POLITECNICO DI TORINO

Master Thesis in Management Engineering

# The role of Value at Risk and Expected Shortfall in Credit Risk Management

A comparative analysis of Basel guidelines and through a Beta Mixture Model application.


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# Table of Contents

# List of Figures

## List of Table

# Abstract

This thesis investigates how Value at Risk differs from Expected Shortfall both in terms of legal and regulatory frameworks and providing a practical example using a static credit model to analyse three different portfolios. In particular, we will use a Beta Mixing Model as a static credit model, by which is possible to model the dependency between defaults of obligors. First, we will analyse Basel Accords from the second to the fourth, and we will investigate how Basel Accords set comprehensive guidelines and standards for risk management and capital adequacy to ensure the stability of the global banking system, introducing the VaR and the ES as different methods to manage credit risk. In addition, we will focus on the use of a Beta Mixture model to capture the dependence between the defaults of borrowers. Then, in order to evaluate how VaR and ES behaves and their robustness in different extreme scenarios, we will evaluate 3 portfolios with different probability of default. Finally, we will conclude that the choice of using VaR and ES will give different results, especially when the correlation between obligors has mid-range values. This implies that, depending on the obligors in a portfolio, VaR and ES can lead to different results in term of credit risk management and capturing extreme market events and tail risks.

# Introduction

The difficulty of assessing and managing risk in finance has strong historical roots, dating back to the late Middle Ages and the Renaissance. During this era, the first banks and financial institutions were created to finance business transactions with equity and depositors' capital. However, the notion of Value-at-Risk (VaR) was only introduced in the 1980s when Dennis Weatherstone, chairman of J.P. Morgan, dissatisfied with the lengthy daily reports, wanted a brief overview of his trading portfolio's overall exposure before the market closing. This request resulted in the construction of the famous '4:15 report,' to remember the time of the first formal exposition of the Value at Risk idea. In 1994, J.P. Morgan published RiskMetrics^TM, a shortened model of its financial risk assessment method, establishing VaR as the industry standard. The combination of the J.P. Morgan publication and the later Basel I accord in 1996, the popularity and the use of VaR in the financial sector accelerated significantly. In particular, Basel I established minimum capital requirements for banks to safeguard their trading books.

From Basel I, credit risk management has been a critical and central concern for almost all financial institutions, due to the potential for significant losses if not properly managed. Now, financial institutions and other companies are legally required to engage in properly management of credit risk. Moreover, effective risk management enhances the stability of an organization and supports its overall economic resilience. However, before managing credit risk, it is important to first find a correct way of modelling the dependency structure between corporate defaults and then quantify it accurately.

The importance of default dependency among companies and its strength is easily seen in extreme events such as the 2008 financial crisis, where credit risk management plays a crucial role. Recent and increasing globalisation, new advanced derivatives that cause strong contagion, and increasing focus on credit risk framework such as the Basel ones, have raised the urgency of an enhanced credit risk management. However, the big challenge when dealing with credit risk management is modelling default dependency: this task is far from simple and straightforward, which led to excessive criticism of the Basel II framework and an oversimplified approach to describing the structure of dependencies. The analysis will then show how Basel II failed to predict the 2008 financial crisis and how the implementation of Basel III aimed to solve and overcome the problems of Basel II. Then, the recent introduction of Basel IV will be discussed, highlighting the key differences and improvements from Basel III. Further, we will investigate how in credit risk management Value at Risk and Expected Shortfall differ in term of definition and results.

The first objective of this research is analysing the Credit risk frameworks and their evolution across the years, and the main root causes of their improvements. Moreover, the thesis aims to provide a clear understanding of static model in credit risk management. In particular, using the Value at Risk and the Expected Shortfall as risk measures, the thesis will analyse three different portfolios with three different probabilities of default, with the aim of understanding how Value at Risk and Expected Shortfall are able to correctly measure the expected loss. The data were collected from S&P Global Ratings Credit Research & Insights and S&P Global Market Intelligence's CreditPro.

To summarise, the present thesis aims to answer the following research questions:

- How Value at Risk and Expected Shortfall have been implemented and why?
- How Value at Risk and Expected Shortfall behave when calculating the expected loss in a portfolio when using a Bernoulli Mixture Model?
- How the correlations between different obligors in a portfolio affect the robustness of the Value at Risk and the Expected Shortfall?

Chapter 1 begins by discussing the principles of risk how managing and measuring it, focusing on our objective for this thesis, the credit risk. Following that, we will investigate the impact of the 2008 financial crisis on regulatory frameworks, including a full examination of Basel II, its flaws, and how the improvements implemented in Basel III address these concerns. Then, Basel IV and its advancements will be discussed. The chapter ends with a discussion of the importance of incorporating the dependence between obligors in a credit portfolio.

Chapter 2 introduce the risk measures and their four properties: translation invariance, subadditivity, positive homogeneity and monotonicity; a risk measure is defined as such if it is consistent with these four properties. The two risk measures we will focus on are Value at Risk and Expected Shortfall: after an introduction to each, we will analyse them, assessing the pros and cons of each and whether one is preferable to the other; this comparison is critical because both measures are common in risk management and regulatory contexts, but they reflect risk in different ways. Even though VaR has been the industry standard for many years due to its simplicity and ease of interpretation, it fails to account for tail risk, which could result in underestimating extreme losses. On the other hand, Expected Shortfall, which has gained popularity, addresses some of the limitations of VaR by presenting a more complete picture of tail risk, at the expense of a higher cost in terms of complexity and calculation. The comparison of these two indicators is crucial to understand their distinct roles in financial risk management and to determine which one may be more appropriate in various contexts. For instance, in a regulatory framework, the choice between VaR and ES can have a substantial impact on capital requirements, affecting both the financial stability and profitability of institutions.

Chapter 3 defines the methodology used to carry out this research. In particular, in order to analyse defaults and dependence between obligors, it will be introduced the Bernoulli

Mixture Model in static credit portfolios and the large portfolio approximation that will be used for our analysis. First, will be highlighted the key differences between static credit risk model and dynamic ones: Static credit risk models are commonly used in credit risk management because they are simple and cost-effective, relying on historical data and fixed assumptions, but they fail to reflect current risks when market conditions change since they do not take account for the timing of defaults or real-time economic fluctuations. Dynamic credit risk models, which incorporate real-time data and respond to changing conditions, provide a more accurate and current evaluation of credit risk, especially when pricing credit securities such as CDOs, but they are more complicated and resource intensive. Finally, after introducing the Bernoulli Mixture model, with Beta as the mixing variable, Section 3.5 will discuss the Value at Risk and Expected Shortfall approximation in large portfolios, which will be used in our analysis in Chapter 4.

Chapter 4 analyses the robustness and behaviour of VaR and ES in predicting losses, considering three different portfolios with different default probability values. First, all relevant data used will be presented, then the Value at Risk and Expected Shortfall for each portfolio will be evaluated. The objective is to assess the robustness of the Value at Risk and Expected Shortfall in correctly measuring the expected loss by changing the correlation between the obligors for each portfolio.

Finally, Chapter 5 addresses the challenges of credit risk management in static portfolios, with a focus on modelling default dependency. The Bernoulli mixture model, using a factor vector of mixing variables, proves to be effective in capturing credit risk in static models. Our analysis of large portfolios (e.g., m=1000 obligors) allowed for valuable approximations in loss distributions, essential for complex models lacking closed-form solutions. The choice between Value at Risk (VaR) and Expected Shortfall (ES) has significant implications, with ES and VaR behaving similarly at low correlations due to diversification benefits, but diverging notably at mid-range correlations (0.5 to 0.6), where joint losses are more probable yet unpredictable.

Accurately estimating key parameters like $\pi$ and $\rho$ remains challenging, as these greatly impact VaR and ES calculations, highlighting variability across industry models. This thesis emphasizes modelling homogeneous portfolios to simplify credit risk analysis, but future research could expand to heterogeneous portfolios by grouping obligors into sub-portfolios. Furthermore, stochastic modelling of loss rates and incorporating dependencies between defaults and economic conditions would offer a dynamic, realistic view of risk. Advanced techniques like copula models or machine learning could enhance VaR and ES estimates by capturing complex dependencies in financial data, ultimately advancing credit risk modelling.

# 1. Credit Risk Management

To understand what Credit Risk means and how banks manage it, it is important to first introduce the definition of risk.

A definition of the Oxford English Dictionary (1655) defines the risk as:

*"The exposure to the possibility of loss, injury, or other adverse or unwelcome circumstance; a chance or situation involving such a possibility."*

When speaking of 'risk', one often refers to the downside, focusing on the potential negative consequences, such as losing something valuable. However, the chance of an upside gain must be taken into consideration as well. In finance, for example, increased risk or volatility is usually associated with higher expected profits, which means that any investment strategy implies a balance between risk and expected return: to achieve higher returns, people have to undertake riskier activities.

After an overview on the different types of risk, the thesis will focus on market and financial risks in the Section 1.1 and on section 1.2 on understanding credit risk and how to manage it. To illustrate these principles in practice, Section 1.3 links the concepts of risk with the Basel Accords and their implementation: the 2008 financial crisis concerning CDOs, Collateralized Debt Obligations, was the turning point in the implementation of more specific risk frameworks, with the aim of preventing subsequent financial collapses.

However, the main focus will be on Section 1.4, where will be discussed the Basel accords, trying to understand why they exists and have changed over time, their differences in terms of credit risk requirements and their different implications.

Finally, the Chapter 1 ends with Section 1.6, which emphasises the "importance of modelling dependence between obligors" (McNeil et. Al 2005).

## 1.1. What is risk

There are various types of risks, which are classified as: business risk, non-business risk (or strategic and operational risk), and financial risk. This thesis focuses specifically on financial risk.

Business risk is referred to threats posed to the focal business stemming from the political, economic, societal and technological environment of the firm (Souder & Bethay, 1993). These risks may stem from unexpected changes in government policies and actions by rival firms.

The nature of risks may involve aspects related to politics, cross-cultural issues, currency as well as both commercial and supply disruption risks (Cavusgil et al., 2012, World Bank, 2013).

In other terms, business risk is the risk undertaken by enterprises to maximize shareholder value and profits, and refers to everything that threatens a company's ability to achieve its financial goals

Non-Business Risk refers to a type of risk that is not directly related to their business operations. For example, risks associated with long-term financing, competitors' actions and technological innovation.

Finally, the financial risk, is defined as any occurrence or activity that may negatively impact an organization's capacity to achieve its goals and execute its strategy, or the quantitative likelihood of loss or lower-than-expected profits. (McNeil et al. 2005).

Usually, Financial Risk is divided into three major components: Market risk, Operational risk and Credit risk (McNeil et al. 2005, Basel Committee on Banking Supervision)

Market risk, or systematic risk, is the potential for losses in a portfolio due to factors that impact the overall performance of financial markets, such as interest rate movements, exchange rate fluctuations, geopolitical events, or recessions. Unlike specific risk, which can be mitigated through diversification, market risk affects the entire market and cannot be eliminated by diversifying a portfolio.

It can be divided into four categories (Credit Risk: A Survey" by Edward Altman and Anthony Saunders, Journal of Banking & Finance):

- Price risk is the risk of adverse fluctuations in the prices of financial assets due to market trend or investors' expectations (Corporate Financial Institute).
- Exchange rate risk also known as currency risk, refers to the potential loss due to fluctuations in the exchange rates between currencies (Corporate Financial Institute).
- Interest rate risk is the probability of a decline in the value of an asset resulting from unexpected fluctuations in interest rates (Corporate Financial Institute).
- Spread risk refers to the risk associated with the fluctuations in the spread between yields of different financial instruments (Corporate Financial Institute).

Operational risk is the second primary category and is defined by Saunders and Cornett (2007, p. 535) as

*"The risk that existing technology or support systems may malfunction, fraud might impact financial activities, and/or external shocks such as hurricanes and floods occur".*

Operational risk includes also model risk, which refers the risk of error due to inadequacies in financial risk measurement and valuation models (European Central Bank), or the risk that the model does not accurately represent real market movements. This thesis will not analyse deeply the operational risk, since the main focus will be evaluating the performance in term of Value-at-Risk and Expected Shortfall of a One-Factor Exchangeable Bernoulli Mixture Model according to his parameters.

Last, Credit risk is the risk of loss that may occur if the borrower does not fulfil the agreed conditions for financial or other reasons, or the possibility that the bank's borrower is unable to meet all or part of its obligations when they fall due. In other words, credit risk is the risk of non-repayment of all debts, the risk of delay in servicing the loan or non-repayment of the loan (Donaldson, 1989).

Credit risk depends on four components which are the amount of loan, the assets of the person who will pay the loan, default payment dates and paid/outstanding debt balance. The factors that cause the occurrence of credit risk in banking activities can be divided into internal or external ones and are summarized in Table 1 (McNeil et al. 2005):

| INTERNAL FACTORS | EXTERNAL FACTORS |
|---|---|
| Credit client-specific factors | Political, economic and social factors |
| Supply, production and marketing structure | Legal regulation changes |
| Competition power | Changes in political structure |
| Managerial skills | Changes in economic policies |
| Product life cycle | Crises |
| Bank-specific factors | Natural and other Factors |
| Financial analysis | Natural disasters |
| Risk assessment capability | Technological development |
| Decision criteria | Customer preferences |

*Table 2. Overview of internal and external factors influencing Credit Risk.*

### 1.2. Measuring and managing the Credit Risk

To manage risk effectively, it is essential to adopt one of several risk measurement methods. Each method incorporates the concept of randomness, since the central aspect of risk is the uncertainty of potential gains or losses. This research focuses on credit risk management when dealing with a portfolio consisting of several obligors, where one or several obligors may default.

From the point of view of a bank's business, i.e. granting loans, understanding the likelihood of borrowers defaulting is essential due to the resulting credit losses in their portfolios. In this context, credit risk exclusively pertains to potential future losses, as gains are not considered. Future losses, ore expected loss is named L, defined as a random variable representing the portfolio's loss (McNeil et.al 2005), whose loss distribution gives information about loss probabilities.

Risk measuring purposes can be bundled as:

- Determining risk capital and capital adequacy: evaluating and assessing the amount of capital a financial institution is required to keep as a buffer against unexpected future losses, to maintain a certain level of solvency of the institution and thus satisfy regulator's satisfaction. (McNeil et al. 2005).
- Management tool often uses risk measures as a tool to limit the level of risk that a firm is allowed to take on. (T. Aven, 2016).
- Insurance premium, that is the amount of money that an individual or business must pay to an insurance company in exchange for coverage against specific risks. This coverage provides financial protection in the event of unexpected losses or damages.

There exist four different approaches to measure the risk a financial position can take (McNeil et al. 2005):

- Notional-amount approach. In this method, the portfolio's risk is calculated by summing the notional values of each security, with each value possibly adjusted by a factor that reflects the risk level of the asset class it belongs to.
- Factor-sensitivity measures. These measures evaluate how a portfolio's value responds to changes in underlying risk factors, often using derivatives like duration for bonds. However, while useful for assessing sensitivity to specific risks, they do not capture the overall riskiness of a position and pose challenges for risk aggregation.
- Risk measures based on the loss distribution. Modern methods, such as Value-at-Risk (VaR) and Expected Shortfall, focus on the statistical distribution of potential losses over a given time horizon. McNeil et al. (2005) point out that these approaches are exposed to inaccuracies due to their over-reliance on historical data and assumptions about market conditions, which may not be applicable under future scenarios. Estimating these distributions, especially for large portfolios, is challenging and often

requires advanced modelling techniques. Critics often point out the oversimplification of assumptions, such as the assumption of a normal distribution, which may underestimate extreme events (Taleb, 2007).

- Scenario-based risk measures. This method assesses portfolio risk by simulating a variety of potential future scenarios and identifying the maximum possible loss across them. It allows some flexibility, such as the weighting of extreme scenarios, but the selection of scenarios and their probability remains subjective, which may introduce bias (Rebonato, 2010).

After explaining what measuring the risk means, let understand the purpose and objectives of risk management.

McNeil, Frey and Embrechts (2005) describe the purpose of the Quantitative Risk Management as follow:

*"An important question we have just addressed concerns the reasons for investing in Quantitative Risk Management as follow. This question can be asked from different perspectives, including those of a financial institution's customer, its shareholders, management, board of directors, regulators, politicians or the general public. Each of these stakeholders may have a different answer and, in the end, a balance between the various interests will have to be found."*

Thus, the performance, positive and negative, of banks, large companies or financial institutions has broad implications beyond its shareholders, affecting also the entire economy and society. This significant influence of banks and financial institutions underlines the importance for stringent regulation and supervisory control. Saunders and Cornett (2007) highlight this as the objective as maximising the social benefits of banking services.

Among challenging regulations, closely linked to credit risk, are those concerning capital requirements, set through the Basel Accords. Basel regulations started in 1974 with 10 central banks forming the Basel Committee in order to have a unified and prudential common set of rules and regulations. The initial focus was to set a minimum capital requirement for different types of risks to set a limit for the level of safety and consistency in banks.

Thus, to guarantee that financial institutions have sufficient capital to pay their obligations and absorb unexpected losses, the Basel Committee on Banking Supervision (BCBS) established the Basel Accords, which began with Basel I in 1988. Basel I aimed to raise the capital adequacy ratio among banks to 8% (capital to risk-weighted assets ratio) by using straightforward methods to calculate the capital charge. This approach was intended to enhance competitiveness among banks and improve coherence by strengthening risk management practices. Through the years, Basel I have been attacked for not meeting its objectives. A significant drawback of this accord was neglecting other critical risk sources, since it focused only on default risk, and the lack of sensitivity in its computational methods.

Basel II expanded on the framework of Basel I by incorporating two additional risk types: market risk and operational risk. It also provided banks with the flexibility to choose between using internally developed models, tailored to their specific circumstances, or adopting a standardized, one-size-fits-all approach. Moreover, Basel II introduced the current three-pillar approach: Pillar 1 established the minimum capital requirement for the three risk types, Pillar 2 set a new assessment of a bank's internal capital adequacy by assessing all risks they can potentially face during their operations and the Pillar 3 aimed to ensure market discipline by requiring full transparency and disclosure of relevant market information.

### 1.3. The importance of Credit Risk Management after the Financial Crisis

The recent 2007's financial crisis of the U.S. housing market, set the basis of the current Basel III framework. An important component of the financial crisis was poor information: on the one hand, banks and investors wrongly believed that they held low-risk assets and, on the other hand, regulators incorrectly believed that banks had sufficient capital to overcome difficult times.

Prior to the crisis, the US housing market had been flourishing for years (Federal Reserve, 2011). Moreover, the housing market was considered extremely safe, and US financial institutions and the US government were overconfident about it. Houses prices kept rising, and lenders provided mortgages to those people whose respected some specific credit requirements. Then, financial institutions decided to mitigate some risk by creating securitized products known as Collateralized Debt Obligations (CDOs). A CDO is essentially a bundle of various loans, both safe and risky (classified from A-Rating to C-rating according to the degree of riskiness), such as student loans, car loans or credit card debts, which can be bought and sold to other investors. Thus, CDOs allowed financial institutions to transfer the majority of the risk to investors.

A CDO was viewed as a highly secured investment, since Rating Agencies gave them a too high rating class, which offered a high expected return to risk ratio. The high rating, combined with the high expected yield, made these CDOs very attractive to investors, who preferred to invest in them rather than in the 1% Treasury bonds of the time.

Moreover, financial institutions viewed mortgages as a safe investment, since when homeowners default on their mortgage the lender gets the house, and houses were always increasing in value. Thus, since lenders are covered if the homeowner's default, they started adding risk to new mortgages not requiring down payments, no proof of income or no documents at all: Banks started using the subprime mortgages. These loans were designed to provide an opportunity for individuals with low income and poor credit history to purchase their own home. In this regard, George Bush said in 2002:

*"We want everybody in America to own their own home. [...] One of the programs is designed to help deserving families who have bad credit histories to qualify for homeownership loans [...] the low-income home buyer can have just as nice a house as anybody else"*

However, in 2007, an increasing amount of people began defaulting on their mortgages, resulting in an oversupply of housing and a steep drop in property values. The first mass defaults occurred with subprime mortgages, which were provided to people with low credit records. The lack of transparency in the derivatives market aggravated the issue by keeping financial institutions ignorant of their true exposure to risky assets, as well as the exposure of other institutions. This uncertainty weakened the market, making banks hesitant or unwilling to engage in short-term interbank lending. The resulting lending freeze caused massive liquidity issues, negatively influencing financial markets and contributing to the start of the global financial crisis.

Consequently, some major financial institutions faced severe solvency risks, culminating in Lehman Brothers filing for bankruptcy on September 15, 2008.

In the next sections we will better understand how the 2007 crisis is linked to the Basel Accords.

## 1.4. The Basel Accords

As already discussed in Section 1.3, the Basel accords have been influential and decisive in centralising banking regulation, supervision and banking regulation, supervision and capital adequacy standards.

### 1.4.1 Basel II

In contrast to Basel I, Basel II is based on three pillars, each targeting a specific segment of the banking system (Basel Committee on Banking Supervision).

Pillar 1 established minimum capital requirements, building on the Basel I capital ratio, to address credit, operational, and market risks, while excluding other types of risks.

For evaluating each risk category, Basel II proposed the options summarized in Table 2.1. Each risk category has different methods of evaluation:

- Credit risk management was based on three methods: the standardised approach, that uses external credit assessments (e.g., ratings from credit rating agencies) to determine the risk weights for different types of exposures. The Internal Rating Approach (IRB) allows banks to use their own internal rating systems to estimate credit risk, by evaluating probability of default (PD), loss given default (LGD) and exposure at default (EAD). Lastly, in Advanced Internal Rating Approach, banks have to estimate credit risk, having more freedom and responsibility.
- Market risk management is based on two methods: the standardised approach, that uses calculations and analysis made by external regulators to assess the market risk associated with trading activities, and the Internal Value at Risk model approach, that is used by the banks' internal models to estimate the potential loss in value of their trading portfolios over a specific time horizon and a certain confidence level.
- Operational risk management is based on three methods. The basic indicator approach evaluates the operational risk capital as a fixed percentage (usually 15%) of the bank's annual gross income over the previous three years. The standardised approach divides a bank's activities into different business lines, each with its own risk factor and then calculates the capital charge by multiplying the gross income of each business line by its respective risk factor. Lastly, the Advanced Measurement Approach requires banks develop their own empirical model based on internal and external data, scenario analysis, risk indicators and advanced risk management systems.

| Credit Risk | Market Risk | Operational Risk |
|---|---|---|
| Standardised approach<br><br>Internal rating approach (IRB)<br>Advanced internal rating approach | Standardised approach<br><br><br>Internal Value at Risk model approach | Basic indicator approach<br><br>Standardised approach<br><br>Advanced measurement approach |

*Table 3. Evaluation measures for Credit Risk, Market Risk and Operational Risk*

Pillar 2 mandates regulators to ensure compliance with the capital requirements set by Pillar 1. Additionally, regulators are tasked with assessing the adequacy of banks' internal controls. Pillar 3 introduces disclosure requirements to enhance transparency. Banks must disclose information about their internal risk management systems and the implementation of Basel II.

Moreover, Basel II allowed banks choosing between using the "internal risk-based" ("IRB") approach that allows banks to make its own assessment of the risk, and the "standardized approach" that relies on CRAs (Credit Risk Agencies) and is designed for smaller banks with less sophisticated risk-modelling and risk-management systems. The IRB option is applicable to banks that already possess advanced risk modelling. A bank can use its own data to assess the risk level of its loans by analysing three key factors:

1. the probability of default within one year
2. the bank's exposure and potential losses in the event of a default
3. the likely repayment timeline if there is no default.

Within the Internal Ratings-Based (IRB) approach, there are two further sub-options: advanced IRB and foundation IRB. The foundation IRB requires more oversight compared to the advanced IRB. Under the foundation approach, banks estimate the probability of default for each asset, while bank supervisors estimate the other three factors. The advanced approach, available only to the most sophisticated banks, allows the banks to make most of these estimates themselves, subject to the approval of bank supervisors.

Criticism to Basel II

Basel II aimed increasing flexibility by assuming that both regulators can adequately police large banks' IRB decisions and that market discipline ensure that banks act prudently. Both these two assumptions have led to criticism.

First, it has been argued by criticisms ("Risk Management and Regulation: Basel II, Basel III and Beyond" by Charles A. Goodhart, "The Basel Handbook: A Guide for Financial Practitioners" by Michael K. Ong) that Basel II overestimates the ability of regulators to adequately supervise large banks. Regulators often focus only on default risk, potentially neglecting other critical risks such as interest rate fluctuations. This limited oversight can leave banks vulnerable to various unsupervised risks, thus weakening the stability that Basel II aims to ensure.

Second, it has been questioned whether the framework relies on market discipline to promote prudent banking practices ("Basel III and Credit Risk Measurement", Matt Schlickenmaier). Basel II requires banks to disclose information on assets and liabilities and methods for measuring credit risk, relying on transparency rather than true market discipline. Critics suggest that to strengthen market discipline, regulators could require banks to issue listed subordinated debt or remove deposit insurance, thus increasing the impact of bank failures and encouraging more prudent behaviour.

Other criticisms were moved against the Standardised Approach and the Credit Rating Agencies role ("Basel III and Credit Risk Measurement", Matt Schlickenmaier). Profit-driven CRAs often face conflicts of interest as they are compensated by the issuers of the securities they rate. This can lead CRAs to inflate ratings to secure future business. Moreover, CRAs sometimes advise issuers on the structuring of securities to obtain desired ratings, especially in the case of complex instruments such as mortgage-backed securities (MBS) and collateralised debt obligations (CDOs). This dual role can blur the line between objective rating and advisory services.

The IRB's approach has led to several criticisms, related to the accuracy of risk models and the integrity of their application. In fact, Banks have an incentive to minimise capital reserves by underestimating the risk of assets, as riskier assets require more capital. Thus, Banking models, especially for complex products such as CDOs and MBS, often lack accuracy and may not take into account atypical or extreme market conditions, underestimating risks in times of extreme stress, thus leading to substantial losses.

Weaknesses of Basel II

As already discussed in the paragraph 1.4, the financial crisis highlighted certain weaknesses in the Banking sector, closely linked to his regulatory framework. The Basel Committee on Banking supervision, 2009b, summarised as:

*"One of the main reasons the economic and financial crisis became so severe was that the banking sectors of many countries had built up excessive on- and off -balance sheet leverage. This was accompanied by a gradual erosion of the level and quality of the capital base. At the same time, many banks were holding insufficient liquidity buffers. The banking system therefore was not able to absorb the resulting systemic trading and credit losses, nor could it cope with the reintermediation of large off-balance sheet exposures that had built up in the shadow banking system. The crisis was further amplified by a procyclical deleveraging process and by the interconnectedness of systemic institutions through an array of complex transactions."*

The first main weakness highlighted by the financial crisis dealt with the amount and the quality of the available capital. Before the 2007-09 financial crisis, many banks maintained high capital ratios mainly through hybrid instruments rather than common equity. For instance, in late 2006, just before the financial crisis, the major European banking groups had an average Tier 1 ratio of 8%. This was significantly higher than the regulatory minimum of 4%. Regulators accepted these instruments as capital, but they proved inadequate for loss absorption. Banks favoured hybrids to avoid diluting control, gain tax benefits, and appeal to specific investors. However, the market perceived hybrids as debt, and banks seldom missed payments on them to protect their reputations, thereby diminishing their capacity to absorb losses. This disconnects between regulatory objectives and market practices indicates a necessity for a greater focus on common equity in bank capital structures.

Another widely recognized shortcoming, especially by the Financial Stability Board and the Basel Committee on Banking Supervision itself, is its tendency to intensify economic cycles, named procyclicality. This arises because rating-based capital requirements increase during recessions and decrease during economic upturns. When banks face pressure on their capital ratios, they respond by restricting loan supplies or reducing assets, thereby worsening the economic downturn. As stated by the Financial Stability Board (2009), this risk was underscored: “The current crisis has demonstrated the disruptive effects of procyclicality – the mutually reinforcing interactions between the financial and real sectors that amplify business cycle fluctuations and exacerbate financial instability.”

Requiring banks to hold more capital during a recession is beneficial from a micro-prudential supervisory perspective, as it mandates individual institutions to maintain higher reserves in the face of increased risks. However, this approach becomes counterproductive when viewed from a macro-prudential standpoint. If all banks tighten credit simultaneously, it exacerbates the recession, increases the risk of defaults, and worsens their financial conditions. Thus, while prudent on a micro level, this strategy can be detrimental at the macro level.

It is quite noticeable that there was a leverage problem. Several large international banks maintained high leverage despite their capital ratios meeting regulatory requirements. This issue, combined with procyclicality, significantly contributed to the recent financial crisis. Many financial institutions were compelled to sell substantial assets to improve their capital ratios, initiating a deleveraging process. Although this ensured individual bank solvency, it also increased financial market instability. Consequently, there's a recognized need to integrate macro-prudential supervision, which aims to prevent systemic crises, with traditional micro-prudential supervision that focuses on maintaining the solvency of individual banks.

Lastly, during the financial crisis, banks faced significant liquidity shortages. Many large banks, heavily reliant on the interbank market's abundant liquidity, managed to survive only because of the central banks' provision of cheap liquidity. Before the crisis, banks did not address the increasing liquidity risk adequately by investing in risk management by improving their human and technology capabilities. This complacency was partly due to the belief that the interbank market's liquidity could resolve potential shortages for any adequately capitalized bank. However, the financial crisis demonstrated that a loss in confidence and an increase in counterparty risk might lead to a liquidity crunch, resulting in severe and unanticipated stress scenarios for individual institutions.

### 1.4.2 Basel III

Between September and December 2010, the Basel Committee on Banking Supervision responded to the failures of the financial crisis by revising its capital rules in an attempt “to strengthen the regulation, supervision and risk management of the banking sector”.

Basel III is similar to Basel II but introduces some significant changes. While Basel III still allows the use of Credit Rating Agencies (CRAs) for reference, it modifies the standardized approach by requiring banks to assess their exposures and evaluate the appropriateness of CRA-based risk estimates. Additionally, Basel III mandates that external credit ratings be "publicly available, on a non-selective basis and free of charge."

To address the goal of strengthen the regulation and supervision of the banking sector, Basel Committee on Banking Supervision introduced several key changes with Basel III:

- a higher quality of capital by strengthening the common equity (core tier 1) requirements, increasing them to 4.5% of risk-weighted assets and introducing several adjustments to the calculation of common equity, such as deferred tax assets and minority interests.
- an additional capital requirement known as the capital conservation buffer. This buffer, equal to 2.5% of risk-weighted assets, is designed to ensure that banks maintain a capital cushion to absorb losses during periods of economic and financial stress.
- an additional countercyclical capital requirement ranging from 0% to 2.5% of risk-weighted assets.
- a non-risk-based maximum leverage ratio, defined as a minimum ratio of tier 1 capital to total assets, set at 3% during the monitoring period.
- two new liquidity ratios, first introduced as monitoring tools and later as compulsory requirements: the liquidity coverage ratio and the net stable funding ratio.

The Basel Committee on Banking Supervision published in 2010 the new requirements associated to Basel III. Table 3 reports the changes of Basel III, highlighting the significant increases associated.

| requirement | % of risk weighted assets | Basel II | Basel III |
|---|---|---|---|
| Common Equity | a. Minimum | 2.0% | 4.5% |
| | b. Conservation buffer | | 2.5% |
| | d. Total (a+b) | | 7.0% |

| | | | |
|---|---|---|---|
| Tier 1 Capital Ratio | c. Minimum<br>e. Total (c+b) | 4.0% | 6.0%<br>8.5% |
| Total Capital Ratio | f. Minimum<br>g. Total (f+b) | 8.0% | 8.0%<br>10.5% |
| Additional macroprudential requirements | h. Anticyclical buffer<br>i. Additional requirements for systemic banks | | 0 - 2.5% |

*Table 4. Overview of Basel Committee on Banking Supervision of Basel II and Basel III requirements*

The first main implementation of Basel III was the restriction of the regulatory capital to improve his level and quality. The financial crisis demonstrated that credit losses are primarily absorbed by retained earnings, which form a key part of a bank's tangible equity. In response, Basel III emphasizes the importance of the highest quality capital, particularly Common Equity, which includes paid-up share capital (ordinary shares) and retained earnings. Basel III introduced new capital requirements based on Tier 1 capital, also known as “going concern capital,” which can absorb losses without necessitating the liquidation of the bank, and on Tier 2 capital.

Tier 1 capital consists of two components:

- Common Equity Tier 1 (CET1): This includes instruments such as common shares, stock surplus, and retained earnings. These instruments are perpetual, meaning they have no maturity date, do not require reimbursement, and do not obligate the bank to pay dividends.
- Additional Tier 1 (AT1): These are subordinated instruments with fully discretionary non-cumulative dividends or coupons, no maturity date, and no incentive for redemption.

On the other hand, Tier 2 capital is designed to absorb losses if a bank undergoes liquidation. It includes revaluation reserves, general provisions, subordinated term debt, and hybrid capital instruments. Together, Tier 1 and Tier 2 capital are known as total capital, which must equal at least 8% of the bank's risk-weighted assets.

Basel III regulations also requires that all new Tier 1 and Tier 2 capital instruments be structured to absorb losses before any government intervention. These 'bail-in' clauses ensure that, in the event of a crisis, the financial burden of bank losses falls primarily on investors in subordinated debt and hybrid capital instruments, rather than on taxpayers.

Moreover, other adjustments from capital have been introduced, mainly regarding Deferred Tax Assets, Minority Interest, Goodwill, Non-Consolidated Investments in Other Financial Institutions and cumulative Gains and Losses from Own Credit Risk on Fair Valued Liabilities.

Basel III introduced two mechanisms against procyclicality. Both strategies aim to boost banks' capital buffers beyond regulatory minimums during periods of economic expansion and high profitability. These measures fall under macroprudential policy tools, designed to ensure the solvency of individual banks while safeguarding the overall financial stability against procyclicality:

The first strategy is the 2,5% common equity capital cushion, additional to the 4,5 % minimum. Failing to respect this total minimum 7% of capital requirements will lead to limits in earnings distribution.

The second mechanism regards the introduction of a countercyclical capital buffer tied to credit growth, based on increasing capital reserve when credit supply exceeds its typical trend and a decreasing during credit contractions. To determine in advance, usually one year, the necessity of this buffer, regulators may consider various indicators, such as the discrepancy between the current bank loans-to-GDP ratio and its long-term average. The activation of this buffer is not automatic. Its primary objective is to safeguard the banking sector against the risks associated with excessive credit growth, which historically has led to systemic risks.

To address leverage issues, the Basel III formally introduced a maximum leverage requirement, formally:

$$\text{Plain Leverage} = \frac{\text{Tier 1}}{\text{On balance sheet and off balance sheet assets}} > 0{,}03$$

This new requirement will be included in the first pillar, aiming both to limit leverage within the banking sector to reduce the risk of destabilizing deleveraging processes that can harm the financial system and economy, and to provide additional protection against model risk and measurement errors by complementing risk-based measures with a straightforward, transparent, and independent risk assessment.

Moreover, the key rules for the new leverage requirement are as follows:

1. The capital measure for the leverage ratio will be based on the revised definition of Tier 1 capital, ensuring that only the highest quality capital is used.

2. The denominator of the leverage ratio will incorporate the bank's total exposures, which include both on-balance sheet and off-balance sheet items, providing a comprehensive view of risk.
3. On-balance sheet exposures, such as loans, must be reported net of specific provisions and valuation adjustments to reflect the bank's actual risk exposure more accurately.
4. Derivative exposures are to be included in the denominator based on their current fair value (current exposure) and an estimate of their potential future exposure, reflecting both present and future risks associated with derivatives.
5. Off-balance sheet (OBS) items, such as commitments, standby letters of credit, trade letters of credit, failed transactions, and unsettled securities, are recognized as significant contributors to leverage. The Basel Committee mandates their inclusion in the leverage ratio calculation to ensure these items are properly accounted for, as they can significantly amplify a bank's overall risk exposure.

Lastly, the Basel Committee responded to the severe liquidity challenges experienced by many international banks during the crisis by introducing two new prudential supervisory requirements. Unlike capital requirements, these regulations mandate that banks maintain a minimum level of liquidity. These new liquidity standards took effect in 2015 and are two:

- LCR, or Liquidity Coverage Ratio
- NSFR, or Net Stable Funding Ratio

LCR is a short-term liquidity ratio, ensuring that banks maintain a sufficient level of unbound and high-quality liquid assets. These assets must be easily convertible into cash during a severe stress scenario to match liquidity needs within a 30-day period.

Formally:

$$LCR = \frac{\mathrm{HQLA}}{\mathrm{COF}_{30}^{\mathrm{S}}} > 1$$

Basel III requires that the ratio of high-quality liquid assets (HQLA), a stock variable, to total net cash outflows over the next 30 days in a stress scenario, a flow variable, must be at least 1. The objective of LCR is guaranteeing that banks have enough HQLA to survive for a thirty-day stress scenario, where banks or supervisors/management are assumed to undertake specific corrective actions.

Basel III introduces the HQLA, that must meet these requirements:

- Low credit and market risk
- Easy and certainty of valuation

- Low correlation with risky assets
- Listed on a developed and recognised exchange market
- Qualified for central banks' intraday and overnight liquidity facilities.
- They must remain unrestricted, providing the bank with unrestricted access and rapid conversion into liquidity to meet funding gaps between inflows and outflows in times of financial stress.

Moreover, Basel III classified HQLA in two level of assets:

- Level 1 assets provide high standards in term of quality and liquidity, and include cash, central bank reserves, and highly marketable securities. Level 1 HQLA are guaranteed by sovereign governments, central banks, or supranational organisations such as the Bank for International Settlements, the International Monetary Fund, and the European Commission. During periods of market crisis, these assets can be swiftly turned into cash with little loss of value.
- Level 2 assets, while still liquid, provide a slightly larger risk than Level 1. They are limited to 40% of total High-Quality Liquid Assets (HQLA) and often include products like corporate bonds and some government securities. These assets may see more dramatic price changes in strained times, yet they remain stable sources of liquidity.

The denominator of LCR, $\mathrm{COF}_{30}^{\mathrm{S}}$, is calculated by definition as the difference between total expected cash outflows and inflows over a stress scenario for the following thirty days.

Total expected cash flows are calculated by multiplying the outstanding balances of the various liabilities and off-balance sheet commitments by the respective run-off or utilisation rates. For instance, customer deposits are split between "stable", and "less stable" categories according to specific factors like duration, a deposit insurance scheme (used to increase stability), and whether they are in transactional accounts (such as salary deposit accounts). Stable deposits are given a minimum run-off rate of 5%, while less stable deposits have a run-off rate of 10% or higher. Higher rates apply to deposits from companies and public entities. Deposits from other banks are assigned a 100% run-off rate, assuming the entire amount will be withdrawn within thirty days.

Total expected cash inflows are calculated by multiplying the outstanding balances of various categories of contractual receivables by their expected inflow rates under the scenario, with a cap of 75% of total expected cash outflows. When assessing available cash inflows, a financial institution has to account for contractual inflows from fully performing outstanding exposures, for which there is no reason to expect a default within the 30-day time horizon (Basel Committee on Banking Supervision)

The Basel Committee also recommends that stress scenarios involve both individual and market-wide shocks. These should include the run-off of retail deposits, partial losses in unsecured and secured financing, additional outflows from credit rating downgrades, increased market volatility affecting collateral, unexpected draws on credit facilities, and the necessity to buy back debt to safeguard the bank's reputation. These scenarios aim mirroring the types of shocks observed during the 2007 financial crisis.

The Liquidity Coverage Ratio (LCR) was introduced in 2015 with an initial value of 60%, which improved by 10% each year until it reached 100% in January 2019.

The Net Stable Funding Ration, NSFR, is a medium and long-term indicator, as it requires a minimum amount of stable funding based on specific liquidity characteristics of a bank' assets and activities over a one-year horizon.

Formally:

$$NSFR = \frac{\mathrm{ASF}}{\mathrm{RSF}} > 1$$

NSFR is the ratio between a bank's available stable funding, ASF, and its required stable funding, RSF. The NSFR seeks to reduce the reliance on short-term wholesale funding to finance medium to long-term assets and operations.

Basel III defined available stable funding as ASF, which refers to the portion of equity and liability financing that is considered to be reliable sources of money over a one-year period under stress conditions.

ASF is defined as:

- Tier 1 and Tier 2 Capital
- Preferred stock with at least one year of maturity
- Liabilities with effective of at least one year of maturity
- the portion of non-maturity liabilities or liabilities with maturities of less than one year that are expected to remain with the institution for a long period during an idiosyncratic stress event.

Each of these ASF components is assigned a stability coefficient (ranging from 100% to 0%), indicating that more stable resources have a greater impact on total ASF than less stable resources.

Table 4 illustrates how the Basel Committee on Banking Supervision, 2010, and International regulatory frameworks for liquidity reports the ASF variables related to a bank's main liabilities.

| ASF Factor | Components of AS Category |
| --- | --- |
| 100% | • Tier 1 and Tier 2 Capital<br>• Preferred shares not counted in Tier 2, with an original maturity of at least one year, factoring in any options that may shorten the maturity to under one year.<br>• Secured and unsecured borrowings, along with liabilities and term deposits, with a remaining maturity of one year or more. |
| 90% | • Stable demand deposits (non-maturity) and term deposits with less than one year to maturity, provided by retail or small business customers.<br>• Stability is determined based on the strength of the customer relationship and the deposit's features. |
| 80% | • Less stable demand deposits (non-maturity) and term deposits with under one year to maturity, also sourced from retail and small business customers.<br>• These are considered less stable due to the likelihood of withdrawal under stress conditions. |
| 50% | • Unsecured wholesale funding, demand deposits, and term deposits with a remaining maturity of less than one year, originating from non-financial corporations, governments, central banks, multilateral development banks, and public sector entities (PSEs). |
| 0% | • Any other liabilities and equity not included in the categories above, including short-term liabilities without substantial liquidity reserves. |

*Table 5. Report of Basel Committee on Banking Supervision, showing the ASF factors associated to each category*

The Required Stable Funding (RSF) follows a similar approach to the ASF: the value of assets being held and funded is multiplied by a designated RSF factor based on the asset type. Additionally, off-balance sheet (OBS) activities, or potential liquidity exposures, must also be factored in by multiplying their value by the corresponding RSF factor. The RSF factor represents the percentage of each OBS asset or exposure that supervisors consider necessary to be supported by stable funding. Thus, more liquid assets, which can provide sustained liquid availability in stressed situations, receive lower RSF values and therefore require less funding. Conversely, less liquid assets are assigned higher RSF factors, indicating a greater need for stable funding. In particular, RSF factors are designed to estimate the portion of an asset that could not be converted into cash or used as collateral in a secured loan during a one-year liquidity event.

Table 5 reports how the Basel Committee on Banking Supervision, 2010, shows the RSF factors for each main assets of a bank.

| RSF Factor | Components of RSF Category |
|---|---|
| 0% | • Immediately available cash that is unencumbered and free for use, not held as collateral or reserved for other obligations (e.g., future salary payments or contingent collateral)<br>• Unsecured short-term financial instruments and transactions with less than one year to maturity that are free of collateral requirements.<br>• Unencumbered securities with maturities under one year, provided there are no embedded options that might extend the maturity .<br>• Unencumbered securities tied to offsetting reverse repurchase agreements<br>• Unencumbered loans to financial institutions with remaining maturities under one year, that cannot be renewed, and where the lender retains an absolute right to recall. |
| 5% | • Non-marketable unencumbered securities with maturity of at least one year, that constitute claims on or guarantees of sovereign entities, central banks, or multilateral development banks, which are assigned a 0% risk-weight. |
| 20% | • Unencumbered corporate or covered bonds rated AA- or above, with a minimum maturity of one year, qualify as Level 2 assets under the Liquidity Coverage Ratio's asset standards.<br>• Unencumbered marketable assets with a maturity of more than one year, indicating claims on sovereigns, central banks, or Public Sector Enterprises, allocated a 20% risk-weight under Basel II, and meeting the Liquidity Coverage Ratio's Level 2 asset standards. |
| 50% | • Unencumbered physical gold reserves.<br>• Unencumbered and not issued by financial institutions shares, traded on a public market and listed in a large-capitalisation stock.<br>• Unencumbered corporate and covered bonds that meet these requirements: (i) eligible for central bank funding, (ii) not issued by financial institutions (except covered bonds), (iii) not emitted by the firm or its affiliates, (iv) rated as low credit risk, and (v) traded in large, active markets with minimal concentration<br>• Unencumbered loans to banks, financial institution, non-financial corporations, or PSEs with less than one year to maturity |
| 65% | • Unencumbered residential mortgages of any maturity that meet the criteria for a 35% or lower risk-weight under the Basel II Standardised Approach |

| | |
|---|---|
| | • Unencumbered loans, excluding those to financial institutions, with a maturity of one year or more, that meet the 35% or lower risk-weight criteria under the Basel II Standardised Approach for credit risk |
| 85% | • Unencumbered loans to retail and small business customers with less than one year to maturity |
| 100% | • Any other assets not captured in the categories above, including assets that may carry higher risk or have lower liquidity value |

*Table 5. Report of Basel Committee on Banking Supervision, showing the RSF factors associated to each category*

The Basel III framework also implemented key changes and innovations to enhance the robustness and reliability of internal models. These improvements address limitations in previous regulatory frameworks, aiming for more accurate risk measurement and greater stability in the banking sector. The most two significant changes, relevant to banks using internal models and embodied in Basel III are the introduction of Stressed VaR and Incremental Risk Charge (IRC).

Stressed VaR aims to address the risk of having a loss during the time of stress conditions. The idea is to simply reproduce the VaR calculation for the bank's current portfolio but under stressed market conditions. Similar to banks with a validated internal model, it uses a 10-day, 99th percentile, one-tailed confidence interval VaR measure. The data are adjusted based on historical information from a continuous year-long period of financial stress that is relevant to the bank's portfolio. The selected period of stress must receive approval from the supervisor and be subject to regular reviews.

A financial institution provided with an approved internal model have to meet a market requirement on a daily basis. This new requirement is a net increase from the situation before Basel III, as before the requirement was simply the max between VaR and Stressed VaR.

This new term, $\mathrm{k_{MKT}}$ is made up of three components and is defined as follows:

$$\mathrm{k_{MKT}} = \max\left[\mathrm{VaR_{99\%,10,t-1}}, \mathrm{m_c} \times \frac{\sum_{i=1}^{60} \mathrm{VaR_{99\%,10,t-1}}}{60}\right] + \left[\mathrm{SVaR_{99\%,10,t-1}}, \mathrm{m_c} \times \frac{\sum_{i=1}^{60} \mathrm{VaR_{99\%,10,t-1}}}{60}\right] + \mathrm{k_{SR}}$$

The first term is the "ante Basel III requirement", that is defined as the larger between the 99% 10-day VaR of the previous day ($\mathrm{VaR_{99\%,10,t-1}}$) and the previous 60 days average VaR multiplied by a multiplying factor $\mathrm{m_c}$ (ranging from 3 to 4 depending on the quality of the VaR model). The second term is defined as the larger between the 10-day, 99% confidence level Stressed VaR of the previous day ($SVaR_{99\%,10,t-1}$)) and the previous 60 days average Stressed VaR multiplied by a multiplying factor $m_c$ (ranging from 3 to 4 depending on the quality of the VaR model). The third term is the $k_{SR}$, that must be added whenever the bank's VaR doesn't capture specific risk.

The Incremental Risk Charge (IRC) emerged as a critical regulatory advancement to address the shortcomings of Value-at-Risk (VaR) models in capturing specific risks within banks' trading portfolios under the Basel framework. Introduced in July 2005, the IRC was initially conceived as an incremental default risk charge to tackle the risks associated with illiquid credit instruments—risks that traditional VaR models failed to identify adequately. The Basel III framework significantly strengthened the IRC, extending its scope to cover both default and downgrading risks. The IRC operates over a one-year horizon at a 99% confidence level, emphasizing the liquidity of individual positions. Banks assign varying liquidity horizons to these positions, with a minimum horizon of three months for the most liquid assets. This methodology allows continuous renegotiation of positions to maintain consistent risk levels.

To implement IRC, banks must decompose their trading portfolios by issuer, credit rating, and maturity, using detailed models to simulate credit spread evolution, default probabilities, and recovery rates, considering issuer correlations. This approach ensures that high-risk positions have appropriate time horizons while maintaining regulatory oversight for comprehensive risk assessment. Following the financial crisis, the 2016 fundamental review of the trading book introduced further revisions to market risk capital requirements, enhancing the robustness and granularity of these measures.

In particular, the innovations introduced were:

- The standardised approach has been adjusted to enhance risk sensitivity, by adding to Delta risk also Vega and Gamma risk.
- A Default Risk Charge has been added to the standardized approach to address the credit risk associated with specific financial instruments.
- The internal model approach has undergone significant revisions, with Expected Shortfall (ES) replacing Value at Risk (VaR) as the primary risk measure to reflect tail risk. Additionally, the stressed ES measure has replaced the previous stressed VaR. Furthermore, the process for approving institutions as internationally active banks has become stricter, with more detailed guidelines for identifying risk factors and stricter constraints on the benefits of hedging and diversification in terms of capital reduction.
- Market illiquidity risk is accounted in the Expected Shortfall (ES) calculations by applying different liquidity horizons for various markets, replacing the previous uniform 10-day horizon that was used for all instruments.

### 1.4.3 Basel IV

Although the Basel III framework was originally intended for major banks, financial authorities in several jurisdictions have already extended essential reform aspects to a broader range of banks. Despite this, the Basel Committee of Banking Supervision decided to work and refine the Basel III framework in order to reduce excessive variability in risk weighted assets (RWAs) and risk-based capital ratios.

Basel IV was published by the Basel Committee on Banking Supervision (BCBS) in December 2017 but has been implemented on the 1st of January 2023. The date has been postponed from the 1st of January 2022 since the Basel Committee on Banking Supervision set out new rules in April 2020 in order to mitigate the impact of Covid 19 on the world banking system.

Basel IV includes new ways to improve the strength and risk sensitivity of credit and operational risk in standardized approaches (SA). Furthermore, it also restricts the application of IRB approaches to credit risk, eliminates internal modelling for operational risk in regulatory capital calculations, and revises credit valuation adjustment guidelines. The changes are intended to enhance the comparison of banks' equity and debt ratios. The framework also contains a finalised leverage ratio for Global Systemically Important Banks (G-SIBs) and a revised capital level of 72.5% for lowering banks' ability to drastically decrease capital requirements using internal risk models (BCBS, 2017a, b). Basel IV is further backed by the Fundamental Review of the Trading Book (FRTB), which updates market risk rules and Pillar 3 disclosure obligations. However, these latter features are outside the focus of this discussion.

Basel IV also restricts the application of internal ratings-based (IRB) approaches to credit risk, eliminates the use of internal modelling for operational risk in calculating regulatory capital, and revises standards related to credit valuation adjustment (CVA). These changes are intended to enhance the comparability of banks' capital ratios. The updated standards also introduce a finalized leverage ratio for Global Systemically Important Banks (G-SIBs) and set a revised capital floor of 72.5% to limit banks' ability to lower capital requirements via internal risk models (BCBS, 2017a, b). Basel IV is complemented by the Fundamental Review of the Trading Book (FRTB), which updates market risk standards, and revised Pillar 3 disclosure requirements. These latter elements, however, fall outside the scope of this thesis.

Basel IV and his implication

Basel IV introduces five main substantial reforms in the calculation of capital requirements for all risk categories, with the aim of resolving several key issues within the overall regulatory framework. One of the main changes is the restriction of the advanced internal ratings-based approach (A-IRB), now limited to credit risks in portfolios with low default rates. This change is intended to ensure more accurate capital assessments for certain types of assets, for which historical default data are poor. In addition, the internal model's approach for credit valuation adjustment risk (CVA) has been replaced by a more robust standardised approach, which further improves the consistency of risk calculations across institutions. Regarding operational risk, Basel IV also eliminates the use of Advanced Measurement Approaches (AMA), replacing them with a revised standardised approach: this ensures that operational risk is assessed more consistently across the industry.  Furthermore, Basel IV introduces a minimum output threshold, ensuring that risk-weighted assets (RWA) calculated with internal models cannot fall below 72.5% of the RWAs determined with the standardised approaches: this threshold is not allowed to fall below 72.5% of the RWAs determined with the standardised approaches. Another significant change is the introduction of a leverage ratio buffer for global systemically important banks (G-SIBs), which requires these institutions to maintain a Tier 1 capital buffer of 50 per cent of their risk-weighted capital buffer.

The Basel Committee on Banking Supervision has indicated that the Basel IV reforms, while reshaping capital requirements, are not expected to dramatically increase overall capital levels in the banking sector (BCBS, 2017,a,b). However, some institutions may experience a significant increase in minimum capital requirements, especially those that rely heavily on internal models for risk assessment. According to the European Banking Authority (EBA,2018), some banks may experience a significant impact on their capital buffers as a result of the full implementation of Basel IV, which is expected to be a complex and costly process for financial institutions.

The challenges posed by Basel IV are significant, especially for banks that have to decide whether to continue using internal models or switch to standardised approaches based on revised risk weight calculations. In addition, banks will have to devise strategies to meet the new capital requirements, which may involve issuing additional capital, preserving profits or reducing RWA. This is particularly difficult in the current economic environment, where low interest rates and ongoing economic stresses caused by the COVID-19 pandemic have led to reduced returns on equity (ROE) across the industry. As Feridun (2020) notes, these factors complicate banks' efforts to meet stricter capital requirements without compromising profitability.

One of the most significant challenges of Basel IV is the revision of credit risk frameworks. Under Basel II, banks had the option of calculating credit risk capital requirements using either the Standardised Approach (SA) or the Internal Ratings Based (IRB) approach. While this flexibility allowed banks to adapt their risk assessments, it also led to inconsistencies in the way risk-weighted assets were calculated. These discrepancies made it difficult to compare capital ratios across banks, raising concerns about the accuracy and transparency of risk assessments.

In response to these concerns, Basel IV aims to improve the comparability of risk-weighted capital ratios by narrowing the scope of internal models and revising the Standardised Approaches to make them more risk-sensitive. The inclusion of higher minimum capital levels and stricter standards for risk modelling is intended to reduce variations in RWA calculations across banks, leading to more consistent regulatory outcomes and improving the overall stability of the financial system. To address these problems, Basel IV introduces a more granular and credit risk sensitive Standardised Approach (SA) (Basel Committee on Banking Supervision, 2017a, b). The main changes include:

- More granularity and risk sensitivity. The Basel IV standardised approach is designed to improve the granularity and risk sensitivity of credit risk assessments. This includes new and recalibrated risk weights for various asset classes, such as retail and commercial real estate exposures. The framework now considers factors such as loan-to-value (LTV) ratio to find more accurate risk weights.
- In order to enhance granularity and risk sensitivity while reducing reliance on external credit ratings, Basel IV introduced a more detailed approach for unrated exposures to banks and corporates. This aims to provide a more accurate reflection of risk, particularly for institutions and loans that do not rely solely on credit ratings, thereby improving the overall resilience of the financial system.
- The treatment of mortgages under Basel IV is significantly more risk sensitive than under Basel II. The framework allows banks to use a split loan (LS) or whole loan (WL) approach to assess risk weights based on LTV ratios. These range between 20% and 105% for residential loans, according to specific loans characteristics, such residential or income-producing loans There is also a range of risk weights between 60% and 150% for commercial real estate exposures, based on LTV ratio,
- Basel IV also introduces significant operational issues, especially regarding loan documentation and assessment. Banks, for example, are now required to submit precise documents demonstrating the ability of the borrower on reimbursing the debt and a full valuation of the real estate. Specifically, for income-producing real estate (IPRE), banks have to consider the property's cash flows in light of the borrower's overall financial status.
- Basel IV introduces a more detailed approach to the management of corporate exposures, in particular by refining the risk weights for small and medium-sized enterprises (SMEs) and distinguishing them from general corporate exposures and specialised lending (SL). Risk weights now vary between 20% and 150%, depending on external assessments, and in jurisdictions where external ratings are not permitted risk weights are typically set between 65% and 100%.

To improve granularity and risk sensitivity while reducing dependence on external ratings, Basel IV introduces more detailed risk procedures for unrated exposures to banks and corporates. The updated methodology adjusts the risk weights for rated exposures and divide between covered bonds, specialised loans and small and medium-sized enterprises (SMEs). This detailed technique allows for a more accurate risk assessment, especially for institutions that do not rely heavily on external ratings, with a better differentiation of credit risk across asset classes.

Given the Basel IV changes, banks can improve capital efficiency by optimising their commercial loan portfolios using the new risk weights. For example, specialised lending (SL) now includes different risk weights according to the type of loan, allowing banks to fine-tune their portfolios. Banks can also choose to minimise their exposure to residential loans with high loan-to-value (LTV) ratios, which have higher risk weights and therefore require more capital. Instead, they may focus on managing portfolios with low LTV loans, which often have lower interest rates and enjoy preferential capital treatment under Basel IV.

Secondly, the impact of Basel IV on banks' capital requirements will be strongly influenced by their current credit risk models, which use the Standardised Approach (SA) rather than the Internal Ratings Based (IRB) approach. Basel IV sets more stringent requirements for the use of IRB models, with the intention of the Basel Committee on Banking Supervision (BCBS) to restore confidence in the calculations of risk-weighted assets (RWAs) and improve the comparability of capital ratios between banks. One of the most significant changes is a greater emphasis on the Foundation IRB (F-IRB) approach, where regulators set important parameters such as Probability of Default (PD), Exposure at Default (EAD) and Loss Given Default (LGD), rather than banks making their own estimates.

For big corporates and financial institutions, where defaults are infrequent and modelling is challenging, Basel IV eliminates the option of adopting the Advanced IRB Approach (A-IRB). Instead, banks must rely on either the F-IRB or the SA approach to manage these exposures. In addition, the use of IRB models for equity exposures has been completely abandoned. However, banks may continue to use both the A-IRB and F-IRB techniques for specialised lending (SL), allowing them to retain some flexibility in this area.

Another important aspect of Basel IV is the implementation of minimum input levels for key factors such as PD, EAD and LGD, which allow greater conservatism in risk modelling. For example, minimum PD levels are currently set between 0.05% and 0.10%, depending on the type of retail exposure, while minimum LGD levels range from 0% to 50%, depending on the exposure and collateral. Furthermore, Basel IV eliminates the 1.06 scaling factor that was applied to RWAs in the Basel II IRB framework, emphasising the more stringent capital requirements imposed by Basel IV.

Thirdly, Basel IV proposes a new Standardised Approach (SA) that replaces the Advanced Measurement Approach (AMA) based on internal models and the previous standardised approaches. The new SA approach applies to all banks and results in a more standardised and risk-sensitive approach for determining capital for operational risk. The calculation of the Basel IV SA Approach is based on two key components: the Business Indicator (BI), which represents a bank's gross income, and the Internal Loss Multiplier (ILM), which is calculated based on the bank's internal loss data over the past ten years. This method ensures that banks with historically higher operating losses have higher capital requirements, strengthening the risk-capital ratio.

Then, the updated Credit Valuation Adjustment (CVA) framework under Basel IV introduces significant adjustments to the computation of capital needs for Credit Valuation Adjustment risk, indicating a shift towards increased risk sensitivity and standardised methodology. Basel IV replaces prior internal model-based techniques with the Standardised Approach for CVA (SA-CVA), which considers a larger range of risk factors, including market risks such as interest rate and credit spread volatility. This move is intended to improve the accuracy and comparability of CVA capital needs across institutions. Furthermore, a simpler Basic Approach for CVA (BA-CVA) is introduced for smaller banks, providing a less sophisticated but still effective technique of capturing CVA risk. The elimination of totally internal models and the inclusion of Securities Financing Transactions (SFTs) under the CVA framework further highlight Basel IV's emphasis on comprehensive and standardised risk management.

Lastly, in order to address problems and challenges about the variability in Risk-Weighted Assets (RWA) caused by banks' use of internal models, Basel IV includes a capital output floor to limit the extent to which banks can decrease their capital requirements using these models. This floor replaces the previous "Basel I floor" and establishes a minimum capital requirement of 72.5% of what would be computed using normal methods. With the adoption of this floor, banks using internal models will be unable to achieve capital reductions greater than 27.5% of the standardised approach estimates, providing a considerable challenge for those who rely largely on internal models.

Basel IV vs Basel III

The Basel III and Basel IV frameworks are critical for the regulation of the global financial sector. Basel III was introduced as a response to the 2008 financial crisis, establishing additional rules for capital adequacy, stress testing, and liquidity management. Basel IV was introduced as a further development of Basel III, imposing stronger restrictions and higher standards, particularly in areas such as risk-weighted assets, credit risk, and the use of technology in risk management. These adjustments aim to further stabilise the financial system and better prepare institutions for future difficulties.

Table 8 summarise the key differences between Basel III and Basel IV (Wolters Kluwer).

| Aspect | Basel III | Basel IV |
|---|---|---|
| Exposure to corporates | • Includes claims on rated corporates, such as insurance companies.<br>• Allows supervisors to assign risk weights (RW) over 100% at their discretion. | • Differentiates between General Corporates and Specialized Lending<br>• Threshold for SMEs with annual sales ≤ €50 million<br>• Risk weights: 65% for investment grade, 85% for SMEs, 100% for others<br>• Introduces Due Diligence requirements |
| Regulatory to Retail Exposure | • Sets a 75% risk weight (RW) for qualifying regulatory retail exposures<br>• Specifies criteria including exposure to individuals or small businesses<br>• Includes product criteria like credit cards, lines of credit, and small loans<br>• Limits maximum aggregated retail exposure to €1 million | • Divides regulatory retail into two categories: non-transactors (which follow the same criteria as Basel III) and transactors (which include facilities with timely repayments and no overdraft).<br>• Other Retail: combines non-transactors and transactors for broader coverage |
| Real Estate Exposure Class | • Classifies real estate as Residential and Commercial property<br>• Lending fully secured by mortgages on residential property<br>• Risk weight (RW) for Residential property is 35%, Commercial is 100%<br>• Splitting approach with different RWs based on loan value | • Introduces detailed classification for regulatory real estate<br>• Regulatory residential and commercial real estate exposures depend on cash flows<br>• Adds other real estate categories with specific risk weights (RW)<br>• New land acquisition development and construction (ADC) exposures |
| Exposure to Banks | • National supervisors apply a consistent approach for all banks<br>• Risk weight assigned is less favourable than sovereign claims<br>• Exception for banks in countries with BB+ to B- ratings capped at 100% RW | • Introduces two approaches: ECRA (based on external ratings) and SCRA (for unrated banks)<br>• SCRA requires classifying banks into risk-weight buckets (Grade A, B, C), respectively 40%, 75% and 150%<br>• Short-term exposures: Grade A (20%), Grade B (50%), Grade C (150%) |

| | | |
|---|---|---|
| Exposure to Covered Bonds | • This asset class was not present in Basel III; treated similarly to bank exposures | • Defines covered bonds as those issued by banks or mortgage institutions under public supervision<br>• Eligible assets include sovereigns, central banks, public sector entities, and multilateral development banks<br>• Specific criteria for residential (LTV ≤ 80%) and commercial real estate (LTV ≤ 60%) claims<br>• Claims by banks with a 30% or lower risk weight are capped at 15% of issuances<br>• Minimum disclosure and asset pool requirements are specified |
| Exposure Multilateral Development Banks (MDS) | • 6 RW buckets, no use of short-term RW allowed<br>• 0% RW for highly rated MDBs | • 6 RW buckets, no use of short-term RW allowed<br>• 50% RW for MDBs in jurisdictions without external ratings |
| Equity | • Standard RW of 100% | • Standard RW of 100%<br>• RW of 400% for speculative unlisted equity exposures<br>• RW of 250% for all other equity holdings, 100% RW at national discretion<br>• Introduces new approaches: Look Through Approach (LTA) and Mandate Based Approach (MBA)<br>• Leverage cap of 1250% on equity investments |
| Subordinated Debt | • Not a separate asset class, standard RW of 100% applies | • RW of 150% to subordinated debt and capital instruments threshold other ran equities |
| Off Balance Sheet | • Converts items to credit exposures using CCF | • No maturity-based treatment, specific rules for OTC derivatives<br>• New capital requirements for credit derivatives |
| SCRA Grading | • Not present | • Introduces SCRA Grading for banks' exposures based on counterparty risk |

| | | |
|---|---|---|
| | | • Grade A refers to exposures to banks, where the counterparty bank has adequate capacity to meet their financial commitments in a timely manner.<br>• Grade B refers to exposures to banks, where the counterparty bank is subject to substantial credit risk, such as repayment capacities that are dependent on stable or favourable economic or business conditions.<br>• Grade C refers to higher credit risk exposures to banks, where the counterparty bank has material default risks and limited margins of safety. |
| Due Diligence | • Not present | • Requires comprehensive due diligence on financial performance<br>• Exemptions for sovereigns and non-central public sector entities |

*Table 6. comparison between Basel III and Basel IV*

### 1.5. The importance of Dependence Modelling

Introducing the interdependence of defaults among the various obligors in the portfolio is probably one of the most important and delicate points playing a crucial role in credit risk modelling. Even if independence of defaults among companies should not be assumed, in the today's interconnected global economy, international companies are increasingly dependent on strong economic regions worldwide: in this case we talk about contagion risk. The market for structural and other securitized products has grown, with these instruments being easily sold internationally. These products attract numerous investors due to their high potential returns and seemingly low risk. However, as the 2008 financial crisis revealed, the lack of transparency means the actual risk is much higher than perceived. This interlinking increases the risk of contagion, impacting both international companies and all-size companies. Therefore, understanding the dependence structure among credit market obligors is crucial, highlighting the need for advanced credit risk models.

However, as David Lando explains in the text “Credit Risk Modelling”:

*“Modelling dependence between default events and between credit quality changes is, in practice one of the biggest challenges of credit risk models. The most obvious reason for worrying about dependence is that it affects the distribution of loan portfolio losses and is therefore critical in determining quantiles or other risk measure used for allocating capital for solvency purposes. (Lando, 2004, p. 213)"*

In credit risk, default dependence plays an important role in the top tail of the loss distribution.

One method for modelling default reliance is to introduce a factor variable. This component variable includes macroeconomic variables like interest rates, GDP growth rates, and unemployment rates that apply to all companies in the portfolio. As this thesis will demonstrate, introducing a factor variable contributes to a more realistic view of how defaults among portfolio businesses are interconnected. This strategy better reflects systemic risk since economic conditions might influence numerous enterprises simultaneously, resulting in correlated defaults. Integrating these macroeconomic elements into default probability models will allow performing more accurate models, ultimately improving our understanding and management of portfolio risk. As we will see in Chapter 4, modelling our portfolios with a One-factor Mixture Model, common to all obligors in the portfolio, will be quite easy and understandable.

## 2. Measures for Risk-Management. VaR and ES

After understanding why measuring and managing risk is so important and the regulation behind Credit Risk management, the aim now is to focus on introducing specific risk measures. Understanding what risk measures are, their pros and cons and why they are used, is the starting point for further analysis of our portfolio in Chapter 4.

Artzner et al. (1999) specified a list of properties, or axioms that any well-defined risk measure have to satisfy to be coherent and robust. In addition, Artzner et al. (1999) observed the properties of commonly used risk measures such as Value at Risk or Expected Shortfall.

This chapter will base the discussion on what Artzner et al. (1999) and McNeil et al. (2005) discussed. Section 2.1 discuss the definition of risk measure and the four axioms of coherence that define a risk measure. Then, Sections 2.2, 2.3 and 2.4 will analyse respectively the VaR, the Expected Shortfall and how they are approximated when dealing with large portfolios, analysing their properties as a risk measure and their possible implication. Section 2.5 will finally understand pros and cons of VaR and ES, why both risk measures are used and when one is preferred over the other.

### 2.1 Risk Measures

Fixing a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and a time horizon $\triangle$, it is possible to denote by $\mathrm{L}^0(\Omega, \mathcal{F}, \mathrm{P})$ the set of all random variables on $(\Omega, \mathcal{F})$ which are almost surely finite. Then, financial risk is represented by a set $\mathrm{M} \subset \mathrm{L}^0(\Omega, \mathcal{F}, \mathrm{P})$of random variables that are interpret as portfolio losses over the time horizon $\triangle$. Furthermore, it is assumed that $M$ is a convex cone which implies that for every $\mathrm{L}_1 \in \mathrm{M}, \mathrm{L}_2 \in \mathrm{M}, \lambda > 0$ it is verified that $\mathrm{L}_1 + \mathrm{L}_2 \in \mathrm{M}$ and $\lambda \cdot \mathrm{L}_1 \in \mathrm{M}$.

Risk measures are real-valued functions $g$: $M \rightarrow \mathbb{R}$ defined on such cone of random variables, satisfying certain properties. Therefore, $g(L)$ is interpreted as the amount of capital that should be added to a position with loss given by $L$

The axioms that define a risk measure $g$: M → R on a convex cone M is 4:

- Translation invariance.
  For all $\mathrm{L} \in \mathrm{M}$. and $\mathrm{l} \in \mathbb{R}$ hold the relationship

$$g(L + l) = g(L) + l \tag{2.1.1}$$

In simple terms, adding or subtracting a certain amount from the position directly impacts the capital requirements in a one-to-one manner, so by applying a fixed quantity $l$ to a position leading to the loss $L$, the capital requirements change by that exact amount $l$.

- Subadditivity: For all $\mathrm{L}_1, \mathrm{L}_2 \in \mathrm{M}$. it is valid the inequality:

$$g(L_1 + L_2) \leq g(\mathrm{L}_1) + g(\mathrm{L}_2). \tag{2.1.2}$$

- Subadditivity ensures that the overall risk of the combined portfolio is not greater than the sum of the risks of the individual portfolios. In practice, this implies that diversification can reduce risk: in simple terms, subadditivity explains the fact that through correlations between the returns of different assets. Moreover, subadditivity is crucial for decentralised risk management systems, as it allows different entities within an organisation (or different financial institutions) to manage their risks separately, while ensuring that total capital requirements to cover potential losses are not overestimated. In decentralised risk management, subadditivity prevents systemic inefficiencies by ensuring that risk is not simply additive but reflects the benefits of diversification.
- Positive homogeneity: For all $\mathrm{L} \in \mathrm{M}$ and every $\lambda > 0$ it is true that

$$g(\lambda L) = \lambda g(L). \tag{2.1.3}$$

Positive homogeneity means that if you scale a portfolio's size by a factor $\lambda$ the risk (capital requirement) will also scale proportionally by $\lambda$. In other words, doubling the size of a position will double its associated risk, and halving the position will halve the risk. This property ensures that if the same financial position is multiplied by a factor (such as leveraging or de-leveraging the portfolio), the risk measure will reflect this by scaling up or down in the same proportion.

- Monotonicity: For $\mathrm{L}_1, \mathrm{L}_2 \in \mathrm{M}$, such that $\mathrm{L}_1 \leq \mathrm{L}_2$., is s always true the relationship:

$$g(\mathrm{L}_1) \leq \mathrm{g}(\mathrm{L}_2) \tag{2.1.4}$$

Monotonicity implies that positions that lead to higher losses require more risk capital.

The axioms outlined do not lead to a single definitive risk measure. Instead, choosing a specific risk measure in quantitative risk management depends on the specific objectives. Key applications of risk measures include:

- Risk Capital and Capital Adequacy. In finance, one of the main objectives of risk management is to determine how much capital a financial institution needs to keep in order to protect itself against unexpected future losses.
- Management Tool. Risk measures act as a management control mechanism, setting limits on the level of risk that different units in a firm can undertake. For example, bank traders are often subject to restrictions on the risk levels they can assume in their positions.
- Insurance Premiums. Insurance companies utilize risk measures to set premiums that compensate them for assuming the risk of insured claims. The premium size reflects the risk associated with these claims.

### 2.2 Value at Risk

Value-at-Risk (VaR) is one of the most used risk measures in financial institutions and was a key component of the Basel II capital adequacy framework. The idea of the VaR is to focus on "maximum loss which is not exceeded with a given high probability", the confidence level, instead of the simply the maximum loss (used in reinsurance).

Given a loss L and a confidence level $\alpha \in (0,1)$, the $\mathrm{VAR}_\alpha(\mathrm{L})$ is the smallest number y such that the probability that L exceeds y is less than or equal to $1-\alpha$;

$$\mathrm{VAR}_\alpha(\mathrm{L}) = \begin{cases} \inf\{\mathrm{y} \in \mathbb{R}\colon \mathrm{P}(\mathrm{L} \geq \mathrm{y}) \leq 1-\alpha\} \\ \inf\{\mathrm{y} \in \mathbb{R}\colon 1-\mathrm{P}(\mathrm{L} \leq \mathrm{y}) \leq 1-\alpha\} \\ \inf\{\mathrm{y} \in \mathbb{R}\colon \mathrm{P}(\mathrm{L} \leq \mathrm{y}) \geq \alpha\} \\ \inf\{\mathrm{y} \in \mathbb{R}\colon \mathrm{F_L}(\mathrm{y}) \geq \alpha\} \end{cases} \tag{2.2.1}$$

If F(x) is a continuous strictly increasing function, we have (McNeil et al. 2005):

$$VaR_\alpha(L) = F_L^{-1}(\alpha) = q_\alpha(F_L). \tag{2.2.2}$$

Where $q_\alpha(F_L)$ is the $\alpha - quantile$ of the loss distribution $F_L$(x)=$\mathbb{P}[L \leq x]$.

VaR is thus the $\alpha$-quantile of the loss distribution. An interpretation of Value at Risk, is as follows:

*"We are α % certain that our loss L will not be bigger than $VaR_\alpha(L)$ dollars at time T"*

In practice, McNeil et al. 2005 defines typical values of α are α = 0.95 or α = 0.99 and the time period T is usually one year for credit risk management. If F(x) is strictly increasing the following, then:

1. $F^{-1}(F(x)) = x$ for all x in its domain.
2. $F^{-1}(F(y)) = y$ for all y in its range.

Following McNeil et al. 2005, this means that finding an expression for the inverse function $F_L^{-1}$, if well defined, it is possible to obtain an expression for $VaR_\alpha(L)$ as:

$$VaR_\alpha(L) = F_L^{-1}(\alpha)$$

$$F_L\big(VaR_\alpha(L)\big) = F_L\big(F_L^{-1}(\alpha)\big) = \alpha$$

$$\mathbb{P}[L \leq VaR_\alpha(L)] = \alpha \tag{2.2.3}$$

In probabilistic terms, VaR is thus simply a quantile of the loss distribution: his means that VaR at the α confidence level provides no information on the severity of losses. △ is usually 1 or 10 days in market risk management, whereas is usually 1 year in credit and operational risk management.

To summarize, there are three important remarks to keep in mind, when talking of VaR:

1. VaR (L) is just the $\alpha$ quantile of the loss distribution $\mathrm{F_L(x)}$, (McNeil et al. ,2005, p. 39).
2. If the distribution function of L is continuous and strictly increasing then $\mathrm{VAR}_\alpha = \mathrm{F_L^{-1}}(\alpha)$, (McNeil et al. ,2005, p. 39).
3. VaR usually have a time horizon.

Moreover, it is possible to demonstrate the three properties that VaR holds. Let $L$, $L_1$, $L_2$, be random variables corresponding to the loss and $\mathrm{x} \in \mathbb{R}$ . Then three properties are valid:

a) Translation Invariance.

$$VaR_\alpha(L + x) = VaR_\alpha(L) + x$$

In fact, we have:

$$\begin{aligned} &VaR_\alpha(L + x) = inf\{y \in \mathbb{R}: P(L + x \leq y) \geq \alpha\} \\ &= \ inf\{y \ \in \mathbb{R}: P(L \leq y - x) \geq \alpha\} = \{let\ y - x = z\} \\ &= \ inf\{z + x \ \in \mathbb{R}: P(L \leq z) \geq \alpha\} = \ inf\{z \ \in \mathbb{R}: P(L \leq z) \geq \alpha\} + x \\ &= \ VaR_\alpha(L) + x \end{aligned}$$

b) Positive homogeneity.

$$VaR_\alpha(xL) = xVaR_\alpha(L)$$

In fact, we have:

$$VaR_\alpha(xL) = inf\{y \in \mathbb{R}: P(x \cdot L \leq y) \geq \alpha\}$$
$$= inf\left\{y \in \mathbb{R}: P\left(L \leq \frac{y}{x}\right) \geq \alpha\right\} = \left\{let\ \frac{y}{x} = z\right\}$$
$$= inf\{z \cdot x \in \mathbb{R}: P(L \leq z) \geq \alpha\} = inf\{z \in \mathbb{R}: P(L \leq z) \geq \alpha\} \cdot x$$
$$= x \cdot VaR_\alpha(L)$$

c) Monotonicity: if $L_1 \leq L_2$ almost surely then $VaR_\alpha(L_1) \leq VaR_\alpha(L_2)$

In fact, we have:
If $L_1 \leq L_2$ a.s. then $F_{L1} \geq F_{L2}$. This implies:
$$VaR_\alpha(L_1) = inf\{y \in \mathbb{R}: F_{L1}(y) \geq \alpha\} \leq inf\{y \in \mathbb{R}: F_{L2}(y) \geq \alpha\} = VaR_\alpha(L_2)$$

What can be concluded is that Value-at-Risk possesses many of the desired properties that an appropriate risk measure should have. However there some drawbacks of VaR. The first drawback regards the non-satisfaction of the subadditivity property. Remember that Artzner et al. (1999) propose a classification scheme for risk measures whereby a risk measure $\rho$ is said to be "coherent" if it satisfies certain conditions.

Subadditivity properties is defined as:

$$\rho(x + y) \leq \rho(x) + \rho(y)$$

According to Artzner et al. (1999), subadditivity is a key property for risk measures since "a merger does not create extra risk". For most of the situations, subadditivity is an attractive feature of a risk measure. Subadditivity ensures that Modern Portfolio Theory's concept of diversification holds since a subadditivity measure always yields lower-risk measures on diversified portfolios than they do on non-diversified ones. On the other hand, Subadditivity implies that the sum of the risks of various divisions within a financial institution exceeds or equals its overall risk.

Subadditivity's violation can cause several problems to financial institutions. Take a scenario where an organization uses VaR but has no idea that it violates subadditivity e.g., when using VaR for ranking investment choices or setting limits for traders in place. In such case, financial organizations may either assume more risks than needed or fail to hedge when necessary.

From the point of view of financial regulations, subadditivity violations might lead financial institutions to hold less capital than desired.

Thus, since VaR doesn't follow the sub-additivity properties, this implies that $VaR_\alpha(L_1 + L_2)$ is not necessarily less than or equal to $VaR_\alpha(L_1) + VaR_\alpha(L_2)$ for arbitrary $L_1\ and\ L_2$.

A simple example that makes easy to demonstrate that VaR violates the subadditivity property is as follow:

Consider two assets X and Y that are usually normally distributed, but subject to the occasional independent shocks:

$$X, Y = \epsilon + \eta$$

Where:

$$\epsilon \sim IID\mathcal{N}(0,1)$$

$$\eta = \begin{cases} 0\ with\ probability\ 0{,}991 \\ -10\ with\ probability\ 0{,}009 \end{cases}$$

For a single asset X (or Y), the 1% VaR is determined by the worst 1% of the distribution. Since the probability of the shock $\eta$ being −10 is only 0.991%, which is less than 1%, it does not affect the 1% VaR. Therefore, the 1% VaR is based on the normal distribution $\epsilon$, and ϵ at 1% quantile is approximately 2.33 (for standard normal distribution).

Considering the additional possible shock, so small adjustments to account for potential outliers, the VaR can be adjusted from 2.33 to 3.1. This is a conservative estimate to ensure that the VaR captures the potential for extreme losses better than just the normal distribution quantile would. It ensures that the 1% VaR not only considers the normal fluctuation but also provides a buffer for the rare, but significant, shock events.

Now, suppose that Y has the same distribution independently from X, and that we formulate an equally weighted portfolio of X and Y. Thus, the probability that neither X nor Y gets the shock is $(0{,}991)^2 \sim 0{,}982081$. Hence, the probability of at least one asset experiencing the shock is $1 - 0{,}982081 \sim 0{,}017919$. Since this probability is greater than 1%, the 1% VaR is significantly influenced by the possible occurrence of −10 for either X or Y.

Given this higher probability of extreme loss in the portfolio, the 1% VaR for the portfolio X+Y is found to be 9.8

Thus, observing that

$$VAR(X+Y) = 9{,}8 > VAR(X) + VAR(Y) = 3{,}1 + 3{,}1 = 6{,}2$$

it is clear that the subadditivity property doesn't hold when using VaR as risk measure.

The second limitation of VaR is that it does not provide information on the severity of losses if the VaR is exceeded, which occurs with probability $1 - \alpha$. Moreover, the sentence "it is possible to be $\alpha$% sure on that we won't lose more than $VaR_\alpha$ in one year" can be wrong and misleading due to the process's reliance on estimates.

The Expected Shortfall is a risk measure that addresses both of the difficulties listed above.

### 2.3 Expected Shortfall

Consider L represent a random variable reflecting the loss, with $\mathbb{E}[\,|L|\,] < \infty$ and a confidence level $\alpha \in (0,1)$. Then, the expected shortfall at confidence level $\alpha$ is defined as:

$$ES_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 q_u F_L \, du \tag{2.3.1}$$

This is taking the average of $q_u(F_L) = F_\alpha^{\leftarrow}(u), the\ quantile\ function\ of\ F_L$, for all confidence levels $u \geq \alpha$ in the loss distribution.

If L is a random variable, from McNeil et al. 2005 it follows that:

$$ES_\alpha(L) \; = \; \mathbb{E}[\, L \mid L \geq \; VaR_\alpha(L)] \tag{2.3.2}$$

Then, the expected shortfall responds to the question:

*"If the VaR is exceeded, how much do we expect to lose?"*

Hence, the Expected shortfall clearly depends on the Value-at-Risk, but, in contrast with VaR, the ES refers to the behaviour of the loss distribution for values higher than the VaR. If the distribution function, $F_L$, is countinuos and strictly increasing, then:

$$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(L)\, du \tag{2.3.3}$$

In fact, we have:

Consider L as a random variable with distribution function $F_L$, and consider U be a random variable which have a standard uniform distribution. Then L and $F_L^{-1}(U)$ have the same distribution. In fact, letting H be the distribution function for $F_L^{-1}(U)$. It is possible to show that

$$H(x) = P(F_L^{-1}(U) \leq x) = P(Ux \leq F_L(x)) = F_U(F_L(x)) = F_L(x)$$

Then:

$$ES_\alpha(L) = \mathbb{E}[L|L \geq VaR_\alpha(L)] = \frac{\mathbb{E}\left[L \cdot \mathrm{I}_{(L \geq VaR_\alpha(L))}(L)\right]}{P\big(L \geq VaR_\alpha(L)\big)}$$
$$= \frac{\mathbb{E}\big[L \cdot \mathrm{I}_{(VaR_\alpha(L),\infty)}(L)\big]}{1-\alpha}$$

Furthermore, we can proceed assuming that L has the same distribution as $F_L^{-1}(U)$ and that $VaR_\alpha(L) = F_L^{-1}(\alpha)$ since $F_L$ is continuous.

Hence, the numerator becomes:

$$\mathbb{E}\left[L \cdot \mathrm{I}_{(L \geq VaR_\alpha(L))}(L)\right] = \mathbb{E}\left[F_L^{-1}(U) \cdot \mathrm{I}_{(F_L^{-1}(\alpha),\infty)} F_L^{-1}(U)\right]$$
$$= \left\{\begin{matrix} I_{[F_L^{-1}(\alpha),\infty]}\big(F_L^{-1}(U)\big) = 1 \\ \leftrightarrow F_L^{-1}(\alpha) \leq F_L^{-1}(U) < \infty \\ \leftrightarrow \alpha \leq U < 1 \end{matrix}\right\} = \mathbb{E}\big[F_L^{-1}(U) \cdot I_{[\alpha,1]}(U)\big]$$
$$= \int_\alpha^1 F_L^{-1}(u) f_u \, du = \int_\alpha^1 VaR_u(L)\, du$$

This identity is useful for calculating the expected shortfall when the distribution function of the loss is continuous and strictly increasing.

### 2.4 Comparative analysis of VaR and Expected Shortfall

As it has been shown in Chapter 1, in financial market industry the Basel Committee on Banking Supervision required banks and other ADIs (Authorized Deposit-taking Institutions) disclose their daily risk forecast at the beginning of each trading day, using one of the existing alternative financial risk models. However, starting from Basel III proposal, the Basel Committee on Banking Supervision moved the quantitative risk metrics system from VaR to Expected Shortfall and decreasing the confidence level from 99% to 97,5%.

The Basel Committee (2013, p. 3) observed that *"a number of weaknesses have been identified in using Value-at-Risk (VaR) for determining regulatory capital requirements, including its inability to capture tail risk".*

However, as shown below, if on one hand the coherence due to the subadditivity property and not taking into account losses beyond the percentile makes the VaR mathematically Inferior to the ES, the latter's practical implementation and greater computational requirements may be challenging for financial institutions.

### 2.4.1 Pros and cons of VaR

Starting with Pros, VaR can be defined as a relatively simple risk management measure. To calculate VaR, a specific amount of consistency must be chosen by selecting the set of worst-case scenarios to be examined, as well as the time horizon for making future earnings predictions. Taking into account a level of 5% and a time horizon of 7 days for simplicity's, often VaR can be wrongly defined as:

*"VaR is the highest possible loss a portfolio could incur in a worst-case scenario of 5% over a period of seven days".*

Instead, VaR can be correctly defined as:

*"VaR is the maximum potential loss that a portfolio can suffer in 95% of the best cases in 7 days"*

Then, VaR can be understood as the amount that can be lost with a certain degree of confidence. VaR is a solitary figure that quantifies risk, as indicated by a specific confidence level, such as $\alpha = 0.95$. Moreover, comparing the VaR of two distributions at the same confidence level allows for ranking. VaR, as opposed to standard deviation, concentrates on the particular region of the distribution that the confidence level designates. In risk management, including banking, nuclear, aerospace, materials science, and different military applications, this is what is frequently required and what has made VaR popular.

Another essential feature of VaR is the stability of estimating processes. Because VaR ignores the tail, it is unaffected by extremely large tail losses, which are typically impossible to quantify.

Dealing with cons, beside what already introduced in the previous paragraphs, the fact that VaR does not account for properties of the distribution beyond the confidence level, $\alpha$, implies that the $VaR_\alpha(L^{(m)})$ can increase considerably with a small increase of $\alpha$. VaR is frequently criticised for failing to effectively reflect tail risk or the danger of significant market volatility. This shortage is most noticeable in tactics like "naked" shorting of deep out-of-the-money options. In these cases, while the trader routinely collects premiums without incurring losses, a sudden negative market action might result in significant losses that VaR may fail to forecast. When VaR-based risk control approaches are used to portfolios with skewed return distributions, the outcomes can be unsatisfactory. Because VaR does not take into account the distribution form beyond the confidence threshold, it may underestimate possible losses.

Lastly, VaR is a nonconvex and discontinuous function, particularly in the case of discrete distributions. This feature complicates the optimisation procedure, making it a difficult computational challenge in financial portfolio management. Modern optimisation methods, such as Portfolio Safeguard (PSG), have been created to address these issues. PSG may optimise portfolios using VaR as a performance indicator and impose numerous VaR limitations at varying confidence levels, resulting in a more flexible and efficient risk management strategy.

### 2.4.2 Pros and cons of Expected Shortfall

The main pro of the Expected Shortfall is having a straightforward engineering interpretation; in fact, ES simply assesses the most negative results. For example, if L is a loss, the constraint $ES(L) < L^{-}$ ensures that the average of $(1-\alpha)\%$ largest losses does not exceed $L^{-}$. Defining $ES(L)$ for all confidence levels $\alpha$ in (0,1) fully specifies the distribution of X. In this regard, it is preferable to standard deviation.

Moreover, Expected Shortfall offers some other appealing mathematical aspects:

- is a coherent risk measure.
- is continuous with regard to α.
- for a convex combination of random variables, $ES_{\alpha}(w_1X_1 + \cdots w_nX_n)$ is a convex function in terms of $(w_1, \cdots, w_n)$ . In finance, the ES of a portfolio is a convex function of its positions. The convexity of ES in terms of portfolio weights $(w_1, \cdots, w_n)$ plays a crucial role in financial risk management. It allows for the use of efficient convex optimization methods to find the optimal asset allocation that minimizes the risk of extreme losses, thereby providing a robust approach to managing portfolio risk.

### 2.4.3 What to use between VaR and ES

Value at Risk (VaR) and Expected Shortfall (ES) measure different characteristics of loss distribution, making them useful for various reasons. The preferred metric is determined by the individual needs of the parties involved.

Traders may prefer VaR to ES because it is less rigid. VaR only considers prospective losses up to a specific confidence level (e.g. 95% or 99%), then excludes extreme losses that exceed that barrier. This means that a trader who prefers higher, uncontrolled risks may choose VaR because it allows for potentially large losses that are not reflected in the risk metric. In addition, traders often do not suffer personal financial consequences in the event of significant losses and can find work elsewhere if they are laid off. Consequently, traders may choose the more lenient risk assessment offered by VaR.

Business owners, on the other hand, prefer ES because they assume the responsibility of covering large losses. ES calculates the average loss in worst-case situations that exceed the VaR threshold, making it a more cautious and comprehensive measure of tail risk. Large losses can have a significant influence on a company's finances, so owners focus on controlling tail events to ensure the company's profitability.

Boards of directors may prefer to disclose VaR to shareholders and regulators because it often produces a lower number than ES at the same confidence level, indicating a less worrisome risk profile. Internally, firms may rely on ES for more effective risk management. Internally, companies may rely on ES for more effective risk management. This technique involves an information asymmetry between internal management and external stakeholders, as the more stringent ES metric is used privately, while the more lenient VaR is disclosed publicly. However, internally, companies may rely on ES to achieve more robust risk.

VaR may be more useful for portfolio optimisation when reliable models of distribution tail occurrences are not available. Since VaR ignores severe tail risks (which are often the most difficult to predict), it can produce more consistent results in optimisation situations, particularly when there is little or no reliable data on uncommon catastrophic events.

Although ES is theoretically preferable because of its sensitivity to high losses, it may yield inferior results in out-of-sample portfolio optimisations if the underlying scenarios are inadequately prepared. This is because historical data may not reliably anticipate future tail events, in particular in financial markets where mean-reversion - the tendency for high returns to be followed by low returns and vice versa - can make ES based on previous data incorrect. In these cases, ES may over- or underestimate risks depending on the characteristics of previous returns.

If an appropriate model for tail events is available, the ES becomes the best option. ES offers superior mathematical properties (such as subadditivity, which supports the assumption that diversification reduces risk) and can be easily integrated into optimisation algorithms and statistical frameworks. ES provides a more realistic assessment of actual risk, especially in the presence of fat-tailed distributions, which increase the probability of extreme events.

When comparing VaR with ES, it is crucial to use the right confidence levels for each. Both are described in terms of confidence level $\alpha$, but measure distinct sections of the loss distribution. VaR represents the highest loss at the quantile $\alpha$, while ES measures the average loss above that quantile. Consequently, comparing VaR and ES at the same confidence level can be misleading because they are different risk exposures. VaR indicates the loss threshold that will not be exceeded with a specific probability, while ES provides a more detailed view of potential losses in severe situations.

# 3. Static Credit Risk Models

In credit risk management, static credit risk models are more commonly employed than dynamic ones, that are typically applied for the pricing of credit securities. The main difference between the two lies in the treatment of default timing: static models neglect the timing of default events, whereas dynamic models attach significant importance to it.

Static credit risk models are based on historical data and are designed with a predetermined set of criteria and assumptions. These models remain constant throughout time, presuming that the relationships between variables are stable, and do not account for real-time changes in the economic environment or borrower behaviour. They are easier and less expensive to execute, and they produce consistent and predictable results that are straightforward to understand. However, because static models cannot adjust to changing conditions, they may fail to appropriately reflect current risks if market conditions, or borrower circumstances change. Static models are primarily concerned with whether an obligor defaulted by a particular time, without regard for the precise timing of the default, making them appropriate for stable environments with fewer frequent changes.

Dynamic credit risk models, on the other hand, are intended to update and adapt in real or near-real time when new data becomes available. Unlike static models, dynamic models use real-time or frequently updated data inputs and allow model parameters to alter over time. This versatility allows dynamic models to respond to changing market conditions and borrower behaviours, resulting in a more accurate and current evaluation of credit risk. Dynamic models are frequently more sophisticated and resource-intensive, requiring advanced statistical approaches and machine learning techniques to capture complex interactions and nonlinear correlations in data. The precise timing of defaults is crucial to dynamic models, which are used to price credit securities such as CDOs. Understanding the precise timing of credit events is critical.

Section 3.1 discusses the critical framework for static portfolio credit risk modelling, forming the foundation of our understanding and application of risk assessment in financial portfolios. The models in this framework are extensively classified into two categories: Mixture Models and Threshold Models. These classifications are critical for organising our investigation and application of credit risk models. Threshold models, while specific examples of mixture models, require special attention due to their distinct methodological peculiarities. Despite their inherent similarities in underlying structure, the different ways these models take to representing defaults justify their respective categories.

This distinction is important because it effects model selection and use in different credit risk scenarios. In this part, we look at the common credit model elements and notations that apply to both model groups. These parts comprise fundamental statistical ideas and parameters that support the operation and correctness of credit risk models. Section 3.2 then concentrates on the binomial model, which is known for its simplicity and fundamental relevance. This model introduces basic credit risk modelling principles, serving as a stepping stone to more complicated models. In Section 3.3, we extend the binomial model by including obligor dependence, resulting in the Bernoulli mixed model. This extension is significant since it addresses the interdependence among obligors, increasing the model's validity and efficiency.

The primary concepts and notation choices in this section are drawn from McNeil et al. (2005) in Quantitative Risk Management and Herbertsson (2009) in his notes from the study on Credit Risk Modelling.

### 3.1 Fundamentals of Static Credit Risk Modelling

Static portfolio in credit risk modelling relies on some common characteristics, problem formulations and set of notations, regardless of the model choice. These models focus on modelling the default of an individual company, with particular emphasis on capturing the defaults of multiple companies within a portfolio.

The default of a company *i* is modelled by a default indicator $Y_i$, which is random variable in $\{0, 1\}$. The default indicator takes the value 1 if the company defaults before time T, and 0 otherwise, i.e.

$$Y_i = \begin{cases} 1 \; if \; obligor \; i \; defaults \; before \; time \; T \\ 0 \; otherwise \end{cases}$$

After introducing the default indicator $Y_i$, in order to model the credit loss of an obligor and then the loss of the whole portfolio of obligors, there are some other variables that have to be introduced:

- let $e_i$ be the exposure of obligor i, or the notional amount held by debtor i. The exposures are considered deterministic, since known a priori

- let $l_i$ be the loss rate for obligor I, which represent the percentage lost if obligors I defaults. The loss rate may be deterministic and consistent across all obligors, but it can also be stochastic as the models evolve. In terms of loss rate and exposure, consider that:

$$l_i \in (0,1) \; and \; e_i > 0.$$

With these notations, the loss $L_i$ from obligor i is given by:

$$L_i = l_i \cdot e_i \cdot Y_i. \tag{3.1.1}$$

To model the loss in a portfolio of m obligors we define the variable $L^{(m)}$ as the sum of the individual losses. $L^{(m)}$ is defined as:

$$L^{(m)} = \sum_{i=1}^{m} L_i = \sum_{i=1}^{m} l_i \cdot e_i \cdot Y_i. \tag{3.1.2}$$

Without loss of generality, it can often be assumed that $e_i$ is equal to 1 for all obligors.

As previously mentioned, loss rates and exposures are typically assumed to be uniform across all obligors. A portfolio in which these quantities are identical for every obligor is referred to as a homogeneous portfolio. Throughout the thesis, this kind of portfolio will be studied, where it is useful set the random variable $N^{(m)}$, corresponding to the number of defaults in the portfolio, that can be defined as

$$N^{(m)} = \sum_{i=1}^{m} Y_i \tag{3.1.3}$$

Moreover, the reason to introduce the new variable $N^{(m)}$ is due to its close relation to $L^{(m)}$. By setting $l_i = l$ and $e_i = 1 \; \forall \; I$, it is possible to obtain:

$$L^{(m)} = \sum_{i=1}^{m} l_i \cdot e_i \cdot Y_i \; = \; l \cdot \sum_{i=1}^{m} 1 \cdot Y_i \; = \; l \cdot N^{(m)} \tag{3.1.4}$$

It can be observed that:

$$P\left(L^{(m)} = l \cdot k\right) = P\left(l \cdot N^{(m)} = l \cdot k\right) = P\left(N^{(m)} = k\right) \tag{3.1.5}$$

In other words, to analyse $L^{(m)}$ in a homogeneous portfolio with constant loss rate it is sufficient to study just the behaviour of $N^{(m)}$. In addition to finding the probability distribution of $N^{(m)}$, expected value and variance must be also evaluated.

### 3.1.1 The Exchangeable Model

The thesis will focus on the exchangeable model, which implies that the individual default probabilities are the same for all companies. Exchangeable models are used to simplify the analysis and rely on the assumption that if the state indicator S is exchangeable than the default indicator Y is exchangeable as well.

Formally, a random vector S is called exchangeable if

$$(S_1, \dots\dots, S_m) \stackrel{\text{def}}{=} \left(S_{\prod(1)}, \dots\dots, S_{\prod(m)}\right)$$

For any permutation $(\prod(1), \; \dots, \prod(m)) \; of \; (1, \dots, \; m)$. This implies in particular that for any $k \; \in \{1, \dots, m-1\}$ of all the $\binom{m}{k}$ possible k-dimensional marginal distribution of S are identical.

In this situation the following notation for default probabilities and joint default probabilities can be introduced:

$$\pi_k = P\left(Y_{i_1} = 1, \dots, Y_{i_k} = 1\right), \{i_1, \dots, i_k\} \subset 1, \dots, m, 1 \leq k \leq m$$

$$\pi = \pi_1 = P(Y_i = 1), i \in 1, \dots, m$$

Thus $\pi_k$, that is the kth order joint default probability, is the probability thaty an arbitrarily selected subgroup of k companies defaults in [0, T].

When default indicators are exchangeable it is possible to get that:

$$E(Y_i) = E(Y_i^2) = P(Y_i = 1) = \pi \tag{3.1.6}$$

$$E\left(Y_i Y_y\right) = P\left(Y_i = 1, Y_j = 1\right) = \pi_2 \tag{3.1.7}$$

Thus, in this setting $\pi = P(Y_i = 1)$ for all obligors in the portfolio. Under the assumption that $Y_i$ are independent, the number of defaults in a portfolio is defined as $N^{(m)} = \sum_{i=1}^{m} Y_i$ and follow the same distribution of $Y_i$. By introducing dependence between obligors in the portfolio, the binomial model can be transformed into a Bernoulli Mixture model: the Bernoulli mixture model is similar to a conditional binomial model, with default reliance based on a random factor. The Bernoulli mixture model is the primary model utilised in the category of mixture models. The second group includes threshold models, where a corporation defaults if a random variable, typically its asset worth, falls below a certain threshold. Threshold models are similar to mixture models, as previously stated.

The thesis will focus only on Bernoulli Mixture Model. However, it is relevant to mention that even if the threshold and mixture models use different ways to model the dependency between the default indicators, to quantify and measure dependency, both models commonly use covariance and standard correlation.

### 3.2 The Binomial Model

The binomial model is a simple way to model obligor defaults in a portfolio. While not practical, this model is useful for intuitive purposes and serves as a foundation for more advanced models.

The Binomial Model assumes a homogeneous portfolio with m obligors, where each obligor can either default or not default according to the previously defined default indicator, $Y_i$. It is assumed that the default indicators $Y_1, Y_2, \ldots, Y_m$ are independent and identically Bernoulli distributed with parameters p, i.e. P $(Y_i = 1) = p$ and $P(Y_i = 0) = 1 - p$.

Further, recall that $N^{(m)} = \sum_{i=1}^{m} Y_i$ is the number of defaults in a portfolio with m obligors. Since $N^{(m)}$ is the sum of m independent Bernoulli random variables, it is binomally distributed with parameters m and p.

Therefore:

$$N^{(m)} \sim Bin(m, p)$$

$$P\left(N^{(m)} = k\right) = \binom{m}{k} \cdot p^k \cdot (1-p)^{m-k}$$

$$E[N_m] = mp$$

In order to analyse the credit portfolio loss, $L^{(m)} = l \cdot N^{(m)}$, where l is defined as the credit loss rate when default occurs, equal for all obligors and $N^{(m)}$ is the number of defaults. Figure 3.1 plots the probability distribution according to the number of defaults with an individual default probability of $\pi = 0{,}05$ and m = 50 obligors.

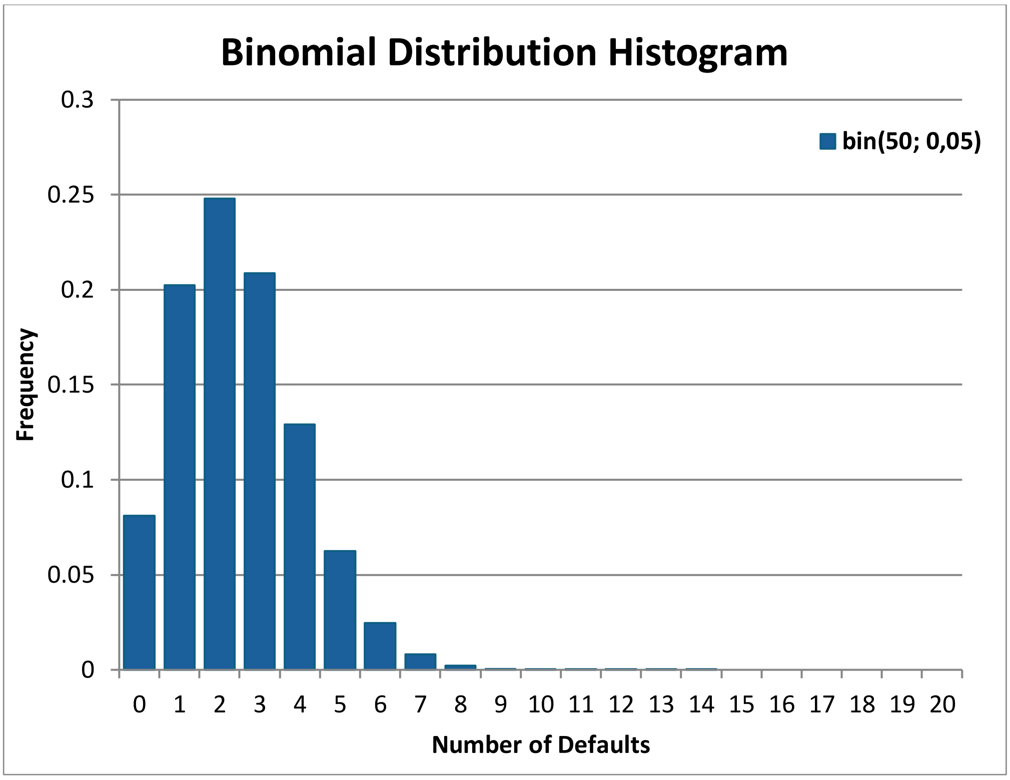


*Figure 1.Binomial distribution graph changing number of defaults, with m=50 and probability of default=0,05.*

As observed in Figure 3.1, the binomial model indicates that the likelihood of experiencing a large number of defaults is low, which implies that the binomial distribution has thin tails.

For example, according to our example

$$P(N^{(m)} \geq 7) \sim 1{,}2\%$$

This means that even if the default probabilities increase, the distribution will still exhibit thin tails. This is a result of the assumption of independence among the default indicators, which limits the probability of observing a large number of simultaneous defaults. But the aim of the thesis is to find and use a model that can create thicker tails while simultaneously accounting for obligor dependence. Thicker tails essentially mean a higher probability of having more extreme loss scenarios, and the binomial model cannot account for thicker tails, which increase the likelihood of extreme loss scenarios.

In the following part, the binomial model will be extended to a Bernoulli mixture model, which addresses both above difficulties. The probability distributions of the number of defaults from the two models will be compared, revealing that the mixture model produces larger tails.

**3.3 The Bernoulli Mixture Model**

The previous section highlights that the independence assumption of default indicators is very simplistic and fails to accurately reflect reality. The Bernoulli mixture model is the first step towards a more complicated static credit portfolio model, introducing dependencies between obligor defaults. The Bernoulli mixture model uses a factor vector to represent the relationship between macroeconomic variables such as interest rates and stock index prices. These macroeconomic variables are also represented stochastically. The obligors' defaults are believed to be independent based on their factor vector.

Definition 3.3.1 is based on McNeil et al. (2005) and describes a Bernoulli mixing model.

Definition 3.3.1

*Given some p < m and a p-dimensional random vector* $\psi = (\Psi_1, \dots, \Psi_p)'$ *, the random vector* $Y = (Y_1, \dots, Y_m)'$ *follows a Bernoulli mixture model with factor vector* $\Psi$ *if there are functions* $p_i: \mathbb{R}^p \to [0,1]$ *,* $1 \leq i \leq m$*, such that conditional on* $\Psi$ *the component Y are independent Bernoulli rvs satisfying* $P(Y_i = 1 \mid \Psi = \psi) = p_i(\psi)$*.*

Definition 3.3.1 states that the default probability of an individual company, given a vector of factors, is a function of those factors, in a Bernoulli Mixture Model. Furthermore, the definition specifies that, conditional on the factors, the components of the default indicator Y are independent.

Let $y = (y_1, \dots, y_m)'$, where $y_i \in \{0,1\}$ for every obligor i, be a factor representing which obligors defaulted and which survived. Since the default indicators are conditionally independent it follows that:

$$P(Y = y \mid \Psi = \psi) = \prod_{i=1}^{m} P(Y_i = y_i \mid \Psi = \psi) = \prod_{i=1}^{m} p_i(\psi)^{y_i} \left(1 - p_i(\psi)\right)^{1-y_i}.$$

Individual default indicators follow a Bernoulli distribution with parameter $p_i(\psi)$. The default indicator's unconditional distribution, P (Y = y), can be calculated by integrating over all possible factor values. This will be discussed further using a one-factor exchangeable Bernoulli mixture model. Bernoulli random variables can be approximated using Poisson random variables.

This thesis does not cover the Poisson mixture model; however, its definition is similar to the Bernoulli mixture model. The primary difference is that the default vector is Poisson distributed rather than Bernoulli distributed, as discussed in the Section 8,4 of McNeil et al. (2005).

### 3.3.1 One-Factor Exchangeable Bernoulli Mixture Model

In this section, default indications for obligors will be based on a single common factor. This is due to more than just the ease of calculating, since calibration of models with multiple factors might be challenging due to limited information. Further, it is assumed an exchangeable model, which means that all individual default probabilities $p_i$ are identical.

To emphasize that companies in the portfolio have the same default functions, a random variable $p(\Psi)$ called mixing variable is defined as $p(\Psi) := p_i(\Psi)\ for\ all\ i$. Thus, $P(\ Y_i = 1 \mid \Psi) = p(\Psi)$ for all obligors, and this implies that the unconditional probability $P\ (Y_i = 1) = \mathbb{E}[\ p(\Psi)\ ]$.

In fact, we have that:

$$
\begin{aligned}
\pi = \ & P(Y_i = 1) = 1 \cdot P(Y_i = 1) + 0 \cdot (P(Y_i = 0) = \mathbb{E}[\ Y_i] \\
& = \mathbb{E}[\ \mathbb{E}\ [\ Y_i \mid \Psi]] = \mathbb{E}[\ 1 \cdot P(\ Y_i = 1 \mid \Psi)\ +\ 0 \cdot \mathrm{P}(\ Y_i = 0 \mid \Psi)] \\
& = \mathbb{E}[\ P(Y_i = 1 \mid \Psi)] \\
& = \mathbb{E}[p(\Psi)]
\end{aligned}
$$

Now the scope is finding the probability of a certain number of defaults, that is finding the unconditional probability $P(N^{(m)} = k)\ for\ k =\ 0,1,\dots,m$ , where $N^{(m)}$ is defined as the number of defaults in a portfolio of m obligors. As previously stated, individual default probabilities given the factors are Bernoulli distributed. Thus, in the one-factor exchangeable model, this implies that the number of defaults $N^{(m)}$ is conditionally bionmial distributed with parameters m and $p(\psi)$, since it is the sum of m independent Bernoulli trials with parameter $p(\psi)$, i.e. we have that:

$$
P\left(N^{(m)} = k \middle| \Psi = \psi\right) = \binom{m}{k} p(\psi)^k \left(1 - p(\psi)\right)^{m-k}
$$

Integrating over $p(\psi)$, between 0 and 1, to find out the unconditiona probability of the number of defaults, follows that:

$$P\left(N^{(m)} = k\right) = \binom{m}{k} \int_0^1 p(\psi)^k \left(1 - p(\psi)\right)^{m-k} \, dG\left(p(\psi)\right)$$

Where G is the distribution function for the mixing variable $p(\psi)$, that is:

$$G(x) = P(p(\psi) \leq x).$$

Frey and McNeil (2003) identified three common distributions for blending variables: beta, probit-normal, and logit-normal. The mixing distributions will be investigated further in the following chapter.

By using the previous equation, it is possible to compute the probability distribution for the number of defaults in a Bernoulli mixture model. For example, taking a mixing variable beta distributed, with parameters $\alpha$ and $\beta$. The equation reduces to:

$$P\left(N^{(m)} = k\right) = \binom{m}{k} \frac{\beta(\alpha + k; \beta + m - k)}{\beta(\alpha, \beta)}, \quad \alpha.\beta > 0$$

$$where \ \beta(\alpha, \beta) = \int_0^1 z^{\alpha-1}(1 - z)^{\beta-1} \, dz, \quad 0 < z < 1$$

The following Figure 2 compares a binomial model to a Bernoulli mixture model with a beta mixing distribution. Both models have the same individual default probability. The mixture model produces thicker tails compared to the binomial model.

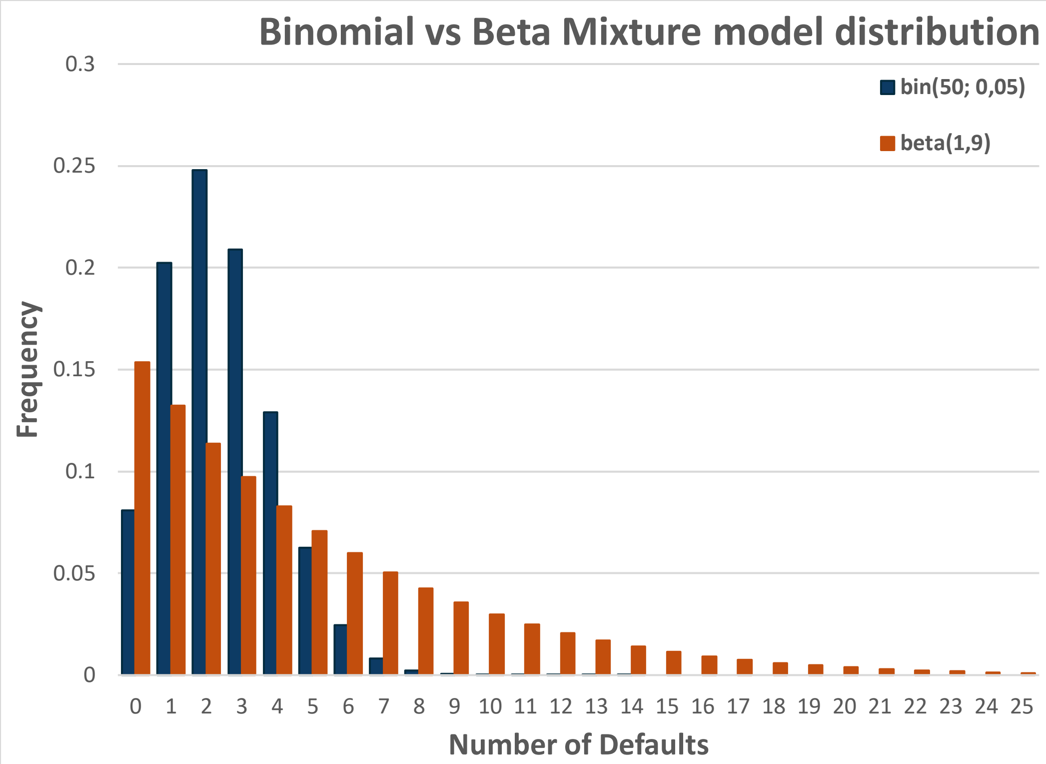


*Figure 2. The probability distribution of the number of defaults for both a binomial model and a Bernoulli mixture model with a beta mixing variable. In both models, the individual default probability is $\pi$=0,05 in both models, and the portfolio consists of m = 50 obligors.*

### 3.3.2 Correlation

The transition from the simple binomial model to the mixture model was made to account for obligor dependence. In the binomial model, there is no association between two default indicators due to their independence. In the mixture model, they are only conditionally independent. The default correlation is non-negative and defined only by the mixing distribution.

Following the notation of McNeil et al. (2005), the general joint default probability for k firms can be defined as:

$$\pi_k = P\left(Y_{i_1} = 1, \dots, Y_{i_k} = 1\right)$$

Where $\{i_1, \dots, i_k\}$ is an arbitrary subset of {1, …, m} for $k \in \{2, \dots, m\}$.

Thus, it is possible to write the correlation between two default indicators as:

$$Corr(Y_i, Y_j) = \frac{Cov(Y_i, Y_j)}{\sqrt{Var(Y_i) \cdot Var(Y_j)}} = \frac{\mathbb{E}[Y_i Y_j] - \mathbb{E}[Y_i] \cdot \mathbb{E}[Y_j]}{Var(Y_i)}$$
$$= \frac{\mathbb{E}[Y_i Y_j] - \mathbb{E}[Y_i] \cdot \mathbb{E}[Y_j]}{\mathbb{E}[Y_i^2] \cdot \mathbb{E}[Y_i]^2} = \frac{\pi_2 - \pi^2}{\pi - \pi^2} \quad (3.3.1)$$

In fact, in the Bernoulli mixture model it is possible to state that:

$$\pi_k = P(Y_{i_1} = 1, \ldots., Y_{i_k}) = \mathbb{E}\big[\mathbb{E}[Y_1 \ldots Y_k | \Psi]\big] = \mathbb{E}[p(\psi)^k] \quad (3.3.2)$$

Consequently, in a Bernoulli mixture framework the correlation is thus given by:

$$Corr(Y_i, Y_j) = \frac{\pi_2 - \pi^2}{\pi - \pi^2} = \frac{\mathbb{E}[p(\Psi)^2] - \mathbb{E}[p(\Psi)]^2}{\mathbb{E}[p(\Psi)] - \mathbb{E}[p(\Psi)]^2} = \frac{Var(p(\Psi))}{\mathbb{E}[p(\Psi)] \cdot (1 - \mathbb{E}[p(\Psi)])} \geq 0$$

Thus, the Bernoulli mixture model reveals a relationship between obligor defaults, which is solely influenced by the mixing variable. The first two moments of the mixing variable totally determine the correlation between $Y_i$ and $Y_j$ for any two pairs $i \neq j$. In Chapter 4, the Bernoulli mixture model will come back and applied in a portfolio credit loss context.

### 3.4 Asymptotic Behaviour in Large Portfolios

This section presents asymptotic results for large portfolios within a Bernoulli mixture framework. In one-factor Bernoulli mixture models, Frey and McNeil (2003) and Herbertsson (2009) demonstrated that the tail behaviour of the mixing distribution plays a crucial role in determining the tail of the loss distribution. Their findings suggest that the characteristics of the mixing distribution heavily influence extreme losses in the portfolio. This important conclusion will be explored in greater depth, alongside other key insights into the behaviour and features of Bernoulli mixture models, such as the impact of portfolio size and the dependence structure among obligors, which can further influence risk in large portfolios. Understanding these aspects is vital for accurately modelling and predicting tail risks in credit portfolios.

To understand the behaviour in large portfolios, we recall the previously introduced default indicator variable $Y_i$, and set $e_i = 1$ for all obligors, to simplify calculations. The loss rate can be assumed random, but this complicates the model significantly. Regarding this, Frey and McNeil (2003), stated that loss rate given default $L_i$ , and default indicators, $Y_i$ , should not be considered independent. It makes sense that corporations would struggle to recover more during a financial crisis than during regular growth. The loss rate, $L_i$, can be predicted based on economic parameter $\Psi$. However, the model considered assumes deterministic and equal loss rates for all obligors, denoted as l.

Bernoulli mixture models are improvements of binomial models, where is present the addition of default dependency (as discussed previously). Using the binomial setup, we can utilise the variable $N^{(m)} = \sum_{i=1}^{m} Y_i$ to get the defaults number in the portfolio. The thesis will be based on an exchangeable model where all obligors have the equal default probability, $p_i(\Psi) = p(\Psi)\ for\ all\ i.$

As stated before, researching $L^{(m)}$ under these assumptions is sufficient to investigate $N^{(m)}$. Since the random variable $N^{(m)}$ s generated by i.i.d. Bernoulli distributed random variables (which are independent when conditioned on the factor vector), it is convenient to apply the law of large numbers. This allows for the analysis of the asymptotic behaviour of the model, particularly as the portfolio size increases. By leveraging the law of large numbers, we can better understand how the proportion of defaults converges to its expected value in large portfolios, providing insight into the long-term behaviour of default probabilities and the overall risk distribution.

Then, it is possible to focus just on the ratio $\frac{N^{(m)}}{m}$, which represents the proportion of defaults in the portfolio (also known as the default fraction). When all obligors have $e_i = 1$, the ratio represents their average loss. To begin, Frey and McNeil (2003) demonstrate that, given that conditional on $\Psi$ the random variables $Y_1, \dots, Y_m$ are i.i.d. with default probability $p(\Psi)$ , the law of large numbers implies:

$$\lim_{m \to \infty} \frac{N^{(m)}}{m} = p(\Psi) \quad a.s. under\ the\ measure\ P(\cdot\,|\Psi)$$

Which implies that the event $\lim_{m \to \infty} \frac{N^{(m)}}{m} = p(\Psi)$ given an outcome $\Psi$ has probabilty 1, i.e.

$$P\left(\lim_{m \to \infty} \frac{N^{(m)}}{m} = p(\Psi) \mid \Psi\right) = 1$$

More specifically, almost sure convergence also implies convergence in distribution, i.e.

$$\lim_{m\to\infty} P\left(\frac{N^{(m)}}{m} \leq\ x\right) = P(p(\Psi)\ \leq x)$$

Inspired by the outline of Lando (2004), this can be show as follows:

$$\lim_{m\to\infty} P\left(\frac{N^{(m)}}{m} \leq\ x \mid \Psi\right) = \begin{Bmatrix} 0\ if\ p(\Psi)\ >\ x \\ 1\ if\ p(\Psi) \leq\ x \end{Bmatrix} = \mathbb{I}_{(p(\Psi)\leq x)} \qquad (3.4.1)$$

It is now possible to derive the unconditional probability as:

$$P\left(\frac{N^{(m)}}{m} \leq x\right) = \mathbb{E}\left[P\left(\frac{N^{(m)}}{m} \leq\ x \mid \Psi\right)\right] \qquad (3.4.2)$$

Finally, letting $m \to \infty$,

$$\lim_{m\to\infty} P\left(\frac{N^{(m)}}{m} \leq\ x\right) = \mathbb{E}\left[\mathbb{I}_{(p(\Psi)\leq x)}\right] = P(p(\Psi) \leq x) = G(x) \qquad (3.4.1)$$

As the number of obligors increases, the distribution of the default fraction converges to the mixing distribution. This convergence allows us to approximate the distribution of the default fraction in large portfolios using the properties of the mixing distribution. More specifically, as the portfolio size grows, individual idiosyncratic risks tend to cancel out, and the influence of the systematic risk factors captured by the mixing distribution becomes dominant. Previous discussions confirmed that the tail behaviour of the mixing distribution primarily determines the distribution of the default fraction, $\frac{N^{(m)}}{m}$, and consequently the tail of the loss distribution, $L^{(m)}$. In other words, if the mixing variable has a heavy or "fat" tail, which allows for more extreme events, the loss distribution will also exhibit a thicker tail, leading to higher probabilities of extreme events and systemic risk. This highlights the importance of understanding the tail properties of the mixing variable, particularly in stress testing and risk management of large credit portfolios.

Frey and McNeil (2003) offer a more generalized framework for analysing the asymptotic behaviour of large portfolios. By relaxing assumptions related to model exchangeability and constant loss rates across obligors, they broaden the scope of their analysis to more realistic settings, such as heterogeneous portfolios where obligors may have different risk profiles or varying exposures. Although this added complexity makes the analysis more intricate, Frey and McNeil still reach the same core conclusion: the tail of the mixing distribution largely governs the behaviour of the loss distribution in large portfolios.

Additionally, Frey and McNeil demonstrate a crucial relationship between the quantiles of the loss distribution and those of the mixing distribution in one-factor Bernoulli mixture models. This means that for high quantiles (such as 99th or 99.9th percentiles), which are critical in risk management, the quantiles of the mixing distribution can be directly linked to those of the loss distribution. This finding is significant for practitioners as it allows for better estimation of extreme losses, making it possible to implement more effective capital allocation and risk mitigation strategies. Understanding this linkage is particularly important for stress testing and calculating risk measures like Value at Risk (VaR) or Expected Shortfall (ES) in large portfolios, where extreme events and tail risk play a critical role in overall portfolio risk.

### 3.5 Approximation of VaR and ES in large portfolios

Considering a Bernoulli mixture model, it is possible to derive an approximation of the VaR and ES when dealing with large portfolios. In our setting, we have $L^{(m)}$ corresponding to the loss distribution of a portfolio with m obligors, where m is considered large, and G be the continuous distribution function of the mixing variable $p(\psi)$.

Then, using a homogeneous portfolio with m obligors, it holds that:

$$VaR_\alpha\left(L^{(m)}\right) \sim l \cdot m \cdot G^{-1}(\alpha) \quad (3.5.1)$$

$$ES_\alpha\left(L^{(m)}\right) \sim \frac{l \cdot m}{1-\alpha} \int_\alpha^1 G^{-1}(u)\, du \quad (3.5.2)$$

In fact, we have:

$$VaR_\alpha\left(L^{(m)}\right) = inf\{y \in \mathbb{R}: P\left(L^{(m)} \leq y\right) \geq \alpha\} = inf\left\{y \ \in \ \mathbb{R}: P\left(\frac{L^{(m)}}{l \cdot m} \leq \frac{y}{l \cdot m}\right) \geq \alpha\right\}$$

$$= \ inf\left\{y \ \in \ \mathbb{R}: P\left(\frac{N^{(m)}}{m} \leq \frac{y}{l \cdot m}\right) \geq \alpha\right\} \xrightarrow{m\to\infty} inf\left\{y \ \in \ \mathbb{R}: G\left(\frac{y}{l \cdot m}\right) \geq \alpha\right\}$$

$$= \left\{let \ y \ \frac{y}{l \cdot m} = x\right\} = inf\{l \cdot m \cdot x \in \mathbb{R}: G(x) \geq \alpha\}$$

$$= \ l \cdot m \cdot inf\{x \in \mathbb{R}: G(x) \geq \alpha\} = \ l \cdot m \cdot G^{-1}(\alpha)$$

In fact, we proved in Section 3.4. that $P\left(\frac{N^{(m)}}{m} \leq \frac{y}{l \cdot m}\right) \to \ G\left(\frac{y}{l \cdot m}\right) \ as \ m \to \infty$. So, as the number of obligors m approaches infinity, the distribution of the average number of losses converges to the distribution of the mixing variable. In other words, the impact of idiosyncratic risk diminishes, and the behaviour of the portfolio loss becomes dominated by the systematic risk factors represented by the mixing variable. This convergence highlights the critical role of the mixing distribution in determining the overall risk profile of large portfolios, especially in terms of extreme losses and tail behaviour.

And the approximation of expected shortfall is derived directly from the VaR-approximation as follows:

$$ES_\alpha\left(L^{(m)}\right) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u\left(L^{(m)}\right) du \sim \frac{1}{1-\alpha} \int_\alpha^1 l \cdot m \cdot G^{(-1)}(u)\, du$$

$$= \frac{l \cdot m}{1-\alpha}\int_\alpha^1 G^{(-1)}(u)\, du$$

# 4 Research Methodology

This Chapter outlines the methodology employed to assess portfolio risks through a One-Factor Exchangeable Bernoulli Mixture Model.

**4.1 Risk Measuring using a One-Factor Exchangeable Bernoulli Mixture Model**

The aim of this chapter is evaluating different types of portfolios, using a One-Factor Exchangeable Bernoulli Mixture Model. After evaluating and discussing the main properties of the Beta Mixture Model, the analysis will focus on a Beta mixing distribution. The Beta distribution is chosen for its flexibility and suitability in modelling diverse correlation structures and default probabilities within the portfolio.

The analysis of this chapter we use different portfolios containing 1000 obligors (i.e. m = 1000), a large enough number (McNeil et al., 2005) so that we can use the large portfolio approximation, and the loss rate given default, denominated as l, set to 60%. The values of m and l are taken equal to the ones from McNeil et al. (2005), in Chapter 8.

Then, from Section 3.1 we have that:

$$L^{(m)} = \ l \cdot N^{(m)} \tag{4.1}$$

The probability of default $\pi$ for each portfolio is taken from S&P Global Ratings Credit Research & Insights and S&P Global Market Intelligence's CreditPro data. When we compare the models, the $\alpha$ of Value-at-Risk is fixed to $\alpha = 0{,}95$, which is common in the industry (Herbertsson, 2014).

Then, three different types of portfolios will be evaluated:

1. A portfolio composed of Global Rating companies of the S&P 500
2. A portfolio composed of Rating B companies of the S&P 500
3. A portfolio composed of Rating CCC companies of the S&P 500

The analysis in Section 4.3 will evaluate Value-at-Risk and Expected Shortfall from 2011 to 2023 in comparison to 1981-2005 values from McNeil (2005). Then, for each of the three different scenarios from 2011 to 2023, the parameters of the Beta distribution will be evaluated through different sensitivity analysis, by changing the probability of default $\pi$ and the correlation between the obligors $\rho$.

The analysis will evaluate how VAR and ES behave and are able to predict by considering different types of portfolios. Section 4.1 defines the Beta distribution, and the parameters $\alpha$, $\beta$ as function of first and second moment-style $\pi$ and $\pi_2$. Section 4.2 will analyse what moment-style estimators are and define two different methods to define them.

**4.2 Properties of Beta distribution**

This section evaluates the selection of a mixing distribution, since as shown in McNeil et al. (2005), the choice of the distribution significantly influences model risk mitigation. Moreover, as stated by Frey and McNeil, 2003, "the tail of the credit loss in large one-factor Bernoulli mixture models is essentially driven by the tail of the mixing variable". Ultimately, this suggests that the portfolio loss distribution is less influenced by the specific choice of mixing distribution and more dependent on estimating the parameters $\pi$, $\pi_2$ and the default correlation $\rho$. Hence, in McNeil et al. (2005) it is also noted that these parameters significantly influence the behaviour of the model tails and are challenging to estimate.

The analysis will therefore concentrate solely on the Beta distribution, as it is commonly applied in credit risk modelling due to its favourable characteristics. Specifically, the Beta distribution's output is confined to the interval (0, 1), which simplifies calculations when working with probabilities by eliminating the need to re-scale the random variable's output.

Let $p(\psi) = \psi$ where the random variable $\psi$ is Beta distributed with parameters $\alpha$ and $\beta$, i.e. $\psi \sim$ Beta ($\alpha$, $\beta$,).

Then, its density function is given by

$$f(x) = \frac{1}{\beta(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \qquad 0 < x < 1;\ \alpha, \beta > 0$$

where the beta function is defined as

$$\beta(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx\,, \quad \alpha, \beta > 0.$$

Having defined the density function of the Beta mixing distribution, we can define the parameters $\pi$, $\pi_2$ and the default correlation $\rho$, recalling the Equations 3.3.1, 3.3.2.

$$\pi = E[p(\psi)] = E[\psi]$$

$$\pi_2 = E[\psi^2]$$

$$\rho = Corr(Y_i, Y_j) = \frac{\pi_2 - \pi^2}{\pi - \pi^2}$$

$\pi$, $\pi_2$ are given by the first and the second moment of the Beta mixing variable, as shown in Section 4.2, and the correlation $\rho$ is calculated from these moments.

To estimate the parameters $\alpha$,$\beta$ we need to use the properties of a Gamma function:

**Definition 4.2**

A random variable X is Gamma distributed with parameters $\lambda$,t > 0, $X \sim Gam\ (\lambda, t)$, if its density function f is given by:

$$f(x) = \frac{1}{\Gamma(t)} \lambda^t x^{t-1} e^{-\lambda t}, \qquad x \geq 0$$

Where $\Gamma(t)$ is the Gamma function defined as

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx$$

One can prove that the gamma function satisfies:

$$\Gamma(t+1) = t\Gamma(t)$$

And the following connection between the Beta function and the Gamma function holds.

$$\beta(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{4.2.1}$$

Now it is possible to define π, $\pi_2$ as function of $\alpha$ and $\beta$. Let $\psi \sim$ Beta (α, β,) for α, β >0 and let $f$ the density function of $\psi$. Then, the first and the second moments order are:

$$1.\ E[\psi] = \frac{\alpha}{\alpha+\beta} \tag{4.2.2}$$

$$2.\ E[\psi^2] = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} \tag{4.2.3}$$

In fact, we have:

$$E[\psi] = \int_0^1 xf(x)\,dx = \int_0^1 \frac{1}{\beta(\alpha,\beta)} xx^{\alpha-1}(1-x)^{\beta-1}\,dx = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha}(1-x)^{\beta-1}\,dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{(\alpha+1)-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \beta(\alpha+1,\beta)$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+1+\beta)} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \frac{\alpha\Gamma(\alpha)}{(\alpha+\beta)\Gamma(\alpha+\beta)} = \frac{\alpha}{\alpha+\beta}$$

Instead, for the second moment we have:

$$E[\psi^2] = \int_0^1 x^2 f(x)\, dx = \int_0^1 \frac{1}{\beta(\alpha,\beta)} x^2 x^{\alpha-1}(1-x)^{\beta-1}\, dx$$

$$= \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha+1}(1-x)^{\beta-1}\, dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{(\alpha+2)-1}(1-x)^{\beta-1} dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \beta(\alpha+2,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+2+\beta)}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \frac{\alpha(\alpha+1)\Gamma(\alpha)}{(\alpha+\beta)(\alpha+\beta+1)\Gamma(\alpha+\beta)} = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

Thus, it is possible to find the Beta distribution parameters, $\alpha \; and \; \beta$, by solving the system of equations:

$$\begin{cases} \pi = \dfrac{\alpha}{\alpha+\beta} \\ \pi_2 = \dfrac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} \end{cases} \tag{4.2.4}$$

Solving the system, it is possible to get $\alpha, \beta$ as:

$$\begin{cases} \alpha = \dfrac{(\pi_2 - \pi)\pi}{\pi^2 - \pi_2} \\ \beta = \dfrac{(\pi_2 - \pi)\pi}{\pi^2 - \pi_2}\left(\dfrac{1}{\pi} - 1\right) \end{cases} \tag{4.2.5}$$

### 4.3 Statistical Interference for Mixture Models

Once having defined the parameters of the Beta distribution $\alpha$, $\beta$ as function of π and $\pi_2$, we need to fit Bernoulli mixture models into historical default data, that means estimating the model parameters π and $\pi_2$ from historical default data.

In finance, the calibration of portfolio credit risk models such as KMV, CreditMetrics, and CreditRisk+ has generally not depended on formal statistical methods due to the lack of historical default data, mainly for higher-rated firms. Industry practices typically estimate default probabilities by aligning them with historical default rates of similar companies, assessed either through standardized credit ratings or proprietary measures like distance-to-default. Furthermore, additional model parameters that describe the dependencies of defaults are often determined through less rigorous methods, involving either economic reasoning or proxy analysis such as equity returns, rather than direct statistical inference. This approach, while pragmatic, leads to significant model risks, especially concerning the accurate assessment of systematic risk components, as evidenced in the variance observed in credit loss distributions' tails. The setting of ad hoc parameters raises substantial concerns about the robustness and reliability of these models, underscoring an urgent need for integrating more formal statistical techniques to enhance precision and reduce potential inaccuracies in credit risk evaluation. Hence, the purpose is to discuss the estimation of default probabilities and default correlations for homogeneous groups, e.g. groups with the same credit rating. This type of data, consisting of defaulting observations on insolvent and non-default companies over several time horizons, is easily accessible to rating agencies.

### 4.3.1 Moment-style estimator methods

Suppose we have n years of data on historical default numbers for a homogeneous group; for j = 1, . . ., n let $m_j$ denote the number of obligors observed in year j and let $M_j$ denote the number that defaulted. Further suppose that these defaults are generated by an exchangeable Bernoulli mixture model so that there exist identically distributed mixing variables $Q_1$, . . ., $Q_n$ and defaults in year j are conditionally independent given $Q_j$. We consider two simple methods for estimating the fundamental parameters π = $\pi_1, \pi_2$ , without taking into account the maximum likelihood method. The definitions are taken from McNeil et. Al 2005.

The first model used is the simple moment-style estimator. For $1 \leq t \leq n$ let $Y_{j,1}, \ldots, Y_{j,m_j}$ be default indicators for $m_j$ companies observed in year j we have:

$$\binom{M_j}{k} = \sum_{i_{1,\dots,i_k}:\{i_1,\dots,i_k\}\subset\{1,\dots,m_j\}} Y_{j,i_1}\dots Y_{j,i_k}$$

That represents the number of possible subgroups of k obligors among the defaulting period j.

By taking expectations we get:

$$E\left(\binom{M_t}{k}\right) = \binom{m_t}{k}\pi_k$$

And hence:

$$\pi_k = \frac{E\left(\binom{M_t}{k}\right)}{\binom{m_t}{k}}$$

It is then possible to find the moment $\pi_k$ by using the average on n years of data:

$$\hat{\pi}_k = \frac{1}{n}\sum_{j=1}^{n}\frac{\binom{M_j}{k}}{\binom{m_j}{k}} = \frac{1}{n}\sum_{j=1}^{n}\frac{M_j(M_j-1)\dots.(M_j-k+1)}{m_j(m_j-1)\dots.(m_j-k+1)}, 1 \le k \le min\{m_1,\dots,m_n\} \quad (4.2.1)$$

Note that for k=1 we get standard default probability estimator:

$$\hat{\pi} = \frac{1}{n}\sum_{t=1}^{n}\frac{M_t}{m_t}$$

The second moment-estimator is the Gordy (2000) moment-style estimator of $\pi_2$ , that takes the form of:

$$\tilde{\pi}_2 = \hat{\pi}^2 + \frac{\frac{1}{n}\sum_{j=1}^{n}\left(\frac{M_j}{m_j}-\hat{\pi}\right)^2 - \frac{\hat{\pi}(1-\hat{\pi})}{n}\sum_{j=1}^{n}\frac{1}{m_j}}{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{m_j}} \qquad (4.2.2)$$

### 4.4 Estimating the parameters of the Beta Mixture Model for our portfolios

In order to find the Beta parameters for our four portfolios, it is collected the Global corporate annual default rate by rating category (%), the Global corporate default rate and the total number of issuers from 1981 to 2023. The data were collected from S&P Global Ratings Credit Research & Insights and S&P Global Market Intelligence's CreditPro and are summarized in Table 8 in Appendix 7.1.

Then, in order to estimate π and $\pi_2$, the total number of defaults $M_j$ and the total issuers $m_j$ for each rating category and for each year are needed. These data are taken from Fitch Ratings Corporate Finance, S&P Global Ratings Credit Research & Insights and S&P Global Market Intelligence's CreditPro. Table 8 in Appendix 7.2 shows the percentage of the defaults by each Rating category from 1981 to 2023. Table 9 in Appendix 7.3 shows the number of issuers for each Rating category.

Additionally, to compare the three portfolios, we need to estimate the parameters of the Beta distribution for each portfolio. Given that the model is a one-factor exchangeable model, we assume that the individual default probabilities of the obligors are identical. The value of $\pi_2$ is determined using the Gordy formula. This approach leverages the statistical properties of the Beta distribution to effectively assess portfolio risk, which is critical for making comparisons across different portfolios. Moreover, the Beta distribution's flexibility allows for a more accurate reflection of the diverse risk characteristics inherent in each portfolio, further enhancing the precision of the VaR and ES estimates. Considering that our sample portfolio consists of m=1000 obligors with a constant loss rate l=60% and given that the Beta distribution is continuous and strictly increasing, we can utilize certain approximations for the Value at Risk (VaR) and Expected Shortfall (ES). Since m is sufficiently large, as indicated by equations 3.5.1 and 3.5.2, we can first recall these approximations to simplify the calculation of portfolio risk measures:

$$VAR_\alpha\left(L^{(m)}\right) \sim l \cdot m \cdot G^{-1}(\alpha) \tag{4.4.1}$$

$$ES_\alpha\left(L^{(m)}\right) \sim \frac{l \cdot m}{1-\alpha} \int_\alpha^1 G^{-1}(u)\, du \tag{4.4.2}$$

Where we can recall that the total loss of the portfolio is noted as $L^{(m)}$, and the distribution function of the Beta mixing variable is noted with G. Then $G^{-1}$, is simply the inverse of the Beta cumulative distribution function, G.

### 4.5 VaR and ES calculations for our portfolios

Then, using the approximations in Equations 4.3.1 and 4.3.2, the calculation of VAR and ES is straightforward. Table 9 presents a summary of the resulting parameter values.

| | S&P Rating | B Rating | CCC Rating |
|---|---|---|---|
| $\pi$ | 0,01313 | 0,016554 | 0,2559 |
| $\pi_2$ | 0,000216 | 0,00043 | 0,0721 |
| $\rho$ | 0,00335 | 0,0096 | 0,349 |
| $\alpha$ | 3,9 | 1,793 | 5,478 |
| $\beta$ | 292,96 | 33,054 | 15,91 |
| VaR | 15,3130 | 71,753 | 252,1396 |
| ES | 18,1703 | 89,5936 | 280,8101 |

*Table 10. Model parameters that approximately equal to a S&P Global corporate Rating, B Rating and CCC Rating*

Figure 3 shows our computation of VAR and ES for the three different portfolios as the confidence level $\alpha$ increases from 0,95 to 1.

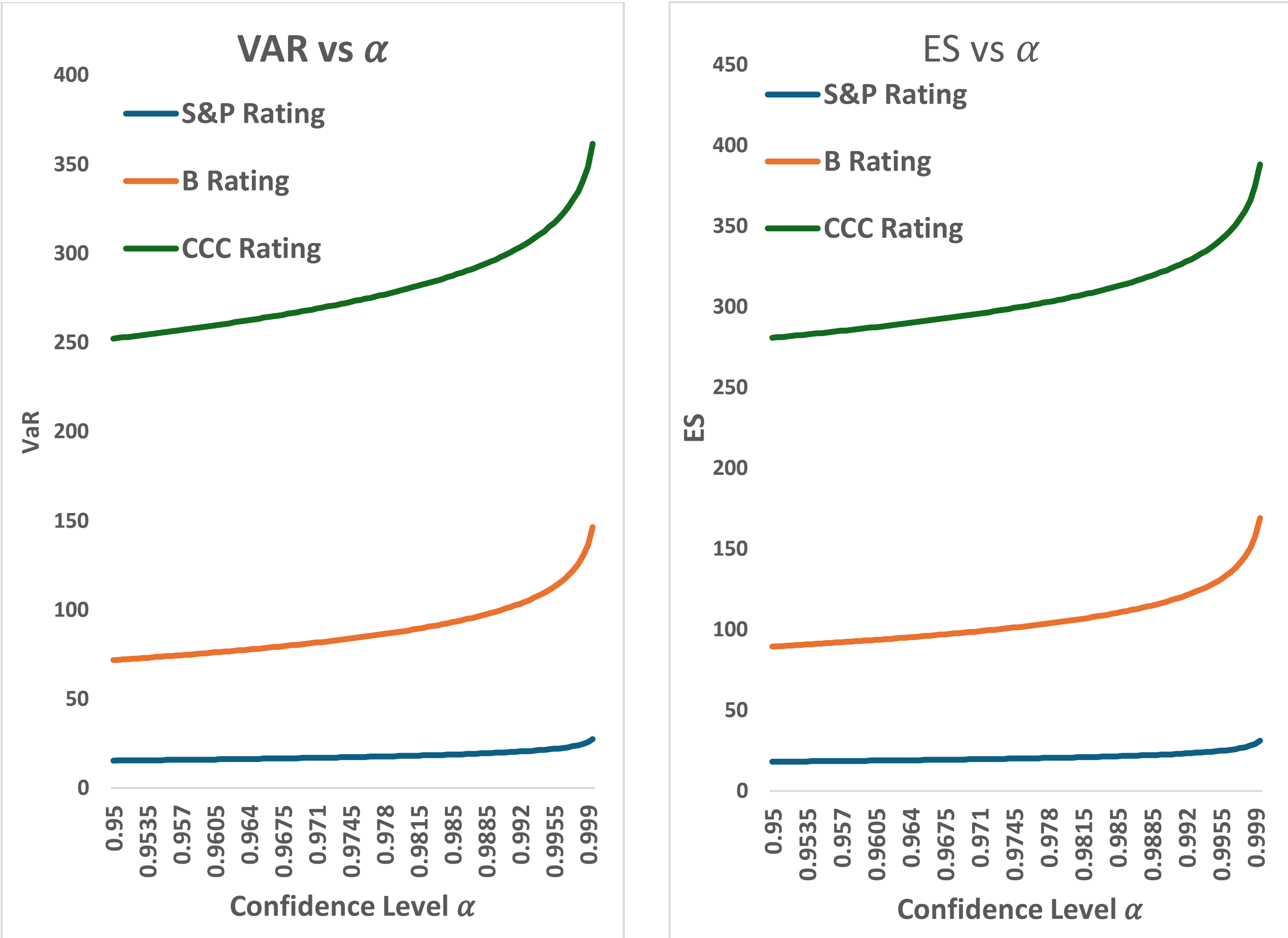


*Figure 3: VaR and ES approximated as functions of the confidence level $\alpha$ in with m=1000 l=60%.*

From Figure 3 can be seen that Value-at-Risk and Expected Shortfall increase exponentially as the confidence level $\alpha$ increases. As was easily predictable, by increasing the probability of default π in our portfolio i.e. using a portfolio more likely to default or with a lower Rating, the Value-at-Risk and Expected Shortfall increases.

Figure 4 compares the VAR of the three portfolios in 2011-2023 to the 1981-2005 values McNeil et al. (2005), as the confidence level $\alpha$ increases.

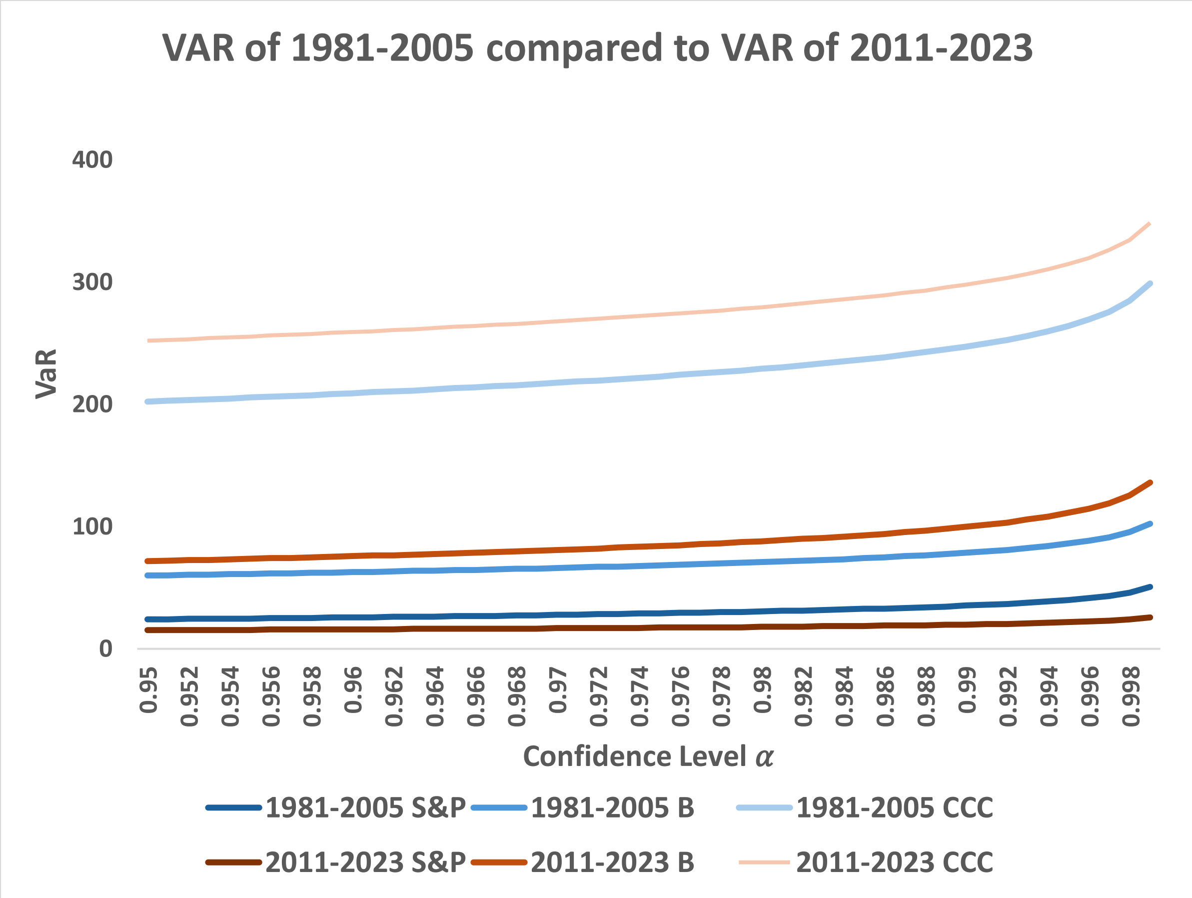


*Figure 4. VaR of 1981-2005 compared to VaR of 2011-2023 for the S&P, B and CCC Rating portfolio.*

There are few considerations that can be made. As expected, all VaR lines show an increasing trend as the confidence level $\alpha$ increases, since higher $\alpha$ correspond to more extreme potential losses

Across both periods, the CCC portfolio exhibits the highest VAR values, followed by the B portfolio, and then the S&P portfolio. This is consistent with the typical risk profiles of these asset classes, where CCC-rated securities are generally the riskiest, followed by B-rated, and then S&P-rated securities. Moreover, this higher value suggests a higher degree of volatility and potential for extreme losses in more recent year

These considerations are crucial for what we discussed previously about risk management, new implementations and Basel III, highlighting the need of more robust risk mitigation options and regulation.

Given the equations recalled in paragraph 4.4 it the aim is now compute the VaR and the ES for different correlation values and look at how VaR and ES are robust in term of loss. We will examine Value at Risk (VaR) and Expected Shortfall (ES) using an indicator, ES-VaR, which is the difference between the ES value and the VaR value. By doing so, we can better understand the tail risk of the portfolio, as this indicator highlights the gap between the two risk measures. Appendix 7.4, 7.5 and 7.6 shows the value of the Value at Risk and Expected Shortfall calculated for the three different portfolios.

Figures 4,5,6 illustrate the ES-VaR graph, when varying the correlation between obligors ρ between 0 and 1, in relation to different values of the probability of default.

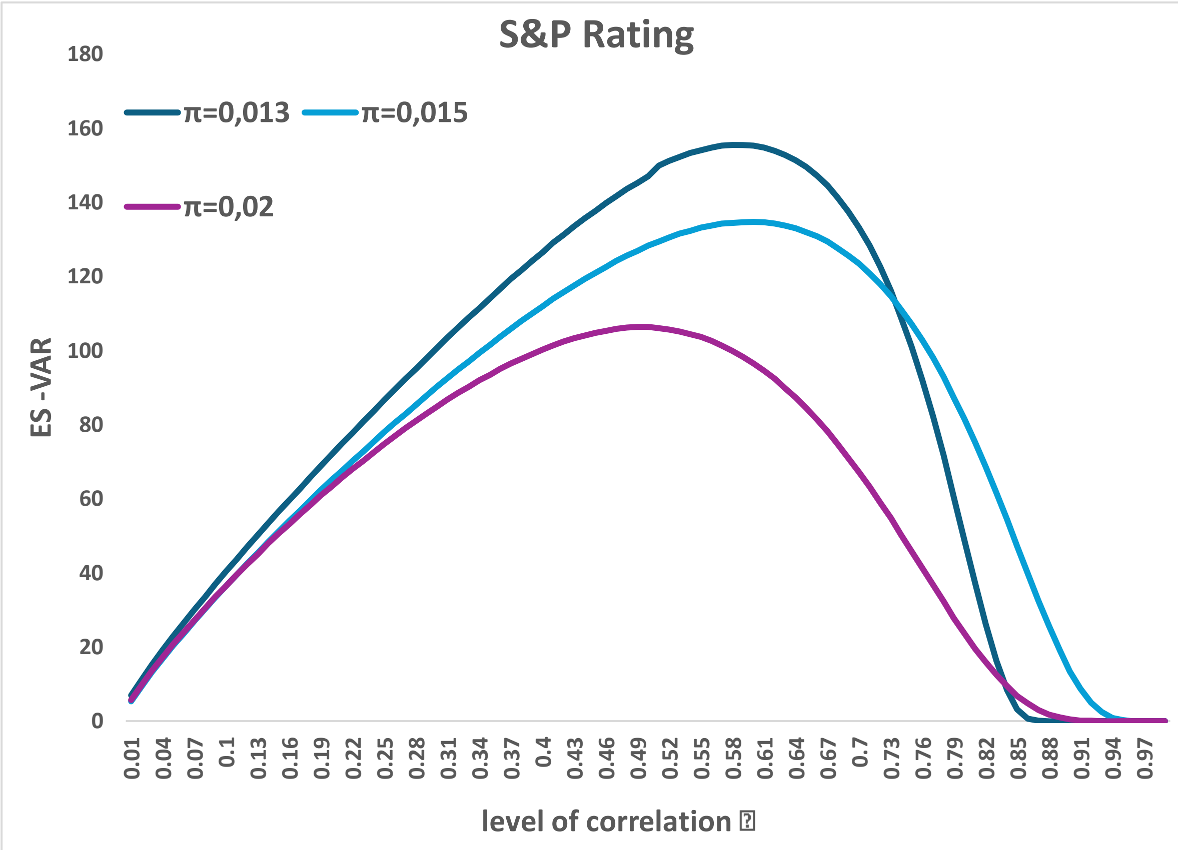


*Figure 5. ES-VaR as a function of ρ in portfolio with m=1000 and l=60%. Each portfolio corresponds to different probability of default π of obligors from the S&P Rating and are driven by a Beta mixing variable.*

As correlation increases, the difference between ES and VaR rises, peaking between correlations of 0.5 to 0.6, and then decline. The first consideration can be made regards the peak point, since it occurs at moderate correlation level. The medium correlation range (0.5 to 0.6) represents a critical zone where the risk concentration is most pronounced. In this range, the assets in the portfolio are sufficiently correlated to cause joint extreme losses but not so correlated that the losses become uniform and predictable. This leads to the maximum difference between ES and VaR, highlighting the most significant tail risk.

Moreover, at very low correlation values (close to 0), the assets in the portfolio are highly diversified. This diversification minimizes the risk of simultaneous large losses, leading to a relatively low difference between Expected Shortfall (ES) and Value-at-Risk (VaR). Both ES and VaR are close to each other because the likelihood of extreme joint losses is low.

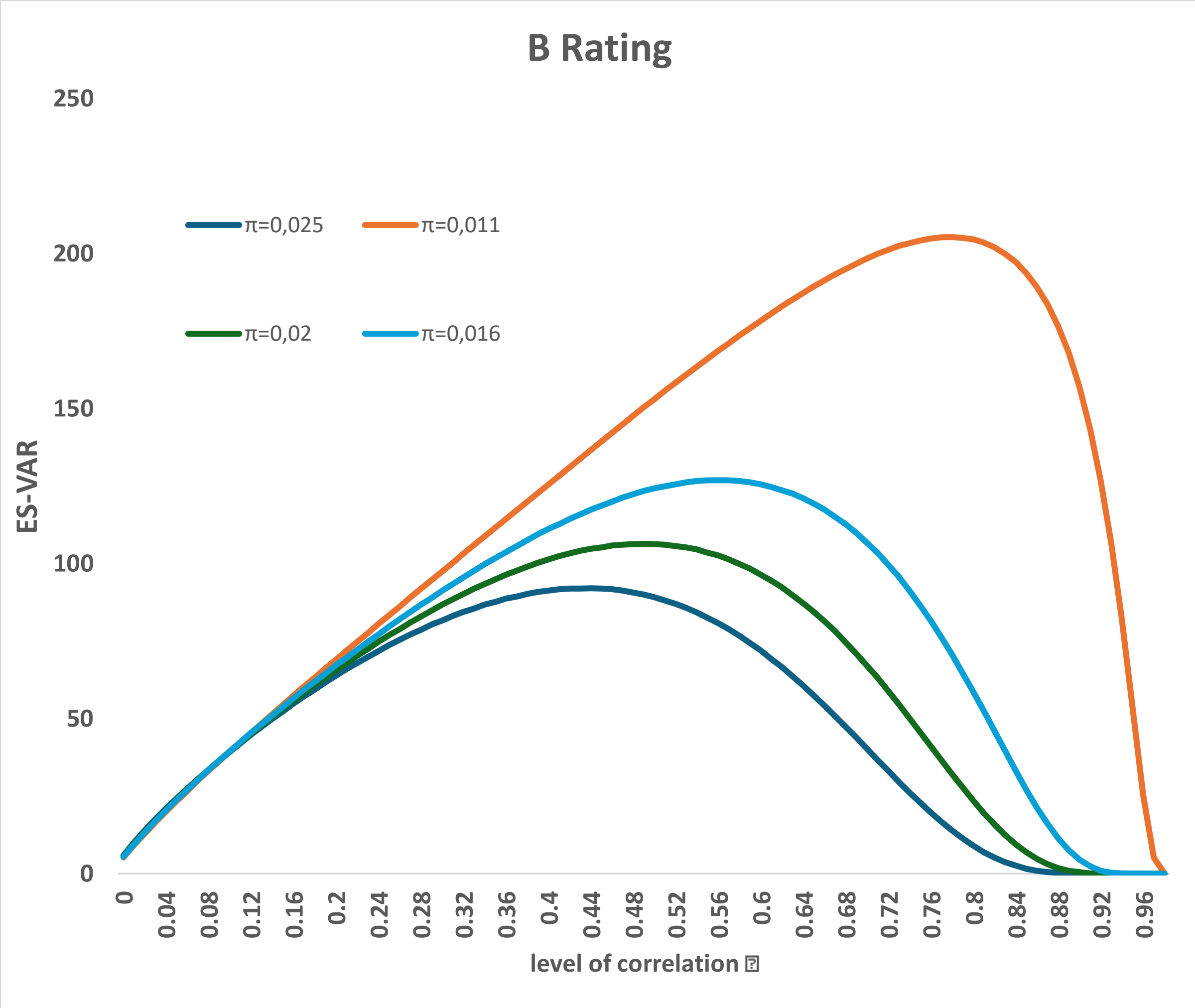


*Figure 6. ES-VaR as a function of $\rho$ in portfolio with m=1000 and l=60%. Each portfolio corresponds to different probability of default $\pi$ of B Rating obligors and are driven by a Beta mixing variable.*

When considering the B rating portfolio, the difference between ES and VaR behave differently: in fact, if we consider a relatively low probability of default, the difference between ES and VaR is high. However, increasing the probability of default, the value ES-VaR appears to be quite similar between the three different $\pi$ values of 0.016, 0.02 and 0.025.

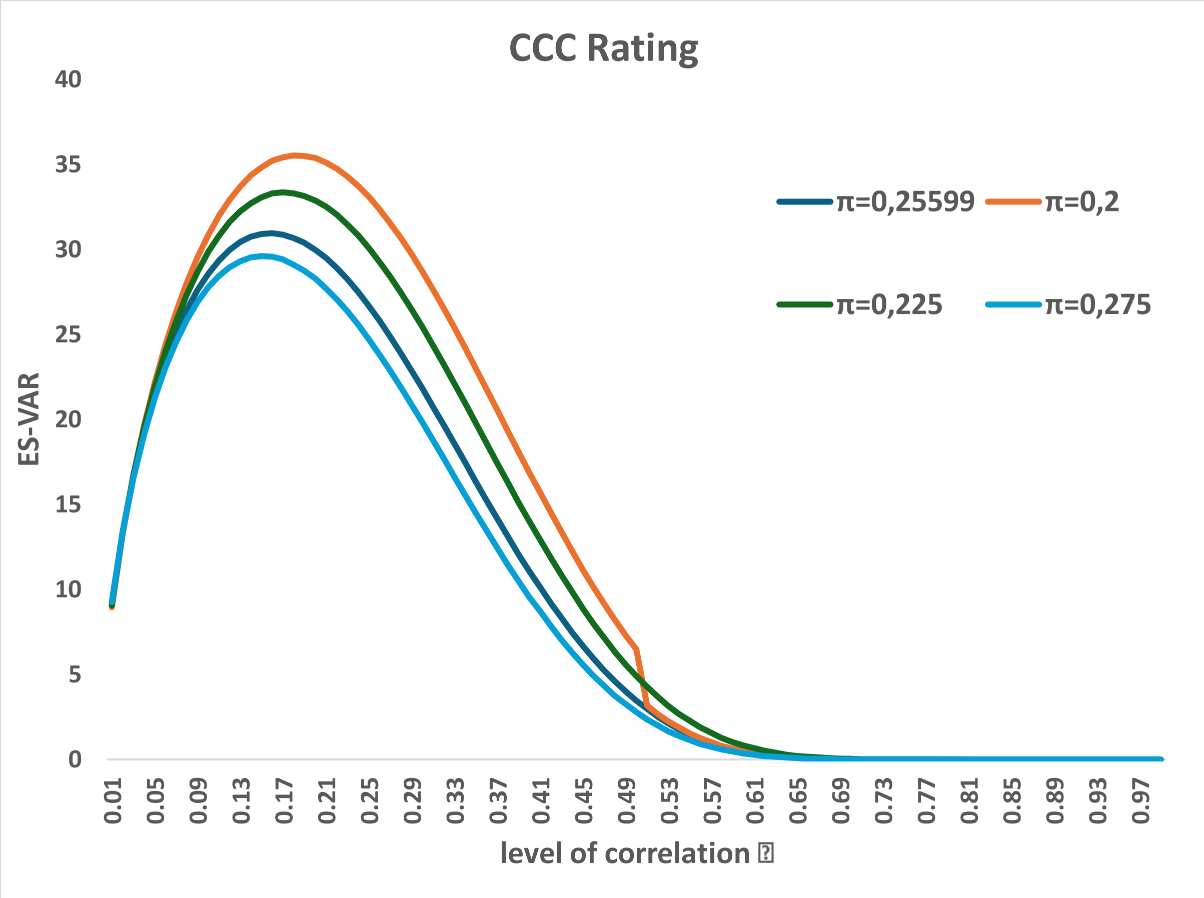


*Figure 7. ES-VaR as a function of $\rho$ in portfolio with m=1000 and l=60%. Each portfolio corresponds to different probability of default π of CCC Rating obligors and are driven by a Beta mixing variable.*

Lastly, CCC rating graph exabits similar results for high probability of default, $\pi$, values. Despite their higher probability values, the difference between ES and VaR remains quite low, peaking around 35. This express that comparable when dealing with higher probability values the VaR and ES are similar in term of expected loss.

Moreover, the 4 curves are almost similar, suggesting that the impact of correlation is almost the same for high level of probability of default.

## 5. Conclusion

We have seen that when talking about credit risk management in static portfolios there are several challenges that we have to face. First, to model the credit risk of the static portfolio, it's important modelling the default dependency, and here lies the main challenge. We have seen that model dependency in credit risk portfolios is the most complex challenge, and it's critical choosing the right mixing variables, as it directly influences the loss distribution and default correlation. We have also seen that the Bernoulli mixture model, which uses a factor vector of mixing variables to model dependence, is the most general model and also the one that perfectly capture credit risk in Static Model. Additionally, we examined portfolio behaviour when the number of obligors is large, in our example m=1000 obligors, so that the portfolio distribution converges to the distribution of the mixing variable. From this asymptotic behaviour, we derived large portfolio approximations for various models. These approximations are highly valuable, as in many models, the portfolio loss distribution cannot be represented in closed form.

However, the choice of the risk measure between Value at Risk and Expected Shortfall has significant implications for risk management practices. The analysis presented in Figures 4, 5, and 6 highlights key insights into the behaviour of ES and VaR under varying conditions. At very low correlation values (close to 0), the assets in the portfolio are highly diversified, thus minimizing the risk of simultaneous large losses and leading to a relatively low difference between ES and VaR. Both ES and VaR are close to each other because the likelihood of extreme joint losses is low, thus demonstrating the importance of diversification in risk management. For all three portfolio ratings, the difference between ES and VaR peaks at mid-range correlation levels, specifically between 0.5 and 0.6, that is a critical zone. In fact, in this range the assets in the portfolio are sufficiently correlated to cause joint extreme losses but not so correlated that the losses become uniform and predictable: this leads to the maximum difference between ES and VaR.

Another significant challenge is obtaining reliable estimates for the parameters $\pi, \pi_2$ and $\rho$, as these parameters heavily influence the calculation of VaR and ES, leading to considerable variability. Additionally, industry models often differ in their approaches to estimating these parameters.

This thesis explores static credit risk modeling, focusing on exchangeable and homogeneous portfolios. For homogeneous portfolios, analyzing default distribution instead of loss distribution simplifies calculations and modeling. Future research could address heterogeneous portfolios by grouping obligors into homogeneous sub-portfolios, and model loss rates as stochastic variables for a more dynamic risk assessment. Introducing dependency between loss rates and defaults would increase model sensitivity to economic conditions, acknowledging that loss rates tend to rise during downturns. Additionally, advanced techniques like copula models and machine learning could enhance the accuracy of risk estimates, capturing complex dependencies and non-linearities in financial data.

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# 7. Appendix

7.1 Appendix 1

| | **S&P Global Corporate** | | | **B** | **CCC** |
|---|---|---|---|---|---|
| | **Default rate** | **Tot. default** | **# issuers** | **Default rate** | **Default rate** |
| **1981** | *0,15%* | *2* | *1349* | *2,33%* | *0%* |
| **1982** | *1,22%* | *18* | *1398* | *3,18%* | *21%* |

| | | | | | |
|---|---|---|---|---|---|
| **1983** | *0,77%* | *12* | *1421* | *4,70%* | *7%* |
| **1984** | *0,93%* | *14* | *1512* | *3,49%* | *25%* |
| **1985** | *1,12%* | *19* | *1603* | *6,53%* | *15%* |
| **1986** | *1,73%* | *34* | *1851* | *8,77%* | *23%* |
| **1987** | *0,94%* | *19* | *2015* | *3,12%* | *12%* |
| **1988** | *1,38%* | *32* | *2103* | *3,68%* | *20%* |
| **1989** | *1,77%* | *44* | *2142* | *3,41%* | *33%* |
| **1990** | *2,71%* | *70* | *2141* | *8,56%* | *31%* |
| **1991** | *3,22%* | *93* | *2078* | *13,84%* | *34%* |
| **1992** | *1,49%* | *39* | *2153* | *6,99%* | *30%* |
| **1993** | *0,60%* | *26* | *2336* | *2,62%* | *13%* |
| **1994** | *0,62%* | *21* | *2562* | *3,07%* | *17%* |
| **1995** | *1,05%* | *35* | *2865* | *4,57%* | *28%* |
| **1996** | *0,51%* | *20* | *3126* | *2,90%* | *8%* |
| **1997** | *0,63%* | *23* | *3486* | *3,50%* | *12%* |
| **1998** | *1,30%* | *57* | *4085* | *4,65%* | *43%* |
| **1999** | *2,14%* | *109* | *4543* | *7,33%* | *34%* |
| **2000** | *2,46%* | *136* | *4713* | *7,68%* | *36%* |
| **2001** | *3,70%* | *229* | *4837* | *11,34%* | *45%* |
| **2002** | *3,53%* | *226* | *4878* | *8,11%* | *44%* |
| **2003** | *1,88%* | *120* | *4885* | *4,03%* | *33%* |
| **2004** | *0,77%* | *56* | *5043* | *1,45%* | *16%* |
| **2005** | *0,60%* | *40* | *5332* | *1,74%* | *9%* |
| **2006** | *0,47%* | *30* | *5494* | *0,81%* | *13%* |
| **2007** | *0,37%* | *24* | *5677* | *0,25%* | *15%* |
| **2008** | *1,79%* | *127* | *5751* | *4,09%* | *27%* |
| **2009** | *4,15%* | *268* | *5637* | *10,87%* | *49%* |
| **2010** | *1,20%* | *83* | *5337* | *0,86%* | *23%* |
| **2011** | *0,80%* | *53* | *5652* | *1,68%* | *17%* |
| **2012** | *1,13%* | *83* | *5835* | *1,57%* | *28%* |
| **2013** | *1,02%* | *81* | *6067* | *1,52%* | *25%* |
| **2014** | *0,69%* | *60* | *6510* | *0,78%* | *17%* |
| **2015** | *1,36%* | *113* | *6911* | *2,40%* | *27%* |
| **2016** | *2,08%* | *163* | *6908* | *3,74%* | *33%* |
| **2017** | *1,21%* | *95* | *6878* | *1,00%* | *27%* |
| **2018** | *1,02%* | *82* | *6948* | *0,94%* | *27%* |
| **2019** | *1,31%* | *118* | *7193* | *1,49%* | *30%* |

| | | | | | |
|---|---|---|---|---|---|
| **2020** | *2,77%* | *226* | *7157* | *3,54%* | *48%* |
| **2021** | *0,85%* | *72* | *7061* | *0,52%* | *11%* |
| **2022** | *0,99%* | *85* | *7202* | *1,10%* | *14%* |
| **2023** | *1,85%* | *153* | *6977* | *1,24%* | *31%* |

*Table 6: Global and by rating corporate default and total number of issuers from 1981 to 2023*

## 7.2 Appendix 2

| | **AAA** | **AA** | **A** | **BBB** | **BB** | **B** | **CCC/C** | **NR** |
|---|---|---|---|---|---|---|---|---|
| **1981** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *100,00%* | *0,00%* | |
| **1982** | *0,00%* | *0,00%* | *0,71%* | *1,19%* | *14,42%* | *10,81%* | *72,87%* | |
| **1983** | *0,00%* | *0,00%* | *0,00%* | *2,64%* | *8,94%* | *36,55%* | *51,87%* | |
| **1984** | *0,00%* | *0,00%* | *0,00%* | *2,24%* | *3,73%* | *11,52%* | *82,51%* | |
| **1985** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *6,33%* | *27,92%* | *65,75%* | |
| **1986** | *0,00%* | *0,00%* | *0,54%* | *1,02%* | *2,65%* | *26,38%* | *69,41%* | |
| **1987** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *2,41%* | *19,77%* | *77,82%* | |
| **1988** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *4,18%* | *14,66%* | *81,16%* | |
| **1989** | *0,00%* | *0,00%* | *0,47%* | *1,59%* | *1,88%* | *8,92%* | *87,14%* | |
| **1990** | *0,00%* | *0,00%* | *0,00%* | *1,32%* | *8,10%* | *19,48%* | *71,10%* | |
| **1991** | *0,00%* | *0,00%* | *0,00%* | *1,10%* | *3,34%* | *27,72%* | *67,83%* | |
| **1992** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *18,80%* | *81,20%* | |
| **1993** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *4,20%* | *15,74%* | *80,06%* | |
| **1994** | *0,00%* | *0,00%* | *0,69%* | *0,00%* | *1,39%* | *15,23%* | *82,69%* | |
| **1995** | *0,00%* | *0,00%* | *0,00%* | *0,50%* | *2,96%* | *13,54%* | *82,99%* | |
| **1996** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *3,96%* | *25,55%* | *70,48%* | |
| **1997** | *0,00%* | *0,00%* | *0,00%* | *1,57%* | *1,19%* | *21,96%* | *75,28%* | |
| **1998** | *0,00%* | *0,00%* | *0,00%* | *0,84%* | *2,00%* | *9,51%* | *87,65%* | |
| **1999** | *0,00%* | *0,40%* | *0,42%* | *0,45%* | *2,23%* | *17,19%* | *79,32%* | |
| **2000** | *0,00%* | *0,00%* | *0,57%* | *0,79%* | *2,53%* | *16,91%* | *79,19%* | |
| **2001** | *0,00%* | *0,00%* | *0,43%* | *0,55%* | *4,83%* | *18,81%* | *75,39%* | |
| **2002** | *0,00%* | *0,00%* | *0,00%* | *1,78%* | *5,04%* | *14,45%* | *78,73%* | |
| **2003** | *0,00%* | *0,00%* | *0,00%* | *0,59%* | *1,53%* | *10,79%* | *87,10%* | |
| **2004** | *0,00%* | *0,00%* | *0,45%* | *0,00%* | *2,47%* | *8,15%* | *88,93%* | |
| **2005** | *0,00%* | *0,00%* | *0,00%* | *0,63%* | *2,78%* | *15,62%* | *80,97%* | |
| **2006** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *2,08%* | *5,61%* | *92,31%* | |
| **2007** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *1,27%* | *1,59%* | *97,13%* | |
| **2008** | *0,00%* | *1,14%* | *1,14%* | *1,47%* | *2,42%* | *12,24%* | *81,60%* | |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **2009** | *0,00%* | *0,00%* | *0,36%* | *0,89%* | *1,21%* | *17,57%* | *79,97%* | |
| **2010** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *2,39%* | *3,54%* | *94,07%* | |
| **2011** | *0,00%* | *0,00%* | *0,00%* | *0,68%* | *0,00%* | *5,32%* | *75,00%* | *19,00%* |
| **2012** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *1,01%* | *4,76%* | *78,57%* | *15,66%* |
| **2013** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *0,38%* | *8,64%* | *75,31%* | *15,67%* |
| **2014** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *10,00%* | *73,33%* | *16,67%* |
| **2015** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *0,55%* | *8,26%* | *80,00%* | *11,19%* |
| **2016** | *0,00%* | *0,00%* | *0,00%* | *0,50%* | *0,00%* | *10,03%* | *84,00%* | *5,47%* |
| **2017** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *0,29%* | *2,00%* | *83,00%* | *14,71%* |
| **2018** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *9,00%* | *83,00%* | *8,00%* |
| **2019** | *0,00%* | *0,00%* | *0,00%* | *0,60%* | *0,00%* | *4,24%* | *80,51%* | *14,65%* |
| **2020** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *4,90%* | *87,60%* | *7,50%* |
| **2021** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *2,80%* | *91,70%* | *5,50%* |
| **2022** | *0,00%* | *0,00%* | *0,00%* | *0,00%* | *2,10%* | *7,21%* | *90,69%* | |
| **2023** | *0,00%* | *0,00%* | *0,00%* | *0,60%* | *0,00%* | *4,60%* | *88,90%* | *5,90%* |

*Table 7: % of default by rating category from 1981 to 2023*

## 7.3 Appendix 3

| | **AAA** | **AA** | **A** | **BBB** | **BB** | **B** | **CCC/C** |
|---|---|---|---|---|---|---|---|
| **1981** | *6,606845* | *79,53196* | *335,5418* | *449,2795* | *222,27* | *217,4829* | *38,18613* |
| **1982** | *6,846826* | *82,42082* | *347,7298* | *465,5988* | *230,3435* | *225,3826* | *39,57317* |
| **1983** | *6,959471* | *83,77681* | *353,4507* | *473,2589* | *234,1332* | *229,0906* | *40,22424* |
| **1984** | *7,405151* | *89,14183* | *376,0854* | *503,5661* | *249,1269* | *243,7614* | *42,80017* |
| **1985** | *7,850832* | *94,50685* | *398,7202* | *533,8733* | *264,1207* | *258,4323* | *45,37611* |
| **1986** | *9,065433* | *109,128* | *460,4062* | *616,4688* | *304,9827* | *298,4143* | *52,39624* |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **1987** | *9,868637* | *118,7968* | *501,1985* | *671,0884* | *332,0044* | *324,854* | *57,03859* |
| **1988** | *10,29962* | *123,985* | *523,0871* | *700,3965* | *346,5039* | *339,0412* | *59,5296* |
| **1989** | *10,49063* | *126,2843* | *532,7877* | *713,3853* | *352,9298* | *345,3287* | *60,63358* |
| **1990** | *10,48573* | *126,2253* | *532,539* | *713,0522* | *352,765* | *345,1675* | *60,60527* |
| **1991** | *10,17719* | *122,5111* | *516,8687* | *692,0703* | *342,3847* | *335,0108* | *58,82193* |
| **1992** | *10,5445* | *126,9328* | *535,5238* | *717,0488* | *354,7422* | *347,1021* | *60,94495* |
| **1993** | *11,44076* | *137,7218* | *581,042* | *777,9963* | *384,8945* | *376,605* | *66,12513* |
| **1994** | *12,54762* | *151,0459* | *637,2559* | *853,2647* | *422,1317* | *413,0402* | *72,52251* |
| **1995** | *14,03159* | *168,9096* | *712,6222* | *954,1778* | *472,056* | *461,8892* | *81,09953* |
| **1996** | *15,30986* | *184,2972* | *777,5417* | *1041,103* | *515,06* | *503,9671* | *88,48766* |
| **1997** | *17,07299* | *205,5214* | *867,0858* | *1161* | *574,3759* | *562,0056* | *98,67817* |
| **1998** | *20,00664* | *240,8362* | *1016,077* | *1360,494* | *673,0711* | *658,5751* | *115,6341* |
| **1999** | *22,24974* | *267,8382* | *1129,997* | *1513,03* | *748,5341* | *732,4129* | *128,5987* |
| **2000** | *23,08233* | *277,8607* | *1172,282* | *1569,647* | *776,5444* | *759,8199* | *133,4109* |
| **2001** | *23,68963* | *285,1713* | *1203,125* | *1610,945* | *796,9754* | *779,8109* | *136,9209* |
| **2002** | *23,89043* | *287,5885* | *1213,323* | *1624,6* | *803,7309* | *786,4208* | *138,0815* |
| **2003** | *23,92471* | *288,0012* | *1215,064* | *1626,931* | *804,8842* | *787,5494* | *138,2797* |
| **2004** | *24,69853* | *297,3163* | *1254,364* | *1679,553* | *830,9173* | *813,0218* | *142,7522* |
| **2005** | *26,11393* | *314,3547* | *1326,248* | *1775,803* | *878,5348* | *859,6138* | *150,9329* |
| **2006** | *26,90734* | *323,9056* | *1366,543* | *1829,757* | *905,227* | *885,7311* | *155,5186* |
| **2007** | *27,8036* | *334,6946* | *1412,061* | *1890,704* | *935,3793* | *915,2339* | *160,6988* |
| **2008** | *28,16602* | *339,0573* | *1430,468* | *1915,35* | *947,572* | *927,1641* | *162,7935* |
| **2009** | *27,6077* | *332,3363* | *1402,112* | *1877,382* | *928,7886* | *908,7852* | *159,5665* |
| **2010** | *26,13842* | *314,6494* | *1327,492* | *1777,468* | *879,3587* | *860,4199* | *151,0744* |
| **2011** | *27,68116* | *333,2207* | *1405,843* | *1882,378* | *931,2601* | *911,2035* | *159,9911* |
| **2012** | *28,57742* | *344,0096* | *1451,361* | *1943,325* | *961,4124* | *940,7064* | *165,1713* |
| **2013** | *29,71366* | *357,6875* | *1509,068* | *2020,592* | *999,6382* | *978,1089* | *171,7385* |
| **2014** | *31,88329* | *383,8051* | *1619,257* | *2168,132* | *1072,63* | *1049,528* | *184,2785* |
| **2015** | *33,84722* | *407,4466* | *1718,999* | *2301,683* | *1138,701* | *1114,177* | *195,6296* |
| **2016** | *33,83253* | *407,2697* | *1718,253* | *2300,684* | *1138,207* | *1113,693* | *195,5447* |
| **2017** | *33,6856* | *405,501* | *1710,791* | *2290,693* | *1133,264* | *1108,857* | *194,6955* |
| **2018** | *34,02843* | *409,6279* | *1728,202* | *2314,006* | *1144,797* | *1120,142* | *196,677* |
| **2019** | *35,22834* | *424,0722* | *1789,142* | *2395,602* | *1185,165* | *1159,64* | *203,6122* |
| **2020** | *35,05203* | *421,9498* | *1780,187* | *2383,613* | *1179,234* | *1153,836* | *202,5931* |
| **2021** | *34,58186* | *416,29* | *1756,309* | *2351,64* | *1163,416* | *1138,359* | *199,8757* |
| **2022** | *35,27242* | *424,6028* | *1791,38* | *2398,6* | *1186,648* | *1161,091* | *203,867* |
| **2023** | *34,17046* | *411,3377* | *1735,415* | *2323,664* | *1149,576* | *1124,817* | *197,4979* |

*Table 8: number of issuers by rating category from 1981 to 2023*

## 7.4 Appendix 4

| | π=0,02 | | π=0,015 | | π=0,013 | |
|---|---|---|---|---|---|---|
| ρ | VaR | ES | VaR | ES | VaR | ES |
| 0,01 | 36,9169693 | 43,8398812 | 33,6498449 | 39,0713662 | 36,9169693 | 43,8398812 |
| 0,02 | 50,2562043 | 61,4867169 | 47,457603 | 56,9567184 | 50,2562043 | 61,4867169 |
| 0,03 | 61,601041 | 76,8846832 | 59,0273099 | 72,3192068 | 61,601041 | 76,8846832 |
| 0,04 | 71,6961025 | 90,8625961 | 69,2896122 | 86,1962731 | 71,6961025 | 90,8625961 |
| 0,05 | 80,8949087 | 103,815121 | 78,6413572 | 99,0307721 | 80,8949087 | 103,815121 |
| 0,06 | 89,4046033 | 115,974499 | 87,3036524 | 111,070106 | 89,4046033 | 115,974499 |
| 0,07 | 97,3611161 | 127,493829 | 95,417499 | 122,472556 | 97,3611161 | 127,493829 |
| 0,08 | 104,860281 | 138,481536 | 103,080546 | 133,348097 | 104,860281 | 138,481536 |
| 0,09 | 111,973123 | 149,018304 | 110,364312 | 143,777518 | 111,973123 | 149,018304 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 0,1 | 118,754265 | 159,166409 | 117,323351 | 153,822601 | 118,754265 | 159,166409 |
| 0,11 | 125,24694 | 168,975284 | 124,000593 | 163,532054 | 125,24694 | 168,975284 |
| 0,12 | 131,486159 | 178,485048 | 130,430655 | 172,945205 | 131,486159 | 178,485048 |
| 0,13 | 137,50082 | 187,728855 | 136,642025 | 182,094421 | 137,50082 | 187,728855 |
| 0,14 | 143,31516 | 196,734516 | 142,658536 | 191,006765 | 143,31516 | 196,734516 |
| 0,15 | 148,949789 | 205,525657 | 148,500418 | 199,705166 | 148,949789 | 205,525657 |
| 0,16 | 154,422444 | 214,122559 | 154,185059 | 208,209264 | 154,422444 | 214,122559 |
| 0,17 | 159,748554 | 222,542796 | 159,727563 | 216,536046 | 159,748554 | 222,542796 |
| 0,18 | 164,941673 | 230,801712 | 165,141185 | 224,700321 | 164,941673 | 230,801712 |
| 0,19 | 170,01381 | 238,912798 | 170,437657 | 232,715092 | 170,01381 | 238,912798 |
| 0,2 | 174,975697 | 246,887981 | 175,627451 | 240,591845 | 174,975697 | 246,887981 |
| 0,21 | 179,836994 | 254,737861 | 180,719981 | 248,340778 | 179,836994 | 254,737861 |
| 0,22 | 184,606461 | 262,471896 | 185,723774 | 255,970985 | 184,606461 | 262,471896 |
| 0,23 | 189,292099 | 270,098556 | 190,646603 | 263,490609 | 189,292099 | 270,098556 |
| 0,24 | 193,901263 | 277,625446 | 195,495597 | 270,906962 | 193,901263 | 277,625446 |
| 0,25 | 198,440757 | 285,05941 | 200,27734 | 278,226627 | 198,440757 | 285,05941 |
| 0,26 | 202,916919 | 292,406617 | 204,997942 | 285,455542 | 202,916919 | 292,406617 |
| 0,27 | 207,335688 | 299,672633 | 209,66311 | 292,599069 | 207,335688 | 299,672633 |
| 0,28 | 211,702665 | 306,862478 | 214,278202 | 299,662057 | 211,702665 | 306,862478 |
| 0,29 | 216,023164 | 313,980681 | 218,848279 | 306,648889 | 216,023164 | 313,980681 |
| 0,3 | 220,302261 | 321,031323 | 223,378143 | 313,563528 | 220,302261 | 321,031323 |
| 0,31 | 224,544829 | 328,018072 | 227,87238 | 320,409548 | 224,544829 | 328,018072 |
| 0,32 | 228,755583 | 334,944214 | 232,335388 | 327,190171 | 228,755583 | 334,944214 |
| 0,33 | 232,939106 | 341,812686 | 236,771408 | 333,908292 | 232,939106 | 341,812686 |
| 0,34 | 237,099886 | 348,626095 | 241,184553 | 340,566501 | 237,099886 | 348,626095 |
| 0,35 | 241,242342 | 355,386739 | 245,578828 | 347,167105 | 241,242342 | 355,386739 |
| 0,36 | 245,37085 | 362,09663 | 249,958156 | 353,712145 | 245,37085 | 362,09663 |
| 0,37 | 249,489776 | 368,757505 | 254,326396 | 360,203413 | 249,489776 | 368,757505 |
| 0,38 | 253,603494 | 375,370845 | 258,687363 | 366,642463 | 253,603494 | 375,370845 |
| 0,39 | 257,716417 | 381,937885 | 263,044846 | 373,030624 | 257,716417 | 381,937885 |
| 0,4 | 261,833022 | 388,459628 | 267,402624 | 379,36901 | 261,833022 | 388,459628 |
| 0,41 | 265,957872 | 394,936856 | 271,764487 | 385,658533 | 265,957872 | 394,936856 |
| 0,42 | 270,095649 | 401,370139 | 276,134246 | 391,899907 | 270,095649 | 401,370139 |
| 0,43 | 274,251178 | 407,759845 | 280,515754 | 398,093655 | 274,251178 | 407,759845 |
| 0,44 | 278,429455 | 414,106147 | 284,912917 | 404,240121 | 278,429455 | 414,106147 |
| 0,45 | 282,635682 | 420,409034 | 289,329715 | 410,339471 | 282,635682 | 420,409034 |
| 0,46 | 286,875298 | 426,668311 | 293,770212 | 416,391698 | 286,875298 | 426,668311 |
| 0,47 | 291,154013 | 432,883613 | 298,238576 | 422,39663 | 291,154013 | 432,883613 |
| 0,48 | 295,477851 | 439,054402 | 302,739091 | 428,353931 | 295,477851 | 439,054402 |
| 0,49 | 299,853186 | 445,179973 | 307,276179 | 434,263105 | 299,853186 | 445,179973 |
| 0,5 | 304,286794 | 451,259455 | 311,854411 | 440,123497 | 304,286794 | 451,259455 |
| 0,51 | 313,358236 | 463,275831 | 316,478525 | 445,934296 | 313,358236 | 463,275831 |
| 0,52 | 318,012098 | 469,210132 | 321,153447 | 451,694536 | 318,012098 | 469,210132 |
| 0,53 | 322,75642 | 475,093143 | 325,884305 | 457,403091 | 322,75642 | 475,093143 |
| 0,54 | 327,600845 | 480,923093 | 330,676447 | 463,05868 | 327,600845 | 480,923093 |
| 0,55 | 332,555811 | 486,697997 | 335,535463 | 468,659859 | 332,555811 | 486,697997 |
| 0,56 | 337,632643 | 492,415629 | 340,467199 | 474,205016 | 337,632643 | 492,415629 |
| 0,57 | 342,843656 | 498,073494 | 345,477779 | 479,692371 | 342,843656 | 498,073494 |
| 0,58 | 348,202264 | 503,668796 | 350,57362 | 485,119959 | 348,202264 | 503,668796 |
| 0,59 | 353,723111 | 509,198393 | 355,761449 | 490,485628 | 353,723111 | 509,198393 |
| 0,6 | 359,422197 | 514,658746 | 361,048323 | 495,787023 | 359,422197 | 514,658746 |
| 0,61 | 365,317029 | 520,04586 | 366,441635 | 501,021569 | 365,317029 | 520,04586 |
| 0,62 | 371,426773 | 525,355214 | 371,949131 | 506,186461 | 371,426773 | 525,355214 |
| 0,63 | 377,772411 | 530,581676 | 377,57891 | 511,278636 | 377,772411 | 530,581676 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 0,64 | 384,376903 | 535,71941 | 383,339425 | 516,294754 | 384,376903 | 535,71941 |
| 0,65 | 391,26533 | 540,761765 | 389,239472 | 521,231172 | 391,26533 | 540,761765 |
| 0,66 | 398,465018 | 545,701149 | 395,288169 | 526,083916 | 398,465018 | 545,701149 |
| 0,67 | 406,005605 | 550,528888 | 401,494916 | 530,848643 | 406,005605 | 550,528888 |
| 0,68 | 413,919024 | 555,235064 | 407,869335 | 535,520614 | 413,919024 | 555,235064 |
| 0,69 | 422,239326 | 559,808337 | 414,421184 | 540,09465 | 422,239326 | 559,808337 |
| 0,7 | 431,002278 | 564,235754 | 421,160231 | 544,565096 | 431,002278 | 564,235754 |
| 0,71 | 440,244572 | 568,502545 | 428,096081 | 548,925778 | 440,244572 | 568,502545 |
| 0,72 | 450,002473 | 572,591917 | 435,237946 | 553,169959 | 450,002473 | 572,591917 |
| 0,73 | 460,309588 | 576,484865 | 442,594325 | 557,290306 | 460,309588 | 576,484865 |
| 0,74 | 471,193334 | 580,16004 | 450,172596 | 561,278849 | 471,193334 | 580,16004 |
| 0,75 | 482,669484 | 583,59372 | 457,978466 | 565,126954 | 482,669484 | 583,59372 |
| 0,76 | 494,733949 | 586,759971 | 466,015264 | 568,825306 | 494,733949 | 586,759971 |
| 0,77 | 507,350667 | 589,631157 | 474,283029 | 572,363915 | 507,350667 | 589,631157 |
| 0,78 | 520,43434 | 592,178996 | 482,777341 | 575,732141 | 520,43434 | 592,178996 |
| 0,79 | 533,826889 | 594,376455 | 491,487856 | 578,918767 | 533,826889 | 594,376455 |
| 0,8 | 547,26772 | 596,200889 | 500,396474 | 581,912113 | 547,26772 | 596,200889 |
| 0,81 | 560,361339 | 597,638768 | 509,475101 | 584,700238 | 560,361339 | 597,638768 |
| 0,82 | 572,554032 | 598,692174 | 518,682977 | 587,27122 | 572,554032 | 598,692174 |
| 0,83 | 583,146865 | 599,386417 | 527,963597 | 589,613572 | 583,146865 | 599,386417 |
| 0,84 | 591,395231 | 599,776378 | 537,241331 | 591,716808 | 591,395231 | 599,776378 |
| 0,85 | 596,758361 | 599,946143 | 546,418026 | 593,572196 | 596,758361 | 599,946143 |
| 0,86 | 599,299159 | 599,993918 | 555,370127 | 595,173724 | 599,299159 | 599,993918 |
| 0,87 | 599,954164 | 599,999867 | 563,947249 | 596,51928 | 599,954164 | 599,999867 |
| 0,88 | 599,999906 | 600 | 571,973684 | 597,61202 | 599,999906 | 600 |
| 0,89 | 600 | 600 | 579,254953 | 598,461806 | 600 | 600 |
| 0,9 | 600 | 600 | 585,592136 | 599,086499 | 600 | 600 |
| 0,91 | 600 | 600 | 590,806725 | 599,512714 | 600 | 600 |
| 0,92 | 600 | 600 | 594,777308 | 599,77545 | 600 | 600 |
| 0,93 | 600 | 600 | 597,484845 | 599,9159 | 600 | 600 |
| 0,94 | 600 | 600 | 599,053592 | 599,976873 | 600 | 600 |
| 0,95 | 600 | 600 | 599,759697 | 599,996111 | 600 | 600 |
| 0,96 | 600 | 600 | 599,969299 | 599,999723 | 600 | 600 |
| 0,97 | 600 | 600 | 599,999006 | 599,999996 | 600 | 600 |
| 0,98 | 600 | 600 | 599,999999 | 600 | 600 | 600 |
| 0,99 | 600 | 600 | 600 | 600 | 600 | 600 |

## 7.5 Appendix 5

| | π=0,025 | | π=0,011 | | π=0,012 | | π=0,016 | |
|---|---|---|---|---|---|---|---|---|
| ρ | VaR | ES | VaR | ES | VaR | ES | VaR | ES |
| 0,01 | 44,5250929 | 50,446063 | 28,7262725 | 33,8947546 | 39,2749785 | 44,9642142 | 34,8134837 | 40,2921645 |
| 0,02 | 60,6422426 | 70,748167 | 41,2214759 | 50,4032738 | 54,3569597 | 64,1838383 | 48,9018268 | 58,4715001 |
| 0,03 | 74,2228537 | 88,1576952 | 51,5988783 | 64,5405652 | 67,050463 | 80,6931469 | 60,7219689 | 74,0901924 |
| 0,04 | 86,3725155 | 103,915661 | 60,7172493 | 77,2595615 | 78,3753334 | 95,6336429 | 71,2225709 | 88,2067587 |
| 0,05 | 97,5445821 | 118,531183 | 68,9516755 | 88,9774114 | 88,7566012 | 109,481316 | 80,8060599 | 101,27063 |
| 0,06 | 107,984964 | 132,280689 | 76,513697 | 99,930399 | 98,427646 | 122,497929 | 89,6959677 | 113,532042 |
| 0,07 | 117,847733 | 145,337523 | 83,5390002 | 110,27086 | 107,535973 | 134,849214 | 98,0345959 | 125,150754 |
| 0,08 | 127,23839 | 157,820425 | 90,1217618 | 120,105096 | 116,1832 | 146,649518 | 105,920405 | 136,237621 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0,09 | 136,233664 | 169,81572 | 96,3309595 | 129,511388 | 124,443545 | 157,982431 | 113,425482 | 146,874009 |
| 0,1 | 144,89185 | 181,388901 | 102,219136 | 138,549683 | 132,373556 | 168,91165 | 120,604811 | 157,122102 |
| 0,11 | 153,258732 | 192,591265 | 107,827536 | 147,267279 | 140,017727 | 179,487253 | 127,501669 | 167,030906 |
| 0,12 | 161,371214 | 203,463996 | 113,189312 | 155,702384 | 147,411968 | 189,749574 | 134,15097 | 176,639971 |
| 0,13 | 169,259681 | 214,040809 | 118,331636 | 163,886458 | 154,585868 | 199,731733 | 140,581456 | 185,981839 |
| 0,14 | 176,949592 | 224,349738 | 123,27714 | 171,845819 | 161,564233 | 209,461359 | 146,817192 | 195,083708 |
| 0,15 | 184,462597 | 234,414397 | 128,044931 | 179,60278 | 168,368162 | 218,961799 | 152,878622 | 203,968612 |
| 0,16 | 191,817343 | 244,254883 | 132,651334 | 187,176472 | 175,015837 | 228,252994 | 158,783326 | 212,65627 |
| 0,17 | 199,030074 | 253,888449 | 137,110441 | 194,583461 | 181,523099 | 237,352133 | 164,546596 | 221,16373 |
| 0,18 | 206,115077 | 263,330011 | 141,434523 | 201,838217 | 187,903887 | 246,27414 | 170,181861 | 229,505843 |
| 0,19 | 213,085032 | 272,592537 | 145,634357 | 208,95347 | 194,170583 | 255,032061 | 175,701023 | 237,695636 |
| 0,2 | 219,951283 | 281,687352 | 149,719474 | 215,940496 | 200,334272 | 263,637355 | 181,114717 | 245,744608 |
| 0,21 | 226,72405 | 290,624379 | 153,69836 | 222,809342 | 206,404955 | 272,100131 | 186,432516 | 253,662953 |
| 0,22 | 233,412606 | 299,412333 | 157,578615 | 229,569003 | 212,391721 | 280,429339 | 191,663103 | 261,459753 |
| 0,23 | 240,025413 | 308,058876 | 161,367087 | 236,227573 | 218,302884 | 288,63292 | 196,814402 | 269,143124 |
| 0,24 | 246,570236 | 316,570745 | 165,069977 | 242,79236 | 224,146096 | 296,717934 | 201,893696 | 276,720342 |
| 0,25 | 253,054242 | 324,953862 | 168,692924 | 249,26999 | 229,928444 | 304,69066 | 206,907715 | 284,197939 |
| 0,26 | 259,484074 | 333,213412 | 172,241089 | 255,666488 | 235,656526 | 312,556685 | 211,86272 | 291,581797 |
| 0,27 | 265,865917 | 341,353928 | 175,719207 | 261,987344 | 241,336521 | 320,320974 | 216,764566 | 298,87721 |
| 0,28 | 272,205555 | 349,379343 | 179,131646 | 268,237576 | 246,974246 | 327,98793 | 221,618759 | 306,088947 |
| 0,29 | 278,508418 | 357,293048 | 182,482453 | 274,421779 | 252,575199 | 335,561446 | 226,430508 | 313,221304 |
| 0,3 | 284,779618 | 365,097935 | 185,77539 | 280,544162 | 258,144609 | 343,044947 | 231,204764 | 320,278142 |
| 0,31 | 291,023984 | 372,796434 | 189,013967 | 286,60859 | 263,687468 | 350,44143 | 235,946261 | 327,26293 |
| 0,32 | 297,24609 | 380,390548 | 192,201478 | 292,618615 | 269,208559 | 357,753491 | 240,659547 | 334,178771 |
| 0,33 | 303,450278 | 387,88188 | 195,34102 | 298,577498 | 274,712494 | 364,983356 | 245,349013 | 341,028431 |
| 0,34 | 309,640675 | 395,27166 | 198,43552 | 304,488235 | 280,203725 | 372,132902 | 250,018921 | 347,814364 |
| 0,35 | 315,821214 | 402,560763 | 201,487753 | 310,353581 | 285,686576 | 379,203681 | 254,673429 | 354,538728 |
| 0,36 | 321,99564 | 409,749731 | 204,500365 | 316,176061 | 291,165256 | 386,196933 | 259,316609 | 361,203407 |
| 0,37 | 328,167524 | 416,838791 | 207,475883 | 321,957992 | 296,643876 | 393,113604 | 263,952471 | 367,810024 |
| 0,38 | 334,340266 | 423,827867 | 210,416736 | 327,701493 | 302,126464 | 399,954362 | 268,58498 | 374,359955 |
| 0,39 | 340,517101 | 430,716597 | 213,325266 | 333,408498 | 307,616978 | 406,719603 | 273,218075 | 380,854341 |
| 0,4 | 346,701101 | 437,504344 | 216,203743 | 339,080771 | 313,119313 | 413,409468 | 277,855682 | 387,294098 |
| 0,41 | 352,89517 | 444,190207 | 219,054375 | 344,71991 | 318,637315 | 420,023846 | 282,501734 | 393,679926 |
| 0,42 | 359,10204 | 450,773032 | 221,879319 | 350,327363 | 324,174781 | 426,562386 | 287,160185 | 400,012317 |
| 0,43 | 365,324267 | 457,251423 | 224,680698 | 355,90443 | 329,735473 | 433,024503 | 291,835023 | 406,291565 |
| 0,44 | 371,564216 | 463,623751 | 227,460605 | 361,452279 | 335,323113 | 439,409386 | 296,530285 | 412,517767 |
| 0,45 | 377,824051 | 469,88816 | 230,221116 | 366,971945 | 340,941389 | 445,715999 | 301,250071 | 418,690832 |
| 0,46 | 384,105717 | 476,042579 | 232,964301 | 372,464344 | 346,593956 | 451,94309 | 305,998557 | 424,810486 |
| 0,47 | 390,410916 | 482,08473 | 235,692235 | 377,93028 | 352,284428 | 458,089193 | 310,78001 | 430,876272 |
| 0,48 | 396,741089 | 488,012134 | 238,407008 | 383,370446 | 358,016379 | 464,15263 | 315,598797 | 436,887559 |
| 0,49 | 403,09738 | 493,822121 | 241,110737 | 388,785439 | 363,793329 | 470,131516 | 320,459403 | 442,843538 |
| 0,5 | 409,480609 | 499,511837 | 243,805574 | 394,17576 | 369,61874 | 476,023761 | 325,366437 | 448,74323 |
| 0,51 | 415,891229 | 505,078256 | 246,493725 | 399,541826 | 375,495992 | 481,827068 | 330,324651 | 454,58548 |
| 0,52 | 422,329283 | 510,518184 | 249,177457 | 404,883972 | 381,428374 | 487,538936 | 335,338945 | 460,36896 |
| 0,53 | 428,794356 | 515,828273 | 251,859113 | 410,202461 | 387,419053 | 493,15666 | 340,41438 | 466,092169 |
| 0,54 | 435,285516 | 521,005032 | 254,541133 | 415,49749 | 393,471047 | 498,677329 | 345,556191 | 471,753426 |
| 0,55 | 441,801254 | 526,044834 | 257,22606 | 420,769194 | 399,587193 | 504,097827 | 350,76979 | 477,350869 |
| 0,56 | 448,339413 | 530,943938 | 259,91657 | 426,017655 | 405,770098 | 509,41483 | 356,060775 | 482,88245 |
| 0,57 | 454,897112 | 535,698498 | 262,615484 | 431,242904 | 412,022091 | 514,624805 | 361,434937 | 488,345924 |
| 0,58 | 461,470659 | 540,304582 | 265,325792 | 436,44493 | 418,345163 | 519,724009 | 366,898257 | 493,738843 |
| 0,59 | 468,055463 | 544,758197 | 268,050685 | 441,623681 | 424,740893 | 524,708485 | 372,456907 | 499,058545 |
| 0,6 | 474,645933 | 549,055309 | 270,793574 | 446,779069 | 431,210362 | 529,574067 | 378,117244 | 504,302143 |
| 0,61 | 481,235372 | 553,191877 | 273,558132 | 451,910973 | 437,754056 | 534,316374 | 383,885792 | 509,466506 |
| 0,62 | 487,815867 | 557,163882 | 276,348326 | 457,019239 | 444,371747 | 538,930815 | 389,769223 | 514,548245 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0,63 | 494,378176 | 560,967368 | 279,168461 | 462,103682 | 451,062362 | 543,412595 | 395,774329 | 519,543695 |
| 0,64 | 500,911609 | 564,598492 | 282,023233 | 467,164083 | 457,823823 | 547,75672 | 401,90797 | 524,44889 |
| 0,65 | 507,403917 | 568,053572 | 284,917783 | 472,200185 | 464,652872 | 551,958011 | 408,177018 | 529,259547 |
| 0,66 | 513,841187 | 571,329152 | 287,857765 | 477,21169 | 471,544867 | 556,01112 | 414,588275 | 533,971035 |
| 0,67 | 520,207741 | 574,422071 | 290,849427 | 482,198248 | 478,493552 | 559,910557 | 421,148363 | 538,578353 |
| 0,68 | 526,486069 | 577,329548 | 293,899701 | 487,159449 | 485,490801 | 563,65072 | 427,863586 | 543,076103 |
| 0,69 | 532,656779 | 580,049266 | 297,016315 | 492,094808 | 492,526333 | 567,225942 | 434,739748 | 547,458466 |
| 0,7 | 538,6986 | 582,579476 | 300,207918 | 497,003748 | 499,587402 | 570,630545 | 441,781926 | 551,719176 |
| 0,71 | 544,588431 | 584,919108 | 303,484242 | 501,885578 | 506,658464 | 573,858918 | 448,994174 | 555,851499 |
| 0,72 | 550,301465 | 587,06789 | 306,856281 | 506,739469 | 513,720835 | 576,905601 | 456,379161 | 559,848219 |
| 0,73 | 555,811409 | 589,026472 | 310,336523 | 511,564422 | 520,752338 | 579,7654 | 463,937709 | 563,70163 |
| 0,74 | 561,090808 | 590,796559 | 313,939213 | 516,359229 | 527,726967 | 582,433524 | 471,668225 | 567,403542 |
| 0,75 | 566,111503 | 592,381041 | 317,680696 | 521,122429 | 534,614601 | 584,905742 | 479,566006 | 570,945304 |
| 0,76 | 570,84524 | 593,784115 | 321,579818 | 525,852255 | 541,380776 | 587,178576 | 487,62238 | 574,317844 |
| 0,77 | 575,264455 | 595,0114 | 325,658436 | 530,546563 | 547,986601 | 589,249517 | 495,823688 | 577,511754 |
| 0,78 | 579,343224 | 596,070026 | 329,942044 | 535,202758 | 554,388839 | 591,11727 | 504,15006 | 580,517398 |
| 0,79 | 583,058409 | 596,968687 | 334,460568 | 539,817686 | 560,540264 | 592,782024 | 512,574005 | 583,325086 |
| 0,8 | 586,390949 | 597,717651 | 339,249357 | 544,387515 | 566,390361 | 594,245736 | 521,058805 | 585,925307 |
| 0,81 | 589,327284 | 598,328704 | 344,350456 | 548,907573 | 571,886482 | 595,51242 | 529,556765 | 588,309043 |
| 0,82 | 591,860819 | 598,815027 | 349,814232 | 553,372149 | 576,975561 | 596,588422 | 538,007416 | 590,46818 |
| 0,83 | 593,99333 | 599,190968 | 355,70148 | 557,774225 | 581,606486 | 597,482648 | 546,335841 | 592,396036 |
| 0,84 | 595,73615 | 599,471729 | 362,086155 | 562,10514 | 585,733176 | 598,206724 | 554,451431 | 594,087999 |
| 0,85 | 597,110935 | 599,672935 | 369,058957 | 566,354128 | 589,318356 | 598,77502 | 562,247511 | 595,542299 |
| 0,86 | 598,14977 | 599,810115 | 376,732042 | 570,507717 | 592,337889 | 599,204515 | 569,602522 | 596,760872 |
| 0,87 | 598,894342 | 599,898105 | 385,24522 | 574,548919 | 594,78533 | 599,514415 | 576,383632 | 597,750283 |
| 0,88 | 599,393971 | 599,950423 | 394,774082 | 578,456143 | 596,676094 | 599,725507 | 582,453872 | 598,522592 |
| 0,89 | 599,702368 | 599,978694 | 405,540454 | 582,201742 | 598,05034 | 599,859202 | 587,683911 | 599,096004 |
| 0,9 | 599,873253 | 599,992202 | 417,825285 | 585,750111 | 598,973283 | 599,936303 | 591,969183 | 599,495049 |
| 0,91 | 599,955364 | 599,997695 | 431,98287 | 589,055302 | 599,531555 | 599,975604 | 595,251919 | 599,749973 |
| 0,92 | 599,987893 | 599,999491 | 448,451612 | 592,058362 | 599,824462 | 599,99256 | 597,545124 | 599,894971 |
| 0,93 | 599,997739 | 599,999926 | 467,746035 | 594,685334 | 599,950332 | 599,998359 | 598,951295 | 599,965062 |
| 0,94 | 599,999759 | 599,999994 | 490,385825 | 596,848728 | 599,990777 | 599,999777 | 599,66322 | 599,991793 |
| 0,95 | 599,999989 | 600 | 516,642057 | 598,459402 | 599,999127 | 599,999986 | 599,931422 | 599,998893 |
| 0,96 | 600 | 600 | 545,808526 | 599,461882 | 599,999975 | 600 | 599,993703 | 599,999943 |
| 0,97 | 600 | 600 | 574,509295 | 599,903807 | 600 | 600 | 599,999882 | 600 |
| 0,98 | 600 | 600 | 594,710383 | 599,996785 | 600 | 600 | 600 | 600 |
| 0,99 | 600 | 600 | 599,956889 | 600 | 600 | 600 | 600 | 600 |

## 7.6 Appendix 6

| | π=0,2 | | π=0,015 | | π=0,013 | |
|---|---|---|---|---|---|---|
| ρ | VaR | ES | VaR | ES | VaR | ES |
| 0,01 | 180,749622 | 189,679376 | 197,758472 | 206,833202 | 206,833202 | 240,163986 |
| 0,02 | 208,495022 | 221,783388 | 226,070078 | 239,446113 | 239,446113 | 273,398552 |
| 0,03 | 230,669386 | 247,409882 | 248,569007 | 265,302003 | 265,302003 | 299,41434 |
| 0,04 | 249,903975 | 269,557835 | 268,000188 | 287,530816 | 287,530816 | 321,557237 |
| 0,05 | 267,222977 | 289,398831 | 285,431571 | 307,355092 | 307,355092 | 341,134023 |
| 0,06 | 283,153335 | 307,537774 | 301,412355 | 325,40597 | 325,40597 | 358,819401 |
| 0,07 | 298,008548 | 324,335848 | 316,269109 | 342,060295 | 342,060295 | 375,016373 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 0,08 | 311,992812 | 340,029226 | 330,214428 | 357,56451 | 357,56451 | 389,988681 |
| 0,09 | 325,247627 | 354,782294 | 343,395554 | 372,090206 | 372,090206 | 403,920178 |
| 0,1 | 337,87555 | 368,71482 | 355,919144 | 385,762469 | 385,762469 | 416,945128 |
| 0,11 | 349,953452 | 381,917155 | 367,865108 | 398,675748 | 398,675748 | 429,165159 |
| 0,12 | 361,540465 | 394,459362 | 379,29489 | 410,903378 | 410,903378 | 440,659442 |
| 0,13 | 372,683005 | 406,396999 | 390,256713 | 422,503619 | 422,503619 | 451,491142 |
| 0,14 | 383,418092 | 417,774953 | 400,789038 | 433,52365 | 433,52365 | 461,711689 |
| 0,15 | 393,775617 | 428,630075 | 410,922938 | 444,002323 | 444,002323 | 471,363716 |
| 0,16 | 403,779957 | 438,993046 | 420,683775 | 453,97211 | 453,97211 | 480,483138 |
| 0,17 | 413,451131 | 448,889739 | 430,09242 | 463,460522 | 463,460522 | 489,100669 |
| 0,18 | 422,805675 | 458,342233 | 439,166159 | 472,49117 | 472,49117 | 497,242946 |
| 0,19 | 431,857292 | 467,369586 | 447,919377 | 481,084566 | 481,084566 | 504,933386 |
| 0,2 | 440,617359 | 475,988433 | 456,364096 | 489,25875 | 489,25875 | 512,192848 |
| 0,21 | 449,095334 | 484,213463 | 464,510393 | 497,029781 | 497,029781 | 519,040151 |
| 0,22 | 457,299068 | 492,05779 | 472,366737 | 504,412129 | 504,412129 | 525,492484 |
| 0,23 | 465,23507 | 499,53327 | 479,940268 | 511,418993 | 511,418993 | 531,565734 |
| 0,24 | 472,908724 | 506,650744 | 487,237022 | 518,062561 | 518,062561 | 537,27476 |
| 0,25 | 480,324466 | 513,420256 | 494,262124 | 524,354223 | 524,354223 | 542,633602 |
| 0,26 | 487,485946 | 519,851223 | 501,019957 | 530,304754 | 530,304754 | 547,655662 |
| 0,27 | 494,396159 | 525,952587 | 507,514302 | 535,924455 | 535,924455 | 552,353845 |
| 0,28 | 501,057565 | 531,732938 | 513,748468 | 541,223285 | 541,223285 | 556,74068 |
| 0,29 | 507,4722 | 537,200623 | 519,725401 | 546,210961 | 546,210961 | 560,828413 |
| 0,3 | 513,641769 | 542,363837 | 525,447789 | 550,897047 | 550,897047 | 564,629078 |
| 0,31 | 519,567737 | 547,230697 | 530,918157 | 555,291025 | 555,291025 | 568,154563 |
| 0,32 | 525,251414 | 551,809312 | 536,138948 | 559,402352 | 559,402352 | 571,416645 |
| 0,33 | 530,694032 | 556,107833 | 541,112602 | 563,240511 | 563,240511 | 574,427027 |
| 0,34 | 535,896818 | 560,134497 | 545,841635 | 566,815042 | 566,815042 | 577,197357 |
| 0,35 | 540,861065 | 563,897668 | 550,328698 | 570,135573 | 570,135573 | 579,739234 |
| 0,36 | 545,588198 | 567,405855 | 554,576646 | 573,211833 | 573,211833 | 582,064211 |
| 0,37 | 550,079836 | 570,667738 | 558,588587 | 576,053666 | 576,053666 | 584,18379 |
| 0,38 | 554,33785 | 573,692175 | 562,36794 | 578,671029 | 578,671029 | 586,109406 |
| 0,39 | 558,364417 | 576,488209 | 565,918476 | 581,073987 | 581,073987 | 587,852406 |
| 0,4 | 562,162074 | 579,065064 | 569,244356 | 583,272701 | 583,272701 | 589,42403 |
| 0,41 | 565,733757 | 581,432136 | 572,350166 | 585,277413 | 585,277413 | 590,835375 |
| 0,42 | 569,082846 | 583,598979 | 575,240939 | 587,098418 | 587,098418 | 592,09737 |
| 0,43 | 572,213197 | 585,575285 | 577,922175 | 588,74604 | 588,74604 | 593,220738 |
| 0,44 | 575,129172 | 587,370858 | 580,39985 | 590,230597 | 590,230597 | 594,215964 |
| 0,45 | 577,835653 | 588,995584 | 582,680416 | 591,562368 | 591,562368 | 595,093254 |
| 0,46 | 580,338061 | 590,459396 | 584,770797 | 592,751552 | 592,751552 | 595,862506 |
| 0,47 | 582,642358 | 591,772239 | 586,678369 | 593,808227 | 593,808227 | 596,533268 |
| 0,48 | 584,755041 | 592,944025 | 588,41094 | 594,742313 | 594,742313 | 597,114712 |
| 0,49 | 586,683128 | 593,984592 | 589,976715 | 595,563524 | 595,563524 | 597,615597 |
| 0,5 | 588,434141 | 594,903661 | 591,384255 | 596,281326 | 596,281326 | 598,044246 |
| 0,51 | 594,826362 | 598,002662 | 592,642425 | 596,904899 | 596,904899 | 598,40852 |
| 0,52 | 595,695588 | 598,384488 | 593,760344 | 597,443095 | 597,443095 | 598,7158 |
| 0,53 | 596,450533 | 598,705764 | 594,747314 | 597,904399 | 597,904399 | 598,972967 |
| 0,54 | 597,100839 | 598,9737 | 595,612759 | 598,296899 | 598,296899 | 599,1864 |
| 0,55 | 597,656076 | 599,195031 | 596,366146 | 598,628248 | 598,628248 | 599,361961 |
| 0,56 | 598,125656 | 599,376004 | 597,016913 | 598,905647 | 598,905647 | 599,505004 |
| 0,57 | 598,51875 | 599,522359 | 597,574391 | 599,135816 | 599,135816 | 599,620371 |
| 0,58 | 598,844203 | 599,639323 | 598,047725 | 599,324982 | 599,324982 | 599,712412 |
| 0,59 | 599,110459 | 599,731611 | 598,4458 | 599,478867 | 599,478867 | 599,784992 |
| 0,6 | 599,325487 | 599,803428 | 598,777167 | 599,602684 | 599,602684 | 599,841513 |
| 0,61 | 599,496724 | 599,858481 | 599,049976 | 599,701138 | 599,701138 | 599,884941 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 0,62 | 599,631022 | 599,9 | 599,271919 | 599,778436 | 599,778436 | 599,917826 |
| 0,63 | 599,73461 | 599,930761 | 599,45017 | 599,838294 | 599,838294 | 599,942342 |
| 0,64 | 599,813067 | 599,953113 | 599,59135 | 599,883963 | 599,883963 | 599,960311 |
| 0,65 | 599,871315 | 599,969012 | 599,701488 | 599,91825 | 599,91825 | 599,973241 |
| 0,66 | 599,913619 | 599,980059 | 599,786007 | 599,943543 | 599,943543 | 599,982362 |
| 0,67 | 599,943606 | 599,987541 | 599,849711 | 599,961849 | 599,961849 | 599,988656 |
| 0,68 | 599,964299 | 599,992464 | 599,896792 | 599,974824 | 599,974824 | 599,992897 |
| 0,69 | 599,978159 | 599,995604 | 599,930846 | 599,983813 | 599,983813 | 599,995682 |
| 0,7 | 599,987138 | 599,997537 | 599,9549 | 599,989885 | 599,989885 | 599,997458 |
| 0,71 | 599,992743 | 599,99868 | 599,971452 | 599,993876 | 599,993876 | 599,998556 |
| 0,72 | 599,996097 | 599,999328 | 599,982516 | 599,996419 | 599,996419 | 599,999211 |
| 0,73 | 599,998012 | 599,999677 | 599,989678 | 599,997985 | 599,997985 | 599,999588 |
| 0,74 | 599,999048 | 599,999854 | 599,99415 | 599,998914 | 599,998914 | 599,999795 |
| 0,75 | 599,999575 | 599,999939 | 599,996833 | 599,999442 | 599,999442 | 599,999903 |
| 0,76 | 599,999825 | 599,999977 | 599,998372 | 599,999729 | 599,999729 | 599,999957 |
| 0,77 | 599,999934 | 599,999992 | 599,99921 | 599,999876 | 599,999876 | 599,999982 |
| 0,78 | 599,999978 | 599,999997 | 599,999642 | 599,999947 | 599,999947 | 599,999993 |
| 0,79 | 599,999994 | 599,999999 | 599,999849 | 599,999979 | 599,999979 | 599,999998 |
| 0,8 | 599,999998 | 600 | 599,999942 | 599,999992 | 599,999992 | 599,999999 |
| 0,81 | 600 | 600 | 599,99998 | 599,999998 | 599,999998 | 600 |
| 0,82 | 600 | 600 | 599,999994 | 599,999999 | 599,999999 | 600 |
| 0,83 | 600 | 600 | 599,999998 | 600 | 600 | 600 |
| 0,84 | 600 | 600 | 600 | 600 | 600 | 600 |
| 0,85 | 600 | 600 | 600 | 600 | 600 | 600 |
| 0,86 | 600 | 600 | 600 | 600 | 600 | 600 |
| 0,87 | 600 | 600 | 600 | 600 | 600 | 600 |
| 0,88 | 600 | 600 | 600 | 600 | 600 | 600 |
| 0,89 | 600 | 600 | 600 | 600 | 600 | 600 |
| 0,9 | 600 | 600 | 600 | 600 | 600 | 600 |
| 0,91 | 600 | 600 | 600 | 600 | 600 | 600 |
| 0,92 | 600 | 600 | 600 | 600 | 600 | 600 |
| 0,93 | 600 | 600 | 600 | 600 | 600 | 600 |
| 0,94 | 600 | 600 | 600 | 600 | 600 | 600 |
| 0,95 | 600 | 600 | 600 | 600 | 600 | 600 |
| 0,96 | 600 | 600 | 600 | 600 | 600 | 600 |
| 0,97 | 600 | 600 | 600 | 600 | 600 | 600 |
| 0,98 | 600 | 600 | 600 | 600 | 600 | 600 |
| 0,99 | 600 | 600 | 600 | 600 | 600 | 600 |