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On Convexity and Stability of Multicommodity Dynamical Flow Networks

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Abstract

Throughout the last few years society had witnessed a drastic change regarding traffic and transportation. The ever-growing congestion and the increasing user's heterogeneity have made clear the importance of mathematical models capable of handling multiple types of flows and minimizing traffic and congestion. In this thesis, multicommodity dynamical flow networks are examined.

In the first part there is a presentation of the model used, which exploits Daganzo's Cell Transmission Model (CTM). This model gives a representation of real life traffic networks which provides a robust tool for modeling and analyzing traffic dynamics. Traffic flow is governed by fundamental principles of mass conservation and the flow-density relationship, identified by supply and demand functions, that limit the flow based on the current density. Examining different models present in the literature led to decide to analyze the set of models (FIFOs and non-FIFOs) such that the cell outflow corresponds exactly to the demand function if the whole system is in freeflow, meaning that the supply constraints are not active. Then the focus shifts to the analysis of the stability, using recent results based on the theory of contractive dynamical systems in single flow networks as a foundation, showing that these results do not extend to multicommodity scenario and providing proofs of the existence of a freeflow equilibrium point and, through the use of Hurwitz stability, of the system's local asymptotic stability.

In the second part the Traffic Assignment Problem, a central problem in transportation systems science, is examined. The objective of the TAP is to find an optimal allocation of traffic so that an aggregate cost function (e.g., the total travel time) of all users in the transportation network is minimized (social optimum). In real-life transportation networks the flow that travels across a road changes over time and this leads to dynamic formulations of the TAP. In particular, the Dynamic Traffic Assignment (DTA) problem is a optimal control problem that entails optimizing dynamic routing and network flows over given transportation network and time horizon. A related problem is the Freeway Network Control (FNC) problem, differing from the DTA problem in that the routing of the users is a known exogenous input, instead of being part of the optimization process. The control must be carried out in a distributed way, so that each road has only access to information about the "neighbouring" ones. However, both these problems are non-convex, but it has been shown that it is possible to relax them through the use of the CTM. By relaxing the supply and demand functions into linear constraints the problems become convex and the flow can be controlled through variable speed limit, ramp metering, and routing. In other words the main objective of the FNC and the DTA is to minimize a given cost function in order to find an optimal dynamic allocation of flow across the network. In this thesis it is studied how these problem can be interpreted and solved in a multi commodity scenario which provides a more accurate representation of real-life traffic networks. For instance, consider a scenario where electric vehicles are permitted to traverse the entire network freely, while petrol cars are restricted from accessing certain roads. In this multi-commodity scenario it is important to control both flows in a separate way, which is achieved by considering two different demand functions and a shared supply function.

This can be interpreted as limiting the amount of flow of a single commodity based on the traffic volume of that commodity in each cell. Moreover, the joint flow entering a cell is also restricted, based on the sum of the traffic volumes of both commodities. The achieved control can then be thought as assigning different variable speed limits to each kind of vehicles based on their density in a certain road. In order to solve both the multi-commodity DTA and FNC, it has been used the Alternating Direction Method of Multipliers (ADMM) introduced by Stephen Boyd as it has already been demonstrated to perform effectively in the single-commodity scenario. The results and simulations show that the selected algorithm is suitable for solving both problems. An optimal value of traffic allocation is reached, which consider different speed limits for the two types of commodities.

Acknowledgements

Before delving into the discussion, I would like to take a moment to express my gratitude to those who have profoundly impacted my journey.

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Chapter 1

Introduction

Technology is advancing at an extremely fast pace and, each year, we witness new innovations across all fields of research. In particular, there is an ongoing revolution concerning transportation and the way society approaches roads and traffic, led not only by progress but also by governments interventions to reduce greenhouse gas emissions. To this end, the European Parliament enacted the Green Deal [1] which is "a package of policy initiatives, which aims to set the EU on the path to a green transition, with the ultimate goal of reaching climate neutrality by 2050". Regarding transportation, it mainly aims to completely reduce CO2 emissions of new cars and light vehicles by 2035, finding in electric vehicles (EVs) the main alternative. Having this objective in mind, companies have focused on the development of new technologies regarding vehicles, such as (semi)autonomous controllers, but also regarding the roads. In fact, the growth of cities and the evergrowing congestion, have made it clear that a rethink of urban design and infrastructure planning is necessary to accommodate cleaner and more efficient forms of transport. In the last few years, we have witnessed the introduction of variable speed limits [2], which adapt on the current amount of traffic volume on a certain road, allowing for higher speeds when it is empty, and, recently, the implementation of freeflow systems [3] which removes toll booths so that drivers can travel without barriers. Moreover, the increasing use of navigation systems, such as Google Maps and Waze, have proven to significantly influence traffic patterns and driver behavior. Using advanced algorithms and real-time data, these systems can suggest alternate routes, avoid congestion, and optimize travel times. However, their widespread use can also lead to unintended traffic consequences. When large numbers of users follow the same recommendations, traffic can be diverted to side streets or residential areas, creating new congestion in places that normally wouldn't experience heavy traffic. However, with the advancement in autonomous vehicles it appears clear how a combination of those along with navigation systems would be capable of reducing congestions and traffic to a groundbreaking minimum. Looking ahead, but closer than 2035, many cities have already limited the entrance to drivers based on their type of car. This is the case of the city of Milan where, since 2022, Euro 2 petrol vehicles and up to Euro 5 diesel vehicles are forbidden to enter "Area B" of Milan [4]. This prohibition raises new challenges since cities are now required to withstand different types of traffic flows, as navigation systems would need to guide unallowed vehicles towards different, and possibly more time consuming, paths. New strategies to direct vehicles and optimize traffic are required, along with a drastic overhaul of cities' infrastructure.

Traffic systems have been widely studied throughout the years. In fact, many studies trace back to 1955 and before. In particular the studies conducted by Lighthill M.J. and Whitham G.B ([5] and [6]) introduced the concept of kinematic waves to describe traffic flow. In their work traffic is modeled to resemble fluid dynamics, where vehicles behave like a continuous flow and changes in traffic density propagate in the form of waves. These waves can move forwards or backwards depending on the traffic conditions. Moreover, their theory explained how traffic jams, known as shockwaves, form and spread on congested roads. The main focus of their research was to provide a mathematical foundation able to predict and analyze traffic behavior, making it a cornerstone of modern traffic flow theory. One of the most important studies that derives from this hydrodynamic theory is the Cell Transmission Model (CTM) ([7] and [8]), introduced by Carlos F. Daganzo in 1994, which has been a great advancement in the field of traffic modeling due to its simplicity and effectiveness in simulating traffic flow on highways and road networks. The CTM is a discretized version of the kinematic wave theory, splitting roadways into small segments, called cells. Each cell represents a section of the road and vehicles flow between these cells based on road's capacity and traffic density. The model keeps track of how many vehicles enter, leave or stay in a cell over discrete time steps. One of the most important feature of the CTM is its ability to efficiently simulate traffic conditions such as congestion, bottlenecks and shockwaves, which makes it a great tool for both theoretical studies and real-world applications, like traffic control and infrastructure planning. Moreover, the CTM has been proven to handle dynamic traffic, while remaining computationally feasible, making it one of the most used models in traffic flow theory. In this work the Cell Transmission Model is used as a starting point, adapting it to handle different types of flow and show how it remains a great tool to model their interaction.

More recent studies have focused not only on the modeling of traffic networks, but on the analysis of their stability to understand how they react to different types of inputs. This helps to analyze the amount of incoming traffic flow each network can sustain, so that if it exceeds a certain limit it can be re-routed towards other roads. Many studies have tackled this problem, but this thesis mainly drew inspiration from the work of E. Lovisari, G. Como and K. Savla [9], in which the authors explores the stability properties of dynamical flow networks, with a focus on systems where the dynamics are governed by monotone interactions. In particular, they investigated how, by defining the traffic network as monotone system, it is possible to introduce a stability region of the exogenous inflows, which is the starting point to their proof of global asymptotically stability. In fact, they prove that by choosing the exogenous inflow within the stability region, the monotone system is locally asymptotically stable. However, by also demonstrating that the system is also non-expansive in the $l1$ norm, it is possible to achieve global asymptotical stability. In this thesis, their approach is generalized, trying to understand wether, with the presence of multiple types of flow, the system remains stable.

Another key aspect of traffic systems studied in recent years, is the optimization of total travel time so that the average time each driver takes to get his destination is reduced to a minimum. A considerable amount of studies have already been conducted regarding static traffic flows merging into two main problems: the System Optimum Traffic Assignment Problem (SO-TAP) and the User Optimum Traffic Assignment Problem (UO-TAP). The main difference between these well-known problem is the fact that in the SO-TAP the optimization is centralized, meaning that the system decides what routes the driver must take so that the total travel time is minimized. On the other hand, in the UO-TAP the optimization is a result of the selfish choices of the users, since each drivers tries to minimize its own travel time. However, one more fascinating field is the study of the dynamic traffic assignement problem, which is able to better model the real world since traffic on a certain road does vary over time. This has made a really interesting topic in the field of traffic networks, leading to the formulation of the Dynamic Traffic Assignment (DTA) and of the Freeway Network Control (FNC), which resemble the SO-TAP and the UO-TAP, respectively, but with time-varying traffic. However, it has already been proven that both problems yield non-convex formulations. Specifically, the work conducted by M. Carey [10] finds the cause in enforcing the FIFO constraints, so that vehicles must follow the order in which they entered the same cell. Nonetheless, further studies have proven how it is possible to reformulate such problems into convex optimal control problems, as in the work conducted by G. Como, E. Lovisari and K. Savla [11]-[12]. The authors of this paper proposed different and convex optimization problems, and then show how an optimal solution of these is also a solution of the original problems. Moreover, thanks to this formulations subsequent studies carried out by Q. Ba, K. Savla and G. Como [13] and C. Rosdahl, G. Nilsson and G.Como [14] have proven how it is possible to solve both problem with the use a slightly modified version of the Alternating Direction Method of Multipliers (ADMM)[15], introduced by S. Boyd, which couples the benefit of dual decomposition and augmented Lagrangian methods. Notably, there are already some studies that try to find a solution to the DTA in a scenario in which different flows can travel through the same network, such as, for example, in S. Samaranayake et al. [10]. However, in that particular study, the DTA is still handled as a non-convex problem, whose solution is obtain through a multi-start approach. In this work, it is evaluated wether it is possible to formulate the DTA and the FNC, in a setting where there are multiple types of flow, as convex optimization problems. Moreover, an iterative distributed algorithm is designed based on the ADMM to find a solution to both problems.

In this thesis, studies regarding standard dynamical flow networks are generalized, so that they can withstand multiple types of flow. Notably, the concept of Multicommodity dynamical flow networks is introduced, where different commodities (like goods, information, or traffic) move through a network and interact with each other, while sharing the same infrastructure. The outline of this thesis is as follows. In Chapter 2 Daganzo's cell transmission model is used as a foundation for this thesis' multicommodity model, providing insightful simulations about system's dynamics. Chapter 3 presents the studies already conducted about the stability of single commodity systems, generalizing them into a multicommodity setting to evaluate stability and the presence of equilibrium points. In Chapter 4 two optimal control problems, the Dynamic Traffic Assignment and the Freeway

Network Control, are introduced, discussing the possibility to achieve a convex formulation and citing previous studies where the solution is obtained despite the non-convexity of the problems. In Chapter 5 the Alternating Direction Method of Multipliers is presented, which is used to solve both the DTA and the FNC, along with simulations to show the improvement obtained with the control.

Chapter 2

Multicommodity Dynamical Flow Networks Model

In the first chapter the model of a multicommodity dynamical flow network, which will be used throughout this thesis, is formulated. Firstly, the concept of transportation network is introduced and then the state and input variables along with the system's dynamics.

2.1 Transportation Network

The network's topology is modeled as a directed multigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with links $i \in \mathcal{E}$ representing sections of the road, called cells, and nodes representing junctions such as ordinary junctions (single incoming and outgoing cell), merge junctions (single outgoing cell), diverge junction (single incoming cell) or mixed junctions (multiple incoming and outgoing cells), as shown in Figure 2.1. Moreover, there are two vectors $\sigma \in \mathcal{V}^{\mathcal{E}}$ and $\tau \in \mathcal{V}^{\mathcal{E}}$ such that σ_i and τ_i are two nodes representing the tail node and head node of a link i in $\mathcal{E}$. Furthermore, there is one special node w in $\mathcal{V}$ which can be interpreted as the external world. It is possible to introduce source and sink cells as the cells whose respectively tail and head node is the external world w in $\mathcal{V}$. Regarding traffic networks source and sink cells can be respectively called on-ramps and off-ramps. The set of source cells is denoted as $\mathcal{R}$ and the set of sink cells as $\mathcal{S}$. Moreover, the set of consecutive cells is defined as $\mathcal{A} = \{(i, j) \in \mathcal{E} \times \mathcal{E} : \tau_i = \sigma_j \neq w\}$.

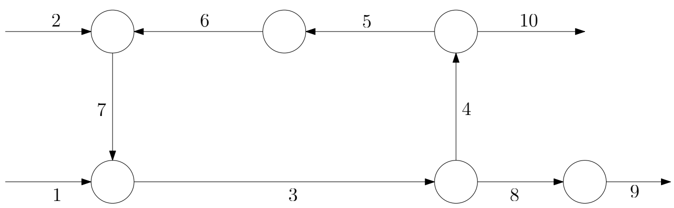


Figure 2.1: Example of Topology

Since this thesis aims at modeling a multicommodity dynamical flow network, it shall allow many different types of flow to traverse the network simultaneously and interact with each other. In order to do so, it is introduced the finite set $\mathcal{K}$ as the set of commodities that are found in the system. A simple example of a network with two commodities can be seen in Figure 2.2.

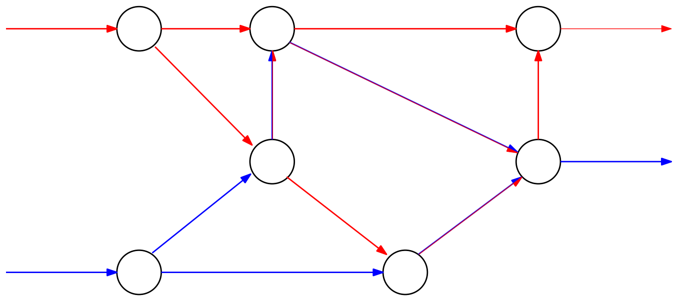


Figure 2.2: Commodities in a network

Then, two functions of great importance in the field of transportation networks are introduced, which are the supply function and the demand function, whose behaviour can be seen in Figure 2.3. They are functions of the traffic volume through the network and they represent the maximum amount of flow that can enter and leave a cell. The demand function $d_i^k(\cdot)$ is piecewise differentiable and increasing, such that $d_i^k(0) = 0$ and defined for each cell i in $\mathcal{E}$ and each commodity k in $\mathcal{K}$. The supply function $s_i(\cdot)$ is differentiable and decreasing, defined for each cell i in $\mathcal{E}$ and shared by commodities.

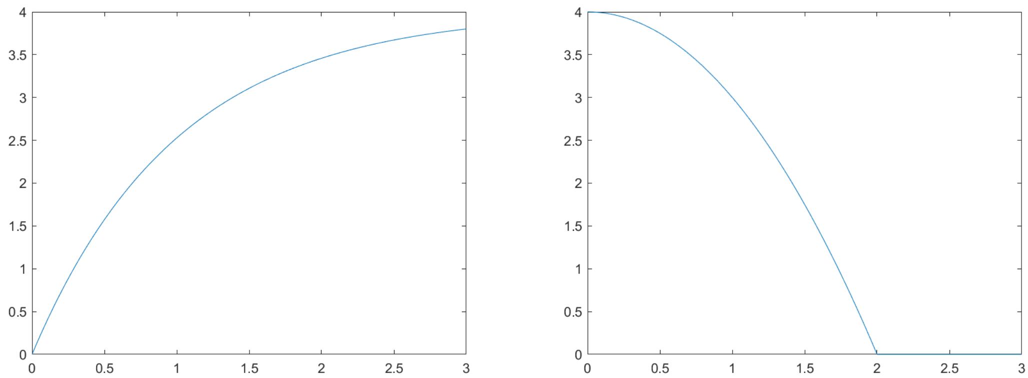


Figure 2.3: Demand and supply functions with respect to the traffic volume

Definition 1. *A Multicommodity Transportation Network (MTN) is the tuple of the multigraph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$*, the set of commodities* $\mathcal{K}$*, demand functions* $\{d_i^k\}_{i\in\mathcal{E},k\in\mathcal{K}}$ *and supply functions* $\{s_i\}_{i\in\mathcal{E}}$.

Then it is possible to define the existence of a *capacity region* of the cells of the MTN, which limits the admissible flow in a cell. In fact, since the supply functions are shared by commodities, if one commodity tries to send more flow the other ones must accordingly send less flow to not exceed the supply function.

Definition 2. *The capacity region of a cell i with demand function $d_i^k(\cdot)$ and supply function $s_i(\cdot)$ is*

$$\mathcal{C}_i = \{z \in \mathbb{R}_+^{\mathcal{K}} : \sum_{k\in\mathcal{K}} z_i^k \leq s_i(\sum_{k\in\mathcal{K}} (d_i^k)^{-1}(z_i^k))\} \tag{2.1}$$

Proposition 1. *If the demand and supply functions are both concave, then the capacity region $\mathcal{C}_i$ of a cell $i \in \mathcal{E}$ is convex.*

Proof. Let's first explicit the condition of the capacity region as

$$\sum_{k\in\mathcal{K}} z_i^k \leq s_i(\sum_{k\in\mathcal{K}} (d_i^k)^{-1}(z_i^k)) \tag{2.2}$$

Since the supply function s_i is concave and strictly decreasing it admits s_i^{-1} also concave and strictly decreasing, which can be applied to both members of (2.2) to obtain the following inequality.

$$s_i^{-1}(\sum_{k\in\mathcal{K}} z_i^k) \geq \sum_{k\in\mathcal{K}} (d_i^k)^{-1}(z_i^k) \tag{2.3}$$

It is then possible to rewrite the equation above as

$$\sum_{k\in\mathcal{K}} (d_i^k)^{-1}(z_i^k) - s_i^{-1}(\sum_{k\in\mathcal{K}} z_i^k) \leq 0 \tag{2.4}$$

Since the demand functions d_i^k are concave and increasing, their inverse $(d_i^k)^{-1}$ are convex and increasing. The supply functions s_i are concave and decreasing, meaning that their inverse s_i^{-1} are concave and decreasing. Therefore, the difference between the two inverse functions is convex and the capacity region $\mathcal{C}_i$ is convex for all i. □

Example 1. *In order to show the capacity region an example is conducted on a very simple network, with two commodities A and B, reported in Figure 2.4 made of only two cells.*

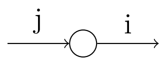


Figure 2.4: Two cells network

In this example the supply is function of the variable x_i^k, representing the traffic volume in each cell for each type of commodity. The objective is to understand what is the form of the capacity region of cell j.

Choosing the demand and supply function of cell j as

$$\begin{aligned} d_j^A(x_j^A) &= 5x_j^A = z_j^A \\ d_j^B(x_j^B) &= 3x_j^B = z_j^B \end{aligned} \tag{2.5}$$

$$s_j(x_j^A + x_j^B) = 10 - 3(x_j^A + x_j^B) \tag{2.6}$$

Notice that it is possible to write the capacity region (2.1) *in this case as*

$$\mathcal{C}_j = \{z \in \mathbb{R}_+^{\mathcal{K}} : z_j^A + z_j^B \leq 10 - 3(\frac{z_j^A}{5} + \frac{z_j^B}{3})\} \tag{2.7}$$

Which can be also rewritten as

$$\mathcal{C}_j = \{z \in \mathbb{R}_+^{\mathcal{K}} : 8z_j^A + 10z_j^B \leq 50\} \tag{2.8}$$

The resulting capacity region is shown in Figure 2.5.

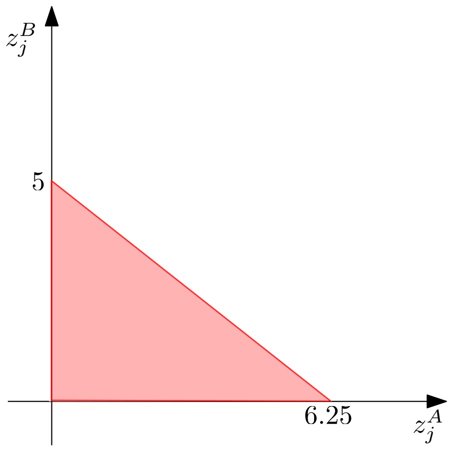


Figure 2.5: Capacity region of a cell with two commodities and linear demand and supply

Example 2. *Another example is taken to show the capacity region assuming the same network shown in Figure 2.4 and the same demand functions in equation* (2.5)*. Moreover, a nonlinear supply constraint is assumed as*

$$s_j((x_j^A)^2 + (x_j^B)^2) = 4 - 2((x_j^A)^2 + (x_j^B)^2) \tag{2.9}$$

Notice that it is possible to write the capacity region (2.1) *in this case as*

$$\mathcal{C}_j = \{z \in \mathbb{R}_+^{\mathcal{K}} : z_j^A + z_j^B \leq 4 - 2((\frac{z_j^A}{5})^2 + (\frac{z_j^B}{3})^2)\} \tag{2.10}$$

Which can be also rewritten as

$$\mathcal{C}_j = \{z \in \mathbb{R}_+^{\mathcal{K}} : 18(z_j^A)^2 + 50(z_j^B)^2 + 225(z_j^A + z_j^B) \leq 900\} \tag{2.11}$$

The resulting capacity region is shown in Figure 2.6.

Remark 1. *In a single commodity case, the capacity region reduces to an interval, which is of course convex.*

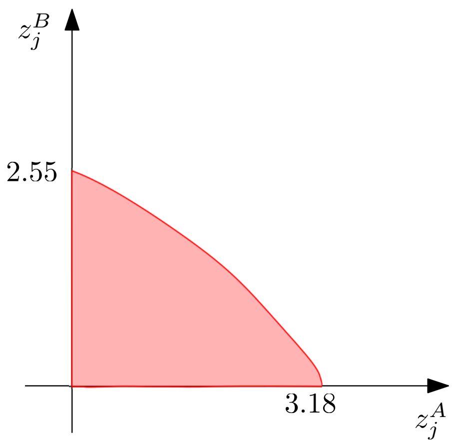


Figure 2.6: Capacity region of a cell with two commodities, linear demand and nonlinear supply

2.2 System's Dynamics and State Variables

The state of the system is identified by the traffic volume (e.g. the amount of vehicles) of each commodity in each cell at time $t \geq 0$

$$x_i^k(t), \qquad i \in \mathcal{E}, \qquad k \in \mathcal{K} \tag{2.12}$$

Moreover, the vector of traffic volumes of the same commodity is

$$x^k(t) \qquad k \in \mathcal{K} \tag{2.13}$$

It is also possible to stack the traffic volume vectors as columns of a matrix, representing the traffic volume of the whole system, which can be called $x(t)$.

The system is subject to exogenous input flows $\lambda_i^k(t) \geq 0$ for k in $\mathcal{K}$ and $t \geq 0$ which represent the amount of traffic coming into the network from the external world. The exogenous inflow is $\lambda_i^k \geq 0$ for each cell i in $\mathcal{R}$ and $\lambda_i^k = 0$ for each cell i in $\mathcal{E} \setminus \mathcal{R}$. It shall be taken into account that vehicles can leave the network and reach the external world which is modeled as an outflow $\mu_i^k(x_i^k) \geq 0$ for k in $\mathcal{K}$ and $t \geq 0$. The outflow is $\mu_i^k \geq 0$ for each cell i in $\mathcal{S}$ and $\mu_i^k = 0$ for each cell i in $\mathcal{E} \setminus \mathcal{S}$. Then the cell-to-cell flow of a commodity $k \in \mathcal{K}$ between two consecutive cells i and j is defined as

$$f_{ij}^k(x) \geq 0, \qquad (i,j) \in \mathcal{A} \qquad k \in \mathcal{K} \tag{2.14}$$

which is 0 if the cell i and cell j are not connected.

It is then possible to introduce a law of mass conservation stating that the variation of traffic volume in a cell depends on the difference between the total inflow y_i^k to and the total outflow z_i^k from the said cell.

$$\begin{aligned} \dot{x}_i^k(t) &= y_i^k(x) - z_i^k(x), \qquad i \in \mathcal{E}, \quad k \in \mathcal{K} \\ y_i^k(x) &= \begin{cases} \sum_{j\in\mathcal{E}} f_{ji}^k(x), & i \in \mathcal{E} \setminus \mathcal{R} \\ \lambda_i, & i \in \mathcal{R} \end{cases} \\ z_i^k(x) &= \begin{cases} \sum_{j\in\mathcal{E}} f_{ij}^k(x), & i \in \mathcal{E} \setminus \mathcal{S} \\ \mu_i^k(x_i^k), & i \in \mathcal{S} \end{cases} \end{aligned} \tag{2.15}$$

In order to close this law of mass conservation it is important carefully define the cell-to-cell flow $f_{ij}^k(x)$. To this end, consider first the demand functions $d_i^k(x_i^k)$ for every i in $\mathcal{E}$ and k in $\mathcal{K}$ and supply functions $s_i(\sum_{k\in\mathcal{K}} x_i^k)$ for every i in $\mathcal{E}$, meaning that the amount of traffic that can enter a cell is limited by the current traffic volume among all commodities.

The definition of the demand and supply functions allows us to introduce two fundamental constraints that limit the total inflow and outflow of each commodity and cell.

Definition 3. *Given a MTN and cell outflow z_i^k, then the demand constraint is*

$$z_i^k(x) \le d_i^k(x_i^k) \quad i \in \mathcal{E} \quad k \in \mathcal{K} \tag{2.16}$$

Definition 4. *Given a MTN and cell inflow y_i^k, then the supply constraint is*

$$\sum_{k\in\mathcal{K}} y_i^k(x) \le s_i(\sum_{k\in\mathcal{K}} x_i^k) \quad i \in \mathcal{E} \tag{2.17}$$

Before moving on, observe that the nonnegative orthant is invariant.

Lemma 1. *The nonnegative orthant $\mathbb{R}_+^n$ is left invariant.*

Proof. Notice that this is always true based on the assumption that if the initial conditions $x(0)$ have all nonnegative entries, so does $x(t)$ for all $t \ge 0$. A sufficient condition for this is the assumption made on $d_i^k(x_i^k)$, which is $d_i^k(x_i^k) = 0$ whenever $x_i^k = 0$. □

Then the routing is introduce through the use of possibly time-varying substochastic matrices $R^k = R^k(t)$, defined as routing matrices, such that

$$R_{ij}^k(t) = 0 \qquad (i,j) \in \mathcal{E} \times \mathcal{E} \setminus \mathcal{A} \tag{2.18}$$

$$\sum_{j\in\mathcal{E}} R_{ij}^k(t) = 1 \qquad i \notin \mathcal{S} \tag{2.19}$$

Equation (2.18) guarantees that the flow leaving a cell is sent only to adjacent cells, while equation (2.19) guarantees that the flow leaving a non-sink cell is distributed among other cells. Moreover, a routing matrix $R^k(t)$ can be interpreted as the routes chosen by the drivers, with their entries $R_{ij}^k(t)$, referred as turning ratios, that represents the percentage of flow from cell i that is directed to cell j.

Definition 5. *Given a MTN and given routing matrices R^k and exogenous inflows λ^k, the Freeflow Region is the region of traffic volume x for which the supply constraints are not active, i.e.*

$$\mathcal{F}(\lambda) = \{x : \sum_{k\in\mathcal{K}} (\lambda_i^k + \sum_{j\in\mathcal{E}} R_{ji}^k d_j^k(x_j^k)) \leq s_i(\sum_{k\in\mathcal{K}} x_i^k), \quad \forall i\} \tag{2.20}$$

The Freeflow constraints are

$$f_{ji}^k(x) = R_{ji}^k d_j^k(x_j^k), \quad \forall x \in \mathcal{F}(\lambda) \quad (j,i) \in \mathcal{A}, \quad k \in \mathcal{K} \tag{2.21}$$

The Freeflow Region is the region for which the supply function of every cell is able to accommodate the whole incoming flow without limiting it and from equation (2.21) it follows that

$$z_i^k(x) = \sum_{j\in\mathcal{E}} f_{ij}^k(x) = \sum_{j\in\mathcal{E}} R_{ij}^k d_i^k(x_i^k) = d_i^k(x_i^k), \qquad i \in \mathcal{E}, \quad k \in \mathcal{K} \tag{2.22}$$

Given all these considerations it is possible to define the Multicommodity Dynamical Flow Network.

Definition 6. *Given a Multicommodity Transportation Network, routing matrices satisfying (2.18) and (2.19) and exogenous inflows λ_i^k, a Multicommodity Dynamical Flow Network (MDFN) is the dynamical system satisfying equations (2.15), (2.16), (2.17) and (2.21).*

Now some relevant examples of Multicommodity Dynamical Flow Networks are introduced.

Example 3. *It is first introduced a FIFO model, that uses the following allocation rule.*

$$\mu_i^k(x) = d_i^k(x_i^k) \qquad i \in \mathcal{S}, \quad k \in \mathcal{K} \tag{2.23}$$

$$f_{ij}^k(x) = \gamma_i^F(x) R_{ij}^k d_i^k(x_i^k), \quad (i,j) \in \mathcal{A}, \quad k \in \mathcal{K} \tag{2.24}$$

where

$$\gamma_i^F = \left\{ \gamma \in [0,1] \ : \ \gamma \cdot \max_{\substack{j\in\mathcal{E} \\ (i,j)\in\mathcal{A}}} \sum_{k\in\mathcal{K}} \sum_{l\in\mathcal{E}} R_{lj}^k \bar{d}_l^k(x_l^k, \alpha_l^k) \leq s_j(\sum_{k\in\mathcal{K}} x_j^k) \right\} \tag{2.25}$$

that is the maximum value in $[0,1]$ such that for each cell j, with (i,j) in $\mathcal{A}$, the supply constraint is satisfied. This means that all the flows sent to downstream cells are scaled by the same value, even if for some cells more flow could be allocated.

In the literature are found examples of FIFOs models in [11], [12], [14].

Example 3.1. *The dynamic of a MDFN is simulated, using the FIFO allocation rule, assuming as the network's topology the directed graph in Figure 2.7.*

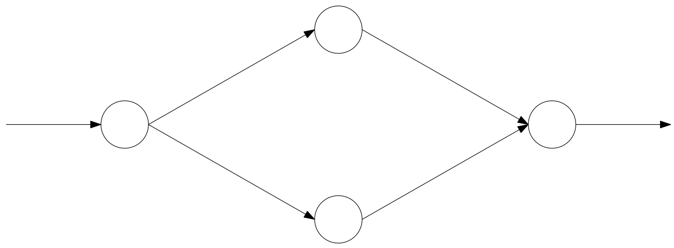


Figure 2.7: Topology of single commodity dynamics

Assuming only two different commodities, namely A and B, and assuming demand and supply functions such as

$$d_i^k(x_i^k) = 3x_i^k \qquad i \in \mathcal{E}, \quad k \in \mathcal{K} \tag{2.26}$$

$$s_i(\sum_{k\in\mathcal{K}} x_i^K) = \begin{cases} 4 - \sum_{k\in\mathcal{K}} x_i^k & i \in \mathcal{E} \setminus \mathcal{R} \\ +\infty & i \in \mathcal{R} \end{cases} \tag{2.27}$$

Moreover, fix the routing matrices R^k as

$$R_A^k = \begin{bmatrix} 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_B^k = \begin{bmatrix} 0 & 0.8 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the exogenous inflow on cell 1 of commodity A to $\lambda_1^A = 2$ and the exogenous inflow on cell 1 of commodity B to $\lambda_1^B = 5$ for the first 15 seconds and set the initial conditions of all cells to zero. Figure 2.8 shows the dynamics of the state of both commodities and it is easily noticeable that almost all cells reach some kind of steady state. However, only the source cell tends to increase due to the congestion in the network, which is quickly solved once the exogenous input has stopped flowing into the network.

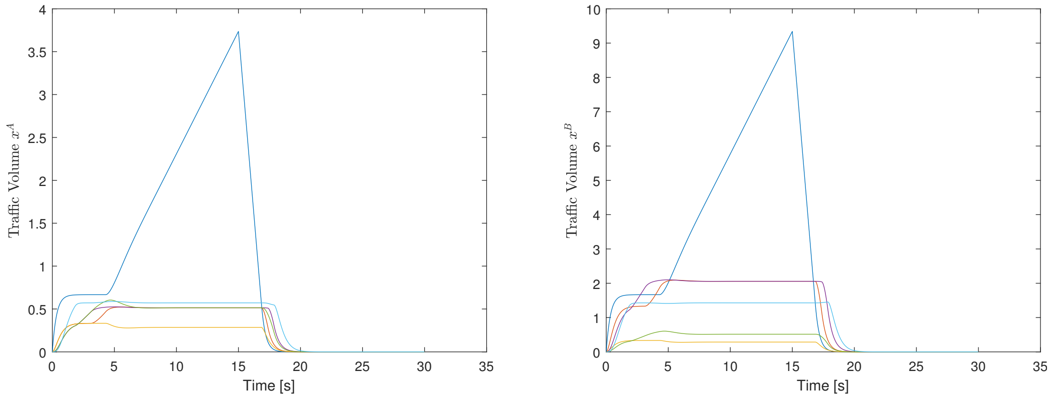
Example 4. *Then it is introduced a non-FIFO model, as the one used in [9], whose allocation rule reads*

$$\mu_i^k(x) = d_i^k(x_i^k) \qquad i \in \mathcal{S}, \quad k \in \mathcal{K} \tag{2.28}$$

$$f_{ji}^k(t,x) = R_{ji}^k(t) d_j^k(x_j^k) \min\left(1, \frac{s_i(\sum_{h\in\mathcal{K}} x_i^h)}{\sum_{l\in\mathcal{E}} \sum_{h\in\mathcal{K}} R_{li}(t)^h d_l^h(x_l^h)}\right), \quad (i,j) \in \mathcal{A}, \quad k \in \mathcal{K} \tag{2.29}$$

which states that when the supply function is not enough to provide for all the incoming flows of all commodities, each flow is properly scaled based on the total amount each cell and commodity want to send.

In the literature are found examples of non-FIFOs models mainly in [9].



Traffic volumes dynamics of commodity A   Traffic volumes dynamics of commodity B

Figure 2.8: Multicommodity system's dynamics

Example 4.1. *One special scenario worth mentioning that has been widely studied in the literature is the single commodity case in which the number of commodities in the network reduces to just one. This thesis' formulation is indeed a generalization of the model employed in [9] that was used to study the stability of dynamical flow networks. In this case the law of mass conservation and allocation rule reduce to*

$$\begin{aligned} \dot{x}_i &= y_i(x) - z_i(x) \\ y_i &= \begin{cases} \sum_{j\in\mathcal{E}} f_{ji}, & i \in \mathcal{E}\setminus\mathcal{R} \\ \lambda_i, & i \in \mathcal{R} \end{cases} \\ z_i &= \begin{cases} \sum_{j\in\mathcal{E}} f_{ij}, & i \in \mathcal{E}\setminus\mathcal{S} \\ d_i(x_i), & i \in \mathcal{S} \end{cases} \end{aligned} \tag{2.30}$$

$$f_{ji}(x) = R_{ji} d_j(x_j) \min\left(1, \frac{s_i(x_i)}{\sum_{l\in\mathcal{E}} R_{li} d_l(x_l)}\right) \tag{2.31}$$

Then the dynamic of a single commodity flow network with topology shown in Figure 2.7 is simulated.

Assume demand and supply functions such as

$$d_i(x_i) = 3x_i \qquad \forall i \tag{2.32}$$

$$s_i(x_i) = \begin{cases} 4 - x_i & i \in \mathcal{E}\setminus\mathcal{R} \\ +\infty & i \in \mathcal{R} \end{cases} \tag{2.33}$$

Moreover, fix the routing matrix R.

$$R = \begin{bmatrix} 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then set the exogenous inflow on cell 1 to $\lambda_1 = 0.5$ *and set the initial conditions of all cells to zero. Figure 2.9 shows the dynamic of the single commodity dynamical flow network. It*

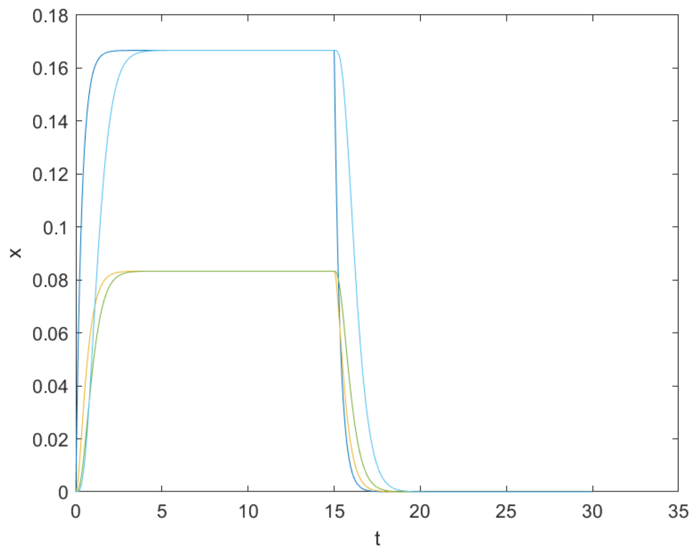


Figure 2.9: Single commodity system's dynamics

is possible to see that, in this case, an equilibrium is reached before the exogenous inflow stops entering the system, thus leading to the network emptying itself.

Example 4.2. *Then simulate the dynamic of a MDFN, using the non-FIFO allocation rule, assuming as the network's topology the directed graph in Figure 2.7.*

Assuming only two different commodities, namely A and B, and assuming demand and supply functions such as

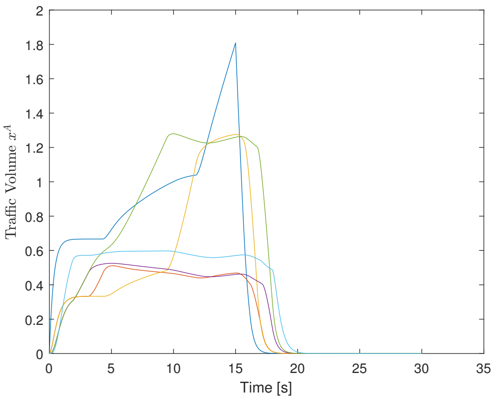
$$d_i^k(x_i^k) = 3x_i^k \qquad i \in \mathcal{E}, \quad k \in \mathcal{K} \tag{2.34}$$

$$s_i(\sum_{k\in\mathcal{K}} x_i^k) = \begin{cases} 4 - \sum_{k\in\mathcal{K}} x_i^k & i \in \mathcal{E} \setminus \mathcal{R} \\ +\infty & i \in \mathcal{R} \end{cases} \tag{2.35}$$

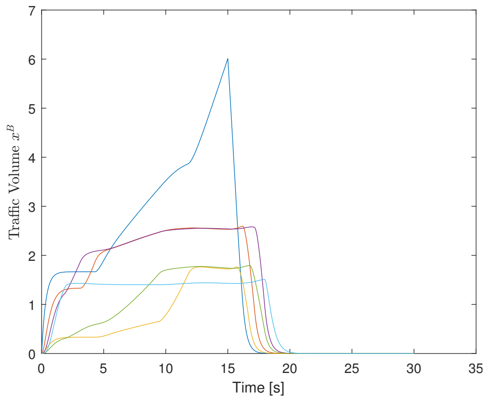
Moreover, fix the routing matrices R^k *as*

$$R_A^k = \begin{bmatrix} 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_B^k = \begin{bmatrix} 0 & 0.8 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the exogenous inflow on cell 1 of commodity A to $\lambda_1^A = 2$ and the exogenous inflow on cell 1 of commodity B to $\lambda_1^B = 5$ for the first 15 seconds and set the initial conditions of all cells to zero. Figure 2.10 shows the dynamics of the state of both commodities and it is easily noticeable that almost no cell reaches a steady state, which means that system could handle more traffic. Moreover, even in this case the traffic volume of the source cells tends to increase, it does so at a slower rate compared to the FIFO model.



Traffic volumes dynamics of commodity A



Traffic volumes dynamics of commodity B

Figure 2.10: Multicommodity system's dynamics

Chapter 3

Stability Analysis

In this chapter the stability of the model introduced in the previous one is investigated. To this end, some common definitions of system stability [16, Chapter 1] are first introduced and then present the work already done to analyze the stability of single commodity dynamical flow networks. Finally, these studies are used to define a generalized formulation which helps us to study the stability of Multicommodity Dynamical Flow Networks.

3.1 Preliminary Stability Concepts

Definition 7. *Given a MDFN with its topology $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, set of commodities $\mathcal{K}$, demand $\{d_i^k\}_{\substack{i\in\mathcal{E}\\k\in\mathcal{K}}}$ and $\{s_i\}_{i\in\mathcal{E}}$ and given constant routing matrices R^k and exogenous inflows λ_i^k, then*

- *The equilibrium point $\bar{x}$ is the solution when the equation (2.15) is set equal to 0;*
- *Given any initial condition $x(0)$, the equilibrium point $\bar{x}$ is stable if*

$$\begin{aligned} \forall \epsilon > 0, \exists \delta > 0 \\ \forall x(0) : ||x(0) - \bar{x}|| < \delta \rightarrow ||x(t) - \bar{x}|| < \epsilon, \quad \forall t \geq 0 \end{aligned} \tag{3.1}$$

- *The equilibrium point $\bar{x}$ is asymptotically stable if*

$$\lim_{t\to\infty} ||x(t) - \bar{x}|| = 0 \tag{3.2}$$

- *The equilibrium point $\bar{x}$ is globally asymptotically stable if it is asymptotically stable for every initial condition x(0);*
- *The equilibrium point $\bar{x}$ is unstable if it is not stable.*

Example 5. *Take into consideration the model proposed in the previous chapter and see how it reacts to different inputs. In order to do so, it is chosen the same topology used in the single commodity example in Figure 2.7. The considered demand function is*

$$d_i^k(x^k(t)) = 3x_i^k(t) \tag{3.3}$$

and the supply function is

$$s_i(\sum_{k\in\mathcal{K}} x_i^k(t)) = 2 - \sum_{k\in\mathcal{K}} x_i^k(t) \tag{3.4}$$

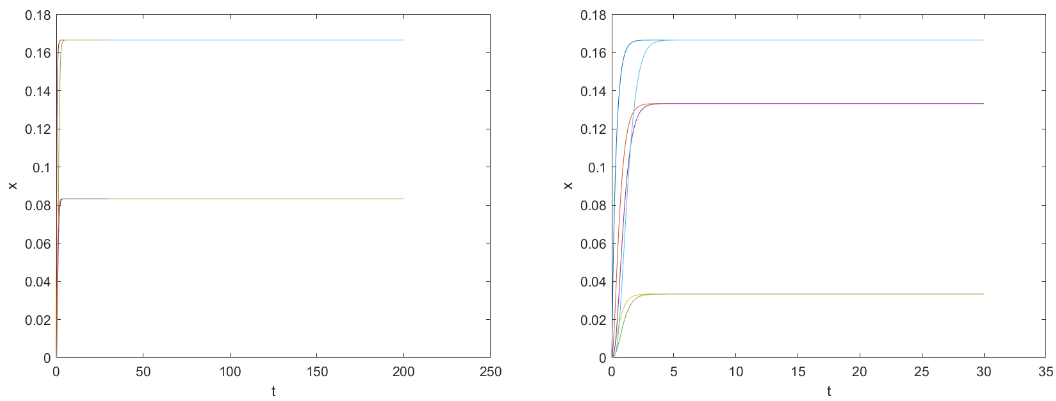
Simulate first the dynamics considering an exogenous inflow

$$\lambda_1^A = 0.5 \qquad \lambda_1^B = 0.5 \tag{3.5}$$

and fixing the routing matrices R^k as

$$R_A^k = \begin{bmatrix} 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_B^k = \begin{bmatrix} 0 & 0.8 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and report the results in Figure 3.1.



Commodity A   Commodity B

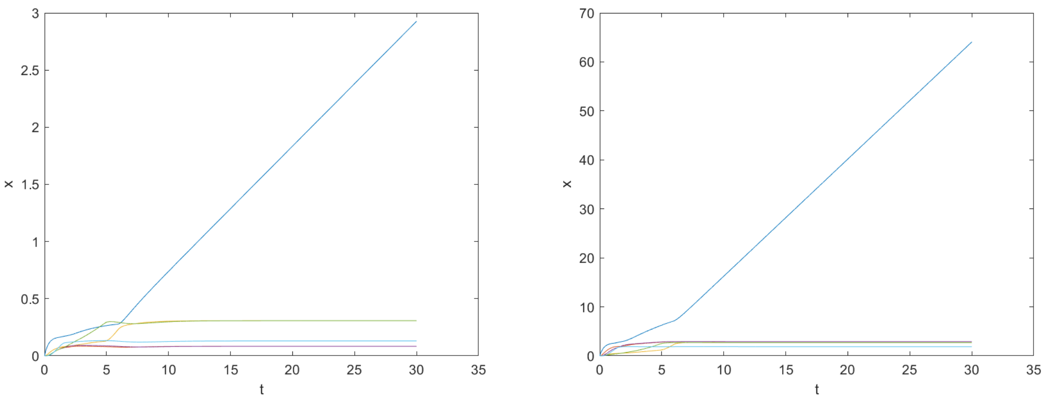
Figure 3.1: Multicommodity stability

Then simulate the dynamics again, keeping fixed the routing matrices R^k but changing the exogenous inflow to

$$\lambda_1^A = 0.5 \qquad \lambda_1^B = 8 \tag{3.6}$$

and report the results in Figure 3.2.

It is easily noticeable how in Figure 3.1 the traffic volumes reach an equilibrium, whereas in Figure 3.2 the trajectories grow unbounded. The main objective in this chapter shall be to understand whether and under what conditions a Multicommodity Dynamical Flow Network is stable. Moreover, it is investigated if there exists one (or more) locally or globally asymptotically stable equilibrium point. However, in the next section are presented the studies already conducted present in the literature, which have been the starting point and inspiration for this research.



Commodity A   Commodity B

Figure 3.2: Multicommodity instability

3.2 Previous results in the single commodity case

Single commodity dynamical flow networks have been widely studied throughout the years and it has already been proven in [9] that under some assumptions it is possible to demonstrate how the system is indeed globally asymptotically stable. In this section their work is presented by stating their results and giving a brief explanation on how they proceeded to prove the global stability. Firstly, they considered a dynamical system defined by the system (2.30) and (2.31) and defining

$$\phi(x) = \dot{x} \tag{3.7}$$

Moreover, they assumed a substochastic and outconnected routing matrix. By combining this assumption with the routing policy in equation (2.31), they call (2.30) as the dynamical flow network with fixed preference rates.

Then, they proceeded by introducing a stability region Λ, defined as the largest set of exogenous inflows for which the system (2.30)-(2.31) is stable

$$\Lambda := \{\lambda \in \mathbb{R}_+^{\mathcal{R}} : \max_{i \in \mathcal{E}} x_i^*(\lambda) < +\infty\} \tag{3.8}$$

Moreover, they defined a set $\mathcal{B} \subset \Lambda$ as the set of λ for which there exists $i \in \mathcal{E} \setminus \mathcal{R}$ such that the summation of all the flows coming into the cell is equal to its supply function

$$\sum_{j \in \mathcal{E}} R_{ji} d_j(x_j^*(\lambda)) = s_i(x_i^*(\lambda)) \tag{3.9}$$

Then for every $\lambda \in \Lambda \setminus \mathcal{B}$ it is possible to introduce the dual graph $\mathcal{G}^d$ associated with $x^*(\lambda)$. Let

$$J_\lambda = \nabla \phi(x)|_{x=x^*(\lambda)} \tag{3.10}$$

Then the dual graph $\mathcal{G}^d = (\mathcal{V}^d, \mathcal{E}^d)$ has set of nodes $\mathcal{V}^d = \mathcal{E}$ and has ana edge (i, j) in $\mathcal{E}^d$ if $[J_\lambda]_{ji} > 0$. Moreover, the dual graph is rooted if for every i in $\mathcal{V}^d$ there exists a directed path from i to an offramp j in $\mathcal{R} \subseteq \mathcal{V}^d$. The main result is defined in [9, Theorem 1] which is reported below for completeness.

Theorem 1. *Consider the dynamical flow network with fixed preference rates (2.30) with inflow vector* $\lambda \in \Lambda \setminus \mathcal{B}$*. Assume that the dual graph* $\mathcal{G}^d$ *is rooted. Then* $x^*(\lambda) = \lim_{t\to\infty} \phi^t(0, \lambda)$ *is a globally asymptotically stable equilibrium.*

In order to prove their result they exploited the theory about monotone and contractive systems. In fact, they proved that the system (2.30)-(2.31) is monotone by using Kamke's theorem which states that a system ϕ is monotone if for almost all x

$$\frac{\partial \phi_i(x)}{\partial x_k} \geq 0 \qquad \forall i \neq k \tag{3.11}$$

The fact that the system is monotone allows the authors of the paper to state a dichotomy because either the system is bounded for each initial condition or each trajectory grows unbounded. Then, after defining the set Λ and introducing the dual graph $\mathcal{G}^d$, they prove that the system is locally asymptotically stable by noticing that the eigenvalues of the sublaplacian of the dual graph are all inside the unit circle. Moreover, they proved global stability by proving that the system is also l_1 non-expansive, thus every locally asymptotically stable equilibrium point is also globally asymptotically stable.

Example 6. *Then the dynamics of the system are simulated, to try to understand how* x *changes when the inflow vector* λ *varies. In order to do so is used the network in Figure 3.3 and a very simple demand function as*

$$d_i(x_i(t)) = x_i(t) \qquad i \in \mathcal{E} \tag{3.12}$$

along with a supply function of the form

$$s_i(x_i(t)) = 2C_i - x_i(t) \qquad i \in \mathcal{E} \tag{3.13}$$

where C_i *is the capacity of cell* i*. Moreover, fix* $\lambda_1 = 0.5$ *and vary* λ_2*.*

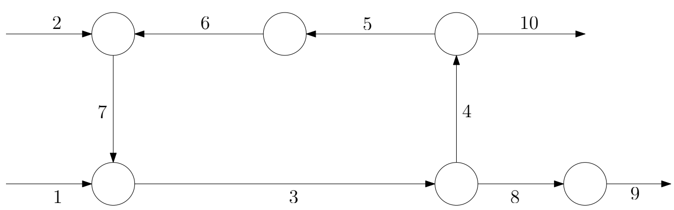


Figure 3.3: Topology for single commodity stability analysis

Then the values of x *under different* λ_2 *are plotted, to understand the stability of the system within and outside the set* Λ*. In Figure 3.4 it is possible to notice the volume of cells 2, 5, 8 and 9 and it is easily noticeable that until* $\lambda_2 \leq 0.125$ *the system as a* *unique globally asymptotically stable equilibrium. At $\lambda_2 = 0.125$ the system has infinite equilibrium points since x_8 can assume many different values. When $\lambda_2 > 0.125$ the system still has one (or more) equilibrium points, but when $\lambda_2 \to 0.8$, ρ tends to increase. Furthermore, when $\lambda_2 = 0.8$ (and even when $\lambda_2 > 0.8$) the system becomes unstable and each trajectory grows unbounded in x_2.*

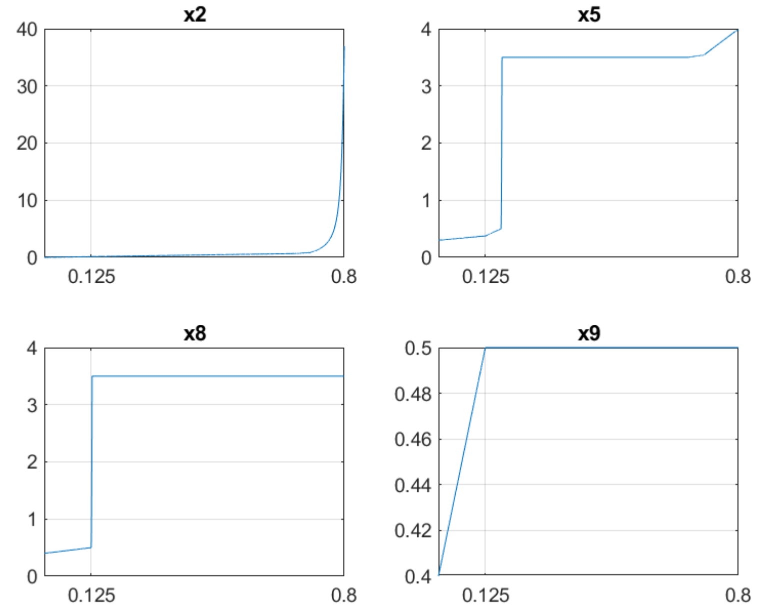


Figure 3.4: Volume values of cells 2, 5, 8 and 9 with respect to different λ

3.3 Stability of Multicommodity Dynamical Flow Networks

In this section it is investigated the stability of the Multicommodity Dynamical Flow Network, by discussing the existence of an equilibrium point inside the Freeflow Region (2.20).

Proposition 2. *Given a MDFN define $\bar{z}_i^k = ((I - (R^k)')^{-1}\lambda^k)_i$ and $\{\bar{z}_i^k\}_{k\in\mathcal{K}}$ as the vectors of the $\bar{z}_i^k$ of all commodities on the same cell i. Then there exists a unique Freeflow equilibrium point if and only if the following holds*

$$\{\bar{z}_i^k\}_{k\in\mathcal{K}} \in \mathcal{C}_i, \quad \forall i \in \mathcal{E} \tag{3.14}$$

Proof. If it is considered a Freeflow equilibrium point, it must satisfy equation (2.15) written in matrix form as

$$\lambda^k + (R^k)' z^k - z^k = 0 \qquad \forall k \tag{3.15}$$

It can be assumed that for each cell $i \in \mathcal{E} \setminus \mathcal{S}$ there exists at least one path to a cell $j \in \mathcal{S}$. This is consistent with the idea that each particle in the network must have a path to leave it. By considering this assumption and noticing that matrices $(I - (R^k)^{-1})$ are Metzler so each one of them is then invertible as in [9] and brings to a unique solution of equation (3.15) as

$$\bar{z}^k = (I - (R^k)')^{-1}\lambda^k \qquad \forall k \tag{3.16}$$

Since the equilibrium is in the Freeflow Region, then there is no congestion, so equation (2.22) can be used to compute the state x^k as the inverse of the demand as

$$\bar{x}_i^k = (d_i^k)^{-1}(\bar{z}_i^k)) = (d_i^k)^{-1}(I - (R^k)')^{-1}\lambda^k)_i \tag{3.17}$$

which is unique. Then, by substituting equations (3.16) and (3.17) in the condition defined by (2.20) the following is obtained

$$\sum_{k\in\mathcal{K}}(\lambda_i^k + \sum_{j\in\mathcal{E}} R_{ji}^k \bar{z}_j^k) \le s_i(\sum_{k\in\mathcal{K}}(d_i^k)^{-1}(\bar{z}_i^k)) \quad \forall i \in \mathcal{E} \tag{3.18}$$

Notice that it is possible to restate the first term of the inequality as

$$\sum_{k\in\mathcal{K}} \bar{z}_i^k \le s_i(\sum_{k\in\mathcal{K}}(d_i^k)^{-1}(\bar{z}_i^k)) \quad \forall i \in \mathcal{E} \tag{3.19}$$

which corresponds to the inequality that defines the capacity region (2.1), thereby validating (3.14).

Moreover, it is immediate to see that if (3.14) holds, then the system is in freeflow. Then for equations (3.15)-(3.17) the equilibrium is unique.

□

In the next definition, it is introduced a stability region to then study the stability of the system given different exogenous inflows λ. Firstly, it is defined a diagonal block matrix M such that it has $(I - (R^k)')^{-1}$ for each commodity k in $\mathcal{K}$ in the diagonal. For example with two commodities, A and B, this matrix would be

$$M = \begin{bmatrix} (I - (R^A)')^{-1} & 0 \\ 0 & (I - (R^B)')^{-1} \end{bmatrix}$$

It is now introduced the concept of Stability Region, which is the region of the exogenous inflows for which the whole system is in freeflow.

Definition 8. *Given a MDFN and its Freeflow Region $\mathcal{F}(\lambda)$. Then the stability region Λ is the set of exogenous inflows such that*

$$\Lambda = \{(M\lambda)_i \in \mathcal{C}_i, i \in \mathcal{E}\} \tag{3.20}$$

Lemma 2. *Given a MDFN. If the demand supply functions are concave, then the stability region Λ is convex.*

Proof. Recall the proof of Proposition 1, that shows the convexity of the capacity region $\mathcal{C}_i$ for each cell i. By extending this result, the Stability region is a product of convex regions, meaning that it is indeed convex. □

Then, properties of the routing matrices are exploited to prove that the Freeflow equilibrium point is stable.

Proposition 3. *Given a MDFN whose routing matrices R^k are both substochastic and out-connected, then the freeflow equilibrium point is locally asymptotically stable.*

Proof. Firstly, it is proven that that R^k is Schur stable. Let λ_R^k be the dominant eigenvalue of R^k. Since $(R^k)'$ has the same eigenvalues of R^k then there exists a nonnegative eigenvector y in $\mathbb{R}_+^n$ such that $(R^k)'y = \lambda_R^k y$. Then, enumerate the elements of the eigenvector and find its minimum element as

$$\mathcal{J} = \{i = 1, \dots, n : y_i > 0\}, \qquad y_* = \min_{j \in \mathcal{J}} y_j > 0 \tag{3.21}$$

Since R^k is substochastic and out-connected

$$\min_{i \in \mathcal{J}} \sum_{j \in \mathcal{J}} R_{ij} = 1 - \gamma < 1 \tag{3.22}$$

Then, it is considered

$$\lambda_R^k \sum_{j \in \mathcal{J}} y_j = \sum_{1 \le i \le n} \sum_{j \in \mathcal{J}} R_{ij}^k y_i = \sum_{i \in \mathcal{J}} y_i \sum_{j \in \mathcal{J}} R_{ij}^k \le \sum_{i \in \mathcal{J}} y_i - y_* \gamma < \sum_{j \in \mathcal{J}} y_j \tag{3.23}$$

proving that $\lambda_R^k < 1$, and thus R^k is Schur stable.

Then, recall the law of mass conservation in matrix form and compute its gradient

$$f^k(x^k) = \lambda^k - (I - (R^k)')z^k(x), \qquad \nabla f^k(x^k) = (I - (R^k)')(z^k)'(x) \tag{3.24}$$

Then, it can be introduced a positive diagonal matrix $D = (z^k)'(x)$ which will be used to prove the stability of matrix $A = (I - (R^k)')D$. First, notice that matrix A is compartmental and let λ_A be its dominant eigenvalue. Using Corollary 1 there exists a nonnegative eigenvector $y \in \mathbb{R}_+^n$ such that $A'y = \lambda_A y$. Then, defining $\mathcal{J}$ and y_* as in (3.21) and let $d_* = \min\{D_i i : i \in \mathcal{J}\}$. Then, since matrix R^k is substochastic and out-connected it is possible to formulate that

$$\min_{i \in \mathcal{J}} \sum_{j \in \mathcal{J}} A_{ij} = \min_{i \in \mathcal{J}} \sum_{j \in \mathcal{J}} D_i i(1 - R_{ij}^k) \le -d_* \gamma < 0 \tag{3.25}$$

Then, it is obtained

$$\lambda_A \sum_{j \in \mathcal{J}} y_j = \sum_{1 \le i \le n} \sum_{j \in \mathcal{J}} A_{ij} y_i = \sum_{i \in \mathcal{J}} y_i \sum_{j \in \mathcal{J}} A_{ij} \le -y_* d_* \gamma < 0 \tag{3.26}$$

thus proving that $\lambda_A < 0$, meaning that matrix A is Hurwitz stable proving that the Freeflow equilibrium point is indeed locally asymptotically stable. □

Example 7. *It is now presented an example to show that the stability region is indeed convex. In order to do so it is taken a network in which the commodities enter and leave the network in different cells.*

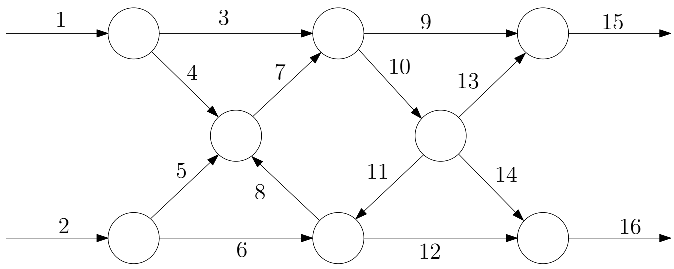


Figure 3.5: Toplogy for the analysis of the convexity of the stability region

Fixing the routing matrix R^A *of commodity A as* $R^A_{1,3} = R^A_{1,4} = 0.5$, $R^A_{10,11} = 0.8$, $R^A_{10,14} = 0.2$, $R^A_{11,8} = 0.6$, $R^A_{11,12} = 0.4$, $R^A_{14,16} = 1$, $R^A_{12,16} = 1$ *and 0 otherwise. Moreover, fixing the routing matrix* R^B *of commodity B as* $R^B_{2,5} = R^B_{2,6} = 0.5$, $R^B_{7,9} = 0.2$, $R^B_{7,10} = 0.8$, $R^B_{10,11} = 0.6$, $R^B_{10,13} = 0.4$, $R^B_{9,15} = 1$, $R_{13,15} = 1$ *and 0 otherwise.*

Choosing again as demand and supply functions the following ones

$$d_i^k(x_i^k) = \gamma_i^k(1 - e^{\alpha_i^k x_i^k}) \qquad i \in \mathcal{E}, \quad k \in \mathcal{K} \tag{3.27}$$

$$s_i(\sum_{k\in\mathcal{K}} x_i^k) = \begin{cases} b_i - \frac{a_i^2}{b_i}\sum_{k\in\mathcal{K}} x_i^k & i \in \mathcal{E}\setminus\mathcal{R} \\ +\infty & i \in \mathcal{R} \end{cases} \tag{3.28}$$

Fixing the parameters to

$$\gamma_i^k = 5 \quad \alpha_i^k = 2 \quad b_i = 4 \quad a_i = 2 \quad i \in \mathcal{E}, \quad k \in \mathcal{K} \tag{3.29}$$

Then iteratively increase the exogenous inflows $\lambda_1 = \lambda_1^A$ *and* $\lambda_2 = \lambda_2^B$ *until they exit from the region defined in (3.20).*

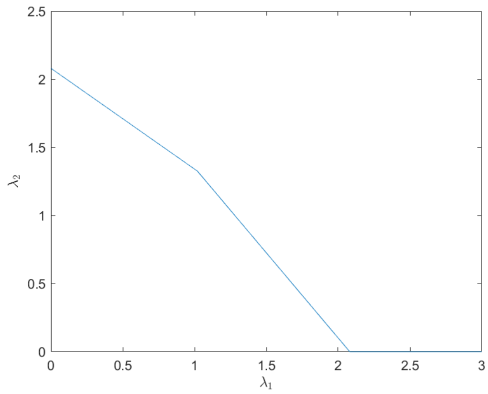


Figure 3.6: Stability Region

Figure 3.6 shows the stability region and it is easily noticeable that such region is indeed convex.

Example 8. *Now simulate the dynamics of the system by considering different initial conditions to evaluate how the system acts when starting inside and outside of the freeflow region. Moreover, it is evaluated how the system acts whether the exogenous inflow is inside or outside the stability region. Set the topology as in Figure 3.3*

Then, define the demand and supply function as

$$d_i^k(x_i^k) = 3x_i^k \qquad i \in \mathcal{E}, \quad k \in \mathcal{K} \tag{3.30}$$

$$s_i(\sum_{k\in\mathcal{K}} x_i^k) = \begin{cases} 2C_i - \sum_{k\in\mathcal{K}} x_i^k & i \in \mathcal{E} \setminus \mathcal{R} \\ +\infty & i \in \mathcal{R} \end{cases} \tag{3.31}$$

Morover, fix the exogenous inflows to $\lambda_1^A = 0.5$ and $\lambda_2^B = 0.2$ and the initial conditions either to $x(0) = 0$ or to $x(0) = 10$, considering routing matrix R^A such that $R_{1,3}^A = 1$, $R_{2,7}^A = 1$, $R_{3,4}^A = 0.5$, $R_{3,8}^A = 0.5$, $R_{4,5}^A = 0.75$, $R_{4,10}^A = 0.25$, $R_{5,6}^A = 1$, $R_{6,7}^A = 1$, $R_{7,3}^A = 1$, $R_{8,9}^A = 1$ and 0 otherwise. Then set the routing matrix R^B such that $R_{1,3}^B = 1$, $R_{2,7}^B = 1$, $R_{3,4}^B = 0.5$, $R_{3,8}^B = 0.5$, $R_{4,5}^B = 0.6$, $R_{4,10}^B = 0.4$, $R_{5,6}^B = 1$, $R_{6,7}^B = 1$, $R_{7,3}^B = 1$, $R_{8,9}^B = 1$ and 0 otherwise.

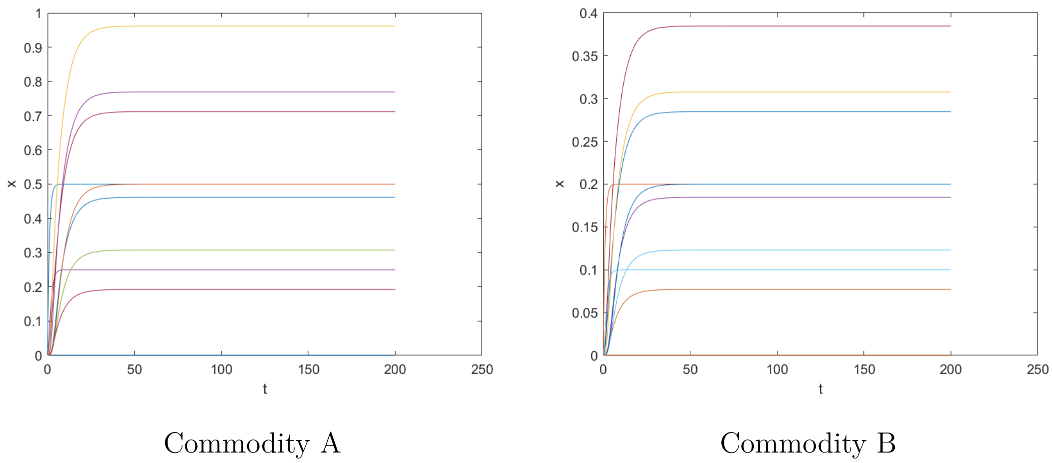
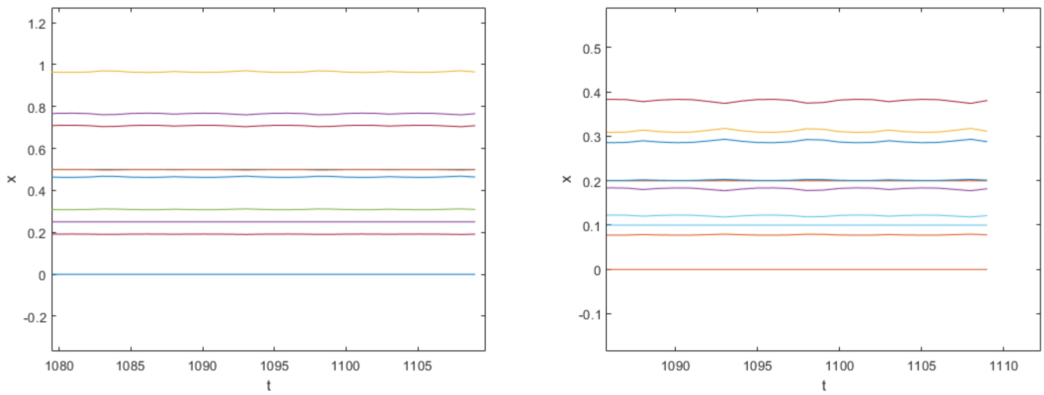


Figure 3.7: Multicommodity System Dynamics with $x(0) = 0$

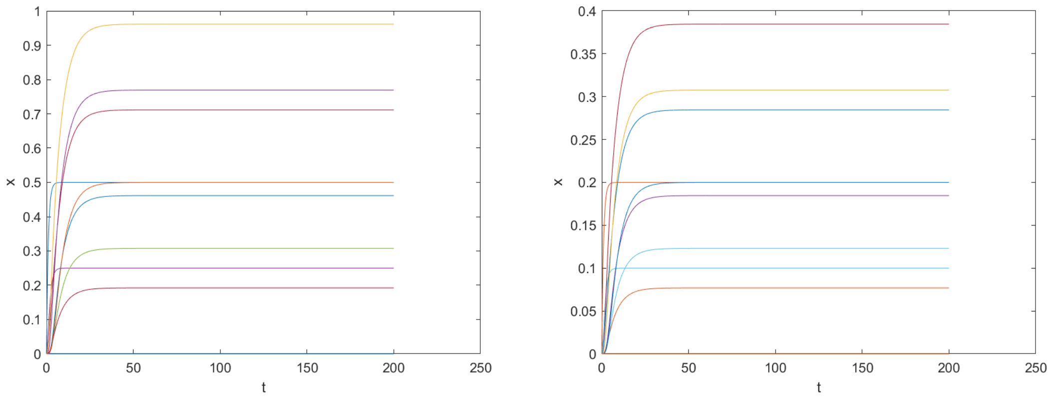
Figure 3.7 and 3.8 shows the dynamic of the system starting from two different initial conditions. It is easily noticeable that in both cases the same equilibrium point is reached even though in the second case the system starts outside the freeflow region this could imply that the freeflow equilibrium point is indeed globally asymptotically stable which could be proven with the use of a suitable Lyapunov function.

Example 9. *Now simulate again the dynamics of the system by considering the initial conditions $x(0) = 0$ and changing the exogenous inflows. In the first case it is considered $\lambda_1^A = 0.5$ and $\lambda_2^B = 0.2$, whereas in the second one fix $\lambda_1^A = 0.5$ and $\lambda_2^B = 1$. Notice that in the second case, it is chosen λ_2^B such that the exogenous inflows are now outside of the convex stability region and this could be helpful to understand the existence of other equilibrium points other than the freeflow equilibrium point.*



Zoom-in Commodity A   Zoom-in Commodity B

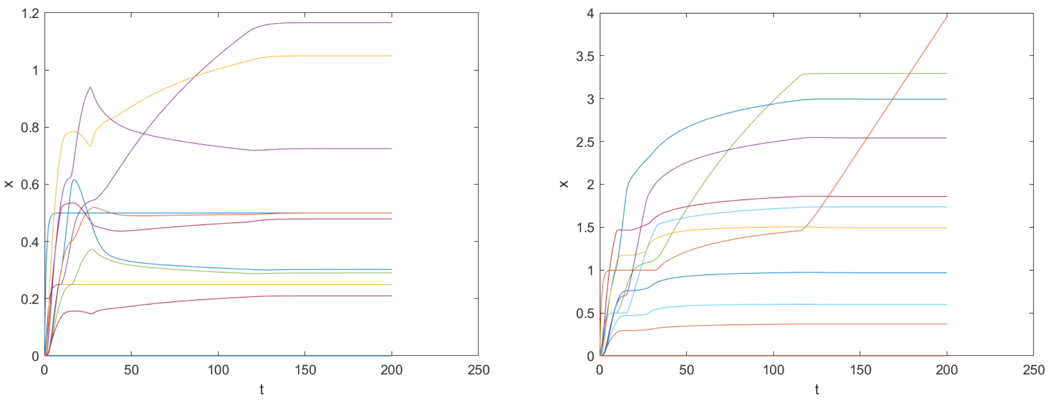
Figure 3.8: Multicommodity System Dynamics with $x(0) = 10$



Commodity A   Commodity B

Figure 3.9: Multicommodity System Dynamics with $\lambda_1^A = 0.5$ and $\lambda_2^B = 0.2$

In figure 3.9 it is possible to notice that by choosing the exogenous inflows such that they are inside of the stability region then the system does indeed converge to the freeflow equilibrium point. However, in Figure 3.10 it is easily noticeable that by choosing the exogenous inflows outside of the stability region the system loses its monotonicity and does not converge anymore to the freeflow equilibrium point. Moreover, it is not possible to talk about a true equilibrium point since, as it can be seen in the picture regarding commodity B, there is one cell volume that diverges (cell 2). This happens because, by choosing such exogenous inflows, congestion is unavoidable and the volume of the cell which receives the exogenous inflow grows unbounded. Furthermore, the other cells do not reach a true equilibrium either since their steady state value is only achieved based on the level of congestion reached.



Commodity A   Commodity B

Figure 3.10: Multicommodity System Dynamics with $\lambda_1^A = 0.5$ and $\lambda_2^B = 1$

3.4 Conjectures and Future Research

In this chapter the stability of Multicommodity Dynamical Flow Networks is examined. It was possible to define a Freeflow Region, the region of the traffic volume such that the supply constraints are not active, and find an equilibrium point within this region if the exogenous inflow is chosen inside a Stability Region. Moreover, it is proven that this freeflow equilibrium point is locally asymptotically stable.

Given the results obtained in this chapter it is possible to formulate some conjectures, supported by simulations, that would be of great interest for future researches. In Proposition 2 is stated that there exists a unique Freeflow Equilibrium point if inequality (3.14) is satisfied. However, this thesis does not evaluate the existence of equilibrium points if the inequality is not satisfied. Many simulations have shown us that, by doing so, the system will not reach an equilibrium since the traffic volume of one (or more) source cell will eventually grow unbounded. Nevertheless, simulations are not enough to state this as a result of this thesis and an analytic proof is not straightforward, leading us to leave this topic open for future studies.

Moreover, in Proposition 3 is proved that the Freeflow Equilibrium point is locally asymptotically stable. Even though I tried to also prove global asymptotical stability, I was not able to found a solution yet. Given the multicommodity nature of this thesis' scenario, it is not possible to use the approach adopted in [9] since the system loses its monotonicity outside the freeflow region. A solution may be found using a suitable Lyapunov function and then making some consideration about its form, possibly exploiting the work done in [17].

Chapter 4

Dynamic Traffic Assignment and Freeway Network Control

This chapter presents two different optimal control problems that aim to optimize allocated flow in multicommodity networks by minimizing a given cost function while satisfying supply and demand constraints.

The Dynamic Traffic Assignment (DTA) has been widely studied throughout the years in a single commodity setting, so the objective is to generalize the formulation for multicommodity networks by achieving control through the use of ramp metering, variable speed limits and variable routing. The Freeway Network Control (FNC) can be seen as a simplification of the DTA in which the control is achieved only through the use of ramp metering and variable speed limits since the routing is a known exogenous input. Since both the DTA and FNC are known to lead to non-convex problems (as shown in [10]), it is then presented a way in which is possible to relax the demand and supply functions in order to achieve a linear program.

4.1 Problems formulation

Firstly, it is presented a controlled version of the Multicommodity Dynamical Flow Network Model presented in Chapter 2. To this end, it is introduced the concept of control parameters $\alpha_i^k(t) \in [0,1]$ that are used to rescale the demand functions $d_i^k(x_i^k)$ and the maximum outflow C_i^k for each source cell i in $\mathcal{R}$. These control parameters are used, on non-source cells, to control the demand functions to obtain variable speed limits (VLS), whereas, on source cells, they control the maximum outflow so that these cells can be thought as frontiers and obtain ramp metering.

Definition 9. *Given a MDFN and given control parameters $\alpha_i^k(t)$, then variable speed limiting corresponds to reduce the demand functions as*

$$\bar{d}_i^k(x_i^k(t), \alpha_i^k(t)) = \alpha_i^k(t) d_i^k(x_i^k(t)) \qquad i \in \mathcal{E} \setminus \mathcal{R}, \quad k \in \mathcal{K} \tag{4.1}$$

Remark 2. *Notice how the previous definition states that each commodity inside the network can have its own variable speed limit. In order to grasp this concept, imagine a* *setting in which on the same roads are present both cars and trucks, where trucks must travel at a much lower speed.*

Definition 10. *Given a MDFN and given control parameters $\alpha_i^k(t)$, then ramp metering corresponds to reduce the maximum outflow of source cells as*

$$\bar{d}_i^k(x_i^k(t), \alpha_i^k(t)) = min(d_i^k(x_i^k(t)), \alpha_i^k(t)C_i^k) \qquad i \in \mathcal{R}, \quad k \in \mathcal{K} \tag{4.2}$$

Remark 3. *Given an initial assignment of traffic volumes*

$$x_i^k(0) = (x_i^0)^k, \qquad i \in \mathcal{E}, \qquad k \in \mathcal{K} \tag{4.3}$$

and for each commodities exogenous inflows $\lambda_k(t)$, routing policies $R^k(t)$ and control parameters $\alpha^k(t)$, the traffic's evolution $x^k(t)$ for $t \geq 0$ is uniquely determined by equations (2.15) *and* (2.21).

Both the FNC and DTA problems, that are yet to be introduced, can be cast as open-loop optimal control problems as a minimization of a cost function $\varphi(x, z)$ over a time interval [0,T] where T > 0 is a given finite time horizon. The cost function is supposed to be convex in (x, z), non-decreasing in x, non-increasing in z and such that $\varphi(0{,}0) = 0$. The cost function can assume many meanings such has

- Total travel time, such that $\varphi_i(x_i, z_i) = \sum_k x_i^k$. Notice that $\int_0^T \sum_k x_i^k(t)\, dt$ can be thought as the total time spent by all kinds of vehicles on cell i over the interval [0,T];
- Total travel distance, such that $\varphi_i(x_i, z_i) = -l_i \sum_k z_i^k$ where l_i is the length of cell i. Notice that $\int_0^T l_i \sum_k z_i^k(t)\, dt$ can be thought as the total distance travelled by drivers in cell i over the interval [0,T]. The minus sign is just to specify that the cost function must be maximized and not minimized.

It is now possible to formulate the DTA and FNC as optimization problems. In the DTA, assume that, given the initial traffic volumes and the exogenous inflow, it is possible to control both the demand functions and the routing matrix within the constraints while in the FNC problem assume that the routing matrix is a known exogenous input.

Definition 11. *Given a MDFN, initial conditions $(x_i^0)^k$, a finite time horizon T and a cost function $\varphi(x, z)$, the multicommodity Dynamical Traffic Assigment can be formulated as*

$$\min_{\substack{\alpha(t) \quad R(t) \\ (2.15),(2.16),(2.17),(2.18),(2.19),(2.21)}} \int_0^T \varphi(x(t), z(t))dt \tag{4.4}$$

Definition 12. *Given a MDFN, initial conditions $(x_i^0)^k$, a finite time horizon T and a cost function $\varphi(x, z)$, the multicommodity Freeway Network Control can be formulated as*

$$\min_{\substack{\alpha(t) \\ (2.15),(2.16),(2.17),(2.21)}} \int_0^T \varphi(x(t), z(t))dt \tag{4.5}$$

4.2 Tight convexifications of the DTA and the FNC problems

In this section it is shown how it is possible to relax both the DTA (4.4) and the FNC (4.5) to obtain a formulation both convex and tight. Following the work done in [11] two convex optimal control problems are presented and it is shown how their solution can be mapped back into the original problems. The problems stated in (4.4) and (4.5) achieve control by acting directly on the control parameters $\alpha^k(t)$ and, only for the DTA, the routing matrices $R^k(t)$. However, the two problems shown in this section act on the cell-to-cell flows $f^k(t)$, the cell inflows $y^k(t)$, the cell outflows $z^k(t)$, the traffic volumes $x^k(t)$ and the outflow $\mu^k(t)$. Intuitively, non-negativity of these variables must be enforced since a negative flow would indicate drivers travelling in the opposite direction, which is inadmissible. So the following constraints are introduced

$$\begin{aligned} f_{ij}^k(t) \geq 0, \quad (i,j) \in \mathcal{A}, & \\ \mu_i^k(t) \geq 0, \quad i \in \mathcal{S}, & \qquad k \in \mathcal{K} \\ \mu_i^k(t) = 0, \quad i \in \mathcal{E} \setminus \mathcal{S}, & \end{aligned} \tag{4.6}$$

where the constraints of $\mu_i^k(t)$ are given by definition of the outflow. Moreover, notice that equation (4.6) implies also non-negativity for both $y_i^k(t)$ and $z_i^k(t)$ and that these are linear constraints. Moreover, given the concave nature of both the supply and demand functions it is possible to verify that the demand (2.16) and supply (2.17) constraints are indeed convex. The non-convexity of the problem, arises from the allocation rule chosen, both in the case of FIFO and non-FIFO models.

So it is now considered a relaxation of the multicommodity Dynamical Traffic Assignment (4.4) which is

$$\min_{\substack{x,y,z,f,\mu \\ (2.15),(4.3),(4.6),(2.16),(2.17)}} \int_0^T \varphi(x(t), z(t))dt \tag{4.7}$$

And it is now possible to obtain a relaxation of the multicommodity Freeway Network Control by ensuring that the flows are split accordingly to the given exogenous routing matrices such that

$$f_{ij}^k(t) = R_{ij}^k(t) z_i^k(t), \qquad i \in \mathcal{E}, \qquad \forall k \tag{4.8}$$

Therefore it is obtained the FNC convex relaxation

$$\min_{\substack{x,y,z,f,\mu \\ (2.15),(4.3),(2.16),(2.17),(4.8)}} \int_0^T \varphi(x(t), z(t))dt \tag{4.9}$$

Proposition 4. *The relaxation of the Dynamic Traffic Assignment (4.7) and of the Freeway Network Control (4.9) are both convex.*

Proof. In order to prove the convexity of the optimal control problems (4.7) and (4.9) notice that it would mean that if $(x^{(0)}(t), y^{(0)}(t), z^{(0)}(t), f^{(0)}(t), \mu^{(0)}(t))$ satisfy the constraints in (4.7) and (4.9) and so does $(x^{(1)}(t), y^{(1)}(t), z^{(1)}(t), f^{(1)}(t), \mu^{(1)}(t))$ then for every β in [0,1], also $(x^{(\beta)}, y^{(\beta)}, z^{(\beta)}, f^{(\beta)}, \mu^{(\beta)})$ does, where $x^{(\beta)} = (1-\beta)x^{(0)} + \beta x^{(1)}$, $y^{(\beta)} = (1-\beta)y^{(0)} + \beta y^{(1)}$ and so on, and

$$\int_0^T \varphi(x^{(\beta)}(t))\,dt \le (1-\beta)\int_0^T \varphi(x^{(0)}(t))\,dt + \beta\int_0^T \varphi(x^{(1)}(t))\,dt \tag{4.10}$$

Notice that the convex nature of the cost function $\varphi(x,z)$ proves inequality (4.10). Moreover, notice that for each choice of routing matrices satisfying (2.18) and (2.19) and of the control parameters $\alpha^k(t)$ the dynamics of the multicommodity dynamical flow network defined by (2.15) and the allocation rule inevitably satisfies (4.6),(2.16), (2.17) and (4.8) thus being a solution of the convex optimal control problems (4.7) and (4.9). □

In order to prove that these relaxations are tight it must be proven that for each solution of the convex optimal control problems (4.7) and (4.9) there exists a choice of control parameters $\alpha^k(t)$ and routing matrices $R^k(t)$ such that (2.21) is satisfied. This can be easily verified by extending Proposition 1 in [11] to the multicommodity setting. In order to do so and to state the next position it is first introduced the concept of variable speed limits and variable routing matrices based on the demand function, capacity and flows in the system. In this thesis refer as variable speed limits $\alpha^k(t)$ obtained to

$$\alpha_i^k(t) = \begin{cases} z_i^k(t)/d_i^k(x_i^k(t)), & i \in \mathcal{E}\setminus\mathcal{R} \\ z_i^k(t)/C_i^k, & i \in \mathcal{R}. \end{cases} \qquad k \in \mathcal{K} \tag{4.11}$$

Moreover assume that $\alpha_i^k(t) = 1$ if $z_i^k(t) = d_i^k(x_i^k(t)) = 0$ on a non-source cell i in $\mathcal{E}\setminus\mathcal{R}$ and that if $z_i^k(t) = 0$ then $R_{ij}^k(t) = |\{k \in \mathcal{E} : (i,k) \in \mathcal{A}\}|^{-1}$ for all (i,j) in $\mathcal{A}$.

Notice that each commodity has its own variable speed limit meaning that two vehicles belonging to different commodities driving in the same road are subjected to different speed limits. Then variable routing matrices $R^k(t)$ are introduced as

$$R_{ij}^k(t) = \begin{cases} f_{ij}^k(t)/z_i^k(t), & (i,j) \in \mathcal{A} \\ 0 & (i,j) \in \mathcal{E}\times\mathcal{E}\setminus\mathcal{A} \end{cases} \tag{4.12}$$

which means that vehicles belonging to different commodities may (or may not) take different routes. Moreover, the variable routing matrices R^k satisfy both equations (2.18) and (2.19)

Proposition 5. *Given a MDFN and initial traffic volumes x^0. Then for any feasible solution $(x(t), y(t), z(t), \mu(t), f(t))$ of the convex optimal control problem (4.7), it is possible to set $\alpha^k(t)$ and $R^k(t)$ as defined in (4.11) and (4.12) respectively, so that $x(t)$ satisfies (2.21) and $(\alpha^k(t), R^k(t))$ is a solution of the original DTA problem (4.4).*

Proof. Let $(x(t), y(t), z(t), \mu(t), f(t))$ be a feasible solution of the convex optimal problem (4.7). It is true that $z_i(t) \le C_i$ for any $t \ge 0$ as defined in the demand constraint in (2.16). Then, when choosing the control parameters as in (4.11), for every non-source cell $i \in \mathcal{E}\setminus\mathcal{R}$

$$z_i^k(t) = \alpha_i^k(t) d_i^k(x_i^k(t)) = \bar{d}_i^k(x_i^k(t), \alpha_i^k(t)) \tag{4.13}$$

Then, for every sink cell $i \in \mathcal{S}$ it follows $\mu_i^k(t) = z_i^k(t) = \bar{d}_i^k(x_i^k(t), \alpha_i^k(t))$. Moreover, for every source cell $i \in \mathcal{R}$ it follows from the choice of control parameters (4.11) and the demand constraint (2.16) that

$$z_i^k(t) = \min\{d_i^k(x_i^k(t)), \alpha_i^k(t)C_i^k\} = \bar{d}_i^k(x_i^k(t), \alpha_i^k(t)) \tag{4.14}$$

So it is possible to say that $z_i^k(t) = \bar{d}_i^k(x_i^k(t), \alpha_i^k(t))$ for every cell $i \in \mathcal{E}$ and from the choice of the routing matrices in (4.12) it follows that

$$f_{ij}^k = R_{ij}^k(t) z_i^k(t) = R_{ij}^k(t) \bar{d}_i^k(x_i^k(t), \alpha_i^k(t)) \qquad \forall (i,j) \in \mathcal{A}, \quad k \in \mathcal{K} \tag{4.15}$$

Therefore, for every cell $j \in \mathcal{E}$

$$\sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{E}} R_{ij}^k(t) \bar{d}_i^k(x_i^k(t), \alpha_i^k(t)) = \sum_{k \in \mathcal{K}} \sum_{(i,j) \in \mathcal{A}} f_{ij}^k(t) = \sum_{k \in \mathcal{K}} y_j^k(t) \le s_j(\sum_k x_j^k(t)) \tag{4.16}$$

where the inequality is the same as in the supply constraint in (2.17). This implies that there is no congestion, meaning that the solution is in freeflow, thus being admissible for each allocation rule. Then, it is easy to verify that the choice of routing matrix (4.12) satisfies equations (2.18) and (2.19). This means that for every feasible solution $(x(t), y(t), z(t), \mu(t), f(t))$ of the convex optimal control problem (4.7), the choices of control parameters (4.11) and routing matrices (4.12) satisfy (2.21), which implies that $(\alpha(t), R(t))$ is a feasible solution of the original DTA problem (4.4). □

Proposition 6. *Given a MDFN, initial traffic volumes x^0 and fix the substochastic routing matrices. Then for any feasible solution $(x(t)^k, y(t)^k, z(t)^k, \mu^k(t), f^k(t))$ of the convex optimal control problem (4.9) it is possible to set $\alpha^k(t)$ as in (4.11) so that $x^k(t)$ satisfies* (2.21)*, such that $\alpha^k(t)$ is a solution of the original FNC problem (4.5).*

Proof. Let $(x(t), y(t), z(t), \mu(t), f(t))$ be a feasible solution of the convex optimal control problem (4.9). From equations (4.8) and (4.11) it follows that

$$f_{ij}^k = R_{ij}^k(t) z_i^k(t) = R_{ij}^k(t) \bar{d}_i^k(x_i^k(t), \alpha_i^k(t)) \qquad (i,j) \in \mathcal{A}, \quad k \in \mathcal{K} \tag{4.17}$$

which implies that

$$\sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{E}} R_{ij}^k(t) \bar{d}_i^k(x_i^k(t), \alpha_i^k(t)) = \sum_{k \in \mathcal{K}} \sum_{(i,j) \in \mathcal{A}} f_{ij}^k(t) = \sum_{k \in \mathcal{K}} y_j^k(t) \le s_j(\sum_k x_j^k(t)) \qquad j \in \mathcal{E} \tag{4.18}$$

which is implied in the supply constraint (2.17). Then, proceed as in the proof of Proposition 5 to satisfy every allocation rule. This means that for every feasible solution $(x(t), y(t), z(t), \mu(t), f(t))$ of the convex optimal control problem (4.9), the choices of control parameters (4.11) satisfies the allocation rule, which implies that $\alpha(t)$ is a feasible solution of the original DTA problem (4.5). □

4.3 Freeway Network Control optimal problem

In this section optimal control techniques are used to study the convex formulation of the Freeway Network Control problem (4.9) by assuming a cost function $\varphi(x(t))$ which is function of the traffic volume only. By recalling that $y_i^k = \lambda_i^k + \sum_j R_{ji}^k z_j^k$ for $i \in \mathcal{E}$ it is possible to restate the law of mass conservation (2.15) as

$$\dot{x}_i^k = \lambda_i^k(t) + \sum_{j \in \mathcal{E}} R_{ji}^k(t) z_j^k(t) - z_i^k(t) \qquad i \in \mathcal{E} \qquad k \in \mathcal{K} \tag{4.19}$$

Considering this and the demand (2.16) and supply (2.17) constraints it is possible to pose the optimal control problem using as control parameters only the outflow from the cells z as such:

$$min \int_0^T \varphi(x(t))dt$$

$$x_i^k(0) = (x_i^0)^k \qquad \dot{x}_i^k = \lambda_i^k(t) + \textstyle\sum_{j\in\mathcal{E}} R_{ji}^k(t) z_j^k(t) - z_i^k(t) \qquad i \in \mathcal{E} \qquad k \in \mathcal{K} \tag{4.20}$$

$$0 \leq z_i^k(t) \leq d_i^k(x_i^k(t)) \qquad \textstyle\sum_{k\in\mathcal{K}} \sum_{j\in\mathcal{E}} R_{ji}^k(t) z_j^k(t) \leq s_i(\sum_{k\in\mathcal{K}} x_i^k(t)) \qquad i \in \mathcal{E}$$

The Hamiltonian associated with problem (4.20) is

$$H(x,z,\zeta) = -\varphi(x) + \sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{E}} \zeta_i^k (\lambda_i(t)^k + \sum_{j\in\mathcal{E}} R_{ji}^k(t) z_j^k(t) - z_i(t)^k) \tag{4.21}$$

where x is the state vector, z is the control parameter and ζ is the adjoint state vector. Then introduce the notation

$$\kappa_i^k(t) = \sum_{i\in\mathcal{E}} R_{ij}^k(t) \zeta_j^k(t) - \zeta_i^k(t) \tag{4.22}$$

So it is possible to rewrite the Hamiltonian as

$$H(x,z,\zeta) = -\varphi(x) + \sum_{k\in\mathcal{K}} (\sum_{i\in\mathcal{E}} \zeta_i^k \lambda_i^k + \sum_{i\in\mathcal{E}} \kappa_i^k z_i^k) \tag{4.23}$$

Then the Pontryagin maximum principle implies that if (x^*, z^*) is an optimal solution of (4.20) then for every t ∈ [0,T]

$$\begin{aligned}
z^*(t) \in & \underset{\substack{(z_i^k)_{i\in\mathcal{E}} \\ \sum_{k\in\mathcal{K}} \sum_{j\in\mathcal{E}} R_{ji}^k(t) z_j^k(t) \leq s_i(\sum_{k\in\mathcal{K}} (x_i^k)^*(t)) \\ z_i^k \leq d_i^k((x_i^k)^*(t))}}{\operatorname{argmax}} H(x^*(t), z, \zeta(t)) \\
= & \underset{\substack{(z_i^k)_{i\in\mathcal{E}} \\ \sum_{k\in\mathcal{K}} \sum_{j\in\mathcal{E}} R_{ji}^k(t) z_j^k(t) \leq s_i(\sum_{k\in\mathcal{K}} (x_i^k)^*(t)) \\ z_i^k \leq d_i^k((x_i^k)^*(t))}}{\operatorname{argmax}} \sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{E}} \kappa_i^k(t) z_i^k \\
= & \underset{\substack{(z_i^k)_{i\in\mathcal{E}} \\ \sum_{k\in\mathcal{K}} \sum_{j\in\mathcal{E}} R_{ji}^k(t) z_j^k(t) \leq s_i(\sum_{k\in\mathcal{K}} (x_i^k)^*(t)) \\ z_i^k \leq d_i^k((x_i^k)^*(t))}}{\operatorname{argmin}} \left(- \sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{E}} \kappa_i^k(t) z_i^k \right)
\end{aligned} \tag{4.24}$$

Denoting with ξ_i and ν_i the multipliers of the demand and the supply constraints then the dual problem of (4.24) reads

$$(\xi^*(t), \nu^*(t)) \in \underset{\substack{(\xi_i,\nu_i)_{i\in\mathcal{E}} \\ \xi_i\geq 0,\ \nu\geq 0,\ \geq 0 \\ \xi_i^k + \sum_{j\in\mathcal{E}} R_{ij}^k v_j^k \geq \kappa_i^k}}{argmax} \quad - \sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{E}} (\xi_i^k d_i^k((x_i^k)^*(t)) + \nu_i^k s_i(\sum_{h\in\mathcal{K}} (x_i^h)^*(t)) \tag{4.25}$$

So the adjoint dynamical equation is

$$\dot{\zeta}_i^k(t) = \frac{\partial}{\partial x_i^k} \varphi(x^*(t)) + (\xi_i^k (d_i^k)^{'}((x_i^k)^*(t)) + \nu_i^k s_i^{'}(\sum_{h\in\mathcal{K}} (x_i^h)^*(t))) \tag{4.26}$$

where $(d_i^k)^{'}$ and $s_i^{'}$ are the derivatives of the demand and supply respectively. The transversality condition is

$$\zeta_i^k(T) = 0 \qquad i \in \mathcal{E} \tag{4.27}$$

4.4 Previous non-convex DTA studies

It is worth mentioning that the Multicommodity Social Optimum Dynamic Traffic Assignment has already been addressed as a non-convex problem by ensuring the FIFO constraints. In fact, the work done by Samaranayake et al. [18] tries to solve this problem by assuming two different types of commodities: the controllable one, whose only origin and destination cells are known and the routes can be freely chosen by the system, and the uncontrollable one which has fixed routing.

In their study they also employed the cell transmission model assuming for each cell i supply function s_i and demand function d_i, assuming that source cells have no supply and sink cells have no demand. As the state of the system for each cell i they decided to use the density of traffic volume x_i computed as the total number of vehicles on a cell divided by its length L_i. Furthermore, they define the density of each commodity x_i^k which must satisfy

$$x_i = \sum_{k\in\mathcal{K}} x_i^k \tag{4.28}$$

Then, they introduced the total inflow and the total outflow computed as the summation of the inflow and outflow of each commodity

$$y_i = \sum_{k\in\mathcal{K}} y_i^k \tag{4.29}$$

$$z_i = \sum_{k\in\mathcal{K}} z_i^k \tag{4.30}$$

Given these definitions they then defined the mass conservation law as

$$x_i^k(t) = \begin{cases} x_i^k(t-1) + \frac{\Delta t}{L_i}(y_i^k(t-1) - z_i^k(t-1)) & \forall i \in \mathcal{E} \setminus (\mathcal{R} \cup \mathcal{S}) \\ x_i^k(t-1) + \frac{\Delta t}{L_i} z_i^k(t-1) \end{cases} \tag{4.31}$$

As previously mentioned, they employed two different types of commodities. A non-controllable commodity defined by a substochastic routing matrix whose elements $R^k_{ij}(t)$ are called split ratios. Moreover, the split ratios are also introduced for the controllable commodity but it assumes a different meaning

$$R^k_{ij}(t) = \begin{cases} 1 & \text{if the path of commodity c contains both cell i and j} \\ 0 & \text{otherwise} \end{cases} \tag{4.32}$$

Moreover, they decided to maintain the non-convexity of the problem by considering the following FIFO condition whenever $x_i(t) \neq 0$

$$z^k_i(t) = z_i(t)\frac{x^k_i(t)}{x_i} \tag{4.33}$$

Furthermore, each commodity outflow must be modulated by the corresponding split ratio

$$f^k_{ij}(t) = R^k_{ij}(t)z^k_i(t) \tag{4.34}$$

The total inflow and the total outflow must also satisfy respectively the supply and demand constraints

$$0 \leq y_i(t) \leq s_i(t) \tag{4.35}$$

$$0 \leq z_i(t) \leq d_i(t) \tag{4.36}$$

Considering the system defined in (4.28)-(4.36) and many more assumptions by employing the Adjoint Method to compute the gradient and solve the problem by reaching a minimum of the total travel time. However, since the problem is non-convex, there might be many minimum points. In order to find the global one it is employed a multi-start strategy, which consists in using the algorithm many times that, if there are not too many minimum points, will eventually reach the global minimum.

Chapter 5

Iterative Distributed Solution

In this chapter it is first presented a discretized version of the problems defined in the previous chapter and then study a method to solve both the Multicommodity Dynamic Traffic Assignment and the Multicommodity Freeway Network Control based on Alternating Direction Method of Multipliers [15] introduced by Sthepen Boyd.

5.1 Discrete time Dynamic Traffic Assignment and Freeway Network Control

In the following section it shall be used a discrete time version of both the DTA and the FNC which are now introduced. The discrete time FNC would read

$$min \sum_{t=0}^{N} \sum_{k} \sum_{i \in \mathcal{E}} \varphi_i^k(x_i^k(t))$$

$$x_i^k(0) = (x_i^k)^0 \quad x_i^k(t+1) = x_i^k(t) + \lambda_i^k(t) + \sum_{j \in \mathcal{E}} R_{ji}^k(t) z_j^k(t) - z_i^k(t) \qquad i \in \mathcal{E}, \quad k \in \mathcal{K}$$

$$0 \le z_i^k(t) \le d_i^k(x_i^k(t)) \qquad \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{E}} R_{ji}^k(t) z_j^k(t) \le s_i(\sum_{k \in \mathcal{K}} x_i^k(t)) \qquad i \in \mathcal{E} \tag{5.1}$$

However it is possible to introduce additional variables $y_i^k(t) = x_i^k(t+1)$ and $f_{ij}^k = R_{ij}^k z_i^k$ to achieve a fully distributed control such as

$$min \sum_{t=0}^{N} \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{E}} \varphi_i^k(x_i^k(t))$$

$$x_i^k(0) = (x_i^k)^0 \quad y_i^k(t) = x_i^k(t) + \lambda_i^k(t) + \sum_{j \in \mathcal{E}} f_{ji}^k - z_i^k(t) \qquad i \in \mathcal{E}, \quad k \in \mathcal{K}$$

$$y_i^k(t) = x_i^k(t+1) \qquad f_{ij}^k = R_{ij}^k(t) z_i^k(t) \qquad i \in \mathcal{E}, \quad k \in \mathcal{K}$$

$$0 \le z_i^k(t) \le d_i^k(x_i^k(t)) \qquad \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{E}} f_{ji}^k(t) \le s_i(\sum_{k \in \mathcal{K}} x_i^k(t)) \qquad i \in \mathcal{E} \tag{5.2}$$

Then it is possible to formulate the discrete time version of the DTA by decoupling the incoming flows f, the outgoing flows g and the exogenous outflow μ such as

$$
\begin{gathered}
min\sum_{t=0}^{N}\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{E}}\varphi_i^k(x_i^k(t))\\
x_i^k(0)=(x_i^k)^0 \quad y_i^k(t)=x_i^k(t)+\lambda_i^k(t)+\sum_{j\in\mathcal{E}}f_{ji}^k(t)-\sum_{j\in\mathcal{E}}g_{ij}^k(t)-\mu_i^k(t) \qquad i\in\mathcal{E},\quad k\in\mathcal{K}\\
y_i^k(t)=x_i^k(t+1) \qquad f_{ij}^k(t)=g_{ij}^k(t) \qquad f_{ij}^k(t)\geq 0 \qquad i\in\mathcal{E},\quad k\in\mathcal{K}\\
0\leq\sum_j g_{ij}^k(t)+\mu_i^k(t)\leq d_i^k(x_i^k(t)) \quad \sum_{k\in\mathcal{K}}(\lambda_i^k(t)+\sum_{j\in\mathcal{E}}f_{ij}^k(t))\leq s_i(\sum_{k\in\mathcal{K}}x_i^k(t)) \quad i\in\mathcal{E}
\end{gathered}
\tag{5.3}
$$

5.2 Solving FNC and DTA with ADMM

The ADMM has already been proven to be a powerful tool to solve both the FNC and the DTA in a single commodity setting. In fact, these problem have been addressed and solved with the ADMM both in [14] and [13]. However, This section shall focus on a formulation of the ADMM algorithm capable of solving both the FNC and DTA problems in a multicommodity scenario and then present some relevant examples.

5.2.1 ADMM with Freeway Network Control

Recall the Multicommodity Freeway Network Control formulation as in (5.2)

$$
\begin{gathered}
min\sum_{t=0}^{N}\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{E}}\varphi_i^k(x_i^k(t))\\
x_i^k(0)=(x_i^k)^0 \quad y_i^k(t)=x_i^k(t)+\lambda_i^k(t)+\sum_{j\in\mathcal{E}}f_{ji}^k(t)-z_i^k(t) \qquad i\in\mathcal{E},\quad k\in\mathcal{K}\\
y_i^k(t)=x_i^k(t+1) \qquad f_{ij}^k(t)=R_{ij}^k(t)z_i^k(t) \qquad i\in\mathcal{E},\quad k\in\mathcal{K}\\
0\leq z_i^k(t)\leq d_i^k(x_i^k(t)) \qquad \sum_{k\in\mathcal{K}}\sum_{j\in\mathcal{E}}f_{ji}^k(t)\leq s_i(\sum_{k\in\mathcal{K}}x_i^k(t)) \qquad i\in\mathcal{E}
\end{gathered}
\tag{5.4}
$$

It is thus possible to formulate the Augmented Lagrangian as

$$
\begin{aligned}
L_\rho(x,y,f,z,\zeta,\chi,\xi,\sigma,\nu,\eta) \;=\; & \textstyle\sum_{t=0}^{N}\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{E}}\varphi_i^k(x_i^k(t))+\zeta_i^k(t)(y_i^k(t)-x_i^k(t)\\
& -\lambda_i^k(t)-\textstyle\sum_{j\in\mathcal{E}}f_{ji}^k(t)+z_i^k(t))+\chi_i^k(t)(y_i^k(t)\\
& -x_i^k(t+1))+\textstyle\sum_{j\in\mathcal{E}}(\xi_{ij}^k(t)(f_{ij}^k(t)-R_{ij}^k(t)z_i^k(t)))\\
& +\sigma_i^k(t)(z_i^k(t)-d_i^k(x_i^k(t)))+\nu_i^k(t)(\textstyle\sum_{h\in\mathcal{K}}\sum_{j\in\mathcal{E}}f_{ji}^h(t)\\
& -s_i(\textstyle\sum_{h\in\mathcal{K}}x_i^h(t)))+\eta_i(t)^k(-z_i^k(t))+(\rho/2)((y_i^k(t)\\
& -x_i^k(t)+\lambda_i^k(t)+\textstyle\sum_{j\in\mathcal{E}}f_{ji}^k(t)-z_i^k(t))^2+(y_i^k(t)\\
& -x_i^k(t+1))^2+\textstyle\sum_{j\in\mathcal{E}}(f_{ij}^k(t)-R_{ij}^k(t)z_i^k(t))^2\\
& +max(0,z_i^k(t)-d_i^k(x_i^k(t),C_i))^2\\
& +max(0,(\textstyle\sum_{h\in\mathcal{K}}\sum_{j\in\mathcal{E}}f_{ji}^h(t)-s_i(\sum_{h\in\mathcal{K}}x_i^h(t))))^2\\
& +max(0,-z_i^k(t))^2)
\end{aligned}
\tag{5.5}
$$

Then it is possible to perform the minimization steps for the variables x, y, f and z as

$$z^{k+1} := \underset{z}{argmin} L_\rho(x^k, y^k, f^k, z, \zeta^k, \chi^k, \xi^k, \sigma^k, \nu^k, \eta^k) \tag{5.6}$$

$$f^{k+1} := \underset{f}{argmin} L_\rho(x^k, y^k, f, z^{k+1}, \zeta^k, \chi^k, \xi^k, \sigma^k, \nu^k, \eta^k) \tag{5.7}$$

$$y^{k+1} := \underset{y}{argmin} L_\rho(x^k, y, f^{k+1}, z^{k+1}, \zeta^k, \chi^k, \xi^k, \sigma^k, \nu^k, \eta^k) \tag{5.8}$$

$$x^{k+1} := \underset{x}{argmin} L_\rho(x, y^{k+1}, f^{k+1}, z^{k+1}, \zeta^k, \chi^k, \xi^k, \sigma^k, \nu^k, \eta^k) \tag{5.9}$$

The solution of the minimization steps (5.6)-(5.9) is equal to solving the following equations

$$\begin{aligned} \frac{\partial L_\rho}{\partial x_i^k(t)} &= (\varphi_i^k)'(t) - \zeta_i^k(t) - \chi_i^k(t-1) + \sigma_i^k(d_i^k)'(x_i^k(t)) - \nu_i^k(t)(s_i)'(\textstyle\sum_{h\in\mathcal{K}} x_i^h(t)) \\ &\quad + \tfrac{\rho}{2}(-2(y_i^k(t) - x_i^k(t) - \lambda_i^k(t) - \textstyle\sum_{j\in\mathcal{E}} f_{ji}^k(t) + z_i^k(t)) - 2(y_i^k(t-1) - x_i^k(t)) \\ &\quad -2(d_i^k(x_i^k(t)))' max(0, z_i^k(t) - d_i^k(x_i^k(t))) \\ &\quad -(s_i)'(\textstyle\sum_{h\in\mathcal{K}} x_i^h(t)) max(0, \sum_{h\in\mathcal{K}}\sum_{j\in\mathcal{E}} \lambda_i^h(t) + f_{ji}^h(t) - s_i(\sum_{h\in\mathcal{K}} x_i^h(t)))) \\ &= 0 \end{aligned} \tag{5.10}$$

$$\begin{aligned} \frac{\partial L_\rho}{\partial f_{iw}^k(t)} &= \zeta_w^k(t) + \xi_{iw}^k(t) + \nu_w^k(t) - \tfrac{\rho}{2}(-2(y_w^k(t) - x_w^k(t) - \lambda_w^k(t) - \textstyle\sum_{j\in\mathcal{E}} f_{jw}^k(t) \\ &\quad + z_w^k(t)) + 2(f_{iw}^k(t) - R_{iw}^k z_i^k(t)) + 2(\textstyle\sum_{h\in\mathcal{K}}\sum_{j\in\mathcal{E}} \lambda_i^h(t) + f_{jw}^h(t) \\ &\quad - s_w(\textstyle\sum_{h\in\mathcal{K}} x_w^h(t)))) \\ &= 0 \end{aligned} \tag{5.11}$$

$$\begin{aligned} \frac{\partial L_\rho}{\partial y_i^k(t)} &= \zeta_i^k(t) + \chi_i^k(t) + \tfrac{\rho}{2}(2(y_i^k(t) - x_i^k(t) - \lambda_i^k(t) - \textstyle\sum_{j\in\mathcal{E}} f_{ji}^k(t) + z_i^k(t)) + 2(y_i^k(t) \\ &\quad - x_i^k(t+1))) \\ &= 0 \end{aligned} \tag{5.12}$$

$$\begin{aligned} \frac{\partial L_\rho}{\partial z_i^k(t)} &= -\zeta_i^k(t) - \textstyle\sum_{j\in\mathcal{E}} \xi_{ij}^k(t) R_{ij}^k + \sigma_i^k(t) - \eta_i^k(t) + \tfrac{\rho}{2}(2(y_i^k(t) - x_i^k(t) - \lambda_i^k(t) \\ &\quad - \textstyle\sum_{j\in\mathcal{E}} f_{ji}^k(t) + z_i^k(t)) - 2(\sum_{j\in\mathcal{E}} R_{ij}^k(f_{ij}^k(t) - R_{ij}^k z_i^k(t))) + 2max(0, z_i^k(t) \\ &\quad - d_i^k(x_i^k(t))) - 2max(0, -z_i^k(t))) \\ &= 0 \end{aligned} \tag{5.13}$$

Once the minimization steps are computed it is necessary to update the dual variables as such

$$\zeta_i^k(t+1) = \zeta_i^k(t) + \rho(y_i^k(t) - x_i^k(t) - \lambda_i^k(t) - \sum_{j\in\mathcal{E}} f_{ji}^k(t) + z_i^k(t)) \tag{5.14}$$

$$\chi_i^k(t+1) = \chi_i^k(t) + \rho(y_i^k(t) - x_i^k(t+1)) \tag{5.15}$$

$$\xi_{ij}^k(t+1) = \xi_{ij}^k(t) + \rho(f_{ij}^k(t) - R_{ij}^k z_i^k(t)) \tag{5.16}$$

$$\sigma_i^k(t+1) = max(0, \sigma_i^k(t) + \rho(z_i^k(t) - d_i^k(x_i^k(t)))) \tag{5.17}$$

$$\nu_i^k(t+1) = max(0, \nu_i^k(t) + \rho(\sum_{h\in\mathcal{K}}\sum_{j\in\mathcal{E}} f_{ji}^k(t) - s_i(\sum_{h\in\mathcal{K}} x_i^h(t)))) \tag{5.18}$$

$$\eta_i^k(t+1) = max(0, \eta_i^k(t) + \rho(-z_i^k(t))) \tag{5.19}$$

By computing the minimization steps and the dual variable updates it is possible to obtain a distributed algorithm capable of solving the FNC problem

Example 10. *It is now applied the ADMM algorithm to an example and evaluate the results obtained. The network chosen for this example is shown in Figure 5.1. Moreover,*

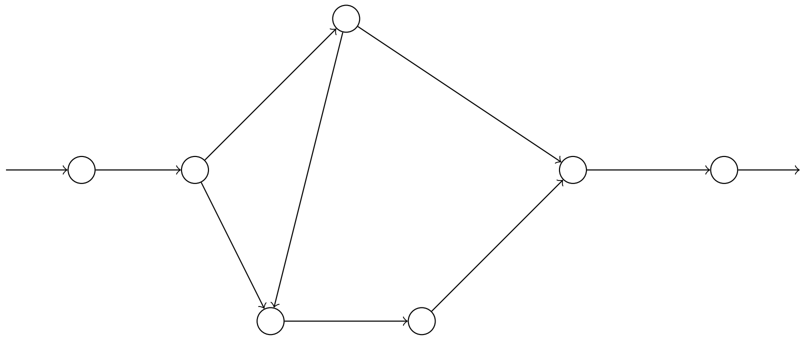


Figure 5.1: Topology for ADMM testing

the cost function chosen is simply the square of the total travel time such as

$$\varphi_i^k(x_i^k(t)) = (x_i^k(t))^2 \tag{5.20}$$

For this example it is chosen to use a linear demand function such as

$$d_i^k(x_i^k(t)) = \alpha_i^k x_i^k(t) \tag{5.21}$$

and affine supply function

$$s_i(\sum_{h\in\mathcal{K}} x_i^h(t)) = b_i - a_i \sum_{h\in\mathcal{K}} x_i^h(t) \tag{5.22}$$

The plots consist of the cost function, the duality gap, interpreted as the difference between the Lagrangian function and the Dual function, and the infeasibility, meaning how much the constraints of the convex optimization problem are not satisfied.

In Figure 5.2 it is easily noticeable that even under different penalty terms the system is able to reach, over many iterations, the same value of the cost function as expected. Moreover, the duality gap tends to zero, as it is possible to imagine, since the result of the Lagrangian function should be the same of that of the Dual Function. Furthermore, Figure 5.3 shows that the infeasibility decreases over iterations until reaching zero, meaning that the control is achieved by satisfying every constraint. Moreover, it is noticeable that when using a bigger penalty parameter ρ it introduces more oscillation caused mainly by an overshoot of the optimal value.

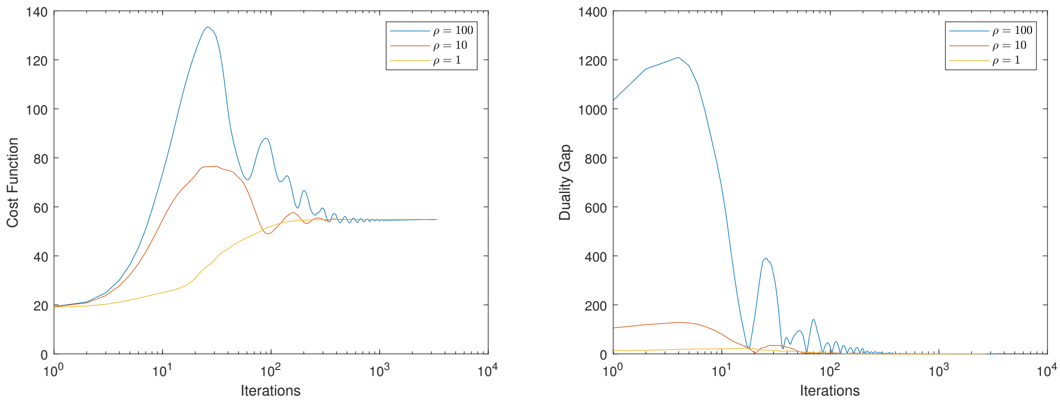


Figure 5.2: FNC: Cost function and Duality Gap under different penalty terms

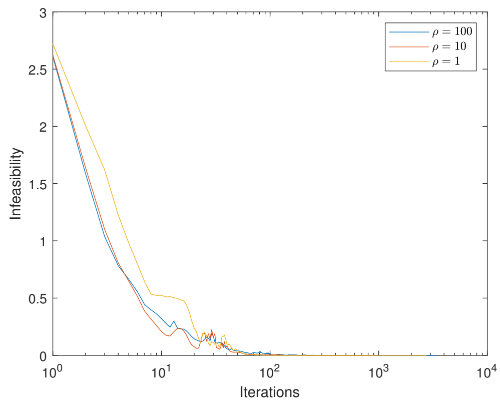


Figure 5.3: FNC: Degree of infeasibility

5.2.2 ADMM with Dynamic Traffic Assignment

Recall the Multicommodity Dynamic Traffic Assignment formulation as in (5.3)

$$min \sum_{t=0}^{N} \sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{E}} \varphi_i^k(x_i^k(t))$$

$$x_i^k(0) = (x_i^k)^0 \quad y_i^k(t) = x_i^k(t) + \lambda_i^k(t) + \sum_{j\in\mathcal{E}} f_{ji}^k(t) - \sum_{j\in\mathcal{E}} g_{ij}^k(t) - \mu_i^k(t) \qquad i \in \mathcal{E}, \quad k \in \mathcal{K}$$

$$y_i^k(t) = x_i^k(t+1) \qquad f_{ij}^k(t) = g_{ij}^k(t) \qquad f_{ij}^k(t) \geq 0 \qquad i \in \mathcal{E}, \quad k \in \mathcal{K}$$

$$0 \leq \sum_{j\in\mathcal{E}} g_{ij}^k(t) + \mu_i^k(t) \leq d_i^k(x_i^k(t)) \quad \sum_{k\in\mathcal{K}} (\lambda_i^k(t) + \sum_{j\in\mathcal{E}} f_{ij}^k(t)) \leq s_i(\sum_{k\in\mathcal{K}} x_i^k(t)) \quad i \in \mathcal{E} \tag{5.23}$$

It is thus possible to formulate the Augmented Lagrangian as

$$\begin{aligned} L_\rho(x,y,f,g,\mu,\zeta,\chi,\xi,\sigma,\nu,\eta) \;=\; & \textstyle\sum_{t=0}^{N}\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{E}}\varphi_i^k(x_i^k(t)) + \zeta_i^k(t)(y_i^k(t) - x_i^k(t) \\ & -\lambda_i^k(t) - \textstyle\sum_{j\in\mathcal{E}} f_{ji}^k(t) + \sum_{j\in\mathcal{E}} g_{ij}^k(t) + \mu_i^k(t)) \\ & +\chi_i^k(t)(y_i^k(t) - x_i^k(t+1)) + \textstyle\sum_{j\in\mathcal{E}}(\xi_{ij}^k(t)(f_{ij}^k(t) \\ & -g_{ij}^k(t))) + \sigma_i^k(t)(\textstyle\sum_{j\in\mathcal{E}} g_{ij}^k(t) + \mu_i(t) - d_i^k(x_i^k(t)) \\ & +\nu_i^k(t)(\textstyle\sum_{h\in\mathcal{K}}\sum_{j\in\mathcal{E}}\lambda_i^h(t) + f_{ji}^h(t) - s_i(\sum_{h\in\mathcal{K}} x_i^h(t))) \\ & +\textstyle\sum_{j\in\mathcal{E}}(\eta_{ij}(t)^k(-f_{ij}^k(t))) + (\rho/2)((y_i^k(t) - x_i^k(t) \\ & +\lambda_i^k(t) + \textstyle\sum_{j\in\mathcal{E}} f_{ji}^k(t) - \sum_{j\in\mathcal{E}} g_{ij}^k(t) - \mu_i^k(t))^2 + (y_i^k(t) \\ & -x_i^k(t+1))^2 + \textstyle\sum_{j\in\mathcal{E}}(f_{ij}^k(t) - g_{ij}^k(t))^2 + max(0,\mu_i^k(t) \\ & +\textstyle\sum_{j\in\mathcal{E}} g_{ij}^k(t) - d_i^k(x_i^k(t)))^2 \\ & +max(0,(\textstyle\sum_{h\in\mathcal{K}}\sum_{j\in\mathcal{E}}\lambda_i^h(t) + f_{ji}^h(t) \\ & -s_i(\textstyle\sum_{h\in\mathcal{K}} x_i^h(t))))^2 + \sum_{j\in\mathcal{E}} max(0,-f_{ij}^k(t))^2) \end{aligned} \tag{5.24}$$

Then it is possible to perform the minimization steps for the variables x, y, f and z as

$$g^{k+1} := \underset{g}{argmin} L_\rho(x^k,y^k,f^k,g,\mu^k,\zeta^k,\chi^k,\xi^k,\sigma^k,\nu^k,\eta^k) \tag{5.25}$$

$$f^{k+1} := \underset{f}{argmin} L_\rho(x^k,y^k,f,g^{k+1},\mu^k,\zeta^k,\chi^k,\xi^k,\sigma^k,\nu^k,\eta^k) \tag{5.26}$$

$$y^{k+1} := \underset{y}{argmin} L_\rho(x^k,y,f^{k+1},g^{k+1},\mu^k,\zeta^k,\chi^k,\xi^k,\sigma^k,\nu^k,\eta^k) \tag{5.27}$$

$$\mu^{k+1} := \underset{\mu}{argmin} L_\rho(x^k,y^{k+1},f^{k+1},g^{k+1},\mu,\zeta^k,\chi^k,\xi^k,\sigma^k,\nu^k,\eta^k) \tag{5.28}$$

$$x^{k+1} := \underset{x}{argmin} L_\rho(x,y^{k+1},f^{k+1},g^{k+1},\mu^{k+1},\zeta^k,\chi^k,\xi^k,\sigma^k,\nu^k,\eta^k) \tag{5.29}$$

Solving the minimization steps (5.25)-(5.29) is the same as solving the following equations

$$\begin{aligned} \frac{\partial L_\rho}{\partial x_i^k(t)} \;=\; & (\varphi_i^k)'(t) - \zeta_i^k(t) - \xi_i^k(t-1) + \sigma_i^k(d_i^k)'(x_i^k(t)) - \nu_i^k(t)(s_i)'(\textstyle\sum_{h\in\mathcal{K}} x_i^h(t)) \\ & +\tfrac{\rho}{2}(-2(y_i^k(t) - x_i^k(t) - \lambda_i^k(t) - \textstyle\sum_{j\in\mathcal{E}} f_{ji}^k(t) + \sum_{j\in\mathcal{E}} g_{ij}^k(t) + \mu_i^k(t)) \\ & -2(y_i^k(t-1) - x_i^k(t)) - 2(d_i^k(x_i^k(t)))'max(0,\mu_i^k(t) + \textstyle\sum_{j\in\mathcal{E}} g_{ij}^k(t) - d_i^k(x_i^k(t))) \\ & -(s_i)'(\textstyle\sum_{h\in\mathcal{K}} x_i^h(t))max(0,\sum_{h\in\mathcal{K}}\sum_{j\in\mathcal{E}}\lambda_i^k(t) + f_{ji}^k(t) - s_i(\sum_{h\in\mathcal{K}} x_i^h(t)))) \\ =\; & 0 \end{aligned} \tag{5.30}$$

$$\begin{aligned} \frac{\partial L_\rho}{\partial f_{iw}^k(t)} \;=\; & \zeta_w^k(t) + \xi_{iw}^k(t) + \nu_i^k(t) - \eta_{iw}^k(t) - \tfrac{\rho}{2}(-2(y_w^k(t) - x_w^k(t) - \lambda_w^k(t) - \textstyle\sum_{j\in\mathcal{E}} f_{jw}^k(t) \\ & +\textstyle\sum_{j\in\mathcal{E}} g_{ij}^k(t) + \mu_w^k(t)) + 2(\sum_{j\in\mathcal{E}} f_{ij}^k(t) - g_{ij}^k(t)) + 2(\sum_{h\in\mathcal{K}}\sum_{j\in\mathcal{E}}\lambda_w^k(t) \\ & +f_{jw}^k(t) - s_w(\textstyle\sum_{h\in\mathcal{K}} x_w^h(t)))) \\ =\; & 0 \end{aligned} \tag{5.31}$$

$$\begin{aligned} \frac{\partial L_\rho}{\partial g_{iw}^k(t)} \;=\; & -\zeta_i^k(t) + -\xi_i^k(t) + \sigma_i^k(t) + \tfrac{\rho}{2}(2(y_i^k(t) - x_i^k(t) - \lambda_i^k(t) - \textstyle\sum_{j\in\mathcal{E}} f_{ji}^k(t) \\ & +\textstyle\sum_{j\in\mathcal{E}} g_{ij}^k(t) + \mu_i^k(t)) - 2(\sum_{j\in\mathcal{E}} f_{ij}^k(t) - g_{ij}^k(t)) + 2max(0,\mu_i^k(t) \\ & +\textstyle\sum_{j\in\mathcal{E}} g_{ij}^k(t) - d_i^k(x_i^k(t)) \\ =\; & 0 \end{aligned} \tag{5.32}$$

$$\begin{aligned}\frac{\partial L_\rho}{\partial y_i^k(t)} &= \zeta_i^k(t)+\chi_i^k(t)+\tfrac{\rho}{2}(2(y_i^k(t)-x_i^k(t)-\lambda_i^k(t)-\textstyle\sum_{j\in\mathcal{E}} f_{ji}^k(t)+\sum_{j\in\mathcal{E}} g_{ij}^k(t)\\ &\quad +\mu_i^k(t))+2(y_i^k(t)-x_i^k(t+1)))\\ &= 0\end{aligned} \tag{5.33}$$

$$\begin{aligned}\frac{\partial L_\rho}{\partial \mu_i^k(t)} &= -\zeta_i^k(t)+\sigma_i^k(t)-\tfrac{\rho}{2}(2(y_i^k(t)-x_i^k(t)-\lambda_i^k(t)-\textstyle\sum_{j\in\mathcal{E}} f_{ji}^k(t)+\sum_{j\in\mathcal{E}} g_{ij}^k(t)\\ &\quad +2max(\mu_i^k(t)+\textstyle\sum_{j\in\mathcal{E}} g_{ij}^k(t)-d_i^k(x_i^k(t)))\\ &= 0\end{aligned} \tag{5.34}$$

Once the minimization steps are computed it is necessary to update the dual variables as such

$$\zeta_i^k(t+1)=\zeta_i^k(t)+\rho(y_i^k(t)-x_i^k(t)-\lambda_i^k(t)-\sum_{j\in\mathcal{E}} f_{ji}^k(t)+\sum_{j\in\mathcal{E}} g_{ij}^k(t)-\mu_i^k(t)) \tag{5.35}$$

$$\chi_i^k(t+1)=\chi_i^k(t)+\rho(y_i^k(t)-x_i^k(t+1)) \tag{5.36}$$

$$\xi_{ij}^k(t+1)=\xi_{ij}^k(t)+\rho(f_{ij}^k(t)-g_{ij}^k(t)) \tag{5.37}$$

$$\sigma_i^k(t+1)=max(0,\sigma_i^k(t)+\rho(\mu_i^k(t)+\sum_{j\in\mathcal{E}} g_{ij}^k(t)-d_i^k(x_i^k(t)))) \tag{5.38}$$

$$\nu_i^k(t+1)=max(0,\nu_i^k(t)+\rho(\sum_{h\in\mathcal{K}}\sum_{j\in\mathcal{E}}\lambda_i^k(t)+f_{ji}^k(t)-s_i(\sum_{h\in\mathcal{K}} x_i^h(t)))) \tag{5.39}$$

$$\eta_{ij}^k(t+1)=max(0,\eta_{ij}^k(t)+\rho(-f_{ij}^k(t))) \tag{5.40}$$

By computing the minimization steps and the dual variable updates it is possible to obtain a distributed algorithm capable of solving the DTA problem.

Example 11. *It is then possible to apply the ADMM algorithm to an example and evaluate the results obtained, with the same network used for the FNC, shown in Figure 5.1. Moreover, the cost function chosen is simply the square of the total travel time such as*

$$\varphi_i^k(x_i^k(t))=(x_i^k(t))^2 \tag{5.41}$$

For this example it is chosen to use a linear demand function such as

$$d_i^k(x_i^k(t))=\alpha_i^k x_i^k(t) \tag{5.42}$$

and affine supply function

$$s_i(\sum_{h\in\mathcal{K}} x_i^h(t))=b_i-a_i\sum_{h\in\mathcal{K}} x_i^h(t) \tag{5.43}$$

The plots consist of the cost function, the duality gap, interpreted as the difference between the Lagrangian function and the Dual function, and the infeasibility, meaning how much the constraints of the convex optimization problem are not satisfied.

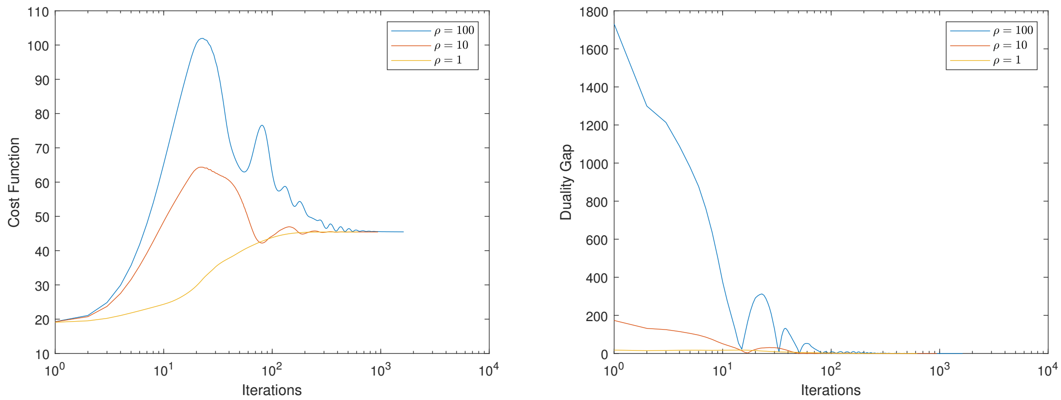


Figure 5.4: DTA: Cost function and Duality Gap under different penalty terms

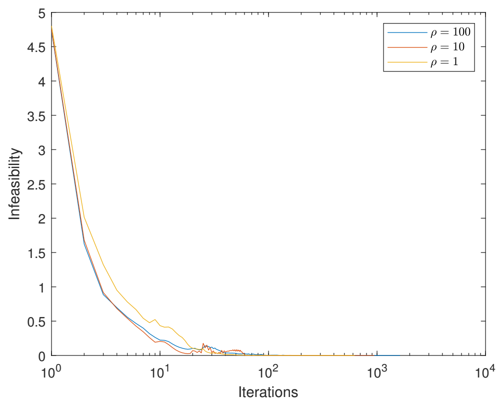


Figure 5.5: DTA: Degree of infeasibility

As found for the FNC problem, the ADMM proves useful to solve the multicommodity DTA. In fact, in Figure 5.4 it is shown that despite the chosen penalty parameter the algorithm reaches the same value of the total travel time. Furthermore, the Duality gap decreases over iterations until it eventually becomes null, meaning that the Lagrangian function and the Dual function become more and more similar the more iterations are considered. Moreover, the ADMM proves to be a great tool to ensure the demand and supply constraints as shown in Figure 5.5 which shows the infeasibility of the system that starts at a high value but decreases over iterations until eventually reaching zero, meaning that the optimal value found actually satisfies all constraints. As for the FNC problem, the use of a bigger penalty parameter ρ introduces some oscillation which can be related to the overshoot of the actual optimal value.

5.3 Los Angeles Network

Proven that the algorithm works on a relatively small network, in this section it is now tested in a real-life scenario. To this end, it is chosen to use the Los Angeles network defined in [12] shown in Figure 5.6.

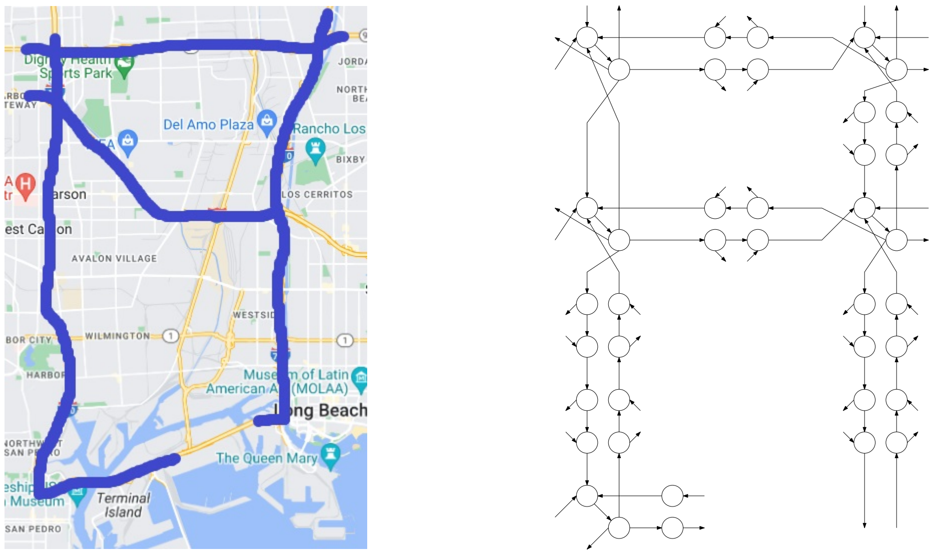


Figure 5.6: Los Angeles Network

In this network its possible to notice different types of cells such as mainlines (the longer cells), intersections, ramps (on and off) and between ramps. For all cells and commodities it is defined the demand function as

$$emd_i^k(x_i^k) = \frac{v_i^k x_i^k}{L_i} \tag{5.44}$$

where v_i^k is the free-flow speed of commodity k in cell i and L_i is the length of cell i. Moreover, it is defined a shared supply function for each cell

$$s_i(\sum_{h\in\mathcal{K}} x_i^h) = \frac{w_i(x_i^{jam} - \sum_{h\in\mathcal{K}} x_i^h)}{L_i} \tag{5.45}$$

where w_i and x_i^{jam} are, respectively, the wave speed and the jam traffic volume on cell i. Fix the variables as shown in Tables 5.1 and 5.2, assuming that the lengths of the cells, the wave speed and the jam traffic volume is independent of commodities. However, two commodities are consider, namely A and B, which have different freeflow speeds. This can be easily thought as the case of cars and heavy vehicles travelling on the same road, with the latter having to proceed at a lower speed for obvious safety reasons. In this example it is assumed that vehicles belonging to commodity A can travel through the network at a maximum of 65 mph, whereas vehicles of commodity B are required to travel at a lower maximum speed of 50 mph.

Considering a time horizon of T = 20 and fixing the exogenous inflows $\lambda_i^k = 0.1$ for every commodity and cell until t = 10. Set the initial condition of each cell and commodity as $x_i^k(0) = 0.5$ and proceeded to simulate choosing the penalty parameter $\rho = 10$.

Table 5.1: Cell Length, wave speed and jam traffic volume

Type of cell	L	w	x^{jam}/L_i
Mainline	various	13 mph	200 veh/mi
Intersection	0.4 mi	13 mph	200 veh/mi
Between ramps	0.4 mi	13 mph	200 veh/mi
Ramps	0.4 mi	25 mph	$+\infty$/200 veh/mi

Table 5.2: Freeflow speed for different commodities

Type of cell	v commodity A	v commodity B
Mainline	65 mph	50 mph
Intersection	65 mph	50 mph
Between ramps	65 mph	50 mph
Ramps	25 mph	25 mph

Example 12. *In this example the FNC is solved on the Los Angeles Network shown in Figure 5.6 and considering the variables fixed in Tables 5.1 and 5.2. Moreover, the routing matrices are fixed for both commodity such that 10% of flows are always sent to the offramps and when considering an intersection, the flow is equally split.*

The results are reported in Figure 5.7 and 5.8 in which it is easily noticeable how even with a more complex network as the Los Angeles one, the iterative distributed algorithm is capable of finding a feasible solution. After few iterations, the Cost function converges to the minimum and also brings the duality gap to zero, meaning that the values of the Cost and Lagrangian functions are equivalent. Moreover, the degree of infeasibility shows how it quickly reaches zero, proving that the solution found is actually feasible and satisfies all the constraints of the system. However, when considering networks whose dimension is comparable to the Los Angeles one, it is expected that the algorithm complexity to increase accordingly. In fact, for the converge of this example it took approximately 40 minutes, before reaching an acceptable value.

Example 13. *In this example, the focus shifts to solving the DTA on the Los Angeles Network shown in Figure 5.6 and considering the variables in Tables 5.1 and 5.2. The results are reported in Figure 5.10 and 5.9 in which it is clear how the iterative algorithm is even able to solve the DTA problem in a more complex setting. After an initial spike the Cost function converges at a much lower value than the one obtained in the FNC. This is expected since the DTA can freely control the routing of the users. Furthermore, both the duality gap and infeasibility reach zero, which mean that the solution obtain is actually feasible and doesn't violate any constraint.*

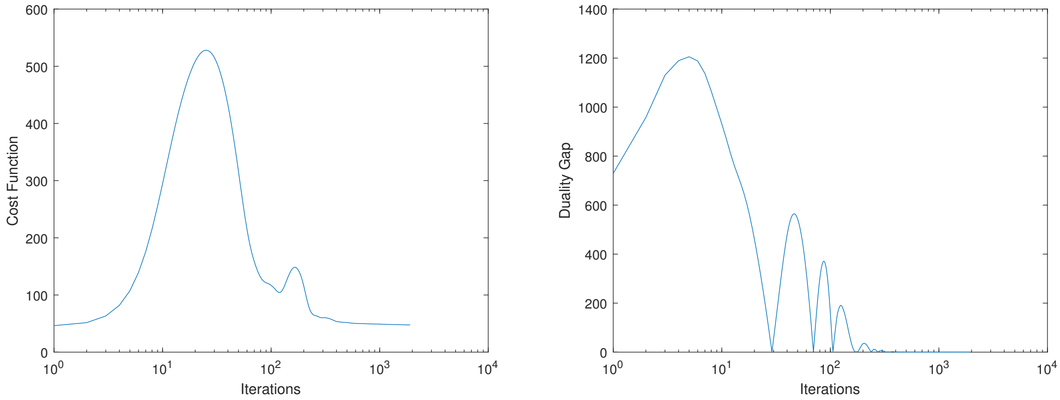


Figure 5.7: Los Angeles FNC: Cost function and Duality Gap with $\rho = 20$

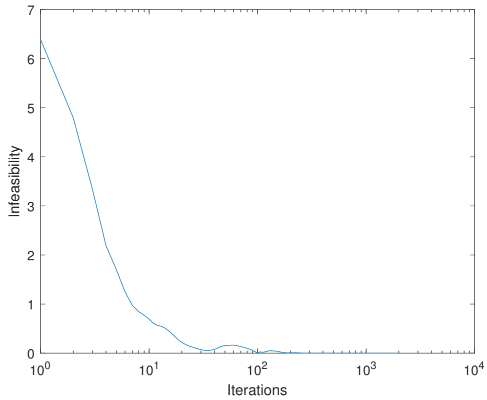


Figure 5.8: Los Angeles FNC: Degree of infeasibility

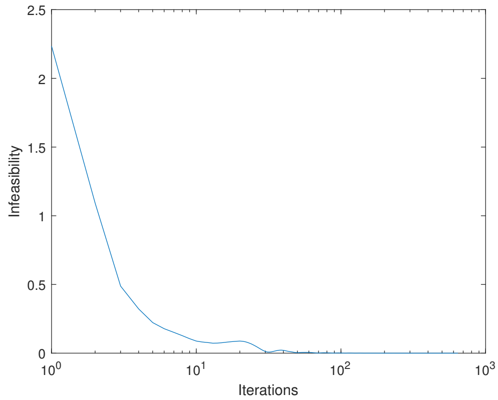


Figure 5.9: Los Angeles DTA: Degree of infeasibility

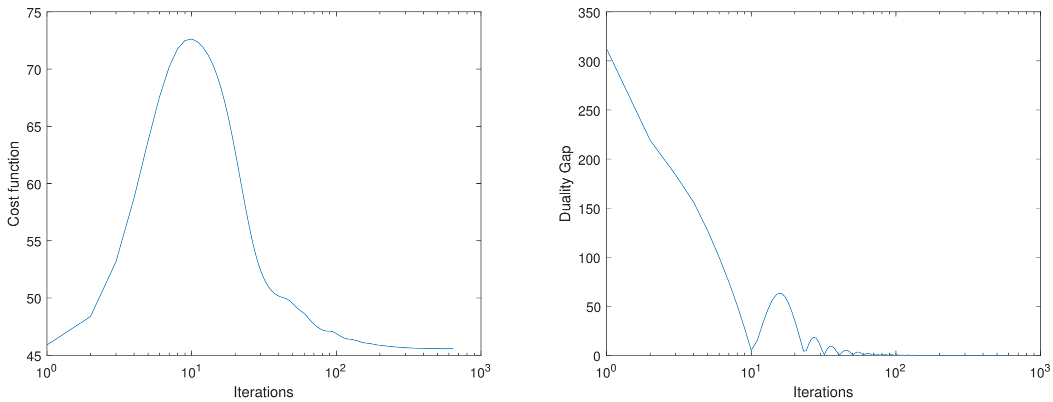


Figure 5.10: Los Angeles DTA: Cost function and Duality Gap with $\rho = 20$

Chapter 6

Conclusions

In this thesis, a model for multicommodity dynamical flow networks was provided and its stability and presence of equilibrium point was analyze. Moreover, the optimization of traffic flows is studied, considering both the cases of a system optimum and of a user selfish optimum.

Daganzo's Cell Transmission Model is initially used as starting point for the formulation of this thesis' model, which splits roadways into cells, each one capable of accommodating multiple types of flow. System's dynamics is governed by a law of mass conservation along with demand functions (maximum outflow) for each commodity and supply functions (maximum inflow) shared by commodities. Moreover, it is then introduced the concept of Freeflow Region, in Definition 5, which is the region of the traffic volumes, given the exogenous inputs, that do not violate the supply constraints. This is useful to define the set of models, both FIFOs and non-FIFOs, that are studied throughout the thesis. Furthermore, the simulations of the system's dynamic show that it is capable of handling real traffic conditions.

In Chapter 3 the model's stability is analyzed drawing inspiration from single commodity studies. It is first introduced the concept of Stability Region, in Definition 8, which is the region of the exogenous inputs for which there exists a unique freeflow equilibrium point. Moreover, in Proposition 2 it is proven that the freeflow equilibrium point is unique.

Furthermore, by considering concave demand and supply functions it is proven that the stability region is also convex, in Lemma 2. After defining the existence of the freeflow equilibrium point, the focus shifted on the study of its stability. Notably, it was possible to prove the equilibrium point is locally asymptotically stable within the Freeflow Region in Proposition 3. The provided simulations are consistent with the formulation presented, but also hint to the possibility that the system might be also globally asymptotically stable which could be a great topic for further research.

In Chapter 4 the thesis focuses on presenting the two known problems of the Dynamic Traffic Assignment and the Freeway Network Control formulated as optimal control problems. Afterwards, by generalizing previous studies it was possible to introduce two further optimal control problems and prove that a solution of those is also a solution of the original ones, both convex and tight. Through the use of optimal control techniques it was possible to formally present the FNC optimal problem, using this as a cornerstone to formulate both the DTA and the FNC in discrete time. Furthermore, in this thesis are presented previous studies that aimed to solve the DTA as a non-convex problem, discussing their methods and solutions.

In Chapter 5 are introduced the concepts of Dual Ascent and Dual Decomposition, along with the methods that can be used to solve convex optimization problems with a, possibly, separable objective function. In order to ensure convergence without assuming strict convexity it is then presented the Augmented Lagrangian Methods and the Method of Multipliers, that uses a slightly modified objective function by considering an additional norm-2 term. Combining these methods together it is possible to obtain the Alternating Direction Method of Multipliers, which is then extended to solve both the DTA and the FNC. In fact, the minimization equations were provided along with the updates of the multipliers, also showing insightful simulations of how the solutions obtained by the algorithm is feasible for the problems considered. It is then analyzed how the algorithm would react to different penalty parameters, used for updating the Lagrangian multipliers, finding that a greater one would imply oscillations before the convergence to the optimal value of the cost function. Moreover, the proposed algorithm has shown great results even in a real-life scenario with the use of the Los Angeles network. In fact, it is possible to find a feasible solution for both the DTA and FNC, managing to optimize the traffic volume within a large scale network.

Future researches regarding this topic could focus on the proof of the global asymptotical stability of the Freeflow equilibrium point in the multicommodity model. Moreover, this extended version of the ADMM has yet to be proven to converge, which leaves a big question unanswered. In the end, it could be possible to investigate the effects of the introduction of feedback control policies to further control the traffic flows within the network.

Appendix A

Background

In this first appendix we shall present some background notion which will be useful throughout this thesis.

A.1 Matrices

We now briefly gather some known definitions and results about matrices.

Definition 13. *A matrix A in $\mathbb{R}^{n\times n}$ is referred to as Metzler if its extra-diagonal entries are nonnegative*

$$A_{ij} \geq 0, \qquad \forall i \neq j \tag{A.1}$$

Theorem 2. *Let M in $\mathbb{R}_+^{n\times n}$ be a nonnegative square matrix. Then, there exist a nonnegative real eigenvalue $\lambda_M \geq 0$ and nonnegative vector $z \neq 0$ such that $Mz = \lambda_M z$ and every eigenvalue λ of M is such that $|\lambda| \leq \lambda_M$.*

Corollary 1. *Let A in $\mathbb{R}^{n\times n}$ be a Metzler matrix. Then, there exist a real eigenvalue λ_A and non negative vector $z \geq 0$ such that $Az = \lambda_A z$ and every eigenvalue λ of A is such that $\mathbb{R}e(\lambda \leq \lambda_A)$.*

We now recall the notion of stochastic, substochastic and compartmental matrices.

Definition 14. *A square matrix M in $\mathbb{R}^{n\times n}$ is*

(i) stochastic if it is nonnegative and such that $M\mathbb{1} = \mathbb{1}$;

(ii) substochastic if it is nonnegative and such that $M\mathbb{1} \leq \mathbb{1}$;

(iii) compartmental if it is Metzler and such that $M\mathbb{1} \leq 0$.

Definition 15. *A substochastic matrix R in $\mathbb{R}_+^{n\times n}$ is out-connected if for every $i_0 = 1, ..., n$ there exists $p \geq 0$ and $i_i, i_2, ..., i_p$ in $\{1, ..., n\}$ such that*

$$\prod_{v=1}^{p} R^k_{i_{v-1}i_v} > 0, \qquad \sum_{i\leq j\leq n} R^k_{pj} \leq 1 \tag{A.2}$$

A.2 Networks and Graphs

A (weighted, directed) multigraph is a triple

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, h) \tag{A.3}$$

such that $\mathcal{V}$ is the set of nodes, $\mathcal{E}$ is the link set and h in $\mathbb{R}_+^{\mathcal{E}}$ is a positive vector of link weights (that can be interpreted as link's capacity) along with two maps $\theta : \mathcal{E} \to \mathcal{V}$ and $\kappa : \mathcal{E} \to \mathcal{V}$ such that $\theta(e)$ and $\kappa(e)$ can be interpreted, respectively, as the head node and the tail node of a link $e \in \mathcal{E}$.

Appendix B

Alternating Direction Method of Multipliers (ADMM)

In this appendix we shall present the Alternating Direction Method of Multipliers [15] introduced by Sthepen Boyd which couples the benefits of dual decomposition and augmented Lagrangian methods for constrained optimization.

B.1 Dual Ascent and Dual Decomposition

Given a convex optimization problem such as

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \end{array} \tag{B.1}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $f : \mathbb{R}^n \to \mathbb{R}$ is convex.

It is then possible to introduce the Lagrangian of problem (B.1) as

$$L(x, y) = f(x) + y'(Ax - b) \tag{B.2}$$

whose dual function is

$$g(y) = \underset{x}{inf} L(x, y) = -f^*(-A'y) - b'y \tag{B.3}$$

where y is the Lagrange multiplier and f^* is the convex conjugate of f.

Given the dual function (B.3) the dual problem reads

$$\text{maximize} \quad g(y) \tag{B.4}$$

with $y \in \mathbb{R}^m$. Then by assuming that strong duality holds the optimal value of the primal problem coincides with the optimal value of the dual problem, meaning that it's possible to obtain a primal optimal point x^* from a dual optimal point y^*

$$x^* = \underset{x}{argmin} L(x, y^*) \tag{B.5}$$

if f is stricly convex.

The dual ascent method is an algorithm to solve constrained convex optimization problems with the use of the gradient ascent. By assuming that the dual function g (B.3) is differentiable, the dual ascent method is composed of two separate steps. Firstly, a x-minimization step by minimizing the Lagrangian function

$$x^{k+1} := \underset{x}{argmin} L(x, y^k) \tag{B.6}$$

where k is the iteration counter. Then, in the second step the dual variable is updated as

$$y^{k+1} := y^k + \alpha^k (Ax^{k+1} - b) \tag{B.7}$$

where $\alpha^k > 0$ is the step size.

One main consequence of the dual ascent method is that it can lead to a decentralized algorithm in case, such as when the function f is separable as

$$f(x) = \sum_{i=1}^{N} f_i(x_i) \tag{B.8}$$

where $x = (x_1, ..., x_N)$ and $x_i \in \mathbb{R}^{n_i}$ It is then possible to partition matrix A as

$$A = [A_1 \dots A_N] \tag{B.9}$$

such that $Ax = \sum_{i=1}^{N} A_i x_i$ thus the Lagrangian function can be rewritten as

$$L(x, y) = \sum_{i=1}^{N} L_i(x_i, y) = \sum_{i=1}^{N} (f_i(x_i) + y' A_i x_i - (1/N) y' b) \tag{B.10}$$

So the x-minimization step becomes

$$x_i^{k+1} := \underset{x_i}{argmin} L_i(x_i, y^k) \tag{B.11}$$

which is made of N separate problems which can be computed in parallel. However, the dual update step doesn't change

$$y^{k+1} = y^k + \alpha^k (Ax^{k+1} - b) \tag{B.12}$$

In this special case the dual ascent method takes the name of dual decomposition.

B.2 Augmented Lagrangians and the Method of Multipliers

In order to ensure convergence to the dual asent method without assume strict convexity the Augmented Lagrangian methods are introduced. The augmented Lagrangian for (B.1) is

$$L_\rho(x, y) = f(x) + y'(Ax - b) + (\rho/2)||Ax - b||_2^2 \tag{B.13}$$

where $\rho > 0$ is the so called penalty parameter. This augmented Lagrangian can be also obtained by slightly modifying problem (B.1) into

$$\begin{aligned} \text{minimize} \quad & f(x) + (\rho/2)||Ax - b||_2^2 \\ \text{subject to} \quad & Ax = b \end{aligned} \tag{B.14}$$

Notice that for problem (B.14) any feasible x implies that the second term is zero thus leading to an equivalent problem of that in (B.1). Furthermore, it is possible to introduce the dual function which is

$$g_\rho(y) = \underset{x}{inf} L_\rho(x, y) \tag{B.15}$$

It is then possible to apply the dual ascent method, as shown in the previous section, to the problem (B.14) yielding

$$x^{k+1} := \underset{x}{argmin} L_\rho(x, y^k) \tag{B.16}$$

$$y^{k+1} := y^k + \rho(Ax^{k+1} - b) \tag{B.17}$$

which is also known as the method of multipliers to solve problem (B.1). The main difference from the standart dual ascent method is that in the x-minimization step the augmented Lagrangian is used and in the dual variable update the penalty parameter ρ is used as the step size α^k.

B.3 The Alternating Direction Method of Multipliers (ADMM)

As previously state the Alternating Direction method of multipliers brings together the benefits of dual decomposition and augmented Lagrangian methods for constrained optimization. In fact, by assuming two convex functions f and g it solves problems such as

$$\begin{aligned} \text{minimize} \quad & f(x) + g(z) \\ \text{subject to} \quad & Ax + Bz = c \end{aligned} \tag{B.18}$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $A \in \mathbb{R}^{p \times n}$, $B \in \mathbb{R}^{p \times m}$ and $c \in \mathbb{R}^p$. Notice that the difference from (B.1) is only that variable x has been split into two variables x and z so that the objective function is separable in those two variables. We shall denote the optimal value of problem (B.18) as

$$p^* = inf\{f(x) + g(z) | Ax + Bz = c\} \tag{B.19}$$

and it is then possible to formulate the augmented Lagrangian as

$$L_\rho(x, z, y) = f(x) + g(z) + y'(Ax + Bz - c) + (\rho/2)||Ax + Bz - c||_2^2 \tag{B.20}$$

. The ADMM algorithm then reads

$$x^{k+1} := \underset{x}{argmin} L_\rho(x, z^k, y^k) \tag{B.21}$$

$$z^{k+1} := \underset{z}{argmin} L_\rho(x^{k+1}, z, y^k) \tag{B.22}$$

$$y^{k+1} := y^k + \rho(Ax^{k+1} + Bz^{k+1} - c) \tag{B.23}$$

where $\rho > 0$.

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