POLITECNICO DI TORINO

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Department of CONTROL AND COMPUTER ENGINEERING (DAUIN)

Master Degree in Mechatronic Engineering

Synchronization in Multi-Agent Systems with Dynamic Interconnections: An Observer-Based Approach

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Summary

In recent years, the study of networked systems has been developed in various fields, from control theory to robotics, computer networks, and biological systems. The ability of individual agents to achieve synchronization is a fundamental property that ensures consensus, i.e. coordinated behavior in systems composed of multiple entities. Such systems are prevalent in many real-world applications, including power grids, robotic swarms and sensor networks. Synchronization and consensus are crucial for the functionality of these networked systems, where agents must work together to achieve common objectives despite operating under different constraints and conditions. The challenge of synchronization becomes particularly complex in multi-agent systems with nonlinear dynamics, where partial information, limited communication, and decentralized control schemes are prevalent. These challenges are even more difficult when the agents are heterogeneous or when the network topology varies over time. In general, achieving synchronization requires robust control strategies that can still satisfy consensus. This thesis explores new methodologies for achieving synchronization in networks of nonlinear agents with dynamical interconnections, with a focus on observer-based control laws. The primary objective is to extend existing theoretical frameworks to develop generalized solutions for synchronization in networks with dynamic and potentially nonlinear interconnections. The core is about the use of high-gain observers to estimate states and design feedback control laws that ensure convergence to a common state.

The Synchronization Problem

In multi-agent systems, synchronization refers to the process of synchronizing the state or output of all agents. This is critical in many applications, such as coordinated control of drones and synchronization of oscillators in power grids. However, several challenges must be addressed:

- *Partial Information*: Agents often have access to limited information about the state of other agents due, for example, to privacy constraints or measurement noise. This necessitates the design of distributed control laws that can operate with partial information.
- *Decentralized Control*: In large-scale networks, centralized control is impractical due to scalability issues. Instead, agents must make decisions based on local information and limited communication with neighboring agents, leading to decentralized and distributed control schemes.
- *Complexity and Scalability*: The complexity of synchronization algorithms increases with the number of agents and with the heterogeneity among them. Effective synchronization methods must therefore be scalable and robust to variations in network size and agent dynamics.

- *Dynamic Topologies*: Real-world networks are often characterized by time-varying or switching topologies, which introduce additional challenges in the synchronization problem. Methods must account for changes in communication links or in the topology of the network.

Our goal in this thesis is to establish control laws that guarantee consensus among the agents and to further investigate the concept of synchronization in nonlinear multi-agent systems.

Outline

In this thesis, we address the synchronization problem in a multi-agent system with the presence of local observers, a critical issue in modern control theory and distributed systems, with several practical applications. The study aims to develop and analyze new methods for achieving synchronization in networks where agents are interconnected through dynamic links. This research contributes to the broader understanding of observer-based synchronization methods and their applications in complex, real-world scenarios.

The thesis is organized as follows:

- **Chapter 1** provides an overview of the fundamental concepts of dynamical systems, including both linear time-invariant (LTI) systems and non-linear systems. It introduces key mathematical tools and theories, such as state-space representation and stability analysis, which form the foundation for the subsequent chapters.

- **Chapter 2** focuses on the design and application of high-gain observers, essential for estimating the state of a system when not all states are directly measurable. This chapter discusses the concepts of observability and detectability and extends the analysis to nonlinear systems, highlighting the advantages of high-gain observer designs.

- **Chapter 3** shifts to multi-agent systems, exploring how these systems can be represented and analyzed using graph theory. It discusses the control of networks of agents, particularly through achieving consensus in homogeneous networks, setting the stage for synchronization strategies discussed later.

- **Chapter 4** presents the core contribution of this thesis: a novel method for synchronization in networks of systems connected via dynamic links using high-gain observers. The chapter details the problem setting, proposed solutions, and possible extensions, such as single-input single-output (SISO) systems, reference tracking, and nonlinear cases. A simulation example demonstrates the effectiveness of the proposed approach.

- **Chapter 5** provides additional simulations to further validate the theoretical findings. This chapter includes practical examples, such as the Van der Pol oscillator, showcasing the versatility and robustness of the synchronization strategies developed in this thesis.

- **Chapter 6** concludes the thesis by summarizing the key findings and contributions. It also discusses potential directions for future research, such as extending the proposed methods to more complex networks or exploring alternative synchronization techniques.

Through this structure, the thesis systematically builds on foundational concepts, introduces advanced observer design techniques, and applies these methods to multi-agent synchronization, thereby providing a comprehensive exploration of the topic.

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Chapter 1

Concepts about dynamical systems

The aim of this chapter is to highlights some features of dynamical systems. First of all, we will analyze briefly the main characteristics of linear time invariant (LTI) systems and afterwards we will address the main properties regarding a class of non linear systems.

1.1 Linear time invariant (LTI) systems

To describe this family of systems we can analyze the two features of them: linearity and time invariance. These two properties are defined below.

Linearity
A linear system represented as $y = f(x)$ is linear if it has the following properties:

$$\begin{aligned} &f(\alpha x) = \alpha f(x) \qquad \text{Scaling} \\ &f(x_1 + x_2) = f(x_1) + f(x_2) \qquad \text{Superposition} \end{aligned} \tag{1.1}$$

where α is a constant. Linearity is a very important concept for LTI system theory because it allows us to use various linear operations and transformations in our system so that we can better understand it or manipulate it. Even if several real systems are not linear, we can often linearize them around a certain point and work with this approximation.

Time invariance
Time invariance describes a function's independence from the time, more precisely a time-invariant system's output will shift in time if its input shifts in time, but otherwise it will remain exactly the same. In other words, a time-invariant function does not care what time it is. The property of time invariance can be described as:

$$\begin{aligned} &y(t) = f(t) \\ &y(t - \delta) = f(t - \delta) \end{aligned} \tag{1.2}$$

With δ being a number representing the time shift. Shift invariance is a crucial property of a system because it allows us to assume that a system will respond in a predictable manner at any time.

1.1.1 State space representation

Roughly speaking, a linear system can be described with a set of ODE (ordinary differential equations). Although a system of ODE lends itself to describing some types of systems better, sometimes it is still difficult to do any math on this system in its raw form. Therefore, a formalized method for representing linear systems of differential equation has been developed, called the *state space formalization*, which exploits matrix notation to represent linear sets of ODE in a mathematically relevant way.

Example: LRC system

A LRC circuit can be described with the following set of differential equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{LC}x_1 + \frac{R}{L}x_2 - \frac{1}{L}V_{src} \end{aligned} \tag{1.3}$$

Where x_1 is the charge inside the capacitor, x_2 is the current flowing in to the inductor, and V_{src} is the battery voltage (considered as an input). We can rewrite this system in matrix form as:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{LC} & \frac{R}{L} \end{bmatrix} x + \begin{bmatrix} 0 \\ -\frac{1}{L} \end{bmatrix} V_{src} \tag{1.4}$$

Supposing we want to measure the current circulating in the circuit we can write the output equation as:

$$y = [0\ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \tag{1.5}$$

The approach can be formalized defining the system defined by:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1.6}$$

In the previous example the matrices A, B, C, D are defined as:

$$A = \begin{bmatrix} 0 & 1 \\ \frac{1}{LC} & \frac{R}{L} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -\frac{1}{L} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix} \tag{1.7}$$

Each of the matrices defined above has a specific name: A is called the space matrix, B the input matrix, C the output matrix and D is the direct feed-through matrix. In many linear systems the matrix D is zero (the output is not affected by the input).

1.1.2 Stability

In order to study the stability of a system, let's start from a system represented in the state space form:

$$\dot{x} = Ax + Bu \tag{1.8}$$

Given an arbitrary system, (for instance, a second order system), the natural response of the system is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B \cdot 0 \tag{1.9}$$

Reminding that for a differential equation the solution assume the form

$$x(t) = e^{rt} \tag{1.10}$$

since the natural exponential function is the only function which is directly proportional to its derivative. Putting (1.9) into (1.10) we end up with:

$$r \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{1.11}$$

i.e.

$$rx = Ax \tag{1.12}$$

which has as solutions exactly the eigenvalues of the matrix A. The overall system response will be a linear combination of all the possible solutions to the set of linear ODEs:

$$y(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + \cdots + a_n e^{\lambda_n t} \tag{1.13}$$

where n is the number of states, $\lambda_1, ..., \lambda_n$ are the eigenvalues of the state matrix, and $a_1, ..., a_n$ are constants which describes the relative weights of each possible solutions with respect to the initial conditions of the system.
Since we proved that the eigenvalues correspond to the exponential constants which constitute the system's natural response, we may infer many of the properties of the system's solutions. Since we are dealing with only causal systems (i.e. $t \geq 0$), the equation:

$$y(t) = e^{\lambda t} \tag{1.14}$$

is stable only when the real component of the exponent is negative. In the end, we can enclose the result in the following definition:

Definition 1. *A system will have a stable natural response if and only if the real component of all the eigenvalues of A have negative real part.*

More precisely we can enunciate this theorem

Theorem 1. *The point $x = 0$ of $\dot{x} = Ax$ is stable if and only if all eigenvalues of A satisfy Re $\lambda_i \leq 0$ and for every eigenvalue with Re $\lambda_i = 0$ and algebraic multiplicity[1] $q_i \geq 2$, rank$(A - \lambda_i I) = n - q_i$, where n is the dimension of x. The point $x = 0$ is asymptotically stable if and only if all eigenvalues of A satisfy Re $\lambda_i < 0$.*

Definition 2. *When all eigenvalues of A satisfy Re$\lambda_i < 0$, A is called a Hurwitz matrix or a stability matrix.*

Referring to a second order system, based on the presence of the imaginary part in the eigenvalues, we will have different responses:

- **Over Damped:** Over damped behavior occurs when both eigenvalues have only real components, and do not have the same numerical value.

[1]The algebraic multiplicity of an eigenvalue is the number of times that it appears as a root of the characteristic polynomial of A

- **Underdamped:** Underdamped behavior occurs when eigenvalues have real and imaginary components, producing an oscillatory response. This is because imaginary components within the exponential translate int a sinusoidal response due to the Euler relation $e^{i\theta} = \cos\theta + i \sin\theta$.

- **Undamped:** A special case of the underdamped responses type occurs when the eigenvalues have no real component. Due to the Euler relation, there will be a sinusoidal component to the system. Without a real component, however, is missing the term related to the decaying exponential. Therefore, the system will oscillate without any loss in overall energy to the system.

- **Critically Damped:** Last case is when the eigenvalues are real and numerically equivalent. This is manifested as a slight modification in the solution of the system, which becomes a second order critically damped system:

$$y(t) = a_1 e^{\lambda t} + a_2 t e^{\lambda t} \tag{1.15}$$

  where λ is the eigenvalue of the system. The presence of the second term multiplied by t allows the contributions of two states to be represented non-redundantly in the solution.

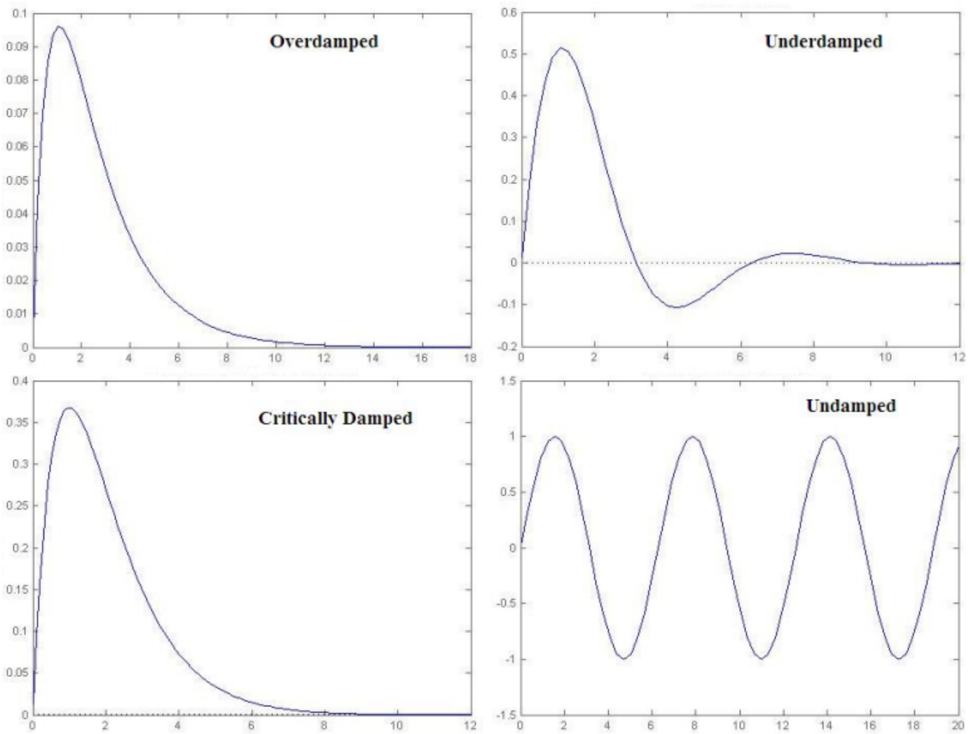


Figure 1.1: Impulse responses for various state-space eigenstructures

1.2 Non linear systems

A general non linear system can be written in the following form:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, ..., x_n, u_1, u_2, ..., u_m) \\
\dot{x}_2 &= f_2(x_1, x_2, ..., x_n, u_1, u_2, ..., u_m) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, x_2, ..., x_n, u_1, u_2, ..., u_m)
\end{aligned}$$

Where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. We will consider autonomous systems, i.e. the case in which f does not depend explicitly on t:

$$\dot{x} = f(x) \tag{1.16}$$

When we deal with non linear systems, we could face particular phenomena. To make a comparison, consider the linear system described by

$$\dot{x} = Ax + Bu \tag{1.17}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$.
The solution is given by

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{1.18}$$

The expression for $x(t)$ is linear in the initial condition $x(0)$ and in the control function $u(\cdot)$. Non linear system are those that do not satisfy these nice properties.
Moreover, the superposition principle no longer holds, and analysis tools involve more advanced mathematics. One technique that is used is to linearize the system about some nominal operating point and analyze the resulting linear model. But, in general, linearization is not sufficient, we must develop tools for the analysis for nonlinear systems. This is because linearization is an approximation in the neighborhood of an operating point, it can only predict the local behavior of the non linear system when we are sufficiently close to the point. In addition, the dynamics of a non linear system is way more richer than the dynamics of a linear system. There are some phenomena that take place only in non linear systems, hence they can not be predicted by a linear system. These are examples of non linear phenomena:

- **Finite escape time**: The state of an unstable linear system can go to infinity as time approaches infinity. A non linear system's state, however, can go to infinity in finite time.

- **Multiple isolated equilibrium points:** A linear system can have only one equilibrium point, and thus only one steady-state operating point that attracts or repels the state of the system irrespective of the initial state. A nonlinear system can have more than one equilibrium point.

- **Chaos:** A nonlinear system can have a more complicated steady-state. behavior that is not equilibrium or periodic oscillation. Some of these chaotic motions exhibit randomness, despite the deterministic nature of the system.

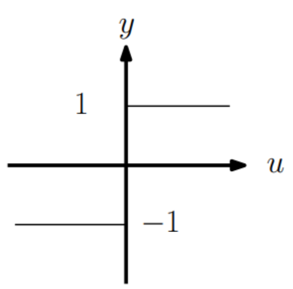
- **Limit cycles:** A linear system can have a stable oscillation if it has a pair of eigenvalues on the imaginary axis. The amplitude of the oscillation will then depend on the initial conditions. A nonlinear system can exhibit an oscillation of fixed amplitude and frequency which appears independently of the initial conditions.

- **Multiple modes of behavior:** A nonlinear system may exhibit very different forms of behaviour depending on external parameter values, or may jump from one form of behaviour to another autonomously. These behaviours cannot be observed in linear systems, where the complete system behaviour is characterized by the eigenvalues of the system matrix A.

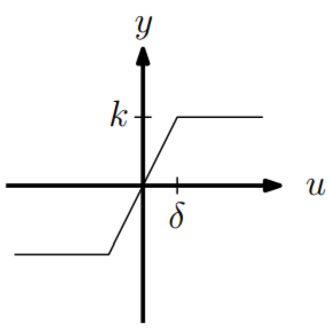
Example: typical nonlinearities

Most common nonlinearities are:

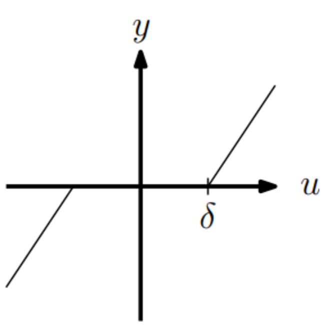
- **Relay:** relays appear when modelling mode changes.
- **Saturation:** saturations appear when modelling variables with hard limits, for instance actuators.
- **Dead Zone:** dead zone appear in connection to actuator or sensor sensitivity.
- **Quantization:** quantization is used to model discrete valued variables, often actuators.



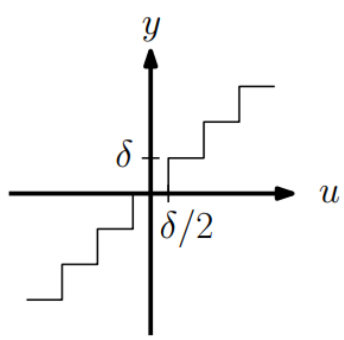
a) Relay



b) Saturation



c) Dead Zone



d) Quantization

An example of non linear system could be the a circuit in which is present a diode, like in the following example:

Example: Diode circuit

Consider the circuit depicted in the below figure. The tunnel diode characteristic $i_r = h(v_r)$ is plotted aside the circuit in the below figure. Choosing $x_1 = v_c$ and $x_2 = i_L$ and $u = E$, we obtian the following equations that describe the system:

$$\begin{aligned} \dot{x}_1 &= \frac{1}{C}(-h(x_1) + x_2) \\ \dot{x}_2 &= \frac{1}{L}(-x_1 - Rx_2 + E) \end{aligned} \tag{1.19}$$

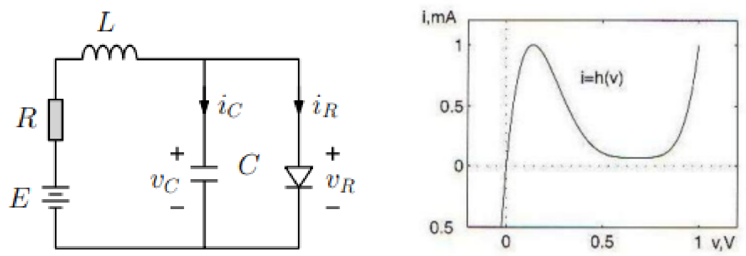
The equilibrium points of this systems are determined by setting $\dot{x}_1 = \dot{x}_2 = 0$, so we obtain

$$\begin{aligned} 0 &= \frac{1}{C}(-h(x_1) + x_2) \\ 0 &= \frac{1}{L}(-x_1 - Rx_2 + E) \end{aligned} \tag{1.20}$$

Therefore, we obtain for the equilibrium points:

$$h(x_1) = \frac{E - x_1}{R} \tag{1.21}$$

for certain values of E and R, this equation has three isolated roots which correspond to three isolated equilibrium points of the system. The number of equilibrium points might change as the values of E and R change.



(a) Diode circuit (b) Diode characteristic curve

1.2.1 A particular case: Second order systems

Second order autonomous systems occupy a very important role in the analysis of nonlinear systems, since their trajectories can be represented in the 2D-plane. This allows us to visualize the behavior of the system. Main aspects of second order systems are:

- Behavior near equilibrium points
- Nonlinear oscillations

- Bifurcations

A second-order autonomous system is represented by two scalar differential equations:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), \quad x_1(0) = x_{10} \\ \dot{x}_2 &= f_2(x_2, x_2), \quad x_2(0) = x_{20} \end{aligned} \tag{1.22}$$

The locus in the (x_1, x_2)-plane of the solution $x(t)$ for all $t \geq 0$ is a curve that passes through the point x_0. This plane is usually called phase plane. The vector f gives the tangent vector to the curve $x(\cdot)$.

Definition 3. *We obtain a vector field diagram by assigning the vector* $(f_1(x), f_2(x))$ *to every point* (x_1, x_2) *in a grid covering the plane.*

Definition 4. *The family of all trajectories of a dynamical system is called the phase portrait.*

To have an example, the following picture show the phase portrait of the circuit analyzed in the previous example:

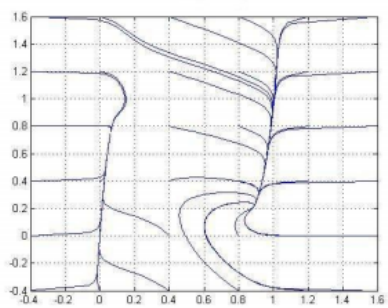


Figure 1.5: Diode phase portrait

1.3 Ordinary Differential Equations

1.3.1 Existence and Uniqueness

In this chapter some results about ordinary differential equations are presented, like existence and uniqueness. These properties are crucial for the state equation $\dot{x} = f(x, t)$ to be an useful mathematical model of a physical system. For the mathematical model to predict the future state of the system from its current state at t_0, the initial-value problem

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0 \tag{1.23}$$

must have a unique solution. It can be shown that existence and uniqueness can be ensured by imposing some constraints on the right-hand side function $f(t, x)$.
With these settings, we can derive sufficient condition to ensure existence and uniqueness for the initial-value problem (1.23). By a solution of (1.23) over an interval $[t_0, t_1]$ we mean a continuous function $x : [t_0, t_1] \to \mathbb{R}^n$ such that $\dot{x}$ is defined and $\dot{x} = f(t, x(t))$ for all $t \in [t_0, t_1]$. If $f(t, x)$ is continuous in x and piecewise continuous in t we can state the solution will be piecewise continuously differentiable.

Generally speaking, we can notice that a differential equation with a given initial condition might have several solutions. Consider, for example the following problem:

$$\dot{x} = x^{1/3}, \qquad x(0) = 0 \tag{1.24}$$

This problem has a solution $x(t) = (2t/3)^{3/2}$. This solution is not unique, since $x(t) \equiv 0$ is another solution for all t. Notice that the right side of (1.24) is continuous in x, so this condition is not enough to ensure the uniqueness of the solution.

Theorem 2. *Let $f(x,t)$ be piecewise continuous in t and satisfy the Lipschitz condition*

$$||f(t,x) - f(t,y)|| \leq L||x-y|| \tag{1.25}$$

$\forall x, y \in B = \{x \in \mathbb{R}^n| \; ||x - x_0|| \leq r\}$, $\forall t \in [t_0, t_1]$. *Then, there exists some $\delta > 0$ such that the state equation $\dot{x} = f(t,x)$ with $x(t_0) = x_0$ has a unique solution over $[t_0, t_0 + \delta]$.*

The key assumption in (2) is the Lipschitz condition. A function satisfying 1.25 is said to be Lipschitz in x, and the positive constant L is called *Lipschitz constant.* We can make a distinction between *locally* Lipschitz and *globally* Lipschitz based on the domain in which the Lipschitz condition holds.

- **Locally Lipschitz function:** A function $f(t,x)$ is said to be locally Lipschitz in x on $[a,b] \times D \subset \mathbb{R} \in \mathbb{R}^n$ if each point $x \in D$ has a neighborhood D_0 such that f satisfies (1.25) on $[a,b] \times D_0$ with some Lipschitz constant L_0. We say that $f(t,x)$ is locally Lipschitz in x on $[t_0, \infty) \in D$ if it is locally Lipschitz in x on $[a,b] \times D$ for every compact interval $[a,b] \subset [t_0, \infty)$. A function $f(t,x)$ is Lipschitz in x on $[a,b] \times W$ if it satisfies (1.25) for al $t \in [a,b]$ and all points in W, with the same constant L.

- **Lipschitz function**: A function is said to be Lipschitz on a set W if it satisfies (1.25) for all points W with the same Lipschitz constant L.

- **Globally Lipschitz function:** A function is said to be globally Lipschitz if it is Lipschitz in $\mathbb{R}^n$.

When $f : \mathbb{R} \to \mathbb{R}$, the Lipschitz condition can be written as

$$\frac{|f(y) - f(x)|}{|y - x|} \leq L \tag{1.26}$$

which implies that on a plot of $f(x)$ versus x, a straight line joining any two points of $f(x)$ cannot have a slope whose absolute value is greater than L. This definition is strictly related to the derivative of a function, indeed if $|f'(x)|$ is bounded by a constant k over the interval of interest, then $f(x)$ is Lipschitz on the same interval with Lipschitz $L = k$.

Lemma 1. *Let $f : [a,b] \times D \to \mathbb{R}^m$ be continuous for some domain $D \subset \mathbb{R}^n$. Suppose that $[\partial f / \partial x]$ exists and is continuous on $[a,b] \times D$. If, for a convex subset $W \subset D$, there is a constant $L \geq 0$ such that*

$$\left|\left|\frac{\partial f}{\partial x}(t,x)\right|\right| \leq L \tag{1.27}$$

on $[a,b] \times W$, then

$$||f(t,x) - f(t,y)|| \leq L||x-y|| \tag{1.28}$$

for all $t \in [a,b]$, $x \in W$, and $y \in W$.

We can now state a sufficient condition for a function to be globally Lipschitz:

Lemma 2. *If $f(t,x)$ and $[\partial f/\partial x](t,x)$ are continuous on $[a,b]\times\mathbb{R}^n$, then f is globally Lipschitz in x on $[a,b]\times\mathbb{R}^n$ if and only if $[\partial f/\partial x](t,x)$ is uniformly bounded on $[a,b]\times\mathbb{R}^n$.*

Example 1

The function

$$f(x) = \begin{bmatrix} -x_1 + x_1x_2 \\ x_2 - x_1x_2 \end{bmatrix} \tag{1.29}$$

is continuously differentiable on $\mathbb{R}^2$. It is not globally Lipschitz since $[\partial f/\partial x]$ is not uniformly bounded on $\mathbb{R}^2$. On any compact subset of $\mathbb{R}^2$, however, is Lipschitz. Let's compute the Lipschitz constant over the convex set $W = \{x \in \mathbb{R}^2 \mid |x_1| \le a_1,\ |x_2| \le a_2\}$. The Jacobian matrix is given by

$$\left[\frac{\partial f}{\partial x}\right] = \begin{bmatrix} -1 + x_2 & x_1 \\ -x_2 & 1 - x_1 \end{bmatrix} \tag{1.30}$$

Using $||.||_\infty$ for vectors in $\mathbb{R}^2$ and the induced matrix norm for matrices, we have

$$\left|\left|\frac{\partial f}{\partial x}\right|\right|_\infty = \max\{|-1+x_2| + |x_1|, |x_2| + |1 - x_1|\} \tag{1.31}$$

All points in W satisfy

$$|-1+x_2| + |x_1| \le 1 + a_2 + a_1 \text{ and } |x_2| + |1-x_1| \le a_2 + 1 + a_1 \tag{1.32}$$

Hence,

$$\left|\left|\frac{\partial f}{\partial x}\right|\right|_\infty \le 1 + a_1 + a_2 \tag{1.33}$$

and a Lipschitz constant can be taken as $L = 1 + a_1 + a_2$.

Example 2

The function

$$f(x) = \begin{bmatrix} x_2 \\ -\mathrm{sat}(x_1 + x_2) \end{bmatrix} \tag{1.34}$$

is not continuously differentiable on $\mathbb{R}^2$. Let us check its Lipschitz property examining $f(x) - f(y)$. Using $||.||_2$ for vectors in $\mathbb{R}^2$ and the fact that the saturation function $\mathrm{sat}(\cdot)$ satisfies

$$|\mathrm{sat}(\eta) - \mathrm{sat}(\xi)| \le |\eta - \xi| \tag{1.35}$$

we obtain

$$\begin{aligned} ||f(x) - f(y)||_2^2 &\le (x_2 - y_2)^2 + (x_1 + x_2 - y_1 - y_2)^2 \\ &= (x_1 - y_1)^2 + 2(x_1 - y_1)(x_2 - y_2) + 2(x_2 - y_2)^2 \end{aligned} \tag{1.36}$$

Example 2: cnt'd

Using the inequality

$$a^2 + 2ab + b^2 \leq 2a^2 + 3b^{2\,a} \leq 3(a^2 + b^2) \tag{1.37}$$

We end up with

$$||f(x) - f(y)||_2 \leq \sqrt{3}||x - y||_2 \tag{1.38}$$

resulting in a Lipschitz constant $L = \sqrt{3}$.

[a] $a^2 + 2ab + b^2 \leq 2a^2 + 3b^2 \to (a^2 + 2b^2 - 2ab) = (a - b)^2 + b^2 \geq 0$

In this two examples we used both the $||.||_\infty$ and $||.||_2$ norms. Due to equivalence of norms, the choice does not affect the Lipschitz property, but only the value of Lipschitz constant. Roughly speaking, theorem (2) is a local theorem since it ensures existence and uniqueness only over an interval $[t_0, t_0 + \delta]$, where δ may be very small. One may pose the following question: when is it granted that the solution can be extended indefinitely? One way to answer this is to require additional conditions which ensure that the solution $x(t)$ will always be in a set where $f(t, x)$ is uniformly Lipschitz in x. The following theorem establishes the existence of a unique solution over $[t_0, t_1]$, where t_1 may be arbitrarily large.

Theorem 3. *Suppose that $f(t, x)$ is piecewise continuous in t and satisfies*

$$||f(t, x) - f(t, y)|| \leq L||x - y|| \tag{1.39}$$

$\forall\ x, y \in \mathbb{R}^n$, $\forall t \in [t_0, t_1]$. *Then, the state equation $\dot{x} = f(t, x)$ with $x(t_0) = x_0$, has a unique solution over* $[t_0, t_1]$.

As the last thing in this chapter we will show by an example that the condition expressed in theorem (3) is only a sufficient condition. Indeed, one can easily construct smooth meaningful examples that do not have the global Lipschitz property, but do have unique global solutions, which is an indication of the conservative nature of the theorem.

Example

Consider the system

$$\dot{x} = -x^3 = f(x) \tag{1.40}$$

The function $f(x)$ does not satisfy a global Lipschitz condition since the Jacobian $\frac{\partial f}{\partial x} = -3x^2$ is not globally bounded. Nevertheless, for any initial state $x(t_0) = x_0$, the equation has the unique solution

$$x(t) = \text{sign}(x_0)\sqrt{\frac{x_0^2}{1 + 2x_0^2(t - t_0)}} \tag{1.41}$$

which is well defined for all $t \geq t_0$.

In view of the conservative nature of the global Lipschitz conditions, it would be useful to have a global existence and uniqueness theorem that requires the function f only to be locally Lipschitz. The following theorem achieves that at the expense of having to know more about the solution of the system.

Theorem 4. *Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and all x in a domain $D \subset \mathbb{R}^n$. Let W be a compact set of D, $x_0 \in W$, and suppose it is known that every solution of*

$$\dot{x} = f(t,x), \qquad x(t_0) = x_0 \tag{1.42}$$

lies entirely in W. Then, there is a unique solution that is defined for all $t \geq t_0$.

Exploiting this theorem we can prove that the system considered in the last example has a unique solution for all $t \geq 0$. To prove that, notice that the function $f(x)$ is locally Lipschitz and at any instant of time in which $x(t)$ is positive, the derivative $\dot{x}$ will be negative. Similarly, if $x(t)$ is negative, $\dot{x}$ will be positive. Therefore, starting from any initial condition $x(0) = a$, the solution cannot leave the compact set $W = \{x \in \mathbb{R} \mid |x| \leq a\}$. Thus, without calculating the solution, we can say that the equation has a unique solution for all $t \geq 0$.

1.4 Stability

Stability is a crucial concept in systems theory and engineering. There are different kinds of stability problems that arise in the study of dynamical systems. The aim of this chapter is to present some results regarding stability of equilibrium points and ISS (input-to-state) stability. Stability of equilibrium points is usually described in the sense of Lyapunov, a Russian mathematician and engineer who laid the foundation of the theory, who now carries his name. An equilibrium point is stable if all solutions starting at nearby points stay nearby; otherwise it is unstable. It is asymptotically stable if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity.
Lyapunov stability theory theorems give sufficient conditions for stability, asymptotic stability, and so on. They do not say whether the given conditions are also necessary. There are some theorems which establish, at least conceptually, that for many of Lyapunov stability theorems, the given conditions are indeed necessary. Such theorems are called converse theorems.

1.4.1 Autonomous Systems

Consider the autonomous system

$$\dot{x} = f(x) \tag{1.43}$$

where $f : D \to \mathbb{R}^n$ is a locally Lipschitz map from a domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Suppose $\bar{x} \in D$ is an equilibrium point of (1.43); that is $f(\bar{x}) = 0$. Our goal is to characterize and study the stability of $\bar{x}$. There is no loss of generality in considering the equilibrium point to be at the origin of $\mathbb{R}^n$ because any equilibrium point can be shifted to the origin via a change of variables. Suppose $\bar{x} \neq 0$ and the change of variables $y = x - \bar{x}$. The derivative of y is given by

$$\dot{y} = \dot{x} = f(x) = f(y + \bar{x}) = g(y), \qquad g(0) = 0 \tag{1.44}$$

In the new variable y, the system has an equilibrium at the origin. Therefore, we will always study the stability of the origin $x = 0$.

Definition 5. *The equilibrium point $x = 0$ of* (1.43) *is*

- *stable if, for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that*

$$||x(0)|| < \delta \implies ||x(t)|| < \varepsilon, \quad \forall\, t \geq 0 \tag{1.45}$$

- *unstable if it is not stable.*
- *asymptotically stable if it is stable and δ can be chosen such that*

$$||x(0)|| < \delta \implies \lim_{t\to\infty} x(t) = 0 \tag{1.46}$$

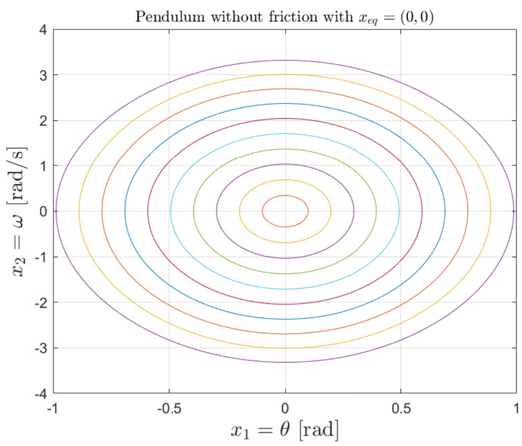
The $\varepsilon - \delta$ requirements say that to demonstrate that the origin is stable, we have to prove that for any possible value of $\varepsilon > 0$ we can find a value of $\delta(\varepsilon)$ such that a trajectory starting in a δ neighborhood of the origin will never leave the ε neighborhood.

Example: Pendulum

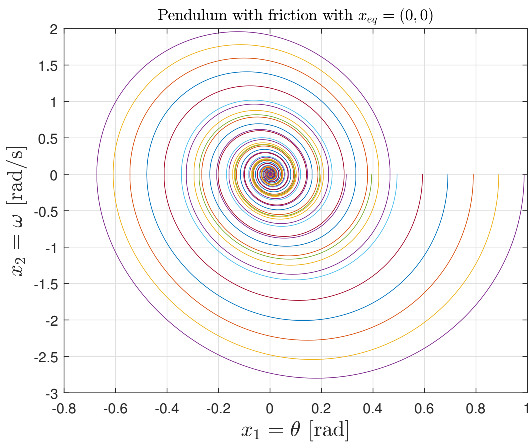
The pendulum can be described by the following equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin x_1 - bx_2 \end{aligned} \tag{1.47}$$

where a is a constant related to the length of the pendulum and b is the friction coefficient. The equation (1.47) has two equilibrium points at $(\bar{x}_{11}, \bar{x}_{12}) = (0, 0)$ and $(\bar{x}_{21}, \bar{x}_{22}) = (\pi, 0)$. Neglecting friction, by setting $b = 0$, the trajectories in the neighborhood of the first equilibrium are closed orbits. Hence, by starting sufficiently close the the equilibrium point, trajectories can be granted to stay within any specified ball centered at the equilibrium point. The equilibrium point, however, is not asymptotically stable since trajectories starting off the equilibrium point do not tend to it eventually
When friction is taken into account ($b > 0$), the equilibrium point at the origin becomes a stable focus, trajectories starting close to the equilibrium tend to it as t tends to ∞. The second equilibrium point is a saddle point and can be seen that this equilibrium point is unstable. In the simulation below are shown the phase portrait of the pendulum system considering the two different equilibrium points and different initial conditions that are close to the equilibrium point under analysis, in order to see the different behaviors.



a) $x_{eq} = (0, 0)$ without friction



b)$x_{eq} = (0, 0)$ with friction

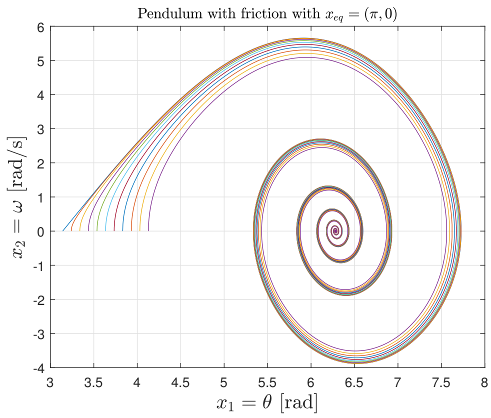


Figure 1.7: $x_{eq} = (\pi, 0)$ with friction

Having defined stability and asymptotic stability of equilibrium points, our task now is to find ways to determine stability. We could have reached the same conclusions using energy concepts. Let us define the energy of the pendulum $E(x)$ as the sum of its potential and kinetic energies with the assumption that $E(0) = 0$; that is

$$E(x) = \int_0^{x_1} a \sin y \; dy + \frac{1}{2}x_2^2 = a(1 - \cos x_1) + \frac{1}{2}x_2^2 \tag{1.48}$$

If friction is neglected ($b = 0$) the system is conservative, indeed we have

$$\frac{dE(x)}{dt} = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = ax_2 \sin x_1 - ax_2 \sin x_1 = 0 \tag{1.49}$$

Where we exploited expression (1.47) in the last passage. This fact means that no energy is dissipated and we can arrive at the conclusion that $x = 0$ is a stable equilibrium point. When friction is accounted, energy will dissipate during the motion, indeed we obtain

$$\frac{dE(x)}{dt} = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = -bx_2^2 \tag{1.50}$$

Since $b > 0$ we can infer that now energy is dissipated during the motion and this show that the trajectory tends to $x = 0$ as $t \to \infty$. Thus, by examining the derivative of $E(x)$ along the trajectories of the system, it is possible to determine the stability of the equilibrium point. Generally speaking, we can extend this result to any function that belongs to a certain class.

Theorem 5. *Let $x = 0$ be an equilibrium point for (1.43) and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that*

$$V(0) = 0 \textit{ and } V(x) > 0 \textit{ in } D - \{0\} \tag{1.51}$$

$$\dot{V}(x) \leq 0 \textit{ in } D \tag{1.52}$$

Then $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \textit{ in } D - \{0\} \tag{1.53}$$

Then $x = 0$ is asymptotically stable.

A function that satisfies (1.51) and (1.52) is called *Lyapunov function.*

Remark 1. *Let $V : D \to \mathbb{R}$ be a continuously differentiable function defined in a domain $D \subset R^n$ that contains the origin. The derivative of V along the trajectories of* (1.43)*, denoted by $\dot{V}(x)$, is given by*

$$\begin{aligned} \dot{V}(x) &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \\ &= \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \cdots, \frac{\partial V}{\partial x_n}\right] \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} = \frac{\partial V}{\partial x} f(x) \end{aligned} \tag{1.54}$$

A function that satisfies (1.51) is said to be *positive definite.* If it satisfies the weaker condition $V(x) \geq 0$ for $x \neq 0$, is said to be *positive semidefinite.* A function $V(x)$ is said to be *negative definite* or *negative semidefinite* if $-V(x)$ is respectively positive definite or positive semidefinite. if $V(x)$ does not have a definite sign as per one of these four cases, it is said to be indefinite. With this terminology we can reformulate theorem (5) to say that *the origin is stable if there is a continuously differentiable positive definite function $V(x)$ such that $\dot{V}(x)$ is negative semidefinite, and it is asymptotically stable if $\dot{V}(x)$ is negative definite.* A class of scalar function for which sign definiteness can be checked easily is the class of functions of the quadratic form

$$V(x) = x^\top P x = \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij} x_i x_j \tag{1.55}$$

Where $P = P^\top \succ 0$. In this particular case $V(x)$ is positive definite (positive semidefinite) if and only if all the eigenvalues of P are positive (nonnegative).

Example: Quadratic function

Consider the following function

$$\begin{aligned} V(x) &= a x_1^2 + 2x_1x_3 + a x_2^2 + 4x_2x_3 + a x_3^2 \\ &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & 0 & 1 \\ 0 & a & 2 \\ 1 & 2 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x^\top P x \end{aligned} \tag{1.56}$$

Computing the eigenvalues of the matrix P we find that $V(x)$ is positive definite if and only if $a > \sqrt{5}$ and negative definite if and only if $a < -\sqrt{5}$. If $a \in (-\sqrt{5}, \sqrt{5})$, P is indefinite.

Unfortunately, there is no systematic way for finding Lyapunov functions. In some cases we can use our intuition and build a Lyapunov function based on the concept of energy of the system under analysis, in other cases it is basically a matter of trial and error.
It's important to remark that the conditions of Lyapunov's theorem are only sufficient, i.e. a failure of a Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not mean that the equilibrium point is not stable or asymptotically stable.

It only means that such stability property can not be established by using the Lyapunov function candidate.
We can now state a result that gives sufficient condition for a equilibrium point to be globally asymptotically stable.

Theorem 6. *Let $x = 0$ be an equilibrium point for (1.43). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that*

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall\, x \neq 0 \tag{1.57}$$

$$||x|| \to \infty \implies V(x) \to \infty \tag{1.58}$$

$$\dot{V}(x) < 0, \quad \forall\, x \neq 0 \tag{1.59}$$

then $x = 0$ is globally asymptotically stable.

If we take a step back and talk about linear systems described by $\dot{x} = Ax$, the asymptotic stability of the system can be also studied by using Lyapunov method. Consider a quadratic Lyapunov function candidate

$$V(x) = x^\top Px \tag{1.60}$$

Where P is a real symmetric positive definite matrix. The derivative of V along the trajectories of the system is given by

$$\dot{V}(x) = x^\top P\dot{x} + \dot{x}^\top Px = x^\top(PA + A^\top P)x = -x^\top Qx \tag{1.61}$$

Where Q is a symmetric matrix defined by

$$PA + A^\top P = -Q \tag{1.62}$$

If Q is positive definite, we can conclude by (5) that the origin is asymptotically stable; that is, Re $\lambda_i < 0$ for all eigenvalues of A.
The next theorem characterizes asymptotic stability of the origin in terms of the solution of the Lyapunov equation.

Theorem 7. *A matrix A is Hurwitz if and only if for any given positive definite symmetric matrix Q there exist a positive definite symmetric matrix P that satisfies the Lyapunov equation (1.62). Moreover, if A is Hurwitz, then P is the unique solution of (1.62).*

1.4.2 Comparison Functions

In order to introduce the concept of ISS (input-to-state stability) we need to define the so called comparison functions, known as class $\mathcal{K}$ and class $\mathcal{KL}$ functions.

Definition 6. *A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.*

Definition 7. *A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if , for each fixed s, the mapping $\beta(r, s)$ belong to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.*

Example: comparison functions

- $\alpha(r) = \tan^{-1}(r)$ is strictly increasing since $\alpha'(r) = 1/(1+r^2) > 0$. It belongs to class $\mathcal{K}$, but not to class $\mathcal{K}_\infty$ since $\lim_{r\to\infty} \alpha(r) = \pi/2 < \infty$.
- $\alpha(r) = r^c$, for any positive real number c, is strictly increasing. Moreover, $\lim_{r\to\infty} \alpha(r) = \infty$; thus, it belongs to class $\mathcal{K}_\infty$.
- $\beta(r, s) = r/(ksr + 1)$ for any positive real number k, is strictly increasing in r since
$$\frac{\partial \beta}{\partial r} = \frac{1}{(ksr+1)^2} > 0$$
and strictly decreasing in s since
$$\frac{\partial \beta}{\partial s} = \frac{-kr^2}{(ksr+1)^2} < 0$$
Moreover, $\beta(r, s) \to 0$ as $s \to \infty$. Therefore, it belongs to class $\mathcal{KL}$.

1.4.3 Input-to-State Stability

Consider the system
$$\dot{x} = f(t, x, u) \tag{1.63}$$
where $f : [0, \infty) \times \mathbb{R}^n \times R^m \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x and u. The input $u(t)$ is a piecewise continuous, bounded function of t for all $t \geq 0$. Suppose that the unforced system
$$\dot{x} = f(t, x, 0) \tag{1.64}$$
has a globally uniformly asymptotically stable equilibrium point at the origin at $x = 0$. What can we say about the behavior of the system (1.63) in the presence of a bounded input $u(t)$? For a LTI system
$$\dot{x} = Ax + Bu \tag{1.65}$$
with a Hurwitz matrix A, we can write the solution as
$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{(t-\tau)A}Bu(\tau)\, d\tau \tag{1.66}$$
and use the bound $||e^{(t-t_0)A}|| \leq ke^{-\lambda(t-t_0)}$ to estimate the solution by
$$\begin{aligned} ||x(t)|| &\leq ke^{-\lambda(t-t_0)}||x(t_0)|| + \int_{t_0}^{t} ke^{-\lambda(t-t_0)}||B||||u(\tau)||\, d\tau \\ &\leq ke^{-\lambda(t-t_0)}||x(t_0)|| + \frac{k||B||}{\lambda} \sup_{t_0 \leq \tau \leq t} ||u(\tau)|| \end{aligned} \tag{1.67}$$

This estimate shows that the zero-input response (the computed response when $u \equiv 0$) decays to zero exponentially fast, while the zero-state response (response when the initial condition of the state are set to 0), is bounded for every bounded input. This estimate shows hat the bound on the zero-state response is proportional to the bound on the input.

For a general nonlinear system, these properties may not hold even if when the origin of the unforced system is globally uniformly asymptotically stable. We can now state the definition of *input-to-state stability*.

Definition 8. *The system* (1.63) *is said to be input-to-state stable if there exist a class* $\mathcal{KL}$ *function* β *and a class* $\mathcal{K}$ *function* γ *such that for any initial state* $x(t_0)$ *and any bounded input* $u(t)$*, the solution* $x(t)$ *exists for all* $t \geq 0$ *and satisfies*

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0) + \gamma \left(\sup_{t_0 \leq \tau \leq t} ||u(\tau)|| \right) \tag{1.68}$$

The inequality (1.68) guarantees that for any bounded input $u(t)$, the state $x(t)$ will be bounded. Moreover, as t increases, the state $x(t)$ will be ultimately bounded by a class $\mathcal{K}$ function of $\sup_{t \geq t_0} ||u(t)||$. It can be proven that if $u(t)$ converges to 0 as $t \to \infty$, so does $x(t)$. Since, if $u(t) \equiv 0$, (1.68) reduces to

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0) \tag{1.69}$$

input-to-state stability implies that the origin of the unforced system (1.67) is globally uniformly asymptotically stable. The notion of ISS stability is defined for the global case where the initial state and the input can be arbitrarily large.

Chapter 2

Highlights On High Gain Observers

In the realm of control systems, the concept of observers plays an important role in enabling accurate estimation of system states. Observers, also known as state estimators, are dynamic systems designed to reconstruct the internal state of a system based on measurements of its inputs and outputs. This reconstruction is essential when not all states are directly measurable, a common scenario in real-world applications due to sensor limitations, cost constraints, or inaccessibility. By leveraging a mathematical model of the system and incorporating available measurements, observers provide an estimate of the full state vector, which can then be used for various control strategies. This approach is particularly valuable in scenarios where direct state feedback is desired but not feasible due to limited sensor information. This chapter aims to present firstly the classic design approach for linear system, and successively the concept and design of high-gain observers, where the main characteristic is the high gain used to reduce the error and make the system converge very quickly.

2.1 Observability and Detectability

Consider the linear system described by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{2.1}$$

In general, the dimension of the observed output, y, may be less than the dimension of the state, x. However, one might assume that if one observed the output over time interval then this might tell us something about the state. Roughly speaking we can say that observability is concerned with the issue of what can be said about the state given measurements of the plant output.

Definition 9. *The state $x_0 \neq 0$ is said to be unobservable if given $x(0) = x_0$, and $u(t) = 0\ \forall t > 0$, then $y(t) = 0,\ \forall t > 0$. The system is said to be completely observable if there exists no non-zero initial state that is unobservable.*

It can be proven that the observability of a system is related to the study of a particular matrix, $\mathcal{M}_o$.

Theorem 8. *Consider the system* (2.1)*. We have*

- *The set of all unobservable states is equal to the null space of the observability matrix* $\mathcal{M}_o$*, defined by*

$$\mathcal{M}_o \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{2.2}$$

- *The system is completely observable if and only if* $\mathcal{M}_o$ *has rank* n*, where* n *is the order of the system.*

If the system is not completely observable we have the following result:

Lemma 3. *If* $rank(\mathcal{M}_o) = k < n$*, there exists a similarity transformation* T *such that* $\bar{x} = T^{-1}x, \bar{A} = T^{-1}AT, \bar{C} = CT$*, then* $\bar{C}$ *and* $\bar{A}$ *take the form*

$$\bar{A} = \begin{bmatrix} \bar{A}_O & 0 \\ \bar{A}_{21} & \bar{A}_{NO} \end{bmatrix} \qquad \bar{C} = \begin{bmatrix} \bar{C}_O & 0 \end{bmatrix} \tag{2.3}$$

where $\bar{A}_O$ *has dimension* k *and the pair* $(\bar{C}_O, \bar{A}_O)$ *is completely observable.*

Bearing in mind this lemma, we can decompose system (2.1) as:

$$\begin{bmatrix} \dot{x}_O(t) \\ \dot{x}_{NO}(t) \end{bmatrix} = \begin{bmatrix} \bar{A}_O & 0 \\ \bar{A}_{21} & \bar{A}_{NO} \end{bmatrix} \begin{bmatrix} \bar{x}_O(t) \\ \bar{x}_{NO}(t) \end{bmatrix} + \begin{bmatrix} \bar{B}_O \\ \bar{B}_{NO} \end{bmatrix} u(t) \tag{2.4}$$

$$y(t) = \begin{bmatrix} \bar{C}_O & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_O(t) \\ \bar{x}_{NO}(t) \end{bmatrix} \tag{2.5}$$

This decomposition allow us to infer properties of the system analyzing the structure of the matrices $\bar{A}_O$ and $\bar{A}_{NO}$.

Definition 10. *The observable subspace of a plant is composed of all states generated through every possible linear combination of the states in* $\bar{x}_O$*. The stability of this subspace is determined by the location of the eigenvalues of* $\bar{A}_O$*.*

Definition 11. *The unobservable subspace of a plant is composed of all states generated through every possible linear combination of the states in* $\bar{x}_{NO}$*. The stability of this subspace is determined by the location of the eigenvalues of* $\bar{A}_{NO}$*.*

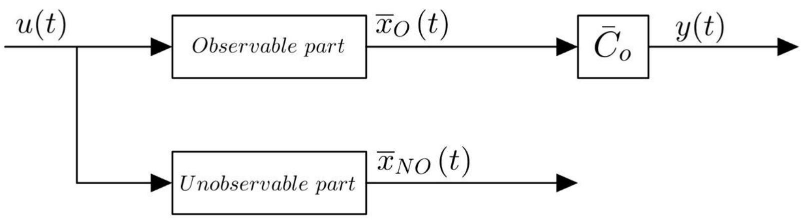


Figure 2.1: Observable–unobservable decomposition

Another important property of a system is detectability. This property deals with the fact that a system might not be observable, but, on the other hand, the unobservable part does not make the whole system to diverge.

Definition 12. *A plant is said to be detectable if its unobservable subspace is stable.*

2.2 Observer Design

In the last section we introduced the main results about observers, observability and detectability. The forthcoming part deals with the design of observer, firstly in the case of linear systems and then we will extend the theory to nonlinear systems.

2.2.1 Linear Systems

Consider again the following state space model:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{2.6}$$

A general linear observer takes the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)) \tag{2.7}$$

where the matrix L is called the *observer gain* and $\hat{x}$ is the *state estimate.* if $L \neq 0$ the term

$$e(t) = \hat{x}(t) - x(t) \tag{2.8}$$

represents the estimation error between the observation and the predicted model output. Next,we compute the dynamics of the estimation error as

$$\begin{aligned} \dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) \\ &= A\hat{x}(t) + \cancel{Bu(t)} - L(C\hat{x}(t) - Cx(t)) - Ax(t) - \cancel{Bu(t)} \\ &= (A - LC)(\hat{x}(t) - x(t)) \\ &= (A - LC)e(t) \end{aligned} \tag{2.9}$$

This is a differential equation, whose solution is given by

$$e(t) = \exp[(A - LC)t]e(0) \tag{2.10}$$

From the above expression we can note that if the matrix $(A - LC)$ is Hurwitz, we would have $e(t) \to 0$, when $t \to \infty$. Bearing this result in mind, the problem has been shifted to the design of the matrix L, in such a way the matrix $(A - LC)$ is Hurwitz.

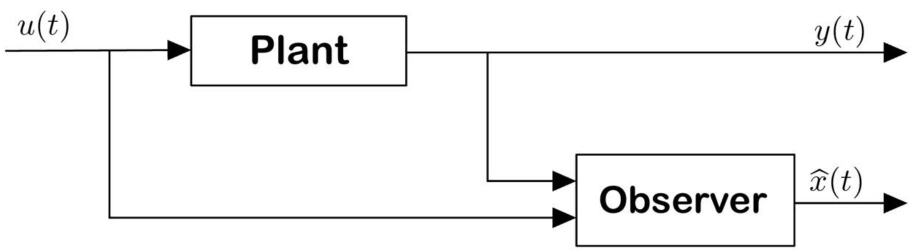


Figure 2.2: Block diagram of a system with observer

Example: Design of a Linear Observer

Consider the linear system described by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{2.11}$$

The matrix $\mathcal{M}_o$ can be constructed as

$$\mathcal{M}_o = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \tag{2.12}$$

and it can be easily verified that the rank of this matrix is 2, i.e. the system is observable. We will design the gain matrix L in such a way the matrix $(A - LC)$ will have $\lambda_1 = -10$ and $\lambda_2 = -20$ as eigenvalues. Since the order of the system is $n = 2$, we can conclude that the matrix L will have the following shape:

$$L = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \tag{2.13}$$

We compute the matrix $(A - LC)$:

$$\begin{aligned} &\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \\ = &\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & \ell_1 \\ 0 & \ell_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 - \ell_1 \\ -1 & 2 - \ell_2 \end{bmatrix} \end{aligned} \tag{2.14}$$

The characteristic polynomial of $(A - LC)$ can be computed as

$$p_{A-LC}(\lambda) = \lambda^2 + (\ell_2 - 2)\lambda + 1 - \ell_1 \tag{2.15}$$

and exploiting the polynomial identity principle we obtain

$$\begin{aligned} 30 &= -2 + \ell_2 \\ 200 &= -\ell_1 + 1 \end{aligned} \tag{2.16}$$

and solving the equations we obtain

$$\ell_1 = -199, \quad \ell_2 = 32 \tag{2.17}$$

2.2.2 Nonlinear Systems : High Gain Observer

In this section, we consider the class of single-input single-output nonlinear systems that can be modeled by equations of the form:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{2.18}$$

in which $x \in \mathbb{R}^n$ and in which $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are smooth maps and $h : \mathbb{R}^n \to \mathbb{R}$ is a smooth function.

We can now introduce the Lipschitz triangular form

$$\begin{aligned}\dot{\xi}_1 &= \xi_2 + \Phi_1(\tilde{u}, \xi_1)\\ &\vdots\\ \dot{\xi}_i &= \xi_{i+1} + \Phi_i(\tilde{u}, \xi_1, ..., \xi_i)\\ &\vdots\\ \dot{\xi}_n &= \Phi_n(\tilde{u}, \xi)\\ y &= \xi_1\end{aligned}$$

where $\tilde{u}$ can be, for instance $\tilde{u} = (u, \dot{u}, \ddot{u}, ...)$. This form is associated with the classical High gain observer. For a system in which $g(x) \equiv 0$, the function $\mathbf{H}_n(x)$ defined by the output and its $n-1$ derivatives, namely

$$\mathbf{H}_n(x) = (h(x), L_f h(x), ..., L_f^{n-1} h(x)) \tag{2.19}$$

maps the system (2.19) into

$$\dot{\xi}_1 = \xi_2 \quad , \quad ... \quad , \quad \xi_i = \xi_{i+1} \quad , ... \quad , \quad \xi_n = L_f^n h(x) \quad , \quad y = \xi_1 \tag{2.20}$$

where $L_f h(x)$ is the Lie-derivative of $h(x)$ with respect to the vector field $f(\cdot)$:

$$L_f h \triangleq \nabla h \cdot f, \qquad \nabla h = \frac{\partial h}{\partial x} = \left[\frac{\partial h}{\partial x_1}, \cdots, \frac{\partial h}{\partial x_n}\right] \tag{2.21}$$

This is a Lipschitz phase-variable form if and only if there exists a function Φ_n Lipschitz on $\mathbb{R}^n$ such that

$$\forall x \in \mathcal{S} \quad , \quad L_f^n h(x) = \Phi_n(\mathbf{H}_n(x)) \tag{2.22}$$

In general, this is a nonlinear changes of coordinates, namely a transformation $\xi = \mathbf{H}_n(x)$, in which $\mathbf{H}_n(\cdot)$ is a map $\mathbb{R}^n \to \mathbb{R}^n$. Such a map qualifies for a change of coordinates if:

- $\mathbf{H}_n(\cdot)$ is a invertible, i.e. there exists a map $\mathbf{H}_n^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathbf{H}_n^{-1}(\mathbf{H}_n(x)) = x$ for all $x \in \mathbb{R}^n$ and $\mathbf{H}_n(\mathbf{H}_n^{-1}(\xi)) = \xi$ for all $\xi \in \mathbb{R}^n$.
- $\mathbf{H}_n(\cdot)$ and $\mathbf{H}_n^{-1}$ are both smooth mappings, i.e., have continuous partial derivatives of any order.

A transformation of this type is called a *global diffeomorphism* on $\mathbb{R}^n$. If $g(x) \neq 0$, the system (2.18) can be transformed into a Lipschitz triangular form (2.19) by a stationary transformation. Before continuing, we need to state some technical definitions.

Definition 13. *We can distinguish between two types of observability, namely:*

- *A system is said to be weakly differentially observable of order m on a subset $\mathcal{S}$ if the function $x \mapsto \mathbf{H}_n(x)$ is injective on $\mathcal{S}$.*
- *A system is said to be strongly differentially observable of order m if the function $x \mapsto \mathbf{H}_n(x)$ is an injective immersion[1] on $\mathcal{S}$.*

[1] $\mathbf{H}_n$ is injective and $\frac{\partial \mathbf{H}_n(x)}{\partial x}$ has full-rank on $\mathcal{S}$

- *A system is said to be uniformly instantaneously observable if it is instantaneously distinguishable, i.e. the state can be uniquely deduced from the output of the system as quickly as we want, on $\mathcal{S}$, for any input u.*

Definition 14. *We call drift system of system* (2.19) *the dynamics with $u \equiv 0$, namely*

$$\dot{x} = f(x), \quad y = h(x) \tag{2.23}$$

We can say that the drift system is weakly (strongly) observable of order m on $\mathcal{S}$ if the function

$$\mathbf{H}_m(x) = (h(x), L_f h(x), ..., L_f^{m-1} h(x)) \tag{2.24}$$

is injective (an injective immersion) on $\mathcal{S}$.

Differential observability of the drift system is weaker than differential observability of the system since it is only for $u \equiv c, c \in \mathbb{R}$. In order to obtain a triangular form, it is necessary to add an assumption of uniform observability:

Theorem 9. *Assume that exists an open subset $\mathcal{S}$ of $\mathbb{R}^n$ such that*

- *System* (2.19) *is uniform instantaneously observable on S*
- *The drift system of system* (2.19) *is strongly differentially observable of order n on $\mathcal{S}$.*

Then, $\mathbf{H}_n$ defined by

$$\mathbf{H}_n(x) = (h(x), L_f h(x), ..., L_f^{n-1} h(x)) \tag{2.25}$$

which is a diffeomorphism on S by assumption, maps system (2.19) *into a Lipschitz triangular form of dimension n on $\mathcal{S}$.*

It is crucial that the order of strong differential observability of the drift system be n (the dimension of the state) to ensure the Lipschitzness of the triangular form in order to use a high gain observer. When this order is larger than n, triangularity is often preserved but the Lipschitzness is lost.

We can now introduce the concept of high gain observer. In order to do so, we will start from an example.

Example: High Gain Observer

Consider the second-order nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \phi(x, u) \\ y &= x_1 \end{aligned} \tag{2.26}$$

where $x = [x_1 \;\; x_2]^\top$. Suppose $u = \gamma(x)$ is a locally Lipschitz state feedback control law that stabilizes the origin $x = 0$ of the closed-loop system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \phi(x, \gamma(x)) \end{aligned} \tag{2.27}$$

Example: cnt'd

To implement this feedback control using only measurements of the input y, we use the observer

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + h_1(y - \hat{x}_1)\\ \dot{\hat{x}}_2 &= \phi_0(\hat{x}, u) + h_2(y - \hat{x}_1)\end{aligned} \tag{2.28}$$

where $\phi_0(x, u)$ is a nominal model of the nonlinear function $\phi(x, u)$. The estimation error can be defined as

$$\tilde{x} = \begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\end{bmatrix} = \begin{bmatrix}x_1 - \hat{x}_1\\ x_2 - \hat{x}_2\end{bmatrix} \tag{2.29}$$

and it satisfies the equation:

$$\begin{aligned}\dot{\tilde{x}}_1 &= -h_1\tilde{x}_1 + \tilde{x}_2\\ \dot{\tilde{x}}_2 &= -h_2\tilde{x}_1 + \phi(x, \gamma(x)) - \phi_0(\hat{x}, \gamma(\hat{x}))\end{aligned} \tag{2.30}$$

The aim is to design the observer gain $H = [h_1 \;\; h_2]^\top$ such that $\lim_{t\to\infty} \tilde{x} = 0$. In the absence of the disturbance term, asymptotic error convergence is achieved designing H such that the matrix

$$A_o = \begin{bmatrix}-h_1 & 1\\ -h_2 & 0\end{bmatrix} \tag{2.31}$$

is Hurwitz. In the presence of the uncertainty on ϕ, we need to design H with the additional goal of rejecting his effects on $\tilde{x}$. This is ideally achieved if the transfer function

$$G_o(s) = \frac{1}{s^2 + h_1 s + h_2}\begin{bmatrix}1\\ s + h_1\end{bmatrix} \tag{2.32}$$

from $\Delta\phi$ to $\tilde{x}$ is identically 0, where $\Delta\phi(x, \hat{x}) = \phi(x, \gamma(x)) - \phi(\hat{x}, \gamma(\hat{x}))$. This is not possible, but we can make $\sup_{\omega\in\mathbb{R}} G_o(j\omega)$ arbitrarily small by choosing $h_2 >> h_1 >> 1$. In particular, we can take

$$h_1 = \frac{\alpha_1}{\varepsilon}, \quad h_2 = \frac{\alpha_2}{\varepsilon^2} \tag{2.33}$$

for some positive constants α_1, α_2, and $\varepsilon << 1$. It can be shown that

$$G_o(s) = \frac{\varepsilon}{(\varepsilon s)^2 + \alpha_1\varepsilon s + \alpha_2}\begin{bmatrix}\varepsilon\\ \varepsilon s + \alpha_1\end{bmatrix} \tag{2.34}$$

Hence, $\lim_{\varepsilon\to 0} G_o(s) = 0$.

If we define the scaled estimation errors (for a second order system), we have

$$\eta_1 = \frac{\hat{x}_1}{\varepsilon}, \quad \eta_2 = \hat{x}_2 \tag{2.35}$$

The newly defined variables satisfy the singularly perturbed equation

$$\begin{aligned}\varepsilon\dot{\eta}_1 &= -\alpha_1\eta_1 + \eta_2\\ \varepsilon\dot{\eta}_2 &= -\alpha_2\eta_1 + \varepsilon\Delta\phi(x, \hat{x})\end{aligned} \tag{2.36}$$

. This equation shows clearly that reducing ε diminishes the effect of $\Delta\phi$. Notice that $\eta_1(0)$ will be $O(1/\varepsilon)$ whenever $x_1(0) \neq \hat{x}_1(0)$. Consequently, the solution of the (2.36) will contain a term of the form $(1/\varepsilon)e^{-at/\varepsilon}$ for some $a > 0$. While this exponential mode decays rapidly, it exhibits an impulsive-like behavior where the transient peaks to $O(1/\varepsilon)$ values before it decays rapidly towards zero. In fact, the function $(1/\varepsilon)e^{-at/\varepsilon}$ approaches an impulse function as ε tends to 0. This behavior is known as the *peaking phenomenon*. It is worth noticing that smaller is the ε higher will be the peak that we can see during the transient.

Example: Peaking Phenomenon

To understand better the peaking phenomenon, let us simulate the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_2^3 + u \\ y &= x_1 \end{aligned} \tag{2.37}$$

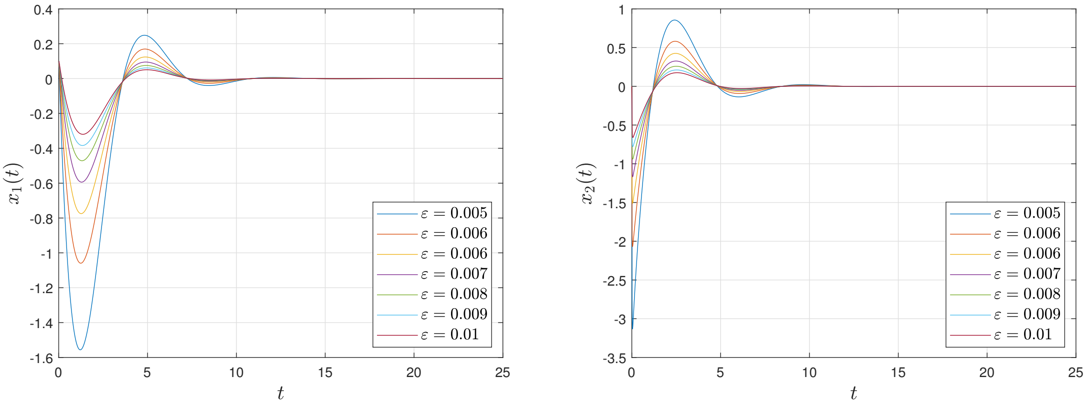
Which can globally stabilized by the state feedback controller

$$u = -x_2^3 - x_1 - x_2 \tag{2.38}$$

The output feedback controller is taken as

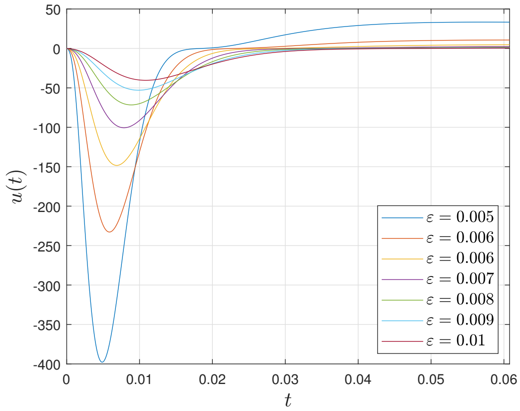
$$\begin{aligned} u &= -\hat{x}_2^3 - \hat{x}_1 - \hat{x}_2 \\ \dot{\hat{x}}_1 &= \hat{x}_2 + (2/\varepsilon)(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= (1/\varepsilon^2)(y - \hat{x}_1) \end{aligned} \tag{2.39}$$

In the pictures below are shown the behavior of the states and of the input for different values of ε, from $\varepsilon = 0.005$ to $\varepsilon = 0.01$. Since it can be noted that the peak is due to the input u that influences the states, we can saturate the input, in order to avoid huge peaking phenomena.

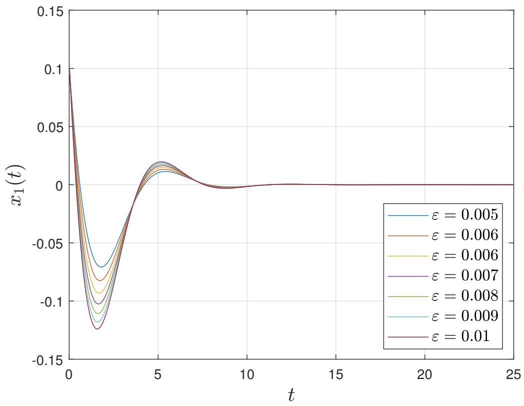


a) $x_1(t)$ with different ε

b) $x_2(t)$ with different ε



c) $u(t)$ with different ε



d) $x_1(t)$ in presence of a saturated $u(t)$

Throughout this simulations the initial conditions are $x_1(0) = 0.1,\ x_2(0) = \hat{x}_1(0) = \hat{x}_2(0) = 0$. The input value is constrained in such a way that $u(t) \in [-10, 10]$.

A special case

As a particular case, consider the nonlinear system described by:

$$\begin{aligned} \dot{x} &= Ax + B\phi(x) + u \\ y &= Cx \end{aligned} \tag{2.40}$$

Where A, B, C are in normal form and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$ and $\phi(x)$ is a globally Lipschitz function with Lipschitz constant L. Define the matrix D_d as

$$D_d = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & d & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & d^{n-1} \end{pmatrix} \tag{2.41}$$

The observer dynamics can be written as

$$\dot{\hat{x}} = A\hat{x} + B\phi(\hat{x}) + K_\alpha D_d(y - C\hat{x}) \tag{2.42}$$

Where K_α is defined as:

$$K_\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \tag{2.43}$$

Where the positive constants α_i are chosen such that the roots of

$$s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1}s + \alpha_n = 0 \tag{2.44}$$

are in the open left-half plane.

Theorem 10. *For all* $T > 0$*, for all* $\varepsilon > 0$*, and for all* $x(0), \hat{x}(0) : |\hat{x}(0) - x(0)| \leq \beta$ *there exists* $d^*(\varepsilon) \geq 1$ *such that for any* $d \geq d^*(\varepsilon)$*:*

$$|x(t) - \hat{x}(t)| \leq \varepsilon, \quad \forall t \geq T \tag{2.45}$$

Proof. In order to prove the result we can make the following change of basis:

$$e_i = \frac{\hat{x}_i - x_i}{d^{i-1}}, \quad i = 1, ..., N \tag{2.46}$$

Where we can see e_i as a scaled version of the observation error. Defining

$$e = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}, \quad \Delta\phi = \phi(\hat{x}) - \phi(x), \quad e = D_d^{-1}(\hat{x} - x) \tag{2.47}$$

we have, since $\phi(x)$ is Lipschitz

$$|\Delta\phi(x)| = |\phi(x + D_d e) - \phi(x)| \leq L d^{n-1}|e| \tag{2.48}$$

since $D_d e \leq d^{n-1}|e|$. Computing the derivative of e we obtain:

$$\dot{e} = d(A - KC)e + \frac{1}{d^{n-1}} B\Delta\phi \tag{2.49}$$

Since the matrix $(A - KC)$ is Hurwitz, there exists P such that

$$P(A - KC) + (A - KC)^\top P = -I \tag{2.50}$$

Consider now the Lyapunov function

$$V = e^\top P e \tag{2.51}$$

Computing the time derivative we have

$$\begin{aligned} \dot{V} &= e^\top P\dot{e} + \dot{e}^\top P e = -d||e||^2 + \frac{2}{d^{n-1}} B\Delta\phi P e \\ &\leq -d||e||^2 + \frac{2}{\cancel{d^{n-1}}}||B||L\cancel{d^{n-1}}||P||||e||^2 \end{aligned} \tag{2.52}$$

And defining

$$c_0 = L||B||||P|| \tag{2.53}$$

we end up with

$$\dot{V} \leq -(d - 2c_0)||e||^2 \tag{2.54}$$

□

2.3 Stabilization of a system with filtered output using an High-Gain observer

2.3.1 Settings of the problem

The aim of this section is to show an implementation of high gain observers to stabilize a system described by the cascade of:

$$\begin{aligned} \dot{x} &= Fx + Gu \\ y &= Hx \end{aligned} \tag{2.55}$$

with

$$\begin{aligned} \dot{w} &= (A + B\varphi)w + By \\ z &= Cw \end{aligned} \tag{2.56}$$

where the control u must be designed only using the information provided by z, i.e. $u = \psi(z)$. With $x \in \mathbb{R}^n, u \in \mathbb{R}, w \in \mathbb{R}^r, y \in \mathbb{R}, z \in \mathbb{R}$, $\varphi \in \mathbb{R}^{1\times r}$ is a Hurwitz polynomial, $(A, B, C)_r$ matrices of appropriate dimensions in prime (or normal) form[2].

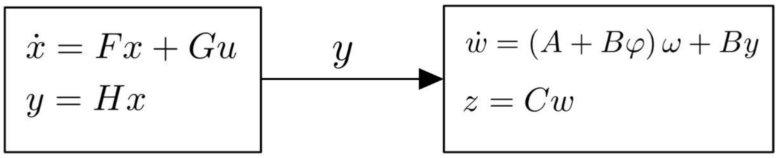


Figure 2.5: Cascade of the two systems described in (2.55), (2.56)

We make the following assumptions.

Assumption 2.1. *There exists a matrix K such that the matrix $(F - GKH)$ is Hurwitz.*

Assumption 2.2. *The matrix $(A + B\varphi)$ is Hurwitz.*

The (2.1) requires that the system (2.55) is controllable[3], while the assumption (2.2) says that the system described by (2.56) is asymptotically stable. In view of the context of application in power grids, system (2.56) can be viewed as a dynamical LC filter that described the dynamics of the transmission line.

2.3.2 Proposed solution

The idea is to design a high gain estimator to estimate y and use it in the output feedback law. In order to do this we build the following estimator:

$$\dot{\hat{w}} = (A + B\varphi)\hat{w} + B\sigma + D_g K_r (Cw - C\hat{w}) \tag{2.57a}$$
$$\dot{\sigma} = g^{r+1} k_{r+1} (Cw - C\hat{w}) \tag{2.57b}$$
$$u = -K\sigma \tag{2.57c}$$

Where σ is an additional variable that we wants to converge to y, $g \in \mathbb{R}$ is a parameter to be chosen, $D_g = \operatorname{diag}(g, \ldots, g^r)$, $K_r = (k_1, \ldots, k_r)$ with $k_1, \ldots, k_r$ to be chosen.

Theorem 11. *Consider system* (2.55), (2.56) *and suppose Assumptions 2.1 and 2.2 hold. Let* $(k_1, \ldots, k_{r+1})$ *be chosen such that the following polynomial*

$$\lambda^{r+1} + k_1\lambda^r + \cdots + \lambda k_r + k_{r+1}$$

[2]Namely $A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{r\times r}$, $B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{r\times 1}$, $C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{1\times r}$

[3]Fixed $(A + B\varphi) = \bar{A}$, the linear system is said to be controllable if the matrix

$$\mathcal{C} = [B \; \bar{A}B \; \bar{A}^2 B \; \ldots \; \bar{A}^{n-1}B]$$

is full rank

is Hurwitz. Then, there exists $g^* \geq 1$ such that, for any $g > g^*$, the closed-loop system (2.55),(2.56) (2.57) *is globally asymptotically stable.*

Proof. The equations (2.57a) and (2.57b) can be written in matrix form in the following way:

$$\begin{bmatrix} \dot{\hat{w}} \\ \dot{\sigma} \end{bmatrix} = \left(\begin{array}{c|c} A & B \\ \hline 0 & 0 \end{array}\right) \left[\begin{array}{c} \hat{w} \\ \hline \sigma \end{array}\right] + \left(\begin{array}{c|c} B\varphi & 0 \\ \hline 0 & 0 \end{array}\right) \left[\begin{array}{c} \hat{w} \\ \hline \sigma \end{array}\right] + \bar{D}_g \bar{K}(Cw - C\hat{w}) \tag{2.58}$$

Where $\bar{D}_g = \text{diag}(g, \dots, g^{r+1})$, $\bar{K} = \text{diag}(k_1, \dots, k_{r+1})$. Consider now a scaled version of the error:

$$e_i = g^{r+1-i}(\hat{w}_i - w_i), \qquad \forall i = 1, ..., r \tag{2.59}$$

$$e_{r+1} = \sigma - y \tag{2.60}$$

Or in matrix form:

$$e = g^{r+1} \bar{D}_g^{-1} \left[\begin{array}{c} \hat{w} - w \\ \hline \sigma - y \end{array}\right] \tag{2.61}$$

Computing the time derivative of e we have:

$$\begin{aligned} \dot{e} =& g^{r+1} \bar{D}_g^{-1} \Bigg\{ \left(\begin{array}{c|c} A & B \\ \hline 0 & 0 \end{array}\right) \left[\begin{array}{c} \hat{w} \\ \hline \sigma \end{array}\right] + \left(\begin{array}{c|c} B\varphi & 0 \\ \hline 0 & 0 \end{array}\right) \left[\begin{array}{c} \hat{w} \\ \hline \sigma \end{array}\right] - \bar{D}_g \bar{K} \bar{C} \frac{1}{g^{r+1}} \bar{D}_g e \\ & - \left(\begin{array}{c|c} A & B \\ \hline 0 & 0 \end{array}\right) \left[\begin{array}{c} w \\ \hline y \end{array}\right] - \left(\begin{array}{c|c} B\varphi & 0 \\ \hline 0 & 0 \end{array}\right) \left[\begin{array}{c} w \\ \hline y \end{array}\right] - \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right] H(Fx - GK\sigma) \Bigg\} \\ &= g(\bar{A} - \bar{K}\bar{C} + \Psi\bar{\varphi}\frac{1}{g^{r+1}}\bar{D}_g)e - \bar{B}H\dot{x} \\ &= g(\bar{A} - \bar{K}\bar{C} + \Psi\bar{\varphi}\frac{1}{g^{r+1}}\bar{D}_g)e - \bar{B}H[(F - GKH)x - GK\bar{B}^T e] \end{aligned} \tag{2.62}$$

Where $(\bar{A}, \bar{B}, \bar{C})_{r+1}$ are in prime form, $\Psi = \left[\begin{array}{c} B \\ \hline 0 \end{array}\right]$ and $\bar{\varphi} = \left[\begin{array}{c|c} \varphi & 0 \end{array}\right]$. The closed loop (2.55), (2.56) (2.57c), (2.62) can be written as:

$$\begin{aligned} \dot{x} &= (F - GKH)x - GK\bar{B}^T e \\ \dot{w} &= (A + B\varphi)w + BHx \\ \dot{e} &= g\left(\bar{A} - \bar{K}\bar{C} + \Psi\bar{\varphi}\frac{1}{g^{r+1}}\bar{D}_g + \bar{B}HGK\bar{B}^T\right)e - \bar{B}H(F - GKH)x \end{aligned} \tag{2.63}$$

Since the matrices $(F - GKH)$, $(A + B\varphi)$, $(\bar{A} - \bar{K}\bar{C})$ are Hurwitz, there exists three symmetric matrices $P, S, Q \succ 0$ such that the following hold:

- $P(F - GKH) + (F - GKH)^\top P = -2I$
- $Q(A + B\varphi) + (A + B\varphi)^T Q = -2I$
- $S(\bar{A} - \bar{K}\bar{C}) + (\bar{A} - \bar{K}\bar{C})^T S = -I$

In order to prove stability of (2.63) we exploit the following quadratic Lyapunov function:

$$V(x, w, e) = ax^T Px + bw^T Qw + e^T Se, \quad a, b > 0 \tag{2.64}$$

Computing its derivative along (2.63) we have:

$$\begin{aligned}\dot{V} &= -2a|x|^2 + 2ax^T PGK\bar{B}^T e - 2b|w|^2 + 2bw^T QBHx - g|e|^2 \\ &+ 2e^T S\Psi\bar{\varphi}\frac{1}{g^r}\bar{D}_g e + 2e^T S\bar{B}HGK\bar{B}Te - 2e^T S\bar{B}H(F-GKH)x \\ &\leq -g|e|^2 + 2|S\bar{B}H(F-GKH)||x||e| - 2a|x|^2 + 2a|PGK\bar{B}^T||x||e| \\ &+ 2|S\bar{B}HGK\bar{B}^T||e|^2 - 2b|w|^2 + 2b|QBH||x||w| + 2|S\Psi\bar{\varphi}\frac{1}{g^r}\bar{D}_g||e|^2\end{aligned} \tag{2.65}$$

Let us fix:

$$c_0 = PGK\bar{B}^T, \quad c_1 = QBH, \quad c_2 = S\Psi\bar{\varphi}\frac{1}{g^r}\bar{D}_g$$
$$c_3 = S\bar{B}HGK\bar{B}^T, \quad c_4 = S\bar{B}H(F-GKH)$$

Using the Young inequality we can write:

$$\begin{aligned}2a|c_0||x||e| &\leq \frac{a|x|^2}{2} + 2a|c_0|^2|e|^2 \\ 2b|c_1||x||w| &\leq \frac{b|w|^2}{2} + 2b|c_1|^2|x|^2 \\ 2|c_4||x||e| &\leq \frac{|x|^2}{2} + 2|c_4|^2|e|^2\end{aligned} \tag{2.66}$$

The inequalities (2.66) turn (2.65) into:

$$\dot{V} \leq -\left(\frac{3a}{2} - \frac{1}{2} - 2b|c_1|^2\right)|x|^2 - \left(g - 2|c_2| - 2|c_3| - 2a|c_0|^2 - 2|c_4|^2\right)|e|^2 - \frac{3}{2}b|w|^2 \tag{2.67}$$

Since $a, b, g > 0$ can be arbitrarily chosen, we can set:

$$g > 2|c_2| + 2|c_3| + 2a|c_0|^2 + 2|c_4|^2, \quad a > \frac{1}{3} + \frac{4}{3}b|c_1|^2$$

and stability of the system (2.63) is achieved. □

2.3.3 Simulation example

For the simulation the following system is considered:

$$\begin{aligned}\dot{x} &= Fx + Gu \\ y &= Hx\end{aligned}$$

In cascade with

$$\begin{aligned}\dot{w} &= (A + B\varphi)w + By \\ z &= Cw\end{aligned}$$

Where

$$F = \begin{bmatrix}-2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1\end{bmatrix}, \quad G = \begin{bmatrix}1 \\ 1 \\ 1\end{bmatrix}$$
$$H = [1\ 2\ 0], \quad \varphi = [-0.1\ -0.1], \quad K = 10$$

With this values the closed loop (2.55),(2.56) is asymptotically stable with eigenvalues

$$\lambda_1 = -1, \quad \lambda_2 = -30.06, \quad \lambda_3 = -0.93$$

For what regard the observer we chose

$$g = 10, \quad \bar{K} = [30\ 275\ 750]^T$$

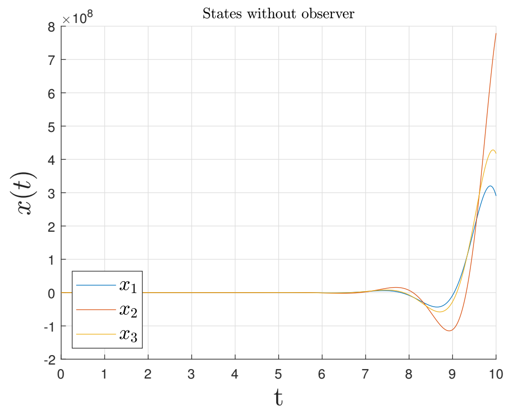
Indeed the polynomial

$$q(\bar{\lambda}) = \bar{\lambda}^3 + 30\bar{\lambda}^2 + 275\bar{\lambda} + 750$$

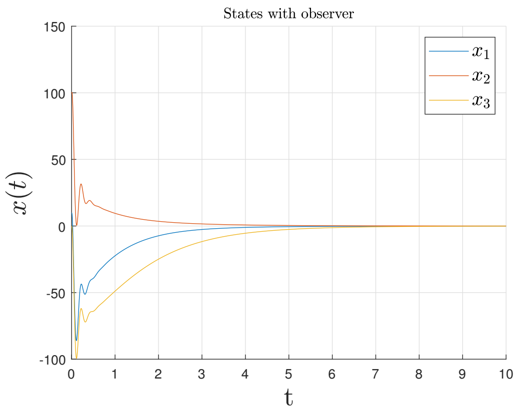
is Hurwitz with roots

$$\bar{\lambda}_1 = -15, \quad \bar{\lambda}_2 = -10, \quad \bar{\lambda}_3 = -5$$

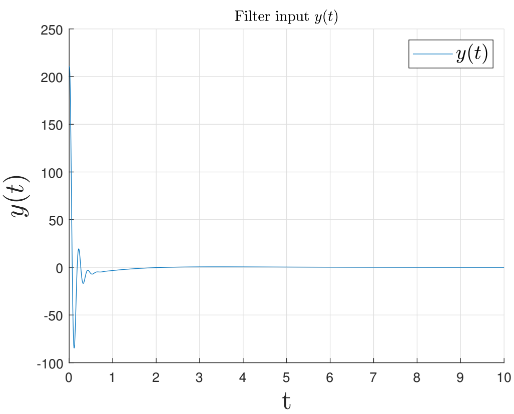
With this choice of the parameters it can be seen in the simulation that without the observer the filter is too slow and it's impossible to the states to settle down. Changing the settings and implementing the observer that estimates the input of the filter $y(t)$, the system becomes stable and the states of the system converge. Below are shown the results of MATLAB simulations:



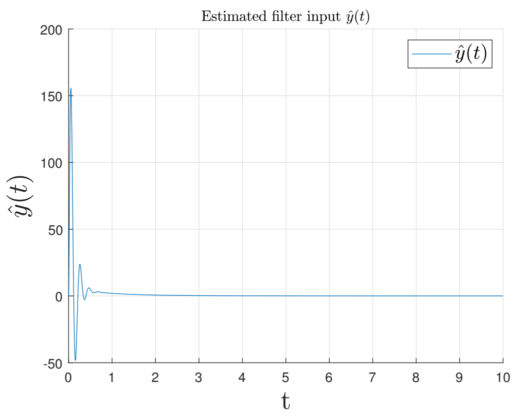
(a) *Evolution of states of* (2.55) *(with filtered output feedback)*



(b) *Evolution of states of* (2.55) *(with observer)*



(a) *Actual input of the filter*



(b) *Estimated input of the filter*

Chapter 3

Highlights of multi-agent systems

The aim of this chapter is to introduce and highlight some characteristics of multi-agent systems. An agent denotes a dynamical system which can be, for example, an aircraft, a satellite, a sensor, and so forth. The objective of control in this environment translates in achieving collective group behaviour via local interaction, that in most of the cases, is the only possibility. Control of multi-agents systems consists of many research topics and problems. One of this problems is called **consensus**, i.e. when a group of agents agree on a common value by communicating with each other. In order to solve this problem, consensus algorithms have been developed and can be divided in two groups, namely, consensus with a leader and consensus without a leader. In the first case all the agents have to follow the behaviour of the leader, in the latter there is no leader, so consensus is achieved when all the agents agree on a certain common trajectory that depends by the network itself.

3.1 Basics of graph theory

A group of agents interacts with each other to achieve some collective objectives. It is useful to model the information flow among the agents by directed and undirected graphs. A directed graph $\mathcal{G}$ is a pair $(\mathcal{V}, \mathcal{E})$, where $V = \{v_1, \cdots, v_n\}$ is the node set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set of ordered pairs of nodes. Graphs can be weighted if there is a weight associated with each edge of the graph. A subgraph $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ of $\mathcal{G}$ is a graph such that both the node set and the edge set are contained in the respective set of $\mathcal{G}$, namely $\mathcal{V}_s \subseteq \mathcal{V}$ and $\mathcal{E}_s \subseteq \mathcal{E}$. In the multi-agent context, an edge (v_i, v_j) indicates that the flow of information is from agent i to agent j, namely agent j can receive information by agent i, but non necessarily viceversa if the graph is directed. The set of neighbours of a generic node v_i is denoted with $\mathcal{N}_i$. A directed graph is strongly connected if there is a directed path from every node to every other node. For an undirected graph, a strong connectdness is simply called connectedness. A directed tree is a directed graph in which every node has exactly one parent except for one node, called the root, which has directed paths to all other nodes. A directed tree is defined as spanning when it connects all the nodes in the graph. A graph contains a directed spanning tree if a subset of the edges forms a directed spanning tree. For undirected graphs, if there exists a directed spanning tree, that means that the graph is connected. A strongly connected graph contains at least one directed spanning tree.
The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ associated with the directed graph $\mathcal{G}$ is defined in a way that $a_{ii} = 0$, $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. If the graph is undirected then $a_{ij} = a_{ji}$, $\forall i \neq j$, i.e. the adjacency matrix is symmetric. The value a_{ij} can be viewed as the

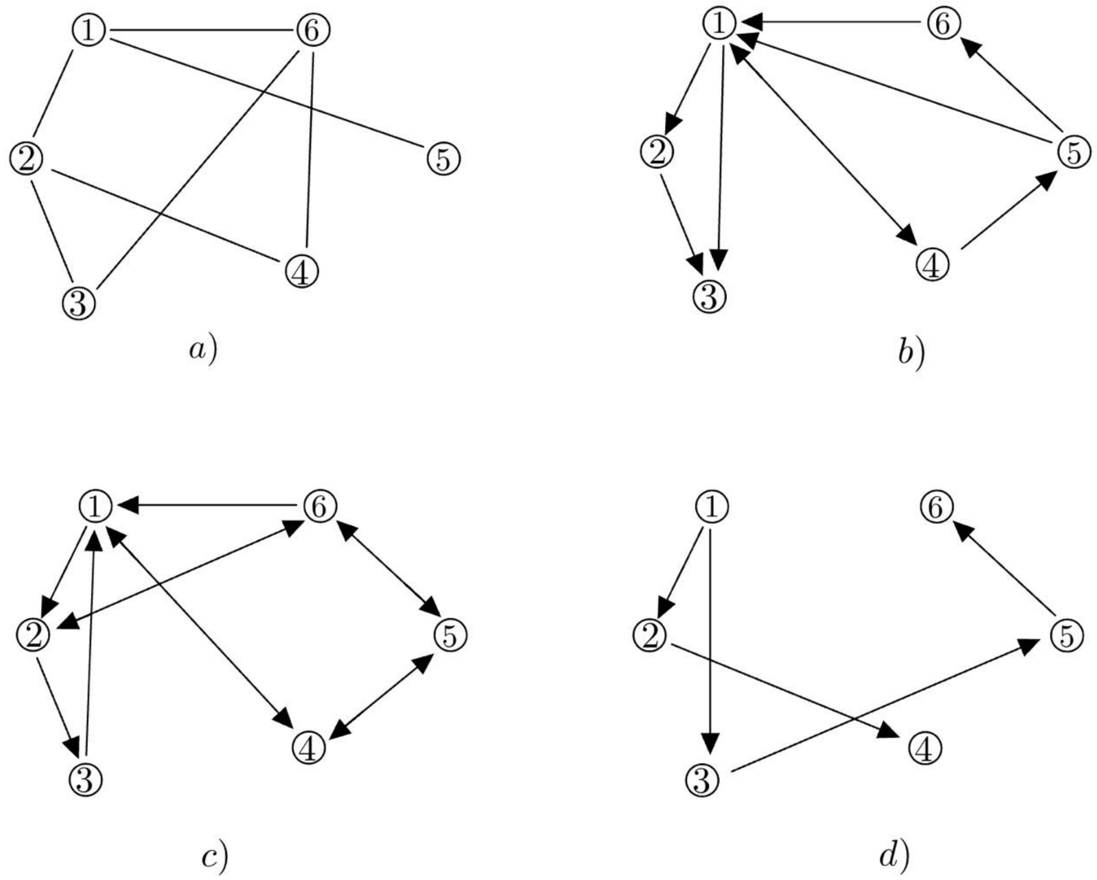


Figure 3.1: Different types of graphs: a) an undirected connected graph, b) a strongly connected graph, c) a balanced and strongly connected graph, d) a directed spanning tree.

weight associated with the edge (v_j, v_i). If the weight is not important, set this value to 1 if $(v_j, v_i) \in \mathcal{E}$. The incidence matrix $\mathcal{I} \in \mathbb{R}^{N \times E}$ with $E = |\mathcal{E}|$ is defined as

$$
\begin{aligned}
\iota_{ij} &= 1, \text{ if edge } e_j \text{ enters vertex } v_i \\
\iota_{ij} &= -1, \text{ if edge } e_j \text{ leaves vertex } v_i \\
i_{ij} &= 0 \text{ otherwise}
\end{aligned}
$$

In case of undirected graphs, we can define the incidence matrix as $i_{ij} = 1$ if the edge e_j is incident on vertex v_i and 0 otherwise. The Laplacian matrix $\mathcal{L} = [\mathcal{L}_{ij}] \in \mathbb{R}^{N \times N}$ of the graph $\mathcal{G}$ is defined as

$$
\begin{aligned}
\mathcal{L}_{ii} &= \sum_{j \neq i} a_{ij} \\
\mathcal{L}_{ij} &= -a_{ij}, \text{ for } i \neq j
\end{aligned}
$$

In case of undirected graphs, the following property holds:

$$
\mathcal{L} = \mathcal{I}\mathcal{I}^\top
$$

3.2 Control of a network of systems

In this section we will see two types of problems related to the control of a network of systems, namely:

- **Leader-follower Coordination Problem**: This is the case in which there are N linear systems with N large. The aim is to design a control law under which the output of each system asymptotically tracks the output of a single autonomous system called leader.
- **Leader-less Coordination Problem**: In this case a set of N agents is modeled, no leader is specified, and a control law is sought in order to achieve consensus to a single trajectory.

In the forthcoming analysis we will focus on the latter case. Suppose that the N systems can be described by:

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i \\ y_i &= Cx_i \end{aligned} \qquad i = 1, ..., N \tag{3.1}$$

Since in this case the matrices A, B, C are the same among the agents, the system is called *homogeneous*. In the leaderless coordination problem, we will consider the case in which the information available for control purpose at the i-th agent has the form:

$$v_i = \sum_{j=1}^{N} a_{ij}(y_j - y_i), \quad i = 1, ..., N \tag{3.2}$$

in which y_i, for $i = 1, ..., N$, is a measurement taken from agent i. This can be equivalently expressed as

$$v_i = \sum_{j \in \mathcal{N}_i} a_{ij}(y_j - y_i), \quad i = 1, ..., N \tag{3.3}$$

Exploiting the definition of the Laplacian $\mathcal{L}$ of a graph we can write the expression (3.3) as

$$v_i = -\sum_{j=1}^{N} \ell_{ij} y_j, \qquad i = 1, ..., N \tag{3.4}$$

By definition, the diagonal entries of $\mathcal{L}$ are nonnegative, the off-diagonal elements are nonpositive and, for each row, the sum of all elements on this row is zero. Letting $\mathbf{1}_N$ denote the "all-ones" N-vector

$$\mathbf{1}_N = col(1, 1, ..., 1) \tag{3.5}$$

this latter property can be written as

$$\mathcal{L}\mathbf{1}_N = 0 \tag{3.6}$$

It is seen from the above that the matrix $\mathcal{L}$ is singular or, what is the same, that $\lambda = 0$ is always an eigenvalue of $\mathcal{L}$. Such eigenvalue is referred to as the trivial eigenvalue of $\mathcal{L}$. Let the other (possibly nonzero) $N-1$ eigenvalues of $\mathcal{L}$ be denoted as $\lambda_2(\mathcal{L}), ..., \lambda_N(\mathcal{L})$. The real parts of these eigenvalues play an important role with respect to the property of connectivity.

Theorem 12. *A graph is connected if and only if its Laplacian matrix $\mathcal{L}$ has only one trivial eigenvalue $\lambda_1 = 0$ and all other eigenvalues $\lambda_2(\mathcal{L}), ..., \lambda_N(\mathcal{L})$ have positive real parts.*

In order to control this system we will use a static controller, in which the control u_i is given by:

$$u_i = Kv_i \tag{3.7}$$

Where K is a matrix of feedback gains to be found. Bearing in mind the expression of v_i, the overall closed loop can be modeled by a set of equations of the form

$$\dot{x}_i = Ax_i - \sum_{j=1}^{N} \ell_{ij} BKCx_j \tag{3.8}$$

Letting n the dimension of x_i, letting $\boldsymbol{x}$ denote the Nn vector

$$\boldsymbol{x} = \mathrm{col}(x_1, ..., x_N) \tag{3.9}$$

And exploiting the Kronecker product of matrices, we can write

$$\dot{\boldsymbol{x}} = (I_N \otimes A)\boldsymbol{x} - (\mathcal{L} \otimes BKC)\boldsymbol{x} \tag{3.10}$$

In what follows, it is assumed that the communication graph is connected. If this is the case, the Laplacian matrix L of the graph has one trivial eigenvalue $\lambda_1 = 0$, for which $\mathbf{1}_N$ is an eigenvector, while all other eigenvalues have positive real part, i.e., there exists a number $\mu > 0$ such that

$$\mathrm{Re}[\lambda_i(\mathcal{L})] \geq \mu, \qquad \forall i = 2, .., N \tag{3.11}$$

Consider the nonsingular matrix

$$T = \begin{bmatrix} 1 & 0_{1\times(N-1)} \\ \mathbf{1}_{N-1} & I_{N-1} \end{bmatrix} \tag{3.12}$$

If we apply the similarity transformation T to $\mathcal{L}$ we end up with:

$$\tilde{\mathcal{L}} = T^{-1}\mathcal{L}T = \begin{bmatrix} 0 & \tilde{\mathcal{L}}_{12} \\ 0 & \tilde{\mathcal{L}}_{22} \end{bmatrix} \tag{3.13}$$

in which $\tilde{\mathcal{L}}_{12}$ is a $1 \times (N-1)$ row vector and $\tilde{\mathcal{L}}_{22}$ is a $(N-1) \times (N-1)$ matrix, whose eigenvalues evidently coincide with the nontrivial eigenvalues $\lambda_2(\mathcal{L}), ..., \lambda_N(\mathcal{L})$ of $\mathcal{L}$. Consider now the controlled system (3.10) and change of coordinates as $\tilde{\boldsymbol{x}} = (T^{-1} \otimes I_n)\boldsymbol{x}$. Then we have:

$$\begin{aligned} \dot{\tilde{\boldsymbol{x}}} &= (T^{-1} \otimes I_n)[(I_N \otimes A) - (\mathcal{L} \otimes BKC)](T \otimes I_n)\tilde{\boldsymbol{x}} \\ &= (T^{-1} \otimes I_n)[(T \otimes A) - (\mathcal{L}\mathcal{T} \otimes BKC)]\tilde{\boldsymbol{x}} \\ &= [(I_N \otimes A) - (\mathcal{L} \otimes BKC)]\, \tilde{\boldsymbol{x}} \end{aligned} \tag{3.14}$$

Observe that, by definition of T, the vector $\tilde{x}$ has the form

$$\tilde{\boldsymbol{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{x}_2 - \tilde{x}_1 \\ \vdots \\ \tilde{x}_N - \tilde{x}_1 \end{pmatrix} \tag{3.15}$$

and hence can be split as

$$\tilde{\boldsymbol{x}} = \begin{pmatrix} \tilde{x}_1 \\ x_\delta \end{pmatrix} \tag{3.16}$$

in which

$$x_\delta = \begin{pmatrix} \tilde{x}_2 - \tilde{x}_1 \\ \vdots \\ \tilde{x}_N - \tilde{x}_1 \end{pmatrix} \tag{3.17}$$

Then, it is easily checked that the system thus obtained has a block-triangular structure, of the following form:

$$\begin{aligned} \dot{\tilde{x}}_1 &= A\tilde{x}_1 - (\tilde{\mathcal{L}}_{12} \otimes BKC)x_\delta \\ \dot{x}_\delta &= \Big[(I_{N-1} \otimes A) - (\tilde{\mathcal{L}}_{22} \otimes BKC)\Big] x_\delta \end{aligned} \tag{3.18}$$

Lemma 4. *Consider the controlled system* (3.10) *and suppose that all the eigenvalues of the matrix*

$$(I_{N-1} \otimes A) - (\tilde{\mathcal{L}}_{22} \otimes BKC) \tag{3.19}$$

have negative real part. Then, for all $i = 2, ..., N$

$$\lim_{t\to\infty} [\tilde{x}_k(t) - \tilde{x}_1(t)] = 0 \tag{3.20}$$

Thus, the states $\tilde{x}_i$ *achieve consensus.*

3.2.1 Consensus in a Homogeneous Network: Design

We consider the case in which the agents are described by (3.1) with the additional assumption that $A \in \mathbb{R}^{n\times n}$, $B \in R^{n\times n}$. In this case the matrix it can be proven that the stability is related to the stability of the matrices

$$A - \lambda_i(\mathcal{L})KC, \quad i = 2, ..., N \tag{3.21}$$

We assume also that the pair (A, C) is observable. This being the case, let $\mu > 0$ such that $\lambda_2 \le \mu$, pick $a > 0$ and let P be the unique positive symmetric solution of the algebraic Riccati equation

$$AP + PA^\top - 2\mu PC^\top CP + aI = 0 \tag{3.22}$$

Set

$$K = PC^\top \tag{3.23}$$

To determine whether the resulting matrix

$$A_i = A - \lambda_i(\mathcal{L})PC^\top C \tag{3.24}$$

has all the eigenvalues with negative real part, it suffices to check that

$$x^*(PA_i^* + A_iP)x < 0, \quad \forall x \neq 0 \tag{3.25}$$

in which the superscript "*" denotes conjugate transpose. By construction we have

$$\begin{aligned} x^*(PA_i^* + A_iP)x &= x^*(PA^\top - \lambda_i^*(\mathcal{L})PC^\top CP + AP - \lambda_i(\mathcal{L})PC^\top CP)x \\ &= x^*(PA^\top + AP - 2\mathrm{Re}[\lambda_i(\mathcal{L})]PC^\top CP)x \\ &\le x^*(PA^\top + AP - 2\mu PC^\top CP)x = -a||x||^2 \end{aligned} \tag{3.26}$$

and hence it is concluded that the choice $K = PC^\top$ solves the problem.

Chapter 4

Synchronization in a network of systems connected via dynamical links using High-Gain observers

4.1 Settings of the problem

The goal of this chapter is to show a way to achieve consensus in a network composed by N agents and M links using an observer in order to get rid of the effects due to presence of the dynamical links (i.e. filters) between the agents.
An example is shown in the following picture where it can be seen an example of graph that depict a possible configuration of the network.

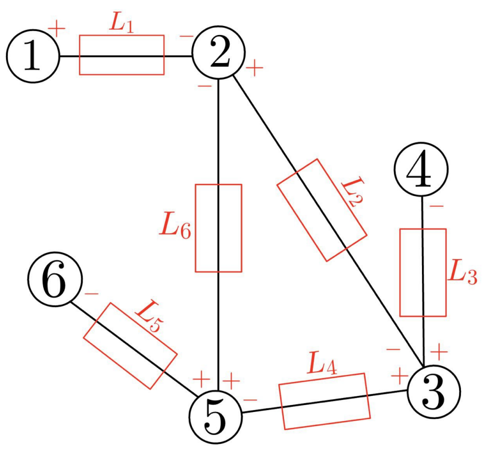


Figure 4.1: Example of a system with 6 agents and 6 links

In the previous image the signs '+' and '−' denote respectively the positive and the negative end of the link that is in between. The graph that represents the system can be described as a triplet $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ in which $\mathcal{V}$ represents the set of nodes, $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$ is a set of edges that describe the flow of information between the nodes and $\mathcal{A} \in \mathbb{R}^{N \times N}$ is the adjacency matrix of the graph describing the network of agents and links. A triplet of matrices (A_n, B_n, C_n) is in prime form, that is $A_n \in \mathbb{R}^{n \times n}$ is a shift matrix (all 1's in the upper diagonal and zero elsewhere), $B_n^\top = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathbb{R}^{n \times 1}$ and $C_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{1 \times n}$.

We define the matrix $D_n(g)$ as :

$$D_n(g) = \text{diag}(g, \dots, g^n)$$

In this context the incidence matrix $\mathcal{I} \in \mathbb{R}^{N \times E}$ with $E = |\mathcal{E}|$ is defined as:

$$\begin{aligned} \iota_{ij} &= 1, \text{ if node } i \text{ is the positive link of link } j \\ \iota_{ij} &= -1, \text{ if node } i \text{ is the negative link of link } j \end{aligned}$$

Since the graph is undirected we can define the Laplacian matrix $\mathcal{L}$ simply as:

$$\mathcal{L} = \mathcal{I}\mathcal{I}^\top$$

The dynamics of the N agents can be described as:

$$\begin{aligned} \dot{x}_j &= A_n x_j + B_n \varphi_x x_j + u_j \\ y_j &= C_n x_j \end{aligned} \tag{4.1}$$

for $j = 1, ..., N$, where $x_j \in \mathbb{R}^n, u_j \in \mathbb{R}^m, y_j \in \mathbb{R}$, A_n, B_n, C_n matrices in prime form and $\varphi_x \in \mathbb{R}^{1 \times n}$. In the following we consider the case $n = m$.
The M links are described by

$$\begin{aligned} \dot{w}_i &= A_l w_i + B_l \varphi_w w_i + B_l \textstyle\sum_{j=1}^N \iota_{ji} y_j \\ z_i &= C_l w_i \end{aligned} \tag{4.2}$$

for $i = 1, ..., M$ with $w_i \in \mathbb{R}^l, z_i \in \mathbb{R}$, (A_l, B_l, C_l) matrices in prime form and $\varphi_w \in \mathbb{R}^{1 \times l}$. For what regards the systems (4.1) and (4.2) we make the following assumptions:

Assumption 4.1. *The agents described by* (4.1) *are neutrally stable, i.e. all the eigenvalues of the matrix* $(A_n + B_n \varphi_x)$ *have real part less or equal to zero, and the ones that have real part equal to zero are simple roots of the characteristic polynomial of the matrix* A.

Assumption 4.2. *The dynamics of the links* (4.2) *is asymptotically stable, i.e. all the eigenvalues associated with the matrix* $(A_l + B_l \varphi_w)$ *have real part strictly less than 0.*

Assumption (4.1) imply that we can deal also with non-trivial (oscillating) asymptotic dynamics for what regards the agents, while assumption (4.2) is reasonable because we can model the link as low pass filters (for example a L-C filter) that can be seen as a model the transmission line.
Bearing in mind this scenario, each agent can only process the information coming from the links to which it is connected to. For the generic agent j we define the available information as:

$$\sigma_j = \sum_{i=1}^{M} \iota_{ji} z_i \tag{4.3}$$

Our goal is to design N (one for each agent) local dynamic regulators of the form:

$$\begin{aligned} \dot{\eta}_j &= f(\eta_j, \sigma_j) \\ u_j &= g(\eta_j) \end{aligned} \tag{4.4}$$

such that for all $i, j = 1, ..., N$ consensus between the agents is achieved, i.e.

$$\lim_{t\to\infty} y_i - y_j = 0$$

It is well known from the literature that if each agent can access the output of his neighbors, the input u_j of the agent j can be constructed as:

$$u_j^\star = K \sum_{i=1}^{N} \ell_{ji} y_i \tag{4.5}$$

with ℓ_{ji} element of the Laplacian matrix $\mathcal{L}$, an opportune design of the parameter K will guarantee consensus of the agents.
A similar problem is presented in [5] where in the dynamics of the link is included a feedthrough matrix D, representing direct relationship between the input u and the ouput y. In our context, we do not have direct access to the outputs of the agents, but we can rely only in a filtered version of them due to the presence of the dynamical links. The idea is similar to what is presented in [5] but is actually an extension of it because in this case we proceed finding a way to retrieve the information used in (4.5) through an observer and use it in the design of the controller (4.4).

4.2 Proposed solution

Define the scalar variable ξ_i associated with the i-th link and his derivative as follows:

$$\xi_i = \varphi_w w_i + \sum_{j=1}^{N} \iota_{ji} y_j \tag{4.6}$$

$$\dot{\xi}_i = \Lambda w_i + \gamma \sum_{j=1}^{N} i_{ji} y_j + \sum_{j=1}^{N} i_{ji} \dot{y}_j \tag{4.7}$$

where $\Lambda = \varphi_w A_l + \varphi_w B_l \varphi_w \in \mathbb{R}^{1\times l}$ and $\gamma = \varphi_w B_l \in \mathbb{R}$ are obtained computing the derivative of (4.6) and exploiting the expression (4.2).
Now we can compute the first $l+1$ derivatives of σ_j obtaining:

$$\begin{aligned}
\sigma_j &= \sum_{i=1}^{M} \iota_{ji} w_{i1} \\
\dot{\sigma}_j &= \sum_{i=1}^{M} \iota_{ji} w_{i2} \\
&\vdots \\
\sigma_j^{(l)} &= \sum_{i=1}^{M} \iota_{ji} \xi_i \\
\sigma_j^{(l+1)} &= \sum_{i=1}^{M} \iota_{ji} \dot{\xi}_i
\end{aligned}$$

defining $\dot{\sigma}_{j_i} = \sigma_j^{(i)}$ we can express the previous derivatives as:

$$\begin{aligned}
\dot{\sigma}_{j_1} &= \sigma_{j2} \\
&\vdots \\
\dot{\sigma}_{j_l} &= \sigma_{j_{l+1}} \\
\dot{\sigma}_j^{l+1} &= \sum_{i=1}^{M} \iota_{ji}\dot{\xi}_i
\end{aligned}$$

Expanding the expression of $\sigma_j^{(l)}$ we obtain:

$$\begin{aligned}
\sigma_j^{(l)} &= \sum_{i=1}^{M} \iota_{ji}\xi_i \\
&= \sum_{i=1}^{M} \iota_{ji} \left(\varphi_w w_i + \sum_{j=1}^{N} \iota_{ji} y_j \right) \\
&= \sum_{i=1}^{M} \iota_{ji}\varphi_w w_i + \sum_{i=1}^{N} \ell_{ji} y_i
\end{aligned} \tag{4.8}$$

Where in the last computation we exploited the fact that, since $\mathcal{L} = \mathcal{I}\mathcal{I}^\top$, we have for each agent $j = 1, ..., N$:

$$\sum_{i=1}^{M} \iota_{ji} \sum_{k=1}^{N} \iota_{ki} = \sum_{i=1}^{N} \ell_{ji}$$

The blue part in (4.8) is exactly the information we want to retrieve in order to build our controller. We can write:

$$\sigma_j^{(l)} - \sum_{i=1}^{M} \iota_{ji}\varphi_w w_i = \sum_{i=1}^{N} \ell_{ji} y_i \tag{4.9}$$

That can be rewritten in a compact way:

$$\sigma_{j_{l+1}} - \varphi_w \sigma_{j[1,l]} = \sum_{i=1}^{N} \ell_{ji} y_i \tag{4.10}$$

Where $\sigma_{j[1,l]} = \mathrm{col}(\sigma_{j1}, \ldots, \sigma_{jl})$. In order to write the dynamic of σ_j in a compact way, we can expand the term $\sigma_j^{(l+1)}$ obtaining:

$$\begin{aligned}
\sigma_j^{(l+1)} &= \sum_{i=1}^{M} \iota_{ji} \left[\Lambda w_i + \gamma \sum_{j=1}^{N} \iota_{ji} y_j + \sum_{j=1}^{N} \iota_{ji} \dot{y}_j \right] \\
&= \sum_{i=1}^{M} \iota_{ji} \left[\Lambda w_i + \gamma \sum_{j=1}^{N} \iota_{ji} y_j + \sum_{j=1}^{N} \iota_{ji} C_n (A_n x_j + u_j) \right]
\end{aligned}$$

Where in the last passage we exploited the expression of the agent dynamics (4.1).
Defining $\boldsymbol{\sigma}_j = \mathrm{col}(\sigma_{j_1}, \ldots, \sigma_{j_{l+1}})$ we can write:

$$\begin{aligned} \dot{\sigma}_j &= (A_{l+1} + \bar{\Lambda})\sigma_j + B_{l+1}\left[\sum_{i=1}^{N} \ell_{ji}\big(\gamma y_i + C_n(A_n x_i + u_j)\big)\right] \\ y_j &= C_{l+1}\sigma_j \end{aligned} \tag{4.11}$$

where $A_{l+1}, B_{l+1}, C_{l+1}$ are in prime form and $\bar{\Lambda}$ is defined as

$$\bar{\Lambda} = \begin{bmatrix} \mathbf{0}_{l\times l} & 0 \\ \Lambda & 0 \end{bmatrix} \in \mathbb{R}^{(l+1)\times(l+1)}$$

Hence we can define the control law (4.4) as an high-gain observer for the dynamic of $\boldsymbol{\sigma}_j$, namely:

$$\begin{aligned} \dot{\hat{\sigma}}_j &= (A_{l+1} + \bar{\Lambda})\hat{\sigma}_j + K_o(\sigma_j - C_{l+1}\hat{\sigma}_j) \\ u_j &= K_c\bar{\varphi}_w\hat{\sigma}_j \end{aligned} \tag{4.12}$$

for $j = 1, ..., N$ with

$$\bar{\varphi}_w = \begin{pmatrix} \varphi_w^\top \\ -1 \end{pmatrix}^\top \in \mathbb{R}^{1\times(l+1)} \tag{4.13}$$

$$\begin{aligned} K_o &= D_{l+1}(d)K_o{}' \in \mathbb{R}^{l+1}, \quad D_{l+1}(d) = \mathrm{diag}(d, ..., d^{l+1}), \quad K_o' = P_0^{-1}C_{l+1}^\top \\ K_c &= D_n(g)K_c' \in \mathbb{R}^n \qquad\quad D_n(g) = \mathrm{diag}(g, ..., g^n), \qquad K_c' = P_cC_n^\top \end{aligned} \tag{4.14}$$

with $d \geq 1$ and $g \geq 1$ to be chosen large enough, and P_0, P_c solutions to following Riccati equations (ARE):

$$P_0A_{l+1} + A_{l+1}^\top P_0 - 2C_{l+1}^\top C_{l+1} = -a_1 I, \quad a_1 > 0 \tag{4.15}$$

$$P_cA_n + A_n^\top P_c - 2\mu C_n^\top C_n = -a_2 I, \quad a_2 > 0 \tag{4.16}$$

where $\mu \leq \lambda_2(L)$.
Letting $\boldsymbol{x} = \mathrm{col}(x_1, ..., x_N)$, $\hat{\boldsymbol{\sigma}} = \mathrm{col}(\hat{\sigma}_1, ..., \hat{\sigma}_N)$, $\boldsymbol{u} = \mathrm{col}(u_1, ..., u_N)$, $\boldsymbol{w} = \mathrm{col}(w_1, ..., w_M)$, $\boldsymbol{\sigma} = \mathrm{col}(\sigma_1, ..., \sigma_N)$, exploiting the properties of Kronecker products and the information we obtained so far, we can write the equations that describe the whole network of agents, links and local regulators as:

$$\begin{aligned} \dot{\boldsymbol{x}} &= [I_N \otimes (A_n + B_n\varphi_x)]\,\boldsymbol{x} + \boldsymbol{u} \\ \dot{\hat{\boldsymbol{\sigma}}} &= \big[I_N \otimes (A_{l+1} + \bar{\Lambda})\big]\,\hat{\boldsymbol{\sigma}} + (I_N \otimes K_oC_{l+1})(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) \\ \boldsymbol{u} &= (I_N \otimes K_c\bar{\varphi}_w)\hat{\boldsymbol{\sigma}} \\ \dot{\boldsymbol{w}} &= [I_M \otimes (A_l + B_l\varphi_w)]\,\boldsymbol{w} + (\mathcal{I}^\top \otimes B_lC_n)\boldsymbol{x} \\ \boldsymbol{\sigma} &= (\mathcal{I} \otimes C_l)\boldsymbol{w} \end{aligned} \tag{4.17}$$

We can now enunciate the following theorem:

Theorem 13. *Consider the system described by* (4.17) *fulfilling assumptions* (4.1) *and* (4.2) *with control parameters K_o and K_c chosen according to* (4.14)*. There exist a $d^\star$ and a $g^\star(d^\star)$*

such that for all $d > d^\star$ and $g > g^\star$, the set:

$$\begin{aligned}\mathcal{S} = \{(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{\sigma}) \in\ & \mathbb{R}^{Nn} \times \mathbb{R}^{lM} \times \mathbb{R}^{(l+1)N} : \\ & x_1 = \dots = x_N, \\ & \quad w_1 = \dots = w_M = 0, \\ & \qquad \sigma_1 = \dots = \sigma_N = 0\}\end{aligned} \tag{4.18}$$

is globally exponentially stable.

Proof. Let's define the observer error $\boldsymbol{e}$ as follows:

$$\boldsymbol{e} = \boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}} \tag{4.19}$$

Computing the time derivative of $\boldsymbol{e}$ we obtain

$$\begin{aligned}\dot{\boldsymbol{e}} &= \left[I_N \otimes (A_{l+1} + \bar{\Lambda})\right] \boldsymbol{\sigma} + \left[\mathcal{L} \otimes B_{l+1}(\gamma C_n + C_n A_n)\right] \boldsymbol{x} + (\mathcal{L} \otimes B_{l+1} C_n)\boldsymbol{u} \\ &- \left[I_N \otimes (A_{l+1} + \bar{\Lambda})\right] \hat{\boldsymbol{\sigma}} - (I_N \otimes K_o C_{l+1})\boldsymbol{e} \\ &= \left[I_N \otimes (A_{l+1} + \bar{\Lambda} - K_o C_{l+1})\right] \boldsymbol{e} + \left[\mathcal{L} \otimes B_{l+1}(\gamma C_n + C_n A_n)\right] \boldsymbol{x} + (\mathcal{L} \otimes B_{l+1} C_n)\boldsymbol{u}\end{aligned} \tag{4.20}$$

Now consider the change of basis described by:

$$\boldsymbol{\varepsilon} = (I_N \otimes D_{l+1}^{-1}(d))\boldsymbol{e} \implies \boldsymbol{e} = (I_N \otimes D_{l+1}(d))\boldsymbol{\varepsilon} \tag{4.21}$$

with this change of coordinates we are rescaling the errors of a factor depending on $D_{l+1}^{-1}(d)$. Computing its time derivative we have:

$$\begin{aligned}\dot{\boldsymbol{\varepsilon}} &= (I_N \otimes D_{l+1}^{-1}(d))\dot{\boldsymbol{e}} \\ &= (I_N \otimes D_{l+1}^{-1}(d))\Big\{ \left[I_N \otimes (A_{l+1} + \bar{\Lambda} - K_o C_{l+1})\right] e \\ &+ \left[\mathcal{L} \otimes B_{l+1}(\gamma C_n + C_n A_n)\right] \boldsymbol{x} + (\mathcal{L} \otimes B_{l+1} C_n)\boldsymbol{u}\Big\}\end{aligned}$$

Exploiting (4.21) and the shape of B_{l+1} we obtain:

$$\begin{aligned}\dot{\boldsymbol{\varepsilon}} &= (I_N \otimes D_{l+1}^{-1}(d)) \left[I_N \otimes (A_{l+1} + \bar{\Lambda} - K_o C_{l+1})\right] (I_N \otimes D_{l+1}^{-1}(d))\boldsymbol{\varepsilon} + \frac{1}{d^{l+1}}\boldsymbol{\Psi}(\boldsymbol{x}, \boldsymbol{u}) \\ &= d\left[I_N \otimes (A_{l+1} - K_o' C_{l+1})\right] \boldsymbol{\varepsilon} + \frac{1}{d^{l+1}}\boldsymbol{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{x}, \boldsymbol{u})\end{aligned} \tag{4.22}$$

where the function $\boldsymbol{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{x}, \boldsymbol{u})$ is defined as:

$$\boldsymbol{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{x}, \boldsymbol{u}) = \left[\mathcal{L} \otimes B_{l+1}(\gamma C_n + C_n A_n)\right] \boldsymbol{x} + (\mathcal{L} \otimes B_{l+1} C_n)\boldsymbol{u} + (I_N \otimes \bar{\Lambda} D_{l+1}(d))\boldsymbol{\varepsilon} \tag{4.23}$$

Since we have chosen K_o accordingly to (4.14) we know that the matrix

$$H_o = (A_{l+1} - K_o' C_{l+1}) \tag{4.24}$$

is Hurwitz. In order to prove the observer convergence and the possibility to tune arbitrarily the error $\boldsymbol{e}$, consider the following quadratic Lyapunov function:

$$V_\varepsilon = \boldsymbol{\varepsilon}^\top (I_N \otimes P_o)\boldsymbol{\varepsilon} \tag{4.25}$$

Computing the time derivative of (4.25) along (4.22) we obtain:

$$\begin{aligned}\dot{V}_\varepsilon &= 2\boldsymbol{\varepsilon}^\top(I_N \otimes P_o)\dot{\boldsymbol{\varepsilon}} = 2\boldsymbol{\varepsilon}^\top(I_N \otimes P_o)\left[d(I_N \otimes P_o\boldsymbol{\varepsilon}) + \frac{1}{d^{l+1}}\Psi(\boldsymbol{\varepsilon}, \boldsymbol{x}, \boldsymbol{u})\right] \\ &= 2d\boldsymbol{\varepsilon}^\top(I_N \otimes P_o H_o)\boldsymbol{\varepsilon} + \frac{2\boldsymbol{\varepsilon}^\top}{d^{l+1}}\left[c_0\boldsymbol{x} + c_1\boldsymbol{u} + c_2\boldsymbol{\varepsilon}\right]\end{aligned} \tag{4.26}$$

With $c_0 = \mathcal{L} \otimes [P_o B_{l+1}(\gamma C_n + C_n A_n)]$, $c_1 = \mathcal{L} \otimes P_o B_{l+1} C_n$ and $c_2 = I_N \otimes P_o \bar{\Lambda} D_{l+1}(d)$. Exploiting the properties of matrix norms we can write:

$$\dot{V}_\varepsilon \leq -a_1 d|\boldsymbol{\varepsilon}|^2 + \frac{2}{d^{l+1}}\left(|c_0||\boldsymbol{x}||\boldsymbol{\varepsilon}| + |c_1||\boldsymbol{u}||\boldsymbol{\varepsilon}| + |c_2||\boldsymbol{\varepsilon}|^2\right)$$

Using the following results (Young inequalities):

$$\begin{aligned}|c_0||\boldsymbol{x}||\boldsymbol{\varepsilon}| &\leq \frac{|c_0||\boldsymbol{x}|^2}{2} + \frac{|c_0||\boldsymbol{\varepsilon}|^2}{2} \\ |c_1||\boldsymbol{u}||\boldsymbol{\varepsilon}| &\leq \frac{|c_1||\boldsymbol{u}|^2}{2} + \frac{|c_1||\boldsymbol{\varepsilon}|^2}{2}\end{aligned}$$

we can write:

$$-\left(a_1 d - \frac{1}{d^{l+1}}\left(|c_0| + |c_1| + |c_2|\right)\right)|\boldsymbol{\varepsilon}|^2 + \frac{1}{d^{l+1}}\left(|c_0||\boldsymbol{x}|^2 + |c_1||\boldsymbol{u}|^2\right) \tag{4.27}$$

Choosing

$$d > \sqrt[l+2]{\frac{1}{a_1}\left(|c_0| + |c_1| + |c_2|\right)} := d^\star$$

We obtain

$$\dot{V}_\varepsilon \leq -a_1 d|\boldsymbol{\varepsilon}|^2 + \frac{1}{d^{l+1}}\left(|c_0||\boldsymbol{x}|^2 + |c_1||\boldsymbol{u}|^2\right) \tag{4.28}$$

Since the dynamics of $\boldsymbol{\varepsilon}, \boldsymbol{x}, \boldsymbol{u}$ are bounded, we conclude that we can make the error $\boldsymbol{\varepsilon}$ (then $\boldsymbol{e}$) arbitrarily small by properly tune the value of d.
Bearing in mind this result we can now prove that agents actually achieve consensus. We can rewrite the input signal u as:

$$\boldsymbol{u} = (I_N \otimes K_c\bar{\varphi}_w)\hat{\boldsymbol{\sigma}} = (I_N \otimes K_c\bar{\varphi}_w)(\boldsymbol{\sigma} - \boldsymbol{e}) \tag{4.29}$$

That, exploiting the definition of σ as in (4.17) can be written as:

$$\boldsymbol{u} = (\mathcal{L} \otimes K_c C_n)\,\boldsymbol{x} - (I_N \otimes K_c\bar{\varphi}_w D_{l+1}(d))\,\boldsymbol{\varepsilon} \tag{4.30}$$

We can now get rid of the dependence on $\boldsymbol{u}$ from $\Psi(\boldsymbol{x}, \boldsymbol{\varepsilon}, \boldsymbol{u})$ writing the $\boldsymbol{x}$ and the $\boldsymbol{\varepsilon}$ dynamics as:

$$\begin{aligned}\dot{\boldsymbol{x}} &= [I_N \otimes (A_n + B_n\varphi_x) + \mathcal{L} \otimes K_c C_n]\,\boldsymbol{x} - (I_N \otimes K_c\bar{\varphi}_w D_{l+1}(d))\,\boldsymbol{\varepsilon} \\ \dot{\boldsymbol{\varepsilon}} &= d\,(I_N \otimes H_o)\,\boldsymbol{\varepsilon} + \frac{1}{d^{l+1}}\Psi(\boldsymbol{x}, \boldsymbol{\varepsilon})\end{aligned} \tag{4.31}$$

With $\Psi(\boldsymbol{x}, \boldsymbol{\varepsilon})$ as follows:

$$\begin{aligned}\Psi(\boldsymbol{x}, \boldsymbol{\varepsilon}) = {} & [\mathcal{L} \otimes B_{l+1}(\gamma C_n + C_n A_n)]\,\boldsymbol{x} + \left(\mathcal{L}^2 \otimes B_{l+1} C_n K_c C_n\right)\boldsymbol{x} \\ & - (\mathcal{L} \otimes B_{l+1} C_n K_c \bar{\varphi}_w D_{l+1}(d))\,\boldsymbol{\varepsilon} + \left(I_N \otimes \bar{\Lambda} D_{l+1}(d)\right)\boldsymbol{\varepsilon}\end{aligned}$$

Since the system is without leader, the consensus trajectory will depends only by the dynamics of the agents and by the initial conditions. In order to study the stability of such a system we have to split the whole dynamics in two parts: an independent one and an "error" one. To do this, let

$$T = \begin{bmatrix} 1 & \mathbf{0}_{1\times(N-1)} \\ \mathbf{1}_{(N-1)\times 1} & I_{N-1} \end{bmatrix} \tag{4.32}$$

Applying the similarity transformation at the Laplacian matrix $\mathcal{L}$ we have:

$$\tilde{\mathcal{L}} = T^{-1}\mathcal{L}T = \begin{bmatrix} 0 & \mathcal{L}_{12} \\ \mathbf{0} & \mathcal{L}_{22} \end{bmatrix} \tag{4.33}$$

With $\mathcal{L}_{12} \in \mathbb{R}^{1\times(N-1)}$ and $\mathcal{L}_{22} \in \mathbb{R}^{(N-1)\times(N-1)}$. Thanks to the structure of the matrix T we have that the non-zero eigenvalues of $\mathcal{L}$ are the $(N-1)$ eigenvalues of the matrix $\mathcal{L}_{22}$. This directly leads to the following change of coordinates:

$$\begin{aligned} \boldsymbol{x'} &= (T^{-1}\otimes I_n)\boldsymbol{x} \implies \boldsymbol{x'} = \begin{pmatrix} x_1 \\ x_2 - x_1 \\ \vdots \\ x_N - x_1 \end{pmatrix} \\ \boldsymbol{\zeta} &= (T^{-1}\otimes I_n)\boldsymbol{\varepsilon} \implies \boldsymbol{\zeta} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 - \varepsilon_1 \\ \vdots \\ \varepsilon_N - \varepsilon_1 \end{pmatrix} \end{aligned} \tag{4.34}$$

Computing the time derivatives of $\boldsymbol{x'}$ and $\boldsymbol{\zeta}$ we obtain:

$$\begin{aligned} \dot{\boldsymbol{x}}' &= (T^{-1}\otimes I_n)\dot{\boldsymbol{x}} = \left[I_N \otimes (A_n + B_n\varphi_x) + \tilde{\mathcal{L}} \otimes K_c C_n\right]\boldsymbol{x'} - (I_N \otimes K_c\bar{\varphi}_w D_{l+1}(d))\,\boldsymbol{\zeta} \\ \dot{\boldsymbol{\zeta}} &= (T^{-1}\otimes I_n)\dot{\boldsymbol{\varepsilon}} = d(I_N\otimes H_o)\boldsymbol{\zeta} + \frac{1}{d^{l+1}}\Psi(\boldsymbol{x}',\boldsymbol{\zeta}) \end{aligned} \tag{4.35}$$

Where

$$\begin{aligned} \Psi(\boldsymbol{x}',\boldsymbol{\zeta}) = &\left[\tilde{\mathcal{L}} \otimes B_{l+1}(\gamma C_n + C_n A_n)\right]\boldsymbol{x}' + (\tilde{\mathcal{L}}^2 \otimes B_{l+1}C_nK_cC_n)\boldsymbol{x}' \\ &- (\tilde{\mathcal{L}} \otimes B_{l+1}C_nK_c\bar{\varphi}_w D_{l+1}(d))\boldsymbol{\zeta} + (I_N \otimes \bar{\Lambda} D_{l+1}(d))\boldsymbol{\zeta} \end{aligned} \tag{4.36}$$

Now we can partition the vectors $\boldsymbol{x}'$ and $\boldsymbol{\zeta}$ as:

$$\boldsymbol{x}' = \begin{bmatrix} x_1' \\ \boldsymbol{x_2'} \end{bmatrix}, \quad \boldsymbol{\zeta} = \begin{bmatrix} \zeta_1 \\ \boldsymbol{\zeta_2} \end{bmatrix} \tag{4.37}$$

Computing the time derivatives we have:

$$\begin{aligned} \dot{x}_1' &= (A_n + B_n\varphi_x)x_1' + (\mathcal{L}_{12}\otimes K_cC_n)\boldsymbol{x_2'} - (K_c\bar{\varphi}_w D_{l+1}(d))\zeta_1 \\ \dot{\boldsymbol{x}}_{\mathbf{2}}' &= \left[I_{N-1}\otimes (A_n + B_n\varphi_x) + (\mathcal{L}_{22}\otimes K_cC_n)\right]\boldsymbol{x_2'} - (I_{N-1}\otimes K_c\bar{\varphi}_w D_{l+1}(d))\zeta_2 \end{aligned} \tag{4.38}$$

$$\begin{aligned} \dot{\zeta}_1 = &\, dH_o\zeta_1 + \frac{1}{d^{l+1}}\Big\{ \left[\mathcal{L}_{12}\otimes B_{l+1}(\gamma C_n + C_nA_n)\right]\boldsymbol{x_2'} + \left[\mathcal{L}_{12}^2 \otimes (B_{l+1}C_nK_cC_n)\right]\boldsymbol{x_2'} \\ &- \left[\mathcal{L}_{12}\otimes (B_{l+1}C_nK_c\bar{\varphi}_w D_{l+1}(d))\right]\boldsymbol{\zeta_2} + (I_{N-1}\otimes\bar{\Lambda}D_{l+1}(d))\boldsymbol{\zeta_2}\Big\} \\ \dot{\boldsymbol{\zeta}}_{\mathbf{2}} = &\, d(I_{N-1}\otimes H_0)\boldsymbol{\zeta_2} + \frac{1}{d^{l+1}}\Psi(\boldsymbol{\zeta_2},\boldsymbol{x_2'}) \end{aligned} \tag{4.39}$$

With

$$\Psi(\boldsymbol{\zeta_2}, \boldsymbol{x_2'}) = \left[\mathcal{L}_{22} \otimes B_{l+1}(\gamma C_n + C_n A_n)\right] \boldsymbol{x_2'} + \left[\mathcal{L}_{22}^2 \otimes (B_{l+1} C_n K_c C_n)\right] \boldsymbol{x_2'} \\ - \left[\mathcal{L}_{22} \otimes (B_{l+1} C_n K_c \bar{\varphi}_w D_{l+1}(d))\right] \boldsymbol{\zeta_2} + (I_{N-1} \otimes \bar{\Lambda} D_{l+1}(d)) \boldsymbol{\zeta_2} \tag{4.40}$$

We can see that after the partitioning we can conclude that if we prove the stability of $\boldsymbol{x_2'}$ and $\boldsymbol{\zeta_2}$ the stability of the whole system is achieved, since the remaining terms that are not dependent by $\boldsymbol{x_2'}, \boldsymbol{\zeta_2}$ and are, by hypothesis, bounded. Before continue with the Lyapunov analysis of the whole system, it is convenient to do a further change of basis, described by:

$$\boldsymbol{\chi} = \left(I_{N-1} \otimes D_n^{-1}(g)\right) \boldsymbol{x_2'} \tag{4.41}$$

Computing the time derivative of (4.41) we obtain:

$$\begin{aligned} \dot{\boldsymbol{\chi}} &= \left[I_{N-1} \otimes D_n^{-1}(g)(A_n + B_n \varphi_x) D_n(g) + \mathcal{L}_{22} \otimes D_n^{-1}(g) K_c C_n D_n(g)\right] \boldsymbol{x_2'} \\ &- \left(I_{N-1} \otimes D_n^{-1}(g) K_c \bar{\varphi}_w D_{l+1}(d)\right) \boldsymbol{\zeta_2} \\ &= g\left[I_{N-1} \otimes A_n + \mathcal{L}_{22} \otimes K_c' C_n\right] \boldsymbol{\chi} + \frac{1}{g^n} \left(I_{N-1} \otimes B_n \varphi_x D_n(g)\right) \boldsymbol{\chi} \\ &- \left(I_{N-1} \otimes D_n^{-1}(g) K_c \bar{\varphi}_w D_{l+1}(d)\right) \boldsymbol{\zeta_2} \end{aligned} \tag{4.42}$$

The dynamics of $\boldsymbol{\zeta_2}$ becomes, exploiting (4.42):

$$\dot{\boldsymbol{\zeta_2}} = d(I_{N-1} \otimes H_o)\boldsymbol{\zeta_2} + \frac{1}{d^{l+1}} \Psi(\boldsymbol{\zeta_2}, \boldsymbol{\chi}) \tag{4.43}$$

With

$$\Psi(\boldsymbol{\zeta_2}, \boldsymbol{\chi}) = \left[\mathcal{L}_{22} \otimes B_{l+1}(\gamma C_n + C_n A_n) D_n(g)\right] \boldsymbol{\chi} + \left[\mathcal{L}_{22}^2 \otimes (B_{l+1} C_n K_c C_n D_n(g))\right] \boldsymbol{\chi} \\ - \left[\mathcal{L}_{22} \otimes (B_{l+1} C_n K_c \bar{\varphi}_w D_{l+1}(d))\right] \boldsymbol{\zeta_2} + (I_{N-1} \otimes \bar{\Lambda} D_{l+1}(d)) \boldsymbol{\zeta_2} \tag{4.44}$$

For what regards the link dynamics, it's possible to reformulate it as:

$$\begin{aligned} \dot{\boldsymbol{w}} &= \left[I_M \otimes (A_l + B_l \varphi_w)\right] \boldsymbol{w} + (\mathcal{I}^\top \otimes B_l C_n) \boldsymbol{x} \\ &= \left[I_M \otimes (A_l + B_l \varphi_w)\right] \boldsymbol{w} + (\mathcal{I}^\top \otimes B_l C_n)(T \otimes I_n) \boldsymbol{x'} \\ &= \left[I_M \otimes (A_l + B_l \varphi_w)\right] \boldsymbol{w} + (\mathcal{I}^\top T \otimes B_l C_n) \boldsymbol{x'} \end{aligned} \tag{4.45}$$

Due to the structure of T it turns out that:

$$\mathcal{I}^\top T = \left[\ \mathbf{0} \mid \mathcal{I}_2\ \right] \tag{4.46}$$

i.e., the first column is always zero. Finally the $\boldsymbol{w}$ dynamics can be written as:

$$\dot{\boldsymbol{w}} = \left[I_M \otimes (A_l + B_l \varphi_w)\right] \boldsymbol{w} + (\mathcal{I}_2^\top T \otimes B_l C_n D_n(g)) \boldsymbol{\chi} \tag{4.47}$$

We can now prove the stability of the whole system studying the interconnection of (4.42), (4.43), (4.47). Consider the following quadratic Lyapunov function:

$$V_T = \frac{1}{2} \boldsymbol{\chi}^\top (I_{N-1} \otimes P_c) \boldsymbol{\chi} + \frac{1}{2} b \boldsymbol{w}^\top (I_M \otimes Q) \boldsymbol{w} + \frac{1}{2} \boldsymbol{\zeta_2}^\top (I_{N-1} \otimes P_o) \boldsymbol{\zeta_2}, \tag{4.48}$$

with some $b > 0$ to be selected and where $Q = Q^\top \succ 0$ is a matrix satisfying

$$(A_l + B_l \varphi_w)^\top Q + Q(A_l + B_l \varphi_w) = -I_m \tag{4.49}$$

Such a matrix exists since we made the assumption (4.2). Computing its time derivative along (4.42), (4.43), (4.47), we have:

$$\begin{aligned}\dot{V}_T &= \boldsymbol{\chi}^\top (I_{N-1}\otimes P_c)\dot{\boldsymbol{\chi}} + b\boldsymbol{w}^\top (I_M\otimes Q)\dot{\boldsymbol{w}} + \boldsymbol{\zeta_2}^\top (I_{N-1}\otimes P_o)\dot{\boldsymbol{\zeta}}_{\mathbf{2}}\\ &= -a_2 g|\boldsymbol{\chi}|^2 + \frac{1}{g^n}\boldsymbol{\chi}^\top \left(I_{N-1}\otimes P_c B_n \varphi_x D_n(g)\right)\boldsymbol{\chi} - \boldsymbol{\chi}^\top \left(I_{N-1}\otimes P_c D_n^{-1}(g) K_c \bar{\varphi}_w D_{l+1}(d)\right)\boldsymbol{\zeta_2}\\ &\quad - b|\boldsymbol{w}|^2 + \boldsymbol{w}^\top (\mathcal{I}_2\otimes QB_l C_n D_n(g))\boldsymbol{\chi} - a_1 d|\boldsymbol{\zeta_2}|^2 + \boldsymbol{\zeta_2}^\top\left[\mathcal{L}_{22}\otimes P_o B_{l+1}(\gamma C_n + C_n A_n) D_n(g)\right]\boldsymbol{\chi}\\ &\quad + \boldsymbol{\zeta_2}^\top\left[\mathcal{L}_{22}^2\otimes P_o B_{l+1} C_n K_c C_n D_n(g)\right]\boldsymbol{\chi} - \boldsymbol{\zeta_2}^\top\left[\mathcal{L}_{22}\otimes (P_o B_{l+1} C_n K_c \bar{\varphi}_w D_{l+1}(d))\right]\boldsymbol{\zeta_2}\\ &\quad + \boldsymbol{\zeta_2}^\top (I_{N-1}\otimes \bar{\Lambda} D_{l+1}(d))\boldsymbol{\zeta_2}\end{aligned} \tag{4.50}$$

Setting:

$$\begin{aligned}&k_0 = \frac{1}{g^n} P_c B_n \varphi_x D_n(g), \quad k_1 = P_c D_n^{-1}(g) K_c \bar{\varphi}_w D_{l+1}(d)\\ &k_2 = \mathcal{I}_2\otimes QB_l C_n D_n(g), \quad k_3 = \mathcal{L}_{22}\otimes P_o B_{l+1}(\gamma C_n + C_n A_n) D_n(g)\\ &k_4 = \mathcal{L}_{22}^2\otimes P_o B_{l+1} C_n K_c C_n D_n(g), \quad k_5 = \mathcal{L}_{22}\otimes (P_o B_{l+1} C_n K_c \bar{\varphi}_w D_{l+1}(d))\\ &k_6 = \bar{\Lambda} D_{l+1}(d)\end{aligned} \tag{4.51}$$

Hence we can bound (4.50) as:

$$\begin{aligned}\dot{V}_T \leq &-a_2 g|\boldsymbol{\chi}|^2 - b|\boldsymbol{w}|^2 - a_1 d|\boldsymbol{\zeta_2}|^2\\ &+ |k_0||\boldsymbol{\chi}|^2 + |k_1||\boldsymbol{\chi}||\boldsymbol{\zeta_2}| + |k_2||\boldsymbol{w}||\boldsymbol{\chi}|\\ &+ |k_3||\boldsymbol{\zeta_2}||\boldsymbol{\chi}| + |k_4||\boldsymbol{\zeta_2}||\boldsymbol{\chi}| + (|k_5| + |k_6|)|\boldsymbol{\zeta_2}|^2\end{aligned} \tag{4.52}$$

The cross-terms can be bounded using Young inequalities:

$$\begin{aligned}|k_1||\boldsymbol{\chi}||\boldsymbol{\zeta_2}| &\leq \frac{|k_1|^2|\boldsymbol{\zeta_2}|^2}{2} + \frac{|\boldsymbol{\chi}|^2}{2}\\ |k_2||\boldsymbol{w}||\boldsymbol{\chi}| &\leq \frac{|k_2|^2|\boldsymbol{w}|^2}{2} + \frac{|\boldsymbol{\chi}|^2}{2}\\ |k_3||\boldsymbol{\chi}||\boldsymbol{\zeta_2}| &\leq \frac{|k_3|^2|\boldsymbol{\zeta_2}|^2}{2} + \frac{|\boldsymbol{\chi}|^2}{2}\\ |k_4||\boldsymbol{\chi}||\boldsymbol{\zeta_2}| &\leq \frac{|k_4|^2|\boldsymbol{\zeta_2}|^2}{2} + \frac{|\boldsymbol{\chi}|^2}{2}\end{aligned} \tag{4.53}$$

Obtaining

$$\begin{aligned}\dot{V}_T \leq &-\left(a_2 g - |k_0| - 2\right)|\boldsymbol{\chi}|^2 - \left(b - \frac{|k_2|^2}{2}\right)|\boldsymbol{w}|^2\\ &-\left[a_1 d - |k_5| - |k_6| - \frac{1}{2}\left(|k_1|^2 + |k_3|^2 + |k_4|^2\right)\right]|\boldsymbol{\zeta_2}|^2\end{aligned} \tag{4.54}$$

Thus, by setting

$$g > \frac{|k_0| + 2}{a_2}, \quad b > \frac{|k_2|^2}{2}, \quad d > \frac{1}{a_1}\left[|k_5| + |k_6| + \frac{1}{2}\left(|k_1|^2 + |k_3|^2 + |k_4|^2\right)\right] \tag{4.55}$$

(4.54) is negative definite, then the dynamics of $(\boldsymbol{\chi}, \boldsymbol{w}, \boldsymbol{\zeta_2})$ is asymptotically stable. To prove that the closed loop defined by (4.17) is globally asymptotically stable we can notice that the dynamics of ζ_1 is asymptotically stable since it depends only on asymptotically stable terms. This implies that also the dynamics of x_1' is stable. Finally, since $\boldsymbol{\chi}$ and $\boldsymbol{\zeta}$ are obtained by linear transformations from $\boldsymbol{x}, \boldsymbol{\varepsilon}$, we can conclude that the whole system (4.17) is globally asymptotically stable [11, pp. 31–36]. □

4.3 Further extensions

4.3.1 SISO case

As an extension of the previous problem, we can consider agents with the following dynamics:

$$\begin{aligned} \dot{x}_j &= Ax_j + Bu_j \\ y_j &= Cx_j \end{aligned} \tag{4.56}$$

in which the matrices (A, B, C) are no more in prime form and they describe a SISO system, i.e. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{n\times 1}$. Keeping the same settings of the former problem, we want to reconstruct the quantity (4.5). It turns out that the closed loop equations now become:

$$\begin{aligned} \dot{\boldsymbol{x}} &= (I_N \otimes A)\,\boldsymbol{x} + (I_N \otimes B)\,\boldsymbol{u} \\ \dot{\hat{\boldsymbol{\sigma}}} &= \left[I_N \otimes (A_{l+1} + \bar{\Lambda})\right]\hat{\boldsymbol{\sigma}} + (I_N \otimes K_o C_{l+1})(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) \\ \boldsymbol{u} &= (I_N \otimes K_c \bar{\varphi}_w)\hat{\boldsymbol{\sigma}} \\ \dot{\boldsymbol{w}} &= \left[I_M \otimes (A_l + B_l \varphi_w)\right]\boldsymbol{w} + (\mathcal{I}^\top \otimes B_l C)\boldsymbol{x} \\ \boldsymbol{\sigma} &= (\mathcal{I} \otimes C_l)\boldsymbol{w} \end{aligned} \tag{4.57}$$

where K_o is computed as before and K_c is obtained through the following set of equations (passivity condition):

$$A^\top P_c + P_c A - 2\mu C^\top R C = -I; \quad a_3 > 0; \quad P_c B = C^\top; \quad K_c = R > 1 \tag{4.58}$$

Proof. Following the same procedure as before we obtain:

$$\begin{aligned} \dot{\boldsymbol{e}} &= \left[I_N \otimes (A_{l+1} + \bar{\Lambda} - K_o C_{l+1})\right]\boldsymbol{e} + \left[\mathcal{L} \otimes B_{l+1}(\gamma C + CA)\right]x + (\mathcal{L} \otimes B_{l+1} CB)\,\boldsymbol{u} \\ \boldsymbol{\varepsilon} &= \left(I_N \otimes D_{l+1}^{-1}(d)\right)\boldsymbol{e} \\ \dot{\boldsymbol{\varepsilon}} &= d\left[I_N \otimes (A_{l+1} - K_o' C_{l+1})\right]\boldsymbol{\varepsilon} + \frac{1}{d^{l+1}}\boldsymbol{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{x}, \boldsymbol{u}) \end{aligned} \tag{4.59}$$

with $\boldsymbol{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{x}, \boldsymbol{u})$ defined as:

$$\boldsymbol{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{x}, \boldsymbol{u}) = \left[\mathcal{L} \otimes B_{l+1}(\gamma C + CA)\right]\boldsymbol{x} + (\mathcal{L} \otimes B_{l+1} CB)\,\boldsymbol{u} + \left[I_N \otimes \bar{\Lambda} D_{l+1}(d)\right]\boldsymbol{\varepsilon}$$

Now, defining:

$$\boldsymbol{u} = (\mathcal{L} \otimes BK_c C)\,\boldsymbol{x} - (I_N \otimes K_c \bar{\varphi}_w D_{l+1}(d))\,\boldsymbol{\varepsilon} \tag{4.60}$$

we can rewrite the agent dynamics and (4.59) as:

$$\begin{aligned} \dot{\boldsymbol{x}} &= (I_N \otimes A + \mathcal{L} \otimes BK_c C)\,\boldsymbol{x} - (I_N \otimes BK_c \bar{\varphi}_w D_{l+1}(d))\,\boldsymbol{\varepsilon} \\ \dot{\boldsymbol{\varepsilon}} &= d\,(I_N \otimes H_o)\,\boldsymbol{\varepsilon} + \frac{1}{d^{l+1}}\boldsymbol{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{x}) \end{aligned} \tag{4.61}$$

Where $H_o = (A_{l+1} - K_o' C_{l+1})$ is an Hurwitz matrix and:

$$\begin{aligned}\boldsymbol{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{x}) = &[\mathcal{L} \otimes B_{l+1}(\gamma C + CA)] + \left(\mathcal{L}^2 \otimes B_{l+1}CBK_cC\right)\boldsymbol{x} \\ &- [\mathcal{L} \otimes B_{l+1}CBK_c\bar{\varphi}_w D_{l+1}(d)]\,\boldsymbol{\varepsilon} + \left(I_N \otimes \bar{\Lambda} D_{l+1}(d)\right)\boldsymbol{\varepsilon}\end{aligned}$$

Scaling the agent and the error dynamics through the matrix T, defined in (4.32) and defining $\tilde{\mathcal{L}} = T^{-1}\mathcal{L}T$, we have:

$$\begin{aligned}\boldsymbol{\chi} = (T^{-1} \otimes I_n)\boldsymbol{x} \implies \boldsymbol{x} = (T \otimes I_n)\boldsymbol{\chi} \\ \boldsymbol{\zeta} = (T^{-1} \otimes I_n)\boldsymbol{\varepsilon} \implies \boldsymbol{\varepsilon} = (T \otimes I_n)\boldsymbol{\zeta}\end{aligned}$$

Computing the derivative we get:

$$\begin{aligned}\dot{\boldsymbol{\chi}} &= \left(I_N \otimes A + \tilde{\mathcal{L}} \otimes BK_cC\right)\boldsymbol{\chi} - \left[\tilde{\mathcal{L}} \otimes B_{l+1}CBK_c\bar{\varphi}_w D_{l+1}(d) + I_N \otimes \bar{\Lambda} D_{l+1}(d)\right]\boldsymbol{\zeta} \\ \dot{\boldsymbol{\zeta}} &= d(I_N \otimes H_o)\boldsymbol{\zeta} + \frac{1}{d^{l+1}}\boldsymbol{\Psi}(\boldsymbol{\chi}, \boldsymbol{\zeta})\end{aligned} \tag{4.62}$$

With $\boldsymbol{\Psi}(\boldsymbol{\chi}, \boldsymbol{\zeta})$ as follows:

$$\begin{aligned}\boldsymbol{\Psi}(\boldsymbol{\chi}, \boldsymbol{\zeta}) = &\left[\tilde{\mathcal{L}} \otimes B_{l+1}(\gamma C + CA)\right] + \left(\tilde{\mathcal{L}}^2 \otimes B_{l+1}CBK_cC\right)\boldsymbol{x} \\ &- \left[\tilde{\mathcal{L}} \otimes B_{l+1}CBK_c\bar{\varphi}_w D_{l+1}(d)\right]\boldsymbol{\varepsilon} + \left(I_N \otimes \bar{\Lambda} D_{l+1}(d)\right)\boldsymbol{\varepsilon}\end{aligned}$$

As before we can split the scaled dynamics of the agents and of the observer error in two parts:

$$\dot{\boldsymbol{\chi}} = \begin{bmatrix}\dot{\chi}_1 \\ \dot{\boldsymbol{\chi}}_2\end{bmatrix} \qquad \dot{\boldsymbol{\zeta}} = \begin{bmatrix}\dot{\zeta}_1 \\ \dot{\boldsymbol{\zeta}}_2\end{bmatrix} \tag{4.63}$$

Making the computation we obtain, for the agent dynamics:

$$\begin{aligned}\dot{\chi}_1 &= A\chi_1 + (\mathcal{L}_{12} \otimes BK_cC)\,\boldsymbol{\chi}_2 - (BK_c\bar{\varphi}_w D_{l+1}(d))\,\zeta_1 \\ \dot{\boldsymbol{\chi}}_2 &= [I_{N-1} \otimes A + (\mathcal{L}_{22} \otimes BK_cC_n)]\,\boldsymbol{\chi}_2 - (I_{N-1} \otimes BK_c\bar{\varphi}_w D_{l+1}(d))\boldsymbol{\zeta}_2\end{aligned} \tag{4.64}$$

and for what regard the error dynamics:

$$\begin{aligned}\dot{\zeta}_1 = &\, dH_o\zeta_1 + \frac{1}{d^{l+1}}\big\{\,[\mathcal{L}_{12} \otimes B_{l+1}(\gamma C + CA)]\,\boldsymbol{\chi}_2 + \left[\mathcal{L}_{12}^2 \otimes (B_{l+1}CBK_cC)\right]\boldsymbol{\chi}_2 \\ &- [\mathcal{L}_{12} \otimes (B_{l+1}CBK_c\bar{\varphi}_w D_{l+1}(d))]\,\boldsymbol{\zeta}_2 + (I_{N-1} \otimes \bar{\Lambda} D_{l+1}(d))\boldsymbol{\zeta}_2\big\} \\ \dot{\boldsymbol{\zeta}}_2 = &\, d(I_{N-1} \otimes H_0)\boldsymbol{\zeta}_2 + \frac{1}{d^{l+1}}\boldsymbol{\Psi}(\boldsymbol{\zeta}_2, \boldsymbol{\chi}_2)\end{aligned} \tag{4.65}$$

With

$$\begin{aligned}\boldsymbol{\Psi}(\boldsymbol{\zeta}_2, \boldsymbol{\chi}_2) = &[\mathcal{L}_{22} \otimes B_{l+1}(\gamma C + CA)]\,\boldsymbol{\chi}_2 + \left[\mathcal{L}_{22}^2 \otimes (B_{l+1}CBK_cC)\right]\boldsymbol{\chi}_2 \\ &- [\mathcal{L}_{22} \otimes (B_{l+1}CBK_c\bar{\varphi}_w D_{l+1}(d))]\,\boldsymbol{\zeta}_2 + (I_{N-1} \otimes \bar{\Lambda} D_{l+1}(d))\boldsymbol{\zeta}_2\end{aligned}$$

As a next step consider the follow quadratic Lyapunov function:

$$V_T = \frac{1}{2}\alpha\boldsymbol{\chi}_2^\top(I_{N-1} \otimes P_c)\boldsymbol{\chi}_2 + \frac{1}{2}\beta\boldsymbol{w}^\top(I_M \otimes Q)\boldsymbol{w} + \frac{1}{2}\boldsymbol{\zeta}_2^\top(I_{N-1} \otimes P_o)\boldsymbol{\zeta}_2 \quad \alpha, \beta > 0 \tag{4.66}$$

Computing its time derivative we obtain:

$$
\begin{aligned}
\dot{V}_T = & -\alpha|\boldsymbol{\chi}_2|^2 + \alpha\boldsymbol{\chi}_2^\top \left(I_{N-1} \otimes P_c B K_c \bar{\varphi}_w D_{l+1}(d)\right)\boldsymbol{\zeta}_2 - \beta|\boldsymbol{w}|^2 + \beta\boldsymbol{w}^\top \left(\mathcal{I}_2 \otimes QC\right)\boldsymbol{\chi}_2 - \\
& a_1 d|\boldsymbol{\zeta}_2|^2 + \frac{1}{d^{l+1}}\boldsymbol{\zeta}_2^\top \left[\mathcal{L}_{22} \otimes P_o B_{l+1}(\gamma C + CA)\right]\boldsymbol{\chi}_2 + \frac{1}{d^{l+1}}\boldsymbol{\zeta}_2^\top \left(\mathcal{L}_{22} \otimes P_o B_{l+1} C B K_c C\right)\boldsymbol{\chi}_2 - \\
& \frac{1}{d^{l+1}}\boldsymbol{\zeta}_2^\top \left(\mathcal{L}_{22}^2 \otimes P_o B_{l+1} C B K_c \bar{\varphi}_w D_{l+1}(d)\right)\boldsymbol{\zeta}_2 + \frac{1}{d^{l+1}}\boldsymbol{\zeta}_2^\top \left(I_{N-1} \otimes P_o \bar{\Lambda} D_{l+1}(d)\right)\boldsymbol{\zeta}_2
\end{aligned}
\tag{4.67}
$$

Where Q and $\mathcal{I}_2$ are defined in (4.49) and (4.46). Grouping the similar terms and exploiting the fact that $|A||B| \geq A \cdot B$ for any vector or matrix, we can obtain a bound for (4.67):

$$
\begin{aligned}
\dot{V}_T \leq & -\alpha|\boldsymbol{\chi}_2|^2 - \beta|\boldsymbol{w}|^2 - a_1 d|\boldsymbol{\zeta}_2|^2 + \alpha|k_0||\boldsymbol{\chi}_2||\boldsymbol{\zeta}_2| + \\
& \beta|k_1||\boldsymbol{w}||\boldsymbol{\chi}_2| + (|k_2| + |k_3|)|\boldsymbol{\zeta}_2||\boldsymbol{\chi}_2| + (|k_4| + |k_5|)|\boldsymbol{\zeta}_2|^2
\end{aligned}
\tag{4.68}
$$

Where:

$$
\begin{aligned}
& k_0 = P_c B K_c \varphi_x D_{l+1}(d), \quad k_1 = \mathcal{I}_2 \otimes QC \\
& k_2 = \frac{1}{d^{l+1}}\left[\mathcal{L}_{22} \otimes P_o B_{l+1}(\gamma C + CA)\right], \quad k_3 = \frac{1}{d^{l+1}}\left[\mathcal{L}_{22} \otimes P_o B_{l+1} C B K_c C\right] \\
& k_4 = \mathcal{L}_{22}^2 \otimes P_o B_{l+1} C B K_c \bar{\varphi}_w D_{l+1}(d), \quad k_5 = P_o \bar{\Lambda} D_{l+1}(d)
\end{aligned}
\tag{4.69}
$$

Using Young inequalities we finally obtain:

$$
\begin{aligned}
\dot{V}_T \leq & -\alpha|\boldsymbol{\chi}_2|^2 + \frac{\alpha}{2}|\boldsymbol{\chi}_2|^2 + |\boldsymbol{\chi}_2|^2 + \frac{\beta}{2}|k_1|^2|\boldsymbol{\chi}_2|^2 - \beta|\boldsymbol{w}|^2 + \frac{\beta}{2}|\boldsymbol{w}|^2 \\
& - a_1 d|\boldsymbol{\zeta}_2|^2 + \frac{\alpha}{2}|k_0|^2|\boldsymbol{\zeta}_2|^2 + \frac{1}{2}\left(|k_2|^2 + |k_3|^2 + 2|k_4| + 2|k_5|\right)|\boldsymbol{\zeta}_2|^2 \\
= & -\left(\frac{\alpha}{2} - \frac{\beta}{2}|k_1|^2 - 1\right)|\boldsymbol{\chi}_2|^2 - \frac{\beta}{2}|\boldsymbol{w}|^2 + \\
& -\left(a_1 d - \frac{\alpha}{2}|k_0|^2 - \frac{1}{2}\left(|k_2|^2 + |k_3|^2\right) - |k_4|^2 - |k_5|^2\right)|\boldsymbol{\zeta}_2|^2
\end{aligned}
\tag{4.70}
$$

So as before we can achieve stability of the system by choosing:

$$
\alpha > 2 + \beta|k_1|^2, \qquad d > \frac{1}{a_1}\left[\frac{\alpha}{2}|k_0|^2 + \frac{1}{2}\left(|k_2|^2 + |k_3|^2\right) + |k_4|^2 + |k_5|^2\right]
$$

□

4.3.2 Adding a reference

As an extension of the previous problem, we can consider agents with the following dynamics:

$$
\begin{aligned}
\dot{x}_j &= A_n x_j + B_n \varphi_x x_j + u_j \\
y_j &= C x_j - r(t)
\end{aligned}
\tag{4.71}
$$

where $r(t)$ is a reference.

In this case we add the following assumption:

Assumption 4.3. *The matrix* $(A_n + B_n\varphi_x)$ *is Hurwitz, i.e. all the eigenvalues have real part strictly less than 0.*

The aim of this part is to show that the system is able also to track a reference, in our case the signal $r(t)$. Following the same idea of the previous proof we can show that the multi-agent system in which the agents are described by (4.71) can be seen as the system described by (4.17) because during the analysis the term $r(t)$ disappears, in particular when we compute the $\boldsymbol{\sigma}$ and the $\boldsymbol{w}$ dynamics. In the end we can conclude that also this system is globally asymptotically stable and that is able to track a reference described by the signal $r(t)$. Further analysis will be done in the following simulations.

4.3.3 Nonlinear case

In this section we want to develop the theory regarding the case in which the N agents can be written as:

$$\begin{aligned} \dot{x}_j &= A_n x_j + B_n\phi(x_j) + u_j \\ y_j &= C x_j \end{aligned} \qquad \forall j = 1, .., N \tag{4.72}$$

Where $x_j \in \mathbb{R}^n$, $u_j \in \mathbb{R}^n$, $y_j \in \mathbb{R}$ and the matrices (A_n, B_n, C_n) are in normal/prime form.

Assumption 4.4. *The function* $\phi(x)$ *is globally Lipschitz, namely there exists a* $\bar{\phi} > 0$ *such that*

$$||\phi(x_1) - \phi(x_2)|| \leq \bar{\phi}||x_1 - x_2|| \qquad \forall x_1, x_2 \in \mathbb{R}^n \tag{4.73}$$

Remark 2. *It can be noticed that the structure of* (4.72) *is equivalent to a requirement of observability for a nonlinear system described by:*

$$\begin{aligned} \dot{x} &= a(x) + u \\ y &= h(x) \end{aligned} \tag{4.74}$$

Indeed, if the system is globally observable, there exists a mapping $\Psi : \mathbb{R}^n \mapsto \mathbb{R}^n$*, defined as:*

$$\Psi(x_k) = \begin{pmatrix} h(x_k) \\ L_f h(x_k) \\ \vdots \\ L_f^{n-1} h(x_k) \end{pmatrix} \tag{4.75}$$

Where with $L_f h(x)$ *we are referring to the Lie derivative of* h *with respect to* f*. This map is the link between* (4.74) *and* (4.72)*.*

The global Lipschitz assumption, that is a very strict assumption in some cases, is motivated by the fact that the result we are aiming to obtain is a global result, i.e. we allow any initial condition for the agents, without restrict them in a compact set. If the initial state of the agents ranges in a fixed set we can weak the assumption and ask the function $\Phi(x)$ to be only *locally* Lipschitz.
Furthermore, it is supposed that the system (4.72) is ISS (input-to-state stable) with respect to the input u_j relative to a compact set, as is stated in the next assumption.

Assumption 4.5. *There exists a compact set $X \subset \mathbb{R}^n$ such that the system*

$$\dot{x}_j = A_n x_j + B_n \phi(x_j) + u_j \qquad \forall j = 1, ..., N \tag{4.76}$$

is ISS stable with respect to u_j relative to X, namely there exist a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class-$\mathcal{K}$ function γ such that [12, pp 174–181]

$$||x_j(t, \bar{x}_j)||_X \leq \beta(||x_j||_X, t) + \gamma(\sup_{\tau \in [0,t]} ||u_j(\tau)||) \tag{4.77}$$

This condition can be equivalently written as:

$$||x_j(t, \bar{x}_j)||_X \leq \bar{\beta}(||x_j||_X, t) + \bar{\gamma}(\sup_{\tau \in [0,t]} ||u_j(\tau)||) \tag{4.78}$$

Where usually $\bar{\beta}(\cdot,\cdot) \neq \beta(\cdot,\cdot)$ and $\bar{\gamma}(\cdot) \neq \gamma(\cdot)$

As in the previous cases, the M links are described by the SISO LTI dynamics:

$$\begin{aligned} \dot{w}_i &= A_l w_i + B_l \left(\varphi_w w_i + \sum_{j=1}^{N} \iota_{ji} y_{ji} \right) \\ z &= C_l w_i \end{aligned} \tag{4.79}$$

for $i = 1, ..., M$ with $w_i \in \mathbb{R}^l, z_i \in \mathbb{R}$, (A_l, B_l, C_l) matrices in prime form and $\varphi_w \in \mathbb{R}^{1 \times l}$. We make the assumptions (4.1) and (4.2) regarding the dynamics of the links.

Following the same steps of the proof of the linear case, we define the scalar variables ξ_i and σ_j as:

$$\begin{aligned} \xi_i &= \varphi_w w_i + \sum_{j=1}^{N} \iota_{ji} y_j \\ \sigma_j &= \sum_{i=1}^{M} \iota_{ji} z_i \end{aligned} \tag{4.80}$$

And computing the derivatives of σ_j we end up with the following equations:

$$\begin{aligned} \dot{\sigma}_j &= (A_{l+1} + \bar{\Lambda})\sigma_j + B_{l+1} \left[\sum_{i=1}^{N} \ell_{ji} \big(\gamma y_i + C_n (A_n x_i + u_j \big) \right] \\ y_j &= C_{l+1} \sigma_j \end{aligned} \tag{4.81}$$

where $A_{l+1}, B_{l+1}, C_{l+1}$ are in prime form and $\bar{\Lambda}$ and γ are defined in the same manner as in (4.7) and (4.11).
We can define the observer-controller law as:

$$\begin{aligned} \dot{\hat{\sigma}}_j &= (A_{l+1} + \bar{\Lambda})\hat{\sigma}_j + K_o(\sigma_j - C_{l+1}\hat{\sigma}_j) \\ u_j &= K_c \bar{\varphi}_w \hat{\sigma}_j \end{aligned} \tag{4.82}$$

for $j = 1, ..., N$ with

$$\bar{\varphi}_w = \begin{pmatrix} \varphi_w^\top \\ -1 \end{pmatrix}^\top \in \mathbb{R}^{1 \times (l+1)} \tag{4.83}$$

$$\begin{aligned} K_o &= D_{l+1}(d) K_o{}' \in \mathbb{R}^{l+1}, \quad D_{l+1}(d) = \mathrm{diag}(d, ..., d^{l+1}), \quad K_o' = P_0^{-1} C_{l+1}^\top \\ K_c &= D_n(g) K_c' \in \mathbb{R}^n, \qquad D_n(g) = \mathrm{diag}(g, ..., g^n), \qquad K_c' = P_c C_n^\top \end{aligned} \tag{4.84}$$

with $d \geq 1$ and $g \geq 1$ to be chosen large enough, and P_0, P_c solutions to following Riccati equations (ARE):

$$P_0 A_{l+1} + A_{l+1}^\top P_0 - 2C_{l+1}^\top C_{l+1} = -a_1 I, \quad a_1 > 0 \tag{4.85}$$

$$P_c A_n + A_n^\top P_c - 2\mu C_n^\top C_n = -a_2 I, \quad a_2 > 0 \tag{4.86}$$

where $\mu \leq \lambda_2(L)$.
Stacking all the vectors can write $\boldsymbol{x} = \mathrm{col}(x_1, ..., x_N)$, $\hat{\boldsymbol{\sigma}} = \mathrm{col}(\hat{\sigma}_1, ..., \hat{\sigma}_N)$, $\boldsymbol{u} = \mathrm{col}(u_1, ..., u_N)$, $\boldsymbol{w} = \mathrm{col}(w_1, ..., w_M)$, $\boldsymbol{\sigma} = \mathrm{col}(\sigma_1, ..., \sigma_N)$, $\Phi(\boldsymbol{x}) = \mathrm{blkdiag}(\{\phi(x_j)\}_{j=1}^n)$. Exploiting the properties of Kronecker products and the information we obtained so far, we can write the equations that describe the whole network of agents, links and local regulators as:

$$\begin{aligned} \dot{\boldsymbol{x}} &= (I_N \otimes A_n)\, \boldsymbol{x} + (I_N \otimes B)\, \Phi(\boldsymbol{x}) + \boldsymbol{u} \\ \dot{\hat{\boldsymbol{\sigma}}} &= \left[I_N \otimes (A_{l+1} + \bar{\Lambda})\right] \hat{\boldsymbol{\sigma}} + (I_N \otimes K_o C_{l+1})(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) \\ \boldsymbol{u} &= (I_N \otimes K_c \bar{\varphi}_w) \hat{\boldsymbol{\sigma}} \\ \dot{\boldsymbol{w}} &= \left[I_M \otimes (A_l + B_l \varphi_w)\right] \boldsymbol{w} + (\mathcal{I}^\top \otimes B_l C_n) \boldsymbol{x} \\ \boldsymbol{\sigma} &= (\mathcal{I} \otimes C_l) \boldsymbol{w} \end{aligned} \tag{4.87}$$

We can now enunciate the following theorem:

Theorem 14. *Consider the system described by* (4.87) *fulfilling assumptions* (4.1) *and* (4.2) *with control parameters K_o and K_c chosen according to* (4.84)*. There exist a $d^\star$ and a $g^\star(d^\star)$ such that for all $d > d^\star$ and $g > g^\star$, the set:*

$$\begin{aligned} \mathcal{S} = \{(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{\sigma}) \in \ & \mathbb{R}^{Nn} \times \mathbb{R}^{lM} \times \mathbb{R}^{(l+1)N} : \\ & x_1 = \cdots = x_N, \\ & w_1 = \cdots = w_M = 0, \\ & \sigma_1 = \cdots = \sigma_N = 0\} \end{aligned} \tag{4.88}$$

is globally exponentially stable.

Proof. The proof is divided in three subsections as follows:

1. First of all, we will show that the dynamics of the closed loop defined in 4.87 is bounded
2. Then, we will show that the observation error can be tuned choosing properly the parameter d
3. Once the observation error is small, we can show that the system actually achieve synchronization

1) Boundness of solutions:
Exploiting the assumption (4.5), there exist a class-$\mathcal{K}$ function $\gamma_x(\cdot) : \mathbb{R}^{nN} \to \mathbb{R}$ and a class-$\mathcal{KL}$ $\beta_x(\cdot, \cdot) : \mathbb{R}^{nN} \to \mathbb{R}$ such that:

$$\begin{aligned} ||\boldsymbol{x}(t, \boldsymbol{x}_0)||_X &\leq \max\{\beta_x(||\boldsymbol{x}_0||, t), \gamma_x(\sup_{\tau \in [0,t]} ||\boldsymbol{u}(\tau)||)\} \\ &\leq \max\{\beta_x(||\boldsymbol{x}_0||, t), \gamma_x(\mathcal{M})\} \end{aligned} \tag{4.89}$$

and there exists a t_1 such that $\forall t > t_1$ we can rewrite (4.89) as:

$$||\boldsymbol{x}(t_1, \boldsymbol{x}_0)||_X \leq \gamma_x(\mathcal{M}) \tag{4.90}$$

and this implies the boundness of $\boldsymbol{x}$-dynamics. About the z-dynamics, we can say that it is also bounded according to assumption (4.2), in fact we have:

$$\begin{aligned} ||\boldsymbol{z}(t, \boldsymbol{z}_0)||_X &\leq \max\{\beta_z(||\boldsymbol{z}_0||, t), \gamma_z(\sup_{\tau\in[0,t]} ||\boldsymbol{x}||)\} \\ &\leq \max\{\beta_z(||\boldsymbol{z}_0||, t), \gamma_z \circ \gamma_x(\mathcal{M})\} \end{aligned} \tag{4.91}$$

This, in turn implies that $\boldsymbol{\sigma}$ is bounded, since for all $j = 1, ..., m$, σ_j is a linear combination of w_i. In the end, boundness of $\boldsymbol{\sigma}$ implies that also the observer state $\hat{\sigma}$ is bounded.

2) Observer convergence:
As usual we start defining the observer error as:

$$\boldsymbol{e} = \hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma} \tag{4.92}$$

Following the same reasoning of (4.20) and defining $\boldsymbol{\varepsilon}$ as in (4.21) we end up with:

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}} &= (I_N \otimes D_{l+1}^{-1}(d)) \left[I_N \otimes (A_{l+1} + \bar{\Lambda} - K_o C_{l+1})\right] (I_N \otimes D_{l+1}^{-1}(d))\boldsymbol{\varepsilon} + \frac{1}{d^{l+1}}\boldsymbol{\Psi}(\boldsymbol{x}, \boldsymbol{u}) \\ &= d\left[I_N \otimes (A_{l+1} - K_o' C_{l+1})\right]\boldsymbol{\varepsilon} + \frac{1}{d^{l+1}}\boldsymbol{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{x}, \boldsymbol{u}) \end{aligned} \tag{4.93}$$

where the function $\boldsymbol{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{x}, \boldsymbol{u})$ is defined as:

$$\boldsymbol{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{x}, \boldsymbol{u}) = \left[\mathcal{L} \otimes B_{l+1}(\gamma C_n + C_n A_n)\right]\boldsymbol{x} + (\mathcal{L} \otimes B_{l+1} C_n)\boldsymbol{u} + (I_N \otimes \bar{\Lambda} D_{l+1}(d))\boldsymbol{\varepsilon} \tag{4.94}$$

Since this is exactly the same expression obtained in (4.22), we can follow the same procedure of the linear case and conclude that we can make small the observer error tuning the parameter d. We can say that with a $\bar{d} - \epsilon$ argument:

Lemma 5. *For all $\epsilon \in \mathbb{R}_{>0}$ and for all $T_s > t_s$, there exists a $\bar{d} \geq d$ such that for all $d > \bar{d}$ we have*

$$||(I_N \otimes D_n(d))\boldsymbol{\varepsilon}|| \leq \epsilon \tag{4.95}$$

This lemma states that the convergence of the observer can be tuned arbitrarily. We can now consider the system when the observation error has converged to an arbitrary small value.

3) Synchronization of the nonlinear agents:
Once the observation error is tuned to be small, we can rewrite the input $\boldsymbol{u}$ as:

$$\boldsymbol{u} = (\mathcal{L} \otimes K_c C_n)\,\boldsymbol{x} - (I_N \otimes K_c \bar{\varphi}_w D_{l+1}(d))\,\boldsymbol{\varepsilon} \tag{4.96}$$

So we can rewrite the $\boldsymbol{x}$ and $\boldsymbol{u}$ dynamics exploiting the former definition of $\boldsymbol{u}$ becoming:

$$\dot{\boldsymbol{x}} = [(I_N \otimes A_n) + (\mathcal{L} \otimes K_c C_n)]\boldsymbol{x} + (I_N \otimes B_n)\Phi(\boldsymbol{x}) - (I_N \otimes K_c \bar{\varphi}_w D_{l+1}(d))\boldsymbol{\varepsilon} \tag{4.97}$$

$$\dot{\boldsymbol{\varepsilon}} = d\,(I_N \otimes H_o)\,\boldsymbol{\varepsilon} + \frac{1}{d^{l+1}}\Psi(\boldsymbol{x}, \boldsymbol{\varepsilon}) \tag{4.98}$$

With $\Psi(\boldsymbol{x}, \boldsymbol{\varepsilon})$ defined as follows:

$$\begin{aligned}\Psi(\boldsymbol{x}, \boldsymbol{\varepsilon}) = & \left[\mathcal{L} \otimes B_{l+1}(\gamma C_n + C_n A_n)\right] \boldsymbol{x} + \left(\mathcal{L}^2 \otimes B_{l+1} C_n K_c C_n\right) \boldsymbol{x} \\ & - \left(\mathcal{L} \otimes B_{l+1} C_n K_c \bar{\varphi}_w D_{l+1}(d)\right) \boldsymbol{\varepsilon} + \left(I_N \otimes \bar{\Lambda} D_{l+1}(d)\right) \boldsymbol{\varepsilon}\end{aligned}$$

We can now exploit these new expressions to prove synchronization. In order to do so, consider the transformation provided by the matrix T, defined as follows:

$$T = \begin{bmatrix} 1 & \mathbf{0}_{1\times(N-1)} \\ \mathbf{1}_{(N-1)\times 1} & I_{N-1} \end{bmatrix} \tag{4.99}$$

Applying the similarity transformation at the Laplacian matrix $\mathcal{L}$ and to the vectors $\boldsymbol{x}$ and $\boldsymbol{\varepsilon}$ we have:

$$\tilde{\mathcal{L}} = T^{-1} \mathcal{L} T = \begin{bmatrix} 0 & \mathcal{L}_{12} \\ \mathbf{0} & \mathcal{L}_{22} \end{bmatrix} \tag{4.100}$$

$$\begin{aligned} \boldsymbol{x'} &= (T^{-1} \otimes I_n)\boldsymbol{x} \\ \boldsymbol{\zeta} &= (T^{-1} \otimes I_n)\boldsymbol{\varepsilon} \end{aligned} \tag{4.101}$$

Computing the time derivative of $\boldsymbol{x'}$ we have:

$$\dot{\boldsymbol{x}}' = \left[(I_N \otimes A_n) + (\tilde{\mathcal{L}} \otimes K_c C_n)\right] \boldsymbol{x'} + (I_N \otimes B_n)\Delta\Phi(\boldsymbol{x'}) - (I_N \otimes K_c \bar{\varphi}_w D_{l+1}(d))\boldsymbol{\zeta} \tag{4.102}$$

Where it turns out that $\Delta\Phi(\boldsymbol{x'})$ is a diagonal matrix:

$$\Delta\Phi(\boldsymbol{x'}) = (I_N \otimes B_n) \begin{pmatrix} \phi(x_1') & 0 & \cdots & 0 \\ 0 & \phi(x_1' + x_2') - \phi(x_1') & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \phi(x_1' + x_N') - \phi(x_1') \end{pmatrix} \tag{4.103}$$

Splitting the dynamics we obtain:

$$\begin{aligned} \dot{x}_1' &= A_n x_1' + (\mathcal{L}_{12} \otimes K_c C_n)\boldsymbol{x}_2' + B_n \Phi(x_1') - (K_c \bar{\varphi}_w D_{l+1}(d))\zeta_1 \\ \dot{\boldsymbol{x}}_2' &= \left[(I_{N-1} \otimes A_n) + (\mathcal{L}_{22} \otimes K_c C_n)\right] \boldsymbol{x}_2' + (I_{N-1} \otimes B_n)\Delta\Phi(x_1', \boldsymbol{x}_2') \\ &\quad - (I_{N-1} \otimes K_c \bar{\varphi}_w D_{l+1}(d))\boldsymbol{\zeta}_2 \\ \dot{\zeta}_1 &= dH_o \zeta_1 + \frac{1}{d^{l+1}}\Big\{ \left[\mathcal{L}_{12} \otimes B_{l+1}(\gamma C_n + C_n A_n)\right] \boldsymbol{x_2'} + \left[\mathcal{L}_{12}^2 \otimes (B_{l+1} C_n K_c C_n)\right] \boldsymbol{x_2'} \\ &\quad - \left[\mathcal{L}_{12} \otimes (B_{l+1} C_n K_c \bar{\varphi}_w D_{l+1}(d))\right] \boldsymbol{\zeta_2} + (I_{N-1} \otimes \bar{\Lambda} D_{l+1}(d))\boldsymbol{\zeta_2}\Big\} \\ \dot{\boldsymbol{\zeta}}_2 &= d(I_{N-1} \otimes H_0)\boldsymbol{\zeta_2} + \frac{1}{d^{l+1}}\Psi(\boldsymbol{\zeta_2}, \boldsymbol{x_2'}) \end{aligned} \tag{4.104}$$

With

$$\begin{aligned}\Psi(\boldsymbol{\zeta_2}, \boldsymbol{x_2'}) = & \left[\mathcal{L}_{22} \otimes B_{l+1}(\gamma C_n + C_n A_n)\right] \boldsymbol{x_2'} + \left[\mathcal{L}_{22}^2 \otimes (B_{l+1} C_n K_c C_n)\right] \boldsymbol{x_2'} \\ & - \left[\mathcal{L}_{22} \otimes (B_{l+1} C_n K_c \bar{\varphi}_w D_{l+1}(d))\right] \boldsymbol{\zeta_2} + (I_{N-1} \otimes \bar{\Lambda} D_{l+1}(d))\boldsymbol{\zeta_2}\end{aligned} \tag{4.105}$$

Where $\Delta\Phi(x_1', \boldsymbol{x}_2')$ is bounded due to assumption (4.4), namely there exists $\bar{\Phi}$ such that:

$$||\Delta\Phi(x_1', \boldsymbol{x}_2')|| \leq \bar{\Phi}||\boldsymbol{x}_2'|| \qquad \forall x_1' \in \mathbb{R}^n,\ \boldsymbol{x}_2' \in \mathbb{R}^{(N-1)n} \tag{4.106}$$

To notice this, note that:

$$\Delta\Phi(x_1',0) = \begin{bmatrix} \Phi(x_1') - \Phi(x_1') \\ \vdots \\ \Phi(x_1') - \Phi(x_1') \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{4.107}$$

So we can rewrite the Lipschitz assumption as:

$$||\Delta\Phi(x_1',x_2') - \Delta\Phi(x_1',0)|| \leq \bar{\Phi}||\boldsymbol{x}_2'|| \tag{4.108}$$

And exploiting (4.107) we obtain (4.106). Furthermore, we rescale the $\boldsymbol{x_2}'$ dynamics as:

$$\boldsymbol{\chi} = (I_{N-1} \otimes D_n^{-1}(g))\boldsymbol{x}_2' \tag{4.109}$$

Computing the time derivative of (4.109) and exploiting the fact that:

$$D_n^{-1}(g) \cdot B_n = \begin{pmatrix} 0 \\ \vdots \\ g^n \end{pmatrix} \in \mathbb{R}^n \tag{4.110}$$

we obtain:

$$\begin{aligned} \dot{\boldsymbol{\chi}} &= g\left[(I_{N_1} \otimes A_n) + (\mathcal{L}_{22} \otimes K_c' C_n)\right]\boldsymbol{\chi} + \frac{1}{g^n}(I_{N-1} \otimes B_n)\Delta\Phi(x_1', \boldsymbol{\chi}) \\ &+ (I_N \otimes D_n^{-1}(g) K_c \bar{\varphi}_w D_{l+1}(d))\boldsymbol{\zeta}_2 \end{aligned} \tag{4.111}$$

The dynamics of $\boldsymbol{\zeta_2}$ becomes, exploiting (4.42):

$$\dot{\boldsymbol{\zeta}}_{\boldsymbol{2}} = d(I_{N-1} \otimes H_o)\boldsymbol{\zeta_2} + \frac{1}{d^{l+1}}\Psi(\boldsymbol{\zeta_2}, \boldsymbol{\chi}) \tag{4.112}$$

With

$$\begin{aligned} \Psi(\boldsymbol{\zeta_2}, \boldsymbol{\chi}) = &\left[\mathcal{L}_{22} \otimes B_{l+1}(\gamma C_n + C_n A_n) D_n(g)\right]\boldsymbol{\chi} + \left[\mathcal{L}_{22}^2 \otimes (B_{l+1} C_n K_c C_n D_n(g))\right]\boldsymbol{\chi} \\ &- \left[\mathcal{L}_{22} \otimes (B_{l+1} C_n K_c \bar{\varphi}_w D_{l+1}(d))\right]\boldsymbol{\zeta_2} + (I_{N-1} \otimes \bar{\Lambda} D_{l+1}(d))\boldsymbol{\zeta_2} \end{aligned} \tag{4.113}$$

For what regards the link dynamics, it's possible to reformulate it as:

$$\left[I_M \otimes (A_l + B_l \varphi_w)\right]\boldsymbol{w} + (\mathcal{I}^\top T \otimes B_l C_n)\boldsymbol{x'} \tag{4.114}$$

Due to the structure of T it turns out that:

$$\mathcal{I}^\top T = \left[\ \mathbf{0} \mid \mathcal{I}_2\ \right] \tag{4.115}$$

i.e., the first column is always zero. Finally the $\boldsymbol{w}$ dynamics can be written as:

$$\dot{\boldsymbol{w}} = \left[I_M \otimes (A_l + B_l \varphi_w)\right]\boldsymbol{w} + (\mathcal{I}_2^\top T \otimes B_l C_n D_n(g))\boldsymbol{\chi} \tag{4.116}$$

In order to prove the asymptotic stability of the system, let us define the following quadratic Lyapunov function:

$$V_T = \frac{1}{2}\boldsymbol{\chi}^\top(I_{N-1} \otimes P_c)\boldsymbol{\chi} + \frac{1}{2}b\boldsymbol{w}^\top(I_M \otimes Q)\boldsymbol{w} + \frac{1}{2}\boldsymbol{\zeta}_2^\top(I_{N-1} \otimes P_o)\boldsymbol{\zeta}_2, \quad b > 0 \tag{4.117}$$

Where the matrix Q is defined as in (4.49). Computing the time derivative of V_T along the trajectories defined by (4.111), (4.112) and (4.116) we have:

$$\begin{aligned}
\dot{V}_T &= \boldsymbol{\chi}^\top (I_{N-1} \otimes P_c)\dot{\boldsymbol{\chi}} + b\boldsymbol{w}^\top (I_M \otimes Q)\dot{\boldsymbol{w}} + \boldsymbol{\zeta}_2^\top (I_{N-1} \otimes P_o)\dot{\boldsymbol{\zeta}}_2 \\
&= -a_2 g|\boldsymbol{\chi}|^2 - b|\boldsymbol{w}|^2 - a_1 d|\boldsymbol{\zeta}_2|^2 + \frac{1}{g^n}\boldsymbol{\chi}^\top (I_{N-1} \otimes P_c B_n)\Delta\Phi(x_1', \boldsymbol{\chi}) \\
&+ \boldsymbol{\chi}^\top (I_{N-1} \otimes P_c D_n^{-1}(g) K_c \bar{\varphi}_w D_{l+1}(d))\boldsymbol{\zeta}_2 + \boldsymbol{w}^\top (\mathcal{I}_2 \otimes Q B_l C_n D_n(g))\boldsymbol{\chi} \\
&+ \boldsymbol{\zeta}_2^\top \left[\mathcal{L}_{22} \otimes P_o B_{l+1}(\gamma C_n + C_n A_n) D_n(g)\right]\boldsymbol{\chi} + \boldsymbol{\zeta}_2^\top \left[\mathcal{L}_{22}^2 \otimes P_o B_{l+1} C_n K_c C_n D_n(g)\right]\boldsymbol{\chi} \\
&- \boldsymbol{\zeta}_2^\top \left[\mathcal{L}_{22} \otimes P_o B_{l+1} C_n K_c \bar{\varphi}_w D_{l+1}(d)\right]\boldsymbol{\zeta}_2 + \boldsymbol{\zeta}_2^\top (I_{N-1} \otimes \bar{\Lambda} D_{l+1}(d))\boldsymbol{\zeta}_2
\end{aligned} \tag{4.118}$$

Since $\boldsymbol{\chi}$ is simply a scaled version of $\boldsymbol{x}_2'$ we can infer that $\Delta\Phi(x_1', \boldsymbol{\chi})$ is Lipschitz, namely there exists a constant $\bar{\Phi}'$ such that:

$$||\Delta\Phi(x_1', \boldsymbol{\chi})|| \le \bar{\Phi}' ||\boldsymbol{\chi}|| \qquad \forall x_1' \in \mathbb{R}^n,\ \boldsymbol{\chi} \in \mathbb{R}^{(N-1)n} \tag{4.119}$$

To provide a bound of the expression (4.118) we define the following variables:

$$\begin{aligned}
&k_0 = \frac{1}{g^n} P_c B_n, \quad k_1 = P_c D_n^{-1}(g) K_c \bar{\varphi}_w D_{l+1}(d) \\
&k_2 = \mathcal{I}_2 \otimes Q B_l C_n D_n(g), \quad k_3 = \mathcal{L}_{22} \otimes P_o B_{l+1}(\gamma C_n + C_n A_n) D_n(g) \\
&k_4 = \mathcal{L}_{22}^2 \otimes P_o B_{l+1} C_n K_c C_n D_n(g), \quad k_5 = \mathcal{L}_{22} \otimes (P_o B_{l+1} C_n K_c \bar{\varphi}_w D_{l+1}(d)) \\
&k_6 = \bar{\Lambda} D_{l+1}(d)
\end{aligned} \tag{4.120}$$

and we proceed using Young inequalities:

$$\begin{aligned}
|k_1||\boldsymbol{\chi}||\boldsymbol{\zeta_2}| &\le \frac{|k_1|^2|\boldsymbol{\zeta_2}|^2}{2} + \frac{|\boldsymbol{\chi}|^2}{2} \\
|k_2||\boldsymbol{w}||\boldsymbol{\chi}| &\le \frac{|k_2|^2|\boldsymbol{w}|^2}{2} + \frac{|\boldsymbol{\chi}|^2}{2} \\
|k_3||\boldsymbol{\chi}||\boldsymbol{\zeta_2}| &\le \frac{|k_3|^2|\boldsymbol{\zeta_2}|^2}{2} + \frac{|\boldsymbol{\chi}|^2}{2} \\
|k_4||\boldsymbol{\chi}||\boldsymbol{\zeta_2}| &\le \frac{|k_4|^2|\boldsymbol{\zeta_2}|^2}{2} + \frac{|\boldsymbol{\chi}|^2}{2}
\end{aligned} \tag{4.121}$$

Therefore we can bound (4.118) as:

$$\begin{aligned}
\dot{V}_T \le &- \left(a_2 g - 2 - |k_0|\bar{\Phi}'\right)|\boldsymbol{\chi}|^2 - (b - \frac{|k_2|^2}{2})|\boldsymbol{w}|^2 \\
&- \left(a_1 d - \frac{1}{2}\left(|k_1|^2 + |k_3|^2 + |k_4|^2\right) - |k_5| - |k_6|\right)|\boldsymbol{\zeta}_2|^2
\end{aligned} \tag{4.122}$$

Choosing the parameters g, b, d as:

$$g > \frac{1}{a_2}\left(2 + |k_0|\bar{\Phi}'\right) \quad b > \frac{|k_2|^2}{2} \quad d > \frac{1}{a_1}\left[\frac{1}{2}\left(|k_1|^2 + |k_3|^2 + |k_4|^2\right) + |k_5| + |k_6|\right] \tag{4.123}$$

(4.118) is negative definite, then the dynamics of $(\boldsymbol{\chi}, \boldsymbol{w}, \boldsymbol{\zeta_2})$ is asymptotically stable. To prove that the closed loop defined by (4.87) is globally asymptotically stable we can notice that the dynamics of ζ_1 is asymptotically stable since it depends only on asymptotically stable terms. This, along with the fact the function $\Phi(x)$ is globally Lipschitz, implies that also the dynamics of x_1' is stable. Finally, since $\boldsymbol{\chi}$ and $\boldsymbol{\zeta}$ are obtained by linear transformations from $\boldsymbol{x}, \boldsymbol{\varepsilon}$, we can conclude that the whole close loop (4.87) is globally asymptotically stable. $\square$

4.4 Simulation example

4.4.1 System and controller settings

In this section a practical example will be considered. The simulation is based on the system described by the graph shown in figure 4.1, therefore $N = M = 6$. The incidence matrix $\mathcal{I} \in \mathbb{R}^{N \times E}$ and the Laplacian matrix $\mathcal{L} \in \mathbb{R}^{N \times N}$ are respectively:

$$\mathcal{I} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\mathcal{L} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 3 & -1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

The eigenvalues of the matrix $\mathcal{L}$ are

$$\lambda_1 = 0, \quad \lambda_2 = 0.69, \quad \lambda_3 = 0.69$$
$$\lambda_4 = 2, \quad \lambda_5 = 4.30, \quad \lambda_6 = 4.30$$

Each agent $x_j, \forall j = 1, ..., 6$ is described by:

$$\begin{bmatrix} \dot{x}_{j_1} \\ \dot{x}_{j_2} \\ \dot{x}_{j_3} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}^{A_3} \begin{bmatrix} x_{j_1} \\ x_{j_2} \\ x_{j_3} \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}^{B_3} \overbrace{\begin{bmatrix} 0 & -2 & -3 \end{bmatrix}}^{\varphi_x} \begin{bmatrix} x_{j_1} \\ x_{j_2} \\ x_{j_3} \end{bmatrix} + \begin{bmatrix} u_{j_1} \\ u_{j_2} \\ u_{j_3} \end{bmatrix} \tag{4.124}$$
$$y_j = \overbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}^{C_3} \begin{bmatrix} x_{j_1} \\ x_{j_2} \\ x_{j_3} \end{bmatrix}$$

The dynamics of each link $w_i, \forall i = 1, ..., 6$ can be written as:

$$\begin{bmatrix} \dot{w}_{i_1} \\ \dot{w}_{i_2} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}^{A_2} \begin{bmatrix} w_{i_1} \\ w_{i_2} \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{B_2} \overbrace{\begin{bmatrix} -1 & -1 \end{bmatrix}}^{\varphi_w} \begin{bmatrix} w_{i_1} \\ w_{i_2} \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{B_2} \sum_{j=1}^{6} \iota_{ji} y_j$$

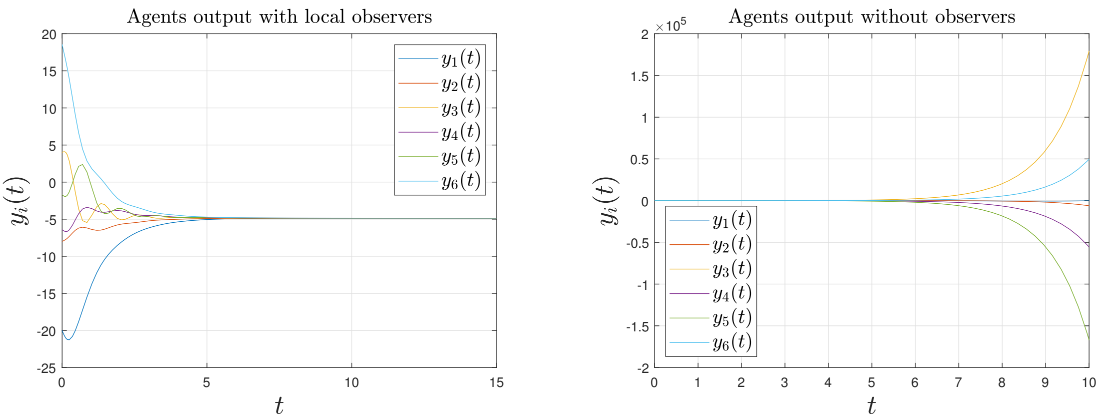
$$z_i = \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{C_2} \begin{bmatrix} w_{i_1} \\ w_{i_2} \end{bmatrix}$$

The parameters d, l are chosen as $d = 8$ and $l = 1$.
Solving the two AREs (4.15) and (4.16), using as parameters: $\mu = 0.5$, $a_1 = 10$, $a_2 = 0.1$ we obtain the following values for K_o and K_c:

$$K_o = \begin{bmatrix} 38.08 \\ 405.33 \\ 1619.08 \end{bmatrix}, \qquad K_c = \begin{bmatrix} 0.74 \\ 0.08 \\ -0.07 \end{bmatrix}$$

4.4.2 Nominal setting

In the nominal context, i.e. without any uncertainty and without any noise, it can be seen from the following pictures that the stability is influenced by the presence of the observers. In particular, without the observers the stability can be achieved only if the filter dynamics is sufficiently fast.



(a) *Evolution of the outputs of the agents* (4.124) *(with local observers)*

(b) *Evolution of states of* (4.124) *(without observers)*

In the picture below is shown the behavior of the agents input:

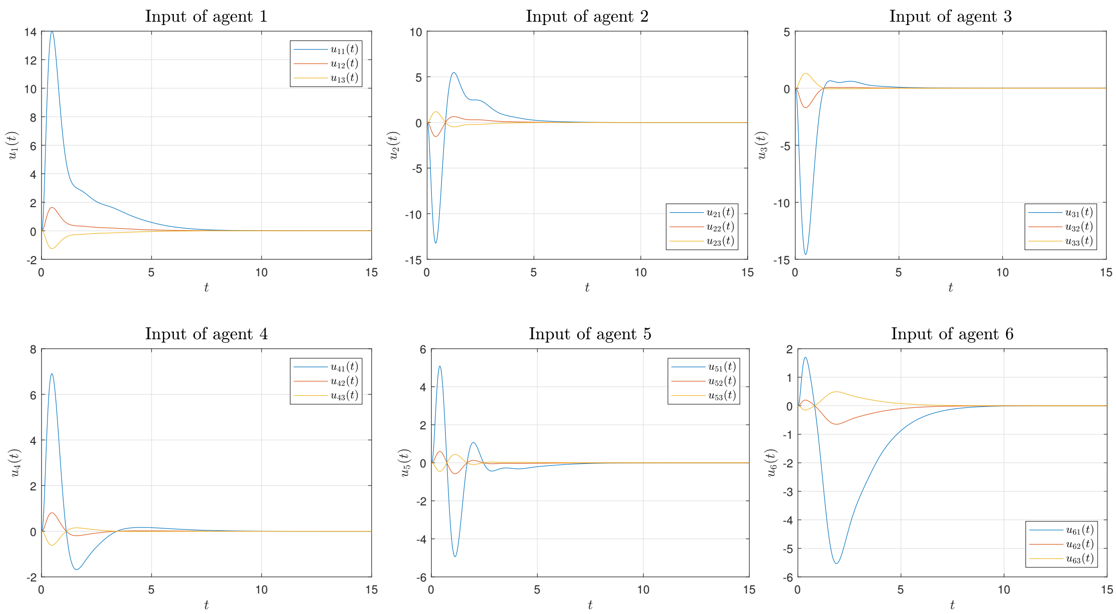


Figure 4.3: Agents input

In some cases we would like to limit the input effort. In order to do so, we can saturate the input between two values $u \in [u_{min}, u_{max}]$. In the following pictures it can be seen that, even tough the inputs are saturated, the consensus is still achieved.

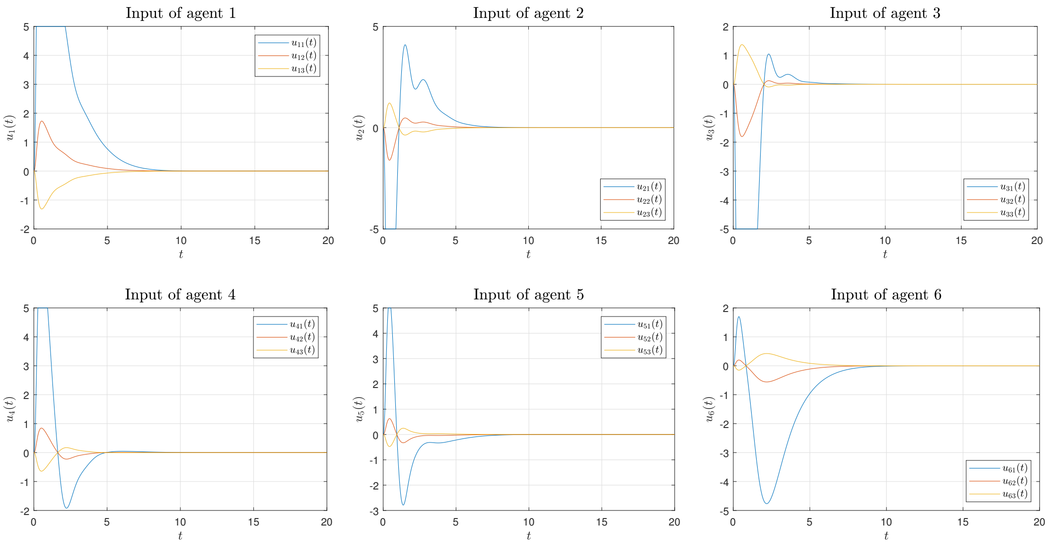


Figure 4.4: Agents input with $u_i \in [-5, 5]$

In the picture below is shown the output behavior when a saturated input is considered. It can be noticed that the consensus is still achieved, even if it requires more time

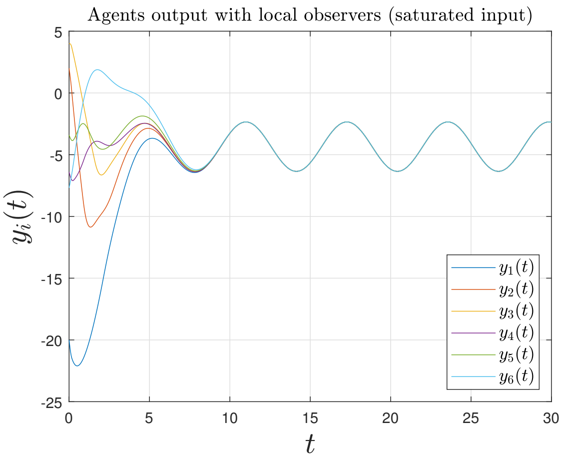
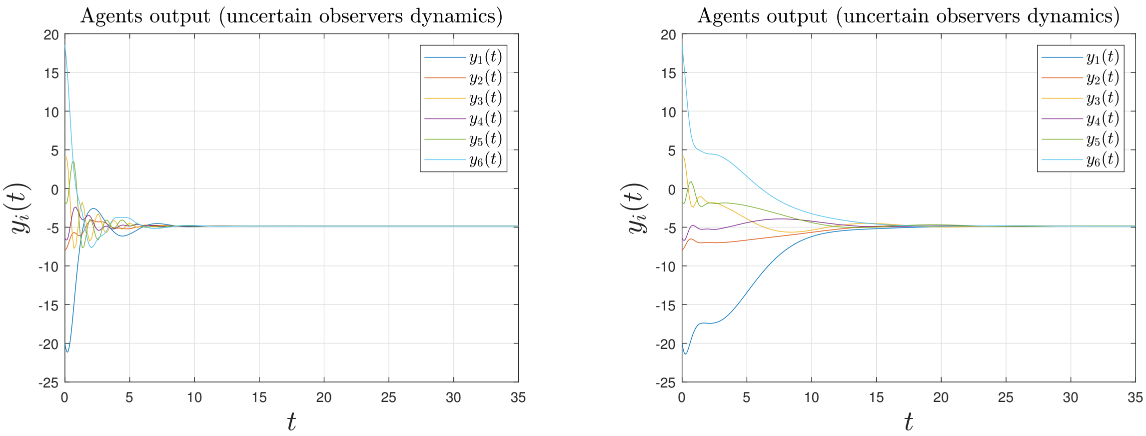


Figure 4.5: Output consensus with saturated input and sinusoidal reference

4.4.3 Robustness analysis

It's worth noticing that the system is robust with respect to filter dynamics uncertainties. In fact if we suppose an uncertain filter dynamics we can't exactly know the dynamics of the observer and this can result in an unstable system. In the following are shown the results for the simulations done modifying the value of the parameter φ_w inside the observer dynamics. Calling the modified parameter $\tilde{\varphi}_w$, we tried with $\tilde{\varphi}_w = 2\varphi_w$ and $\tilde{\varphi}_w = 0.1\varphi_w$. In both cases it can be seen that consensus is achieved, despite some differences in the transient phase due to the fact that the dynamics of the filter is not exactly known.



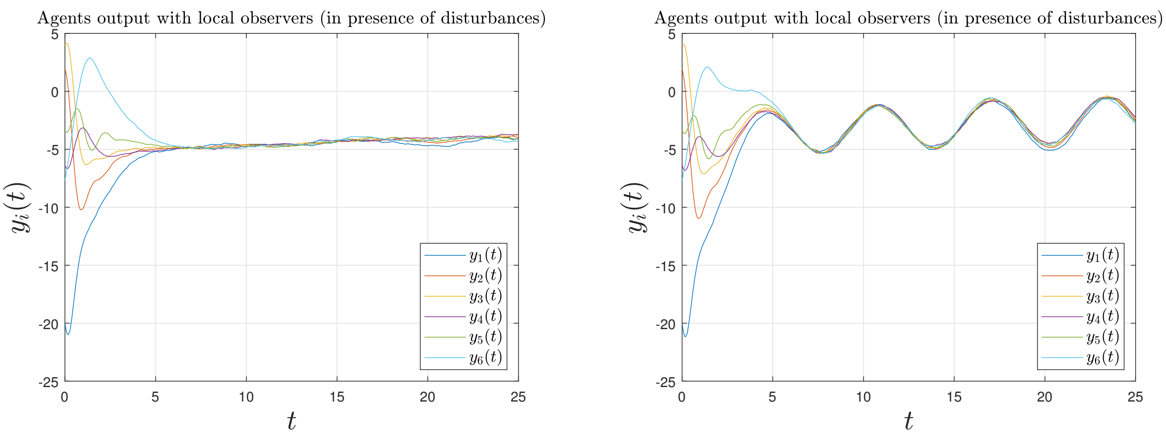
(a) *Evolution of the outputs of the agents with* $\tilde{\varphi}_w = 2\varphi_w$

(b) *Evolution of the outputs of the agents with* $\tilde{\varphi}_w = 0.1\varphi_w$

Another test that we can do is to simulate the system injecting a white noise in the dynamics of each agent. With these settings the new equations for the agent dynamics become:

$$\begin{aligned} \dot{x}_j &= A_n x_j + B_n \varphi_x x_j + u_j + d_j \\ y_j &= C x_j \end{aligned} \tag{4.125}$$

When $d(t)$ is the injected disturbance. It can be seen from the following pictures that the consensus is still achieved after some seconds:

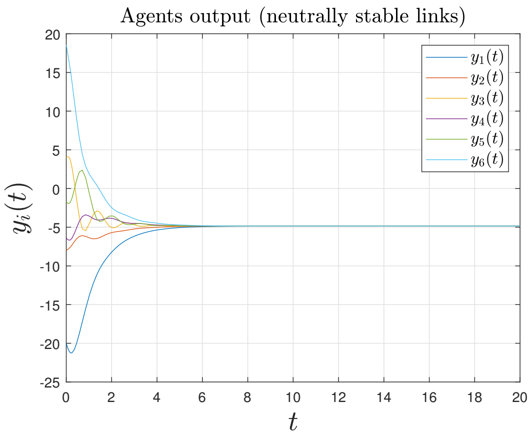


(a) *Evolution of the outputs of the agents in presence of white noise (no reference)*

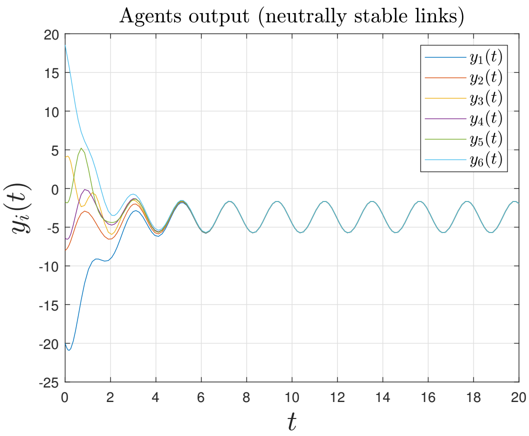
(b) *Evolution of the outputs of the agents in presence of white noise and a sinusoidal reference*

In the previous simulation the disturbance is a white noise with a height of its power spectral density (PSD)[1] equal to 0.1.
We can also simulate the system dropping the assumption of asymptotic stability of the links, by making the assumption of neutrally stable links. Again, the consensus is achieved, as can be seen from the following plots:

[1]The power spectral density describes how power is distributed over the frequency content of the random process



(a) *Evolution of the outputs of the agents in presence of neutrally stable links (no reference)*



(b) *Evolution of the outputs of the agents in presence of neutrally stable links and a sinusoidal reference*

Finally we can simulate the system when a sudden change of reference happens after a certain amount of time. In the following example the amplitude of the sinusoidal reference change after 25 s.

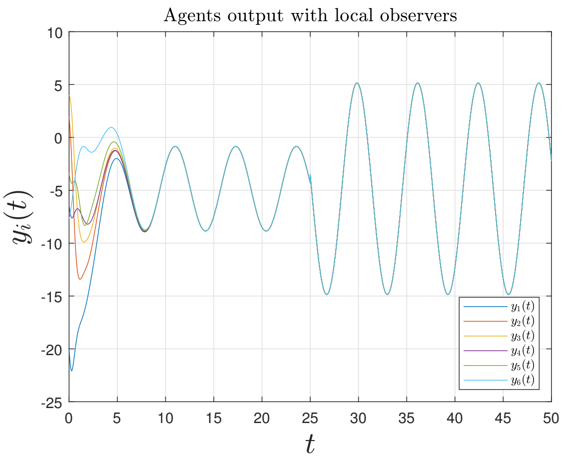


Figure 4.9: Output consensus with variable reference

It can be seen that the consensus is maintained despite the sudden change in amplitude of the reference signal.

Chapter 5

Further Simulations

In this chapter we present some simulations results in the context of multi-agent systems. First of all, we will simulate the previous control law when each agent is modeled as a Van der Pol oscillator. In second place, we will consider a special type of oscillator in which each agent dynamics is described by:

$$\dot{x}_i = A_i x_i - kx \arctan\left(||x||_i^2 - R_i^2\right) + u_i, \quad \forall i = 1, ..., N \tag{5.1}$$

5.1 Simulation 1: Van der Pol oscillator

In this case each agent is modeled as a Van der Pol oscillator:

$$\begin{aligned} \dot{x}_{i,1} &= x_{i,2} \\ \dot{x}_{i,2} &= \mu(1 - x_{i,1}^2)x_{i,2} - x_{i,1} + u_i \end{aligned} \qquad \forall i = 1, ..., N \tag{5.2}$$

Where $\mu \in \mathbb{R}$ is a parameter. The input u_i is computed following the same procedure shown in the previous chapters. In our simulations $\mu = 0.5$.

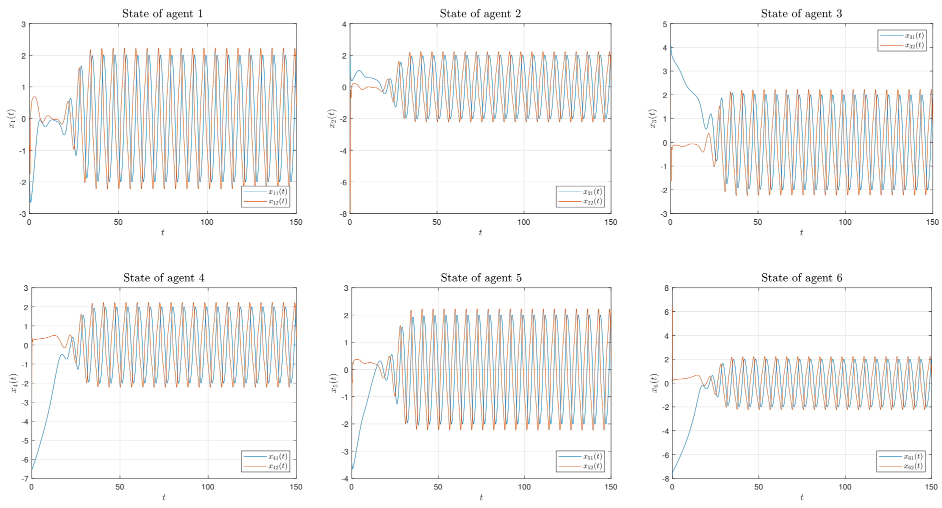


Figure 5.1: Van der Pol oscillators: state consensus

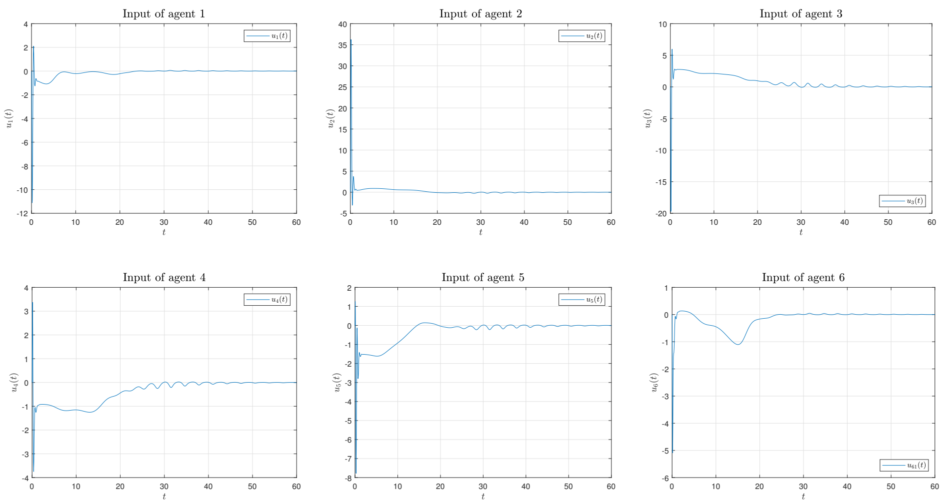


Figure 5.2: Van der Pol oscillators: agents input

From the following picture it can be seen that the output consensus is also achieved:

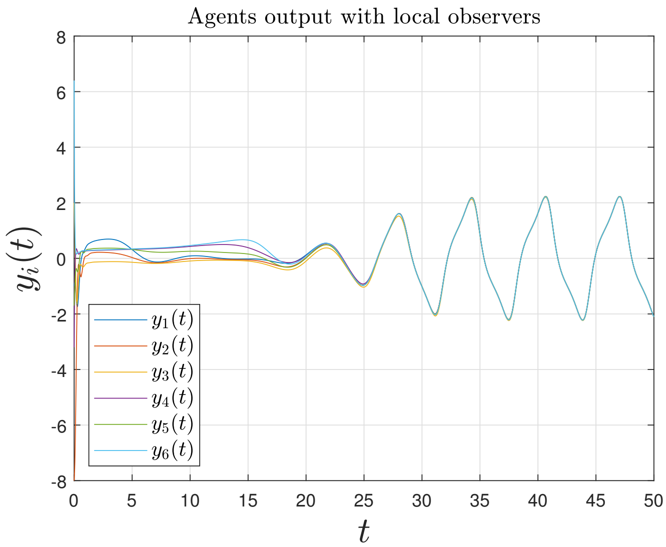


Figure 5.3: Van der Pol oscillators: output consensus

5.2 Simulation 2

In this section we consider agents described by:

$$\begin{aligned} \dot{x}_i &= Sx_i - kx \arctan\left(||x||_i^2 - R^2\right) + u_i \qquad \forall i = 1, ..., N \\ y &= Cx_i \end{aligned} \tag{5.3}$$

Where

$$S = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \qquad \omega = \frac{2\pi}{T} \qquad \phi(x) = \arctan\left(||x||^2 - R^2\right) \tag{5.4}$$

Regarding this system, the following properties holds:

1. The function $f(x) = x\phi(x) = x \arctan(||x||^2 - R^2)$ is globally Lipschitz.
2. The unforced system admit a T-periodic orbit of radius R which is asymptotically stable with domain of attraction $\mathbb{R}^2 \setminus \{0\}$ and locally exponentially stable.
3. The system (5.1) is input-to-state stable with respect to the set $\{x \in \mathbb{R}^2 : ||x|| = R\}$.

Proof. 1. Exploiting the properties of the arctan function we have:

$$\begin{aligned}
|f(e+x) - f(x)| &= |\phi(e+x)(e+x) - \phi(x)x| \\
&\leq |\phi(e+x)x - \phi(x)x| + |\phi(e+x)e| \\
&\leq \left| \int_0^1 \frac{\partial \phi}{\partial e}(x+se)xds\ e \right| + |\phi(e+x)||e| \\
&\leq \left| \int_0^1 \frac{k(x+se)^\top x}{1 + ((x+se)^\top (x+se) - R^2)^2} ds \right| |e| + \left| k \arctan\left(|e+x|^2 - R^2\right) \right| |e| \\
&\leq k \sup_{s \in \mathbb{R}} \left| \frac{s}{1 + (s^2 - R^2)^2} \right| |e| + k \sup_{s \in \mathbb{R}} \left| \arctan\left(s^2 - R^2\right) \right| |e| \\
&\leq k \left(1 + \frac{\pi}{2}\right) |e|
\end{aligned} \tag{5.5}$$

Furthermore, we compute

$$\frac{\partial f}{\partial x}(x) = \frac{\partial}{\partial x}(x\phi(x)) = I\phi(x) + 2\phi'(x)xx^\top \tag{5.6}$$

2. Transform the system in polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \end{cases} \tag{5.7}$$

Then, the unforced system is transformed into

$$\dot{\theta} = 1 \qquad \theta \in \mathbb{R} \tag{5.8}$$

$$\dot{r} = -r\phi(r) \qquad r \in \mathbb{R}_{\geq 0}. \tag{5.9}$$

In this coordinates, the two variables θ and r are independent. Concerning the r coordinate, we have two equilibria which are $r = 0$ and $r = R$ that implies $\phi(r) = 0$. The equilibrium $r = 0$ coincide with the origin $x = 0$. We are interested in showing that the second equilibrium $r = R$ is asymptotically stable. For this, consider the Lyapunov function $V = (r - R)^2$. On its domain of definition $r \in [0, \infty)$, we obtain

$$\dot{V} = -(r-R)r\phi(r) = -kr(r-R)\arctan\left(r^2 - R^2\right) < 0 \quad \forall r \in [0, R) \cup (R, \infty)$$

where the latter is a consequence of the fact that

$$(r-R)\arctan\left(r^2 - R^2\right) > 0 \quad \forall r \neq R$$

This shows asymptotic stability of the point $r = R$. Furthermore, the orbit $r = R$ is also locally exponentially stable. To see this, compute the linearization of the r-dynamics around the equilibrium $r = R$:

$$\frac{\partial}{\partial r}(-r\phi(r))_{r=R} = -\phi(R) - R^2\phi'(R) = -kR^2 < 0$$

which conclude the proof. Alternatively, we can establish exponential stability in the original coordinates by computing the approximation of the Jacobian of the unforced system around $|x| = R$. By using (5.6), we obtain

$$\frac{\partial}{\partial x}[Sx - f(x)]_{|x|=R} = \left[S - I\phi(x) - 2\phi'(x)xx^\top\right]_{|x|=R} = \left[S - 2kRxx^\top\right]_{|x|=R} \tag{5.10}$$

Since $|x| = R$, then $x \neq 0$. Therefore, for any x, $\left(S, x^\top\right)$ is an observable pair. It turns out that $S - 2kRxx^\top$ is Hurwitz for any $x \neq 0$ and any $k > 0$. Hence local exponential stability is proven.

3. First of all, consider the Lyapunov function $V_0 = x^\top x$. Its derivative satisfies

$$\begin{aligned} \dot{V}_0 &= 2x^\top(Sx - \phi(x)x) \\ &= -2x^\top x\phi(x) \\ &\leq -\phi(x)V_0 \end{aligned}$$

For $|x| > R$, we obtain $\phi(x) > \varepsilon$. Therefore, the previous inequality can be bounded as $\dot{V}_0 \leq -\varepsilon V_0 + |d|^2$ showing that trajectories remain ultimately bounded. In order to obtain a suited ISS Lyapunov function that holds everywhere, let consider the Lyapunov function

$$V(x) = x^\top P x$$

with P satisfying

$$\dot{P} = -P\left(S + xx^\top\right) - \left(S + xx^\top\right)^\top P + 2\phi(x)I - P^2, \quad P(0) = P(0)^\top > 0 \tag{5.11}$$

Therefore, there exists $0 < \underline{p} < \bar{p} < \infty$ so that

$$\underline{p}|x|^2 \leq V \leq \bar{p}|x|^2$$

Its derivative along the unforced system satisfies

$$\begin{aligned} \dot{V} &= 2x^\top P(Sx - \phi(x)I) + x^\top \dot{P}x \\ &= 2x^\top P(Sx - \phi(x)I) + x^\top\left(-P\left(S + xx^\top\right) - \left(S + xx^\top\right)^\top P + \phi(x)I - P^2\right)x \\ &= -2x^\top xx^\top Px - x^\top P^2 x \\ &\leq -2x^\top xV \end{aligned}$$

When, $|x| \geq R$, we obtain $\dot{V} \leq -2R^2$.

□

Chapter 6

Conclusions and Perspectives

This thesis has explored the problem of achieving consensus (i.e. synchronization) in multi-agent systems in the presence of dynamic interconnections, exploiting an observer-based control strategy through the use of high-gain observers.
The methods proposed and analyzed demonstrate the potential for robust synchronization in networks of nonlinear agents, even when subject to noise. The core of this thesis is the extension of existing theoretical results to systems with dynamic interconnections, achieving synchronization in a wide range of networks. An important application is related to power grid systems, where maintaining synchronization is essential for stability and efficiency. In modern power grids, especially decentralized systems that exploit renewable energy sources, synchronization of inverters is crucial to ensure that all components operate at the same frequency, typically 50Hz, to balance supply and demand.
As a perspective, the methodologies developed through this work could be useful in achieving consensus in a context in which each inverter is seen as an agent. This approach ensures that all inverters synchronize to the grid frequency while simultaneously responding to demand fluctuations from agents such as houses, factories, and other loads. This work has significant implications in the context of grid-forming inverters, which establish and maintain the voltage and frequency of the grid. Indeed, the results obtained can be applied in this context, ensuring consensus between the agents along the same trajectory, as is shown by simulations throughout the thesis. Beyond power grids, there are several other important applications:

- **Robotic Swarms [24], [4]:** The synchronization techniques could be applied to ensure coordination in robotic swarms, where multiple robots work together to achieve a common objective. We can see them as agents in a dynamic environment, where the aim is to achieve and maintain synchronization in their movements, even in the presence of noise and some dynamics between them.

Figure 6.1: Example of Robotic Swarms

- **Autonomous Vehicle Networks [19], [26]:** The synchronization of autonomous vehicles, whether in drones, cars, or maritime vessels, is crucial for safe and efficient operation in traffic systems. Applying observer-based synchronization methods could help vehicles dynamically adjust their routes or speeds, ensuring coordinated and collision-free operation.

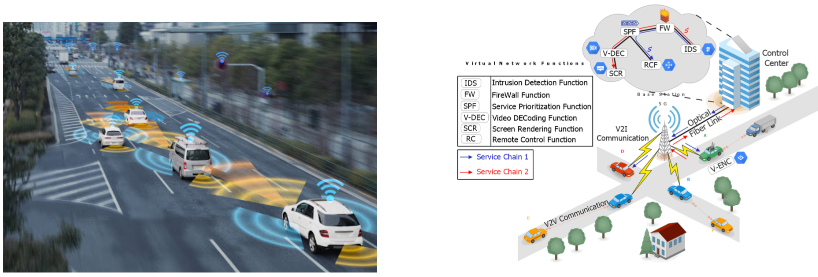


Figure 6.2: Example of Autonomous Vehicle Networks

- **Sensor Networks [1], [13]:** In distributed sensor networks, ensuring that sensors across a large area synchronize data collection is critical for applications like environmental monitoring, surveillance, and smart cities. The techniques developed in this thesis could be applied to maintain consistent communication and synchronization between sensors, despite changes in network connectivity or environmental conditions.

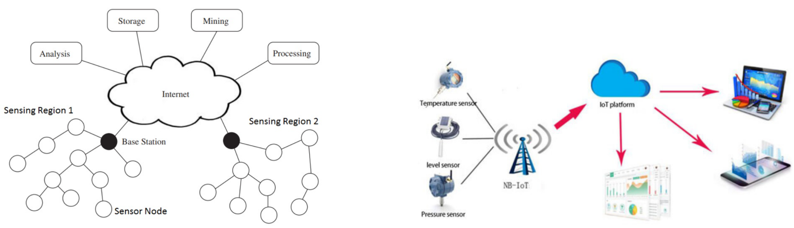


Figure 6.3: Example of Sensor Networks

- **Biological Systems [16], [17]:** The observer-based synchronization methods could also find applications in modeling and controlling biological systems, such as neuronal networks or biochemical processes, where multiple agents interact dynamically. Ensuring synchronization in such systems could help in understanding complex biological phenomena or developing treatments for disorders related to synchronization failures.

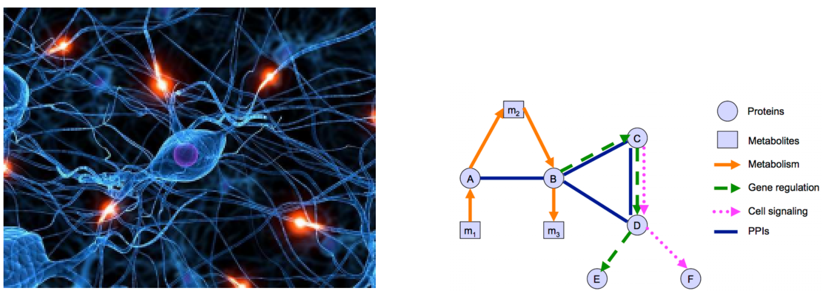


Figure 6.4: Example of Biological Networks

In future work, further investigation will be required to test these theoretical results in large-scale practical implementations, particularly in power grids and other decentralized systems. This research provides a strong foundation for developing control laws tailored to the complexities of modern systems, such as handling noise, nonlinearities, and a transmission line that involves particular dynamics.
In conclusion, the synchronization techniques presented in this thesis represent a crucial step toward more robust and sustainable applications in the context of multi-agent systems. By enabling better synchronization among agents in applications like power grids, robotic swarms, and autonomous vehicles, these methods hold great promise for a wide range of applications.

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