POLITECNICO DI TORINO

Master's Degree in Communications Engineering

Master's Degree Thesis

Fundamental Limits of Integrated Sensing and Communication without Perfect Channel State Information

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Summary

This thesis explores the theoretical underpinnings and practical implications of Integrated Sensing and Communication (ISAC) within future wireless networks. As a transformative technology, ISAC merges sensing and communication functionalities to optimize the utilization of spectrum and hardware resources, a necessity driven by the increasing demands of modern applications such as autonomous vehicles and immersive technologies.

The research addresses the critical need to understand the fundamental tradeoffs between sensing and communication within ISAC systems. It investigates the Bayesian Cramér-Rao Bound (BCRB) as a metric for evaluating the performance limits of ISAC systems under various conditions. This bound is particularly relevant given the non-realistic nature of assuming known state information in dynamic channel environments, necessitating continuous estimation.

This thesis evaluates the performance of different transmission strategies by developing a comprehensive theoretical model. It balances the conflicting requirements of sensing and communication, thus providing insights into the optimal allocation of power and resources. The simulation results validate the theoretical predictions, demonstrating how ISAC can enhance spectral, energy, and economic efficiencies in next-generation wireless networks.

The findings contribute significantly to the field by laying the groundwork for future research to integrate further and optimize sensing and communication. The methodologies and insights presented in this work are expected to drive the development of advanced ISAC solutions, pushing the boundaries of what is achievable in wireless communication systems.

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Simone

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Acronyms

ISAC
: Integrated Sensing And Communication

MIMO
: Multiple Input Multiple Output

CRB
: Cramér-Rao Bound

BCRB
: Bayesian Cramér-Rao Bound

bpcu
: Bits per Channel Use

pdf
: Probability Density Function

LoS
: Line of Sight

MI
: Mutual Information

BS
: Base Station

EU
: End User

UE

User Equipment

ULA

Uniform Linear Antenna

i.i.d.

Independent and Identically Distributed

BFIM

Bayesian Fisher Information Matrix

Chapter 1

Introduction

Integrated Sensing and Communication (ISAC) is emerging as a critical technology for future wireless networks, blending sensing and communication to efficiently utilize spectrum and hardware resources. This convergence is driven by the demands of applications such as autonomous vehicles and immersive technologies, alongside advancements in millimeter wave and massive MIMO technologies. ISAC promises to mitigate spectrum congestion and enhance spectral, energy, and economic efficiencies. While initial research has focused on practical ISAC system designs, understanding the theoretical limits of ISAC is crucial for bridging the gap between current capabilities and potential performance peaks. This involves exploring the trade-offs between sensing and communication functions to guide the development of more effective ISAC solutions. As a cornerstone of next-generation networks, ISAC represents a synergistic approach that significantly boosts both sensing and communication capabilities, ensuring the adaptability and efficiency of future wireless systems.

Extensive research has been done on the fundamental tradeoff of ISAC [1] under the assumption of known state information from a communication point of view. This is, however, non-realistic as channels often change and need to be continuously estimated.

We start by reviewing the literature on ISAC and related topics used in this dissertation. The model on which we focus is then shown and presented. We proceed with a detailed explanation of what has been done to achieve the results, which are commented on in the end.

1.1 Notation Convention

We denote with a, $\mathbf{a}$, and $\mathbf{A}$ scalar random variables, random vectors, and random matrices, respectively. With A, we denote known quantities, e.g., P is the power.

With $\mathbb{E}_{\mathbf{x}}\{\cdot\}$, we denote the expected value of the argument with respect to $\mathbf{x}$. With $\mathrm{tr}\{\cdot\}$, we denote the trace of a (square) matrix. We use $|\cdot|$ for the absolute value of a scalar and $\|\cdot\|_p$ for the l_p norm of a vector which is the Euclidean norm when the subscript is omitted. With $\mathrm{diag}(\mathbf{b})$, we denote a diagonal matrix with $\mathbf{b}$ being its diagonal. The notations $[\cdot]^*$ and $[\cdot]^{\mathrm{H}}$ are used to indicate the complex conjugate and the Hermitian transpose of the argument, respectively. We denote with $\Re\{\cdot\}$ and $\Im\{\cdot\}$ the operators returning their argument's real and imaginary parts, respectively. The notation $\mathbb{C}^{M\times N}$ denotes the set of complex-valued matrices with M rows and N columns. The notation $\hat{\mathbf{a}}$, denotes the unit norm vector obtained by dividing the vector $\mathbf{a}$ by its norm.

1.2 Concepts and Background

This section will present the main concepts needed to understand this dissertation, with a short theoretical explanation.

1.2.1 Bayesian Cramér-Rao Bound

The Cramér-Rao Bound (CRB) is a lower bound on the variance ($\mathrm{Var}\{\cdot\}$) of any unbiased parameter estimator [2]. In simpler terms, it tells us the best accuracy we can achieve when estimating a parameter from noisy observations. Mathematically, for an unbiased estimator $\hat{\theta}$ of a parameter θ, the CRB states that:

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1}{I(\theta)}, \tag{1.1}$$

where $I(\theta)$ is the Fisher Information and can be computed as:

$$\mathcal{I}(\theta) = -\mathbb{E}\left[\frac{\partial^2 \log L(\theta)}{\partial\theta^2}\right], \tag{1.2}$$

where $L(\theta)$ is likelihood function.

The Bayesian Cramér-Rao Bound (BCRB) [3] can be used when prior information about the parameter is available. This bound incorporates both the prior distribution of the parameter and the likelihood of the observations, giving a more general lower bound on the estimator's variance. It integrates over all possible values of the parameter:

$$\mathbb{E}[(\hat{\theta}-\theta)^2] \geq \left(\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\ln p(y|\theta)\right] + \mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\ln p(\theta)\right]\right)^{-1}, \tag{1.3}$$

where $p(y|\theta)$ is the likelihood function and $p(\theta)$ is the prior distribution.

1.2.2 Communication Rate and Mutual Information

The communication rate is the rate at which information can be transmitted over a communication channel, usually measured in bits per channel use (bpcu). It is constrained by the channel capacity, which can be found by maximizing the mutual information between the transmitted and received signal.

Mutual information measures the amount of information one random variable contains about another [4]. In the context of communication, it quantifies the amount of information transmitted from the sender to the receiver. For a channel with input X and output Y, the mutual information $I(X;Y)$ is given by:

$$I(X;Y) = h(X) - h(X|Y) \tag{1.4}$$

where $h(X)$ is the differential entropy of X and $h(X|Y)$ is the conditional differential entropy of X given Y. The differential entropy can be defined as:

$$h(X) = -\int_{-\infty}^{\infty} f_X(x) \log f_X(x) dx, \tag{1.5}$$

where $f_X(x)$ is the pdf of X.

1.2.3 Von Mises Distribution

The Von Mises Distribution [5] is a probability distribution for circular data analogous to the normal distribution for linear data. It is used when dealing with angles or periodic phenomena. The probability density function (pdf) of the Von Mises distribution is:

$$f(\theta; \mu, \kappa) = \frac{e^{\kappa \cos(\theta-\mu)}}{2\pi I_0(\kappa)}, \tag{1.6}$$

where μ is the mean direction, κ is the concentration parameter (analogous to the inverse of the variance), and $I_0(\kappa)$ is the modified Bessel function of the first kind. The support of the pdf is an interval of length 2π centered at μ. Note that $1/\kappa$ plays a role similar to σ^2 (the variance) in the Gaussian distribution.

The von Mises distribution is particularly useful for modeling data that are directional or cyclical, such as phases, angles, or the time of day.

1.2.4 Steering Vectors and MIMO Antennas

Steering Vectors are used in array signal processing to describe the phase shifts required to steer the beam of an antenna array toward a specific direction. In a far-field Line-of-Sight (LoS) MIMO model [6], the steering vector $\mathbf{a}(\theta)$ for a direction θ is usually defined by:

$$\mathbf{a}(\theta) = \frac{1}{\sqrt{M}} \left[1, e^{jkd\cos(\theta)}, e^{j2kd\cos(\theta)}, \ldots, e^{j(M-1)kd\cos(\theta)}\right]^T, \tag{1.7}$$

where k is the wavenumber, d is the distance between antennas, and M is the number of antennas. This normalization by $\sqrt{M}^{-1}$ ensures that the steering vector has unit length.

Note that the steering vectors of an antenna with M elements, where $M = 2\widetilde{M}+1$, can be written with indexes $n \in \left\{-\widetilde{M}, -\widetilde{M}+1, \dots, \widetilde{M}-1, \widetilde{M}\right\}$. Namely:

$$\mathbf{a}(\theta) = \frac{1}{\sqrt{M}} \left[e^{j(-\widetilde{M})kd\cos(\theta)}, e^{j(-\widetilde{M}+1)kd\cos(\theta)}, \dots, 1, \dots, e^{j(\widetilde{M})kd\cos(\theta)}\right]^T. \tag{1.8}$$

This formulation above ensures the orthogonality between the steering vector and its derivative, $\dot{\mathbf{a}}^{\mathrm{H}}(\theta)\mathbf{a}(\theta) = 0$.

Multiple-input multiple-output (MIMO) systems use multiple antennas at both the transmitter and receiver to improve communication performance. MIMO technology exploits spatial diversity and spatial multiplexing to increase data rates and reliability.

1.3 Goals of Dissertation

This dissertation aims to study the fundamental limits of ISAC systems under a specific set of assumptions. In particular, we focus on deriving the BCRB for the joint estimation of two targets. We then proceed with the study of the optimal sensing and communication points. Finally, we study various transmission strategies to achieve an inner bound for the BCRB-Rate region and identify the ISAC tradeoff in our scenario.

We attempt to obtain similar results as in [1] in a more realistic scenario. Hence, we communicate with targets that must be sensed and assume only partially known channel state information at the receiver. A Base Station (BS) estimates the angle of arrivals of targets in the vicinity. Already acquired targets are communication receivers, while other targets may appear on the scene and thus must be estimated and characterized by different prior distributions. Our goal is to derive the fundamental Rate-BCRB tradeoff region.

Under these different assumptions, we study the new tradeoffs in the system, proposing a simple outer bound and several communication strategies to achieve some inner bounds.

Chapter 2

Previous Work

The state-of-the-art ISAC reflects a variety of approaches, ranging from theoretical foundations to practical implementations and standards. The conceptualization of ISAC can be traced back to the necessity for coexistence and coordination between radar and communication systems within the same spectral resources. Early works focused on understanding and defining the potential synergies between these functionalities. Paper [7] provides an extensive review of the fundamental limits of ISAC, exploring how traditional radio sensing and emerging ISAC approaches can be unified under a comprehensive framework. Similarly, the work in [8] delineates a novel relationship between mutual information and the minimum mean-square error in Gaussian channels. This fundamental concept underpins the theoretical limits of ISAC performance.

As we transition from 5G to 6G, ISAC is identified as a key technology that exploits dense cellular infrastructures to create highly perceptive networks. Paper [9] highlights how ISAC could integrate within 6G networks, enhancing both the performance and functionality of RANs by incorporating sensing capabilities. This vision aligns with the trends in IoT, where ISAC could redefine the architectural layers. Paper [10] discusses the transition towards a unified signaling layer in IoT, driven by ISAC technologies, indicating a shift towards more integrated and efficient network frameworks. Practical implementations of ISAC have been explored in various domains, including vehicular networks and cooperative systems. Paper [11] introduces a predictive beamforming approach in vehicular networks, showcasing how ISAC can effectively reduce overhead while improving the accuracy of vehicle localization and communication. This application demonstrates the practical benefits and efficiency gains from deploying ISAC in dynamic environments.

ISAC can be broadly classified into two categories: device-free ISAC and device-based ISAC [7]. In device-free ISAC, the sensing targets cannot transmit or receive signals, or the sensing procedure does not rely on the target's transmit/receive capabilities. A typical example is radar sensing, where a radar transmits a probing signal, and the echo reflected from the target is used for sensing. In device-based ISAC, the sensing functionality is achieved by device-based sensing, where the sensing targets can transmit and receive signals. A common example is localization and communication in cellular networks.

Despite the promising advances, ISAC still faces significant challenges, particularly in balancing the dual-functional performance and optimizing resource allocation. The research highlighted in [10] and [7] identifies these challenges and proposes future research directions, including the development of more sophisticated integration techniques and the exploration of optimal trade-offs between sensing and communication capabilities. We focus on the latter problem studied in [1] under the model of having some targets to sense and others to communicate with under known state information at both the transmitter and receiver. The theoretical studies allow us to understand the limits one can try to achieve in practical scenarios. For this reason, we try to fill the gap in the literature caused by the lack of more realistic settings.

The work in [1], which we will reference the most, stands as one of the first to address the fundamental trade-off in ISAC from both information-theoretic and estimation-theoretic perspectives. It demonstrates that the optimal sensing performance is achieved when the sample covariance matrix $\mathbf{R_X} = \frac{1}{T}\mathbf{X}\mathbf{X}^H$ has a deterministic trace, and the distribution $p(\mathbf{R_X})$ (and hence $p(\mathbf{X})$) is restricted to the optimal solution set of a deterministic CRB minimization problem. If the solution is unique, the sensing-optimal sample covariance matrix $\mathbf{R_X}$ itself should be deterministic. The authors define the CRB-rate region as the set of all achievable communication rate and sensing CRB pairs. They propose a pentagon inner bound of the CRB-rate region that can be achieved through a simple time-sharing strategy. Within this framework, they study ISAC performance at the two corner points of the CRB-rate region: P_{CS} (minimum achievable CRB constrained by maximum communication rate) and P_{SC} (maximum achievable communication rate constrained by minimum CRB). The paper derives the high-SNR communication capacity for the sensing-optimal point P_{SC}. It proves that it can be asymptotically achieved by a strategy based on uniform sampling over the set of semi-unitary matrices (the Stiefel manifold). Additionally, they provide lower and upper bounds for the sensing CRB at the communication-optimal point P_{CS}. The paper reveals a two-fold trade-off in ISAC systems:

- Subspace Trade-Off (ST): Balances resource allocation between the subspaces spanned by sensing and communication channels.
- Deterministic-Random Trade-Off (DRT): Depicts the exploitable degrees of freedom (DoFs) in ISAC signals.

The authors propose an outer bound and various inner bounds for the CRB-rate region based on these trade-offs. We will expand upon their work by changing the assumptions made on the model, considering an incomplete knowledge of the channel state information. We also consider a case with two targets to be jointly sensed, where one of the two is also a communication target. By studying this new setting, we provide a step forward to completely understanding ISAC tradeoff and its limits in real-life scenarios.

Chapter 3

System Model

For the sake of simplicity, we consider a two-dimensional (i.e., no z-coordinate) scenario with one communication target and one additional separate sensing target, as illustrated in Fig. 3.1.

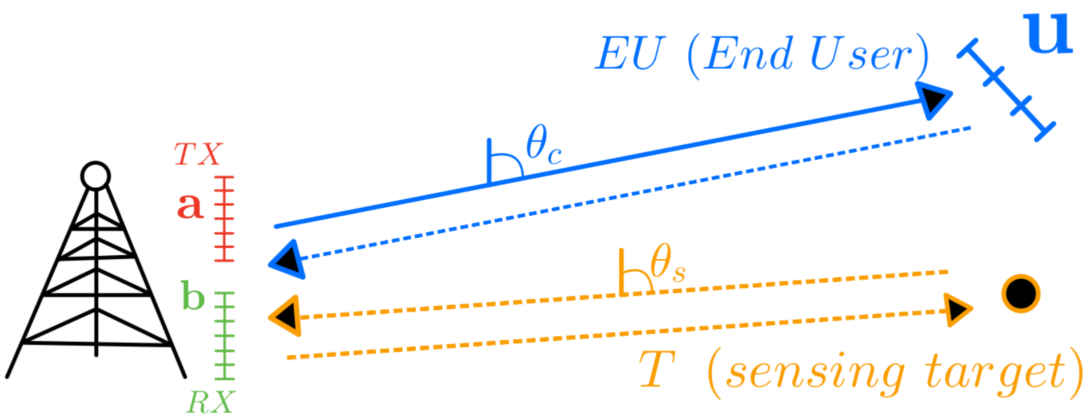

Figure 3.1: Representation of setting considered in this dissertation, with one communication target (end-user) and one sensing target.

The Base Station (BS) is simultaneously a transmitter for the communication receivers and a mono-static radar for the sensing targets. The BS comprises a transmitting uniform linear antenna array (ULA) with M_{TX} elements, characterized by the steering vector $\mathbf{a} \in \mathbb{C}^{M_{\mathrm{TX}} \times 1}$, and a receiving ULA with M_{RX} elements, characterized by the steering vector $\mathbf{b} \in \mathbb{C}^{M_{\mathrm{RX}} \times 1}$. The BS operates in full-duplex mode. We make the idealized assumption that there is no self-interference on the TX side from the RX side. Future work will investigate even more realistic scenarios.

The communication receiver, or UE for 'user equipment,' has a ULA with M_{UE} elements, characterized by the steering vector $\mathbf{u} \in \mathbb{C}^{M_{\mathrm{UE}}\times 1}$. The UE reflects the BS-transmit signal back to the BS-receiver, which enables it to estimate their angle of arrival.

3.1 Channel Model

The BS transmits signal is denoted by $\mathbf{X} \in \mathbb{C}^{M_{\mathrm{TX}}\times T}$, where T is the channel coherence time. We assume the sensing parameters vary synchronously with the communication channel parameters every T channel uses. At the same time, the BS receives the signal $\mathbf{Y}_s \in \mathbb{C}^{M_{\mathrm{RX}}\times T}$ from the reflections of targets and objects present in the surroundings. The UE receives the signal $\mathbf{Y}_c \in \mathbb{C}^{M_{\mathrm{UE}}\times T}$. We write the received signals as

$$\mathbf{Y}_c = \mathbf{H}_c\mathbf{X} + \mathbf{Z}_c, \quad \mathbf{H}_c = \alpha\ \mathbf{u}(\theta_1)\mathbf{a}^H(\theta_1), \tag{3.1}$$

$$\mathbf{Y}_s = \mathbf{H}_s\mathbf{X} + \mathbf{Z}_s, \quad \mathbf{H}_s = \sum_{\ell=1}^{N_s} \beta_\ell\ \mathbf{b}(\theta_\ell)\mathbf{a}^H(\theta_\ell), \tag{3.2}$$

where:

- $\mathbf{X} \in \mathbb{C}^{M_{\mathrm{TX}}\times T}$ is the transmit signal subject to

$$\mathbb{E}\left\{\mathrm{tr}\left\{\mathbf{R}_{\mathbf{X}}\right\}\right\} \leq P_{\mathrm{TX}}M_{\mathrm{TX}}, \tag{3.3}$$

  where $\mathbf{R}_{\mathbf{X}} := T^{-1}\mathbf{X}\mathbf{X}^{\mathrm{H}}$ is the 'sample covariance matrix.' Note that P_{TX} has the meaning of average power constraint per transmit antenna.

- $\mathbf{Z}_c$ has entries assumed i.i.d., circularly symmetric, complex Gaussian, with zero mean and variance σ_c^2.

- $\mathbf{Z}_s$ has entries assumed i.i.d., circularly symmetric, complex Gaussian, with zero mean and variance σ_s^2.

- $\mathbf{H}_c \in \mathbb{C}^{M_{\mathrm{UE}}\times M_{\mathrm{TX}}}$ is the down-link communication channel matrix. $\theta_1 \in [0,2\pi]$ denotes the angle of arrival of the transmit signal, and $\alpha \in \mathbb{C}$ is the channel attenuation, both assumed perfectly known at the UE.

- $\mathbf{H}_s \in \mathbb{C}^{M_{\mathrm{RX}}\times M_{\mathrm{TX}}}$ is the sensing channel matrix, where N_s is the number of sensing targets, assumed known at the BS. The angles of arrival $(\theta_1, \theta_2, \ldots, \theta_{N_s}) \in [0,2\pi]^{N_s}$ must be estimated. $(\beta_1, \beta_2, \ldots, \beta_{N_s}) \in \mathbb{C}^{N_s}$ is the vector of channel gains from the targets to the BS. We assume that the prior distribution on the target parameters factorizes as

$$P_{\theta_1,\theta_2,\ldots,\theta_{N_s},\beta_1,\beta_2,\ldots,\beta_{N_s}} = \prod_{\ell=1}^{N_s} P_{\theta_\ell} P_{\Re\{\beta_\ell\}} P_{\Im\{\beta_\ell\}}. \tag{3.4}$$

3.2 Sensing Task

The sensing task consists in estimating the angle of arrivals in $\mathbf{H}_s$ in (3.2). As a metric for this sensing task, we use the BCRB [1], a lower bound for the Mean Squared Error (MSE) of weakly unbiased estimators. The BCRB is defined as

$$\epsilon := \mathbb{E}_{\mathbf{X}} \left\{ \mathrm{tr} \left\{ \mathbf{J}_{\boldsymbol{\theta}|\mathbf{X}}^{-1} \right\} \right\}, \tag{3.5}$$

where $\mathbf{J}_{\boldsymbol{\theta}|\mathbf{X}}$ is the Bayesian Fisher Information Matrix (BFIM) of the parameters we wish to estimate. The BFIM of the parameters $\boldsymbol{\theta}$, is given by [12]:

$$\begin{aligned} \mathbf{J}_{\boldsymbol{\theta}|\mathbf{X}} := \mathbb{E} & \left\{ \frac{\partial \ln p_{\mathbf{Y}_{\mathrm{s}}|\mathbf{X},\boldsymbol{\theta}} \left(\mathbf{Y}_{\mathrm{s}} \mid \mathbf{X}, \boldsymbol{\theta} \right)}{\partial \boldsymbol{\theta}} \frac{\partial \ln p_{\mathbf{Y}_{\mathrm{s}}|\mathbf{X},\boldsymbol{\theta}} \left(\mathbf{Y}_{\mathbf{s}} \mid \mathbf{X}, \boldsymbol{\theta} \right)}{\partial \boldsymbol{\theta}^{\mathrm{T}}} \middle| \mathbf{X} \right\} \\ & + \ \mathbb{E} \left\{ \frac{\partial \ln p_{\boldsymbol{\theta}}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \frac{\partial \ln p_{\boldsymbol{\theta}}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^{\mathrm{T}}} \right\}. \end{aligned} \tag{3.6}$$

3.3 Communication Task

The communication task consists of reliably transmitting information to the UE with received signal $\mathbf{Y}_c$ in (3.1). As a metric for this communication task, we use the ergodic achievable rate: [1]

$$R := \frac{1}{T} I(\mathbf{X}; \mathbf{Y}_c | \mathbf{H}_c), \ \text{s.t.} \ \mathbb{E} \left\{ \mathrm{tr} \left\{ \mathbf{R}_{\mathbf{X}} \right\} \right\} \leq P_{\mathrm{TX}} M_{\mathrm{TX}}. \tag{3.7}$$

Note that the channel $\mathbf{H}_c$ is a random variable with known prior distribution; hence the expected value implied by the definition of mutual information is to be considered with respect to the prior probability density function of $\mathbf{H}_c$.

3.4 ISAC Region

Overall, we are interested in obtaining the lowest possible estimation error and the highest possible rate. The minimum sensing error is obtained by minimizing (3.5) with respect to the pdf of the transmit signal $\mathbf{X}$. On the other hand, the maximum achievable rate is obtained by maximizing (3.7) with respect to the pdf of the transmit signal $\mathbf{X}$. These two tasks are conflicting, so we define the ISAC region where the power constraint on $\mathbf{X}$ is introduced.

The ISAC region is defined as the set of achievable pairs (ϵ, R), where ϵ is the BCRB for the joint estimation of all the angles we need to estimate, and R is the ergodic communication rate. This region can be mathematically expressed as:

$$\mathcal{R}_{\text{ISAC}} = \bigcup_{p_{\mathbf{X}}(\mathbf{X}) \in \mathcal{X}(P_{\text{TX}} M_{\text{TX}})} \left\{ (\epsilon, R) \,\middle|\, \epsilon = \mathbb{E}_{\mathbf{X}} \left\{ \text{tr} \left\{ \mathbf{J}_{\boldsymbol{\theta}|\mathbf{X}}^{-1} \right\} \right\},\ R = \frac{1}{T} I(\mathbf{X}; \mathbf{Y}_c | \mathbf{H}_c) \right\} \tag{3.8}$$

where $\mathcal{X}(P_{\text{TX}} M_{\text{TX}})$ is the set of all possible distributions for the input signal that meet the average power constraint $P_{\text{TX}} M_{\text{TX}}$, i.e., $\mathbb{E}\{\text{tr}\{\mathbf{R}_{\mathbf{X}}\}\} \leq P_{\text{TX}} M_{\text{TX}}$.

Chapter 4

Fundamental Quantities for the Characterization of ISAC Scheme

In this chapter, we dive deeper into the definition of the errors for the joint estimation of two targets and the general formulation of the communication rate. A general form was derived in [13]; here, we apply the general results to obtain the case of interest of two targets.

4.1 Bayesian Cramér-Rao Bound for Two Targets

It is key for our study to evaluate the BCRB for the joint estimation of the unknowns in the sensing channel matrix: the angles and the complex amplitudes. We start by defining the following quantities based on the parameters in (3.2):

$$\mathbf{B} = \begin{bmatrix}\mathbf{b}(\theta_1) & \mathbf{b}(\theta_2)\end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix}\mathbf{a}(\theta_1) & \mathbf{a}(\theta_2)\end{bmatrix}, \tag{4.1}$$

$$\boldsymbol{\theta} = \begin{bmatrix}\theta_1 & \theta_2\end{bmatrix}^\top, \quad \boldsymbol{\beta} = \begin{bmatrix}\beta_1 & \beta_2\end{bmatrix}^\top, \quad \boldsymbol{\mathcal{B}} = \mathrm{diag}(\boldsymbol{\beta}). \tag{4.2}$$

We also define

$$\dot{\mathbf{B}} = \begin{bmatrix}\frac{\partial \mathbf{b}(\theta_1)}{\partial \theta_1} & \frac{\partial \mathbf{b}(\theta_2)}{\partial \theta_2}\end{bmatrix}, \quad \dot{\mathbf{A}} = \begin{bmatrix}\frac{\partial \mathbf{a}(\theta_1)}{\partial \theta_1} & \frac{\partial \mathbf{a}(\theta_2)}{\partial \theta_2}\end{bmatrix}. \tag{4.3}$$

According to [13], we can write the Fisher Information matrix as

$$\mathbf{F} = 2\begin{bmatrix}\Re(\mathbf{F}_{11}) & \Re(\mathbf{F}_{12}) & -\Im(\mathbf{F}_{12}) \\ \Re^\top(\mathbf{F}_{12}) & \Re(\mathbf{F}_{22}) & -\Im(\mathbf{F}_{22}) \\ -\Im^\top(\mathbf{F}_{12}) & -\Im^\top(\mathbf{F}_{22}) & \Re(\mathbf{F}_{22})\end{bmatrix}, \tag{4.4}$$

where

$$
\begin{aligned}
\mathbf{F}_{11} = & \frac{T}{\sigma^2}\left(\dot{\mathbf{B}}^{\mathrm{H}}\dot{\mathbf{B}}\right) \odot \left(\mathcal{B}^{*}\mathbf{A}^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{A}\boldsymbol{\mathcal{B}}\right) + \frac{T}{\sigma^2}\left(\dot{\mathbf{B}}^{\mathrm{H}}\mathbf{B}\right) \odot \left(\mathcal{B}^{*}\mathbf{A}^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\dot{\mathbf{A}}\mathcal{B}\right) \\
& + \frac{T}{\sigma^2}\left(\mathbf{B}^{\mathrm{H}}\dot{\mathbf{B}}\right) \odot \left(\mathcal{B}^{*}\dot{\mathbf{A}}^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{A}\mathcal{B}\right) + \frac{T}{\sigma^2}\left(\mathbf{B}^{\mathrm{H}}\mathbf{B}\right) \odot \left(\mathcal{B}^{*}\dot{\mathbf{A}}^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\dot{\mathbf{A}}\mathcal{B}\right), \qquad (4.5)
\end{aligned}
$$

$$
\mathbf{F}_{12} = \frac{T}{\sigma^2}\left(\dot{\mathbf{B}}^{\mathrm{H}}\mathbf{B}\right) \odot \left(\mathcal{B}^{*}\mathbf{A}^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{A}\right) + \frac{T}{\sigma^2}\left(\mathbf{B}^{\mathrm{H}}\mathbf{B}\right) \odot \left(\mathcal{B}^{*}\dot{\mathbf{A}}^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{A}\right), \qquad (4.6)
$$

$$
\mathbf{F}_{22} = \frac{T}{\sigma^2}\left(\mathbf{B}^{\mathrm{H}}\mathbf{B}\right) \odot \left(\mathbf{A}^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{A}\right). \qquad (4.7)
$$

For clarity, we simplify the notation during the derivation by using $\mathbf{b}_1 = \mathbf{b}(\theta_1)$, $\mathbf{b}_2 = \mathbf{b}(\theta_2)$, $\mathbf{a}_1 = \mathbf{a}(\theta_1)$ and $\mathbf{a}_2 = \mathbf{a}(\theta_2)$. We can thus write:

$$
\begin{aligned}
\mathbf{F}_{11} = & \frac{T}{\sigma^2}\begin{bmatrix} \|\beta_1\|^2 & \beta_1^{*}\beta_2 \\ \beta_2^{*}\beta_1 & \|\beta_2\|^2 \end{bmatrix} \\
& \odot \left(\begin{bmatrix} \|\dot{\mathbf{b}}_1\|^2 & \dot{\mathbf{b}}_1^{\mathrm{H}}\dot{\mathbf{b}}_2 \\ \dot{\mathbf{b}}_2^{\mathrm{H}}\dot{\mathbf{b}}_1 & \|\dot{\mathbf{b}}_2\|^2 \end{bmatrix} \odot \begin{bmatrix} \mathbf{a}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_1 & \mathbf{a}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_2 \\ \mathbf{a}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_1 & \mathbf{a}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_2 \end{bmatrix} \right. \\
& + \begin{bmatrix} \dot{\mathbf{b}}_1^{\mathrm{H}}\mathbf{b}_1 & \dot{\mathbf{b}}_1^{\mathrm{H}}\mathbf{b}_2 \\ \dot{\mathbf{b}}_2^{\mathrm{H}}\mathbf{b}_1 & \dot{\mathbf{b}}_2^{\mathrm{H}}\mathbf{b}_2 \end{bmatrix} \odot \begin{bmatrix} \mathbf{a}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\dot{\mathbf{a}}_1 & \mathbf{a}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\dot{\mathbf{a}}_2 \\ \mathbf{a}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\dot{\mathbf{a}}_1 & \mathbf{a}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\dot{\mathbf{a}}_2 \end{bmatrix} \qquad (4.8) \\
& + \begin{bmatrix} \mathbf{b}_1^{\mathrm{H}}\dot{\mathbf{b}}_1 & \mathbf{b}_1^{\mathrm{H}}\dot{\mathbf{b}}_2 \\ \mathbf{b}_2^{\mathrm{H}}\dot{\mathbf{b}}_1 & \mathbf{b}_2^{\mathrm{H}}\dot{\mathbf{b}}_2 \end{bmatrix} \odot \begin{bmatrix} \dot{\mathbf{a}}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_1 & \dot{\mathbf{a}}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_2 \\ \dot{\mathbf{a}}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_1 & \dot{\mathbf{a}}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_2 \end{bmatrix} \\
& \left. + \begin{bmatrix} \|\mathbf{b}_1\|^2 & \mathbf{b}_1^{\mathrm{H}}\mathbf{b}_2 \\ \mathbf{b}_2^{\mathrm{H}}\mathbf{b}_1 & \|\mathbf{b}_2\|^2 \end{bmatrix} \odot \begin{bmatrix} \dot{\mathbf{a}}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\dot{\mathbf{a}}_1 & \dot{\mathbf{a}}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\dot{\mathbf{a}}_2 \\ \dot{\mathbf{a}}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\dot{\mathbf{a}}_1 & \dot{\mathbf{a}}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\dot{\mathbf{a}}_2 \end{bmatrix} \right),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F}_{12} = & \frac{T}{\sigma^2}\begin{bmatrix} \beta_1^{*} & 0 \\ 0 & \beta_2^{*} \end{bmatrix} \\
& \odot \left(\begin{bmatrix} \dot{\mathbf{b}}_1^{\mathrm{H}}\mathbf{b}_1 & \dot{\mathbf{b}}_1^{\mathrm{H}}\mathbf{b}_2 \\ \dot{\mathbf{b}}_2^{\mathrm{H}}\mathbf{b}_1 & \dot{\mathbf{b}}_2^{\mathrm{H}}\mathbf{b}_2 \end{bmatrix} \odot \begin{bmatrix} \mathbf{a}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_1 & \mathbf{a}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_2 \\ \mathbf{a}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_1 & \mathbf{a}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_2 \end{bmatrix} \right. \qquad (4.9) \\
& \left. + \begin{bmatrix} \mathbf{b}_1^{\mathrm{H}}\mathbf{b}_1 & \mathbf{b}_1^{\mathrm{H}}\mathbf{b}_2 \\ \mathbf{b}_2^{\mathrm{H}}\mathbf{b}_1 & \mathbf{b}_2^{\mathrm{H}}\mathbf{b}_2 \end{bmatrix} \odot \begin{bmatrix} \dot{\mathbf{a}}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_1 & \dot{\mathbf{a}}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_2 \\ \dot{\mathbf{a}}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_1 & \dot{\mathbf{a}}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_2 \end{bmatrix} \right),
\end{aligned}
$$

$$
\mathbf{F}_{22} = \frac{T}{\sigma^2}\begin{bmatrix} \mathbf{b}_1^{\mathrm{H}}\mathbf{b}_1 & \mathbf{b}_1^{\mathrm{H}}\mathbf{b}_2 \\ \mathbf{b}_2^{\mathrm{H}}\mathbf{b}_1 & \mathbf{b}_2^{\mathrm{H}}\mathbf{b}_2 \end{bmatrix} \odot \begin{bmatrix} \mathbf{a}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_1 & \mathbf{a}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_2 \\ \mathbf{a}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_1 & \mathbf{a}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^{*}\mathbf{a}_2 \end{bmatrix}. \qquad (4.10)
$$

We can further simplify the expressions using the Hermitian operation and matrix properties. Then:

$$\begin{aligned}\mathbf{F}_{11} =& \frac{T}{\sigma^2}\begin{bmatrix} \|\beta_1\|^2 & \beta_1^*\beta_2 \\ \beta_2^*\beta_1 & \|\beta_2\|^2 \end{bmatrix} \\ &\odot \left(\begin{bmatrix} \|\dot{\mathbf{b}}_1\|^2 & \dot{\mathbf{b}}_1^{\mathrm{H}}\dot{\mathbf{b}}_2 \\ \dot{\mathbf{b}}_2^{\mathrm{H}}\dot{\mathbf{b}}_1 & \|\dot{\mathbf{b}}_2\|^2 \end{bmatrix} \odot \begin{bmatrix} \mathbf{a}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^*\mathbf{a}_1 & \mathbf{a}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^*\mathbf{a}_2 \\ \mathbf{a}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^*\mathbf{a}_1 & \mathbf{a}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^*\mathbf{a}_2 \end{bmatrix} \right. \\ &+ \begin{bmatrix} 2\mathrm{Re}\left\{\mathbf{b}_1^{\mathrm{H}}\dot{\mathbf{b}}_1\dot{\mathbf{a}}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^*\mathbf{a}_1\right\} & \mathbf{b}_1^{\mathrm{H}}\dot{\mathbf{b}}_2\left(\dot{\mathbf{a}}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^*\mathbf{a}_2 - \frac{\sin\theta_1}{\sin\theta_2}\mathbf{a}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^*\dot{\mathbf{a}}_2\right) \\ \mathbf{b}_2^{\mathrm{H}}\dot{\mathbf{b}}_1\left(\dot{\mathbf{a}}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^*\mathbf{a}_1 - \frac{\sin\theta_2}{\sin\theta_1}\mathbf{a}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^*\dot{\mathbf{a}}_1\right) & 2\mathrm{Re}\left\{\mathbf{b}_2^{\mathrm{H}}\dot{\mathbf{b}}_2\dot{\mathbf{a}}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^*\mathbf{a}_2\right\} \end{bmatrix} \\ &\left. + \begin{bmatrix} \|\mathbf{b}_1\|^2 & \mathbf{b}_1^{\mathrm{H}}\mathbf{b}_2 \\ \mathbf{b}_2^{\mathrm{H}}\mathbf{b}_1 & \|\mathbf{b}_2\|^2 \end{bmatrix} \odot \begin{bmatrix} \dot{\mathbf{a}}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^*\dot{\mathbf{a}}_1 & \dot{\mathbf{a}}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^*\dot{\mathbf{a}}_2 \\ \dot{\mathbf{a}}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^*\dot{\mathbf{a}}_1 & \dot{\mathbf{a}}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^*\dot{\mathbf{a}}_2 \end{bmatrix} \right),\end{aligned} \tag{4.11}$$

$$\mathbf{F}_{12} = \frac{T}{\sigma^2}\begin{bmatrix} \beta_1^*\left(\dot{\mathbf{b}}_1^{\mathrm{H}}\mathbf{b}_1\mathbf{a}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^*\mathbf{a}_1 + \mathbf{b}_1^{\mathrm{H}}\mathbf{b}_1\dot{\mathbf{a}}_1^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^*\mathbf{a}_1\right) & 0 \\ 0 & \beta_2^*\left(\dot{\mathbf{b}}_2^{\mathrm{H}}\mathbf{b}_2\mathbf{a}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^*\mathbf{a}_2 + \mathbf{b}_2^{\mathrm{H}}\mathbf{b}_2\dot{\mathbf{a}}_2^{\mathrm{H}}\mathbf{R}_{\mathbf{X}}^*\mathbf{a}_2\right) \end{bmatrix}. \tag{4.12}$$

By including prior information, the BFIM can be written as

$$\mathbf{J}_{\boldsymbol{\theta}|\mathbf{X}} = \mathbf{F} + \mathbf{J}^P, \tag{4.13}$$

where, from (3.4), the prior Fisher information matrix is a matrix of 2×2 matrices along its diagonal and zeroes everywhere else. The matrices along the diagonal express the prior information of the angles and the real and imaginary parts of the complex amplitudes, namely:

$$\mathbf{J}^P = \begin{bmatrix} \mathbf{J}_{\boldsymbol{\theta}}^P & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{\Re\{\beta\}}^P & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J}_{\Im\{\beta\}}^P \end{bmatrix}. \tag{4.14}$$

Since we are only interested in the estimation of the angles of arrival, we consider the equivalent Bayesian Fisher information matrix (BFIM) [1][12] by treating the complex amplitudes as nuisance parameters, namely,

$$\begin{aligned}\mathbf{J}_{\mathrm{e}}(\boldsymbol{\theta}) =& 2\mathbb{E}_{\beta}\left\{\mathbf{F}_{11}\right\} + \mathbf{J}_{\boldsymbol{\theta}}^P \\ &- 4\mathbb{E}_{\beta}\left\{\mathbf{F}_{12}^*\right\}\left(2\mathbb{E}_{\beta}\left\{\mathbf{F}_{22}\right\} + \mathbf{J}_{\boldsymbol{\beta}}^P\right)^{-1}\mathbb{E}_{\beta}\left\{\mathbf{F}_{12}\right\}.\end{aligned} \tag{4.15}$$

We also work under the assumption that the complex amplitudes $\beta_1,\ \beta_2$ are circularly symmetric, which implies that $\mathbb{E}\{\beta_i\} = \mathbb{E}\left\{{\beta_i}^*\right\} = 0$, and that $J^{\mathrm{P}}_{\mathrm{Re}\{\beta_i\}} = J^{\mathrm{P}}_{\mathrm{Im}\{\beta_i\}} = J^{\mathrm{P}}_{\beta_i}$ [1]. This also implies that $\mathbb{E}_{\boldsymbol{\beta}}\{\mathbf{F}_{12}^*\} = \mathbb{E}_{\boldsymbol{\beta}}\{\mathbf{F}_{12}\} = \mathbf{0}$.

We can thus focus on $\mathbf{F}_{11}$ to obtain the CRB. We now introduce the assumption of uncorrelated complex coefficients, hence $\mathbb{E}\{\beta_1\beta_2\} = \mathbb{E}\{\beta_1\}\mathbb{E}\{\beta_2\}$. When taking the expected value (which is with respect to the nuisance parameter $\boldsymbol{\beta}$), due to the circularly symmetric assumption of the complex amplitudes, we have the following:

$$\mathbb{E}_{\beta}\left\{\left[\begin{array}{cc} \|\beta_1\|^2 & \beta_1^*\beta_2 \\ \beta_2^*\beta_1 & \|\beta_2\|^2 \end{array}\right]\right\} = \left[\begin{array}{cc} \mathbb{E}_{\beta_1}\{\|\beta_1\|^2\} & 0 \\ 0 & \mathbb{E}_{\beta_2}\{\|\beta_2\|^2\} \end{array}\right]. \tag{4.16}$$

By choosing the phase reference point of the transmitting and receiving arrays such that $\mathbf{b}_1^{\mathrm{H}}\dot{\mathbf{b}}_1 = \dot{\mathbf{b}}_1^{\mathrm{H}}\mathbf{b}_1 = \mathbf{b}_2^{\mathrm{H}}\dot{\mathbf{b}}_2 = \dot{\mathbf{b}}_2^{\mathrm{H}}\mathbf{b}_2 = 0$, we arrive at:

$$\mathbb{E}\{\mathbf{F}_{11}\} = \frac{T}{\sigma_s^2}\begin{bmatrix} \mathbb{E}\{|\beta_1|^2\}\ f_1(\mathbf{R_X}) & 0 \\ 0 & \mathbb{E}\{|\beta_2|^2\}\ f_2(\mathbf{R_X}), \end{bmatrix} \tag{4.17}$$

where

$$\begin{aligned} f_i(\mathbf{R_X}) := \mathbb{E}_{\theta_i}\{&\|\dot{\mathbf{b}}(\theta_i)\|^2 tr\left\{\mathbf{a}(\theta_i)\mathbf{a}^{\mathrm{H}}(\theta_i)\mathbf{R}_{\mathbf{X}}^*\right\} \\ &+ \|\mathbf{b}(\theta_i)\|^2 tr\left\{\dot{\mathbf{a}}(\theta_i)\dot{\mathbf{a}}^{\mathrm{H}}(\theta_i)\mathbf{R}_{\mathbf{X}}^*\right\}\}. \end{aligned} \tag{4.18}$$

An alternative way to express $f_i(\mathbf{R_X})$ is obtained by defining $\overline{\mathbf{M}}_i = \mathbb{E}_{\theta_i}\{\mathbf{M}^*(\theta_i)\}$, with $\mathbf{M}(\theta_i)$ being

$$\mathbf{M}(\theta_i) = \|\dot{\mathbf{b}}(\theta_i)\|^2\mathbf{a}(\theta_i)\mathbf{a}^{\mathrm{H}}(\theta_i) + \|\mathbf{b}(\theta_i)\|^2\dot{\mathbf{a}}(\theta_i)\dot{\mathbf{a}}^{\mathrm{H}}(\theta_i). \tag{4.19}$$

It is key to highlight that the matrix $\mathbf{J}_{\boldsymbol{\theta}}^P$ defines the prior knowledge on the distribution of the angles θ_1 and θ_2. Such matrix is assumed diagonal; hence, the priors of the angles are uncorrelated, with entries $J_{\theta_1}^P$ and $J_{\theta_2}^P$. The two entries are the inverse of the known variances of the prior distribution for the two angles and can be either approximated using $\sigma^2 \approx \frac{1}{\kappa}$ or by using the more accurate approximation $\sigma^2 = 1 - \frac{I_1(\kappa)}{I_0(\kappa)}$, where $I_i(\kappa)$ is the modified Bessel function of order i.

It is possible to invert the resulting diagonal matrix and finally obtain:

$$\begin{aligned} \epsilon = &\underbrace{\mathbb{E}_{\mathbf{X}}\left\{\left(\frac{2T}{\sigma_s^2}\mathbb{E}_{\beta_1}\{|\beta_1|^2\}\ f_1(\mathbf{R_X}) + J_{\theta_1}^P\right)^{-1}\right\}}_{\epsilon_{\theta_1}} \\ &+ \underbrace{\mathbb{E}_{\mathbf{X}}\left\{\left(\frac{2T}{\sigma_s^2}\mathbb{E}_{\beta_2}\{|\beta_2|^2\}\ f_2(\mathbf{R_X}) + J_{\theta_2}^P\right)^{-1}\right\}}_{\epsilon_{\theta_2}}. \end{aligned} \tag{4.20}$$

The derived expression for ϵ is the sum of two errors: the estimation of θ_1 and θ_2. We can keep this distinction and study how the rate changes in relation to these two errors. We also remark that the BCRB depends on the transmit signal $\mathbf{X}$ through $\mathbf{R_X}$.

We can rewrite the errors as follows:

$$\epsilon_{\theta_i} = \mathbb{E}_{\mathbf{X}} \left\{ \left(\frac{2T\mathbb{E}\left\{|\beta_i|^2\right\}}{\sigma^2} \operatorname{tr}\left\{\overline{\mathbf{M}}_i \mathbf{R}_{\mathbf{X}}\right\} + J^{\mathrm{P}}_{\theta_i} \right)^{-1} \right\}. \tag{4.21}$$

If we only focus on estimating the angle θ_i, the optimal choice of $\mathbf{R}_{\mathbf{X}}$ is known from previous work [1]. However, a closed form optimal solution for the joint estimation of multiple angles is not available. An optimal deterministic signal $\mathbf{X}$ can be found numerically by using convex optimization tools such as CVX [14][15].

4.2 Optimal Communication and Sensing Points

In ISAC, we are trying to share resources between the sensing and communication tasks to optimize our transmission. Before diving into the proposed strategies, it is good to consider the optimal achievable points if the two tasks are not done jointly. Hence, here we study the deterministic optimal choice of $\mathbf{X}$ that minimizes the total estimation error and the optimal $\mathbf{X}$ that maximizes the ergodic rate. The goal is to find in the next chapter an achievable ISAC region that interpolates between these points.

4.2.1 Optimal Sensing Point

It is known [1] that to minimize the BCRB ϵ_{θ_i} of a single angle, hence falling back to the problem of single target estimation, we can achieve the optimal result by transmitting in the direction of the eigenvector corresponding to the largest eigenvalue of the matrix $\overline{\mathbf{M}}_i$.

However, finding an analytical solution is not as straightforward when investigating the joint minimization of both angles, which we have seen corresponds to minimizing the sum of the estimation error on each angle. It is left for future works to find a closed-term optimal solution for this problem, while for this dissertation to provide a reliable achievable point, we rely on a numerical optimization for any angle pair.

Since [1] provided the result that the rank of the optimal solution to this issue is equal to 2 and that by analyzing our resulting matrix $\mathbf{R}_{\mathbf{X}}$ it has two main eigenvalues, we select the two main directions (the two eigenvectors corresponding to the two largest eigenvalues) of our numerical evaluation of the matrix $\mathbf{R}_{\mathbf{X}}$. We will refer to these two directions as $\hat{\mathbf{r}}_{1,opt}$ and $\hat{\mathbf{r}}_{2,opt}$.

4.2.2 Optimal Communication Point

The optimal communications strategy for the communication-only point is obtained by solving

$$C = \max_{\mathbf{R_X}:\ \mathbb{E}\{\mathrm{tr}\{\mathbf{R_X}\}\} \leq P_{\mathrm{TX}} M_{\mathrm{TX}}} R; \tag{4.22}$$

$$R := \mathbb{E}_{\mathbf{H}_c}\left[\log_2\left|I + \tfrac{1}{\sigma_c^2}\mathbf{H}_c^{\mathrm{H}}\mathbb{E}_{\mathbf{X}}[\mathbf{R_X}]\mathbf{H}_c\right|\right] = \mathbb{E}_{\alpha,\theta_c}\left[\log\left(1 + \tfrac{|\alpha|^2}{\sigma_c^2}\mathbf{a}^H(\theta_c)\mathbb{E}_{\mathbf{X}}[\mathbf{R_X}]\mathbf{a}(\theta_c)\right)\right]; \tag{4.23}$$

which is again solvable with a numerical optimizer such as CVX.

Our simulations consider a LoS channel with a single receiver antenna; hence, the channel can be written as $\mathbf{h}_c$. If we consider the uncertainty on $\mathbf{h}_c$ small, the optimal solution is simply to transmit Gaussian information alongside the direction that is the hermitian of the channel. We will refer to this direction as $\hat{\mathbf{r}}_{c,opt}$.

4.3 Ways to Characterize the ISAC Region

In this dissertation, we have studied the optimal sensing and communication points, practically providing an outer bound for the BCRB-Rate region, and are now interested in characterizing an inner bound. This problem can be solved in different ways:

1. Evaluation of parameterized achievable scheme and computation of the convex closure of achievable pair (R, ϵ).

2. Solve the constrained optimization problem:

$$\max_{p_{\mathbf{X}}(\mathbf{X}) \in \mathcal{X}(P_{\mathrm{TX}} M_{\mathrm{TX}})} I(\mathbf{X}; \mathbf{Y}_c \mid \mathbf{H}_c) \qquad \text{s.t. } \epsilon = \epsilon_{\theta_1} + \epsilon_{\theta_2} \leq \epsilon_{fix},\ \forall \epsilon_{fix} \in [\epsilon_{\min}, 1], \tag{4.24}$$

   where $\mathcal{X}(P_{\mathrm{TX}} M_{\mathrm{TX}})$ is the set of all possible distributions for the input signal that meet the average power constraint $P_{\mathrm{TX}} M_{\mathrm{TX}}$, i.e., $\mathbb{E}\{\mathrm{tr}\{\mathbf{R_X}\}\} \leq P_{\mathrm{TX}} M_{\mathrm{TX}} = P_{\mathrm{TX}} M_{\mathrm{TX}}$; and $\epsilon_{\min}$ is the error obtained with (4.20) by using the $\mathbf{R_X}$ obtained via CVX as described in section 4.2.1.

3. Solve the constrained optimization problem:

$$\min_{p_{\mathbf{X}}(\mathbf{X}) \in \mathcal{X}(P_{\mathrm{TX}} M_{\mathrm{TX}})} \epsilon \qquad \text{s.t. } I(\mathbf{X}; \mathbf{Y}_c \mid \mathbf{H}_c) \leq R_{fix},\ \forall R_{fix} \in [0, R_{\max}], \tag{4.25}$$

   where $R_{\max} = \max_{p_{\mathbf{X}}(\mathbf{X}) \in \mathcal{X}(P_{\mathrm{TX}} M_{\mathrm{TX}})} I(\mathbf{X}; \mathbf{Y}_c \mid \mathbf{H}_c)$.

4. Solve the constrained optimization problem:

$$\max_{p_{\mathbf{X}}(\mathbf{X}) \in \mathcal{X}(P_{\mathrm{TX}} M_{\mathrm{TX}})} \left\{ I(\mathbf{X}; \mathbf{Y}_c \mid \mathbf{H}_c) - \zeta\epsilon \right\}, \tag{4.26}$$

   where $\mathcal{X}(P_{\mathrm{TX}} M_{\mathrm{TX}})$ is the set of all possible distributions for the input signal that meet the average power constraint $P_{\mathrm{TX}} M_{\mathrm{TX}}$, i.e., $\mathbb{E}\{\mathrm{tr}\{\mathbf{R_X}\}\} \leq P_{\mathrm{TX}} M_{\mathrm{TX}}$, and the parameter $\zeta \in [0, +\infty)$ is making it possible to optimize the curve so that for any given value of the rate the lowest error on the joint estimation of the angles is achieved, and that for any given value of ϵ the maximum rate is achieved.

However, only the first way to solve the problem provides clear interpretability of the results regarding the optimal transmission strategy. For this reason, we focus on the first one in the next chapter.

Chapter 5

Achievable Transmission Strategies for Characterization of ISAC Tradeoff

5.1 General Communicating Strategy to the UE

We transmit a signal $\mathbf{X}$ that needs to be optimized to obtain the lowest possible estimation error on the angles and the highest possible communication rate. In the signal $\mathbf{X}$, we must decide the direction(s) we want to send the signal and how we want to deliver information.

The next sections will apply different strategies to study the ISAC tradeoff. All of the strategies proposed in this work can be written in a general way as follows:

$$\mathbf{X} = [\mathbf{X}_1, \dots, \mathbf{X}_T] : \mathbf{X}_t = \sum_{b=1}^{N_B} \sqrt{P_{s;b,t}}\ \mathbf{s}_{b,t} + \sqrt{P_{c;b,t}}\ \mathbf{c}_{b,t} G_{b,t}, \tag{5.1}$$

where N_B is the number of beams one wishes to use, $\mathbf{s}_{b,t}$ is a deterministic vector for sensing of unit length, and $\mathbf{c}_{b,t}$ is a communication-beamforming vector which is normalized and 'modulated' by the information-carrying i.i.d. $\mathcal{N}(0,1)$ Gaussian random variable $G_{b,t}$. The signal $\mathbf{X}$ is subject to the power constraint $\sum_{b=1}^{N_B} P_{s;b,t} + P_{c;b,t} \leq P_{\mathrm{TX}} M_{\mathrm{TX}}$, which ensures that $\mathbb{E}[\mathrm{tr}\,\{\mathbf{R}_{\mathbf{X}}\}] \leq P_{\mathrm{TX}} M_{\mathrm{TX}}$.

We will analyze specific choices of the various parameters on the proposed ISAC scheme, with specific channel parameters, to provide some case studies. We will use for each proposed strategy different choices for the directions of sensing and communication. Those directions will be spanned through selection of parameters λ_i, such that $\sum_i \lambda_1 \leq 1$ and each $\lambda_i \in [0,1]$.

5.1.1 Communication Rate for Given Strategy

It is equally important in this study to evaluate the rate for any given structure of the signal we want to transmit. Based on the generic transmitted signal in (5.1), we can compute the rate as:

$$R(\epsilon_{fix}) = \mathbb{E}_{\mathbf{h}_c}\left[\sum_{t=1}^{T} \log\left(1 + \frac{1}{T\sigma_c^2}\left(\sum_{b=1}^{N_B} P_{c;b,t}|\mathbf{h}_c^{\mathrm{H}}\mathbf{c}_{b,t}|^2\right)\right)\right]. \tag{5.2}$$

5.1.2 BCRB for Given Strategy

To compute the BCRB for the given strategy, we take the transmitted signal $\mathbf{X}$ and directly apply:

$$\epsilon = \epsilon_{\theta_1} + \epsilon_{\theta_2}, \tag{5.3}$$

where:

$$\epsilon_{\theta_i} = \mathbb{E}_{\mathbf{X}}\left\{\left(\frac{2T\mathbb{E}\left\{|\beta_i|^2\right\}}{\sigma^2}\operatorname{tr}\left\{\overline{\mathbf{M}}_i\mathbf{R}_{\mathbf{X}}\right\} + J_{\theta_i}^{\mathrm{P}}\right)^{-1}\right\}. \tag{5.4}$$

Note that we refer to θ_1 as the angle of the sensing only target and to θ_2 as the angle of the communication target to send data to and to sense at the same time.

5.2 Strategy 1: One Information Carrying Beam

For the first strategy, we send only in the direction we send i.i.d. Gaussian random variables, hence $P_{s,t} = 0$. We also transmit in the same direction for all $t \in [1, T]$. The transmit signal is then:

$$\mathbf{X} = [\mathbf{X}_1, \ldots, \mathbf{X}_T] : \mathbf{X}_t = \sqrt{P_{\mathrm{TX}}M_{\mathrm{TX}}}\ \mathbf{c}_{\mathrm{first}}G_t. \tag{5.5}$$

For this strategy, a closed form solution of (4.21) is derived in [1]:

$$\epsilon_{\theta_i,\mathrm{first}} = \left(2(T-1)\,\mathrm{SNR}_{\mathrm{s}}\,\mathbf{c}_{\mathrm{first}}^{\mathrm{H}}\overline{\mathbf{M}}_i\mathbf{c}_{\mathrm{first}}\right)^{-1}(1 + r_\zeta), \tag{5.6}$$

where $\mathrm{SNR}_s = M_{\mathrm{TX}}P_{\mathrm{TX}}\mathbb{E}\left\{|\beta|^2\right\}\sigma_s^{-2}$. The correction term r_ζ is given by

$$r_\zeta = \sum_{n=1}^{T-2}\frac{(-1)^n\zeta^n}{\prod_{i=1}^{n}(T-i-1)} + \underbrace{(-1)^{T-1}\cdot\frac{e^\zeta\zeta^{T-1}\Gamma(0,\zeta)}{\Gamma(T-1)}}_{O(\zeta^{T-1}\log\zeta)}, \tag{5.7}$$

where $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}\ dt$, denotes the incomplete Gamma function, and the variable ζ is computed as $\zeta = J_{\theta_i}^P \left(2\,\mathrm{SNR}_s\, \mathbf{c}_{\text{first}}^{\mathrm{H}} \overline{\mathbf{M}} \mathbf{c}_{\text{first}}\right)^{-1}$.

For what concerns the communication rate, this strategy yields:

$$R = \mathbb{E}_{\mathbf{h}_c}\left[\log_2\left(1 + \mathbb{E}_{\mathbf{h}_c}[\|\mathbf{h}_c\|^2]^{-1}\left|\mathbf{c}_{\text{first}}^{\mathrm{H}}\mathbf{h}_{\mathrm{c}}\right|^2 \mathrm{SNR}_{\mathrm{c}}\right)\right], \tag{5.8}$$

where $\mathrm{SNR}_{\mathrm{c}} = M_{\mathrm{TX}} P_{\mathrm{TX}} \mathbb{E}_{\mathbf{h}_c}[\|\mathbf{h}_c\|^2]\sigma_c^{-2}$.

5.2.1 Choice 1: Spanning among the two main eigenvectors of $\overline{\mathbf{M}}_1$, and the two main eigenvectors of $\overline{\mathbf{M}}_2$

For the direction $\mathbf{c}_{\text{first}}$, we start by spanning between the two eigenvectors corresponding to the largest eigenvalues of the matrices $\overline{\mathbf{M}}_1$ and $\overline{\mathbf{M}}_2$. The choice is motivated by the fact that the error on the estimation of each angle is minimized by choosing as direction the "main" eigenvector corresponding to the largest eigenvalue of the corresponding $\overline{\mathbf{M}}_i$ matrix [1], and since if the angle distribution is very narrow the rank of the matrices is 2, by choosing the two main directions we try to see if for the case of joint target angles estimation, using a direction different than the main one can help. Denoting by $\mathbf{v}_1$ and $\mathbf{v}_2$ the two main eigenvectors of $\overline{\mathbf{M}}_1$, and as $\mathbf{v}_3$ and $\mathbf{v}_4$ the ones of $\overline{\mathbf{M}}_2$, we can write the direction of transmission as:

$$\mathbf{r} = \sqrt{\lambda_1}\mathbf{v}_1 + \sqrt{\lambda_2}\mathbf{v}_2 + \sqrt{\lambda_3}\mathbf{v}_3 + \sqrt{\lambda_4}\mathbf{v}_4, \qquad \mathbf{c}_{\text{first}} = \frac{\mathbf{r}}{\|\mathbf{r}\|}. \tag{5.9}$$

5.2.2 Choice 2: Spanning among the Directions of the Optimal Sensing and the Direction of Optimal Communication

Another strategy is to span between the two optimal directions obtained via CVX for the overall minimization of the BCRB and the optimal direction of communication. We construct our direction of transmission as follows:

$$\mathbf{r} = \sqrt{\lambda_1}\hat{\mathbf{r}}_{1,opt} + \sqrt{\lambda_2}\hat{\mathbf{r}}_{2,opt} + \sqrt{\lambda_3}\hat{\mathbf{r}}_{c,opt} \qquad \mathbf{c}_{\text{first}} = \frac{\mathbf{r}}{\|\mathbf{r}\|}. \tag{5.10}$$

5.3 Strategy 2: One Information-less Beam

For the second strategy, we send only in the direction without Gaussian random variables, hence $P_{c,t} = 0$. We also transmit in the same direction for all $t \in [1, T]$. The transmit signal is then:

$$\mathbf{X} = [\mathbf{X}_1, \ldots, \mathbf{X}_T] : \mathbf{X}_t = \sqrt{P_{\mathrm{TX}} M_{\mathrm{TX}}}\ \mathbf{s}_{\text{second}}. \tag{5.11}$$

For this strategy, a closed form solution of 4.21 is derived in [1]:

$$\epsilon_{\theta_i,second} = \left(2T\,\mathrm{SNR}_s\,\mathbf{s}_{\mathrm{second}}^{\mathrm{H}}\overline{\mathbf{M}}_i\mathbf{s}_{\mathrm{second}} + J_{\theta_i}^{P}\right)^{-1}. \tag{5.12}$$

Due to the lack of information, the resulting rate will be zero. Given that this strategy returns a zero rate, it is presented to provide some insights into the minimum values for the joint estimation of the two angles.

5.3.1 Choice 1: Spanning among the two main eigenvectors of $\overline{\mathbf{M}}_1$, and the two main eigenvectors of $\overline{\mathbf{M}}_2$

The first spanning choice uses the same direction as the first direction choice in 5.2.1 to compare them. Hence, the direction is chosen as follows:

$$\mathbf{r} = \sqrt{\lambda_1}\mathbf{v}_1 + \sqrt{\lambda_2}\mathbf{v}_2 + \sqrt{\lambda_3}\mathbf{v}_3 + \sqrt{\lambda_4}\mathbf{v}_4, \qquad \mathbf{s}_{\mathrm{second}} = \frac{\mathbf{r}}{\|\mathbf{r}\|}. \tag{5.13}$$

5.3.2 Choice 2: Spanning among the main eigenvector of $\overline{\mathbf{M}}_1$, the main eigenvector of $\overline{\mathbf{M}}_2$ and the Directions of Optimal Sensing

The second choice uses the main components of the $\overline{\mathbf{M}}_i$ matrices and the optimal directions for minimizing the BCRB obtained via CVX. The direction is then chosen as follows:

$$\mathbf{r} = \sqrt{\lambda_1}\hat{\mathbf{r}}_{1,opt} + \sqrt{\lambda_2}\hat{\mathbf{r}}_{2,opt} + \sqrt{\lambda_3}\mathbf{v}_1 + \sqrt{\lambda_4}\mathbf{v}_3 \qquad \mathbf{s}_{\mathrm{second}} = \frac{\mathbf{r}}{\|\mathbf{r}\|}. \tag{5.14}$$

5.4 Strategy 3: Both One Information Carrying Beam and One Information-less Beam

For the third strategy, we send in both the directions highlighted in the general transmission signal (5.1). We transmit in the same directions or all $t \in [1, T]$. The transmit signal is then:

$$\mathbf{X} = [\mathbf{X}_1, \ldots, \mathbf{X}_T] : \mathbf{X}_t = \sqrt{P_s}\,\mathbf{s}_{\mathrm{third}} + \sqrt{P_c}\,\mathbf{c}_{\mathrm{third}}G_t. \tag{5.15}$$

Note that power allocation becomes a key factor for this strategy. When spanning among different directions we will also be affecting the power allocation towards P_s and P_c, respectively.

For what concerns the communication rate, this strategy yields:

$$R = \mathbb{E}_{\mathbf{h}_c}\left[\log_2\left(1 + \mathbb{E}_{\mathbf{h}_c}[\|\mathbf{h}_c\|^2]^{-1}\left|\mathbf{c}_{\mathrm{third}}^{\mathrm{H}}(\lambda)\mathbf{h}_{\mathrm{c}}\right|^2 \mathrm{SNR}_{\mathrm{c}}\right)\right], \tag{5.16}$$

where $\mathrm{SNR_c} = P_c \mathbb{E}_{\mathbf{h}_c}[\|\mathbf{h}_c\|^2]\sigma_c^{-2}$. We will express P_c as a function of the maximum power $P_{\mathrm{TX}} M_{\mathrm{TX}}$ based on the directions used.

5.4.1 Choice 1: Spanning among the main eigenvector of $\overline{\mathbf{M}}_1$, the main eigenvector of $\overline{\mathbf{M}}_2$ and the Direction of Optimal Communication

As a first choice, we use for the sensing direction $\mathbf{s}$ the main eigenvector corresponding to the largest eigenvalue of $\overline{\mathbf{M}}_1$ and $\overline{\mathbf{M}}_2$, which are $\mathbf{v}_1$ and $\mathbf{v}_3$ respectively. The optimal communication direction $\hat{\mathbf{r}}_{c,opt}$ is used for the communication direction $\mathbf{c}$. We can write the directions as:

$$\mathbf{r}_1 = \sqrt{\lambda_1}\mathbf{v}_1 + \sqrt{\lambda_2}\mathbf{v}_3, \qquad \mathbf{s}_{\text{third}} = \frac{\mathbf{r}_1}{\|\mathbf{r}_1\|}, \tag{5.17}$$

$$\mathbf{r}_2 = \sqrt{\lambda_3}\hat{\mathbf{r}}_{c,opt}, \qquad \mathbf{c}_{\text{third}} = \frac{\mathbf{r}_2}{\|\mathbf{r}_2\|}. \tag{5.18}$$

The spanning choice allows us to compute the power devoted to the sensing-focused direction P_s and communication-focused direction as $P_s = (\lambda_1 + \lambda_2) P_{\mathrm{TX}} M_{\mathrm{TX}}$ and $P_c = \lambda_3 P_{\mathrm{TX}} M_{\mathrm{TX}}$, respectively.

5.4.2 Choice 2: Spanning among the Directions of the Optimal Sensing and the Direction of Optimal Communication

As a second choice, we choose the sensing direction $\mathbf{s}$ to span between the two optimal directions for sensing obtained via CVX. For the communication side, we utilize instead the optimal direction $\hat{\mathbf{r}}_{c,opt}$. We can write the directions as follows:

$$\mathbf{r}_1 = \sqrt{\lambda_1}\hat{\mathbf{r}}_{1,opt} + \sqrt{\lambda_2}\hat{\mathbf{r}}_{2,opt}, \qquad \mathbf{s}_{\text{third}} = \frac{\mathbf{r}_1}{\|\mathbf{r}_1\|}, \tag{5.19}$$

$$\mathbf{r}_2 = \sqrt{\lambda_3}\hat{\mathbf{r}}_{c,opt}, \qquad \mathbf{c}_{\text{third}} = \frac{\mathbf{r}_2}{\|\mathbf{r}_2\|}. \tag{5.20}$$

The spanning choice allows us to compute the power devoted to the sensing-focused direction P_s and communication-focused direction as $P_s = (\lambda_1 + \lambda_2) P_{\mathrm{TX}} M_{\mathrm{TX}}$ and $P_c = \lambda_3 P_{\mathrm{TX}} M_{\mathrm{TX}}$, respectively.

5.4.3 Choice 3: Spanning among the main eigenvector of $\overline{\mathbf{M}}_1$, the main eigenvector of $\overline{\mathbf{M}}_2$, the Directions of the Optimal Sensing, and the Direction of Optimal Communication

As a third and final choice we use a combination of the previous choices in 5.4.1 and 5.4.2. For the sensing-focused direction $\mathbf{s}$, we use a combination of both the main eigenvector corresponding to the largest eigenvalue of $\overline{\mathbf{M}}_1$ and $\overline{\mathbf{M}}_2$, and also the two optimal directions for sensing obtained via CVX. For the communication-focused direction, we use optimal communication direction $\hat{\mathbf{r}}_{c,opt}$. We can write the directions as follows:

$$\mathbf{r}_1 = \sqrt{\lambda_1}\mathbf{v}_1 + \sqrt{\lambda_2}\mathbf{v}_3 + \sqrt{\lambda_3}\hat{\mathbf{r}}_{1,opt} + \sqrt{\lambda_4}\hat{\mathbf{r}}_{2,opt}, \qquad \mathbf{s}_{\text{third}} = \frac{\mathbf{r}_1}{\|\mathbf{r}_1\|}, \tag{5.21}$$

$$\mathbf{r}_2 = \sqrt{\lambda_5}\hat{\mathbf{r}}_{c,opt}, \qquad \mathbf{c}_{\text{third}} = \frac{\mathbf{r}_2}{\|\mathbf{r}_2\|}. \tag{5.22}$$

The spanning choice allows us to compute the power devoted to the sensing-focused direction P_s and communication-focused direction as $P_s = (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)P_{\text{TX}}M_{\text{TX}}$ and $P_c = \lambda_5 P_{\text{TX}}M_{\text{TX}}$, respectively.

Chapter 6

Simulation Results

In this chapter, we provide different simulations for each of the strategies and choices of direction presented in the previous chapter.

We start with a batch of simulations sharing the same parameters specified in 6.1 to be able to compare them effectively. Note that we assume colocated TX and RX antennas; hence, the spacing and number of antennas are the same.

Table 6.1: Parameters Used to Compare Different Strategies

Configuration	**Value**
No. antennas ($M_{\mathrm{TX}} = M_{\mathrm{RX}}$)	10
Antennas spacing	1/2 wavelength
Max. Sensing Receiving SNR	20dB per Antenna
Max. Communication Receiving SNR	33dB per Antenna
Coherence Time (T)	3
Mean Sensing Angle (θ_s)	30°
Concentration of Sensing Angle (κ)	131.8
Mean Communication Angle (θ_c)	42°
Concentration of Communication Angle (κ)	131.8
Number of Simulated Sensing Angles	10,000
Number of Simulated Communication Angles	1,000
Number of Simulated Gaussians	10,000

Since the theoretical formulas for the first strategy are available, the theoretical plots are provided and compared with the simulated results. The presented plots can either represent the total BCRB, given by the sum of the ones on each angle, against the rate or the relationship between each BCRB on the x and y axes, respectively, with the rate represented through colors. Since the total rate is zero for the second transmission strategy, representing the total error vs. the rate does not provide insights.

6.1 Strategy 1: One Information Carrying Beam

6.1.1 Choice 1: Spanning among the two main eigenvectors of $\overline{\mathbf{M}}_1$, and the two main eigenvectors of $\overline{\mathbf{M}}_2$

The results shown in 6.1 confirm the validity of our simulation compared to the theoretical results in 6.2. It is possible to appreciate how, by using the eigenvectors of the matrices $\overline{\mathbf{M}}_1$ and $\overline{\mathbf{M}}_2$, it is possible to achieve the lowest errors on the single angle estimation. However, it is in our interest to study the joint estimation; hence, having more points in the low-left corner of both figures would be desirable. It is also important to highlight that using this strategy, it is possible to achieve optimal communication rates, as confirmed by the results in 6.3 and 6.4. However, the choice of this strategy does not allow us to reach the minimum overall achievable error on the estimation, which requires a deterministic signal. This can be confirmed by the result in 6.5, in which the outer bound of the scatter plot in 6.4 is taken. It is possible to appreciate the difference between the overall minimum achievable error and the one we can achieve with the first strategy. Note how there is a gap between the total error we obtain with this choice of direction and the minimum achievable error for the first strategy. This is because, as mentioned, there is a lack of points that produce low errors on both the estimation of θ_1 and θ_2.

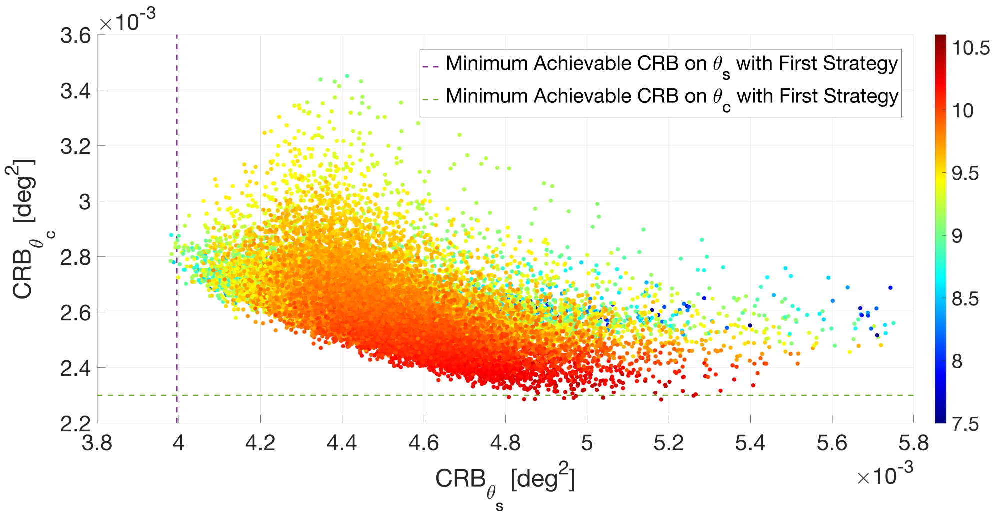

Figure 6.1: First Strategy, First Direction Choice, BCRB-Rate Simulated Points.

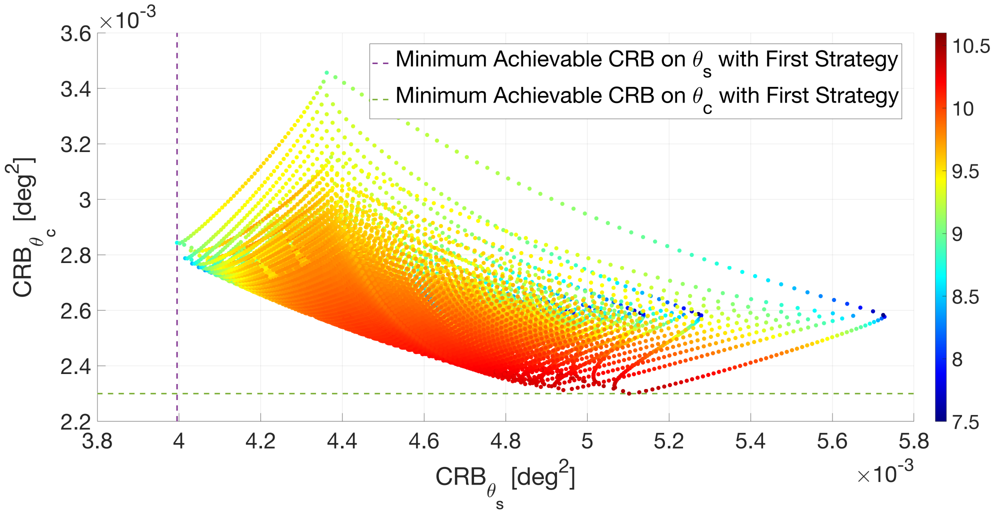

Figure 6.2: First Strategy, First Direction Choice, BCRB-Rate Theoretical Points.

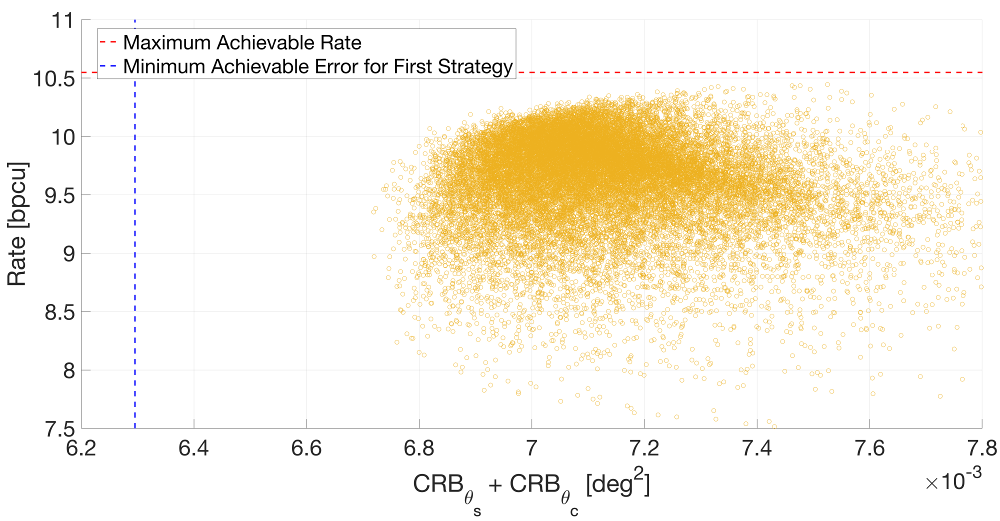

Figure 6.3: First Strategy, First Direction Choice, Total BCRB-Rate Simulated Points.

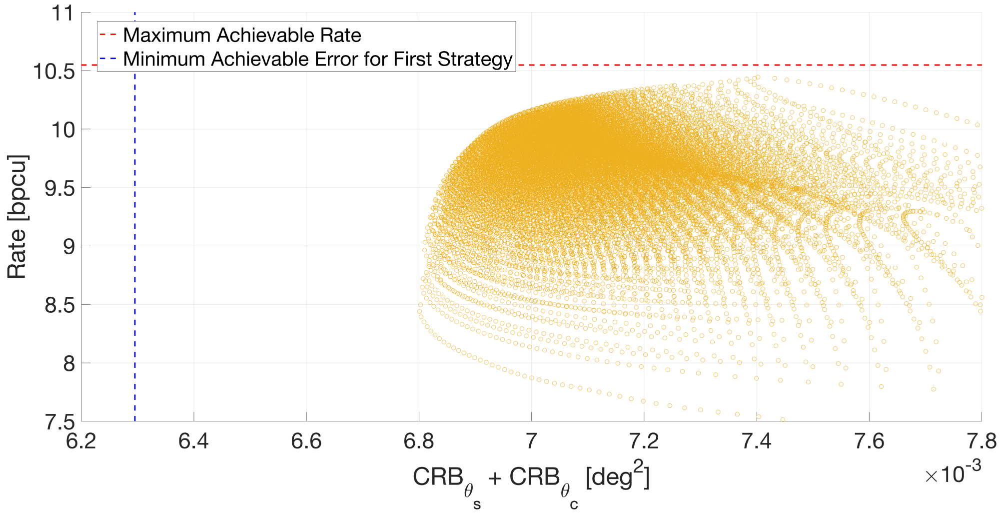

Figure 6.4: First Strategy, First Direction Choice, Total BCRB-Rate Theoretical Points.

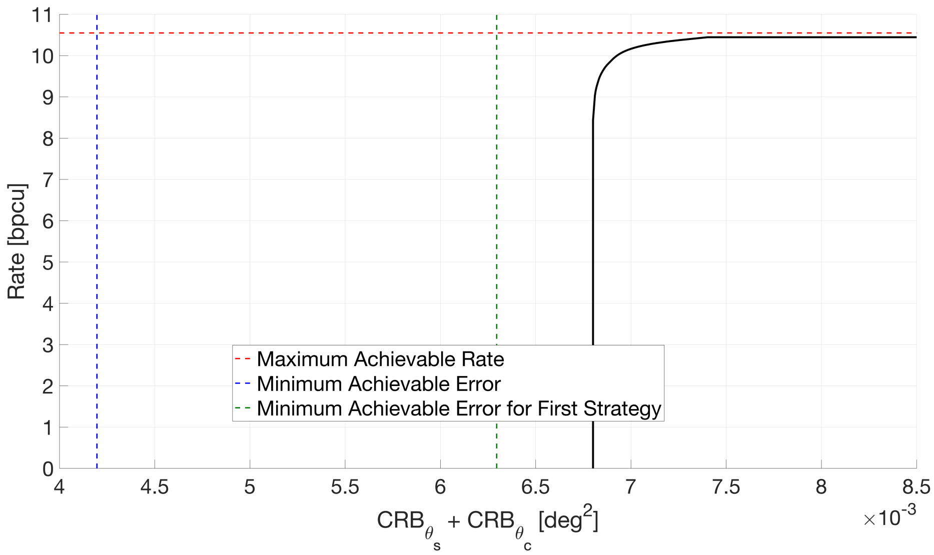

Figure 6.5: First Strategy, First Direction Choice, Total BCRB-Rate Outer Bound.

6.1.2 Choice 2: Spanning among the Directions of the Optimal Sensing and the Direction of Optimal Communication

We mentioned the importance of having points in the low-left of the figures representing the CRB of each angle in the x and y-axis, respectively. This is possible using the directions obtained via CVX, as seen in 6.6 and 6.7. As expected, as a consequence of this, it is also able to achieve a lower total error than the previous choice, as it can be seen in 6.8 and 6.9. The comparison is even more appreciable in 6.11, where the choices for this strategy are compared.

Overall, from this first strategy, we could say that using the eigenvectors of the matrices $\overline{\mathbf{M}}_1$ and $\overline{\mathbf{M}}_2$ is more suitable if interested in one specific angle. Using a more optimal solution for the joint estimation is certainly more suitable for the overall minimization of the total error.

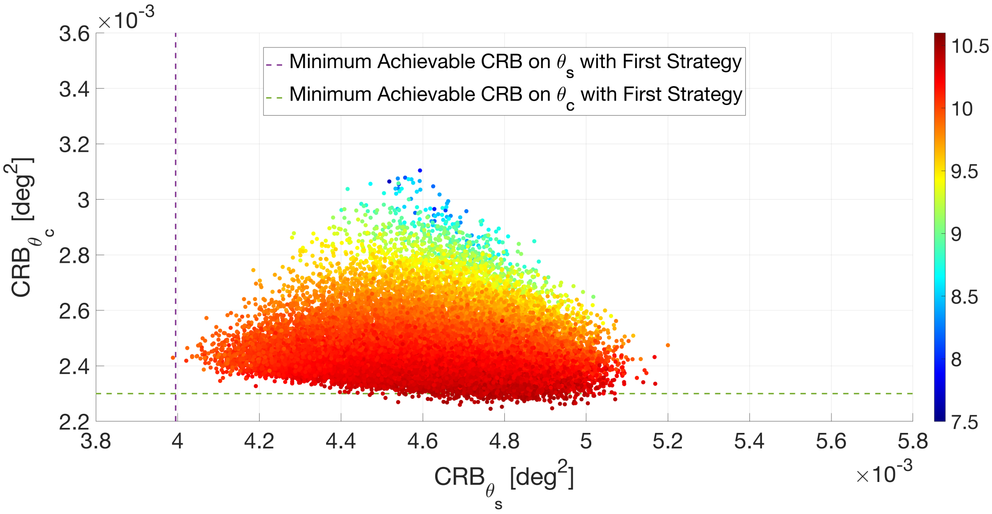

Figure 6.6: First Strategy, Second Direction Choice, BCRB-Rate Simulated Points.

6.2 Strategy 2: One Information-less Beam

6.2.1 Choice 1: Spanning among the two main eigenvectors of $\overline{\mathbf{M}}_1$, and the two main eigenvectors of $\overline{\mathbf{M}}_2$

The second strategy, returning a zero rate, focuses on the sensing side of our study. Since we are now sending a deterministic signal, achieving the optimal estimation errors is possible. In particular, with this first choice of direction, it can be seen

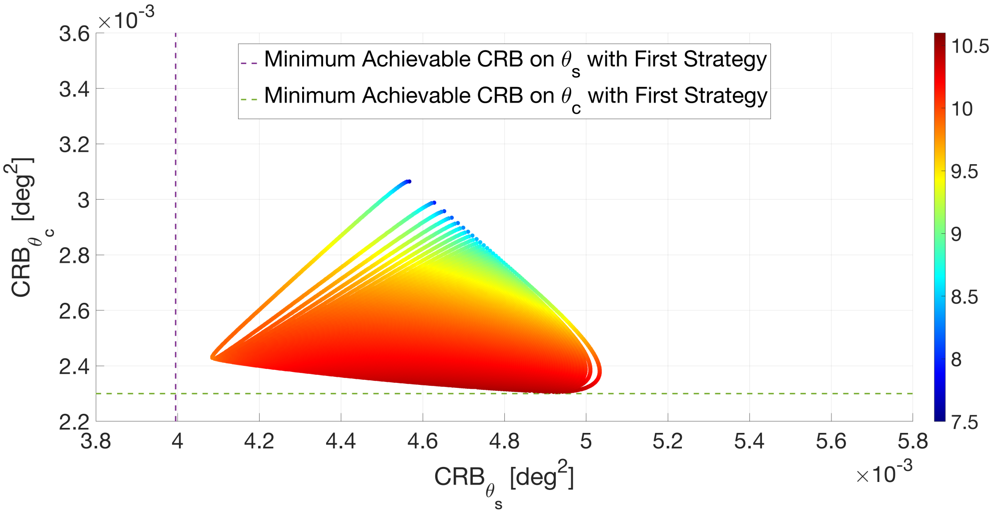

Figure 6.7: First Strategy, Second Direction Choice, BCRB-Rate Theoretical Points.

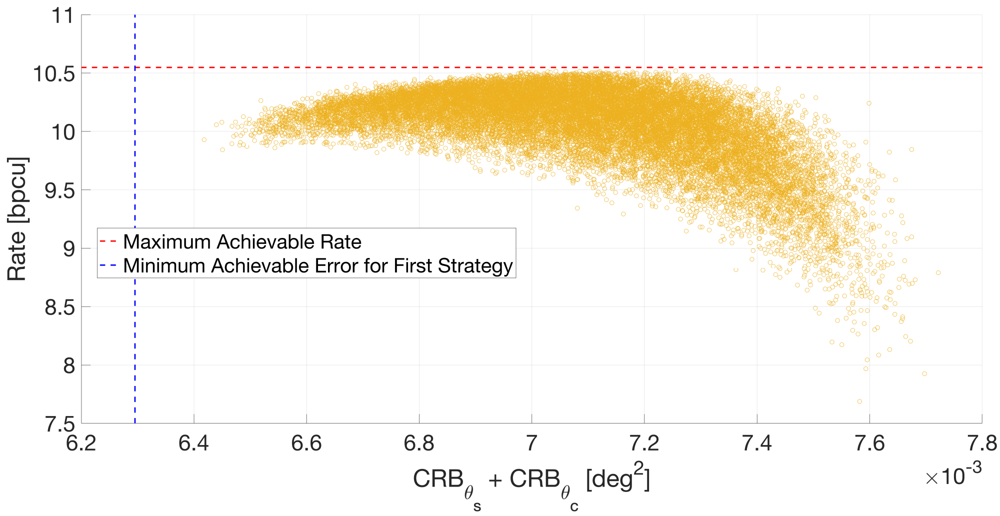

Figure 6.8: First Strategy, Second Direction Choice, Total BCRB-Rate Simulated Points.

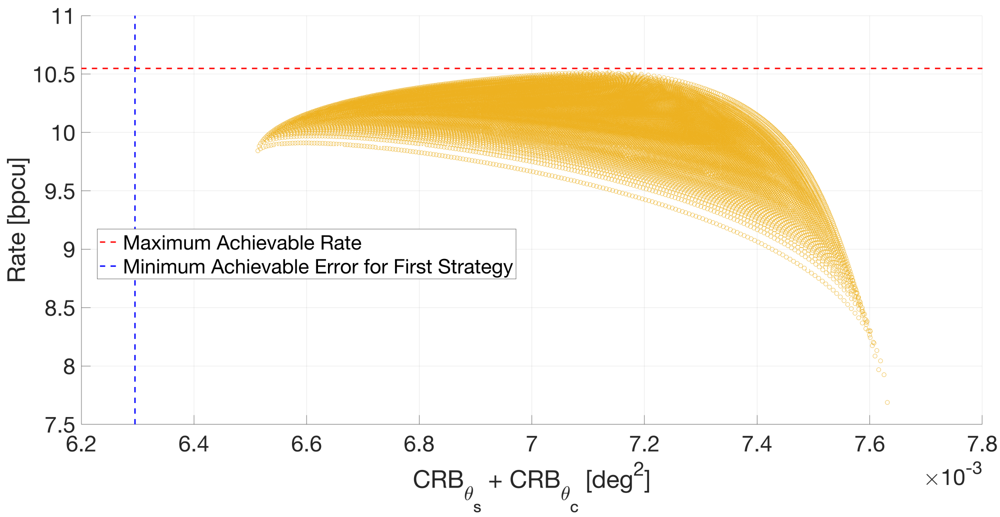

Figure 6.9: First Strategy, Second Direction Choice, Total BCRB-Rate Theoretical Points.

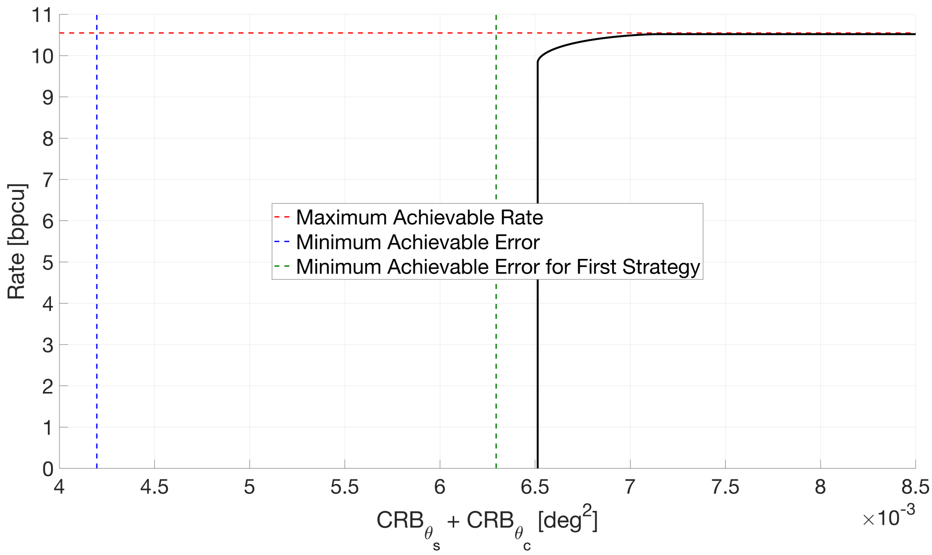

Figure 6.10: First Strategy, Second Direction Choice, Total BCRB-Rate Outer Bound.

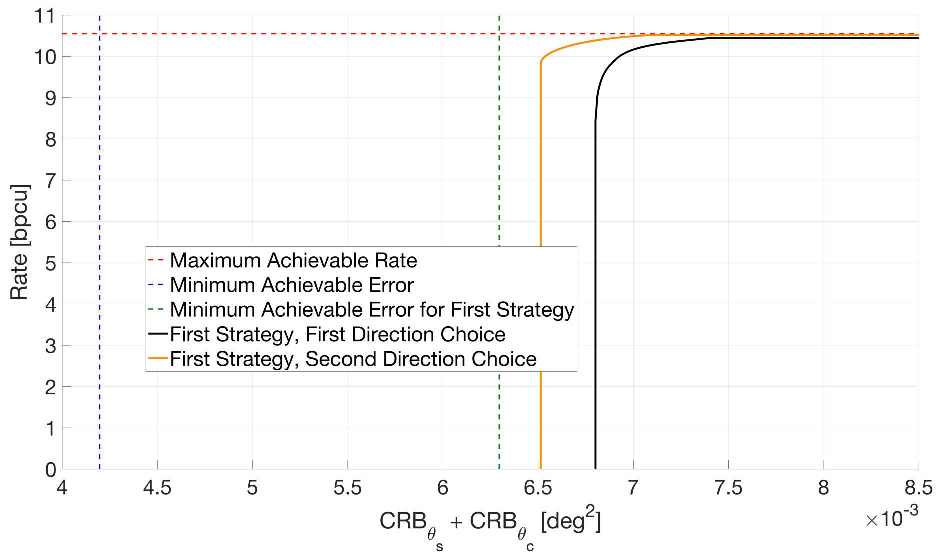

Figure 6.11: First Strategy, Comparison of Direction Choice, Total BCRB-Rate Outer Bound.

in 6.12 that by using the eigenvectors of $\overline{\mathbf{M}}_1$ and $\overline{\mathbf{M}}_2$ it is possible to reach the optimal values for the estimation of each of the angles by themselves. However, as in the previous strategy, having points in the low-left region would be desirable if focusing on minimizing the joint estimation of the angles.

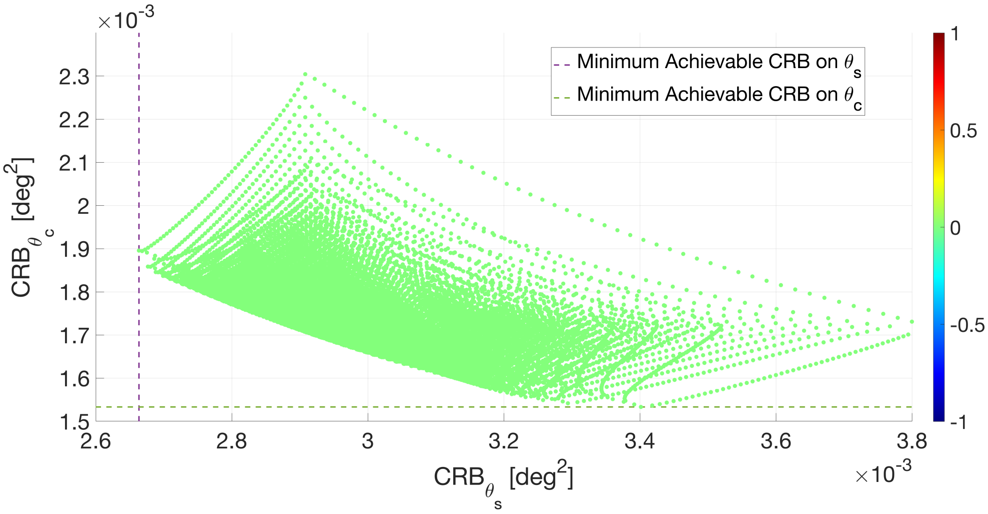

Figure 6.12: Second Strategy, First Direction Choice, BCRB-Rate Simulated Points.

6.2.2 Choice 2: Spanning among the main eigenvector of $\overline{\mathbf{M}}_1$, the main eigenvector of $\overline{\mathbf{M}}_2$ and the Directions of Optimal Sensing

As per the results for the previous strategy, it is possible to see in this scenario as well how by using the optimal directions found via CVX, it is possible to fill the plot's lower-left portion in 6.13.

In this case, since we are also spanning among the eigenvectors of $\overline{\mathbf{M}}_1$, we reach the minimum estimation error for θ_1, which was not reached in 6.7 as we were spanning on the optimal directions or the joint sensing only.

6.3 Strategy 3: Both One Information Carrying Beam and One Information-less Beam

This strategy combines the things we could notice from the previous two. Without sending Gaussian i.i.d. random variables (information-less beam), we can achieve

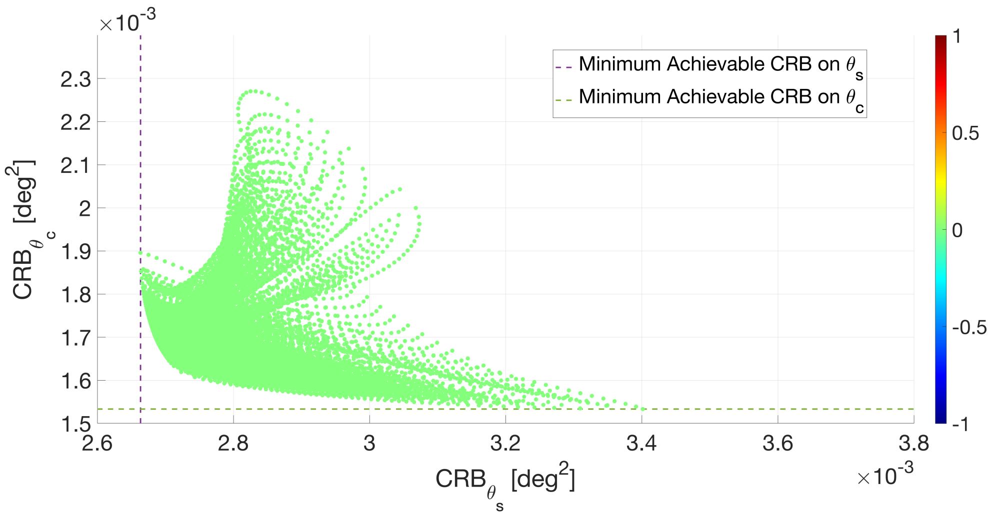

Figure 6.13: Second Strategy, Second Direction Choice, BCRB-Rate Simulated Points.

the minimum BCRB point, while we need the information carrying beam to achieve the highest possible rate.

6.3.1 Choice 1: Spanning among the main eigenvector of $\overline{\mathbf{M}}_1$, the main eigenvector of $\overline{\mathbf{M}}_2$ and the Direction of Optimal Communication

As expected, in 6.14, we can see that by using the eigenvectors of $\overline{\mathbf{M}}_1$ and $\overline{\mathbf{M}}_2$, we can achieve the lowest error on the single angle estimation of θ_1 or θ_2. Using this strategy, we can span between points that optimize the single angle estimation and points that optimize the communication rate. This could provide a valid strategy for using ISAC in this two-angle scenario by using proper power allocation between the beams if interested in minimizing the single-angle estimation error.

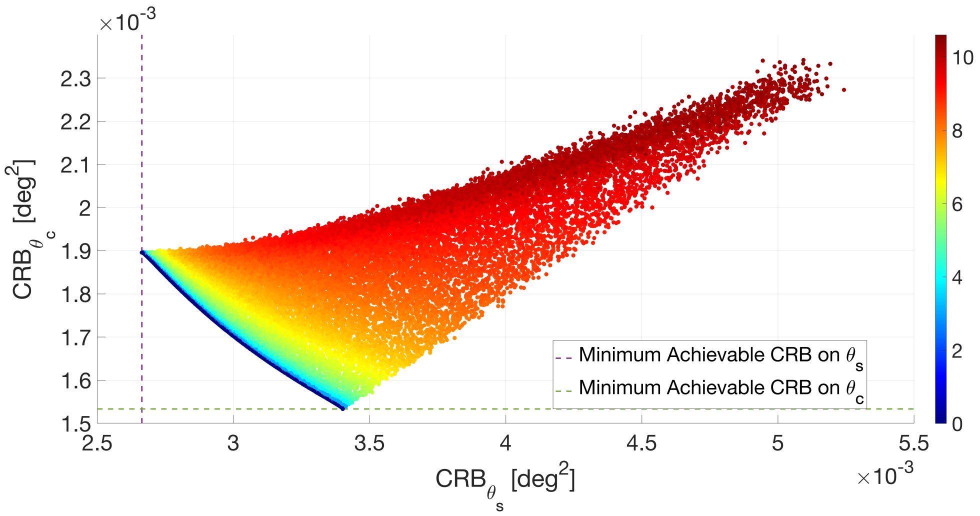

Figure 6.14: Third Strategy, First Direction Choice, BCRB-Rate Simulated Points.

6.3.2 Choice 2: Spanning among the Directions of the Optimal Sensing and the Direction of Optimal Communication

Using the optimal directions for the joint estimation, it is possible to have more points in the low-left region in 6.17 compared to the previous choice (in 6.14). This leads to an overall lower error when considering the joint estimation of the angles as in 6.18. However, we again highlight that the minimum points for the single angle estimations have not been reached.

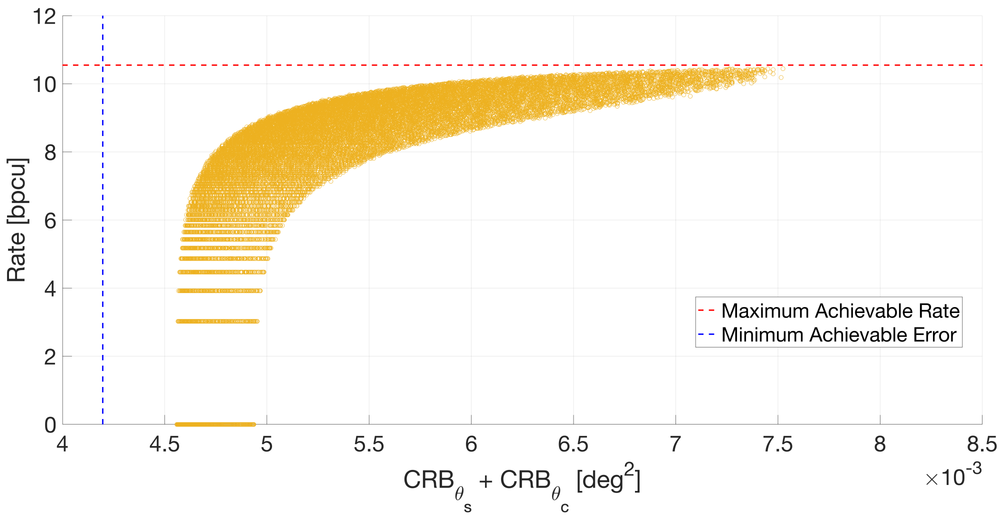

Figure 6.15: Third Strategy, First Direction Choice, Total BCRB-Rate Simulated Points.

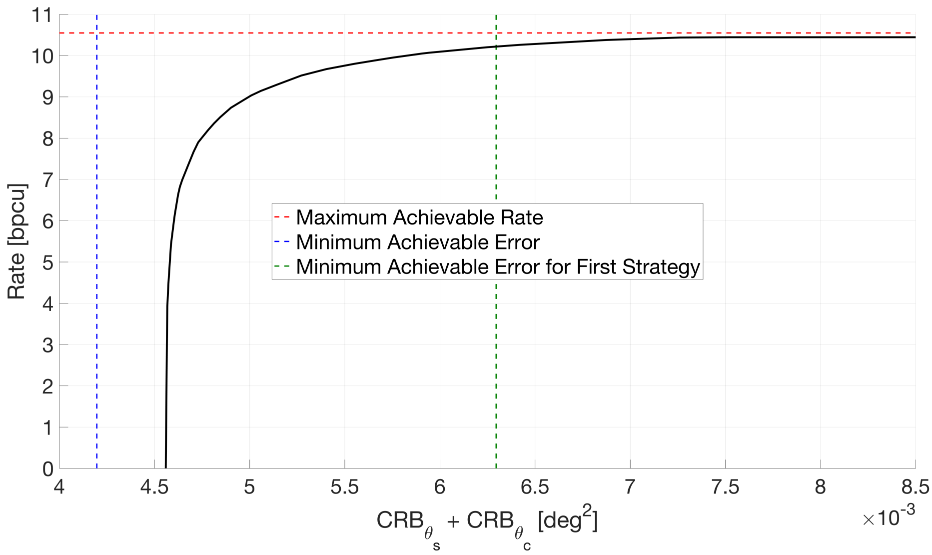

Figure 6.16: Third Strategy, First Direction Choice, Total BCRB-Rate Outer Bound.

This could provide a valid strategy for using ISAC in this two-angle scenario, using proper power allocation between the beams if we want to minimize the joint angle estimation error.

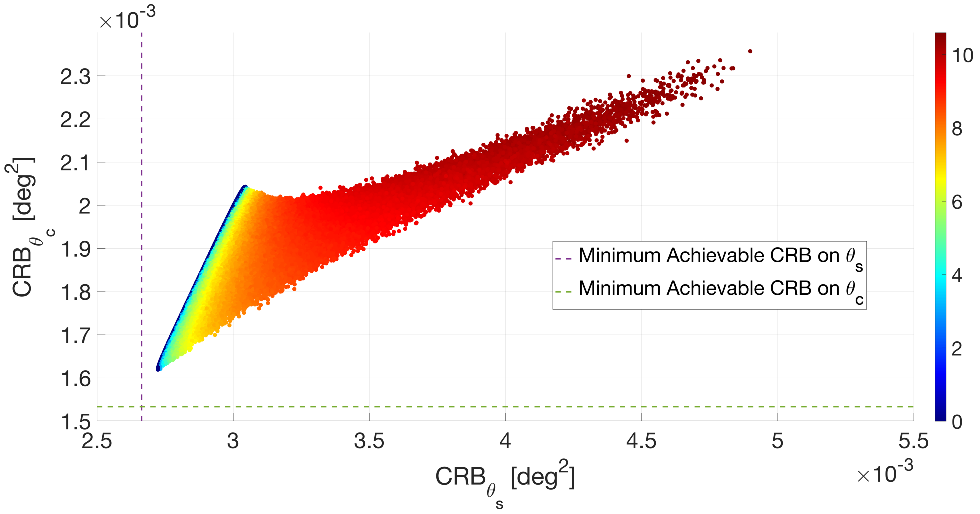

Figure 6.17: Third Strategy, Second Direction Choice, BCRB-Rate Simulated Points.

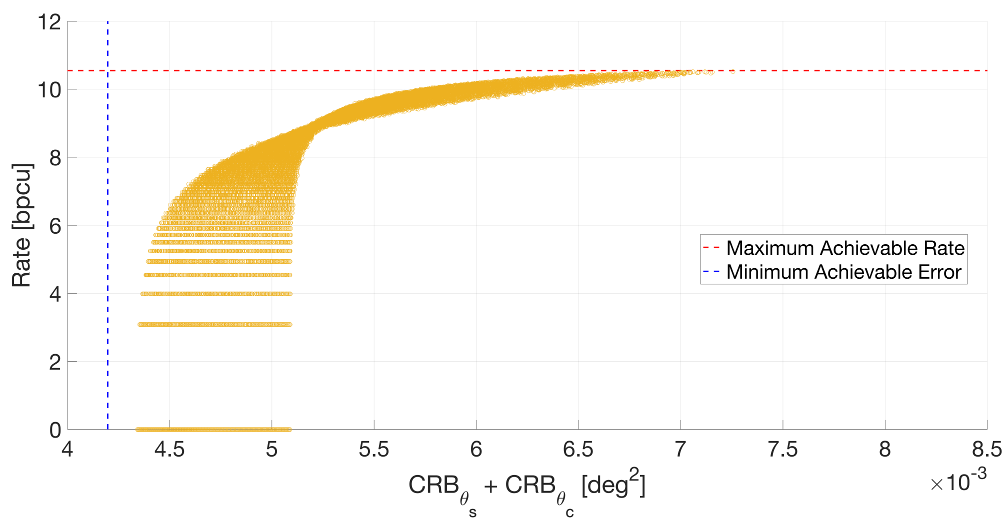

Figure 6.18: Third Strategy, Second Direction Choice, Total BCRB-Rate Simulated Points.

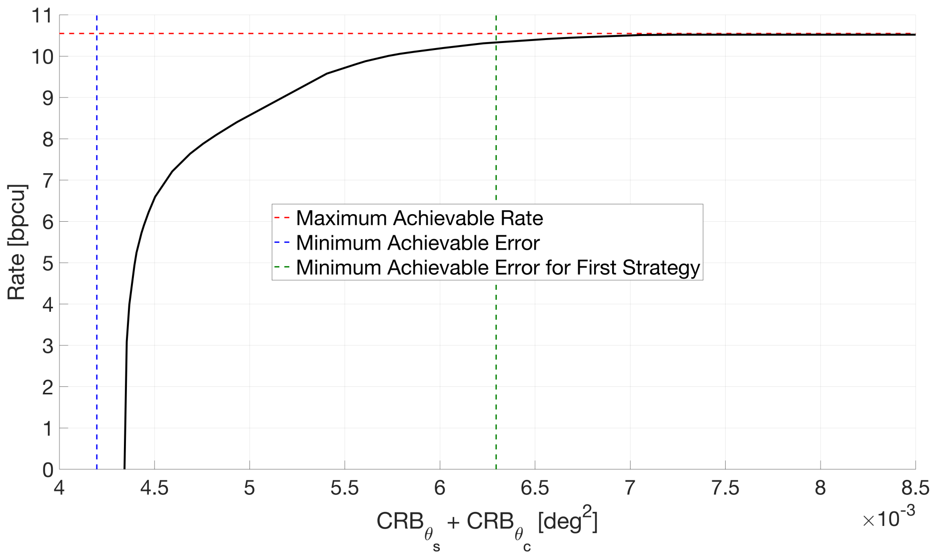

Figure 6.19: Third Strategy, Second Direction Choice, Total BCRB-Rate Outer Bound.

6.3.3 Choice 3: Spanning among the main eigenvector of $\overline{\mathbf{M}}_1$, the main eigenvector of $\overline{\mathbf{M}}_2$, the Directions of the Optimal Sensing, and the Direction of Optimal Communication

Finally, we combine the previous choices of directions seen in this strategy to obtain what we expect to be the more comprehensive inner bound for ISAC performance (in terms of BCRB-Rate tradeoff) in this scenario.

In 6.20, we can appreciate that the optimal points for single angle estimation and points focusing on the joint estimation are now reached.

It is key to compare in 6.23 the difference among the different direction choices while using this transmission strategy. These results provide a more comprehensive inner bound for the considered ISAC scenario.

Finally, it is possible to compare in 6.24 the overall comparison between the choices presented for the first strategy and those presented for the third one, further validating what has been discussed and commented on.

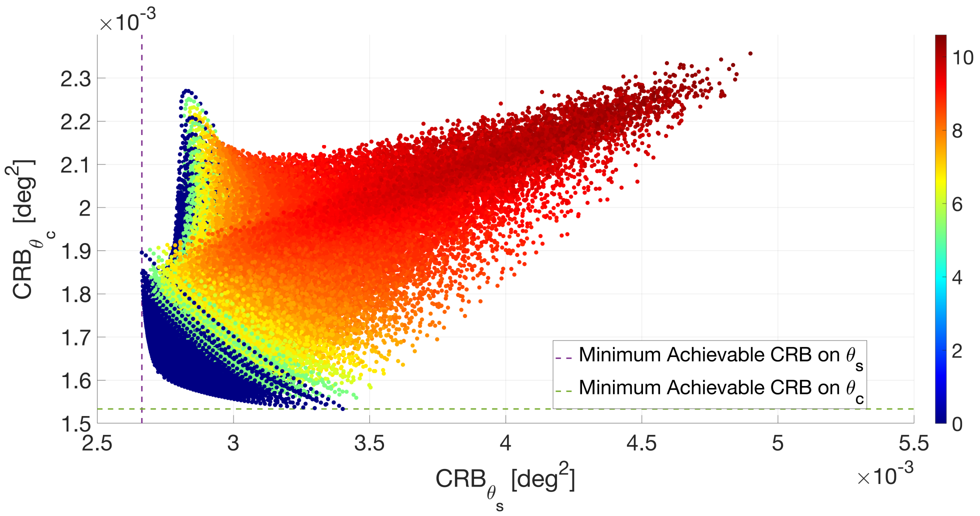

Figure 6.20: Third Strategy, Third Direction Choice, BCRB-Rate Simulated Points.

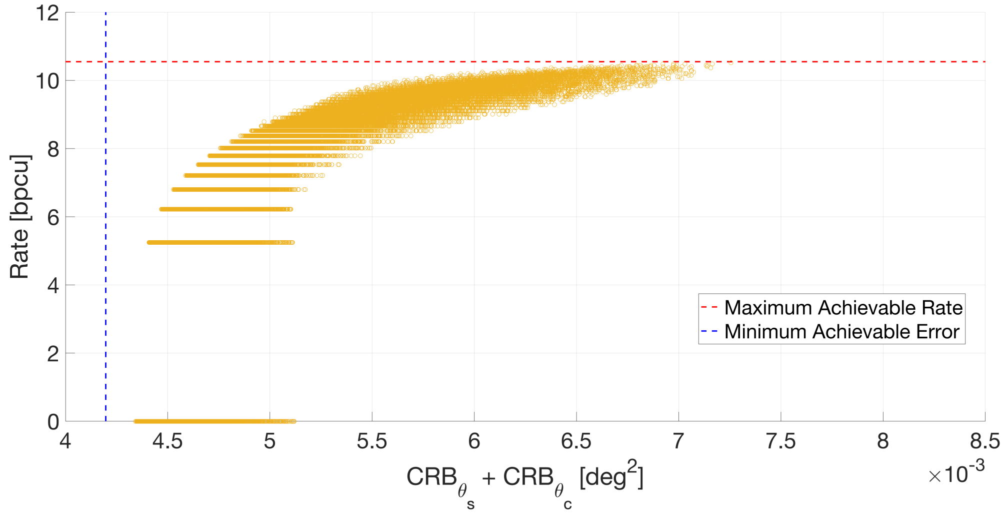

Figure 6.21: Third Strategy, Third Direction Choice, Total BCRB-Rate Simulated Points.

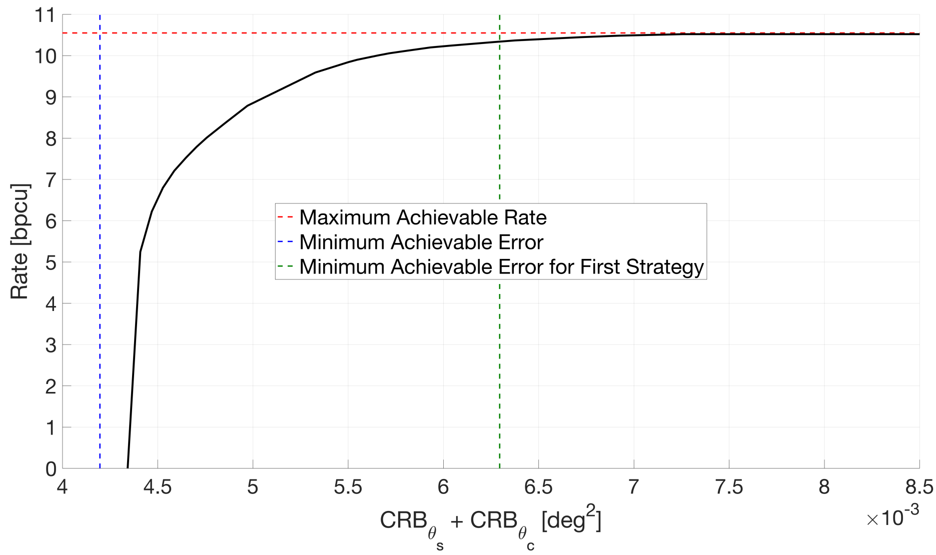

Figure 6.22: Third Strategy, Third Direction Choice, Total BCRB-Rate Outer Bound.

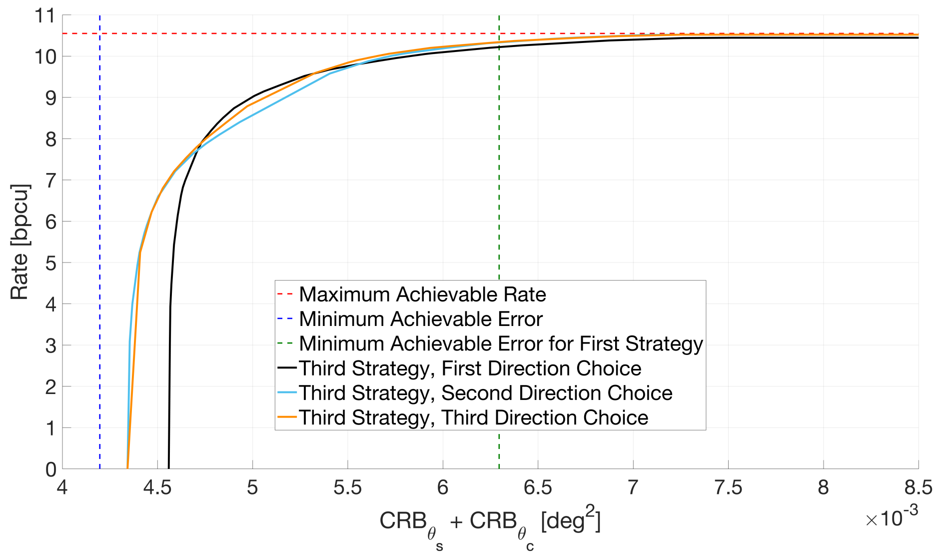

Figure 6.23: Third Strategy, Comparison of Direction Choice, Total BCRB-Rate Outer Bound.

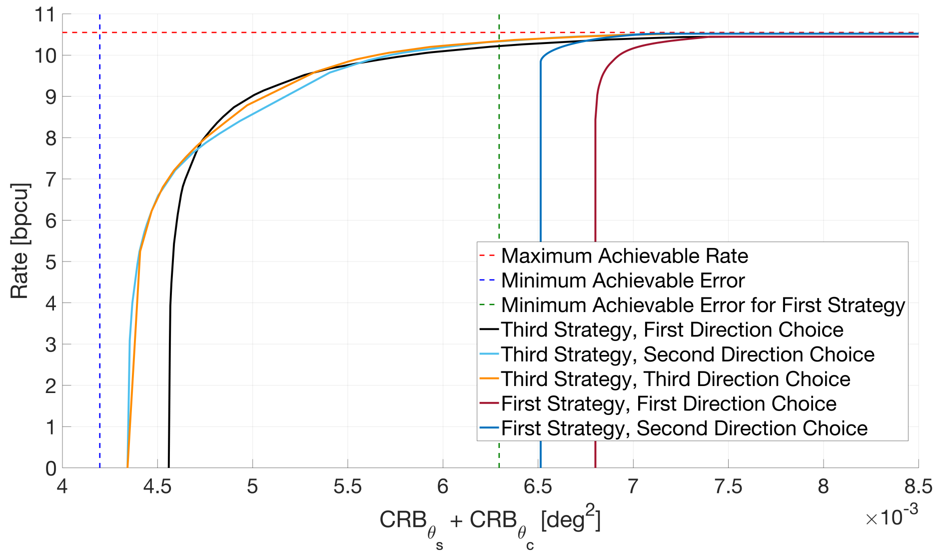

Figure 6.24: First and Third Strategy, Comparison of Direction Choice, Total BCRB-Rate Outer Bound.

Chapter 7

Conclusion

In this dissertation, we have conducted a comprehensive analysis of the ISAC tradeoff in a practical scenario involving the joint estimation of two targets, where one target is also an end-user for communication. This scenario represents a crucial step towards generalizing ISAC systems to more complex environments with multiple targets.

This work contributes by deriving the Bayesian Cramer-Rao Bound (BCRB) for the joint estimation of two angles of arrival representing the sensing and communication targets. Under certain assumptions, we demonstrated that the BCRB can be reduced to a sum of single-target estimation bounds, simplifying the analysis. Furthermore, we employed convex optimization techniques (CVX) to obtain the optimal sensing solution.

We have characterized the outer bound for the BCRB-Rate region, which describes the fundamental tradeoff between sensing accuracy and communication rate in ISAC systems. This was achieved by identifying the optimal sensing and communication points, providing valuable insights into the limits of ISAC performance.

We explored several achievable inner bounds through different transmission strategies to further understand and characterize these limits. These strategies involved one or two beams aimed at the sensing direction, communication direction, or a combination thereof. We identified a promising transmission strategy that allows achieving the optimal sensing and communication points based on the power allocation, effectively balancing the sensing and communication requirements.

This work represents a significant step forward in understanding and fully characterizing the ISAC tradeoff, a critical aspect of exploiting the full potential of this emerging technology. The theoretical foundations established in this dissertation pave the way for more efficient and effective integration of sensing and communication in future wireless networks, enabling a wide range of applications that demand seamless convergence of these functionalities.

Looking ahead, the insights and methodologies developed in this research can serve as a foundation for further exploration and generalization to more complex ISAC scenarios, ultimately driving the development of innovative solutions that push the boundaries of what is achievable in integrated sensing and communication systems.

Appendix A

Matlab Code

```
clc
close all
clear all
format longG

%% Simulation Parameters
theta_1 = deg2rad(30);        % Sensing target
theta_2 = deg2rad(42);        % Communication target
theta_1_var = deg2rad(5)^2;
theta_2_var = deg2rad(5)^2;

optimize_with_solver = false;
number_vectors_span_over = 4;    % Remember to change granularity too
granularity = 1/40;             % Step size for discretization
transmission_strategy = 'third';     % Select 'first', 'second', or '
    third'

n_sim_theta_s = 1e4;
n_sim_theta_c = 1e2;
n_sim_gauss = 1e4;

N_T = 10;
k = 2*pi;
d_lambda = 0.5;
T = 3;
SNR_s = 20;                % maximum SNR of sensing RX in dB/antenna
SNR_c = 33;                % maximum SNR of communication RX in dB/
    antenna

save_figures = 0;
name_path = 'ResultsJune5'; % specify your path here
```

```
sim_type = '1_1_strategy'; % specify your simulation type string here

%% Plotting Parameters
FontAxis = 25;
FontSizenum = 25;
FontTitle = 25;
if isequal(transmission_strategy, 'first')
    transmission_description = 'One Direction with Gaussian, e.g. \bf X\rm = \surd{P_2}[\bf c\rm g_1,\bf c\rm g_2, ..., \bf c\rm g_T]';
elseif isequal(transmission_strategy, 'second')
    transmission_description = 'One Direction without Gaussian, e.g. \bf X\rm = \surd{P_1}[\bf s\rm,\bf s\rm, ..., \bf s\rm]';
elseif isequal(transmission_strategy, 'third')
    transmission_description = 'One Direction with Gaussian and One Without, e.g. \bf X\rm = [\surd{P_1}\bf s\rm + \surd{P_2}\bf c\rm g_1, ..., \surd{P_1}\bf s\rm + \surd{P_2}\bf c\rm g_T]';
else
    error('Incorrect Choice of Transmission Strategy')
end

num_fig = 0;
fig_names = {};

%% Precomputation of Useful Variables
rng(665093593)              % Fixing seed
zero_tol = 1e-12;           % To check computations with Matlab's precision
N_T_vec = -(N_T-1)/2:(N_T-1)/2;
SNR_s_lin = db2pow(SNR_s);
SNR_c_lin = db2pow(SNR_c);

if isequal(transmission_strategy, 'second')
    n_sim_theta_c = 1;
    n_sim_gauss = 1;
end

%% Initialization of Angle's Distributions and Priors
k_VonMises_1 = double(find_k_VonMises(theta_1_var))*2;
theta_1_vec = circ_vmrnd(theta_1, k_VonMises_1, [1, n_sim_theta_s]);
I_0 = besseli(0,k_VonMises_1);
I_1 = besseli(1,k_VonMises_1);
J_theta_P_1 = (1-I_1^2/I_0^2)^(-1);

k_VonMises_2 = double(find_k_VonMises(theta_2_var))*2;
theta_2_vec = circ_vmrnd(theta_2, k_VonMises_2, [1, n_sim_theta_s]);
theta_c_vec = circ_vmrnd(theta_2, k_VonMises_2, [1, n_sim_theta_c]);
I_0 = besseli(0,k_VonMises_2);
I_1 = besseli(1,k_VonMises_2);
J_theta_P_2 = (1-I_1^2/I_0^2)^(-1);
```

```
J_theta_P = [J_theta_P_1, 0;
                   0, J_theta_P_2];

%% Communication Steering Vector
h_c = zeros(N_T, n_sim_theta_c);
for ind_theta_c = 1:n_sim_theta_c
    h_c(:,ind_theta_c) = exp(1i * N_T_vec * k * d_lambda * cos(
    theta_c_vec(ind_theta_c))).';    % rx steering vector (comm)
end
h_c_true = exp(1i * N_T_vec * k * d_lambda * cos(theta_2)).';
h_c_true_hat = h_c_true/norm(h_c_true);
max_rate_theo = find_Rate(1, h_c_true_hat, h_c, n_sim_theta_c,
    SNR_c_lin);

%% Sensing Steering Vectors
M_1 = zeros(N_T, N_T, n_sim_theta_s);
M_2 = zeros(N_T, N_T, n_sim_theta_s);

for iter = 1:n_sim_theta_s
    a_1 = exp(-1i * N_T_vec * k * d_lambda * cos(theta_1_vec(iter)))
    .';    % tx steering vector
    a_2 = exp(-1i * N_T_vec * k * d_lambda * cos(theta_2_vec(iter)))
    .';    % rx steering vector (comm)
    a_1_dot = a_1 .* (1i * N_T_vec * k * d_lambda * sin(theta_1_vec(
    iter))).';
    a_2_dot = a_2 .* (1i * N_T_vec * k * d_lambda * sin(theta_2_vec(
    iter))).';

    M_1(:, :, iter) = norm(a_1_dot)^2 * (a_1 * a_1') ...
                + norm(a_1)^2 * (a_1_dot * a_1_dot');
    M_2(:, :, iter) = norm(a_2_dot)^2 * (a_2 * a_2') ...
                + norm(a_2)^2 * (a_2_dot * a_2_dot');

    if n_sim_theta_s == 1
        a_1 = exp(-1i * N_T_vec * k * d_lambda * cos(theta_1)).';
    % tx steering vector
        a_2 = exp(-1i * N_T_vec * k * d_lambda * cos(theta_2)).';
    % rx steering vector (comm)
        a_1_dot = a_1 .* (1i * N_T_vec * k * d_lambda * sin(theta_1))
    .';
        a_2_dot = a_2 .* (1i * N_T_vec * k * d_lambda * sin(theta_2))
    .';

        M_1 = norm(a_1_dot)^2 * (a_1 * a_1') ...
            + norm(a_1)^2 * (a_1_dot * a_1_dot');
        M_2 = norm(a_2_dot)^2 * (a_2 * a_2') ...
            + norm(a_2)^2 * (a_2_dot * a_2_dot');
    end
```

```
end

M_1_bar = mean(conj(M_1), 3);
M_2_bar = mean(conj(M_2), 3);

[V_M_1_bar, D_M_1_bar] = eig(M_1_bar);
[V_M_2_bar, D_M_2_bar] = eig(M_2_bar);

v_1_hat = V_M_1_bar(:, N_T-1);
v_2_hat = V_M_1_bar(:, N_T);

v_3_hat = V_M_2_bar(:, N_T-1);
v_4_hat = V_M_2_bar(:, N_T);

if optimize_with_solver
    R_x_opt = R_x_opt_solver(N_T, T, SNR_s_lin, J_theta_P, M_1_bar,
    M_2_bar);
    [V_opt, D_opt] = eig(R_x_opt);
    v_1_opt = V_opt(:, N_T);
    v_2_opt = V_opt(:, N_T-1);
end

%% Finding theoretical boundaries
min_eps_1_theo = rad2deg(rad2deg(find_CRB_theo(v_2_hat, v_2_hat,
    M_1_bar, J_theta_P(1,1), SNR_s_lin, T, N_T, 'second')));
min_eps_2_theo = rad2deg(rad2deg(find_CRB_theo(v_4_hat, v_4_hat,
    M_2_bar, J_theta_P(2,2), SNR_s_lin, T, N_T, 'second')));
tot_min_eps_theo = min_eps_1_theo + min_eps_2_theo;

first_eps_1_theo = rad2deg(rad2deg(find_CRB_theo(v_2_hat, v_2_hat,
    M_1_bar, J_theta_P(1,1), SNR_s_lin, T, N_T, 'first')));
first_eps_2_theo = rad2deg(rad2deg(find_CRB_theo(v_4_hat, v_4_hat,
    M_2_bar, J_theta_P(2,2), SNR_s_lin, T, N_T, 'first')));
tot_first_eps_theo = first_eps_1_theo + first_eps_2_theo;

%% Simulation
Lambdas = generateCombinations(number_vectors_span_over, granularity)
    ;
[row_Lambdas, ~] = size(Lambdas);
eps_1_theo = zeros(1, row_Lambdas);
eps_2_theo = zeros(1, row_Lambdas);
eps_1_avg_gauss = zeros(1, row_Lambdas);
eps_2_avg_gauss = zeros(1, row_Lambdas);
rate = zeros(1, row_Lambdas);

for ind_l = 1:row_Lambdas
    lambda = Lambdas(ind_l,:);

    r_1_tx = zeros(N_T,1);
```

```
    r_2_tx = zeros(N_T,1) + sqrt(lambda(1)) * v_1_hat + sqrt(lambda(2)) * v_2_hat + sqrt(lambda(3)) * v_3_hat + sqrt(lambda(4)) * v_4_hat;
    p_1_tx = 0;
    p_2_tx = lambda(1) + lambda(2) + lambda(3) + lambda(4);

    if isequal(transmission_strategy, 'first') && p_1_tx~=0
        error('Inconsistent choice of power with transmission strategy. With first strategy p_1_tx should always be 0.')
    elseif isequal(transmission_strategy, 'second') && p_2_tx~=0
        error('Inconsistent choice of power with transmission strategy. With second strategy p_2_tx should always be 0.')
    end

    r_1_tx_hat = normalized(r_1_tx);
    r_2_tx_hat = normalized(r_2_tx);
    if isnan(r_1_tx_hat)
        r_1_tx_hat = zeros(N_T, 1);
    elseif isnan(r_2_tx_hat)
        r_2_tx_hat = zeros(N_T,1);
    end

    [err_1_theo, err_2_theo, err_1_avg_gauss, err_2_avg_gauss, rate_out] = myfun(p_1_tx, p_2_tx, r_1_tx_hat, r_2_tx_hat,...
                        M_1_bar, M_2_bar, T, N_T, SNR_s_lin, J_theta_P, n_sim_gauss, h_c, n_sim_theta_c, SNR_c_lin, transmission_strategy);

    eps_1_theo(ind_l) = err_1_theo;
    eps_2_theo(ind_l) = err_2_theo;
    eps_1_avg_gauss(ind_l) = err_1_avg_gauss;
    eps_2_avg_gauss(ind_l) = err_2_avg_gauss;
    rate(ind_l) = rate_out;
end

%% Finding minimum error and maximum rate points
tot_eps_gauss = eps_1_avg_gauss + eps_2_avg_gauss;
vec_eps_theo_2D = linspace(min(tot_eps_theo), max(tot_eps_theo), 2);
vec_eps_1_theo_3D = linspace(min(eps_1_theo), max(eps_1_theo), 2);
vec_eps_2_theo_3D = linspace(min(eps_2_theo), max(eps_2_theo), 2);
vec_eps_gauss_2D = linspace(min(tot_eps_gauss), max(tot_eps_gauss), 2);
vec_eps_1_gauss_3D = linspace(min(eps_1_avg_gauss), max(eps_1_avg_gauss), 2);
vec_eps_2_gauss_3D = linspace(min(eps_2_avg_gauss), max(eps_2_avg_gauss), 2);
vec_rate = linspace(min(rate), max(rate), 2);

%% Plotting 2D Thoretical
```

```
if isequal(transmission_strategy, 'first')
    num_fig = num_fig + 1;
    fig_names{end + 1} = 'BCRB_tot-Rate_Theo';
    figure
    hold on
    plot(vec_eps_theo_2D, max_rate_theo*ones(1,2), 'r--', LineWidth=2)
    plot(tot_min_eps_theo*ones(1,2), vec_rate, 'b--', LineWidth=2)
    scatter(eps_1_theo + eps_2_theo, rate)
    grid on
    set (gca, 'FontSize', FontAxis);
    title(['Total BCRB-Rate Theoretical Points for \theta_s = ', num2str(round(rad2deg(theta_1))) ,...
            '^o and \theta_c = ', num2str(round(rad2deg(theta_2))), '^o, T = ', num2str(T)], 'FontSize',FontTitle)
    subtitle(['Transmission Strategy: ', transmission_description], 'FontSize', FontTitle-2)

    xlabel('CRB_{\theta_1} + CRB_{\theta_2} [deg^2]', 'FontSize', FontSizenum)
    ylabel('Rate [bpcu]', 'FontSize', FontSizenum)
end
%% Plotting 2D Gaussian
if ~isequal(transmission_strategy, 'second')
    num_fig = num_fig + 1;
    fig_names{end + 1} = 'BCRB_tot-Rate_Gauss';
    figure
    hold on
    plot([tot_min_eps_theo-0.001, tot_min_eps_theo+0.01], max_rate_theo*ones(1,2), 'r--', LineWidth=2)
    plot(tot_min_eps_theo*ones(1,2), [-1,max_rate_theo+5], 'b--', LineWidth=2)
    scatter(eps_1_avg_gauss + eps_2_avg_gauss, rate)
    grid on
    legend('Maximum Achievable Rate', 'Minimum Achievable Error', 'Location', 'best')
    set (gca, 'FontSize', FontAxis);
    title(['Total BCRB-Rate Simulated Points for \theta_s = ', num2str(round(rad2deg(theta_1))) ,...
            '^o and \theta_c = ', num2str(round(rad2deg(theta_2))), '^o, T = ', num2str(T)], 'FontSize',FontTitle)
    subtitle(['Transmission Strategy: ', transmission_description], 'FontSize', FontTitle-2)

    xlabel('CRB_{\theta_1} + CRB_{\theta_2} [deg^2]', 'FontSize', FontSizenum)
    ylabel('Rate [bpcu]', 'FontSize', FontSizenum)
    xlim([4, 8.5]*1e-3)
    ylim([0,12])
```

```
end
%% Plotting 3D Theoretical with Colors
if ~isequal(transmission_strategy, 'third')
    num_fig = num_fig + 1;
    fig_names{end + 1} = 'BCRB-Rate_Theo';
    figure
    hold on
    plot(vec_eps_1_theo_3D, min_eps_2_theo*ones(1,2), '--', 'color',
    [0.1, 0.1, 0.1], LineWidth=2)
    plot(min_eps_1_theo*ones(1,2), vec_eps_2_theo_3D, '--', 'color',
    [0.5,0.5,0.5], LineWidth=2)
    scatter(eps_1_theo, eps_2_theo, [], rate, 'filled')
    colormap(jet)
    colorbar
    grid on
    set (gca, 'FontSize', FontAxis);
    title(['BCRB-Rate Theoretical Points for \theta_s = ', num2str(
    round(rad2deg(theta_1))) ,...
            '^o and \theta_c = ', num2str(round(rad2deg(theta_2))), '
    ^o, T = ', num2str(T)], 'FontSize',FontTitle)
    subtitle(['Transmission Strategy: ', transmission_description], '
    FontSize', FontTitle-2)

    xlabel('CRB_{\theta_1} [deg^2]', 'FontSize', FontSizenum)
    ylabel('CRB_{\theta_2} [deg^2]', 'FontSize', FontSizenum)
    zlabel('Rate [bpcu]', 'FontSize', FontSizenum)
end
%% Plotting 3D Gaussian with Colors
num_fig = num_fig + 1;
fig_names{end + 1} = 'BCRB-Rate_Gauss';
figure
hold on
plot(vec_eps_1_gauss_3D, min_eps_2_theo*ones(1,2), '--', 'color',
    [0.1, 0.1, 0.1], LineWidth=2)
plot(min_eps_1_theo*ones(1,2), vec_eps_2_gauss_3D, '--', 'color',
    [0.5,0.5,0.5], LineWidth=2)
scatter(eps_1_avg_gauss, eps_2_avg_gauss, [], rate, 'filled')
colormap(jet)
colorbar
grid on
set (gca, 'FontSize', FontAxis);
title(['BCRB-Rate Simulated Points for \theta_s = ', num2str(round(
    rad2deg(theta_1))) ,...
        '^o and \theta_c = ', num2str(round(rad2deg(theta_2))), '^o,
    T = ', num2str(T)], 'FontSize',FontTitle)
subtitle(['Transmission Strategy: ', transmission_description], '
    FontSize', FontTitle-2)
% caxis([6.2, 10.8])
% xlim([min(eps_1_theo)-0.3 * 1e-3, max(eps_1_theo)+0.3 * 1e-3])
```

```
% ylim([min(eps_2_theo)-0.3 * 1e-3, max(eps_2_theo)+0.3 * 1e-3])

xlabel('CRB_{\theta_1} [deg^2]', 'FontSize', FontSizenum)
ylabel('CRB_{\theta_2} [deg^2]', 'FontSize', FontSizenum)
zlabel('Rate [bpcu]', 'FontSize', FontSizenum)

%% Plotting Outer Bound
if ~isequal(transmission_strategy, 'second')
    if isequal(transmission_strategy, 'first')
        x = tot_eps_theo;
    else
        x = tot_eps_gauss;
    end
    y = rate;

    % Combine x and y into a matrix of points
    points = [x(:), y(:)];
    % Compute the convex hull
    hullIndices = convhull(points(:,1), points(:,2));
    % Extract the hull points
    hullPoints = points(hullIndices, :);
    % Get the left and top side points
    boundaryPoints = getLeftTopSidePoints(hullPoints);

    figure
    num_fig = num_fig + 1;
    fig_names{end + 1} = 'Outer_Bound';
    hold on
    plot([tot_min_eps_theo-0.01, tot_min_eps_theo+0.01], max_rate_theo*ones(1,2), 'r--', LineWidth=2)
    plot(tot_min_eps_theo*ones(1,2), [-1,max_rate_theo+5], 'b--', LineWidth=2)
    plot(boundaryPoints(:,1), boundaryPoints(:,2), 'k-', LineWidth=3)
    grid on
    legend('Maximum Achievable Rate', 'Minimum Achievable Error', 'Location', 'best')
    set (gca, 'FontSize', FontAxis);
    title(['Total BCRB-Rate Simulated Bound for \theta_s = ', num2str(round(rad2deg(theta_1))) ,...
          '^o and \theta_c = ', num2str(round(rad2deg(theta_2))), '^o, T = ', num2str(T)], 'FontSize',FontTitle)
    subtitle(['Transmission Strategy: ', transmission_description], 'FontSize', FontTitle-2)

    xlabel('CRB_{\theta_1} + CRB_{\theta_2} [deg^2]', 'FontSize', FontSizenum)
    ylabel('Rate [bpcu]', 'FontSize', FontSizenum)
    xlim([4, 8.5]*1e-3)
    ylim([0,12])
```

```
end
%% Save figures
if save_figures
    % Loop over each figure
    for i = 1:num_fig
        % Create a figure if it doesn't exist (comment this line if figures already exist)
        figure(i); % this line is just to create sample figures

        % Resize the figure to full screen or a specific size
        set(gcf, 'Units', 'normalized', 'OuterPosition', [0 0 1 1]);

        % Generate the full path for the figure
        file_name = strcat(name_path, '/', fig_names{i}, '/', sim_type);

        savefig(gcf, [file_name '.fig'])
        print(gcf, file_name, ['-d', 'png'], '-r300');

    end

end

%% Functions
function [err_1_theo, err_2_theo, err_1_avg_gauss, err_2_avg_gauss, rate_out] = myfun(p_1_tx, p_2_tx, r_1_tx, r_2_tx,...
                    M_1_bar, M_2_bar, T, N_T, SNR_s_lin, J_theta_P, n_sim_gauss, h_c, n_sim_theta_c, SNR_c_lin, option_theo)
    [err_1_theo] = find_CRB_theo(r_1_tx, r_2_tx, M_1_bar, J_theta_P(1,1), SNR_s_lin, T, N_T, option_theo);
    [err_2_theo] = find_CRB_theo(r_1_tx, r_2_tx, M_2_bar, J_theta_P(2,2), SNR_s_lin, T, N_T, option_theo);

    [err_1_avg_gauss] = find_CRB_avg(p_1_tx, p_2_tx, r_1_tx, r_2_tx, M_1_bar, J_theta_P(1,1), SNR_s_lin, T, N_T, n_sim_gauss);
    [err_2_avg_gauss] = find_CRB_avg(p_1_tx, p_2_tx, r_1_tx, r_2_tx, M_2_bar, J_theta_P(2,2), SNR_s_lin, T, N_T, n_sim_gauss);

    rate_out = find_Rate(p_2_tx, r_2_tx, h_c, n_sim_theta_c, SNR_c_lin);

    err_1_theo = rad2deg(rad2deg(err_1_theo));
    err_2_theo = rad2deg(rad2deg(err_2_theo));

    err_1_avg_gauss = rad2deg(rad2deg(err_1_avg_gauss));
    err_2_avg_gauss = rad2deg(rad2deg(err_2_avg_gauss));
end
```

```
function [err_theo] = find_CRB_theo(r_1_tx, r_2_tx, M_bar, J_theta,
    SNR_s_lin, T, N_T, option_theo)
     if isequal(option_theo, 'first')
         R_x = r_2_tx * r_2_tx';
         corr_term = 0;
         psi = real(J_theta * (2*SNR_s_lin*trace(M_bar*R_x))^(-1));
         if T==1
              err_theo = real( (SNR_s_lin * trace(M_bar * R_x))^(-1) *
    psi^(T-1) * exp(psi) * gammainc(psi, 1-T) );
         else
              for n=1:T-2
                   prod_denom = 1;
                   for i=1:n
                        prod_denom = prod_denom * (T-i-1);
                   end
                   new_term = ((-1)^n * psi^n) / (prod_denom) + (-1)^(T
    -1)* (exp(psi)*psi^(T-1)*gammainc(psi, 0)) / (gamma(T-1));
                   corr_term = corr_term + new_term;
              end
              err_theo = real(( 2 * (T-1) * SNR_s_lin * trace(M_bar *
    R_x))^(-1) * (1+corr_term));
         end
     elseif isequal(option_theo, 'second')
         R_x = r_1_tx * r_1_tx';
         err_theo = real((2 * T * SNR_s_lin * trace(M_bar * R_x) +
    J_theta)^(-1));
     elseif isequal(option_theo, 'third')
         R_x = r_1_tx * r_1_tx';
         err_theo = real((2 * T * SNR_s_lin * trace(M_bar * R_x) +
    J_theta)^(-1));
     end
end

function [err_gauss_avg] = find_CRB_avg(p_1_tx, p_2_tx, r_1_tx,
    r_2_tx, M_bar, J_theta, SNR_s_lin, T, N_T, n_sim_gauss)
     err_gauss = zeros(1, n_sim_gauss);
     gauss_rnd = (1/sqrt(2)) * (randn(T, n_sim_gauss) + 1i * randn(T,
    n_sim_gauss));
     for ind_gauss = 1:n_sim_gauss
         X_tx = zeros(N_T, T);
         for t=1:T
              X_tx(:,t) = sqrt(p_1_tx)*r_1_tx + sqrt(p_2_tx)*r_2_tx *
    gauss_rnd(t,ind_gauss);
         end
         R_x_gauss = 1/T * (X_tx * X_tx');

         err_gauss(ind_gauss) = real((2 * T * SNR_s_lin * trace(M_bar
    * R_x_gauss) + J_theta)^(-1));

```

```
    end
    err_gauss_avg = mean(err_gauss);
end

function rate_avg = find_Rate(p_2_tx, r_2_tx, h_c, n_sim_theta_c,
    SNR_c_lin)
    rate = zeros(1, n_sim_theta_c);
    R_x = r_2_tx * r_2_tx';
    if p_2_tx == 0
        rate_avg = 0;
    else
        for ind_theta_c = 1:n_sim_theta_c
            rate(ind_theta_c) = log2(det(1 + norm(h_c(:,ind_theta_c))
    ^(-2) * real(h_c(:,ind_theta_c)' * R_x * h_c(:,ind_theta_c)) *
    SNR_c_lin * p_2_tx));
        end
        rate_avg = mean(rate);
    end
end

function [R_x_opt] = R_x_opt_solver(N_T, T, SNR_s_lin, J_theta_P,
    M_1_bar, M_2_bar)
    % Initialize CVX
    cvx_begin sdp
        % Variable definition
        variable X(N_T, N_T) hermitian
        variable eps_1
        variable eps_2
        variable t1
        variable t2
        minimize ( eps_1 + eps_2 )

        % Constraints
        subject to
            trace(X) == 1
            X >= 0 % X must be positive semidefinite
            2 * T * SNR_s_lin * trace(M_1_bar * X) + J_theta_P(1,1)
    >= t1
            2 * T * SNR_s_lin * trace(M_2_bar * X) + J_theta_P(2,2)
    >= t2
            eps_1 >= inv_pos(t1)
            eps_2 >= inv_pos(t2)
    cvx_end

    % Optimal solution
    R_x_opt = X;
end

% Function to find the left and top side points
```

```
function boundaryPoints = getLeftTopSidePoints(hullPoints)
    % Sort the points by x-coordinate (ascending)
    sortedPoints = sortrows(hullPoints, 2);
    x = sortedPoints(:,1);
    y = sortedPoints(:,2);

    [min_x, ind_min_x] = min(x);
    [max_y, ind_max_y] = max(y);
    max_x = x(ind_max_y);
    min_y = y(ind_min_x);

    boundaryPoints = [];
    maxsize = size(sortedPoints, 1);

    % Loop through sorted points to find leftmost points
    for i = 1:maxsize
        if sortedPoints(i,1) >= min_x && sortedPoints(i,1) <= max_x && sortedPoints(i,2)>= min_y && sortedPoints(i,2)<=max_y
            if i>1
                if sortedPoints(i,2) == boundaryPoints(end,2)
                else
                    boundaryPoints = [boundaryPoints; sortedPoints(i, 1), sortedPoints(i, 2)];
                end
            else
                boundaryPoints = [boundaryPoints; sortedPoints(i, 1), sortedPoints(i, 2)];
            end
        end
    end

    % if min(y)~=0
    %     boundaryPoints = [min_x, 0; boundaryPoints];
    % end
    % if max_x < 8.5*1e-3
    %     boundaryPoints = [boundaryPoints; 8.5*1e-3, max_y];
    % end
end

function y = normalized(x)
   y = x/norm(x);
end
```

Bibliography

[1] Yifeng Xiong, Fan Liu, Yuanhao Cui, Weijie Yuan, Tony Xiao Han, and Giuseppe Caire. «On the Fundamental Tradeoff of Integrated Sensing and Communications Under Gaussian Channels». In: *IEEE Transactions on Information Theory* 69.9 (Sept. 2023), pp. 5723–5751. DOI: 10.1109/TIT.2023.3284449 (cit. on pp. 1, 4, 6, 10, 14, 16, 20–22).

[2] Harald Cramér. *Mathematical Methods of Statistics*. Princeton, NJ: Princeton University Press, 1946. ISBN: 0-691-08004-6 (cit. on p. 2).

[3] Harry L. Van Trees. *Detection, Estimation, and Modulation Theory, Part I: Detection, Estimation, and Linear Modulation Theory*. 2nd. Hoboken, NJ, USA: Wiley, 2004, p. 716. ISBN: 978-0-471-46382-5 (cit. on p. 2).

[4] Abbas El Gamal and Young-Han Kim. *Network Information Theory*. Cambridge, UK: Cambridge University Press, 2011. ISBN: 978-0521761451 (cit. on p. 3).

[5] Milton Abramowitz and Irene A. Stegun, eds. *Handbook of Mathematical Functions*. Reprinted by Dover Publications, 1965. National Bureau of Standards, 1964. ISBN: 0-486-61272-4 (cit. on p. 3).

[6] Yuanwei Liu, Zhaolin Wang, Jiaqi Xu, Chongjun Ouyang, Xidong Mu, and Robert Schober. «Near-Field Communications: A Tutorial Review». In: *IEEE Open Journal of the Communications Society* 4 (2023), pp. 1999–2049. DOI: 10.1109/OJCOMS.2023.3305583 (cit. on p. 3).

[7] An Liu et al. «A Survey on Fundamental Limits of Integrated Sensing and Communication». In: *IEEE Communications Surveys & Tutorials* 24.2 (2022), pp. 994–1034. DOI: 10.1109/COMST.2022.3149272 (cit. on pp. 5, 6).

[8] Dongning Guo, S. Shamai, and S. Verdu. «Mutual information and minimum mean-square error in Gaussian channels». In: *IEEE Transactions on Information Theory* 51.4 (2005), pp. 1261–1282. DOI: 10.1109/TIT.2005.844072 (cit. on p. 5).

[9] Fan Liu, Yuanhao Cui, Christos Masouros, Jie Xu, Tony Xiao Han, Yonina C. Eldar, and Stefano Buzzi. «Integrated Sensing and Communications: Toward Dual-Functional Wireless Networks for 6G and Beyond». In: *IEEE Journal on Selected Areas in Communications* 40.6 (2022), pp. 1728–1767. DOI: `10.1109/JSAC.2022.3156632` (cit. on p. 5).

[10] Yuanhao Cui, Fan Liu, Xiaojun Jing, and Junsheng Mu. «Integrating Sensing and Communications for Ubiquitous IoT: Applications, Trends, and Challenges». In: *IEEE Network* 35.5 (2021), pp. 158–167. DOI: `10.1109/MNET.010.2100152` (cit. on pp. 5, 6).

[11] Weijie Yuan, Fan Liu, Christos Masouros, Jinhong Yuan, Derrick Wing Kwan Ng, and Nuria González-Prelcic. «Bayesian Predictive Beamforming for Vehicular Networks: A Low-Overhead Joint Radar-Communication Approach». In: *IEEE Transactions on Wireless Communications* 20.3 (2021), pp. 1442–1456. DOI: `10.1109/TWC.2020.3033776` (cit. on p. 5).

[12] Yuan Shen, Henk Wymeersch, and Moe Z. Win. «Fundamental Limits of Wideband Localization— Part II: Cooperative Networks». In: *IEEE Transactions on Information Theory* 56.10 (2010), pp. 4981–5000. DOI: `10.1109/TIT.2010.2059720` (cit. on pp. 10, 14).

[13] Jian Li, Luzhou Xu, Petre Stoica, Keith W. Forsythe, and Daniel W. Bliss. «Range Compression and Waveform Optimization for MIMO Radar: A CramÉr–Rao Bound Based Study». In: *IEEE Transactions on Signal Processing* 56.1 (2008), pp. 218–232. DOI: `10.1109/TSP.2007.901653` (cit. on p. 12).

[14] Michael Grant and Stephen Boyd. *CVX: Matlab Software for Disciplined Convex Programming, version 2.1*. `https://cvxr.com/cvx`. Mar. 2014 (cit. on p. 16).

[15] Michael Grant and Stephen Boyd. «Graph implementations for nonsmooth convex programs». In: *Recent Advances in Learning and Control*. Ed. by V. Blondel, S. Boyd, and H. Kimura. Lecture Notes in Control and Information Sciences. `http://stanford.edu/~boyd/graph_dcp.html`. Springer-Verlag Limited, 2008, pp. 95–110 (cit. on p. 16).