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Physics of Complex Systems

Schelling-type agent-based models to study competition between individual and collective behaviour

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Abstract

This work aims to describe how the behaviour of an agent-based system changes while the will to act collectively of its components is tuned. By tuning this quantity, the equilibrium state that is reached by the system of agents changes, some theoretical phase diagrams that show this result for a variant of the Schelling model are already present in the literature. The main result of this work is the computational verification of these theoretical phase diagrams and a first step in the direction of the generalization of the model with which the diagrams have been produced. The framework in which this work is set is socio-physics: the model which has been used as the reference agent-based model throughout our analysis is a Schelling-inspired model for the problem of urban residential segregation. The analysis we performed includes the definition of a useful order parameter for the segregation/crowding problem and the construction of an algorithm which simulates the behaviour of such a system. A possible application to economics and a more general discussion on the topic will be given at the end of the report.

Contents

Introduction

Agent-based models (ABMs) are computational models that try to simulate the behaviour of a complex system describing its dynamics at the level of the individual element, the agent. This very general definition makes it very easy to use this type of models in many different fields of science. From ecology to economics, from social science to physics, with the more available computational power of machines, ABMs started to spread from the 1950s onward. To simulate an ABM means to create a virtual ensemble of agents that evolve following some rules. As stated in [1], the outcome of ABMs can be either a cycle or an equilibrium distribution for the agents' state, characterized by its shape and the eventual presence of macroscopic structures or patterns. The emerging of properties at the macroscopic/system level comes out from the direct or indirect interactions between the individuals we considered in the model. The dynamics of the agents is autonomous: there exists either an evolution equation for the single agent's state or a utility function that the agent has to maximize/minimize. The main difference with respect to system-based models (such as the Ising model) is that in ABMs each individual component tries to maximize its own utility, not the system's one.

Usually physical systems behave in a system-based way: that happens for example in a thermodynamic gas or in a magnet. In the work of V. Venkatasubramanian [2], the author explains that a physical interacting system can be treated as an agent-based model if one can find the proper individual evolution law, that is the one which explains the macroscopic emergent properties of the global system. With this method one could consider even thermodynamics as a game of many different individuals with the proper utility function. In [2] these physical systems are referred to as 0% purposeful agent-based models, meaning that the agents (in this case molecules or spins) do not have any intention in their actions. This is not the case in a society of completely rational agents (100% purposeful agents). If indeed physics is the typical framework in which we can find non purposeful agents, economics and game theory are the classical environment for fully rational and intentional individuals.

Game theory is the theory which considers the most pure form of rational agents' behaviour, with agents analyzing the possibilities of all the other individuals playing and moving just towards a rise of their own utility function. This decision criterion may be relaxed introducing some noise or randomness such as in evolutionary game theory. Furthermore, for example the description of active matter is instead individual-based but usually not competitive and rational, it therefore represents a middle way between the physical and the game-theoretical cases [2].

With this perspective on these different approaches one deduces that a whole spectrum of models would be covered just by the tuning of a parameter which describes the competition between collective (physics-like) and purely competitive (game theory-like) dynamics (α in [3]). The study of the behaviour of a system while tuning this parameter will be the main focus of this work. During the analysis we will firstly introduce the problem of segregation in society and the associated famous Schelling agent-based model, which will act as our reference ABM. Once obtained some conclusion and result on a variant of this problem following [3], we will discuss the phase transition which is driven by a competition between the collective and individual components of the agents' dynamics. We will then propose an algorithm to recover the theoretical phase diagrams present in the literature. Finally some generalization results that could improve the applicability of the model to a real situation will be obtained and the more economics-related topic of Pigouvian taxes will be discussed. A general discussion on the transition between bottom-up approach and reductionism will be present in the conclusion of this work as a wide generalization of the thesis' topic.

Premises

1 The segregation problem and the Schelling model

Even if the main focus of this work will not be the analysis of the Schelling model, it is important to understand from what context and with what purpose this ABM has been created and why it has become important in the physical context too.

Social segregation is the phenomenon characterized by the separation of two or more social groups, which, depending on the cases, may have limited contacts, relations and interactions. This phenomenon could cause an uneven distribution of individuals in cities (spatial segregation) and an inhomogeneous use of services between different wage classes, nationalities or other social groups. The issues arising from segregation are mostly the emergence of criminality in isolated neighborhoods and the improved difficulty of social mobility [4]. This problem has therefore been studied in order to improve political policies that might eradicate it. Historical examples of the segregation problems are for example the Apartheid [4] or the racial segregation in the United States. In particular, an example of why it is important to study this problem still today is given by the situation in the USA. Even if american formal segregation is finished de jure in 1964 with the "Civil rights act"[5], it is still de facto present in the residential distribution in cities and in the access of people to the health care system in America [6]. This is true also in other contexts in which effective segregation in the past has caused a less visible residential segregation which is still present.

The problem of segregation has in the recent years spread on the virtual platforms too, the known phenomenon of echo-chambers in social networks creates de facto segregated communities of people with respect to their access to information [7].

1.1 A brief history of the Schelling model

The most important model for the study of socio-spatial segregation is the Schelling model [8]. The relevance of the Schelling model in the description of the segregation phenomenon is due to the manifest importance that the complex character of the described system assumes in this model. The model is able to describe a phase transition between the mixed homogeneous case and the segregated state for a schematized city; so it is able to correctly predict the emergence of the macroscopic behaviour of the system.

This is an important result both from the physical perspective and from the sociological point of view; that's why, since when it has been proposed, literature about the Schelling model has always been produced, introducing slight changes in the model and finding new results. We will follow the work of Antoine Lucquiaud [9] in order to summarize the history of the Schelling model in a few lines and in the next paragraph we will describe how the Schelling model works. It's fundamental to understand that most of the papers produced in the recent years actually describe and study some slight variations of the original Schelling model [8], which are introduced in order to be able to obtain stronger results.

H.P. Young [10] is the first one which studies the Schelling model using game theory and Markov chains' properties following the original evolution rules. Then a statistical dynamics approach for slight variants of the model have been proposed both by M. Pollicott et H. Weiss [11] and L. Gauvin et al. [12], in particular L.Gauvin et al. could obtain a proper empirical phase diagram. In [13] Christina Brandt et al. find a proper probabilistic and mathematical description of a one dimensional Schelling segregation, and Hagen Echzell et al. in [14] introduce the possibility of having more than 2 types of agents in the city. After having introduced the scheme of the Schelling model, in this section we will focus on the result of L.Gauvin [12] (obtained in 2009), which is probably the most significant one for our scopes, then in the rest of the paper we will mainly talk about the version introduced in [3], which is more focused on the competition between collective and individual behaviour.

1.2 The Schelling model and its phase diagram

The Schelling model as presented by its author in [8] is an evolution algorithm for the positions of two types of agents placed randomly on some square grid representing a city. As said, the total population is divided into two groups and placed on a stylized two-dimensional area leaving some empty space, everyone's membership in its group is permanent and recognizable. Everybody cares about the type of people living in its neighbourhood in the same way, moreover everybody in the model is capable of moving to an empty space if he is dissatisfied with the neighbourhood in which he is. In order to define and quantify when an agent is discontent or unsatisfied with its neighbourhood a tolerance parameter T is defined. This parameter depends on the number of opposed-type and same-type agents present in the neighbourhood, as well as on the number of empty spaces. There are many different ways in which tolerance can be defined: one could ask that in a neighbourhood a minimum number of same-type agents is required in order for an agent to be satisfied or one could define a content agent as one which has up to some threshold number of opposed-type agents in its surroundings. In any case in the original model a discontent individual moves to the nearest vacant spot that surrounds him with a neighborhood that meets his demands.

By iterating the process we reach a final state which can be either segregated or mixed depending on the tolerance of the agents. At this level we do not quantitatively define the distinction between segregated and mixed state as this depends on the order parameter of segregation one considers. We will talk about this in section 3.2. The main result of the model proposed by Schelling can be stated basically in one phrase: "a low tolerance is not necessary for the emergence of segregation". This is a very strong result from the sociological point of view as it explains that segregation can emerge also in a society which is highly tolerant. The more physics-related result is instead the presence of a phase transition at a threshold level of the tolerance control parameter. We can have a clearer view of this transition in the reported phase diagram: in figure 1 we show the resulting phase diagram obtained by L. Gauvin et al. for a slightly varied version of the original Schelling model. Even if this is not the diagram of the original Schelling model, it is representative for many Schelling-like models at least for the different phases that there appear.

In this model also content individuals are able to move to empty spaces and randomly chosen agents could move to any vacancy which had a satisfying neighbourhood, not just the nearest one. Moreover a content agent is in this case defined as one which has a number of unlike agents N_d lower than (or equal to) a fixed proportion T of all the agents in the neighborhood, where T is the aforementioned tolerance control parameter.

The diagram refers to 2 different parameters: the tolerance T and the vacancies density ρ, which quantifies the number of empty places in the grid.

The importance of this result is that it shows the existence of four phases:

1. segregated state;
2. mixed state;
3. diluted segregation state;
4. frozen state

Moreover it describes the transitions between all of them (see [12]).
In order to recover this result some definitions of the proper order parameters for the segregated state have been introduced. We will talk about the problem of the order parameter in segregation in section 3.2, focusing on the variant of the Schelling model proposed in [3].

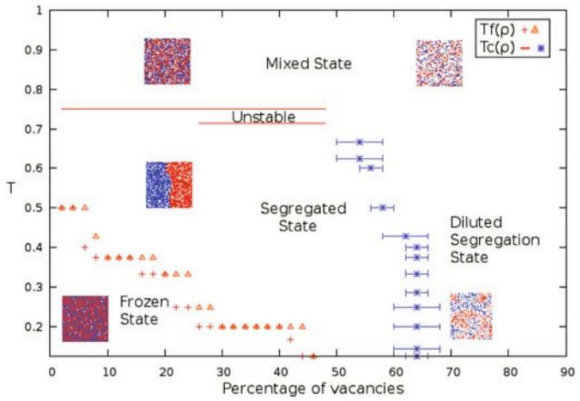


Figure 1: Phase diagram of the Schelling model as found in [12]

2 Competition between individual and collective behaviour

After having introduced the Schelling model and its main features, we will now present a variant of it which has been proposed in [3]. As we said before, we will use this variant of the Schelling model as a reference ABM in order to study the transition that arise while tuning the parameter that describe the collective will of the agents. In this version, the segregation problem will be put aside in order to focus more on the main topic of this work. In this section we will explain all the theoretical reasoning that brought S. Grauwin et al. in [3] to solve this variant of the Schelling model, after this we will recover computationally the theoretical results and we will work in the direction of the generalization of the model.

2.1 A Schelling-like model for the crowding problem

The main difference of the Schelling-like model introduced in [3] is that just one type of agent (and not 2 or more) is present on the grid. That is because it is easier to find results for a 1-type population and then generalize them to a population made up of a higher number of types of agents, this has also been done in the same paper; in the final section the authors introduce a second type of agent and find results about the segregation problem with this model. However, as we will mainly focus on the results obtained for 1 type of agents it would be hard to think about the segregation problem with just 1 type of individuals, we will consider this version as a possible model of what we call a **crowding problem**, which we present here below.

Imagine that in a city agents would like to live in a neighbourhood in which the density has an optimal value which is around half of the maximum number of people that a district can host. Let's also say that it is in general better for the agents to stay in an overcrowded neighbourhood rather than in a very sparse one and that preference is quantified by an asymmetric parameter m.

Then the simplest possible utility function that can describe the preferences of an agent would be the one represented in fig. 2.

The modeled city is divided into Q blocks, each block containing H cells or flats, each cell can contain at most one agent, the number n_q of agents in the block q satisfies $n_q \leq H$, the density of agents is $\rho_q = \frac{n_q}{H}$. The state $x = \{\rho_q\}$ represents the coarse-grained configuration of the city; we will see afterwards that this is the only useful information for the dynamics of the agents because u depends just on ρ and not on all the possible configuration of agents one may have inside a district.

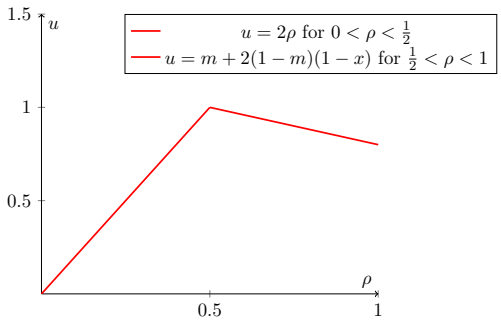


Figure 2: Utility $u(\rho)$

The interaction with the other agents is in this model taken into account by the fact that the utility function of an agent depends on the density of agents in the same neighbourhood, that is to say that the presence of other agents in the same state may influence an agent's behaviour.

In this case the analogue of the segregated state will be the **concentrated state**, meaning that the population in that case will be mostly concentrated in a few number of neighbourhoods, and the analogue of the mixed state will be a homogeneous state over all the districts of the city.

As we will see, we will be able to show a transition between the two states which occurs when we tune a parameter which represents the aim of an agent to act considering the effects of its action on the others. Of course in this 1-type ABM we will not consider the tolerance anymore, because of the fact that we just have one type of agent. α is confined between 0 and 1 and in the two extreme cases we have either all the people moving just according to their own utility function ($\alpha = 0$) or people moving just according to the improvement of the collective utility function ($\alpha = 1$), which is defined as the sum of all the utilities of the agents.

In the general case agents agents will have to consider doing a move according to the quantity:

$$G = \Delta u + \alpha(\Delta U - \Delta u) \tag{1}$$

That means that they move if G is positive.

If we want to introduce some irrationality/noise in the agents' choice we can introduce also the temperature parameter T (not to be confused with the tolerance parameter we talked about before). This parameter quantifies the irrationality of the agents'choices, meaning that if T is large, even if G is positive an agent may refuse to move (or viceversa) according to the probability law defined below. At $T = 0$ the probability of accepting a proposed move from neighbourhood x to neighbourhood y is 1 if $G > 0$ and 0 if $G < 0$.

At $T \neq 0$, the probability for an agent to move from a neighbourhood x to a neighbourhood y is chosen to be:

$$P_{xy} = \frac{1}{1 + e^{-G/T}} \tag{2}$$

This probability distribution has been chosen according to some criteria:

1. the probability should be equal to 1 when $G > 0$ and $T \longrightarrow 0$
2. the probability should be equal to $\frac{1}{2}$ for $T \longrightarrow \infty$ as in that case agents would just act irrationally and G would not mean anything anymore
3. the exponential behaviour of such an acceptance rule for the proposed move between district x and y is the typical one for Metropolis Monte Carlo algorithm, which will be the one we will use in the computational analysis

A very strong and important point of the analysis which has been done in [3] is the fact that we can re-write the quantity G as a difference of 2 different values of a potential function

$G = V(y) - V(x)$. We have been clearing this step in the theoretical demonstration in the appendix section "Congestion and potential games".

By assigning a potential energy to the different configurations, we can make the analogy with the physics of systems in thermal equilibrium. In particular we can define the following function F, analogue to an effective free energy except the sign:

$$F(\mathbf{x}) = V(\mathbf{x}) + TS(\mathbf{x}) \tag{3}$$

where $S(x)$ is the entropy of the configuration $\mathbf{x} = \{\rho_q\}$ (not to be confused with the neighbourhood x we considered before), which represents the logarithm of the number of possible rearrangements of agents in the cells, given $\mathbf{x}$.

The function $V(x)$ is a potential related to a "link" function $L(x)$, connecting the individual and collective levels, such that $\Delta u = \Delta L$. The function V would thus be given by

$$V(x) = (1-\alpha)L(x) + \alpha U(x) \tag{4}$$

The existence of a potential is not in general guaranteed, but in the case of the model the game is a congestion game, hence it is a potential game (again see appendix). By analogy to the entropy, we assume that $L(x)$ can be written as a sum over the blocks, namely $L(x) = H\sum_q l(\rho_q)$. Considering a move from a block at density ρ_1 to a block at density ρ_2, ΔL reduces in the large H limit to $l'(\rho_2) - l'(\rho_1)$, where l' is the derivative of l. The condition $\Delta u = \Delta L$ then leads to the identification $l'(\rho) = u(\rho)$, from which the expression of $l(\rho)$ follows:

$$l(\rho) = \int_0^\rho u(\rho')d\rho' \tag{5}$$

From this result we directly get that

$$V(x) = \alpha H \sum_q \rho_q u(\rho_q) + (1-\alpha)\int_0^\rho u(\rho')d\rho' \tag{6}$$

The quantity $F(\mathbf{x})$ is important because if we consider all the possible configurations with the same density distribution in the districts as equal, then the equilibrium probability of any of this configurations is:

$$\Pi(\mathbf{x}) = \frac{1}{Z}e^{F(\mathbf{x})/T} \tag{7}$$

where Z is the partition function which normalizes the probability distribution.

That is because the dynamics characterized by the transition probability defined in (2) satisfies detailed balance with respect to this probability distribution for the states. There are indeed $e^{S(\mathbf{x})}$ configurations of the system with the coarse grained configuration $\mathbf{x}$, then the total probability of passing from one particular configuration with coarse grained configuration $\mathbf{x}$ to any of the single configuration that have $\mathbf{y}$ as coarse grained configuration one is: $e^{S(\mathbf{y})}\Pi(\mathbf{x})P_{xy}$, detailed balance writes:

$$e^{S(\mathbf{y})}\Pi(\mathbf{x})P_{xy} = \Pi(\mathbf{y})P_{yx}e^{S(\mathbf{x})} \tag{8}$$

from which, by using the definition of $F(\mathbf{x})$ we find the function $\Pi(\mathbf{x})$ as defined above.

$$\frac{\Pi(\mathbf{x})}{\Pi(\mathbf{y})} = e^{(F(\mathbf{x})-F(\mathbf{y}))/T} \tag{9}$$

The entropy has for large H the standard expression $S(\mathbf{x}) = H\sum_q s(\rho_q)$, with:

$$s(\rho) = -\rho\ln\rho - (1-\rho)\ln(1-\rho) \tag{10}$$

The expression of $F(\mathbf{x})$ is therefore:

$$F(\mathbf{x}) = H \sum_q f_q(\rho_q) \tag{11}$$

$$f(\rho) = -T\rho\ln(\rho) - T(1-\rho)\ln(1-\rho) + \alpha\rho u(\rho) + (1-\alpha)\int_0^\rho u(\rho')d\rho' \tag{12}$$

The probability $\Pi(\mathbf{x})$ is dominated by the configurations $\mathbf{x} = \{\rho_q\}$ that maximize the $\sum_q f(\rho_q)$ under the constraint of fixed $\rho_0 = Q^{-1}\sum_{q=1}^{Q}\rho_q$.

The objective here is to know if the stationary state, so the state associated to the absolute maximum of the free energy, is homogeneous or not; in order to compute this we will use the double tangent method, a common tool used to find transitions points in binary systems [15]. The state with the global density ρ_0 is unstable if there exist ρ_1 and ρ_2 such that:

$$\gamma f(\rho_1) + (1-\gamma)f(\rho_2) > f(\rho_0) \tag{13}$$

this does not mean that f is convex in the interval $[\rho_1, \rho_2]$, because for that we would need for any couple (ρ_1, ρ_2) inequality 13 to be true; but it corresponds to say that $f(\rho)$ is a non concave function.

The parameter γ corresponds to the number of neighbourhoods with density ρ_1. The values of ρ_1 and ρ_2 are obtained by optimizing the value of the quantity:

$$\gamma f(\rho_1') + (1-\gamma)f(\rho_2') \tag{14}$$

under the condition:

$$\gamma\rho_1' + (1-\gamma)\rho_2' = \rho_0 \tag{15}$$

Deriving 14 w.r.t. ρ_1' and ρ_2', we get to 2 equations that the function f has to satisfy.

$$\frac{\partial\gamma}{\partial\rho_1'}f(\rho_1') + \gamma\frac{\partial f}{\partial\rho_1'} - f(\rho_2')\frac{\partial\gamma}{\partial\rho_1'}\bigg|_{\rho_1'=\rho_1} = 0 \tag{16}$$

where from 15

$$\frac{\partial\gamma}{\partial\rho_1} = \frac{(\rho_2-\rho_0)}{(\rho_2-\rho_1)^2} \tag{17}$$

which gives

$$\frac{f(\rho_2)-f(\rho_1)}{\rho_2-\rho_1} = \frac{\partial f}{\partial\rho_1} \tag{18}$$

Identical computations give the result for the derivative w.r.t. ρ_2

$$\frac{f(\rho_2)-f(\rho_1)}{\rho_2-\rho_1} = \frac{\partial f}{\partial\rho_2} \tag{19}$$

From this 2 equations it is possible to find the optimal values ρ_1 and ρ_2, note that the condition given by the 2 equations is the same as requiring that the tangent to f at the values of ρ ρ_1 and ρ_2 is the same, an illustration is shown just below.

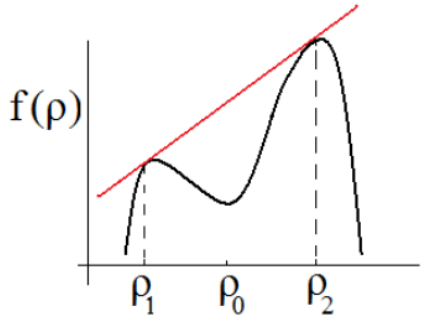


Figure 3: Geometrical illustration of the double tangent method, ρ_1 and ρ_2 for which there exists the same tangent are the optimal values of $\gamma f(\rho_1') + (1-\gamma) f(\rho_2')$, credits: [3]

In conclusion we can say that the maximum value of $\gamma f(\rho_1) + (1-\gamma) f(\rho_2)$ is given by ρ_1 and ρ_2 which satisfy the conditions 18 19 and the segregated state is favoured if for the given ρ_0, 13 is valid.

2.2 the $T = 0$ case

The function $f(\rho)$ for the particular asymmetric shape we have chosen for the function u in Fig 2 is given by:

$$f(\rho) = -T(\rho \ln \rho + (1-\rho)\ln(1-\rho)) + (1+\alpha)\rho^2 \tag{20}$$

if $\rho < \frac{1}{2}$ and by:

$$f(\rho) = -T(\rho \ln \rho + (1-\rho)\ln(1-\rho)) - (1+\alpha)(1-m)\rho^2 + (2-m)\rho - (1-\alpha)(2-m)/4 \tag{21}$$

if $\rho > \frac{1}{2}$.

If we concentrate on the $T = 0$ case the 2 expressions simplify into:

$$f(\rho) = (1+\alpha)\rho^2 \tag{22}$$

if $\rho < \frac{1}{2}$ and by:

$$f(\rho) = -(1+\alpha)(1-m)\rho^2 + (2-m)\rho - (1-\alpha)(2-m)/4 \tag{23}$$

if $\rho > \frac{1}{2}$.

As the function f is in this case convex in $[0, \frac{1}{2}]$, inequality 13 is satisfied for any $\rho_0 \in [\rho_1, \rho_2]$ with $\rho_1 < \rho_2$ and $\rho_1, \rho_2 \in [0, \frac{1}{2}]$. Hence if ρ_0 belongs to the interval $[0, \frac{1}{2}]$, then it is always possible to find $\rho_1, \rho_2 \in [0, \frac{1}{2}]$ such that inequality 13 is valid, and so in this case the stationary state is the segregated/concentrated one. Moreover in order to maximize the value of the free energy $\gamma f(\rho_1) + (1-\gamma) f(\rho_2)$ we take the extreme $\rho_1 = 0$ for ρ_1 and we take $\rho_2 >= \frac{1}{2}$, this is because the highest value of $\gamma f(\rho_1) + (1-\gamma) f(\rho_2)$ is reached by taking the extremes (at least) of the range in which f is convex. Now that ρ_1 is fixed to 0, we can compute the exact value of ρ_2 by using eq. 19, we get:

$$\rho_2 = \frac{1}{2}\sqrt{\frac{(1-\alpha)(2-m)}{(1+\alpha)(1-m)}} \tag{24}$$

Moreover, equation 13 becomes just:

$$f(\rho_2) > f(\rho_0) \tag{25}$$

The function f is in this case ($T = 0$) monotonously increasing, therefore if $\rho_0 < \rho_2$ the favoured state is the segregated one. The value of ρ_2 is in this case always bigger than $\frac{1}{2}$ because the function f is convex in $[0, \frac{1}{2}]$.

ρ_2 is in the range $\frac{1}{2} < \rho_2 < 1$ if (and only if) the following conditions are satisfied:

$$\frac{3m-2}{6-5m} = \alpha_t(m) < \alpha < \alpha_c = \frac{1}{3-2m} \tag{26}$$

Hence for $\alpha \geq \alpha_c(m)$, ρ_2 sticks to the value $\rho_2 = \frac{1}{2}$ and for $\alpha \leq \alpha_t(m)$, one has $\rho_2 = 1$.

The conclusion for the $T = 0$ case is that for $\rho_0 < \rho_2(\alpha, m)$ the segregated state is the equilibrium one, so in the city we will have just empty blocks and blocks of density ρ_2. The phase diagram representing this situation for $m = 0.8$ is shown in the top left corner of Fig. 4.

In the case $\rho_0 < \rho_2(\alpha, m)$ the value of the normalized collective utility function increases with α and it is:

$$U^*(x) = \gamma u(\rho_1) + (1-\gamma)u(\rho_2) = u(\rho_2) = \begin{cases} m, & \alpha \leq \alpha_t \\ 2 - m - \sqrt{\frac{1-\alpha}{1+\alpha}(2-m)(1-m)}, & \alpha_t \leq \alpha \leq \alpha_c \\ 1, & \alpha > \alpha_c \end{cases} \tag{27}$$

In the case $\rho_0 > \rho_2(\alpha, m)$, the equilibrium state is the homogeneous one and the value of $U^*(x)$ is:

$$U^*(x) = u(\rho_0) = 2 - m - 2(1-m)\rho_0 \tag{28}$$

2.3 the $T \neq 0$ case

Let's now describe the $T \neq 0$ case starting from the high T case. The high T case is the simplest to analyze. If T is high enough the second derivative of the function f never becomes positive. Indeed for

$$\frac{2T}{1+\alpha} \geq \max_{[0,1]} 4\rho(1-\rho) = 1 \tag{29}$$

f is concave on the whole interval $[0, 1]$ as it is concave on the two intervals $[0, 1/2[$ and $]1/2, 1]$ where it is regular and at the singular point $\rho = \frac{1}{2}$, $f'(\frac{1}{2}^+) > f'(\frac{1}{2}^-)$.

So in the case $T > \frac{1+\alpha}{2}$ the equilibrium phase is the homogeneous one, with density ρ_0. In the opposite case $0 < \frac{T}{1+\alpha} < 1/2$ the function f is convex just on the interval for which

$$\frac{1}{2}\left(1 - \sqrt{1 - \frac{2T}{1+\alpha}}\right) < \rho < \frac{1}{2} \tag{30}$$

As the function is convex in this interval and it has a positive (negative) infinite slope for $\rho = 0$ ($\rho = 1$), then if we search for the optimal values ρ_1 and ρ_2 for sure we will have that: $\rho_1 < \frac{1}{2}\left(1 - \sqrt{1 - \frac{2T}{1+\alpha}}\right)$ and $\rho_2 > \frac{1}{2}$.

In general the 2 values of the densities ρ_2 and ρ_1 are coupled by the 2 nonlinear equations 18 and 19 that can be numerically solved.

Once we have the values of ρ_1 and ρ_2 at a given temperature T, then the equilibrium state will be the segregated one with these 2 densities if 13 is satisfied.

From the theoretical analysis, the results for the phase diagram that we get are the following:

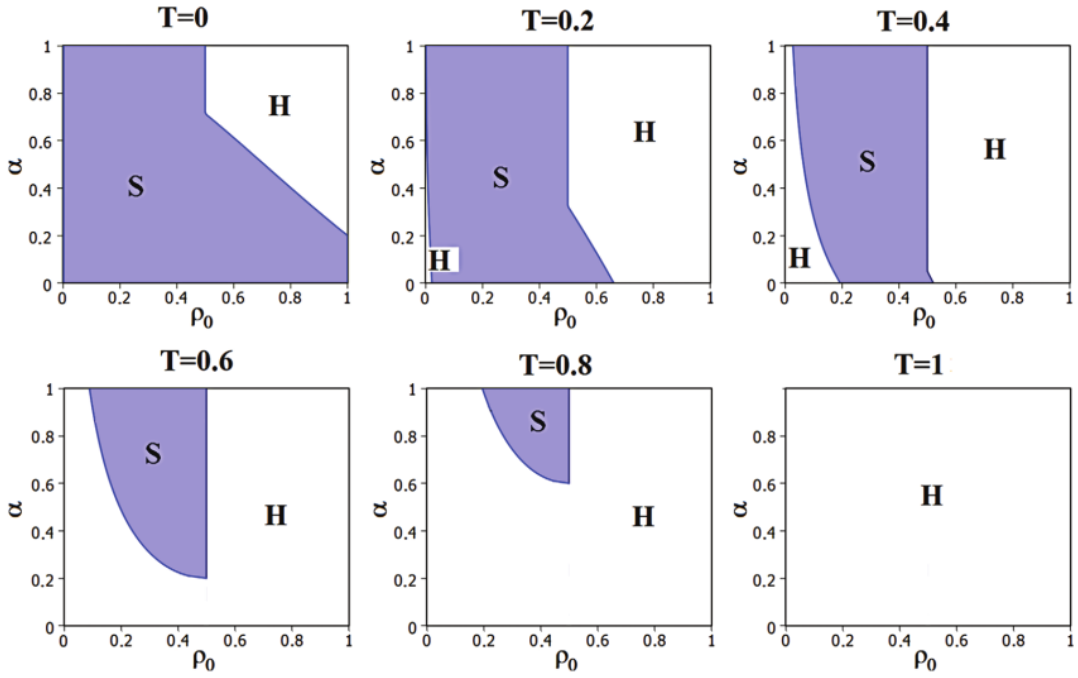


Figure 4: Theoretical phase diagrams obtained for different values of the temperature T at $m = 0.8$, credits: [3]

If we compare the results obtained for low T with the $T = 0$ case the main difference is the appearance of another homogeneous phase for $\rho_0 < 1/2$ for small values for the parameter T. But whereas for $\rho_0 > 1/2$ homogeneity corresponds to the best interest of the agents, for $\rho_0 < 1/2$, collective utility is not maximized in a homogeneous city. This homogeneous region is here purely induced by noise/temperature/irrational choices of the agents. Note that an increase in α tends to reduce this noise region, while it tends to increase the homogeneous domain for $rho_0 > 1/2$.

In the rest of our work we will not change the value of the asymmetry parameter m as we don't want to focus on the consequences of the tuning of this parameter, but on the ones caused by the tuning of α.

Methods and Results

3 The computational analysis

The first step of our work is related to recover the phase diagram of the model by simulating the behaviour of the agent-based system. While doing this, we will see that some enrichment in the description is needed in order to correctly come to the same result: in particular we will have to face the problem of the dependence of the result from the initial configuration of the system and the non-trivial definition of the right order parameters for the segregation/crowding problem.

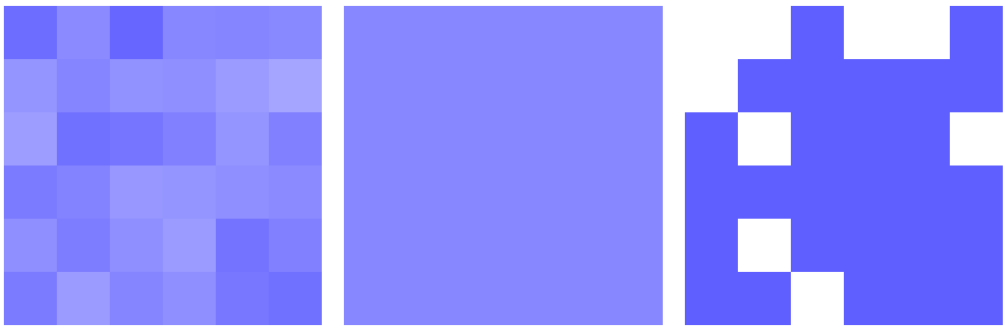
First of all we will try to recover the results for the case $T = 0$ and after that we will introduce the parameter T in the model.

3.1 The simulation

The first simulation of the model will make use of a vector of Q components, each of which represents a district of the city, that can have a maximum value of H, which is the maximum number of people that a district can host. The number of people in each neighborhood will be initialized according to a random placement of a total number of $\rho_0 \cdot H \cdot Q$ agents in a neighbourhood. ρ_0 defines the initial density of the whole city without specifying if it is homogeneously distributed or not, the definition of ρ sets also the total number of people in the city, which does not change all over the simulation. For the relative high values of the parameters Q and H we will use ($Q = 36$, $H = 100$), the randomized initial assignement of the district to each agent is basically equivalent to a uniform starting distribution of the agents onto the different neighbourhoods; we will see in a few lines why the initial distribution we consider for the agents has a strong relevance.

After having initialized the system, the agent's moves from a district to another are proposed randomly and accepted using the probability rule 2 as it usually happens when using Metropolis Monte Carlo algorithms [16].

If we run a simulation and plot each component of the vector on a square grid we would obtain results according to the parameters we set. We show here the results obtained starting from the configuration 1 in 2 different cases after 100 runs (i.e. a total number of $100 \cdot Q$ moves) with $Q = 36$,$H = 100$, $m = 0.8$, $\rho_0 = 0.6$, $T = 0$.



(a) Initial density distribution (random uniform)

(b) Result of the run with $\alpha = 1$ (non concentrated)

(c) Result of the run with $\alpha = 0$(concentrated state)

By changing the values of α and ρ_0 from 0 to 1 we can get the empirical phase diagram if we are able to reach the equilibrium for our system and if we find an order parameter to use to describe how much a population is segregated/concentrated.

3.2 The order parameter problem: how to measure segregation?

The problem of measuring segregation/concentration has been already considered in the literature, it is indeed not easy to quantify segregation even if one may find trivial to decide if a city is in the segregated state or not.

In [17] a full analysis of all the indices that should be taken into account in order to correctly describe segregation is available. In particular Massey and Denton in [17] conclude that the following quantities (with related quantitative indices) give a complete description of the phenomenon.

1. evenness: it quantifies the differential distribution of 2 social groups among areal units in a city
2. clustering: refers to the degree of assemblage of same-type agents in space
3. concentration: it refers to the space occupied by a minority in the city
4. exposure: it represents the degree of potential contact between 2 members of 2 different groups

We will not consider centralization, which is instead present in [17], because no preference has been given in our model to one district with respect to another.

From [9] we take the following diagram showing in a qualitative way the importance of each index in the description of the different realizations of segregation.

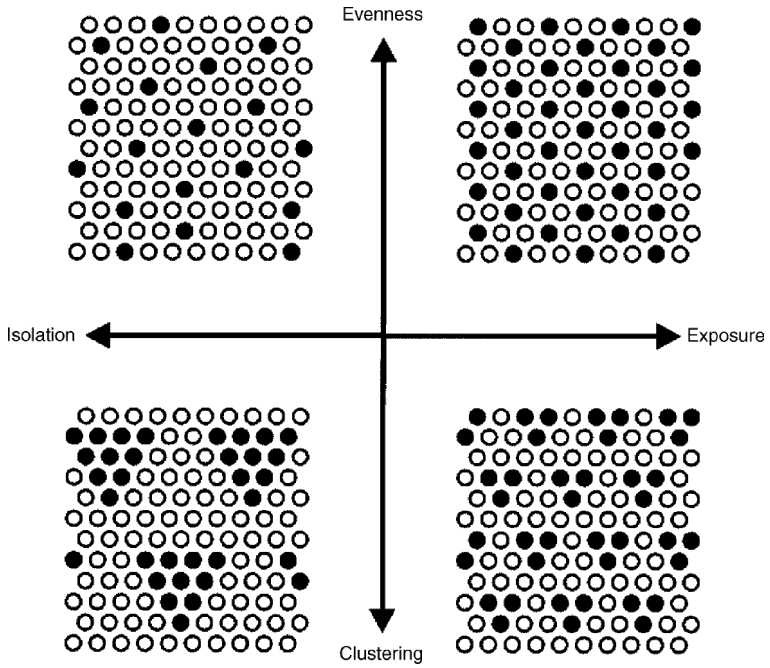


Figure 6: Qualitative representation of the meaning of the order parameters of segregation as found in [9]

In order to have a simpler (with a lower number of order parameters) description of the level of segregation we could also consider other types of quantities.

In [12] Gauvin et al. combine 2 quantities in order to distinguish between the 4 different phases we saw in 1.

The first order parameter is linked to the definition of segregation as the grouping of agents of the same type and the exclusion of the other type in a given area. It basically quantifies what

we firstly referred to as "clustering" and in particular it is defined in the following way:

$$s = \frac{2}{(L^2(1-\rho))^2} \sum_c n_c^2 \tag{31}$$

Where $N_{tot} = L^2(1-\rho)$ is the total number of agents, n_c is the number of agents in cluster c and a cluster is defined such that 2 agents belong to the same cluster if they are nearest neighbors. This order parameter is equal to 1 if there are just 2 clusters (complete segregation), it vanishes if the cluster dimensions remain finite when the system dimension L goes to infinity. The second one (which we do not report here) instead allows to distinguish between the homogeneous state at low tolerance and the one at high tolerance. All the parameters we have been considering up to now refer to a 2-types agents' system, they are not therefore useful to describe the problem as we considered it in chapter 2: what we called the crowding problem.

In the theoretical analysis we discussed above, there was no need to introduce an order parameter which quantifies a segregation state because just two possibilities were considered: either the system is in the segregated state if there exist 2 different densities ρ_1 and ρ_2 for which Eq.13 is valid, or the system is in the homogeneous state. There were no intermediate states and therefore a system with $\rho_1 = 0.49$ and $\rho_2 = 0.51$ would be considered segregated as much as a system with $\rho_1 = 0$ and $\rho_2 = 1$. In order to effectively quantify the segregation from the output of our simulation there are several types of order parameters one can think of.

We will consider 2 parameters which may quantify segregation by relating it to the "distance" from the homogeneous state.

The average distance of the local density: this parameter is an order parameter which we propose as a possible measure of segregation/concentration of agents. It is defined in the following way:

$$D = \frac{1}{Q} \sum_q |\rho_q - \rho_0| \tag{32}$$

where ρ_q is the density of district q obtained after the simulation and ρ_0 is the density of the city, which should be reached in any district in the homogeneous case.

The distance from the global homogeneous utility: this parameter is a possible order parameter for concentration/segregation and it is defined as:

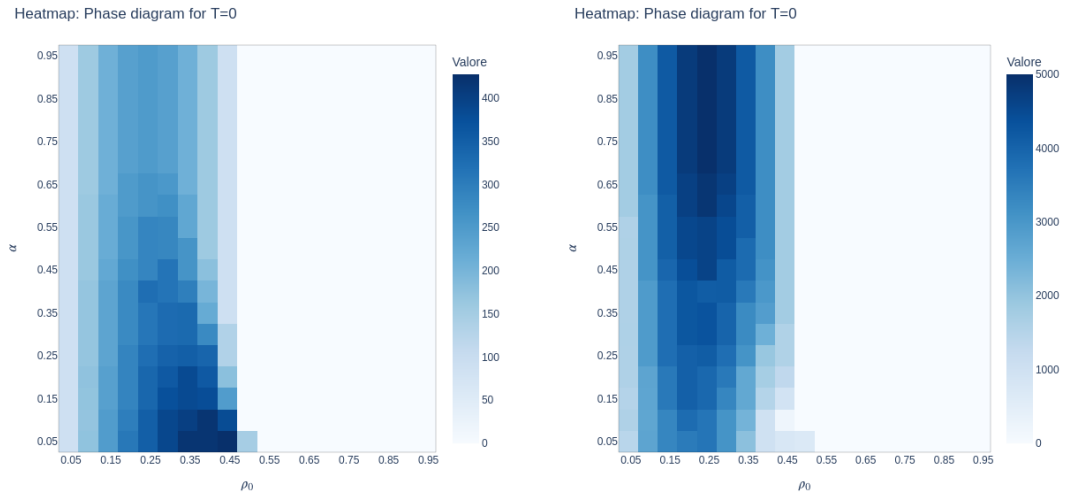
$$B = |U - U_{hom}| \tag{33}$$

where U is the global utility of the agents after the simulation (sum of all the individual utilities) and U_{hom} is the global utility in the homogeneous case, so

$$U_{hom} = Q \cdot H \cdot \rho_0 \cdot u(\rho_0) \tag{34}$$

3.3 First results for $T = 0$

Once defined the order parameters and the algorithm of the code, after some tuning of the fixed parameters we should get all the right results. However if we run the code as defined in section 3.1 with $m = 0.8$, $H = 1000$, $Q = 36$, what we obtain are the following phase diagrams (each for one of the 2 order parameters):



(a) Phase diagram with D as order parameter (b) Phase diagram with B as order parameter

Figure 7: Simulated phase diagrams for $T = 0$ (uniform initial distribution)

As we can see from the resulting diagrams, we do not actually get the same results as the theoretical ones. This may be caused by several factors, as we discuss in the next section.

In order to get a more quantitative evidence of the difference of this result from the theoretical one, we introduce a new parameter C as follows:

$$C = \sum_{\{\alpha\},\{\rho\}} |U^*_{sim,\alpha,\rho} - U^*_{th,\alpha,\rho}| \tag{35}$$

where the quantities we introduced are defined in the following way:

1. $\{\alpha\}$: the discrete set of values of α we considered in the simulation
2. $\{\rho\}$: the discrete set of values of ρ we considered in the simulation
3. $U^*_{sim,\alpha,\rho}$: the normalized global utility function obtained with the simulation
4. $U^*_{th,\alpha,\rho}$: the theoretical normalized global utility function, as in (27)

The higher the value of C, the higher the effective distance from the result we got from the one we would like to reproduce.

In the case of Fig. 7 we get the value $C = 17.723$.

If we plot the quantity $|U^*_{sim,\alpha,\rho} - U^*_{th,\alpha,\rho}|$ for all the values of $\{\alpha\}$ and $\{\rho\}$ we get the following result:

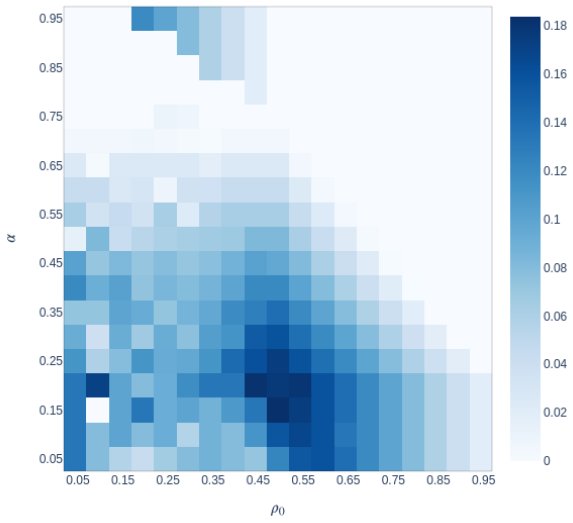


Figure 8: Plot of the quantity $|U^*_{sim,\alpha,\rho} - U^*_{th,\alpha,\rho}|$ for $T = 0$, $m = 0.8$

As we can conclude from this last plot, the fact that C is not equal to 0 is basically due to the fact that in particular in the region of high ρ_0 and small α the simulated and theoretical diagram do not coincide.

3.4 Discussion

One of the differences between our simulation results and the theoretical diagram reported here on the right is the dependence on α of the equilibrium state, we can indeed see that from our results it seems that if the final state is segregated/concentrated or homogeneous depends just on the parameter ρ_0. The main causes of this may be:

1. a dependence on the initial distribution of agents that may have made the system end up in a local minimum
2. the fact that in our simulation $H \not\longrightarrow \infty$, but it is fixed at 1000
3. the order parameter does not represent correctly the idea of segregation of the theoretical result
4. the equilibrium state hasn't been reached.

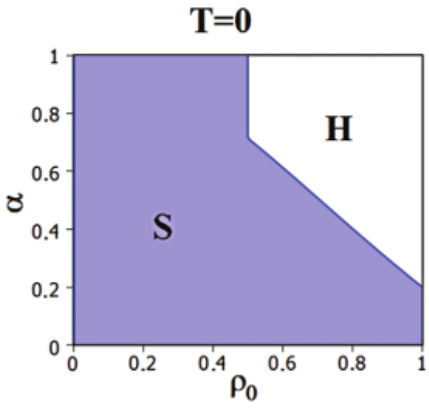


Figure 9: Theoretical phase diagram obtained for $T = 0$, $m = 0.8$, $H \longrightarrow \infty$

We can already discard the second hypothesis by performing a simulation with a higher number of H. The diagram with a higher value of H turns out identical to figure 7, hence we do not

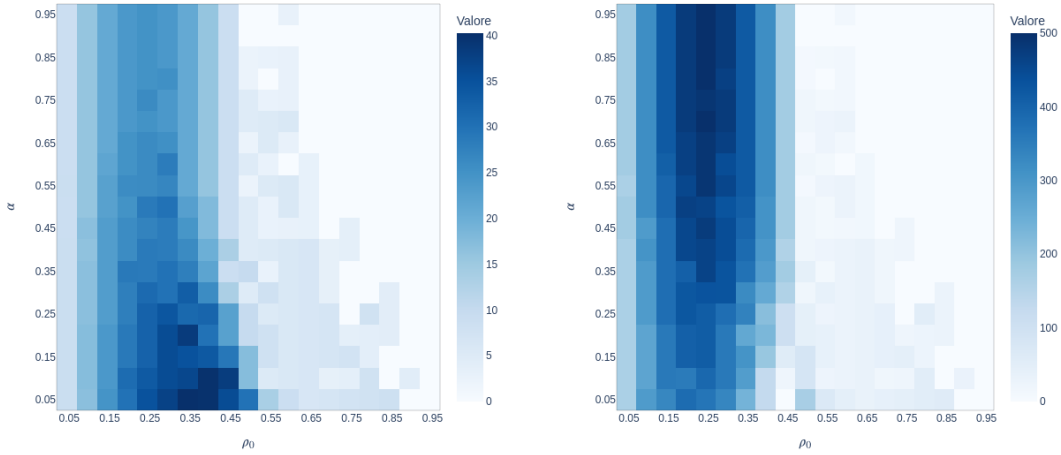
represent it here, but we can conclude that the reason behind such different results is not the finite value of H.

Talking about the order parameters, as we can see from the graph, the parameter D is equal to 0 in a region in which the city should have the segregated state as equilibrium state. If the parameter D is 0, then we are sure to be in a homogeneous city, because it means that no district has a different density than ρ_0. This does not mean that the parameter D is the perfect parameter in the description of segregation, because when it is high it is not given that we are in a segregated/concentrated state as there may be few non-homogeneous districts which make the value of D increase. However we can conclude that in this case the problem is not due to the choice of the order parameter, because surely the region with low α and high ρ in the simulation is associated to the homogeneous case, which is not what we would expect from theory. Also, increasing the runs (and so the number of proposed moves for the agents) the diagram does not change, therefore we exclude also the third reason. The only possibility is that with the simulation we have been ending in a local minimum.

The idea is that if a system is already in the homogeneous case, it will hardly segregate/concentrate as in order to segregate it has to empty a lot of districts (that is because at $T = 0$ $\rho_1 = 0$). That is not difficult when ρ_0 is low, because there are less people in each neighbourhood, so that it is easy to empty them and it is easier to accumulate them in other district as the threshold of H people is further from the starting density. If instead ρ_0 is higher, let's say it is equal to 0.95, it is very unlikely to empty an entire district and also all the other neighbourhoods are already almost full, therefore the system is most likely frozen in its uniform state. We can think at this as the 1-type agent generalization of what we described for the Schelling model in Figure 1.

In order to see if this problem is really related to the starting conditions, we try to run a code with a different starting distribution, in particular a truncated normal distribution, centered in the central districts with a variance of $\frac{3}{2}Q$.

The result we get in this case is the following ($H = 100$):



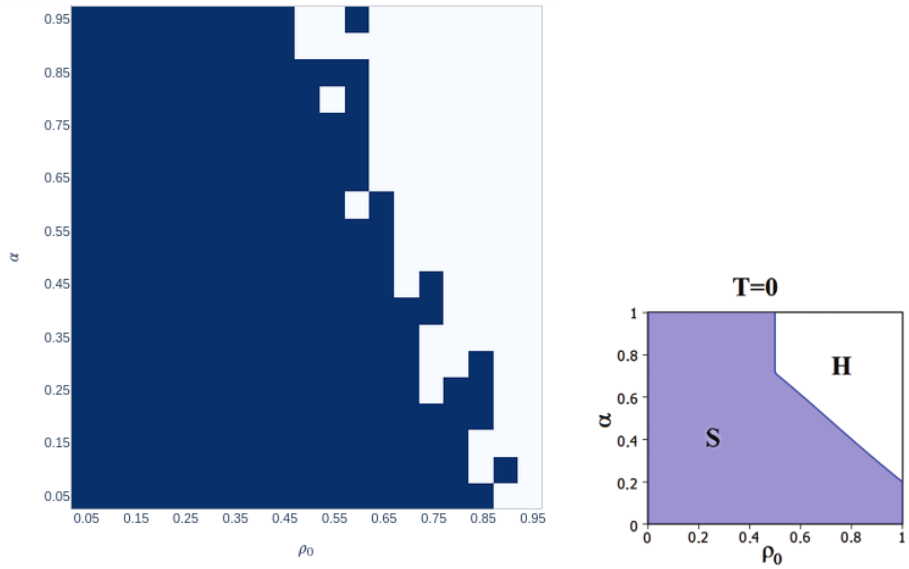
(a) Phase diagram with D as order parameter (b) Phase diagram with B as order parameter

Figure 10: Simulated phase diagrams for $T = 0$ (truncated normal starting distribution)

If we print the diagrams with truncated normal starting distribution we can see that in

the region in which we should have the resulting segregated state (according to the theory) we obtain a slight concenetration, so according to our order parameters, a slight distance from the homogeneous case.

As in the theoretical diagram, even a very slight change from the homogeneous case is considered segregation, then we have to change the way in which we represent this diagram in order to compare it to the theoretical one. In particular the following picture represents better the improve we made by starting from a normal distribution.



(a) Phase diagram with D as order parameter, the white part represents values for which $D = 0$

(b) Theoretical phase diagram for T=0

Figure 11: Simulated phase diagram for $T = 0$ (truncated normal starting distribution, saturated) and comparison with the theoretical one

As we can now more clearly see, at least qualitatively we get actually closer to the theoretical result found in [3]. We now have a dependence of the equilibrium state from the parameter α and also a very similar slope of the segregated equilibrium distribution.

Quantitatively the quantity C is now equal to $C = 17.130$ and the diagram of the error is:

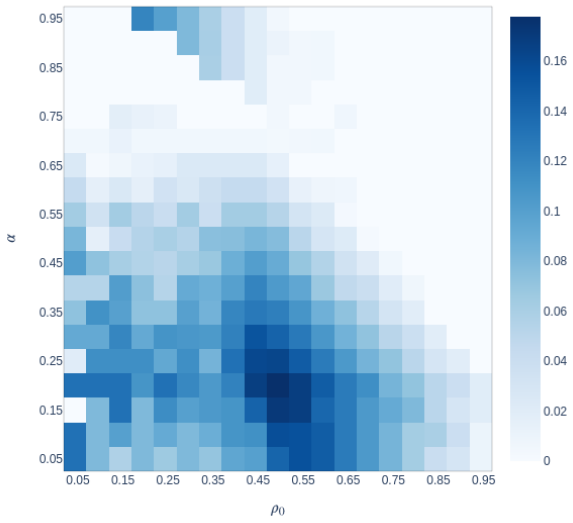


Figure 12: Plot of the quantity $|U^*_{sim,\alpha,\rho} - U^*_{th,\alpha,\rho}|$ for $T = 0$, $m = 0.8$, Truncated Norm

Even if there is an improvement in the result, it is not quantitatively meaningful. However the most important result of this trial is that it manifestly shown the very strong dependence of the phase diagram on the initial distribution of the system. We are not yet sure we have reached the equilibrium distribution associated to the global minimum of the free energy though, we therefore will study in the next section how to reach this state using Simulated Annealing (SA).

3.5 Results for $T = 0$ with Simulated Annealing

There are several algorithms that face the problem of reaching the global equilibrium distribution of some system [18]. One of those is probably the most known physics-inspired algorithm: Simulated annealing. The simulated annealing algorithm is inspired by the annealing process, a heat treatment used in metallurgy that alters the physical properties of a material to increase its ductility and reduce its hardness, so that it becomes more workable.

If the cooling of the system happens slowly enough, at each temperature T the system is able to reach equilibrium. If instead the cooling is too fast then the system does not reach equilibrium at each step and some defects crystallize inside it giving rise to a amorphous phase, in a meta-stability condition. The simulated annealing algorithm is nothing more than a sequence of Metropolis algorithms evaluated for decreasing values of a parameter, which assumes the role of the physical temperature. This parameter in our case is T, which is associated to the irrationality/noise in the agents' choices.

In order to better define how the code with simulated annealing will work, we have to determine some parameters:

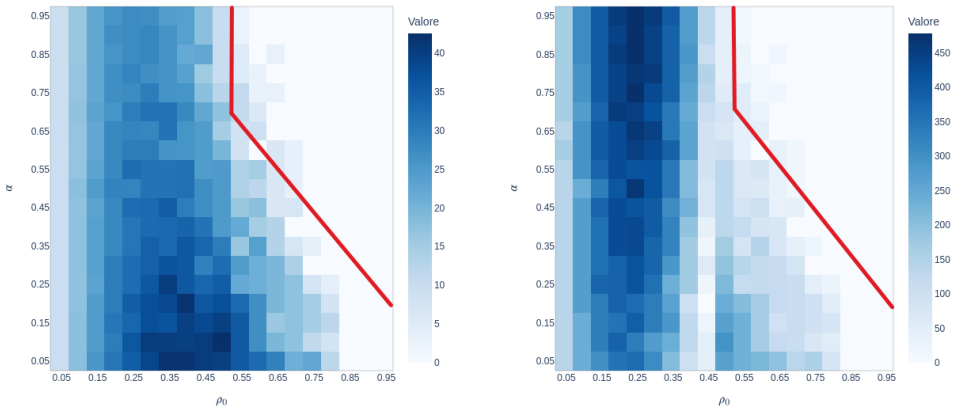
1. **initial temperature**: the initial temperature should be high enough in order to be sure that starting from any point in the phase space, at that T any other point in reachable
2. **decreasing step** ΔT:the decreasing of the temperature should follow a function which ensures that in particular at low T the systems cools slowly

3. **number of iterations at a fixed temperature** T:the number of iterations at each temperature T should be high for low values of T, where it is more important to actually reach the optimal value and lower for high values of T, where it is more important to select the right region of the phase diagram.

According to the theory, an initial temperature of $T = 1$ should be high enough as from what we can see in the phase diagrams in Fig. 4, at that value of T the system is in its maximal disordered phase, with no case stabilizing in the segregated state.

In order to satisfy point 2 and 3 after some trials and some tuning of these parameters, we choose an exponential behaviour for the decreasing temperature, with a linear increase in the number of iterations while the temperature decreases.

If we run the simulation with those values of the parameters and $m = 0.8$, $Q = 40$, $H = 100$, we obtain the following result:



(a) Phase diagram with D as order parameter and comparison with the expected theoretical result

(b) Phase diagram with B as order parameter and comparison with the expected theoretical result

Figure 13: Phase diagrams for $T = 0$ obtained through simulated annealing

In this case $C = 14.981$, it has a significant improvement with respect of before. This is also recognizable in the error diagram:

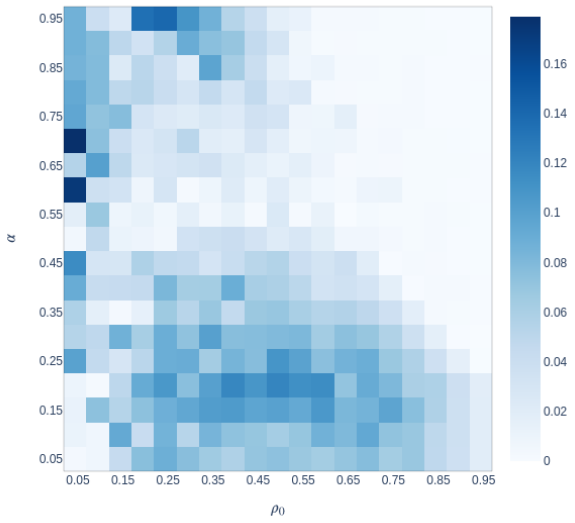


Figure 14: Plot of the quantity $|U^*_{sim,\alpha,\rho} - U^*_{th,\alpha,\rho}|$ for $T = 0$, $m = 0.8$, SA

3.6 Discussion of the results for $T = 0$ with SA

The result we got is not qualitatively different from what we obtained in the case of a starting normal distribution of the agents, but in this case we obtain high values of segregation order parameters also in the region of $\rho_0 > 0.5$. This is the best result we can get with the limited computational power. The obtained diagram is qualitatively similar to the theoretical one, the main differences may be due to the following considerations:

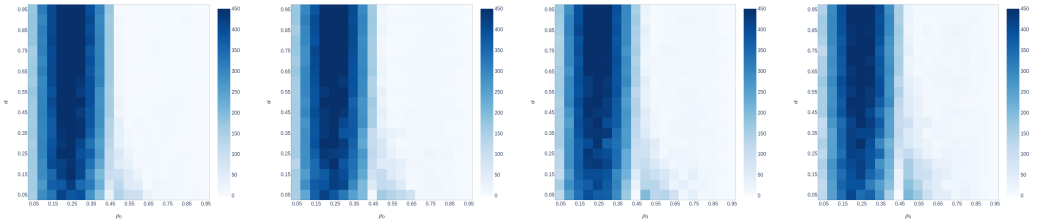
1. the parameters of the simulated annealing do not bring the system to the right equilibrium state
2. H and Q are finite, so for systems at very high ρ_0 it is very unlikely to reach the segregation state.

To improve the result one should increase the values of H and Q and refine the temperature decrease. This increases a lot the time required by the simulation to finish.

3.7 Results and discussion for $T \neq 0$

We will try in this section to recover the phase diagrams for $T > 0$ still using the same code we used before (without simulated annealing as $T \neq 0$ now). If the temperature is different from 0, entropy gives an important contribution to the potential to minimize, so the system may now behave differently with respect to before.

Actually if we perform some simulation considering $T \neq 0$ what we obtain is not very similar with respect to the theoretical results. We show our results in the following figures:



(a) T=0.4 (b) T=0.6 (c) T=0.8 (d) T=1

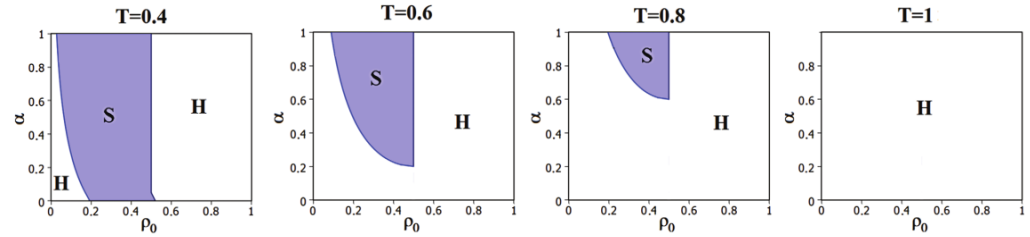
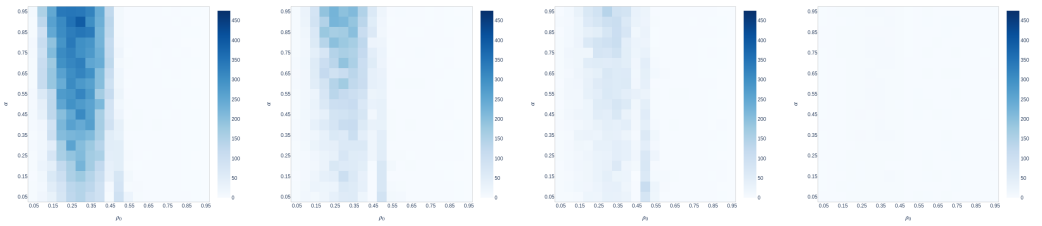


Figure 16: Comparison between theoretical and computational results for the phase diagrams at $T \neq 0$ with the same code used for the case $T = 0$ and B as order parameter

That is due to the fact that by just considering a neighbourhood as we did in the case for $T = 0$, so by associating it to just the number of residents, we actually lose the difference between a state with a number of residents equal to H (just one possibility) and a state in which the number of residents is equal to $\frac{H}{2}$ (many possibilities, maximum entropy). We are in this case *de facto* neglecting entropy, which was not really a problem before as T was equal to 0, but it is an issue now. In order to avoid this we will now run a code in which to each district it is associated a vector (and not anymore just a number) of possible places, so a string of values 1 and 0 representing the occupied places and the empty ones in the neighbourhood; the placement in a place with respect to another in the same district is chosen randomly.

If we run the code exploiting this we get the following results:



(a) T=0.4 (b) T=0.6 (c) T=0.8 (d) T=1

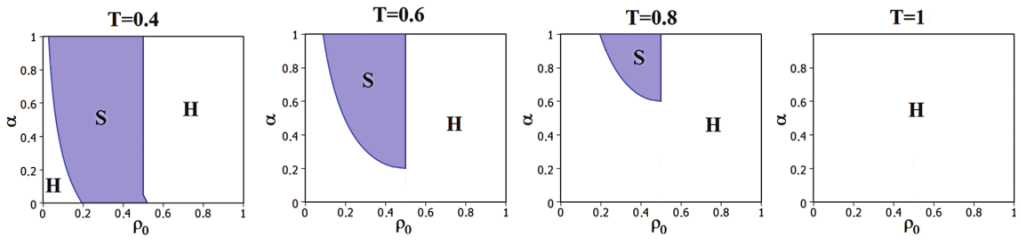


Figure 18: Comparison between theoretical and computational results for the phase diagrams at $T \neq 0$ with the new code and B as order parameter

In this case, we can actually see that there is a better match between theoretical and numerical results. Moreover, here we have not considered "saturated" diagrams because ρ_1 is not anymore equal to 0 as $T \neq 0$.

We perform again the whole analysis at $T = 0$ using this code in order to check that the non-consideration of entropy does not imply a qualitative change in the result obtained with the simulated annealing algortihm.

As the result we obtain with the new code for $T = 0$ is the same with respect to the one obtained with in [**fig:seconda2**], we do not report it here and we will continue using the old code while at $T = 0$ as the one considering entropy is much more computationally expensive.

4 Generalization of the model to non-identical agents

In this section we will focus mostly on the second goal of this work: the generalization of the model to non identical agents. Because we are studying the phase transition between the segregation/concentration state and the homogeneous one by tuning the α parameter, in this section by non-identical agents we do not mean to include a difference between agents in the "segregation sense", so to create different types of agents, but just to include some possible difference in the willing of agents of acting in a collective way to admit that different agents may have a different value of α. This is very important in order to build a model which closer represents a real social system. If we think at a society of individuals making everyday choices according to some global known condition, we can surely understand that it is very unlikely that they all have the same will of acting in favour of the community. Taking for example the frame of climate change: people have different opinion about it, a vegetarian acting in order to reduce the pollution coming from the intensive farming in this model would have a higher value of α with respect to the others as their acting would cause a positive externality (see chapter 5) on the rest of the population.

What we want to do with this work is to qualitatively and computationally show that the same result we obtained before for the phase diagram at $T = 0$ may be recovered if the control parameter α represents the average value of the collective will of the agents and therefore it is not a fixed number for the whole population.

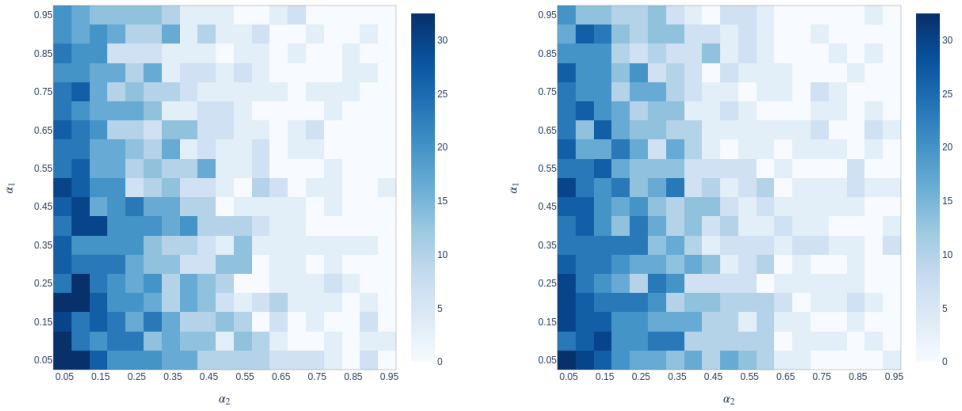
This would give also a first clue that a generalization of the model which considers the average α as control parameter may be possible.

4.1 Results and discussion for 2 types of agents

Before considering many different α values distributed according to some probability rule, in order to have a first insight on the problem, we may want to consider a population made up by 2 types of individuals, each with a different fixed value of the parameter α. We will analyze the

phase diagrams we get in 2 different situations. In the first case we will have a 2-type population: one part of the agents will have $\alpha = \alpha_1$ and the other part will have $\alpha = \alpha_2$, in the second case we will have a 1-type population with a value of α which is the weighted average of the 2 values of the first case.

In the following diagrams we will fix the proportion between agents with $\alpha = \alpha_1$ and agents with $\alpha = \alpha_2$ to $\frac{1}{4}$ and we will represent the results for the segregation/concentration order parameter obtained for every value of α_1 and α_2. The results of the comparison between the 2 diagrams for different values of the proportion between the 2 populations are very similar. In order to better see the dependence form the α parameter, we take a value of the initial/global density ρ_0 which is fixed to $\rho_0 = 0.6$ in this simulation; this value of the density ensures (based on the results obtained in the case of a 1-type population at $T = 0$ in the previous paragraphs) that there is a visible dependence on α. The simulation makes use of the Simulated Annealing method in this case too.



(a) Segregation values for a population made up by 20% of the agents with $\alpha = \alpha_1$ and 80% of the agents with $\alpha = \alpha_2$

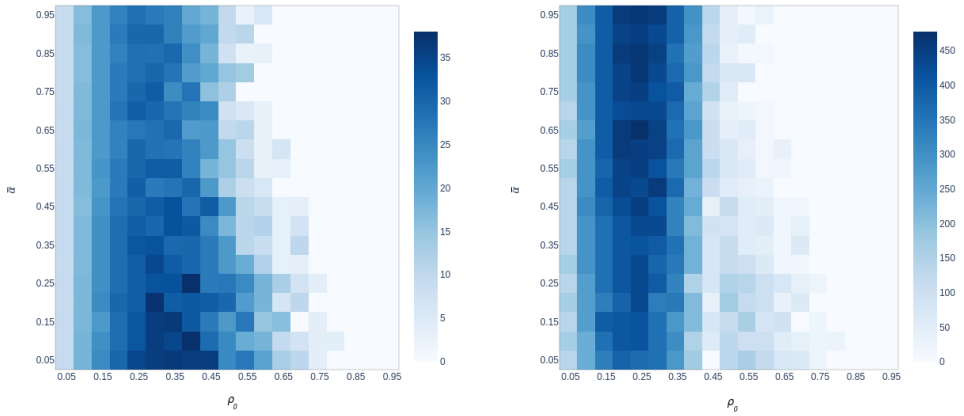
(b) Segregation values for a population made up by just 1 type of agents with $\alpha = \alpha_1 \cdot 0.20 + \alpha_2 \cdot 0.80 \ \forall \alpha_1, \alpha_2$

This result shows that between the case of a 2-types population and the 1-type one there is not (at least qualitatively) a big difference in the final equilibrium state reached for different values of these parameters. This is an important step in our generalization procedure, as it shows that we could go further and make each agent have a different value of α, distributed according to some probability rule and that the important parameter to study the transition may be the average value of α and not the fixed α value.

4.2 Results and discussion for highly heterogeneous agents

If we don't require anymore the agents to have the same α, but we just impose its distribution to be a truncated normal distribution with fixed average $< \alpha >= \bar{\alpha}$, then we obtain a phase diagram which is very similar to the one we obtain for a 1-type population with fixed α.

We can see this result here for the case in which the variance σ of the normal distribution is $\sigma = 0.25$:



(a) Phase diagram with D as order parameter obtained for different agents with $< \alpha >= \bar{\alpha}$

(b) Phase diagram with B as order parameter obtained for different agents with $< \alpha >= \bar{\alpha}$

Figure 20: Resulting phase digram for an ensamble of different agents with $< \alpha >= \bar{\alpha}$

The value of σ is chosen so that the distribution is not too much spread. Indeed as the truncated gaussian must have a domain which is $[0, 1]$, a high spread of the distribution would cause the impossibility of having values of the average of α very close to the extremes of the domain. Even with the chosen σ some values of $\bar{\alpha}$ have been excluded, but the excluded domain is small enough to still be able to get the necessary result. The 2 phase diagrams are once again very similar to the ones obtained in Fig. 13. This suggests that we could generalize the theoretical model in order to obtain a more interesting result, moreover with respect to an application to social science, in which the constraint of identical agents would be very limiting. In order to actually conclude that the control parameter might be $\bar{\alpha} =< \alpha >$ instead of α, some modification to the theoretical model should be done. What we found is a strong clue that a more general result may be obtained. This could be a possible future work to improve the validity range of the model.

It is clear that a different value of α for each agent would complicate a lot the theoretical analysis. But a clue to start working on this model with heterogeneity may be to relate it with other models in which parameters are distributed according to some probability law, such as the Random Field Ising Model (RFIM). Of course this would be just a clue which may exploit the theory of disordered systems. The evident difference would be that in this case the probability distribution is related directly with the control parameter and not to a different parameter present in the model such as in the RFIM.

5 A possible application: the economic concept of Pigouvian tax

This section will be dedicated to the description of the economic concept of Pigouvian tax. Economics indeed is another possible field in which the main characteristics of the model we used for segregation/crowding may be used. The objective of this work was indeed to describe the transition in the final equilibrium state by tuning a parameter which describes how much the agents act in a collective way; this aspect may be easily generalized in many fields. In particular, the control parameter we are considering (α) from an economic point of view may

be considered as a way in which what is called "internalization of externalities" is quantified.

Let's better describe this concept before even defining an economics-related problem.

An **externality** is an indirect cost or benefit for an agent which is caused by the action of others in an indirect way. The typical example of externality is pollution. In a society everyone is damaged by the production of pollution of some factory, but neither the producer of the good neither the consumer pay in order to compensate the collective damage it is causing: this is what is called a negative externality. An example of positive externality can be found instead in [19]: people living in an apartment which is placed above a bakery get free heat during winter without compensating back the baker. In this example it is not the collectivity to benefit from the action of a private, but another private.

There are many other examples in society in which externalities appear. Positive externalities are caused for example by a beekeeper, as it contributes to the pollination of the surrounding environment, by the restoration of historical buildings, as it may cause an increase in the commercial activity of the neighbourhood. Negative externalities may also be passive smoking or traffic congestion. In order to take these aspects into consideration governments decide to act by introducing a re-equilibrating taxation or subsidy, which has as objective the internalization of such externalities. By taxating a factory which produces indirect costs for the collectivity one should be able to limit the production of such externalities and at the same time it should gather enough money to compensate the collectivity of the damage caused. These type of taxes are called Pigouvian taxes as they were firstly introduced by the british economist Arthur Cecile Pigou in his "The economics of welfare" [20] in 1920.

Pigouvian taxes (in order to be fare and to compensate the externalities caused by some action) are set equal to the external marginal cost of the negative externalities, so to the change in the cost to parties other than the producer or buyer of a good due to the production of an additional unit of the good or service.

Let's consider the example of pollution: the factory considers the pollution as an input of the production and as usual this input has a price. This price is equal to 0 if no tax has been imposed. By imposing a taxation, so that the price of the pollution increases, the factory will produce less of it. In order to impose the right taxation and find the optimal quantity of pollution we have to internalize all the externalities according to the price of one unit of pollution.

The goal is to minimize the total societal cost, which is the sum of the cost of reducing pollution and the total damage caused by the pollution. If x is the quantity of pollution, $D_i(x)$ is the damage caused by the pollution on agent i and $C(x)$ is the cost of the reduction of pollution. The quantity to minimize (total societal cost) is:

$$C(x) + \sum_{i=1}^{n} D_i(x) \tag{36}$$

This quantity is minimized when the marginal cost of the reduction of one pollution unit is equal to the marginal damage of one unit of pollution more (derivative equal to 0)

$$\sum_{i=1}^{n} D_i'(x*) = -C'(x^*) \tag{37}$$

The marginal cost is indeed equal to the derivative of the function C and the same is true for the function D. The optimal pollution level (which takes into account the negative externalities is therefore x^*. The Pigouvian tax p^* is set equal to the marginal cost of reducing pollution at the optimal level $x^* : p^* = -C'(x^*)$

This tax effectively internalizes the externality by making the polluter bear the cost of the marginal damage their pollution causes. Thus, the polluter has an economic incentive to reduce pollution to the optimal level x^*.

5.1 The relation with the model

The concept of Pigouvian tax is closely related to our model because in some sense the parameter α acts as a taxation parameter, which if set to 1 means the actual internalization of all the externalities and prevents the agent of acting individualistically. α can be seen not just as the natural willing of agents of acting in a way such that the collectivity ameliorates, but as the sum of 2 different aspects: one that starts from each agent and it is its natural propension to work for the collectivity and one imposed tax which aims to set a minimum value for the collective will an agent can have.

As in our model we have come to the result that even with many different agents what counts is the average value of α, the introduction of a tax on the most individualistic agents would be useful just in the shift in the average value it causes; it would not make a difference to apply such a tax at random or to the less altruistic agents.

By introducing the concept of Pigouvian tax we have seen how there is the possibility of tuning such a lower bound in many different possible systems. Moreover, if as we said in the beginning of the work, an α value of 1 is associated to physical systems, we might also conclude that a Pigouvian taxation aims to bring an agent free-market model more close to how a physical system behaves.

If one wants to build up a more economics-oriented model which has the Pigouvian taxation t as parameter, they may find that by tuning it there are phase transitions occuring. One may therefore find what is the optimal value of the taxation in order to come to the wanted final state and study its stability.

In the theoretical model we used for example one result could be the consideration that it is convenient to increase the value of α just in some cases and in any case just to a threshold level. After this value the increase of the parameter does not change the configuration anymore.

By the way in our model an increase of the value of α just leads to a better result for the collectivity, this might not be the case in a more complex economic model in which the individual-collective behaviour parameter actually describes the Pigouvian tax. The building up of such a model may be an idea for a future work describing an agent-based society which includes recompensation for indirect damages.

Conclusions and future work

Even if the project has started by considering the segregation problem and the Schelling model, its focus was not just oriented towards the description of that particular situation. The last paragraph of the report (chapter 5) is crucial to understand the generality and the potentiality of the description of the behaviour of a social system while regulating the will of its agents to act collectively. Indeed it not only shows an application to economics, but also a way in which the parameter we have been tuning in the whole work could be effectively modified and controlled in a real society.

The main goal of this work was to simulate a system whose components behave in the same way as in [3] in order to verify the theoretical results referred mostly to the phase diagrams of the model. We worked mainly on 2 aspects that in the theoretical analysis had not been taken into account: the problem of measuring segregation by defining a proper order parameter and the problem of reaching the global minimum/maximum computationally by avoiding the local stationary points. The second aspect we worked on was the generalization of the computational model to the case in which the agents are not all equal, but have different aims of acting collectively. The result for 2 and for many different values of α have been shown: qualitatively the phase diagram does not change. That means that if we have a population made up by just 1 type of agents (with the same α) or a population of many different agents with α distributed has a Gaussian with average $\bar{\alpha}$ there seem not to be a difference in the phase diagram. This result is very important as it gives some hope that the theoretical model may be generalized

in such a way that it becomes closer to reality. Indeed it is unlikely that a real social system could be well described as an ensemble of identical individuals. An example may be how much a person changed their behaviour according to some external issue that involve everyone such as the climate crisis. Of course not every person has the same sensibility on this topic so there may be people with a very high level of collective aim and people with a lower one. In order to describe such a situation of course the model we used should be modified as ours was still related to the crowding/segregation issue, but in any social model in which a collective aim may be taken into consideration by the agents what we did may be helpful. An idea of a future work may be therefore to explain from the theoretical point of view the results obtained by our simulation towards a more general model with respect to the one proposed in [3].

Another idea would be to build up a model on a particular economics-related topic which actually aims to find what is the best Pigouvian tax (the tuning parameter α) in a society of producers and consumers in order to have an economic growth which limits the production of pollution (which would represent the externality).

For any complex system, it is of fundamental importance to describe any situation in which the state of the global system is taken into consideration by the single agent. As we read in [2], it is possible, using the frame of potential game theory (see appendix), to do the opposite of what we have seen in the theoretical analysis, finding what rule the single follows given that the whole shows some emergent phenomenon. This could give some insight on the perception of the individual and could be an important thing to look in order to understand the choice of purposeful agents. Interesting are the cases presented in [2], in which, starting from a probability distribution, one could recover the agents' utility function of the corresponding game. This is what is typically called the bottom-up approach, which aims to derive the behaviour of the single from the macroscopic phenomena we observe in the global system; it is in open contrast with the reductionist approach which is usually considered in science. Both the approaches may be of importance in different situations, with a non trivial extension the transition we described could be interpreted as a transition between a parts-to-whole model to a whole-to-parts one. In the case $\alpha = 0$ the agents act as selfish individual, the whole's behaviour comes after their choices, but it doesn't determine them. Instead in the case $\alpha = 1$ all the choices of the agents are completely determined by the whole system, it is the system as a whole that has a physical relevance in this case, not the agents anymore. That's why the transition we described is not constrained to the particular topic where it firstly arose, but it is very general and may be also useful in a process that may describe the genesis of a complex system, starting from the "aggregation" of individual parts.

Appendix

Congestion and potential games

In the theoretical analysis of the model we considered, many concepts related to game theory have been used. In particular in our work we have passed from an agent-based view (by the definition of a utility/payoff function) to a system-based one (with the introduction of a potential to maximize). This is a general method used for a particular type of games, which are called potential games, and it is very useful when the number of players is huge. In potential games even if the will of the agents leads them to improving their payoff, the global result from the point of view of the system is the maximization of a potential function. The maximum of this potential corresponds to the Nash equilibrium of the game. However, it is not possible to find such a potential for any game. Let's give a definition of potential game and report some theorem in order to understand why in the case we considered we could say that such a potential existed.

A potential game is a particular type of game in game theory in which it is possible to find a potential function that captures the behaviour of the agents being directly linked to their payoff

function. In a game $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ with:

1. N: set of players
2. S_i: set of possible strategies for player i
3. u_i: utility function (payoff) for player i

is an ordinal potential game if there exist $\Phi : S \to \mathbb{R}$, where $S = S_1 \times S_2 \times \cdots \times S_N$ is the set of combined strategies of the players such that for any player i and for any couple of strategies $s_i, s_i' \in S_i$ with the strategies of the other players fixed $s_{-i} \in S_{-i}$ we have that:

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \longrightarrow \Phi(s_i', s_{-i}) > \Phi(s_i, s_{-i}) \tag{38}$$

it is an exact potential game if under the same assumptions:

$$u_i(s_i', s_{-i}) - u_i(s_i, s_{-i}) = \Phi(s_i', s_{-i}) - \Phi(s_i, s_{-i}) \tag{39}$$

it is a weighted potential game if:

$$u_i(s_i', s_{-i}) - u_i(s_i, s_{-i}) = \omega_i(\Phi(s_i', s_{-i}) - \Phi(s_i, s_{-i})) \tag{40}$$

This implies that a variation of the utility function of the agent is linked to a variation in the global potential function, with the different relations we have shown above according to the different type of potential game. Even with this definition it is not easy to verify that in our case we have a potential game, but luckily some theorems help us with this. In particular theorem 3.1 in [21] states: "Every congestion game is a potential game"; the proof is shown in the same paper. If we prove that our agent-based model is a congestion game, we have then reached the goal of proving that to find a potential is always possible for our model. A congestion game is a game where each player's payoff depends on how congested some shared resources are. In a congestion game other then the set of N players and the set of strategies S_i defined as before, one has to define a set of resources $R = \{r_1, r_2, ..., r_m\}$ and the utility function u becomes the sum of all the congestion functions c_r for each resource r. The congestion function depends on how many agents are already "consuming" that resource.

In our case the resources are the places in neighbourhood q, and we require our agents to just choose one type r of resource (direct correspondence between strategy and resources), then the utility function of the agents coincides with the congestion function of that resource. Moreover, in our case all the congestion functions (for different resources/neighbourhoods) are equal, then the utility is just the unique congestion function defined as in Fig. 2. As the utility in our model depends just on the occupation number of the district, that is on how much the resource "places in that district" is congested, our game is a congestion game.

Let's now see how it is possible to find the potential of our game.

As we read in [2], when the utility function of an agent on site q just depends on the density ρ_q of that site (so when $\alpha = 0$), we have that for an exact potential it is possible to find the potential in the folowing way (case $T = 0$):

$$\Phi = \sum_q \int_0^{\rho} u(\rho_q') d\rho_q \tag{41}$$

where we suppose that $u(\rho_q)$ is already derived with respect to the strategy q. So when the utility function is $u(\rho_q)$ ($\alpha = 0$) the potential is $\sum_q \int_0^{\rho} u(\rho_q') d\rho_q \propto L(x)$, with $L(x)$ defined as in the first paragraph of the appendix.

If then we introduce the parameter $\alpha \neq 0$, it is not anymore possible to find the potential Φ in such a way. Let's see how the added term to u changes the global potential.

If u changes in the following way:

$$\tilde{u}_i(x) = u_i(x) + a\sum_{j=1}^{N} u_j(x) \tag{42}$$

the potential changes accordingly:

$$\tilde{P}(x) = P(x) + a\sum_{j=1}^{N} u_j(x) \tag{43}$$

The modified game with the new utility functions $\tilde{u}_i(x)$ is still an exact potential game as:

$$\tilde{u}_i(x'_i, x_{-i}) - \tilde{u}_i(x_i, x_{-i}) = \left[u_i(x'_i, x_{-i}) + a\sum_{j=1}^{N} u_j(x'_i, x_{-i})\right] - \left[u_i(x_i, x_{-i}) + a\sum_{j=1}^{N} u_j(x_i, x_{-i})\right] \tag{44}$$

Becomes:

$$\tilde{u}_i(x'_i, x_{-i}) - \tilde{u}_i(x_i, x_{-i}) = [u_i(x'_i, x_{-i}) - u_i(x_i, x_{-i})] + a\left[\sum_{j=1}^{N} u_j(x'_i, x_{-i}) - \sum_{j=1}^{N} u_j(x_i, x_{-i})\right]$$

where $u_i(x'_i, x_{-i}) - u_i(x_i, x_{-i})$ is equal to $P(x'_i, x_{-i}) - P(x_i, x_{-i})$. So the new relation for the potential is:

$$\tilde{u}_i(x'_i, x_{-i}) - \tilde{u}_i(x_i, x_{-i}) = [P(x'_i, x_{-i}) - P(x_i, x_{-i})] + a\left[\sum_{j=1}^{N} u_j(x'_i, x_{-i}) - \sum_{j=1}^{N} u_j(x_i, x_{-i})\right]$$

which is what we found in the first chapter.

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