) = l$. To compute $\mu(S_r(x_0)) = \sum_{x \in \mathcal{V}: d(x,x_0)=r} q^{l(x)}$ we have to count how many vertices with distance r from x_0 are contained in $\mathcal{V}$ for every level.

First we give a useful definition.

Definition 4.3.1. We say that y *lies above* x if

$$l(x) = l(y) - d(x, y).$$

In this case we also say that x *lies below* y.

Now fix x_0 and $r > 0$ and denote by l the level of x_0. We can see that:

- there are 0 vertices x s.t. $d(x, x_0) = r$ at level $l(x) > l + r$.
- there is exactly 1 vertex x s.t. $d(x, x_0) = r$ at level $l(x) = l + r$.

  This is the vertex that one reaches starting from x_0 and moving for r times to the vertex above the current one.
- there are $(q - 1)$ vertices x s.t. $d(x, x_0) = r$ at level $l(x) = (l + r) - 2$.

  These are the vertices that can be reached from x_0 moving $r - 1$ times to the only vertex above the current one (reaching level $(l+r)-1$) and then taking one step down to a child node not already visited (there are $q - 1$ such nodes).

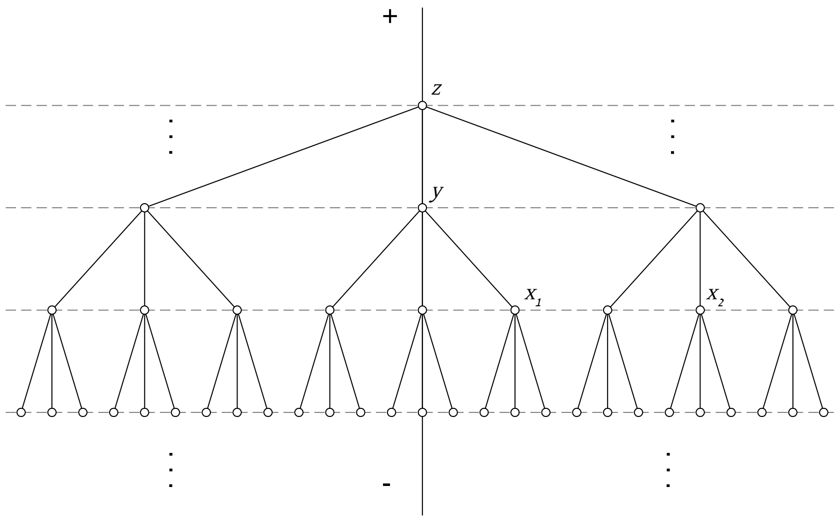


Figure 4.4: In this example, with $q = 3$, y lies above x_1 and below z, while it lies nor above nor below of x_2.

- there are $q(q-1)$ vertices x s.t. $d(x, x_0) = r$ at level $l(x) = (l+r) - 4$.

  These are the vertices that can be reached from x_0 moving $r-2$ times to the only vertex above the current one (reaching level $(l+r)-2$), then taking one step down to a child node not already visited (there are $q-1$ such nodes) and finally another step down to any of the q children.

  $\vdots$

- there are q^r vertices x s.t. $d(x, x_0) = r$ at level $l(x) = l - r$.

  These are the vertices that one reaches by moving r steps down starting from x_0 (every time there are q possible choices for the child node).

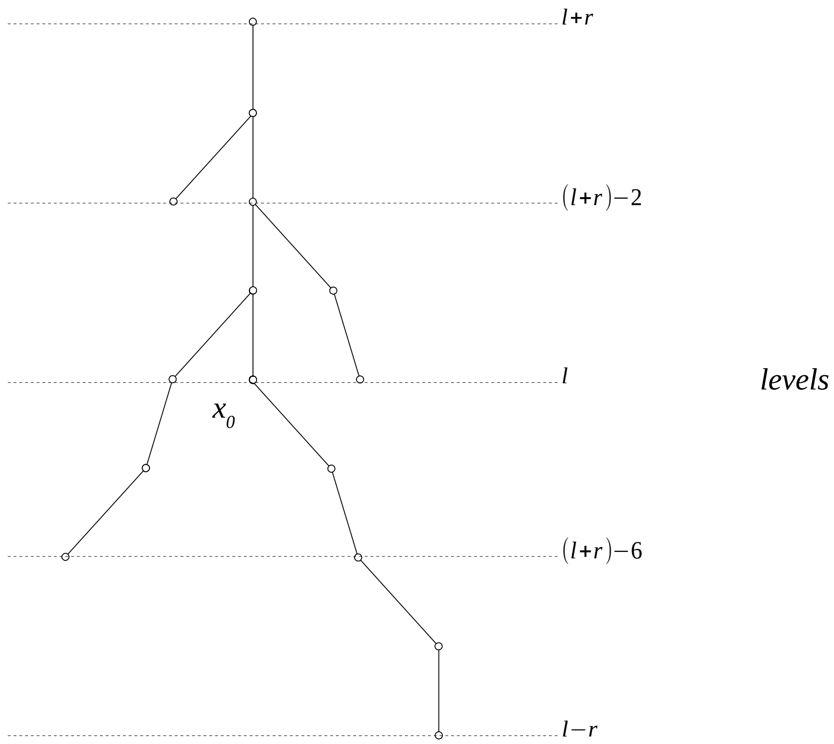


Figure 4.5: Representation of one element of the sphere $S_r(x_0)$ for every level ($q = 3$ and $r = 4$)

In general, if we denote with $n_{r,x_0}(L)$ the number of vertices with distance r from x_0 and level L we have:

$$n_{r,x_0}(L) = \begin{cases} 1 & \text{if} \quad L = l + r \\ (q-1)q^k & \text{if} \quad L = (l+r-2) - 2k = (l+r) - 2(k+1) \\ & \quad \text{with} \quad k = 0, 1, \ldots, r-2 \\ q^r & \text{if} \quad L = l - r \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for $r \geq 2$:

$$\begin{aligned} \mu(S_r(x_0)) &= \sum_{x \in \mathcal{V}: d(x,x_0)=r} q^{l(x)} \\ &= \sum_{L=l-r}^{l+r} n_{r,x_0}(L) q^L \\ &= q^{r+l-r} + q^{l+r} + \sum_{k=0}^{r-2} (q-1) q^k q^{(l+r-2)-2k} \\ &= q^l(q^r + 1) + (q-1) q^{l+r-2} \sum_{k=0}^{r-2} q^{-k} \\ &= q^l(q^r + 1) + (q-1) q^{l+r-2} \frac{1 - (\frac{1}{q})^{r-1}}{1 - \frac{1}{q}} \\ &= q^l(q^r + 1) + (q-1) q^{l+r-2} \frac{1 - q^{(1-r)}}{q-1} q \\ &= q^l(q^r + 1) + q^{(l+r-1)}(1 - q^{(1-r)}) \\ &= q^l(q^r + 1) + q^{(l+r-1)} - q^l \\ &= q^l(q^r + 1 + q^{(r-1)} - 1) \\ &= q^{l+r-1}(1 + q). \end{aligned}$$

For $r = 0$ and $r = 1$ we can compute directly:

$$\begin{aligned} \mu(S_0(x_0)) &= q^l, \\ \mu(S_1(x_0)) &= q^l(1 + q), \end{aligned}$$

so we see that the formula found for $r \geq 2$ holds also for $r = 1$.

We notice that $\mu(S_r(x_0))$ depends on the level of the center x_0.

For example, if $l(x_0) = l = 0$:

$$\mu(S_r(x_0)) = q^{r-1}(1+q) = \#S_r(x_0),$$

i.e. the μ-measure of the sphere coincides with its counting measure.

Now, to compute $\mu(B_r(x_0))$ we can exploit the fact that $B_r(x_0)$ is the union of the disjoint spheres with radius from 0 to r and obtain:

$$\begin{aligned}
\mu(B_r(x_0)) &= \mu(S_0(x_0)) + \sum_{j=1}^{r} \mu(S_j(x_0)) \\
&= q^l + \sum_{j=1}^{r} q^{l+j-1}(1+q) \\
&= q^l \left(1 + (1+q)\sum_{j=1}^{r} q^{j-1}\right) \\
&= q^l \left(1 + (1+q)\sum_{k=0}^{r-1} q^{k}\right) \\
&= q^l \left(1 + (1+q)\frac{1-q^r}{1-q}\right) \\
&= q^l \left(\frac{1-q+1+q-q^r-q^{r+1}}{1-q}\right) \\
&= q^l \frac{2-q^r-q^{r+1}}{1-q} \\
&= q^l \frac{q^{r+1}+q^r-2}{q-1}.
\end{aligned}$$

4.3.2 The doubling property

We notice that $\mu(B_r(x_0))$ grows exponentially with respect to the radius r.

Definition 4.3.2 (Doubling Property)**.** A metric measure space (X, ρ, ν) is said to be doubling if:

$$\exists C > 0 \text{ s.t. } \quad \nu(B_{2r}(x_0)) \le C\nu(B_r(x_0)) \quad \forall r \ge 0, \forall x_0 \in X. \tag{4.11}$$

Conversely, we say that (X, ρ, ν) is not doubling if $\nexists C > 0$ satisfying (4.11).

An example of doubling metric measure space is Euclidean n-dimensional space endowed with the Lebesgue measure. In that case:

$$\lambda(B_{2r}(x_0)) = (2r)^n B_1(x_0) \leq 2^n r^n B_1(x_0) = 2^n \lambda(B_r(x_0))$$

so 4.11 is verified by $C = 2^n$.

Proposition 4.3.1. The space $(\mathcal{V}, d, \mu)$ is not doubling.

Proof. Recall that

$$\mu(B_r(x_0)) = q^l \, \frac{q^{r+1} + q^r - 2}{q - 1}.$$

The doubling property 4.11 implies that $\exists C > 0$ such that:

$$\frac{\mu(B_{2r}(x_0))}{\mu(B_r(x_0))} \leq C \quad \forall r > 0, \forall x_0 \in \mathcal{V}.$$

We show that this is not possible by computing:

$$\begin{aligned}
\lim_{r\to\infty} \frac{\mu(B_{2r}(x_0))}{\mu(B_r(x_0))} &= \lim_{r\to\infty} \frac{q^l \, \frac{q^{2r+1}+q^{2r}-2}{q-1}}{q^l \, \frac{q^{r+1}+q^r-2}{q-1}} \\
&= \lim_{r\to\infty} \frac{q^{2r}(q+1) - 2}{q^r(q+1) - 2} \\
&= \lim_{r\to\infty} \frac{q^{2r}\left[(q+1) - \frac{2}{q^{2r}}\right]}{q^r\left[(q+1) - \frac{2}{q^r}\right]} \\
&= \lim_{r\to\infty} q^r \cdot \lim_{r\to\infty} \frac{(q+1) - \frac{2}{q^{2r}}}{(q+1) - \frac{2}{q^r}} \\
&= \lim_{r\to\infty} q^r = \infty.
\end{aligned}$$

□

We observe that the counting measure $(\mathcal{V}, d, \#)$ is not doubling too, as one can see repeating the preceding proof with $\#(B_r(x_0)) = \frac{q^{r+1}+q^r-2}{q-1}$.

This is due to the fact that in both cases the measure of the ball increases exponentially with respect to its radius.

We can define a weaker condition which is satisfied by these spaces.

Definition 4.3.3 (Locally doubling property)**.** A metric measure space (X, ρ, ν) is said to be locally doubling if:

$$\forall R > 0\,, \exists C_R > 0 \text{ s.t. } \quad \nu(B_{2r}(x_0)) \leq C_R \nu(B_r(x_0)) \quad \forall r \leq R, \forall x_0 \in X. \tag{4.12}$$

Proposition 4.3.2. The space $(\mathcal{V}, d, \mu)$ is locally doubling.

Proof. Fix $R > 0$ and consider $r \leq R$.

$$\begin{aligned}
\mu(B_{2r}(x_0)) &= q^l \,\frac{q^{2r+1} + q^{2r} - 2}{q-1} \\
&= q^l \frac{\frac{q^{2r+1}+q^{2r}-2}{q-1}}{\frac{q^{r+1}+q^r-2}{q-1}} \frac{q^{r+1} + q^r - 2}{q-1} \\
&\leq q^l \frac{\frac{q^{2R+1}+q^{2R}-2}{q-1}}{\frac{q^{R+1}+q^R-2}{q-1}} \frac{q^{r+1} + q^r - 2}{q-1} \\
&= q^l \frac{q^{r+1} + q^r - 2}{q-1} C_R \\
&= \mu(B_r(x_0)) C_R.
\end{aligned}$$

with $C_R = \frac{q^{2R+1}+q^{2R}-2}{q^{R+1}+q^R-2} = \frac{\mu(B_{2R}(x_0))}{\mu(B_R(x_0))} > 0$ not depending on x_0.

□

The same property holds for the counting measure. In both cases this is due to the fact that the ratio $\frac{\mu(B_{2r}(x_0))}{\mu(B_r(x_0))}$ does not depend on the center of the ball x_0 but only on the radius r.

4.3.3 The isoperimetric property

We now study another relevant property that links measure and metrics. It is expressed by the following definition which captures the fact that the region close to the boundary of a set A gives a significant contribution to the measure of A.

Let (X, ρ, ν) be a metric measure space.

For any subset $A \subset X$ and each $k \in \mathbb{R}^+$ we denote:

$$\begin{aligned}
A_k &= \{x \in A : \rho(x, A^c) \leq k\} \\
A^k &= \{x \in A : \rho(x, A^c) > k\}\,.
\end{aligned}$$

Definition 4.3.4 (Isoperimetric property)**.** The metric measure space (X, ρ, ν) has the isoperimetric property if

$$\exists k_0, C \in \mathbb{R}^+ \quad \text{such that } \forall A \subset X \text{ bounded open set it holds}$$
$$\nu(A_k) \geq Ck\nu(A) \quad \forall k \in (0, k_0]. \tag{4.13}$$

Definition 4.13 is the standard way to express such concept in the context of metric measure spaces. However we show that it is not the most suitable for the present case, since it is trivially false for every measure when the distance is discrete (i.e. it assumes only integer values).

Proposition 4.3.3. The space $(\mathcal{V}, d, \mu)$ has not the isoperimetric property (4.13).

Proof. We show that the converse of 4.13 is true, i.e.

$$\forall k_0, \forall C > 0 \quad \exists A \subset X \text{ bounded open set and } \exists k \leq k_0 \text{ such that}$$
$$\nu(A_k) < Ck\nu(A). \tag{4.14}$$

Indeed, it is sufficient to take $k < 1$ arbitrarily if $k_0 \geq 1$ or $k = \frac{k_0}{2}$ if $k_0 < 1$ and we have that $A_k = \emptyset$, since there are no points in A with distance less than 1 from A^c. Then, since $\mu(\emptyset) = 0$ (4.14) is satisfied. □

Proof of Proposition 4.3.3 does not rely on specific feature of d and μ: we just used the general fact that $\mu(\emptyset) = 0$ (which is true for every measure) and that every vertex that does not belong to a set $A^c \subset \mathcal{V}$ has distance not less that 1 from everyone of its point (which is true for every discrete distance on $\mathcal{V}$).

So Property (4.13) does not express a link between d and μ and does not provide any insight.

For this reason it is convenient to introduce a new definition that express the same concept as 4.13 but is adapted to a discrete setting. The following definition refers to [HLW06].

Definition 4.3.5 (Isoperimetric property - discrete case)**.** The metric measure space (X, ρ, ν) with discrete distance ρ has the discrete isoperimetric property if

$$\exists C \in \mathbb{R}^+ \quad \text{such that } \forall A \subset X \text{ bounded set it holds}$$
$$\nu(A_1) \geq C\nu(A). \tag{4.15}$$

In Definition 4.3.5 we compare the measure of A with the measure of the boundary of A (that is the set of vertices of A that have a neighbor in A^c or, in other words, that have distance equal to 1 from A^c).

The fact that $(\mathcal{V}, d, \mu)$ has not the isoperimetric property 4.3.5 is now more interesting since it is not trivial.

Proposition 4.3.4. The space $(\mathcal{V}, d, \mu)$ has not the discrete isoperimetric property 4.3.5.

Proof. We show that the converse of (4.15) is true, i.e.

$$\forall C > 0 \quad \exists A \subset X \text{ bounded set such that}$$

$$\frac{\nu(A_1)}{\nu(A)} < C. \tag{4.16}$$

In particular, we show that for every $C > 0$ we can find an admissible trapezoid satisfying 4.16.

Admissible trapezoids will be formally introduced in Definition 5.1.1 and deeply investigated in the next chapter since they are fundamental in the construction of the Calderón–Zygmund theory for the tree. For the moment we just need to consider an admissible trapezoid as a set of vertices

$$R = \{x \in \mathcal{V} : x \text{ lies below } x_R, h \leq l(x_R) - l(x) < 2h\}$$

for given x_R and $h = h(R)$.

The boundary of R is made of the two bases of R, that is the set of vertices in R with level equal to $l(x_R) - h$ or $l(x_R) - 2h + 1$. More formally, we can define

$$b_R = \{x \in R : l(x) = l(x_R) - h\}$$
$$B_R = \{x \in R : l(x) = l(x_R) - 2h + 1\}$$

and we have that

$$A_1 = B_R \cup b_R.$$

We will show that $\mu(R) = h(R)q^{l(x_R)}$ (see 5.1) and that each level of a trapezoid has measure equal to the measure of the root x_R, i.e.

$$\mu(B_R) = q^{2h-1}q^{l(x_R)-2h+1} = q^{l(x_R)}$$
$$\mu(b_R) = q^h q^{l(x_R)-h} = q^{l(x_R)}$$

since in B_R there are q^{2h-1} vertices and each of them has level $l(x_R)-2h+1$, while in b_R there are q^h vertices and each of them has level $l(x_R)-h$. Then

$$\begin{aligned}\frac{\mu(A_1)}{\mu(A)} &= \frac{\mu(B_R)+\mu(b_R)}{\mu(A)} \\ &= \frac{2q^{l(x_R)}}{h(R)q^{l(x_R)}} \\ &= \frac{2}{h(R)}\,.\end{aligned}$$

We can make this ratio arbitrary small by choosing a trapezoid of appropriate height:

$$\frac{\mu(A_1)}{\mu(A)} < C \quad \Leftrightarrow \quad h(R) > \frac{2}{C}\,.$$

That trapezoid plays the role of A in 4.16 and the proof is concluded. □

We underline that Proposition 4.3.4 is not trivial as 4.3.3 was. To show that we give an example of a family of sets $A \subset \mathcal{V}$ for which the property 4.3.5 is satisfied.

Example We set $A = B_r(x_0)$. The boundary of A is $A_1 = S_r(x_0)$.Then:

$$\begin{aligned}\frac{\mu(A_1)}{\mu(A)} &= \frac{\mu(S_r(x_0))}{\mu(B_r(x_0))} \\ &= \frac{q^{l+r-1}(1+q)}{q^l\,\frac{q^{r+1}+q^r-2}{q-1}} \\ &= (q^2-1)\frac{q^{r-1}}{q^{r+1}+q^r-2} \\ &= \frac{(q^2-1)}{q}\frac{q^r}{q^r(1+q)-2} \\ &\geq \frac{(q^2-1)}{q}\frac{q^r}{q^r(1+q)} \\ &= \frac{(q^2-1)}{q}\frac{1}{(1+q)} \\ &= \frac{(q-1)}{q}\end{aligned}$$

So for every ball $A = B_r(x_0)$ it holds

$$\frac{\mu(A_1)}{\mu(A)} \geq C \quad \text{with } C = \frac{(q-1)}{q}.$$

Chapter 5

Calderón–Zygmund theory for the weighted tree

In their work [HS03], Hebish and Steger outline the proof of a covering lemma for the weighted tree, based on the introduction of a family of sets called admissible trapezoids, thus showing that the tree possesess the abstract Calderón–Zygmund property described in Chapter 3. We develope in great detail their proof and also the one concerning the weak-type $(1,1)$ boundedness of the maximal function M introduced therein.

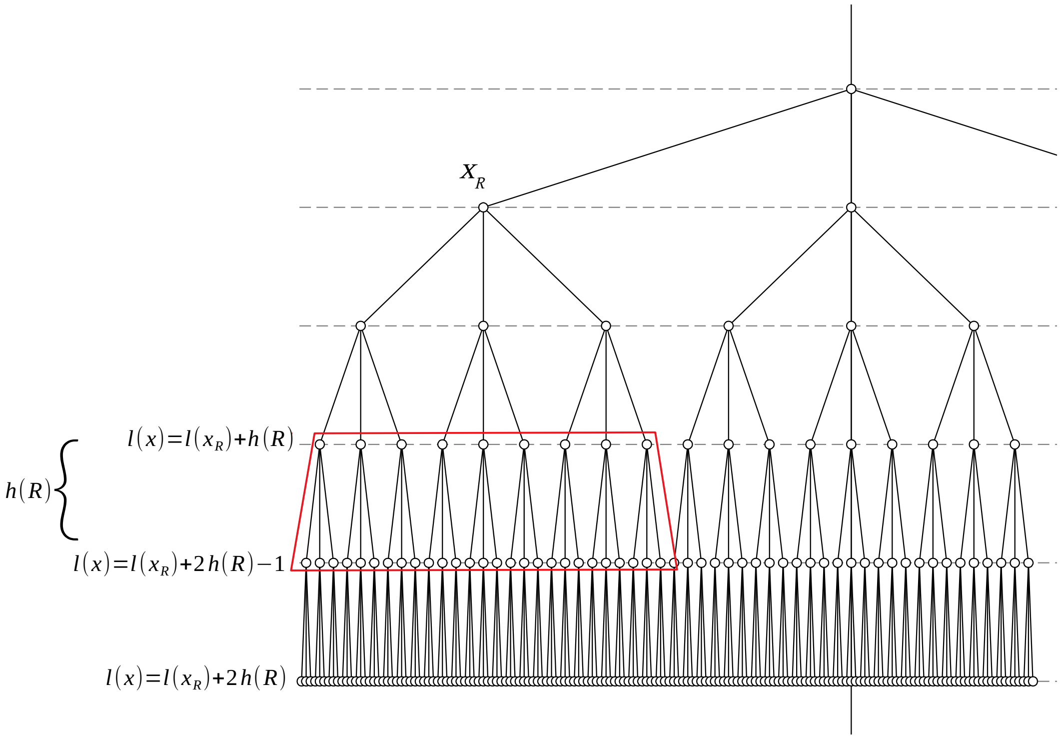
5.1 Admissible trapezoids

Definition 5.1.1. $R \subset \mathcal{V}$ is an admissible trapezoid if and only if one of the following occurs:

- $R = \{x_R\}$ with $x_R \in \mathcal{V}$, that is R consists of a single vertex ;
- $\exists x_R \in \mathcal{V}$, $\exists h \in \mathbb{N}^+$ such that

$$R = \{x \in \mathcal{V} : x \text{ lies below } x_R\,, h \leq l(x_R) - l(x) < 2h\}\,.$$

We set $h(R) = 1$ in the first case and $h(R) = h$ in the second case. In both cases we will refer to x_R as the root node of the trapezoid.



Representation of an admissible trapezoid with $h(R) = 2$ $(q = 3)$

For an admissible trapezoid R, $h(R)$ can be interpreted as the height of the trapezoid and coincides with the number of levels spanned by R.

Definition 5.1.2. We call width of the trapezoid R the quantity

$$w(R) = q^{l(x_R)}.$$

It holds that:

$$\mu(R) = h(R)q^{l(x_R)} = h(R)w(R). \tag{5.1}$$

To prove this we distinguish two cases:

- $R = \{x_R\}$, that is R consists of a single vertex. In this case:

$$\mu(R) = \sum_{x \in R} q^{l(x)} = q^{l(x_R)} = h(R)q^{l(x_R)}.$$

- R is not a degenerate trapezoid. In this case:

$$\begin{aligned}
\mu(R) = \sum_{x\in R} q^{l(x)} &= \sum_{l=l(x_R)-2h+1}^{l(x_R)-h} \sum_{x\in R: l(x)=l} q^l \\
&= \sum_{l=l(x_R)-2h+1}^{l(x_R)-h} q^l q^{l(x_R)-l} \\
&= \sum_{l=l(x_R)-2h+1}^{l(x_R)-h} q^{l(x_R)} \\
&= q^{l(x_R)} \left(l(x_R) - h - (l(x_R) - 2h + 1) + 1\right) \\
&= q^{l(x_R)} h = q^{l(x_R)} h(R).
\end{aligned}$$

Definition 5.1.3. Let R be an admissible trapezoid. We define its envelope $\tilde{R}$ as follows:

- if R consists of a single vertex, then $\tilde{R} = R$;
- otherwise

$$\tilde{R} = \left\{ x \in \mathcal{V} : x \text{ lies below } x_R\,, \frac{h}{2} \leq l(x_R) - l(x) < 4h \right\}.$$

Proposition 5.1.1. Let R be an admissible trapezoid. Then it holds:

$$\mu(\tilde{R}) \leq 4\mu(R).$$

Proof. As usual, we distinguish between the degenerate and non-degenerate case.

- If R consists of a single vertex $R = \{x_R\}$ then $\tilde{R} = R = \{x_R\}$ and

$$\mu(\tilde{R}) = \sum_{x\in\tilde{R}} q^{l(x)} = h(R) q^{l(x_R)} = \mu(R) \leq 4\mu(R).$$

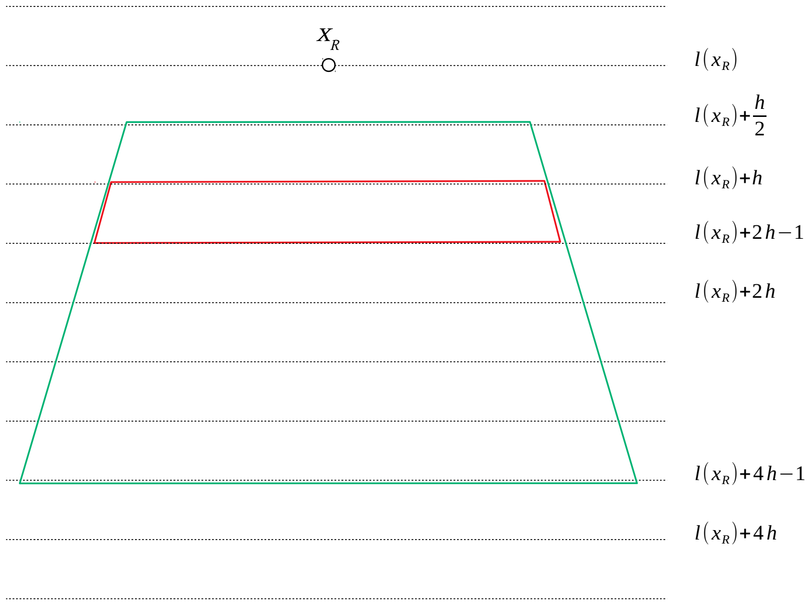


Figure 5.1: Representation of the envelope $\tilde{R}$ (green) of the admissible trapezoid R (red) when $h = h(R) = 2$

- In the non degenerate case

$$\begin{aligned}\mu(\tilde{R}) &= \sum_{x\in\tilde{R}} q^{l(x)} = \sum_{l=l(x_R)-4h+1}^{\lfloor l(x_R)-\frac{h}{2}\rfloor} \sum_{x\in\tilde{R}:l(x)=l} q^l \\ &= \sum_{l=l(x_R)-4h+1}^{\lfloor l(x_R)-\frac{h}{2}\rfloor} q^l q^{l(x_R)-l} = \sum_{l=l(x_R)-4h+1}^{\lfloor l(x_R)-\frac{h}{2}\rfloor} q^{l(x_R)} \\ &= q^{l(x_R)}\left(\lfloor l(x_R)-\tfrac{h}{2}\rfloor - (l(x_R)-4h+1)+1\right) \\ &\le q^{l(x_R)}\left(l(x_R)-\tfrac{h}{2}-l(x_R)+4h\right) \\ &\le \left(q^{l(x_R)}h\right)(4-\tfrac{1}{2}) \\ &\le 4\mu(R).\end{aligned}$$

□

Proposition 5.1.2. Let R_1 and R_2 be two admissible trapezoids. If

$$R_1 \cap R_2 \neq \emptyset \text{ and } w(R_1) \geq w(R_2)\,,$$

then

$$R_2 \subset \tilde{R}_1.$$

Proof. We distinguish four cases:

- 1) both R_1 and R_2 consist of a single vertex;
- 2) R_1 is composed of a single vertex, while R_2 is not;
- 3) R_2 is composed of a single vertex, while R_1 is not;
- 4) nor R_1 neither R_2 is composed of a single vertex.

Case 1): In this case

$$R_1 = \{x_{R_1}\}\,, R_2 = \{x_{R_2}\}\ .$$

Thus,

$$\begin{aligned} R_1\cap R_2 \neq \emptyset \quad \Longrightarrow \quad x_{R_1} = x_{R_2} \quad \Longrightarrow \quad R_2 = R_1 = \tilde{R}_1 \\ \Longrightarrow\ R_2 \subset \tilde{R}_1.\end{aligned}$$

Case 2): This case is inconsistent with the hypothesis.

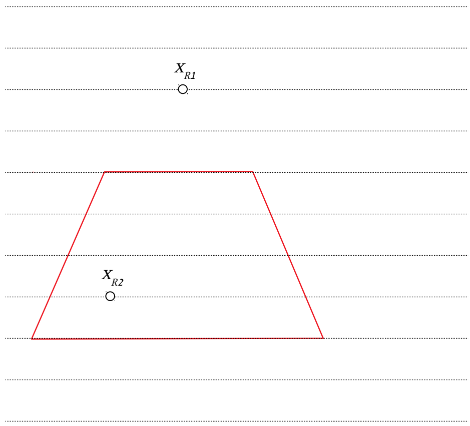
$R_1 = \{x_{R_1}\}$, let x_{R_2} be the root node of R_2. Then

$$w(R_1) = q^{l(x_{R_1})} \geq q^{l(x_{R_2})} = w(R_2) \quad \Longrightarrow \quad l(x_{R_1}) \geq l(x_{R_2})$$

$$\Longrightarrow x_{R_1} \notin R_2 \quad \Longrightarrow \quad R_1 \cap R_2 = \emptyset.$$

Therefore this case cannot happen.

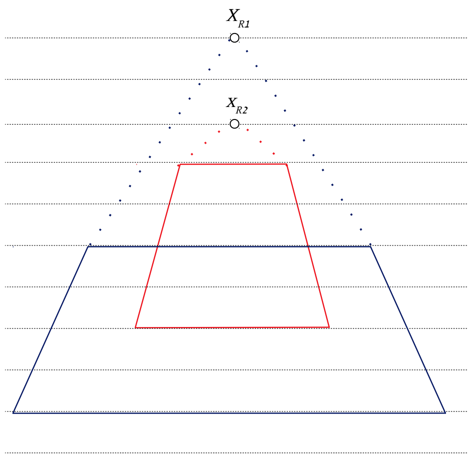
Case 3): In this case $R_2 = \{x_{R_2}\}$.



Let x_{R_1} be the root node of R_1.

$$R_1 \cap R_2 \neq \emptyset \quad \Longrightarrow \quad x_{R_2} \in R_1 \quad \Longrightarrow \quad R_2 \subset R_1 \subset \tilde{R}_1.$$

Case 4)



Let x_{R_1} and x_{R_2} be the two root nodes of R_1 and R_2, respectively. Then

$$w(R_1) = q^{l(x_{R_1})} \geq q^{l(x_{R_2})} = w(R_2) \qquad \Longrightarrow \qquad l(x_{R_1}) \geq l(x_{R_2}).$$

If x_{R_2} were not below x_{R_1}, then it would hold $R_1 \cap R_2 = \emptyset$ (since all vertices of R_1 lie below x_{R_1} and all vertices of R_2 lie below x_{R_2}). This means that x_{R_2} is below x_{R_1} and so is every vertex of R_2. In the following we denote $h_1 = h(R_1)$ and $h_2 = h(R_2)$.

$$\begin{aligned} R_1 &= \{x \in \mathcal{V} : x \text{ below } x_{R_1}\,, h_1 \leq l(x_{R_1}) - l(x) < 2h_1\} \\ &= \{x \in \mathcal{V} : x \text{ below } x_{R_1}\,, l(x_{R_1}) - 2h_1 < l(x) \leq l(x_{R_1}) - h_1\}\,, \end{aligned}$$

$$\begin{aligned} R_2 &= \{x \in \mathcal{V} : x \text{ below } x_{R_2}\,, h_2 \leq l(x_{R_2}) - l(x) < 2h_2\} \\ &= \{x \in \mathcal{V} : x \text{ below } x_{R_2}\,, l(x_{R_2}) - 2h_2 < l(x) \leq l(x_{R_2}) - h_2\}\,. \end{aligned}$$

Let $\hat{x} \in R_1 \cap R_2 \neq \emptyset$. Then, since $l(\hat{x})$ satisfies both sets of inequalities we obtain the following constraints:

$$\begin{cases} l(x_{R_2}) - 2h_2 + 1 \leq l(\hat{x}) \leq l(x_{R_1}) - h_1 \\ l(x_{R_1}) - 2h_1 + 1 \leq l(\hat{x}) \leq l(x_{R_2}) - h_2 \end{cases}$$

$$\Longrightarrow \quad \begin{cases} l(x_{R_1}) - l(x_{R_2}) \geq h_1 - 2h_2 + 1 \\ l(x_{R_1}) - l(x_{R_2}) \leq 2h_1 - h_2 - 1\,. \end{cases}$$

These last inequalities can be interpreted in the following way. For the intersection $R_1 \cap R_2$ not to be empty

- the lower base of R_2 cannot be higher than the highest base of R_1,
- the higher base of R_2 cannot be lower than the lowest base of R_1.

Let $x \in R_2$. Then

- x lies below $x_{R_2} \quad \Longrightarrow \quad x$ lies below x_{R_1},
- $h_2 \leq l(x_{R_2}) - l(x) \leq 2h_2 - 1$.

To conclude that $x \in \tilde{R}_1$ we must estimate the quantity $l(x_{R_1}) - l(x)$.

$$\begin{aligned} l(x_{R_1}) - l(x) &= [l(x_{R_1}) - l(x_{R_2})] + [l(x_{R_2}) - l(x)] \\ &\leq [2h_1 - h_2 - 1] + [2h_2 - 1] \\ &\leq 2\,[2h_1 - h_2 - 1] + [2h_2 - 1] \\ &= 4h_1 - 2h_2 - 2 + 2h_2 - 1 \\ &= 4h_1 - 3 < 4h_1. \end{aligned}$$

$$\begin{aligned} l(x_{R_1}) - l(x) &= [l(x_{R_1}) - l(x_{R_2})] + [l(x_{R_2}) - l(x)] \\ &\geq [h_1 - 2h_2 + 1] + [h_2] \\ &\geq \frac{1}{2}\,[h_1 - 2h_2 + 1] + [h_2] \\ &= \frac{h_1}{2} + \frac{1}{2} > \frac{h_1}{2}. \end{aligned}$$

Summing up,

$$\frac{h_1}{2} \leq l(x_{R_1}) - l(x) < 4h_1 \qquad \forall x \in R_2$$

$$\Longrightarrow \qquad R_2 \subset \tilde{R}_1.$$

□

5.2 The maximal function M

Definition 5.2.1. We define the maximal function M as follows:

$$Mf(x) = \sup_{R:x\in R} \mu(R)^{-1} \sum_{y\in R} |f(y)| q^{l(y)}$$

where $f \in C(\mathcal{V})$ and the supremum is taken over all admissible trapezoids R containing x.

Theorem 5.2.1. *The maximal function M is of weak-type $(1,1)$.*

Proof. Let $f \in L^1(\mathcal{V}, \mu)$ and $\lambda > 0$.
Define S_0 as the family of all admissible trapezoids R such that

$$\sum_{x\in R} |f(x)| q^{l(x)} \geq \lambda \mu(R).$$

Since S_0 is countable, we can introduce an ordering in S_0.
All trapezoids in S_0 have:

- bounded measure:

$$\forall R \in S_0, \quad \mu(R) \leq \frac{1}{\lambda} \sum_{x\in R} |f(y)| q^{l(y)} \leq \frac{1}{\lambda} \|f\|_{L^1(\mathcal{V},\mu)} \ ;$$

- bounded width:

$$\forall R \in S_0, \quad w(R) = \frac{\mu(R)}{h(R)} \leq \mu(R) \leq \frac{1}{\lambda} \|f\|_{L^1(\mathcal{V},\mu)} \ .$$

So it is possible to choose in S_0 a trapezoid R_0 of largest width (in case of ties, we choose that trapezoid of largest width which occurs earliest in the ordering).

Then we proceed inductively:

- S_{i+1} is the family of all admissible trapezoids $R \in S_i$ disjointed from $R_0, \dots, R_i$;

- R_{i+1} is the trapezoid of largest width in S_{i+1} which occurs earliest in the ordering.

Let $R \in S_0$, that is R satisfies $\sum_{x\in R} |f(x)|q^{l(x)} \geq \lambda\mu(R)$. Then by construction R intersects some R_i with $w(R_i) \geq w(R)$.

Indeed, there exists a number $j \in \{0, 1, 2, \dots\}$ such that $R \in S_j$ and $R \notin S_{j+1}$, i.e. in the previous construction there exists a step j in which one of the following occurs:

1. either R is the trapezoid of largest width that occurs earliest in the ordering, and then R is selected and $R_j = R$, so that $R \cap R_i \neq \emptyset$ for $i = j$
2. either R isn't the trapezoid of largest width that occurs earliest in the ordering and it intersects R_j. Then R will not be present in S_i $\forall i \geq j+1$ and $R \cap R_i \neq \emptyset$ for $i = j$.

To assure that there is some j with the stated property it is sufficient to avoid that S_0 can contain an infinite number of trapezoids with the same width that do not intersect each other.

If this were true, it could happen that for some step k in the construction the set S_k contains an infinite number of trapezoids $\{T_1, T_2, \dots\}$ all with the same width equal to the maximum width in S_k and all disjoint with respect to each other and to R.

Then it would hold:

$$\begin{aligned}
R_k &= T_1 \text{ and } T_i \in S_{k+1} \quad \forall i > 1 \\
R_{k+1} &= T_2 \text{ and } T_i \in S_{k+2} \quad \forall i > 2 \\
R_{k+2} &= T_3 \text{ and } T_i \in S_{k+3} \quad \forall i > 3 \\
&\vdots
\end{aligned}$$

and in this case, if $w(R) < w(t_i)$, nor 1. nor 2. would ever happen.

This possibility is excluded observing that:

$$\begin{aligned}
\sum_i \mu(R_i) &\leq \frac{1}{\lambda} \sum_i \sum_{x\in R_i} |f(x)|q^{l(x)} \leq \\
&\leq \frac{1}{\lambda} \sum_{x\in\mathcal{V}} |f(x)|q^{l(x)} = \frac{1}{\lambda} \|f\|_{L^1(\mathcal{V},\mu)} < \infty \,,
\end{aligned}$$

while if there was among the R_i's an infinite number of trapezoids with constant width w

$$\sum_i \mu(R_i) \geq \sum_{n=1}^{\infty} w = \infty.$$

Summing up,

$$\forall R \in S_0, \quad \exists i : R \cap R_i \neq \emptyset \text{ and } w(R_i) \geq w(R).$$

By proposition 5.1.2, this implies $R \subset \tilde{R}_i$.

We put:

$$E := \bigcup_i \tilde{R}_i$$

and it follows from what we have just seen that for every trapezoid $R \in S_0$, $R \subset E$.

Then it holds that:

$$(Mf)(x) \leq \lambda \quad \forall x \notin E.$$

Indeed:

$$(Mf)(x) = \sup_{R: x \in R} \mu(R)^{-1} \sum_{y \in R} |f(y)| q^{l(y)}.$$

For every admissible trapezoid R containing $x \in E^c$, $R \notin S_0$. Indeed, if there was an admissible trapezoid $R \in S_0$ such that $x \in R$, then we would have $x \in R \subset E$ but by hypothesis $x \notin E$.

So,

$$\begin{aligned} \forall x \in E^c, \quad (Mf)(x) &= \sup_{R \notin S_0 : x \in R} \mu(R)^{-1} \sum_{y \in R} |f(y)| q^{l(y)} \\ &\leq \sup_{R \notin S_0 : x \in R} \mu(R)^{-1} \lambda \mu(R) \leq \lambda. \end{aligned}$$

Moreover:

$$\mu(E) = \mu\left(\bigcup_i \tilde{R}_i\right) \leq \sum_i \mu(\tilde{R}_i) \leq 4 \sum_i \mu(R_i) \leq \frac{4 \, \|f\|_{L^1(\mathcal{V},\mu)}}{\lambda}.$$

This shows that $Mf \in L^{1,\infty}(\mathcal{V}, \mu)$, because:

$$\|Mf\|_{L^{1,\infty}(\mathcal{V},\mu)} = \inf\left\{c > 0 : d_{Mf}(\lambda) \leq \frac{c}{\lambda}\right\}$$

with

$$d_{Mf}(\lambda) = \mu\{x : (Mf)(x) > \lambda\}$$

and we have that

$$\{x : (Mf)(x) > \lambda\} \subset E$$

since $\forall x \in E^c$, $(Mf)(x) \leq \lambda$.

So:

$$d_{Mf}(\lambda) \leq \mu(E) \leq \frac{4\,\|f\|_{L^1(\mathcal{V},\mu)}}{\lambda}$$
$$\implies \|Mf\|_{L^{1,\infty}(\mathcal{V},\mu)} \leq 4\,\|f\|_{L^1(\mathcal{V},\mu)}$$
$$\implies M \text{ is of weak-type } (1,1).$$

□

5.3 The Calderón–Zygmund property for the tree

We pass now to the proof of the main theorem of this chapter.

Theorem 5.3.1. *The space $(\mathcal{V}, d, \mu)$ has the Calderón–Zygmund property.*

Proof. In the following we use the same notation used in the proof of 5.2.1. Take $f \in L^1(\mathcal{V}, \mu)$ and $\lambda > 0$. Consider the sets R_i and $\tilde{R}_i$ constructed as in the proof of Theorem 5.2.1. To construct the functions f_i of the Calderón–Zygmund decomposition 3.1 of f at the level λ we first define auxiliary sets U_i and functions h_i as follows:

$$U_i := \tilde{R}_i - \bigcup_{j<i} \tilde{R}_j,$$
$$h_i(x) := \begin{cases} f(x) & x \in U_i \\ 0 & x \notin U_i \end{cases} .$$

It holds that:

$$\sum_{x\in\mathcal{V}} |h_i(x)| q^{l(x)} \leq 6q\lambda\mu(\tilde{R}_i).$$

Indeed it is possible to find three admissible trapezoids P_1, P_2, P_3 with the following properties:

- $w(P_k) > w(R_i)$ for $k = 1, 2, 3$;
- $\mu(P_k) \leq 2q\mu(\tilde{R}_i)$ for $k = 1, 2, 3$;

- $\tilde{R}_i \subset P_1 \cup P_2 \cup P_3$.

For example, for P_1, P_2, P_3 one can take the trapezoids obtained in the following way.

P_1, P_2, P_3 are admissible trapezoids with the same root equal to the only parent vertex of x_i, the root of R_i, which we denote by x_v.

$$l(x_v) = l(x_i) + 1 \quad \Longrightarrow \quad w(P_k) > w(R_i) \quad \text{for } k = 1, 2, 3\,.$$

We denote h_k the height of P_k for $k = 1, 2, 3$ and h the height of R_i and observe that P_k contains all vertices x below x_v such that:

$$h_k \leq l(x_v) - l(x) < 2h_k \quad \Leftrightarrow \quad h_k - 1 \leq l(x_i) - l(x) < 2h_k - 1.$$

We want that the union of P_k covers $\tilde{R}_i$, which contains all vertices x below x_i such that

$$\frac{h}{2} \leq l(x_i) - l(x) < 4h$$

so we require that:

- P_1 covers the higher part of $\tilde{R}_i$:

$$h_1 = \lfloor \frac{h}{2} \rfloor + 1.$$

  In this way P_1 covers $\tilde{R}_1$ at least from level (measured with respect to the root x_i) $\frac{h}{2}$ to h;

- P_2 covers the middle part of $\tilde{R}_i$:

$$h_2 = h + 1.$$

  In this way P_2 covers $\tilde{R}_1$ from level (measured with respect to the root x_i) h to $2h + 1$;

- P_3 covers the lower part of $\tilde{R}_i$:

$$h_3 = 2h + 1.$$

  In this way P_2 covers $\tilde{R}_1$ from level (measured with respect to the root x_i) $2h$ to $4h + 1$.

By construction $\tilde{R}_i \subset P_1 \cup P_2 \cup P_3$.

To show that $\mu(P_k) \leq 2q\mu(\tilde{R}_i)$ for $k = 1, 2, 3$ it is sufficient to make a check for P_3 since it is the largest one: all P_k have the same width $w(P_k) = q^{l(x_v)}$ while $h(P_3) = h_3 = 2h + 1 \geq h(P_k)$ for $k = 1, 2$ and $\forall h \geq 1$.

So we compute:

$$\mu(P_3) = w(P_3)h(P_3) = q^{l(x_i)+1}(2h+1) = 2q\mu(R_i) + q^{l(x_i)+1}$$

$$\begin{aligned}
\mu(\tilde{R}_i) &= \sum_{l=l(x_i)-4h+1}^{l(x_i)-\lceil \frac{h}{2} \rceil} \sum_{x \in \tilde{R}_i : l(x)=l} q^l = \sum_{l=l(x_i)-4h+1}^{l(x_i)-\lceil \frac{h}{2} \rceil} q^{l(x_i)-l} q^l \\
&= \sum_{l=l(x_i)-4h+1}^{l(x_i)-\lceil \frac{h}{2} \rceil} q^{l(x_i)} = q^{l(x_i)} \left(4h - 1 - \lceil \frac{h}{2} \rceil + 1 \right) \\
&= \begin{cases} \frac{7}{2} h q^{l(x_i)} \text{ if } h \text{ even} \\ \frac{7}{2} h q^{l(x_i)} - \frac{1}{2} q^{l(x_i)} \text{ if } h \text{ odd} \end{cases} \\
&= \begin{cases} \frac{7}{2} \mu(R_i) \text{ if } h \text{ even} \\ \frac{7}{2} \mu(R_i) - \frac{1}{2} q^{l(x_i)} \text{ if } h \text{ odd} \end{cases} .
\end{aligned}$$

When h is even:

$$2q\mu(\tilde{R}_i) = 7q\mu(R_i).$$

Then

$$\begin{aligned}
&\mu(P_3) \leq 2q\mu(\tilde{R}_i) \\
&\Leftrightarrow 2q\mu(R_i) + q^{l(x_i)+1} \leq 7q\mu(R_i) \\
&\Leftrightarrow q^{l(x_i)+1} \leq 5q\mu(R_i) \\
&\Leftrightarrow q^{l(x_i)} \leq 5\mu(R_i) = 5q^{l(x_i)}h \\
&\Leftrightarrow \frac{1}{5} \leq h \quad \text{ true because } h \geq 1.
\end{aligned}$$

When h is odd:

$$2q\mu(\tilde{R}_i) = 7q\mu(R_i) - q^{l(x_i)+1}.$$

Then

$$\begin{aligned}
&\mu(P_3) \le 2q\mu(\tilde{R}_i)\\
&\Leftrightarrow 2q\mu(R_i) + q^{l(x_i)+1} \le 7q\mu(R_i) - q^{l(x_i)+1}\\
&\Leftrightarrow 2q^{l(x_i)+1} \le 5q\mu(R_i)\\
&\Leftrightarrow 2q^{l(x_i)} \le 5\mu(R_i) = 5q^{l(x_i)}h\\
&\Leftrightarrow \frac{2}{5} \le h \quad \text{ true because } h \ge 1.
\end{aligned}$$

For trapezoids P_k with the previous properties we have that:

$$\sum_{x\in P_k} |h_i(x)|q^{l(x)} \le \lambda\mu(P_k) \qquad k = 1, 2, 3.$$

Indeed:

- if P_k is such that
$$\sum_{x\in P_k} |f(x)|q^{l(x)} \ge \lambda\mu(P_k)$$
then by definition $P_k \in S_0$ with $w(P_k) > w(R_i)$. For what already shown, P_k intersects some R_j with $w(R_j) \ge w(P_k) > w(R_i)$ and so $\exists j < i : P_k \cap R_j \ne \emptyset$ and then $P_k \subset \tilde{R}_j$.

  This implies that:
$$h_i = 0 \text{ on } P_k$$
because $U_i = \tilde{R}_i - \bigcup_{j<i} \tilde{R}_j$ and so $P_k \cap U_i = \emptyset$.

- conversely, if P_k is such that
$$\sum_{x\in P_k} |f(x)|q^{l(x)} < \lambda\mu(P_k)$$
then
$$\sum_{x\in P_k} |h_i(x)|q^{l(x)} \le \sum_{x\in P_k} |f(x)|q^{l(x)} \le \lambda\mu(P_k).$$

So in both cases we have $\sum_{x\in P_k} |h_i(x)|q^{l(x)} \le \lambda\mu(P_k)$.

Consequently, since $\mathrm{supp}(h_i) \subset U_i \subset \tilde{R}_i \subset (P_1 \cup P_2 \cup P_3)$:

$$\begin{aligned}\sum_x |h_i(x)| q^{l(x)} &\leq \sum_{k=1}^{3} \sum_{x \in P_k} |h_i(x)| q^{l(x)} \\ &\leq \sum_{k=1}^{3} \lambda \mu(P_k) \\ &\leq \sum_{k=1}^{3} \lambda 2q \mu(\tilde{R}_i) \\ &= 6q\lambda\mu(\tilde{R}_i)\end{aligned}$$

as stated.

Now we put:

$$\begin{aligned} f_i &= h_i - \left(\sum_{x \in \mathcal{V}} h_i(x) q^{l(x)} \right) \frac{\chi_{R_i}}{\mu(R_i)}, \\ g &= f - \sum_i f_i, \\ Q_i &= \tilde{R}_i, \\ r_i &= \frac{h(R_i)}{4}, \\ x_i &\in R_i \text{ chosen arbitrarily} . \end{aligned}$$

Then the following conditions hold by definition:

- $f = g + \sum_i f_i$.
- $f_i = 0$ outside Q_i .
  Indeed:
  $$\mathrm{supp}\, f_i \subset (\mathrm{supp}\, h_i \cup \mathrm{supp}\, \chi_{R_i}) \subset (U_i \cup R_i) \subset \tilde{R}_i = Q_i.$$
- $Q_i \subset B(x_i, 32r_i) \quad \Leftrightarrow \quad \tilde{R}_i \subset B(x_i, 8h(R_i))$.
  Indeed, starting from an arbitrary vertex $x_i \in \tilde{R}_i$ it is possible to reach every other vertex in $\tilde{R}_i$ passing through at most $[(4h-1)-0]\,2 = 8h-2$ edges, going up (i.e. moving to the only node above the current one) at most $4h-1$ times and the down (i.e. moving to a child node of the current one) at most $4h-1$ times.

So if we center in x_i a ball of radius $8h$ we cover $\tilde{R}_i$ because all vertex x that can be reached as just described have distance $d(x, x_i) \leq 8h - 2 < 8h$.

- $\int_{\mathcal{V}} f_i d\mu = 0$, i.e. f_i is a zero-mean function.

  Indeed:

$$\begin{aligned}
\int_{\mathcal{V}} f_i d\mu &= \int_{Q_i} f_i d\mu \\
&= \int_{\tilde{R}_i} f_i d\mu \\
&= \sum_{x \in \tilde{R}_i} \left[h_i(x) - \left(\sum_{y \in \mathcal{V}} h_i(y) q^{l(y)} \right) \frac{\chi_{R_i}(x)}{\mu(R_i)} \right] q^{l(x)} \\
&= \sum_{x \in \tilde{R}_i} h_i(x) q^{l(x)} - \left(\sum_{y \in \mathcal{V}} h_i(y) q^{l(y)} \right) \sum_{x \in \tilde{R}_i} \frac{\chi_{R_i}(x)}{\mu(R_i)} q^{l(x)} \\
&= \sum_{x \in (\tilde{R}_i - \cup_{j<i} \tilde{R}_j)} h_i(x) q^{l(x)} - \sum_{y \in \mathcal{V}} h_i(y) q^{l(y)} \\
&= \sum_{x \in U_i} h_i(x) q^{l(x)} - \sum_{y \in U_i} h_i(y) q^{l(y)} = 0 \ ,
\end{aligned}$$

where for the first integral we have used the fact that $\operatorname{supp} h_i \subset U_i = \tilde{R}_i - \cup_{j<i} \tilde{R}_j \subset \tilde{R}_i$ while for the second one we expoited the fact that $R_i \subset \tilde{R}_i$ and $\int_{\tilde{R}_i} \chi_{R_i}(x) d\mu(x) = \int_{R_i} \chi_{R_i}(x) d\mu(x) = \mu(R_i)$.

- $\mu(Q_i^*) \leq 2\mu(Q_i)$

  where

$$\begin{aligned}
Q_i^* &= \{x \in \mathcal{V} : d(x, Q_i) < r_i\} \\
&= \left\{ x \in \mathcal{V} : d(x, \tilde{R}_i) < \frac{h(R_i)}{4} \right\} \\
&= \left\{ x \in \mathcal{V} : d(x, \tilde{R}_i) \leq \lceil \frac{h(R_i)}{4} \rceil - 1 \right\} .
\end{aligned}$$

To show that, we observe that the only way of "going out" of $\tilde{R}_i$ is passing through the bases of trapezoid $\tilde{R}_i$, that we can define as:

$$b = \left\{ x \in \tilde{R}_i : l(x) = l(x_{R_i}) - \lceil \frac{h}{2} \rceil \right\},$$
$$B = \left\{ x \in \tilde{R}_i : l(x) = l(x_{R_i}) - 4h + 1 \right\}.$$

So

$$\{x \in \mathcal{V} : d(x, Q_i) < r_i\} = \left\{ x \in \mathcal{V} : x \in \tilde{R}_i \vee d(x, b) < r_i \vee d(x, B) < r_i \right\}.$$

More precisely,

- the nodes x that are above $\tilde{R}_i$ and have dinstance $d(x, b) < r_i$ are the nodes above $\tilde{R}_i$ with level

$$l(b) < l(x) < l(b) + r_i,$$

$$\begin{aligned} l(x) &\leq l(x_{R_i}) - \lceil \frac{h}{2} \rceil + \lceil \frac{h}{4} \rceil - 1 \\ &\leq l(x_{R_i}) - \frac{h}{2} + \frac{h}{4} + 1 - 1 = \\ &= l(x_{R_i}) - \frac{h}{4}. \end{aligned}$$

- the nodes x that lie below $\tilde{R}_i$ and have $d(x, B) < r_i$ are the nodes below $\tilde{R}_i$ with level

$$\begin{aligned} l(x) &\geq l(x_{R_i}) - 4h + 1 - \left(\lceil \frac{h}{4} \rceil - 1 \right) \\ &= l(x_{R_i}) - 4h - \lceil \frac{h}{4} \rceil + 2 \\ &\geq l(x_{R_i}) - 4h - \frac{h}{4} - 1 + 2 \\ &= l(x_{R_i}) - \frac{17}{4} h + 1. \end{aligned}$$

So Q_i^* is a (non admissible) trapezoid containing nodes below x_{R_i} having level l satisfying the constraint:

$$\frac{h}{4} \leq l(x_{R_i}) - l \leq \frac{17}{4} h - 1$$

which means it contains at most $\frac{17}{4}h - 1 - \frac{h}{4} + 1 = \frac{16}{4}h = 4h$ levels.

Each level has measure equal to $\mu(\{x_{R_i}\}) = q^{l(x_{R_i})}$, then the measure of Q_i^* satisfies:

$$\mu(Q_i^*) \leq 4hq^{l(x_{R_i})} \leq (7h-1)q^{l(x_{R_i})} \leq 2\mu(Q_i)\,,$$

since

$$2\mu(Q_i) = 2\mu(\tilde{R}_i) = \begin{cases} 7hq^{l(x_{R_i})} & \text{if h even} \\ (7h-1)q^{l(x_{R_i})} & \text{if h odd} \end{cases} \geq (7h-1)q^{l(x_{R_i})}\,.$$

- $$\begin{aligned} \sum_i \mu(Q_i^*) &\leq \sum_i 2\mu(Q_i) = 2\sum_i \mu(\tilde{R}_i) \\ &\leq 2\sum_i 4\mu(R_i) \leq 8\sum_i \mu(R_i) \\ &\leq \frac{8\,\|f\|_{L^1(\mathcal{V},\mu)}}{\lambda}. \end{aligned}$$

- $$\begin{aligned} \sum_i \int_{\mathcal{V}} |f_i| d\mu &= \sum_i \int_{\mathcal{V}} \left| h_i - \left(\sum_{x\in\mathcal{V}} h_i(x) q^{l(x)} \right) \frac{\chi_{R_i}}{\mu(R_i)} \right| d\mu \\ &\leq \sum_i \left(\int_{\mathcal{V}} |h_i| d\mu + \left| \sum_{x\in\mathcal{V}} h_i(x) q^{l(x)} \right| \int_{\mathcal{V}} \frac{|\chi_{R_i}|}{\mu(R_i)} d\mu \right) \\ &\leq \sum_i \left(\int_{\mathcal{V}} |h_i| d\mu + \sum_{x\in\mathcal{V}} |h_i(x)| q^{l(x)} \right) \\ &= 2\sum_i \int_{\mathcal{V}} |h_i| d\mu = \\ &= 2\sum_i \int_{U_i} |f| d\mu \\ &\leq 2\int_{\mathcal{V}} |f| = 2\,\|f\|_{L^1(\mathcal{V},\mu)} \\ \implies \sum_i \|f_i\|_{L^1(\mathcal{V},\mu)} &\leq 2\,\|f\|_{L^1(\mathcal{V},\mu)}\,. \end{aligned}$$

- The function g is bounded.

  To show that, we observe that

- $g = f$ outside $E = \bigcup_i \tilde{R}_i$.

  Indeed,
  $E = \bigcup_i \tilde{R}_i = \bigcup_i U_i$ where $U_i = \tilde{R}_i - \bigcup_{j<i} \tilde{R}_j \supset \operatorname{supp} h_i$.
  If $x \notin E \implies x \notin \operatorname{supp} h_i \; \forall i$ then

$$
\begin{aligned}
g(x) &= f(x) - \sum_i f_i(x) = \\
&= f(x) - \sum_i \left(h_i(x) - \left[\sum_y h_i(y) q^{l(y)} \right] \frac{\chi_{R_i}(x)}{\mu(R_i)} \right) \\
&= f(x)
\end{aligned}
$$

  where $\chi_{R_i}(x) = 0$ because $R_i \subset \tilde{R}_i$ so $x \notin \tilde{R}_i \; \forall i \implies x \notin R_i \; \forall i$.

- $g = \sum_i \left(\sum_{x \in \mathcal{V}} h_i(x) q^l(x) \right) \frac{\chi_{R_i}}{\mu(R_i)}$ on $E = \bigcup_i \tilde{R}_i$.

  Indeed $E = \bigcup_i \tilde{R}_i = \bigcup_i U_i$ where the $\tilde{R}_i$ are not necessarily disjoint while the U_i are disjoint by construction.

  Then if $x \in E$, x belongs exactly to one of the U_i which we denote by $U_{\bar{i}}$. We have:

$$
h_i(x) = \begin{cases} 0 & \forall i \neq \bar{i} \\ f(x) & \text{if } i = \bar{i} \end{cases}
$$

  and then

$$
\begin{aligned}
g(x) &= f(x) - \sum_i f_i(x) \\
&= f(x) - \sum_i \left(h_i(x) - \left[\sum_y h_i(y) q^{l(y)} \right] \frac{\chi_{R_i}(x)}{\mu(R_i)} \right) \\
&= f(x) - h_{\bar{i}}(x) + \sum_i \left(\sum_y h_i(y) q^{l(y)} \right) \frac{\chi_{R_i}(x)}{\mu(R_i)} \\
&= \sum_i \left(\sum_y h_i(y) q^{l(y)} \right) \frac{\chi_{R_i}(x)}{\mu(R_i)}.
\end{aligned}
$$

- $|g(x)| \leq \lambda$ outside E.

Indeed, if $x \notin E$:

$$\begin{aligned}|g(x)| &= |f(x)| \\ &\leq \mu(\bar{R})^{-1} \sum_{y \in \bar{R}} |f(y)| q^{l(y)} \\ &\leq \sup_{R : x \in R} \mu(R)^{-1} \sum_{y \in R} |f(y)| q^{l(y)} \\ &= (Mf)(x) \leq \lambda\end{aligned}$$

where $\bar{R} = \{x\}$ with $\mu(\bar{R}) = q^{l(x)}$ is an admissible trapezoid.

– $\sup_{x \in E} |g(x)| \leq 24q\lambda$.
Indeed,

$$\begin{aligned}\sup_{x \in E} |g(x)| &\leq \sup_{x \in E} \left| \sum_i \left(\sum_y h_i(y) q^{l(y)} \right) \frac{\chi_{R_i}(x)}{\mu(R_i)} \right| \\ &= \sup_i \left| \frac{\sum_y h_i(y) q^l(y)}{\mu(R_i)} \right| \\ &\leq \sup_i \frac{\sum_y |h_i(y)| q^{l(y)}}{\mu(R_i)} \\ &\leq \sup_i \frac{6q\lambda \mu(\tilde{R}_i)}{\mu(R_i)} \\ &\leq \sup_i \frac{6q\lambda 4 \mu(R_i)}{\mu(R_i)} = 24q\lambda.\end{aligned}$$

where the second inequality follow from the fact that the R_i are disjoint and the function to be evaluated is constant on each R_i.

The previous considerations about g show that g is bounded, more precisely:

$$\exists c : |g| \leq c\lambda$$

for example, one can take $c = 24q$.

Summing up, we have proved that $\forall f \in L^1(\mathcal{V}, \mu)$ and $\forall \lambda > 0$ it exists a decomposition of f that satisfies the properties 1) to 6) of the definition 3.1.1.

This concludes the proof of the theorem. □

Chapter 6

Main results

Calderón–Zygmund theory developed in previous chapters can be used to study boundedness properties of some integral operators. In particular we are interested in operators related to the Laplacian $\mathcal{L}$ on the infinite homogeneous tree, namely spectral multipliers of $\mathcal{L}$ and Riesz transform.

6.1 Spectral multipliers

From the functional calculus recalled in Section 2.2 and for the self-adjointness property of the Laplacian $\mathcal{L}$ proved in Chapter 4, we know that $H(\mathcal{L})$ is a bounded linear operator on $L^2(\mathcal{V}, \mu)$ for every continuous function H defined on the spectrum $\sigma(\mathcal{L}) = [0, 2]$. In this section we present conditions on the function H which guarantee that H is an L^p-spectral multiplier for the Laplacian for $p \in (1, \infty)$, i.e. that the restriction of $H(\mathcal{L})$ to $L^p(\mathcal{V}, \mu) \cap L^2(\mathcal{V}, \mu)$ can be extended to a bounded operator on $L^p(\mathcal{V}, \mu)$ for $1 < p < \infty$, and that $H(\mathcal{L})$ is a weak type $(1, 1)$ operator.

The main result is stated in Theorem 6.1.5 and is obtained as an application of the abstract Calderón–Zygmund Theorem 3.2.2 in the context of the tree. Before stating it and giving its proof, we need some preliminary results.

6.1.1 Preliminary results

Let F be a real function with support $\operatorname{supp} F \subset [0,2)$. By [FTP83] and [HS03] we know that $F(\mathcal{L})$ is an integral operator with kernel

$$F(\mathcal{L})(x,y) = Re\left(K(x,y)\int_0^\pi F(1-\cos\theta)e^{i\theta d(x,y)}\eta(\theta)\sin\theta d\theta\right) \tag{6.1}$$
$$= Re\left(K(x,y)E_F(d(x,y))\right) \tag{6.2}$$

where

$$K(x,y) = q^{\frac{-l(x)-l(y)-d(x,y)}{2}}, \tag{6.3}$$
$$\eta(\theta) = \frac{2}{\pi i\left(e^{-i\theta}-\frac{1}{q}e^{i\theta}\right)}, \tag{6.4}$$
$$E_F(k) = \int_0^\pi F(1-\cos\theta)e^{i\theta k}\eta(\theta)\sin\theta d\theta\,, \tag{6.5}$$

for $x,y \in \mathcal{V}, k \in \mathbb{N}$, $\theta \in [0,\pi)$.

Definition 6.1.1. For $s \geq 0$, the Sobolev space H^s is defined by:

$$H^s = \left\{F : [0,\pi) \to \mathbb{C} : \sum_{k\in\mathbb{Z}} |\hat{F}(k)|^2(1+|k|)^{2s} < \infty\right\}$$

where

$$\hat{F}(k) = \frac{1}{\sqrt{\pi}}\int_0^\pi F(t)e^{itk}dt\,.$$

We denote by $\|\cdot\|_{H^s}$ the norm on H^s defined by

$$\|F\|_{H^s} = \left(\sum_{k\in\mathbb{Z}} |\hat{F}(k)|^2(1+|k|)^{2s}\right)^{\frac{1}{2}}.$$

Lemma 6.1.1. *Let $s \geq 0$, $m \in \mathbb{N}$ such that $m \geq s$. Let $a < b$, $c < d$ fixed constants.*

Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is in C^m and is an increasing function such that $\phi(c) < a$, $\phi(d) > b$, $\phi' > 0$. Then there is a positive constant C depending only on $a, b, c, d, s, \|\frac{1}{\phi'}\|_{L^\infty}, \|\phi\|_{C^m}$ such that

$$\|F\circ\phi\|_{H^s} \leq C\|F\|_{H^s} \qquad \forall F \text{ with } \operatorname{supp} F \subset [a,b]\,.$$

Definition 6.1.2. For $t > 0$ and $H : \mathbb{R} \to \mathbb{C}$, we define the dilation operator D_t by the following formula:

$$(D_t H)(\lambda) = H(t\lambda) \qquad \forall \lambda \in \mathbb{R}\,.$$

Lemma 6.1.2. *Fix $\epsilon \in (0, 1]$. If $s > \frac{3}{2} + \epsilon$ then there is a constant $C > 0$ such that for each integer $n \in \mathbb{N}$ and for each function $F : \mathbb{R} \to \mathbb{C}$ such that*

$$\operatorname{supp} F \subset [2^{-2n-1}, 2^{-2n+2}] \cap [0, 2] \tag{6.6}$$

it holds

$$\sum_{k=0}^{\infty} |E_F(k)|(1 + k)(1 + 2^{-n}k)^{\epsilon} \leq C\|D_{2^{-2n}}F\|_{H^s} \tag{6.7}$$

$$\sum_{k=0}^{\infty} |E_F(k)|(1 + 2^{-n}k)^{\epsilon} \leq C2^{-n}\|D_{2^{-2n}}F\|_{H^s} \tag{6.8}$$

$$\sum_{k=0}^{\infty} |E_F(k + 1) - E_F(k)|(1 + k)(1 + 2^{-n}k)^{\epsilon} \leq C2^{-n}\|D_{2^{-2n}}F\|_{H^s}\,. \tag{6.9}$$

Proof. We define G as:

$$G(x) = (D_{2^{-2n}}F)(x) = F(2^{-2n}x) \qquad \forall x \in \mathbb{R}$$

that is

$$F(x) = (D_{2^{2n}}G)(x) = G(2^{2n}x) \qquad \forall x \in \mathbb{R}.$$

We consider formula 6.5 and we observe that $1 - \cos\theta$ is monotone increasing over $[0, \pi]$. Moreover:

$$\begin{aligned} 1 - \cos(2^{-n}) &= 2\sin^2(2^{-n-1}) \\ &\leq 2(2^{-n-1})^2 = 2^{-2n-1}\,. \end{aligned}$$

Then by 6.6, $F(1 - \cos\theta) = 0$ in the interval $(0, 2^{-n})$. Similarly:

$$1 - \cos(2^{-n}\pi) \geq \min(2^{-2n+2}, 2)\,.$$

Then by 6.6, $F(1 - \cos\theta) = 0$ in the interval $(2^{-n}\pi, \pi)$. Then we can restrict the integral in 6.5 to $(2^{-n}, 2^{-n}\pi)$ and perform the change of variables $t = 2^n\theta$

to obtain

$$\begin{aligned}
E_F(k) &= \int_{2^{-n}}^{2^{-n}\pi} F(1-\cos\theta)e^{i\theta k}\eta(\theta)\sin\theta d\theta \\
&= \int_1^{\pi} F(1-\cos(2^{-n}t))e^{i2^{-n}tk}\eta(2^{-n}t)\sin(2^{-n}t)2^{-n}dt \\
&= 2^{-n}\int_1^{\pi} F(2\sin^2(2^{-n-1}t))e^{i2^{-n}tk}\eta(2^{-n}t)\sin(2^{-n}t)dt \\
&= 2^{-n}\int_1^{\pi} G(2^{2n+1}\sin^2(2^{-n-1}t))e^{i2^{-n}tk}\eta(2^{-n}t)\sin(2^{-n}t)dt \\
&= 2^{-2n}\int_1^{\pi} H(t)\psi_l(t)e^{itm}dt \\
&= 2^{-2n}\int_1^{\pi} \tilde{H}_l(t)e^{itm}dt\,, \qquad (6.10)
\end{aligned}$$

where we have introduced

$$k = 2^n m + l \quad \text{with } 0 \le l < 2^n\,, \qquad (6.11)$$

$$\psi_l(t) = e^{i2^{-n}tl}2^n\sin(2^{-n}t)\eta(2^{-n}t)\,, \qquad (6.12)$$

$$H(t) = G(2^{2n+1}\sin^2(2^{-n-1}t))\,, \qquad (6.13)$$

$$\tilde{H}_l(t) = H(t)\psi_l(t)\,. \qquad (6.14)$$

We now claim that ψ_l and its derivatives are bounded by a constant that does not depend on n, l, t in $[0,\pi]$.

Now we verify the claim for ψ_l.

$$\begin{aligned}
|\psi_l(t)| &= 2^n\sin(2^{-n}t)|\eta(2^{-n}t)| \\
&\le 2^n(2^{-n}t)\frac{2}{\pi|e^{-i2^{-n}t}-\frac{1}{q}e^{i2^{-n}t}|} \\
&\le t\frac{2}{\pi(1-\frac{1}{q})} \\
&\le \frac{2}{1-\frac{1}{q}}\,.
\end{aligned}$$

For the derivatives of ψ_l the proof is similar.

We also claim that the function $\phi_n(t) := 2^{2n+1}\sin^2(2^{-n-1}t)$, $t\in[0,\pi]$ and all its derivatives are bounded by a constant that does not depend on n. Indeed,

$$\begin{aligned}|\phi_n(t)| &= 2^{2n+1}\sin^2(2^{-n-1}t)\\ &\le 2^{2n+1}(2^{-n-1}t)^2\\ &= \frac{t^2}{2} \le \frac{\pi^2}{2}\,,\end{aligned}$$

and similarly for all derivatives of ϕ_n.

This implies that

$$\|\phi_n\|_{C^j} \le C < \infty \qquad \forall j \in \mathbb{N}$$

with C not depending on n.

Moreover $|\phi_n'|$ is bounded from below by a positive constant independent of n on the set $\phi_n^{-1}(\operatorname{supp} G)$, where $\operatorname{supp} G \subset [\frac{1}{2}, 4]$. Indeed, if $t \in \phi_n^{-1}(\operatorname{supp} G)$, then

$$\begin{aligned}2^{2n+1}\sin^2(2^{-n-1}t) &\ge \frac{1}{2} = 2^{-1}\\ \implies \sin^2(2^{-n-1}t) &\ge 2^{-2n-2}\\ \implies 2^{-n-1}t \ge \sin(2^{-n-1}t) &\ge 2^{-n-1}\end{aligned}$$

which implies that $t \ge 1$. Moreover,

$$\begin{aligned}&2^{2n+1}\sin^2(2^{-n-1}t) \le 4 = 2^2\\ &\implies \sin^2(2^{-n-1}t) \le 2^{2-2n-1} = 2^{1-2n}\\ &\implies \sin(2^{-n-1}t) \le 2^{-n+\frac{1}{2}} \le 2^{-\frac{1}{2}} \quad \text{for } n \ge 1.\end{aligned}$$

This implies that $2^{-n-1}t \le \frac{\pi}{4} = \arcsin(2^{-\frac{1}{2}})$.

Thus:

$$\begin{aligned}|\phi_n'(t)| &= 2^{2n+1}2\sin(2^{-n-1}t)\cos(2^{-n-1}t)2^{-n-1}\\ &= 2^{n+1}\cos(2^{-n-1}t)\sin(2^{-n-1}t)\\ &\ge 2^{n+1}2^{-\frac{1}{2}}\frac{2^{-n-1}}{2}\\ &\ge 2^{n+1}2^{-\frac{1}{2}}2^{-n-2} = 2^{-\frac{3}{2}} > 0\,.\end{aligned}$$

Then $\|\frac{1}{\phi'}\|_{L^\infty}$ is bounded by a constant that does not depend on n.

It is now possible to apply Lemma 6.1.1 to conclude that:

$$\begin{aligned}\|\tilde{H}_l\|_{H^s} &\le C\|H\|_{H^s}\|\psi_l\|_{C^m}\\ &= C\|G\circ\phi_n\|_{H^s}\\ &\le C\|G\|_{H^s}\,. \end{aligned} \tag{6.15}$$

Then:

$$\sum_{k=0}^{\infty} |E_F(k)|(1+k)(1+2^{-n}k)^{\epsilon} = \\ = \sum_{l=0}^{2^n-1} \sum_{m=0}^{\infty} |E_F(2^n m + l)|(1 + 2^n m + l)(1 + m + 2^{-n} l)^{\epsilon}.$$

We notice that:

$$\begin{aligned}(1 + 2^n m + l) &= 2^n(m + 2^{-n}(1 + l)) \\ &\leq 2^n(m + 1).\end{aligned}$$

$$\begin{aligned}(1 + m + 2^{-n} l)^{\epsilon} &\leq (1 + m + 2^{-n}(2^n - 1))^{\epsilon} \\ &= (1 + m + 1 - 2^{-n})^{\epsilon} \\ &= (2 + m - 2^{-n})^{\epsilon} \\ &\leq (2 + m)^{\epsilon} \\ &\leq (2 + 2m)^{\epsilon} \\ &= 2^{\epsilon}(m + 1)^{\epsilon} \\ &\leq 2(m + 1)^{\epsilon}.\end{aligned}$$

Then by applying Cauchy-Schwarz inequality in the inner sum we get:

$$\begin{aligned}\sum_{k=0}^{\infty} |E_F(k)|(1+k)(1+2^{-n}k)^{\epsilon} &= \sum_{l=0}^{2^n-1} \sum_{m=0}^{\infty} |E_F(2^n m + l)|(m+1)2^n(m+1)^{\epsilon} 2 \\ &= 2^{n+1} \sum_{l=0}^{2^n-1} \sum_{m=0}^{\infty} |E_F(2^n m + l)|(m+1)^{1+\epsilon} \\ &\leq 2^{n+1} \sum_{l=0}^{2^n-1} \left(\sum_{m=0}^{\infty} (|E_F(2^n m + l)|(1+m)^s)^2 \right)^{\frac{1}{2}} \left(\sum_{m=0}^{\infty} (1+m)^{2(1+\epsilon-s)} \right)^{\frac{1}{2}}.\end{aligned}$$

The last sum converges since $s > \frac{3}{2} + \epsilon$ so $2(1 + \epsilon - s) < -1$. Using 6.10 and 6.15 we get:

$$
\begin{aligned}
&\sum_{k=0}^{\infty}|E_F(k)|(1+k)(1+2^{-n}k)^{\epsilon}\\
&\qquad \le C2^{n+1}\sum_{l=0}^{2^n-1}\left(\sum_{m=0}^{\infty}\left(2^{-2n}\left|\int_1^{\pi}\tilde{H}_l(t)e^{itm}dt\right|(1+m)^s\right)^2\right)^{\frac{1}{2}}\\
&\qquad \le C2^{n+1}2^{-2n}\sum_{l=0}^{2^n-1}\|\tilde{H}_l\|_{H^s}\\
&\qquad \le C2^{-n+1}(2^n-1)\|G\|_{H^s}\\
&\qquad = C\|G\|_{H^s}\\
&\qquad = C\|D_{2^{-2n}}F\|_{H^s}.
\end{aligned}
$$

This proves (6.7). The proof of (6.8) is similar:

$$
\begin{aligned}
\sum_{k=0}^{\infty}|E_F(k)|(1+2^{-n}k)^{\epsilon} &\le \sum_{l=0}^{2^n-1}\sum_{m=0}^{\infty}|E_F(2^n m+l)|(m+1)^{\epsilon}2\\
&= 2\sum_{l=0}^{2^n-1}\sum_{m=0}^{\infty}|E_F(2^n m+l)|(m+1)^{\epsilon}\\
&\le 2\sum_{l=0}^{2^n-1}\left(\sum_{m=0}^{\infty}(|E_F(2^n m+l)|(1+m)^s)^2\right)^{\frac{1}{2}}\left(\sum_{m=0}^{\infty}(1+m)^{2(\epsilon-s)}\right)^{\frac{1}{2}}\\
&\le C\sum_{l=0}^{2^n-1}\left(\sum_{m=0}^{\infty}\left(2^{-2n}\left|\int_1^{\pi}\tilde{H}_l(t)e^{itm}dt\right|(1+m)^s\right)^2\right)^{\frac{1}{2}}\\
&\le C2^{-2n}\sum_{l=0}^{2^{n-1}}\|\tilde{H}_l\|_{H^s}\\
&\le C2^{-2n}(2^n-1)\|G\|_{H^s}\\
&\le C2^{-n}\|G\|_{H^s}\\
&= C2^{-n}\|D_{2^{-2n}}F\|_{H^s}\,,
\end{aligned}
$$

where we applied 6.10 and 6.15.

We now pass to 6.9. First we compute:

$$\begin{aligned}
&E_F(k+1) - E_F(k)\\
&= 2^{-n}\int_1^{\pi} G(2^{2n+1}\sin^2(2^{-n-1}t))\eta(2^{-n}t)\sin(2^{-n}t)\left[e^{i(k+1)2^{-n}t} - e^{ik2^{-n}t}\right] dt\\
&= 2^{-n}\int_1^{\pi} \left[G(2^{2n+1}\sin^2(2^{-n-1}t))\eta(2^{-n}t)\sin(2^{-n}t)e^{ik2^{-n}t}\right]\left(e^{i2^{-n}t} - 1\right) dt
\end{aligned}$$

which implies

$$\begin{aligned}
|E_F(k+1) - E_F(k)| &\leq \sup_{t\in(1,\pi)}\left(\left|e^{i2^{-n}t} - 1\right|\right)|E_F(k)|\\
&\leq 2^{-n}\pi|E_F(k)|\,.
\end{aligned}$$

In conclusion we apply 6.8 to deduce that:

$$\begin{aligned}
&\sum_{k=0}^{\infty}|E_F(k+1) - E_F(k)|(1+k)(1+2^{-n}k)^{\epsilon}\\
&\qquad\qquad \leq 2^{-n}\pi\sum_{k=0}^{\infty}|E_F(k)|(1+k)(1+2^{-n}k)^{\epsilon}\\
&\qquad\qquad \leq C2^{-n}\|D_{2^{-2n}}F\|_{H^s}\,.
\end{aligned}$$

□

Lemma 6.1.3. *Take $y, z \in \mathcal{V}$ such that $d(y,z) = 1$ and $l(z) = l(y) - 1$, and let $k \in \mathbb{N}$. Then:*

$$\sum_{x:d(x,y)=k} K(x,y)q^{l(x)} = \begin{cases} 1 & \text{if } k = 0\\ 2 + \frac{q-1}{q}(k-1) & \text{if } k > 0\,;\end{cases} \tag{6.16}$$

$$\sum_{x:d(x,y)=k} |K(x,y) - K(x,z)|q^{l(x)} < 1\,; \tag{6.17}$$

$$\sum_{x:d(x,y)=k} |\nabla_x K(x,y)|q^{l(x)} \leq q - \frac{1}{q}\,. \tag{6.18}$$

Proof. We first prove 6.16.

$$\begin{aligned}
\sum_{x:d(x,y)=k} K(x,y)q^{l(x)} &= \sum_{x:d(x,y)=k} q^{\frac{-l(y)-l(x)-d(x,y)}{2}}q^{l(x)}\\
&= \sum_{x:d(x,y)=k} q^{\frac{-l(y)+l(x)-k}{2}}\,.
\end{aligned}$$

Keeping in mind the characterization of the nodes of the sphere $S_k(y)$ explained in Section 4.3.1, we split the sum in three components according to their level:

$$\sum_{x:d(x,y)=k} K(x,y)q^{l(x)} =$$
$$= q^k q^{\frac{-k-k}{2}} + \left(\sum_{p=1}^{k-1}(q-1)q^{k-p-1}q^{\frac{2p-k-k}{2}}\right) + q^{\frac{2k-k-k}{2}}$$
$$= 1 + \frac{q-1}{q}\left(\sum_{p=1}^{k-1} 1\right) + 1$$
$$= 2 + \frac{q-1}{q}(k-1).$$

The last equality holds for $k-1 \geq 1$ i.e. $k \geq 2$. When $k = 1$ direct computation shows that 6.16 equals 2 so the result is still valid. When $k = 0$ the only term in the sum appearing in 6.16 is for $x = y$ and $K(y,y)q^{l(y)} = 1$.

We now prove 6.17. First we show that if x does not lie below z, then $K(x,y) = K(x,z)$.

Let suppose x does not lie below z.

$$K(x,z) = q^{\frac{-l(z)-l(x)-d(x,z)}{2}}$$
$$= q^{\frac{-(l(y)-1)-l(x)-(d(x,y)+1)}{2}}$$
$$= q^{\frac{-l(y)-l(x)-d(x,y)}{2}} = K(x,y).$$

Then we have that:

$$\sum_{x:d(x,y)=k} |K(x,y) - K(x,z)|q^{l(x)} = \sum_{x:d(x,y)=k,\ x \text{ below } z} |K(x,y) - K(x,z)|q^{l(x)}$$
$$= \sum_{x:d(x,y)=k,\ x \text{ below } z} \left|q^{\frac{-l(y)+l(x)-d(x,y)}{2}} - q^{\frac{-l(z)+l(x)-d(x,z)}{2}}\right|$$
$$= q^{k-1}\left|q^{-k-k} - q^{-k+1-k+1}\right|$$

where the last equality is due to the fact that:

$$l(x) - l(y) = -k, \quad l(x) - l(z) = -k+1, \quad l(z) = l(y) - 1$$
$$d(x,y) = k, \quad d(x,z) = k-1.$$

It follows that:

$$\begin{aligned}\sum_{x:d(x,y)=k} |K(x,y) - K(x,z)| q^{l(x)} &= q^{k-1} \left| q^{-2k} - q^{-2k+2} \right| \\ &= q^{k-1-2k} |1 - q^{+2}| \\ &= q^{-k-1} |1 - q^{+2}| \\ &\leq \frac{q^2-1}{q^{k+1}} \\ &\leq \frac{q^2-1}{q^2} < 1\end{aligned}$$

if $k \geq 1$.

If $k = 0$, the sum in 6.17 consist of just one term and we can compute it explicitly:

$$\begin{aligned}|K(y,y) - K(y,z)| q^{l(y)} &= \left| 1 - q^{\frac{l(y)-l(z)-d(y,z)}{2}} \right| \\ &= \left| 1 - q^{\frac{1-1}{2}} \right| \\ &= 0 < 1 \,.\end{aligned}$$

In conclusion, we prove 6.18. We recall that

$$\nabla_x K(x,y) = \sum_{w:d(w,x)=1} |K(w,y) - K(x,y)| \,.$$

We have to distinguish two cases. To this aim we adopt the following notation: we label vertices in the neighborhood of x in such a way that w_0 is the father of x and $w_1, \ldots, w_q$ are the children of x.

Case 1): $d(w_0, y) < d(x, y)$, i.e. y can be reached from x going up. In this case we have:

$$\begin{aligned}K(w_0, y) &= q^{\frac{-l(y)-l(w_0)-d(y,w_0)}{2}} \\ &= q^{\frac{-l(y)-(l(x)+1)-(d(y,x)-1)}{2}} \\ &= q^{\frac{-l(y)-l(x)-d(x,y)}{2}} = K(x,y) \,.\end{aligned}$$

In the same case, if we consider a neighbor $w \neq w_0$.

$$\begin{aligned}K(w, y) &= q^{\frac{-l(y)-l(w)-d(y,w)}{2}} \\ &= q^{\frac{-l(y)-(l(x)-1)-(d(y,x)+1)}{2}} \\ &= q^{\frac{-l(y)-l(x)-d(x,y)}{2}} = K(x,y) \,.\end{aligned}$$

So in Case 1) we have that

$$\nabla_x K(x,y) = 0.$$

Case 2): $d(w_0, y) > d(x, y)$, i.e. y can be reached from x going down. We denote by w^* the child of x such that $d(w^*, y) < d(x, y)$. In this case we have:

$$\begin{aligned}
K(w^*, y) &= q^{\frac{-l(y)-l(w^*)-d(y,w^*)}{2}} \\
&= q^{\frac{-l(y)-(l(x)-1)-(d(y,x)-1)}{2}} \\
&= q^{1+\frac{-l(y)-l(x)-d(x,y)}{2}} = qK(x,y)\,,
\end{aligned}$$

$$\begin{aligned}
K(w_0, y) &= q^{\frac{-l(y)-l(w_0)-d(y,w_0)}{2}} \\
&= q^{\frac{-l(y)-(l(x)+1)-(d(y,x)+1)}{2}} \\
&= q^{-1+\frac{-l(y)-l(x)-d(x,y)}{2}} = \frac{1}{q}K(x,y)\,.
\end{aligned}$$

In the same case, if we consider a neighbor $w \neq w^*, w_0$ we have that:

$$\begin{aligned}
K(w, y) &= q^{\frac{-l(y)-l(w)-d(y,w)}{2}} \\
&= q^{\frac{-l(y)-(l(x)-1)-(d(y,x)+1)}{2}} \\
&= q^{\frac{-l(y)-l(x)-d(x,y)}{2}} = K(x,y)\,.
\end{aligned}$$

Summing up, in Case 2) we have:

$$\begin{aligned}
\nabla_x K(x,y) &= |K(w^*, y) - K(x,y)| + |K(w_0, y) - K(x,y)| \\
&= |qK(x,y) - K(x,y)| + |\frac{1}{q}K(x,y) - K(x,y)| \\
&= (q-1)K(x,y) + (1 - \frac{1}{q})K(x,y) \\
&= K(x,y)(q - \frac{1}{q}).
\end{aligned}$$

In the sum over $x : d(x, y) = k$ appearing in 6.18, $\nabla_x K(x, y) \neq 0$ only if y lies below x: this happens for a single node $\hat{x}$ with $l(\hat{x}) = l(y) + k$. Then we have:

$$\begin{aligned}\sum_{x:d(x,y)=k}|\nabla_x K(x,y)|q^{l(x)} &= \nabla_x K(\hat{x},y)q^{l(\hat{x})}\\ &= K(\hat{x},y)(q-\frac{1}{q})q^{l(y)+k}\\ &= q^{\frac{-l(y)-l(\hat{x})-d(y,\hat{x})}{2}}(q-\frac{1}{q})q^{l(y)+k}\\ &= q^{\frac{-2l(y)-k-k}{2}}q^{l(y)+k}(q-\frac{1}{q}) = q-\frac{1}{q}.\end{aligned}$$

This concludes the proof.

□

Lemma 6.1.4. *Fix $\epsilon \in (0,1]$. If $s > \frac{3}{2}+\epsilon$ then there is a constant $C > 0$ such that for each integer $n \in \mathbb{N}$ and for each function F such that*

$$\operatorname{supp} F \subset [2^{-2n-1}, 2^{-2n+2}] \cap [0,2]$$

it holds

$$\sum_x |F(\mathcal{L})(x,y) - F(\mathcal{L})(x,z)|q^{l(x)} \leq C2^{-n}d(y,z)\|D_{2^{-2n}}F\|_{H^s}\,; \tag{6.19}$$

$$\sum_x |F(\mathcal{L})(x,y)|(1-2^{-n}d(x,y))^{\epsilon}q^{l(x)} \leq C\|D_{2^{-2n}}F\|_{H^s}\,; \tag{6.20}$$

$$\sum_x |\nabla_x F(\mathcal{L})(x,y)|(1-2^{-n}d(x,y))^{\epsilon}q^{l(x)} \leq C2^{-n}\|D_{2^{-2n}}F\|_{H^s}\,. \tag{6.21}$$

Proof. We start proving 6.19. It is sufficient to prove it for F real. Moreover it is sufficient to study the case $d(y,z)=1$. Indeed, suppose that 6.19 holds for points whose distance is 1. For y and z having distance greater than 1 we can denote by γ the only path joining y and z having length $d(y,z)$:

$$\gamma = (y = v_0, v_1, \ldots, v_{d(y,z)} = z).$$

Then we can write:

$$\begin{aligned}
&\sum_x |F(\mathcal{L})(x,y) - F(\mathcal{L})(x,z)| q^{l(x)} = \\
&\quad = \sum_x \left| \sum_{j=0}^{d(y,z)-1} F(\mathcal{L})(x,v_j) - F(\mathcal{L})(x,v_{j+1}) \right| q^{l(x)} \\
&\quad \leq \sum_x \sum_{j=0}^{d(y,z)-1} |F(\mathcal{L})(x,v_j) - F(\mathcal{L})(x,v_{j+1})| \, q^{l(x)} \\
&\quad = \sum_{j=0}^{d(y,z)-1} \sum_x |F(\mathcal{L})(x,v_j) - F(\mathcal{L})(x,v_{j+1})| \, q^{l(x)} .
\end{aligned}$$

Since $d(v_j, v_{j+1}) = 1$ we can apply 6.19 to each term obtaining:

$$\begin{aligned}
\sum_x |F(\mathcal{L})(x,y) - F(\mathcal{L})(x,z)| q^{l(x)} &\leq \sum_{j=0}^{d(y,z)-1} C 2^{-n} \|D_{2^{-2n}} F\|_{H^s} \\
&= C 2^{-n} d(y,z) \|D_{2^{-2n}} F\|_{H^s} .
\end{aligned}$$

Lastly we can assume that $l(z) = l(y) - 1$. Indeed, if we have $l(z) = l(y) + 1$ it is sufficient to exchange the roles of z and y, since 6.19 is symmetric with respect to this swap.

We can now prove the claim for $y, z \in \mathcal{V}$ s.t. $d(y,z) = 1$, $l(z) = l(y) - 1$.

$$\begin{aligned}
&\sum_x |F(\mathcal{L})(x,y) - F(\mathcal{L})(x,z)| q^{l(x)} = \\
&\quad = \sum_x \left| Re\left(K(x,y)E_F(d(x,y))\right) - Re\left(K(x,z)E_F(d(x,z))\right)\right| q^{l(x)} \\
&\quad = \sum_x \left| Re\left(K(x,y)E_F(d(x,y)) - K(x,z)E_F(d(x,z))\right)\right| q^{l(x)} \\
&\quad \le \sum_x \left| K(x,y)E_F(d(x,y)) - K(x,z)E_F(d(x,z))\right| q^{l(x)} \\
&\quad \le \sum_x |K(x,y)E_F(d(x,y)) - K(x,z)E_F(d(x,y)) + \\
&\qquad\qquad + K(x,z)E_F(d(x,y)) - K(x,z)E_F(d(x,z))| q^{l(x)} \\
&\quad = \sum_x |K(x,y) - K(x,z)||E_F(d(x,y))| q^{l(x)} + \\
&\qquad\qquad + \sum_x K(x,z)|E_F(d(x,y)) - E_F(d(x,z))| q^{l(x)} \\
&\quad = S_1 + S_2.
\end{aligned}$$

We estimate the two terms S_1 and S_2 separately.

$$\begin{aligned}
S_1 &= \sum_{k=0}^{\infty} \sum_{x:d(x,y)=k} |K(x,y) - K(x,z)||E_F(d(x,y))| q^{l(x)} \\
&= \sum_{k=0}^{\infty} |E_F(k)| \left(\sum_{x:d(x,y)=k} |K(x,y) - K(x,z)| q^{l(x)} \right).
\end{aligned}$$

We can apply 6.17 to the quantity in brackets, to obtain:

$$\begin{aligned}
S_1 &\le \sum_{k=0}^{\infty} |E_F(k)| \\
&\le \sum_{k=0}^{\infty} |E_F(k)|(1 + 2^{-n}k)^{\epsilon} \\
&\le C 2^{-n} \|D_{2^{-2n}} F\|_{H^s}.
\end{aligned}$$

The last inequality is due to 6.8.

To estimate S_2 we observe that if $d(x,y)=k$ then $d(x,z)=k\pm 1$ (since $d(y,z)=1$). Then:

$$
\begin{aligned}
&|E_F(d(x,y)) - E_F(d(x,z))| \le \\
&\qquad \le |E_F(k) - E_F(k+1)| + |E_F(k) - E_F(k-1)|\,,
\end{aligned}
$$

since the left hand side is equal to exactly one of the terms on the right hand side and the other one is positive. We formally put $E_F(-1)=E_F(0)$ so that the inequality holds also for $k=0$.

$$
\begin{aligned}
S_2 &= \sum_{k=0}^{\infty} \sum_{x:d(x,y)=k} K(x,z)|E_F(d(x,y)) - E_F(d(x,z))|q^{l(x)} \\
&\le \sum_{k=0}^{\infty} \left(|E_F(k+1) - E_F(k)| + |E_F(k-1) - E_F(k)|\right) \sum_{x:d(x,y)=k} K(x,z)q^{l(x)}\,.
\end{aligned}
$$

We can apply 6.16 to the last factor, oberving that $2+\frac{q-1}{q}(k-1) \le 2+k-1 = 1+k$. So we obtain:

$$
\begin{aligned}
S_2 &\le \sum_{k=0}^{\infty} |E_F(k+1) - E_F(k)|(1+k) + \sum_{k=0}^{\infty} |E_F(k-1) - E_F(k)|(1+k) \\
&\le \sum_{k=0}^{\infty} |E_F(k+1) - E_F(k)|(1+k) + \sum_{k=1}^{\infty} |E_F(k-1) - E_F(k)|(1+k) \\
&= \sum_{k=0}^{\infty} |E_F(k+1) - E_F(k)|(1+k) + \sum_{j=0}^{\infty} |E_F(j+1) - E_F(j)|(2+j) \\
&= \sum_{k=0}^{\infty} |E_F(k+1) - E_F(k)|(1+k) + 2\sum_{j=0}^{\infty} |E_F(j+1) - E_F(j)|(1+j) \\
&= 3\sum_{k=0}^{\infty} |E_F(k+1) - E_F(k)|(1+k) \\
&\le 3\sum_{k=0}^{\infty} |E_F(k+1) - E_F(k)|(1+k)(1+2^{-n}k)^{\epsilon} \\
&\le C2^{-n}\|D_{2^{-2n}}F\|_{H^s}\,,
\end{aligned}
$$

where the last inequality follows applying 6.9. This concludes the proof of 6.19.

We pass now to the proof of 6.20.

$$\begin{aligned}\sum_x |F(\mathcal{L})(x,y)|(1-2^{-n}d(x,y))^\epsilon q^{l(x)} \leq \\ &\leq \sum_x |K(x,y)E_F(d(x,y))|(1-2^{-n}d(x,y))^\epsilon q^{l(x)} \\ &= \sum_{k=0}^{\infty} \sum_{x:d(x,y)=k} K(x,y)|E_F(k)|(1-2^{-n}k)^\epsilon q^{l(x)} \\ &= \sum_{k=0}^{\infty} |E_F(k)|(1-2^{-n}k)^\epsilon \sum_{x:d(x,y)=k} K(x,y)q^{l(x)} \,.\end{aligned}$$

We can apply 6.16 to the last factor. We get:

$$\begin{aligned}\sum_x |F(\mathcal{L})(x,y)|(1-2^{-n}d(x,y))^\epsilon q^{l(x)} &\leq \sum_{k=0}^{\infty} |E_F(k)|(1-2^{-n}k)^\epsilon (1+k) \\ &\leq C\|D_{2^{-2n}}F\|_{H^s},\end{aligned}$$

where the last inequality follows from 6.7.

Lastly we prove 6.21.

$$\begin{aligned}&\sum_x |\nabla_x F(\mathcal{L})(x,y)|(1-2^{-n}d(x,y))^\epsilon q^{l(x)} \leq \\ &\leq \sum_x \sum_{w:d(w,x)=1} |K(w,y)E_F(d(w,y)) - K(x,y)E_F(d(x,y))|(1-2^{-n}d(x,y))^\epsilon q^{l(x)} \\ &= \sum_x \sum_{w:d(w,x)=1} |K(w,y)E_F(d(w,y)) - K(w,y)E_F(d(x,y)) + \\ &\qquad + K(w,y)E_F(d(x,y)) - K(x,y)E_F(d(x,y))|(1-2^{-n}d(x,y))^\epsilon q^{l(x)} \\ &= \sum_x \sum_{w:d(w,x)=1} |K(w,y)||E_F(d(w,y)) - E_F(d(x,y))|(1-2^{-n}d(x,y))^\epsilon q^{l(x)} + \\ &\qquad \sum_x \sum_{w:d(w,x)=1} |K(w,y) - K(x,y)||E_F(d(x,y))|(1-2^{-n}d(x,y))^\epsilon q^{l(x)} \\ &= \Sigma_2 + \Sigma_1.\end{aligned}$$

We consider the two terms individually.

$$\begin{aligned}
\Sigma_1 &= \sum_x \sum_{w:d(w,x)=1} |K(w,y)-K(x,y)||E_F(d(x,y))|(1-2^{-n}d(x,y))^\epsilon q^{l(x)} \\
&= \sum_x \left(\sum_{w:d(w,x)=1} |K(w,y)-K(x,y)| \right) |E_F(d(x,y))|(1-2^{-n}d(x,y))^\epsilon q^{l(x)} \\
&= \sum_{k=0}^{\infty} \sum_{x:d(x,y)=k} \nabla_x K(x,y) q^{l(x)} |E_F(k)|(1-2^{-n}k)^\epsilon \\
&\le (q-\frac{1}{q}) \sum_{k=0}^{\infty} |E_F(k)|(1-2^{-n}k)^\epsilon \\
&\le C2^{-n} \|D_{2^{-2n}}F\|_{H^s}
\end{aligned}$$

where the last inequality is due to 6.8 ,

Now we consider the second term.

$$\Sigma_2 = \sum_{k=0}^{\infty} \sum_{x:d(x,y)=k} \sum_{w:d(w,x)=1} |K(w,y)||E_F(d(w,y))-E_F(k)|(1-2^{-n}k)^\epsilon q^{l(x)} .$$

Since $d(w,y)$ is either $k+1$ or $k-1$, we can make the following estimation:

$$\Sigma_2 \le \sum_{k=0}^{\infty} \left(|E_F(k+1)-E_F(k)| + |E_F(k-1)-E_F(k)|\right)(1-2^{-n}k)^\epsilon \times \\ \times \left(\sum_{x:d(x,y)=k} \sum_{w:d(w,x)=1} |K(w,y)| q^{l(x)} \right) .$$

We evaluate separately the term in round brackets:

$$\begin{aligned}
&\sum_{x:d(x,y)=k} \sum_{w:d(w,x)=1} |K(w,y)|q^{l(x)} = \\
&= \sum_{x:d(x,y)=k, x\neq\hat{x}} (q+1)K(x,y)q^{l(x)} + q^{l(\hat{x})}\left((q-1)K(\hat{x},y) + qK(\hat{x},y) + \frac{1}{q}K(\hat{x},y)\right) \\
&\le \sum_{x:d(x,y)=k} (2q+1)K(x,y)q^{l(x)} \\
&\le (2q+1)(1+k) ,
\end{aligned}$$

where in the last inequality we exploited 6.16 and again $\hat{x}$ denotes the single node with $d(\hat{x}, y) = k$ and $l(\hat{x}) = l(y) + k$. We can go back to Σ_2:

$$\Sigma_2 \leq C \sum_{k=0}^{\infty} |E_F(k+1) - E_F(k)|(1+k)(1-2^{-n}k)^{\epsilon} + \\ + C \sum_{k=0}^{\infty} |E_F(k-1) - E_F(k)|(1+k)(1-2^{-n}k)^{\epsilon} .$$

The first sum is less than $C2^{-n}\|D_{2^{-2n}}F\|_{H^s}$ thanks to 6.9. For the second sum we change the index $\tilde{k} = k+1$ (then we drop the tilde):

$$C \sum_{k=0}^{\infty} |E_F(k) - E_F(k+1)|(2+k)(1-2^{-n}(k+1))^{\epsilon} \\ \leq C \sum_{k=0}^{\infty} |E_F(k) - E_F(k+1)| 2(1+k) 2(1-2^{-n}k)^{\epsilon} \\ = C \sum_{k=0}^{\infty} |E_F(k) - E_F(k+1)|(1+k)(1-2^{-n}k)^{\epsilon} .$$

In conclusion:

$$\Sigma_2 \leq C2^{-n}\|D_{2^{-2n}}F\|_{H^s} + C \sum_{k=0}^{\infty} |E_F(k) - E_F(k+1)|(1+k)(1-2^{-n}k)^{\epsilon} \\ \leq C2^{-n}\|D_{2^{-2n}}F\|_{H^s} ,$$

where the last inequality follows from 6.9. This concludes the proof. □

6.1.2 Spectral multipliers theorem

Given a continuous $H : [0, 2) \to \mathbb{C}$ the spectral theorem allows us to define the operator $H(\mathcal{L})$ which is bounded on $L^2(\mathcal{V}, \mu)$.

Definition 6.1.3. H is called an L^p-spectral multiplier for $\mathcal{L}$, with $p \in (1, \infty)$, if $H(\mathcal{L})$ extends to a bounded operator on $L^p(\mathcal{V}, \mu)$, i.e. the restriction $H(\mathcal{L})_{L^p \cap L^2}$ of the operator $H(\mathcal{L}) : L^2(\mathcal{V}, \mu) \to L^2(\mathcal{V}, \mu)$ can be extended to a bounded operator on $L^p(\mathcal{V}, \mu)$.

Definition 6.1.4. Let H be a function with $\operatorname{supp} H \subset [0, 2)$. H satisfies a Mikhlin-Hörmander condition of order s if

$$\sup_{t>0} \|(D_t H)\phi\|_{H^s} < \infty \tag{6.22}$$

for some $\phi \in C_c^{\infty}([\frac{1}{2}, 4])$, $\phi \neq 0$.

Remark: let ϕ_1 and ϕ_2 be any two functions in $C_c^\infty([\frac{1}{2}, 4])$. If condition 6.22 is satisfied for ϕ_1, then it is satisfied also for ϕ_2.

In particular, we can fix $\phi \in C_c^\infty([\frac{1}{2}, 4])$ such that $\phi = \psi^2$ with $\psi \in C_c^\infty([\frac{1}{2}, 4])$ and such that

$$\forall x > 0 \qquad \sum_{n=-\infty}^{+\infty} \phi(2^{2n}x) = 1.$$

We are now ready to state the main theorem of this section.

Theorem 6.1.5. *Let H be a continuous function with* $\operatorname{supp} H \subset [0, 2)$ *which satisfies a Mikhlin-Hördmander condition of order s for some $s > \frac{3}{2}$. Then H is an L^p-spectral multiplier for the Laplacian $\mathcal{L}$ for $1 < p < \infty$ and $H(\mathcal{L})$ is a weak type $(1, 1)$ operator.*

Proof. We can represent H as:

$$\begin{aligned} H(x) &= \left(\sum_{n=-\infty}^{+\infty} \phi(2^{2n}x) \right) H(x) \\ &= \sum_{n=-\infty}^{+\infty} \phi(2^{2n}x) H(x) \\ &= \sum_{n=-\infty}^{+\infty} G_n(x) \end{aligned}$$

where we have defined $G_n(x) = \phi(2^{2n}x)H(x)$.

We observe that:

$$\operatorname{supp} \phi \subset [\tfrac{1}{2}, 4] \quad \leftrightarrow \quad \operatorname{supp} D_{2^{2n}}\phi \subset [2^{-2n-1}, 2^{-2n+2}].$$

Then we have:

$$\operatorname{supp} G_n \subset \operatorname{supp} H \cap \operatorname{supp} D_{2^{2n}}\phi \subset [0, 2) \cap [2^{-2n-1}, 2^{-2n+2}].$$

For $n < 0$, $2^{-2n-1} \geq 2$, so in this case:

$$[0, 2) \cap [2^{-2n-1}, 2^{-2n+2}] = \emptyset$$

and $G_n(x) = 0$, $\forall x$.

So the representation obtained for H can be reduced to:

$$H(x) = \sum_{n=0}^{\infty} G_n(x).$$

It follows that the operator $H(\mathcal{L})$ admits the representation:

$$H(\mathcal{L}) = \sum_{n=0}^{\infty} G_n(\mathcal{L}).$$

where $G_n(\mathcal{L})$ is an integral operator whose kernel admits a representation of the kind described in 6.2. We show that $G_n(\mathcal{L})$ satisfies the hypothesis of Theorem 3.2.2. By hypothesis

$$\exists s > \tfrac{3}{2} : \quad \sup_{t>0} \|(D_t H)\phi\|_{H^s} < \infty.$$

Moreover:

$$\begin{aligned} D_{2^{-2n}} G_n(x) &= G_n(2^{-2n}x) \\ &= \phi(x) H(2^{-2n}x) \\ &= (D_{2^{-2n}} H)(x)\phi(x)\,. \end{aligned}$$

So it holds:

$$\exists s > \tfrac{3}{2} : \quad \|D_{2^{-2n}} G_n\|_{H^s} = \|(D_{2^{-2n}} H)\phi\|_{H^s} \leq C < \infty.$$

We set $\epsilon \in (0, 1]$ such that $s > \frac{3}{2} + \epsilon$. We can apply Lemma 6.1.4 (using G_n in the role of F) to obtain:

$$\sum_x |G_n(\mathcal{L})(x, y)|(1 + 2^{-n} d(x, y))^{\epsilon} q^{l(x)} \leq C\|D_{2^{-2n}} G_n\|_{H^s} \leq C$$

where the last constant is independent on n. Then the first assumption 3.6 of Theorem 3.2.2 is satisfied. Similarly, the second assumption 3.7 of Theorem 3.2.2 follows from 6.19. Thus $H(\mathcal{L})$ extends to an operator of weak type $(1, 1)$ and bounded on $L^p(\mathcal{V}, \mu)$, $1 < p \leq 2$.

The result for $p > 2$ follows by duality. In particular, we can proceed similarly to prove that the adjoint operator $H(\mathcal{L})^* = \bar{H}(\mathcal{L})$ extends to an operator of weak type $(1, 1)$ and bounded on $L^p(\mathcal{V}, \mu)$, $1 < p \leq 2$. Then the transpose operator $\bar{H}(\mathcal{L})^t = \bar{\bar{H}}(\mathcal{L}) = H(\mathcal{L})$ is bounded on $L^{p'}$, $p' > 2$.

□

6.2 The Riesz transform

As another application of the abstract Calderón–Zygmund Theorem 3.2.2 we study the boundedness of the Riesz transform operator $\nabla\mathcal{L}^{-1/2}$, which is the analogue of the classical Riesz transform in this setting.

Theorem 6.2.1. *The Riesz transform operator* $\nabla\mathcal{L}^{-1/2}$ *is of weak type* $(1,1)$ *and bounded on* $L^p(\mathcal{V},\mu)$ *for* $1 < p \leq 2$ *.*

Proof. We represent the function $\frac{1}{\sqrt{t}}$ as follows:

$$\begin{aligned}\frac{1}{\sqrt{t}} &= \sum_n \frac{\phi(2^{2n}t)}{\sqrt{t}}\\ &= \sum_n U_n(t)\\ &= \sum_n \frac{\psi(2^{2n}t)}{\sqrt{t}}\psi(2^{2n}t)\\ &= \sum_n V_n(t)W_n(t).\end{aligned}$$

where again we have fixed $\phi \in C_c^\infty([\frac{1}{2},4])$ such that $\phi = \psi^2$ with $\psi \in C_c^\infty([\frac{1}{2},4])$ and such that

$$\forall x > 0 \qquad \sum_{n=-\infty}^{+\infty} \phi(2^{2n}x) = 1.$$

We now write $(\mathcal{L})^{-\frac{1}{2}} = \sum_n U_n(\mathcal{L}) = \sum_n V_n(\mathcal{L})W_n(\mathcal{L})$. Each of the operators $U_n(\mathcal{L}), V_n(\mathcal{L}), W_n(\mathcal{L})$ is an integral operator and the respective kernels $U_n(\mathcal{L})(x,y), V_n(\mathcal{L})(x,y), W_n(\mathcal{L})(x,y)$ admit a representation as the one given for $F(\mathcal{L})(x,y)$ in 6.2.

We make some preliminary estimates:

- $$\forall s > 0, \quad \|D_{2^{-2n}}U_n\|_{H^s} \leq C2^n\,. \tag{6.23}$$

  Indeed:

  $$\begin{aligned}D_{2^{-2n}}U_n(t) &= \frac{\phi(t)}{\sqrt{2^{-2n}t}}\\ &= \frac{\phi(t)}{2^{-n}\sqrt{t}}\\ &= 2^n\frac{\phi(t)}{\sqrt{t}} = 2^nU_0(t)\,.\end{aligned}$$

Then we can compute:

$$\begin{aligned}\|D_{2^{-2n}}U_n\|_{H^s} &= \|2^nU_0\|_{H^s}\\ &= 2^n\|U_0\|_{H^s}\\ &= 2^n\|\frac{\phi(t)}{\sqrt{t}}\|_{H^s} \le C2^n\end{aligned}$$

since $\frac{\phi(t)}{\sqrt{t}} \in C_c^\infty([\frac{1}{2},4])$ and so $\frac{\phi(t)}{\sqrt{t}} \in H^s$.

- $$\forall s>0, \quad \|D_{2^{-2n}}V_n\|_{H^s} = 2^nC\,. \tag{6.24}$$

  Indeed,

  $$\begin{aligned}D_{2^{-2n}}V_n(t) &= \frac{\psi(t)}{\sqrt{2^{-2n}t}}\\ &= 2^n\frac{\psi(t)}{\sqrt{t}} = 2^nV_0(t)\,.\end{aligned}$$

  Then we have:

  $$\|D_{2^{-2n}}V_n\|_{H^s} = \|2^nV_0\|_{H^s} = 2^nC\,.$$

- $$\forall s>0, \quad \|D_{2^{-2n}}W_n\|_{H^s} = \|W_0\|_{H^s} = C\,. \tag{6.25}$$

  Indeed,

  $$D_{2^{-2n}}W_n(t) = \psi(t) = W_0(t).$$

Using 6.21 we obtain:

$$\begin{aligned}\sum_x|\nabla_xV_n(\mathcal{L})(x,y)|q^{l(x)} &\le \sum_x|\nabla_xV_n(\mathcal{L})(x,y)|(1+2^{-n}d(x,y))^\epsilon q^{l(x)}\\ &\le C2^{-n}\|D_{2^{-2n}}V_n\|_{H^s}\\ &= C2^{-n}2^nC = C\,,\end{aligned} \tag{6.26}$$

where we have used 6.24 and the last constant does not depend on n. Then for $\nabla_x U_n(\mathcal{L})(x,y) = \int_{\mathcal{V}} \nabla_x V_n(\mathcal{L})(x,w) W_n(w,y) d\mu(w)$ we have:

$$
\begin{aligned}
&\sum_x |\nabla_x U_n(\mathcal{L})(x,y) - \nabla_x U_n(\mathcal{L})(x,z)| q^{l(x)} = \\
&\quad \leq \int\int |\nabla_x V_n(\mathcal{L})(x,w)||W_n(\mathcal{L})(w,y) - W_n(\mathcal{L})(w,z)| d\mu(w) d\mu(x) \\
&\quad = \int\int |\nabla_x V_n(\mathcal{L})(x,w)||W_n(\mathcal{L})(w,y) - W_n(\mathcal{L})(w,z)| d\mu(x) d\mu(w) \\
&\quad = \int |W_n(\mathcal{L})(w,y) - W_n(\mathcal{L})(w,z)| \left(\int |\nabla_x V_n(\mathcal{L})(x,w)| d\mu(x) \right) d\mu(w) \\
&\quad \leq \left(\sup_w \int |\nabla_x V_n(\mathcal{L})(x,w)| d\mu(x) \right) \int |W_n(\mathcal{L})(w,y) - W_n(\mathcal{L})(w,z)| d\mu(w) \\
&\quad \leq C 2^{-n} d(y,z) \| D_{2^{-2n}} W_n \|_{H^s} \\
&\quad \leq C 2^{-n} d(y,z).
\end{aligned}
$$

where in the last two steps we have used 6.26, 6.19, 6.25.

This shows that the second assumption 3.7 of Theorem 3.2.2 is satisfied by $\nabla U_n(\mathcal{L})$. The first assumption 3.6 is satisfied as a direct consequence of 6.21 and 6.23, which upon the substitution $F \to U_n$ give:

$$
\sum_x |\nabla_x U_n(\mathcal{L})(x,y)| (1 - 2^{-n} d(x,y))^{\epsilon} q^{l(x)} \leq C 2^{-n} \| D_{2^{-2n}} U_n \|_{H^s} \tag{6.27}
$$

$$
\leq C\,. \tag{6.28}
$$

This concludes the proof. □

Previous result can be reformulated for the operators $X_j \mathcal{L}^{-1/2}$, $j = 0, \ldots, q$, i.e. in terms of directional derivatives instead of the gradient.

Theorem 6.2.2. *The operator* $X_j \mathcal{L}^{-1/2}$ *if of weak type* $(1,1)$ *and bounded on* $L^p(\mathcal{V}, \mu)$ *for* $1 < p \leq 2$.

Proof. We prove that the restriction of $X_j \mathcal{L}^{-1/2}$ to $L^p(\mathcal{V}, \mu) \cap L^2(\mathcal{V}, \mu)$ can be extended to a bounded operator on $L^p(\mathcal{V}, \mu)$. The other part of the claim can be proved similarly.

We take a function $f \in L^p(\mathcal{V}, \mu)$ and consider a sequence f_n, $n \in \mathbb{N}$ of functions in $L^p(\mathcal{V}, \mu) \cap L^2(\mathcal{V}, \mu)$ converging to f, i.e.:

$$
\|f_n - f\|_{L^p(\mathcal{V},\mu)} \to 0, \text{ when } n \to \infty\,.
$$

The sequence $X_j\mathcal{L}^{-1/2}f_n$ converges in $L^p(\mathcal{V},\mu)$. Indeed consider $m, l > n$:

$$\begin{aligned}\|X_j\mathcal{L}^{-1/2}f_l - X_j\mathcal{L}^{-1/2}f_m\|_{L^p(\mathcal{V},\mu)} &= \|X_j\mathcal{L}^{-1/2}(f_l - f_m)\|_{L^p(\mathcal{V},\mu)} \\ &\leq \|\nabla\mathcal{L}^{-1/2}(f_l - f_m)\|_{L^p(\mathcal{V},\mu)} \\ &\leq \|\nabla\mathcal{L}^{-1/2}\|\|f_l - f_m\|_{L^p(\mathcal{V},\mu)}\end{aligned}$$

where the last quantity tends to zero when $n \to \infty$ since $\nabla\mathcal{L}^{-1/2}$ is bounded on L^p and f_n is a Cauchy sequence.

Then $X_j\mathcal{L}^{-1/2}f$ is the $L^p(\mathcal{V},\mu)$ function defined as

$$X_j\mathcal{L}^{-1/2}f = \lim_{n\to\infty} X_j\mathcal{L}^{-1/2}f_n\,.$$

This is a good definition since the limit does not depend on the choice of the sequence f_n. Indeed, consider another sequence g_n, $n \in \mathbb{N}$ of functions in $L^p(\mathcal{V},\mu) \cap L^2(\mathcal{V},\mu)$ converging to f.

$$\begin{aligned}\|X_j\mathcal{L}^{-1/2}f_n - X_j\mathcal{L}^{-1/2}g_n\|_{L^p(\mathcal{V},\mu)} &= \|X_j\mathcal{L}^{-1/2}(f_n - g_n)\|_{L^p(\mathcal{V},\mu)} \\ &\leq \|\nabla\mathcal{L}^{-1/2}(f_n - g_n)\|_{L^p(\mathcal{V},\mu)} \\ &\leq \|\nabla\mathcal{L}^{-1/2}\|\|f_n - g_n\|_{L^p(\mathcal{V},\mu)}\,.\end{aligned}$$

The last quantity tends to zero when $n \to \infty$ since $\|f_n - g_n\|_{L^p(\mathcal{V},\mu)}$ tends to zero, so that:

$$\lim_{n\to\infty} X_j\mathcal{L}^{-1/2}f_n = \lim_{n\to\infty} X_j\mathcal{L}^{-1/2}g_n\,.$$

Moreover:

$$\begin{aligned}\|X_j\mathcal{L}^{-1/2}f\|_{L^p(\mathcal{V},\mu)} &= \|\lim_{n\to\infty} X_j\mathcal{L}^{-1/2}f_n\|_{L^p(\mathcal{V},\mu)} \\ &= \lim_{n\to\infty}\|X_j\mathcal{L}^{-1/2}f_n\|_{L^p(\mathcal{V},\mu)} \\ &\leq \lim_{n\to\infty}\|\nabla\mathcal{L}^{-1/2}f_n\|_{L^p(\mathcal{V},\mu)} \\ &\leq \lim_{n\to\infty}\|\nabla\mathcal{L}^{-1/2}\|\|f_n\|_{L^p(\mathcal{V},\mu)} \\ &= \|\nabla\mathcal{L}^{-1/2}\|\|\lim_{n\to\infty} f_n\|_{L^p(\mathcal{V},\mu)} \\ &= \|\nabla\mathcal{L}^{-1/2}\|\|f\|_{L^p(\mathcal{V},\mu)}\end{aligned}$$

and then $X_j\mathcal{L}^{-1/2}$ is bounded on $L^p(\mathcal{V},\mu)$ and its L^p-norm is bounded by the L^p-norm of $\nabla\mathcal{L}^{-1/2}$. □

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