POLITECNICO DI TORINO

Master degree
in Mathematical Engineering

Master's Degree Thesis

A priori estimates for Boltzmann-type equations on graphs

Supervisor
Prof. Tommaso Lorenzi

Candidate
Debora Bisoglio

Academic Year 2023-2024

A Sara,

Acknowledgements

The moment has come for me to stop and draw conclusions after this first chapter of mine. When I look around, I see lots of beautiful people, lots of wonderful stories that are worth listening to, and it fills my heart with joy when I think that I am surrounded by people I can be proud of, and I hope they would say the same about me. I would like to thank who helped me during this journey.

First of all, I would like to thank Professor Lorenzi and Professor Tosin, who enabled me to experience what doing Mathematics "putting my hands in the dough" is like, for their patience and for answering my endless doubts and curiosities.

I would like to thank Dr. Capoano for the wonderful journey, because if today I am here with this awareness it is thanks to her.

I would like to thank my mom, the first person I call when something good happens and the only voice I want to hear when I feel like everything is falling apart, my solace and my strength.

I would like to thank my dad, my model of good heart, who taught me to always fight for the things I believe in, because if it was easy everybody would be doing that (cit.).

I would thank my grandparents Domenica and Giuseppe for their love and for all the afternoons spent talking, because when I am with them time seems to fly. A special mention must be made about my grandma Anna, who is always sending a beautiful butterfly to control me (I graduated grandma, I did it).

I would like to thank my uncle Roberto and my aunt Clotilde for their support, and the rest of my huge family.

I would like to thank Carlo, Mirella, Gloria and my goddaughter Eleonora, a second family to me.

I would like to thank Felix and Edo, who have always been there for me, always with a smile.

I would like to thank my Swiss family, Susanna with Suela and Ranju, because from the first moment I was there I felt at home and I could not have been luckier, and all the friends I got there, especially Andrea and Àlex.

I would like to thank who helped me from here, especially Alessandro, Lorenzo and Tommaso, and of course Dima, Delgiu, Fra and Liuzzi. Super mention about Monica, who cared about me as if I was her daughter.

I would like to thank Francesca and Asia, because they were there when it all began, and they still are.

I would like to thank all the beautiful people I met at university and outside, especially Didos, Luca, Sergio, Edo, Pinna, Luca, Davide and Monica, with whom I spared lots of laughter (and food).

I would like to thank Mery, who taught me that sometimes we just need to breathe, believe in ourselves and try, even if it seems scary.

I would like to thank Erika, because she is always there for me, for the endless calls, for helping me feeling less lonely when I needed it.

I would like to thank Beatrice for sharing so many moments with me (in so many years), for the laughter on the balcony, for the bubble teas, for spending hours with me checking this work and for the aperitivi at 9, because I could not have had a better flatmate.

I would like to thank Federico because he handles me, he makes me laugh, and he helps me being a better person every day, because he taught me to believe in myself, and I will never stop thanking him for this.

Lastly, I would like to thank the most important person to me, the best gift life could ever give me, my sister Sara, for her bravery and her contageous laughter, that even after 17 years still remains the best sound I have ever heard. I wanted to dedicate my work to her, wishing her, in a future, to celebrate a goal knowing to be sourrounded by lots of beautiful people, as I am doing today.

Summary

This work is devoted to study the asymptotic behaviour of a Boltzmann-type equation on a graph. A Boltzmann-type equation aims to describe the macroscopic effect of microscopic interactions in a set of elements. By applying it to a graph, as introduced in [6], the interacting agents are supposed to be divided into N groups, each one represented by a node: interactions can occur only between elements of the same node. Moreover, agents can jump from a node to another and then interact with the elements of their new group. In this work, the uniqueness and continuous dependence on the initial data of an a-priori solution of this adapted equation is proved. In the last chapter, its asymptotic behaviour is analysed using a Fourier-based metric. This type of approach is suggested by the fact that the Boltzmann-type equation models the microscopic interaction with a generalised convolution: the Fourier transform turns a convolution into a product, a much easier form to handle. In this work, we follow the same steps as in [8], where a Fourier-based metric is used to perform an asymptotic analysis on a spatially homogeneous Boltzmann-type equation. It is proved that, if the parameters satisfy a specific requirement, every couple of solutions will collide one on the other, heading towards a unique equilibrium distribution, provided it exists. In Pareschi's work, the physical meaning of this requirement is shown (and it is reported here in a dedicated section): if the parameters satisfy this condition, the finiteness of a physical quantity is proved - the quantity to be analysed change according to one of the parameters. Along the same lines, in this work the Fourier-based metric is adapted to the graph structure leading to an analogous result.

Contents

Chapter 1

Introduction

A Boltzmann-type equation connects the macroscopic and the microscopic worlds, in fact, the macroscopic behaviour is derived from the study of the interactions that occur at a microscopic level.

Although its first application was to study the statistical behaviour of a thermodynamic system not in a state of equilibrium, nowadays it is also used to model several types of interactions between humans. The phenomenon of using mathematical and physical tools to describe human interactions goes under the name of *social physics* (or *sociophysics*), while its application to the economic world is called *econophysics*. Boltzmann-type equations are used, among other things, to model the opinion exchange - as in [3], where interesting features, such as an opinion leader or political segregation, are added. Other applications are the viral charge during the pandemic (as in [1], where the kinetic description is influenced by a classical SIR-type compartmental model), and finance (i.e. in [5], in which the actions of High-Frequency Traders are investigated).

The main feature of a Boltzmann-type equation lays in the collisional term, in the shape of a generalised convolution. Hence, transforming the equation in its Fourier form seems a natural choice. As a consequence, a Fourier-based metric has been introduced in connection with the study of the large-time asymptotics of the Boltzmann equation for Maxwell molecules (see [4]). An interesting overview of the properties and studies that can be computed using this tool can be found in [2], where this metric is compared to more traditional ones and applied to find some non-trivial results in the case of dissipative Boltzmann equations. In [9], the decay analysis performed with this tool on a Boltzmann-type equation for a granular gas subject to dissipative collisions is compared to the decay of the Kac equation. In the same field, in [1] the same study is performed on spatially uniform freely cooling inelastic Maxwell molecules. In [6] and [7] this type of equation is applied to a graph structure, hence the idea of performing an asymptotic study with an adjustment of the Fourier-based metric on this spacial structure.

Let us now focus on the structure of this thesis.
In the second Chapter, Boltzmann-type equations are derived and the metric is introduced. Furthermore, the asymptotic study on a spatially homogeneous Boltzmann-type equation

is reported and the equation is adapted to the graph structure.
The third Chapter focuses on the proof of uniqueness and continuous dependence on the initial data of an a-priori solution using the L^2-norm adjusted to the spatial setting.
The adaptation of the Fourier-based metric is used in the fourth Chapter, where some asymptotic results are shown.

Chapter 2

Preliminaries

The aim of this chapter is to recall some definitions and results about Boltzmann-type equations. In Section 2.1 a Boltzmann-type equation is presented, in Section 2.2 and 2.3 a Fourier-based metric is respectively introduced and applied to analyse the behaviour of a solution to a Boltzmann-type equation. Furthermore, in Section 2.4, the graph structure is introduced and the equation is adapted to the new environment.

2.1 Boltzmann-type equations

A Boltzmann-type equation describes a system in which each element is characterised by its position x and its state v. This type of equations evaluate the probability that a particle occupies a certain position with a certain state at a certain time. For what concerns this work, in the position x, the hypothesis of spatial homogeneity is made: hence, the state v does not depend on the position. In addition, as statistically multiple interactions are less likely to occur, the interaction is supposed to be between two individuals at a time (interactions in a group of elements are thought as several interactions between couples that happen subsequently, parted by a small interval of time Δt).

One individual with state V_t interacts with a second individual with state V_t^* with probability $\Theta \sim \text{Bernoulli}(\Delta t)$, and the state of the first becomes

$$V_t' = V_t + I\left(V_t,\, V_t^*\right) + D\left(V_t,\, V_t^*\right)\eta \tag{2.1}$$

where

- $I : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the interaction function, the deterministic component of the interaction;
- $D : \mathbb{R} \to \mathbb{R}$ is the amplitude of the stochastic fluctuation;
- η is a fluctuation, with average 0.

Hence,

$$V_{t+\Delta t} = (1-\Theta)\, V_t + \Theta V_t^{'}.$$

This work focuses on linear interactions of the following type:

$$V_t' = pV_t + qV_t^*. \tag{2.2}$$

where $p := 1 - \nu_1 + \eta$ and $q := \nu_2$, with

- $\nu_1,\, \nu_2 \in [0,\, 1]$;
- η is a stochastic coefficient, with $\langle \eta \rangle = 0$, $\eta \geq \nu_1 - 1$ since $p \in \mathbb{R}_+$.

Calling

$$f = f(v,\, t) : \mathbb{R} \times [0,\, +\infty] \to \mathbb{R}_+$$

the probability density function of v and introducing an arbitrary test function $\varphi(v,\, t)$, its expected value can be computed as follows:

$$\begin{aligned}\langle \varphi\left(V_{t+\Delta t}\right)\rangle &= \langle \varphi\left(V_t\right)\rangle + \langle \Theta\left[\varphi\left(V_t'\right) - \varphi\left(V_t\right)\right]\rangle \\ &= \langle \varphi\left(V_t\right)\rangle + 1\cdot\Delta t\cdot\left[\langle \varphi\left(V_t'\right)\rangle - \langle \varphi\left(V_t\right)\rangle\right] + 0\cdot(1-\Delta t)\left[\langle \varphi\left(V_t'\right)\rangle - \langle \varphi\left(V_t\right)\rangle\right],\end{aligned} \tag{2.3}$$

that means

$$\frac{\langle \varphi\left(V_{t+\Delta t}\right)\rangle - \langle \varphi\left(V_t\right)\rangle}{\Delta t} = \langle \varphi\left(V_t'\right)\rangle - \langle \varphi\left(V_t\right)\rangle\,. \tag{2.4}$$

In the limit of $\Delta t \to 0$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \varphi\left(v\right) f\left(v,\, t\right) \mathrm{d}v = \int_{\mathbb{R}}\int_{\mathbb{R}} \langle \varphi\left(v'\right)\rangle\, f\left(v,\, t\right) f\left(v^*,\, t\right) \mathrm{d}v\mathrm{d}v^* - \int_{\mathbb{R}} \varphi\left(v\right) f\left(v,\, t\right) \mathrm{d}v \tag{2.5}$$

that is the weak form of a Boltzmann-type equation under the spatial homogeneity hypothesis.

2.2 A Fourier-based metric

Let f and g be two probability densities that solve (2.5) defined on $\mathbb{R}$ and let $\hat{f}$ and $\hat{g}$ be their Fourier transforms

$$\hat{f}\left(\xi\right) = \int_{\mathbb{R}} e^{-i\xi v} f(v)\, \mathrm{d}v, \qquad \hat{g}\left(\xi\right) = \int_{\mathbb{R}} e^{-i\xi v} g(v)\, \mathrm{d}v. \tag{2.6}$$

As shown in [8], we define

$$d_s(f,\, g) \;=\; \sup_{\xi \in \mathbb{R}\setminus\{0\}} \frac{\left|\hat{f}\left(\xi\right) - \hat{g}\left(\xi\right)\right|}{\left|\xi\right|^s}. \tag{2.7}$$

Proposition 2.1. *If f and g have equal moments up to n, $d_s(f,\, g)$ is finite for $s \leq n+1$.*

Proof. First of all, we need to prove that the numerator is bounded. Since f is a probability density it holds that

$$\left|\hat{f}\,(\xi)\right| = \left|\int_{\mathbb{R}} e^{-i\xi v} f(v)\,\mathrm{d}v\right| \leq \int_{\mathbb{R}} \left|e^{-i\xi v}\right| f(v)\,\mathrm{d}v = \int_{\mathbb{R}} f(v)\,\mathrm{d}v = 1 \tag{2.8}$$

and analogously $|\hat{g}| \leq 1$. This proves that the numerator of $d_s(f,\, g)$ is bounded, in fact

$$\left|\hat{f}\,(\xi) - \hat{g}\,(\xi)\right| \leq \left|\hat{f}\,(\xi)\right) + |\hat{g}\,(\xi)| = 2. \tag{2.9}$$

On the other hand, the denominator goes to 0 as $\xi \to 0$. In order to write the MacLaurin expansion of $d_s(f,\, g)$, we first need to compute the derivatives of $\hat{f}$ and $\hat{g}$ in $\xi = 0$.
By definition,

$$\hat{f}^{(k)}\,(\xi) = \frac{\mathrm{d}^k}{\mathrm{d}\xi^k}\int_{\mathbb{R}} e^{-i\xi v} f(v)\,\mathrm{d}v = \int_{\mathbb{R}} (-iv)^k\, e^{-i\xi v} f(v)\,\mathrm{d}v = (-i)^k \int_{\mathbb{R}} v^k e^{-i\xi v} f(v)\,\mathrm{d}v \tag{2.10}$$

and for the same reason

$$\hat{g}^{(k)}\,(\xi) = (-i)^k \int_{\mathbb{R}} v^k e^{-i\xi v} g(v)\,\mathrm{d}v. \tag{2.11}$$

We start by writing the MacLaurin expansion of $\hat{f}\,(\xi) - \hat{g}\,(\xi)$, that is

$$\hat{f}\,(\xi) - \hat{g}\,(\xi) = \sum_{k=0}^{\infty} \left(\hat{f} - \hat{g}\right)^{(k)} (0)\,\frac{\xi^k}{k!} = \sum_{k=0}^{\infty} \left(\hat{f}^{(k)}\,(0) - \hat{g}^{(k)}\,(0)\right)\frac{\xi^k}{k!}. \tag{2.12}$$

Since f and g have, by hypothesis, equal moments up to n, it holds that

$$\hat{f}^{(k)}\,(\xi) = (-i)^k \int_{\mathbb{R}} v^k e^{-i\xi v} f(v)\,\mathrm{d}v = (-i)^k \int_{\mathbb{R}} v^k e^{-i\xi v} g(v)\,\mathrm{d}v = \hat{g}^{(k)}\,(\xi) \tag{2.13}$$

for $k \leq n$. The Equation (2.12) becomes

$$\hat{f}\,(\xi) - \hat{g}\,(\xi) = \sum_{k=n+1}^{\infty} \left(\hat{f}^{(k)}\,(0) - \hat{g}^{(k)}\,(0)\right)\frac{\xi^k}{k!} = \left(\hat{f}^{(n+1)}\,(0) - \hat{g}^{(n+1)}\,(0)\right)\frac{\xi^{n+1}}{(n+1)\,!} + \mathrm{o}\left(\xi^{n+2}\right). \tag{2.14}$$

Then,

$$\frac{\left|\hat{f}\left(\xi\right)-\hat{g}\left(\xi\right)\right|}{\left|\xi\right|^{s}}\leq\frac{\left|\left(\hat{f}^{(n+1)}\left(0\right)-\hat{g}^{(n+1)}\left(0\right)\right)\right|}{\left|\xi\right|^{s}}\frac{\left|\xi\right|^{n+1}}{\left(n+1\right)!}+\mathrm{o}\left(\left|\xi\right|^{n+2-s}\right) \tag{2.15}$$

that is finite for $n+1 \geq s$, and so is $d_s(f,\, g)$. □

Proposition 2.2. *If f and g have equal moments up to n and $s \leq n+1$, then $d_s(f,\, g)$ is a metric.*

Proof. In order to be a metric, $d_s\left(f,\, g\right)$ must be finite, positive, symmetric and must satisfy the triangular inequality. The finiteness has already been discussed. The positivity and the symmetry are automatically satisfied due to the presence of the the absolute values. The triangular inequality is easily proved: suppose there is another probability density h, then

$$\begin{aligned}
d_s\left(f,\, g\right) &= \sup_{\xi\in\mathbb{R}\setminus\{0\}}\frac{\left|\hat{f}\left(\xi\right)-\hat{g}\left(\xi\right)\right|}{\left|\xi\right|^{s}}\\
&= \sup_{\xi\in\mathbb{R}\setminus\{0\}}\frac{\left|\hat{f}\left(\xi\right)-\hat{h}\left(\xi\right)+\hat{h}\left(\xi\right)-\hat{g}\left(\xi\right)\right|}{\left|\xi\right|^{s}}\\
&\leq \sup_{\xi\in\mathbb{R}\setminus\{0\}}\left[\frac{\left|\hat{f}\left(\xi\right)-\hat{h}\left(\xi\right)\right|}{\left|\xi\right|^{s}}+\frac{\left|\hat{h}\left(\xi\right)-\hat{g}\left(\xi\right)\right|}{\left|\xi\right|^{s}}\right]\\
&\leq \sup_{\xi\in\mathbb{R}\setminus\{0\}}\frac{\left|\hat{f}\left(\xi\right)-\hat{h}\left(\xi\right)\right|}{\left|\xi\right|^{s}}+\sup_{\xi\in\mathbb{R}\setminus\{0\}}\frac{\left|\hat{h}\left(\xi\right)-\hat{g}\left(\xi\right)\right|}{\left|\xi\right|^{s}}\\
&= d_s\left(f,\, h\right)+\ d_s\left(h,\, g\right).
\end{aligned} \tag{2.16}$$

□

2.3 Application of the metric to the Boltzmann-type equation

In order to use the metric introduced in the previous section, the Boltzmann-type equation must be transformed in its Fourier form. The choice of the Fourier form is induced by the fact that it turns the convolution into a product, significatively easier to be dealt with.

2.3.1 Fourier form of a Boltzmann-type equation

Since φ is arbitrary, in Equation (2.5) it can be chosen equal to $e^{-i\xi v}$, leading to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} e^{-i\xi v}\left(v\right) f\left(v,\, t\right) \mathrm{d}v = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\xi v'} f\left(v,\, t\right) f\left(v^{*},\, t\right) \mathrm{d}v\mathrm{d}v^{*} - \int_{\mathbb{R}} e^{-i\xi v}\left(v\right) f\left(v,\, t\right) \mathrm{d}v \tag{2.17}$$

At a microscopic state, from (2.2) it follows that $v' = pv + qv^{*}$ and the second term becomes

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\xi v'} f\left(v,\, t\right) f\left(v^{*},\, t\right) \mathrm{d}v\mathrm{d}v^{*} &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\xi\left(pv+qv^{*}\right)} f\left(v,\, t\right) f\left(v^{*},\, t\right) \mathrm{d}v\mathrm{d}v^{*} \\ &= \int_{\mathbb{R}} e^{-i\left(p\xi\right)v} f\left(v,\, t\right) \mathrm{d}v \int_{\mathbb{R}} e^{-i\left(q\xi\right)v^{*}} f\left(v^{*},\, t\right) \mathrm{d}v^{*}.\end{aligned} \tag{2.18}$$

Then, according to Definitions (2.6), Equation (2.17) can be written as

$$\frac{\partial}{\partial t}\hat{f}\left(\xi,\, t\right) = \hat{f}\left(p\xi,\, t\right)\hat{f}\left(q\xi,\, t\right) - \hat{f}\left(\xi,\, t\right), \tag{2.19}$$

that is the *Kinetic Equation* for $\hat{f}$.

2.3.2 Uniqueness and continuous dependence on the initial data of a solution of a Boltzmann-type equation

The metric introduced in 2.2 can act as an extremely useful tool for studying the behaviour of a solution of a Boltzmann-type equation in its Fourier form.

Theorem 2.3. *If a solution to a Boltzmann-type equation exists, then it is unique and it depends continuously on the initial data.*

Proof. Let f_1, f_2 be two solutions of Equation (2.19) with initial values $f_{1,0}$ and $f_{2,0}$ such that

$$\int_{\mathbb{R}} f_{1,0}\left(v\right) \mathrm{d}v = 1, \qquad \int_{\mathbb{R}} v f_{1,0}\left(v\right) \mathrm{d}v = 0, \qquad \int_{\mathbb{R}} v^{2} f_{1,0}\left(v\right) \mathrm{d}v = 1, \quad i = 1,\, 2. \tag{2.20}$$

Fixing $|\xi| > 0$ and a constant $s > 0$, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\hat{f}_1\left(\xi,\, t\right) - \hat{f}_2\left(\xi,\, t\right)}{|\xi|^{s}}\right) = \frac{\hat{f}_1\left(p\xi,\, t\right)\hat{f}_1\left(q\xi,\, t\right) - \hat{f}_2\left(p\xi,\, t\right)\hat{f}_2\left(q\xi,\, t\right)}{|\xi|^{s}} - \frac{\hat{f}_1\left(\xi,\, t\right) - \hat{f}_2\left(\xi,\, t\right)}{|\xi|^{s}} \tag{2.21}$$

that can be rewritten as

$$\frac{\partial}{\partial t}\left(\frac{\hat{f}_1(\xi,t)-\hat{f}_2(\xi,t)}{|\xi|^s}\right)+\frac{\hat{f}_1(\xi,t)-\hat{f}_2(\xi,t)}{|\xi|^s}=\frac{\hat{f}_1(p\xi,t)\,\hat{f}_1(q\xi,t)-\hat{f}_2(p\xi,t)\,\hat{f}_2(q\xi,t)}{|\xi|^s}. \tag{2.22}$$

Computing the absolute value on both sides of (2.22), we get

$$\left|\frac{\partial}{\partial t}\left(\frac{\hat{f}_1(\xi,t)-\hat{f}_2(\xi,t)}{|\xi|^s}\right)+\frac{\hat{f}_1(\xi,t)-\hat{f}_2(\xi,t)}{|\xi|^s}\right|\le\frac{\left|\hat{f}_1(p\xi,t)\,\hat{f}_1(q\xi,t)-\hat{f}_2(p\xi,t)\,\hat{f}_2(q\xi,t)\right|}{|\xi|^s}. \tag{2.23}$$

The second term satisfies

$$\begin{aligned}
&\frac{\left|\hat{f}_1(p\xi,t)\,\hat{f}_1(q\xi,t)-\hat{f}_2(p\xi,t)\,\hat{f}_2(q\xi,t)\right|}{|\xi|^s}\\
&=\frac{\left|\hat{f}_1(p\xi,t)\,\hat{f}_1(q\xi,t)-\hat{f}_1(p\xi,t)\,\hat{f}_2(q\xi,t)+\hat{f}_1(p\xi,t)\,\hat{f}_2(q\xi,t)-\hat{f}_2(p\xi,t)\,\hat{f}_2(q\xi,t)\right|}{|\xi|^s}\\
&\le\frac{\left|\hat{f}_1(p\xi,t)\left(\hat{f}_1(q\xi,t)-\hat{f}_2(q\xi,t)\right)\right|}{|\xi|^s}+\frac{\left|\left(\hat{f}_1(p\xi,t)-\hat{f}_2(p\xi,t)\right)\hat{f}_2(q\xi,t)\right|}{|\xi|^s}\\
&=\left|\hat{f}_1(p\xi,t)\right|\frac{\left|\hat{f}_1(q\xi,t)-\hat{f}_2(q\xi,t)\right|}{|\xi|^s}+\frac{\left|\hat{f}_1(p\xi,t)-\hat{f}_2(p\xi,t)\right|}{|\xi|^s}\left|\hat{f}_2(q\xi,t)\right|\\
&\le\frac{\left|\hat{f}_1(q\xi,t)-\hat{f}_2(q\xi,t)\right|}{|q\xi|^s}q^s+\frac{\left|\hat{f}_1(p\xi,t)-\hat{f}_2(p\xi,t)\right|}{|p\xi|^s}p^s\\
&\le\sup_{\xi\in\mathbb{R}\setminus\{0\}}\frac{\left|\hat{f}_1(\xi,t)-\hat{f}_2(\xi,t)\right|}{|\xi|^s}\left(p^s+q^s\right).
\end{aligned} \tag{2.24}$$

Multiplying this equation by e^t it becomes

$$\left|\frac{\partial}{\partial t}\left(\frac{\hat{f}_1(\xi,t)-\hat{f}_2(\xi,t)}{|\xi|^s}\right)+\frac{\hat{f}_1(\xi,t)-\hat{f}_2(\xi,t)}{|\xi|^s}\right|e^t\le\sup_{\xi\in\mathbb{R}\setminus\{0\}}\frac{\left|\hat{f}_1(\xi,t)-\hat{f}_2(\xi,t)\right|}{|\xi|^s}\left(p^s+q^s\right)e^t. \tag{2.25}$$

Investigating the first term we obtain that

$$\begin{aligned}
&\left|\frac{\partial}{\partial t}\left(\frac{\hat{f}_1(\xi,t)-\hat{f}_2(\xi,t)}{|\xi|^s}\right)+\frac{\hat{f}_1(\xi,t)-\hat{f}_2(\xi,t)}{|\xi|^s}\right|e^t\\
&=\left|e^t\frac{\partial}{\partial t}\left(\frac{\hat{f}_1(\xi,t)-\hat{f}_2(\xi,t)}{|\xi|^s}\right)+e^t\frac{\hat{f}_1(\xi,t)-\hat{f}_2(\xi,t)}{|\xi|^s}\right|\\
&=\left|\frac{\partial}{\partial t}\left(e^t\frac{\hat{f}_1(\xi,t)-\hat{f}_2(\xi,t)}{|\xi|^s}\right)\right|
\end{aligned} \tag{2.26}$$

and using the inequality $|\partial_t(\cdot)| \geq \partial_t |(\cdot)|$ Equation (2.17) becomes

$$\frac{\partial}{\partial t}\left(e^t \frac{\left|\hat{f}_1(\xi,\, t) - \hat{f}_2(\xi,\, t)\right|}{|\xi|^s}\right) \leq \sup_{\xi\in\mathbb{R}\setminus\{0\}} \frac{\left|\hat{f}_1(\xi,\, t) - \hat{f}_2(\xi,\, t)\right|}{|\xi|^s}(p^s + q^s)\, e^t. \tag{2.27}$$

Choosing $\varphi = 1$ the mass is proved to be conserved, in fact

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} f(v,\, t)\, \mathrm{d}v = \int_{\mathbb{R}\times\mathbb{R}} (1-1)\, f(v,\, t)\, f(v^*,\, t)\, \mathrm{d}v\mathrm{d}v^* = 0. \tag{2.28}$$

This being said, we can estimate

$$\left|\hat{f}_i(\xi,\, t)\right| = \left|\int_{\mathbb{R}} e^{-i\xi v} f_i(v,\, t)\, \mathrm{d}v\right| \leq \int_R \left|e^{-i\xi v}\right| f_i(v,\, t)\, \mathrm{d}v \leq$$
$$\int_{\mathbb{R}} \left|e^{-i\xi v}\right| f_i(v,\, t)\, \mathrm{d}v = \int_{\mathbb{R}} \left|e^{-i\xi v}\right| f_{i,0}(v,\, t)\, \mathrm{d}v = 1 \quad i = 1, 2.$$

Calling

$$h(\xi) := \frac{\left|\hat{f}_1(\xi,\, t) - \hat{f}_2(\xi,\, t)\right|}{|\xi|^s}$$

Equation (2.27) can be written as

$$\frac{\partial}{\partial t}\left(e^t h(\xi,\, t)\right) \leq (p^s + q^s)\, \|e^t h(\xi)\|_\infty (t). \tag{2.29}$$

Integrating in time from 0 to t we obtain

$$e^\tau h(\xi,\, \tau)\Big|_0^t \leq \int_0^t (p^s + q^s)\, \|he^\tau\|_\infty (\tau)\, \mathrm{d}\tau. \tag{2.30}$$

Hence,

$$h(\xi,\, t)\, e^t \leq h(\xi,\, 0) + \int_0^t (p^s + q^s)\, \|he^\tau\|_\infty (\tau)\, \mathrm{d}\tau. \tag{2.31}$$

If it holds for every ξ then it holds also for its *supremum*, and calling $H := \|h(\cdot,\, t)\, e^t\|_\infty$, then

$$H(t) \leq H(0) + \int_0^t (p^s + q^s)\, H(\tau)\, \mathrm{d}\tau. \tag{2.32}$$

Using Gronwall Lemma we get

$$H(t) \leq H(0)\, exp\left\{(p^s + q^s)\, t\right\}. \tag{2.33}$$

Since $H(t) = e^t \|h(\cdot,\, t)\|_\infty = e^t d_s(f_1,\, f_2)$, then

$$d_s(f_1,\, f_2)(t) \leq d_s(f_{1,0},\, f_{2,0})\, exp\{(p^s + q^s - 1)\, t\} \tag{2.34}$$

This proves the uniqueness of the solution and its continuous dependence on the initial data. □

Since the first three moments of $f_{1,0}$ and $f_{2,0}$ coincide, $d_s(f_{1,0},\, f_{2,0})$ is surely bounded for $s \leq 3$. Consequently, it holds also for $d_s(f_1,\, f_2)(t)$ for a finite t. Furthermore, this result states that if $p^s + q^s < 1$ the two solutions f_1 and f_2 reach the same equilibrium when $t \to \infty$ and, since no further characterisation of f_1 and f_2 is involved in the computation, every density function f with the same first n moments has the same asymptotic behaviour.

2.3.3 Considerations on the parameters

As analysed in [8], the sign of $p^s + q^s - 1$ leads to some considerations based on the value of s. First of all, by choosing $\varphi(v) = v$ in Equation (2.5) we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} vf(v,\, t)\,\mathrm{d}v &= \int_{\mathbb{R}}\int_{\mathbb{R}} \langle v'\rangle f(v,\, t)\, f(v^*,\, t)\,\mathrm{d}v\mathrm{d}v^* - \int_{\mathbb{R}} vf(v,\, t)\,\mathrm{d}v \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} (pv + qv^*)\, f(v,\, t)\, f(v^*,\, t)\,\mathrm{d}v\mathrm{d}v^* - \int_{\mathbb{R}} vf(v,\, t)\,\mathrm{d}v \\ &= \int_{\mathbb{R}} pvf(v,\, t)\,\mathrm{d}v \int_{\mathbb{R}} f(v^*,\, t)\,\mathrm{d}v^* \\ &\quad + \int_{\mathbb{R}} qv^* f(v^*,\, t)\,\mathrm{d}v^* \int_{\mathbb{R}} f(v,\, t)\,\mathrm{d}v - \int_{\mathbb{R}} vf(v,\, t)\,\mathrm{d}v\end{aligned} \tag{2.35}$$

and supposing

$$\int_{\mathbb{R}} vf(v,\, t)\,\mathrm{d}v = 1$$

as in (2.20) it becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} vf(v,\, t)\,\mathrm{d}v = (p + q - 1)\int_{\mathbb{R}} vf(v,\, t)\,\mathrm{d}v. \tag{2.36}$$

Defining the momentum

$$m(t) = \int_{\mathbb{R}} vf(v,\, t)\,\mathrm{d}v \tag{2.37}$$

Equation (2.36) leads to

$$m(t) = m(0)\exp\left\{\Big(p + q - 1\Big)t\right\}. \tag{2.38}$$

This means that, if the initial momentum is not 0, then

- it tends to zero if $p+q$ is less than 1;
- its value remains constant if $p+q$ is equal to 1;
- it grows to $+\infty$ if $p+q$ is greater than 1;

In case the hypotheses in (2.20) hold, then $m(t) = m(0) = 0$ at every time for every value of the parameters p and q.

Introducing the energy

$$E(t) = \int_{\mathbb{R}} v^2 f(v,\, t)\, \mathrm{d}v \tag{2.39}$$

and, after analogous computations, we obtain that

$$E(t) = E(0)\exp\left\{\left(p^2+q^2-1\right)t\right\}. \tag{2.40}$$

Hence, the behaviour of the energy can be described as the one of the momentum, substituting $p+q$ with p^2+q^2.

In general, we can conclude that the asymptotic behaviour of the k-th moment, provided it is finite, depends on the sign of the quantity p^k+q^k-1.

2.4 Problem on the graph

Let the individuals that we are considering be divided into groups - each denoted by a label - and let us assume that they can change the group they belong to. This circumstance can be seen as a weighted directed graph, where each node represents a group of individuals and the weight of the edge connecting the j-th node to the i-th one is the probability that a member of the j-th group would join the i-th one in the next timestep denoted as $P_{ij} \in [0,\, 1]$.

Suppose these individuals carry - other than the label $i \in I \subset \mathbb{N}$ of the group they belong to - a state $v \in \mathbb{R}_+$ that can change with the interactions as previously analysed. The label changes according to a variable $\Xi \sim \text{Bernoulli}(\chi\Delta t)$, while an interaction occurs according to $\Theta \sim \text{Bernoulli}(\mu\Delta t)$, where χ is the mobility rate and μ is the social contact rate. This is intended for a timestep Δt that satisfies $\Delta t \leq \min\left\{\frac{1}{\chi},\, \frac{1}{\mu}\right\}$. Interactions can occur only among individuals of the same group.

The label and the state evolve according to

$$\begin{cases} X_{t+\Delta t} &= (1-\Xi)\, X_t + \Xi J_t \\ V_{t+\Delta t} &= \left(1-\Theta\delta_{X_t,\, X_t^*}\right) V_t + \Theta\delta_{X_t,\, X_t^*} V_t' \end{cases} \tag{2.41}$$

where v_t' satisfies (2.1) while $J_t \in I$ is a random variable returning the new vertex after a jump. This is a Markov-type jump process: the probability to switch from the current label j to a new label i does not depend on how the agent previously reached the label j. The *transition matrix* P_{ij} has two properties:

- it is left-stochastic, i.e. $\sum_i P_{ij} = 1 \quad \forall j \in I$
- it is strongly connected, i.e. there a direct path from every node to each other.

Analogously to what previously done in Section 2.1 for a Boltzmann-type equation in the spatially homogeneous case, an arbitrary test function $\phi(i, v, t)$ is introduced, and its expected value can be written as

$$\langle \phi\left(X_{t+\Delta t},\, V_{t+\Delta t}\right)\rangle = \langle \phi\left(\left(1-\Xi\right)X_t + \Xi J_t,\, \left(1-\Theta\delta_{X_t,\,X_t^*}\right)V_t + \Theta\delta_{X_t,\,X_t^*}V_t'\right)\rangle\,.$$

Substituting Θ and Ξ with their expected values and exploiting the linearity it becomes

$$\begin{aligned}\langle \phi\left(X_{t+\Delta t},\, V_{t+\Delta t}\right)\rangle &= \langle \phi\left(\left(1-\chi\Delta t\right)X_t + \chi\Delta t J_t,\, \left(1-\mu\Delta t\delta_{X_t,\,X_t^*}\right)V_t + \mu\Delta t\delta_{X_t,\,X_t^*}V_t'\right)\rangle \\ &= \langle\left(1-\chi\Delta t\right)\left(1-\mu\Delta t\delta_{X_t,\,X_t^*}\right)\phi\left(X_t,\, V_t\right)\rangle \\ &\quad + \langle\left(1-\chi\Delta t\right)\mu\Delta t\delta_{X_t,\,X_t^*}\phi\left(X_t,\, V_t'\right)\rangle \\ &\quad + \langle\chi\Delta t\left(1-\mu\Delta t\delta_{X_t,\,X_t^*}\right)\phi\left(J_t,\, V_t\right)\rangle + \langle\chi\Delta t\mu\Delta t\delta_{X_t,\,X_t^*}\phi\left(J_t,\, V_t'\right)\rangle \\ &= \left(1-\chi\Delta t\right)\langle\phi\left(X_t,\, V_t\right)\rangle - \mu\Delta t\left(1-\chi\Delta t\right)\langle\delta_{X_tX_t^*}\phi\left(X_t,\, V_t\right)\rangle \\ &\quad + \left(1-\chi\Delta t\right)\mu\Delta t\langle\delta_{X_t,\,X_t^*}\phi\left(X_t,\, V_t'\right)\rangle + \chi\Delta t\langle\phi\left(J_t,\, V_t\right)\rangle \\ &\quad - \chi\Delta t\mu\Delta t\langle\delta_{X_tX_t^*}\phi\left(J_t,\, V_t\right)\rangle + \chi\Delta t\mu\Delta t\langle\delta_{X_t,\,X_t^*}\phi\left(J_t,\, V_t'\right)\rangle\,.\end{aligned} \tag{2.42}$$

Neglecting higher order terms, we get

$$\begin{aligned}\langle \phi\left(X_{t+\Delta t},\, V_{t+\Delta t}\right)\rangle &= \left(1-\chi\Delta t\right)\langle\phi\left(X_t,\, V_t\right)\rangle - \mu\Delta t\langle\delta_{X_tX_t^*}\phi\left(X_t,\, V_t\right)\rangle \\ &\quad + \mu\Delta t\langle\delta_{X_t,\,X_t^*}\phi\left(X_t,\, V_t'\right)\rangle + \chi\Delta t\langle\phi\left(J_t,\, V_t\right)\rangle\end{aligned} \tag{2.43}$$

Dividing by Δt we obtain

$$\begin{aligned}\frac{\langle \phi\left(X_{t+\Delta t},\, V_{t+\Delta t}\right)\rangle - \langle\phi\left(X_t,\, V_t\right)\rangle}{\Delta t} &= -\chi\langle\phi\left(X_t,\, V_t\right)\rangle - \mu\langle\delta_{X_tX_t^*}\phi\left(X_t,\, V_t\right)\rangle \\ &\quad + \mu\langle\delta_{X_t,\,X_t^*}\phi\left(X_t,\, V_t'\right)\rangle + \chi\langle\phi\left(J_t,\, V_t\right)\rangle\end{aligned} \tag{2.44}$$

Considering the limit of $\Delta t \to 0$ it shows that

$$\frac{\partial}{\partial t}\langle\phi\left(X_t,\, V_t\right)\rangle = -\chi\langle\phi\left(X_t,\, V_t\right)\rangle - \mu\langle\delta_{X_tX_t^*}\phi\left(X_t,\, V_t\right)\rangle + \mu\langle\delta_{X_t,\,X_t^*}\phi\left(X_t,\, V_t'\right)\rangle + \chi\langle\phi\left(J_t,\, V_t\right)\rangle \tag{2.45}$$

Let

$$f(i,\, v,\, t)\,\mathrm{d}v = \sum_i f_i(v,\, t)\,\mathrm{d}v$$

be the distribution function, such that

$$\sum_i \int_{\mathbb{R}_+} f_i(v,\, t)\,\mathrm{d}v = 1.$$

Then the expected values can be computed as follows:

i.

$$\langle \phi(X_t,\, V_t)\rangle = \sum_i \int_{\mathbb{R}_+} \phi(i,\, v)\, f_i(v,\, t)\,\mathrm{d}v \tag{2.46}$$

ii.

$$\begin{aligned}\langle \delta_{X_t X_t^*} \phi(X_t,\, V_t')\rangle &= \sum_i \sum_j \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \langle \phi(i,\, v')\rangle\, f_i(v,\, t)\, f_j(v^*,\, t)\, \delta_{ij} \mathrm{d}v \mathrm{d}v^* \\ &= \sum_i \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \langle \phi(i,\, v)\rangle\, f_i(v,\, t)\, f_i(v^*,\, t)\,\mathrm{d}v\mathrm{d}v^*\end{aligned} \tag{2.47}$$

iii.

$$\langle \phi(J_t,\, V_t)\rangle = \sum_i \int_{\mathbb{R}_+} \langle \phi(i,\, t)\rangle\, \mathrm{Prob}(\cdot \to i)\,\mathrm{d}v = \sum_i \sum_j \int_{\mathbb{R}_+} \phi(i,\, v)\, P_{ij} f_j(v,\, t)\,\mathrm{d}v \tag{2.48}$$

iv.

$$\begin{aligned}\langle \delta_{X-tX_t^*} \phi(X,\, V_t)\rangle &= \sum_i \sum_j \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \phi(i,\, v)\, \delta_{ij} f_i(v\, t)\, f_j(v^*,\, t)\,\mathrm{d}v\mathrm{d}v^* \\ &= \sum_i \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \phi(i,\, v)\, f_i(v,\, t)\, f_i(v^*,\, t)\,\mathrm{d}v\mathrm{d}v^*\end{aligned} \tag{2.49}$$

Substituting these expressions in (2.45), this becomes

$$
\begin{aligned}
\frac{\partial}{\partial t}\sum_i\int_{\mathbb{R}_+}\phi\left(i,\,v\right)f_i\left(v,\,t\right)\mathrm{d}v =\ & \chi\sum_i\sum_j\int_{\mathbb{R}_+}\phi\left(i,\,v\right)P_{ij}f_j\left(v,\,t\right)\mathrm{d}v \\
& -\chi\sum_i\int_{\mathbb{R}_+}\phi\left(i,\,v\right)f_i\left(v,\,t\right)\mathrm{d}v \\
& +\mu\sum_i\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\left\langle\phi\left(i,\,v'\right)\right\rangle f_i\left(v,\,t\right)f_i\left(v^*,\,t\right)\mathrm{d}v\mathrm{d}v^* \\
& -\mu\sum_i\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\phi\left(i,\,v\right)f_i\left(v,\,t\right)f_i\left(v^*,\,t\right)\mathrm{d}v\mathrm{d}v^* \\
=\ & \chi\sum_i\int_{\mathbb{R}_+}\phi\left(i,\,v\right)\left[\sum_j P_{ij}f_j\left(v,\,t\right)-f_i\left(v,\,t\right)\right]\mathrm{d}v \\
& +\mu\sum_i\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\left\langle\phi\left(i,\,v'\right)-\phi\left(i,\,v\right)\right\rangle f_i\left(v,\,t\right)f_i\left(v^*,\,t\right)\mathrm{d}v\mathrm{d}v^*
\end{aligned}
\tag{2.50}
$$

Since the test function $\phi\left(i,\,v\right)$ is arbitrary, it is possible to choose $\phi\left(i,\,v\right)=\psi\left(i\right)\varphi\left(v\right)$ with $\psi\left(i\right)=\delta_{ij}$ for a fixed i. This leads to

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}_+}\varphi\left(v\right)f_i\left(v,\,t\right)\mathrm{d}v =\ & \chi\int_{\mathbb{R}_+}\varphi\left(v\right)\left[\sum_j P_{ij}f_j\left(v,\,t\right)-f_i\left(v,\,t\right)\right]\mathrm{d}v \\
& +\mu\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\left\langle\varphi\left(v'\right)-\varphi\left(v\right)\right\rangle f_i\left(v,\,t\right)f_i\left(v^*,\,t\right)\mathrm{d}v\mathrm{d}v^*,
\end{aligned}
\tag{2.51}
$$

a Boltzmann-type equation in its weak form extended to a graph. It holds for every function $\varphi\left(v\right)$.

Chapter 3

A priori estimates on the solution on the graph

The aim of this chapter is to prove the uniqueness of the solution and its continuous dependence on the initial data. In order to do that, the L^2-norm is extended to the graph and in Section 3.1 the L^2-norm of the solution is estimated. Then in Section 3.2 the uniqueness is proved. The choice of the L^2-norm is suggested by the simplicity of the Boltzmann-type equation in its Fourier transform form: since the L^2-norm of a function and of its Fourier transform coincide, this appears to be the natural norm to be chosen. Thus, the norm that will be used is

$$\sum_{i=1}^{N} \left\| f_i \right\|_2^2 \tag{3.1}$$

where N is the number of nodes the graph has.

3.1 Estimate of the L^2-norm of the solution

Let us assume f is a solution of the Boltzmann-type equation on the graph. First of all, we are interested in investigating its behaviour as time evolves, mainly we want to prove that its norm is bounded from above at every finite time t. Thus, its L^2-norm can be estimated as follows.

Theorem 3.1. *If f is a solution of (2.51), then*

$$\sum_i \left\| \hat{f}_i\left(\cdot,\, t\right) \right\|_2^2 \leq \sum_i \left\| \hat{f}_i\left(\cdot,\, 0\right) \right\|_2^2 \exp\left\{ 2\left[\chi\left(N+1\right) + \mu\left(\frac{1}{\sqrt{|p|}} + 1 \right) \right] t \right\}. \tag{3.2}$$

Proof. Choosing $\varphi(v) = e^{-i\xi v}$ as in Section 2.3, Equation (2.51) becomes

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathbb{R}_+} e^{-i\xi v} f_i(v,\,t)\,\mathrm{d}v = \chi\int_{\mathbb{R}_+} e^{-i\xi v}\left[\sum_j P_{ij} f_j(v,\,t) - f_i(v,\,t)\right]\mathrm{d}v \\ + \mu\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\langle e^{-i\xi v'} - e^{-i\xi v}\rangle f_i(v,\,t)\, f_i(v^*,\,t)\,\mathrm{d}v\mathrm{d}v^*\end{aligned} \tag{3.3}$$

and recalling that $v' = pv + qv^*$, it reads

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathbb{R}_+} e^{-i\xi v} f_i(v,\,t)\,\mathrm{d}v &= \chi\int_{\mathbb{R}_+} e^{-i\xi v}\left[\sum_j P_{ij} f_j(v,\,t) - f_i(v,\,t)\right]\mathrm{d}v \\ &\quad + \mu\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\langle e^{-i\xi(pv+qv^*)} - e^{-i\xi v}\rangle f_i(v,\,t)\, f_i(v^*,\,t)\,\mathrm{d}v\mathrm{d}v^* \\ &= \chi\sum_j P_{ij}\int_{\mathbb{R}_+} e^{-i\xi v} f_j(v,\,t)\,\mathrm{d}v - \chi\int_{\mathbb{R}_+} e^{-i\xi v} f_i(v,\,t) \\ &\quad + \mu\int_{\mathbb{R}_+} e^{-i\xi pv} f_i(v,\,t)\,\mathrm{d}v\int_{\mathbb{R}_+} e^{-i\xi(qv^*)} f_i(v^*,\,t)\,\mathrm{d}v^* \\ &\quad - \mu\int_{\mathbb{R}_+} f_i(v^*,\,t)\,\mathrm{d}v^*\int_{\mathbb{R}_+} e^{-i\xi v} f_i(v,\,t)\,\mathrm{d}v.\end{aligned} \tag{3.4}$$

Recalling the Definition in (2.6) and introducing the probability density in the i-th node

$$\varrho_i^f := \int_{\mathbb{R}_+} f_i(v,\,t)\,\mathrm{d}v, \tag{3.5}$$

Equation (3.4) can be written as

$$\frac{\partial}{\partial t}\hat{f}_i(\xi,\,t) = \chi\sum_j P_{ij}\hat{f}_j(\xi,\,t) - \chi\hat{f}_i(\xi,\,t) + \mu\hat{f}_i(p\xi,\,t)\,\hat{f}_i(q\xi,\,t) - \mu\varrho_i^f\hat{f}_i(\xi,\,t)\,. \tag{3.6}$$

First of all, we apply the absolute value to this equation, obtaining

$$\begin{aligned}\left|\frac{\partial}{\partial t}\hat{f}_i(\xi,\,t)\right| &= \left|\chi\sum_j P_{ij}\hat{f}_j(\xi,\,t) - \chi\hat{f}_i(\xi,\,t) + \mu\hat{f}_i(p\xi,\,t)\,\hat{f}_i(q\xi,\,t) - \mu\varrho_i^f\hat{f}_i(\xi,\,t)\right| \\ &\le \chi\sum_j P_{ij}\left|\hat{f}_j(\xi,\,t)\right| + \chi\left|\hat{f}_i(\xi,\,t)\right| + \mu\left|\hat{f}_i(p\xi,\,t)\right|\left|\hat{f}_i(q\xi,\,t)\right| + \mu\varrho_i^f\left|\hat{f}_i(\xi,\,t)\right|.\end{aligned} \tag{3.7}$$

Using inequality $|\partial_t(\cdot)| \ge \partial_t|(\cdot)|$ and since $P_{ij} \le 1$, $\left|\hat{f}_i(q\xi,\,t)\right| \le 1$ and $\varrho_i^f \le 1$, the previous equation becomes

$$\frac{\partial}{\partial t}\left|\hat{f}_i(\xi,t)\right| \leq \chi\sum_j\left|\hat{f}_j(\xi,t)\right| + \chi\left|\hat{f}_i(\xi,t)\right| + \mu\left|\hat{f}_i(p\xi,t)\right|\left|\hat{f}_i(q\xi,t)\right| + \mu\left|\hat{f}_i(\xi,t)\right|. \tag{3.8}$$

Then, multiplying the equation by $\left|\hat{f}_i\right|$, we get

$$\frac{1}{2}\frac{\partial}{\partial t}\left|\hat{f}_i(\xi,t)\right|^2 \leq \chi\sum_j\left|\hat{f}_j(\xi,t)\right|\left|\hat{f}_i(\xi,t)\right| + \chi\left|\hat{f}_i(\xi,t)\right|^2 + \mu\left|\hat{f}_i(p\xi,t)\right|\left|\hat{f}_i(\xi,t)\right| + \mu\left|\hat{f}_i(\xi,t)\right|^2. \tag{3.9}$$

Summing on all the nodes we obtain

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\sum_i\left|\hat{f}_i(\xi,t)\right|^2 &\leq \chi\sum_j\left|\hat{f}_j(\xi,t)\right|\sum_i\left|\hat{f}_i(\xi,t)\right| + \chi\sum_i\left|\hat{f}_i(\xi,t)\right|^2 \\ &\quad + \mu\sum_i\left[\left|\hat{f}_i(p\xi,t)\right|\left|\hat{f}_i(\xi,t)\right|\right] + \mu\sum_i\left|\hat{f}_i(\xi,t)\right|^2 \\ &= \chi\left[\sum_i\left|\hat{f}_i(\xi,t)\right|\right]^2 + (\chi+\mu)\sum_i\left|\hat{f}_i(\xi,t)\right|^2 + \mu\sum_i\left[\left|\hat{f}_i(p\xi,t)\right|\left|\hat{f}_i(\xi,t)\right|\right]\end{aligned} \tag{3.10}$$

and using the Cauchy-Schwarz inequality for sums it reads

$$\left(\sum_i\left|\hat{f}_i(\xi,t)\right|\right)^2 = \left(\sum_i\left[\left|\hat{f}_i(\xi,t)\right||1|\right]\right)^2 \leq \left(\sum_i\left|\hat{f}_i(\xi,t)\right|^2\right)\left(\sum_i|1|^2\right) = N\sum_i\left|\hat{f}_i(\xi,t)\right|^2. \tag{3.11}$$

Therefore, the previous inequality results in

$$\frac{1}{2}\frac{\partial}{\partial t}\sum_i\left|\hat{f}_i(\xi,t)\right|^2 \leq \left[\chi(N+1)+\mu\right]\sum_i\left|\hat{f}_i(\xi,t)\right|^2 + \mu\sum_i\left[\left|\hat{f}_i(p\xi,t)\right|\left|\hat{f}_i(\xi,t)\right|\right]. \tag{3.12}$$

Integrating in ξ, we get

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\sum_i\int_{\mathbb{R}}\left|\hat{f}_i(\xi,t)\right|^2 \mathrm{d}\xi &\leq \left[\chi(N+1)+\mu\right]\sum_i\int_{\mathbb{R}}\left|\hat{f}_i(\xi,t)\right|^2 \mathrm{d}\xi \\ &\quad + \mu\sum_i\int_{\mathbb{R}}\left[\left|\hat{f}_i(p\xi,t)\right|\left|\hat{f}_i(\xi,t)\right|\right] \mathrm{d}\xi.\end{aligned} \tag{3.13}$$

Using the Cauchy-Schwarz inequality for integrals, this becomes

$$\int_{\mathbb{R}} \left|\hat{f}_i\left(p\xi,\, t\right)\right| \left|\hat{f}_i\left(\xi,\, t\right)\right| \,\mathrm{d}\xi \leq \left(\int_{\mathbb{R}} \left|\hat{f}_i\left(p\xi,\, t\right)\right|^2 \,\mathrm{d}\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left|\hat{f}_i\left(\xi,\, t\right)\right|^2 \,\mathrm{d}\xi\right)^{\frac{1}{2}} \tag{3.14}$$

and exploiting the fact that

$$\int_{\mathbb{R}} \left|\hat{f}_i\left(p\xi,\, t\right)\right|^2 \,\mathrm{d}\xi = \frac{1}{|p|} \int_{\mathbb{R}} \left|\hat{f}_i\left(\xi,\, t\right)\right|^2 \,\mathrm{d}\xi. \tag{3.15}$$

Defining

$$\left\|\hat{f}_i\left(\cdot,\, t\right)\right\|_2^2 := \int_{\mathbb{R}} \left|\hat{f}_i\left(\xi,\, t\right)\right|^2 \,\mathrm{d}\xi$$

and using Gronwall's Lemma we obtain the thesis.

□

This estimate can be sharpened by substituting p with $\max\{p,\, q\}$, but since it does not bring any added knowledge, it won't be necessary. It can be noted that the structure of the exponent recalls the one of the starting equation and that the effects of the two processes occurring can be easily analysed separately.

3.2 Uniqueness and continuous dependence on the initial data

In this section, the L^2-norm is used to prove uniqueness and continuous dependence on the initial data. In this study, this norm is preferred to an adaptation of the metric $d_s\left(f,\, g\right)$ because the presence of $|\xi|^s$ at the denominator leads to a more difficult computation (and to a weaker estimate of the difference of two solutions).

Theorem 3.2. *If a solution f of (2.5) exists, it is unique and depends continuously on the initial data.*

Proof. Let us suppose there are two solutions to (2.5), namely f and g. They must satisfy Equation (3.6). From now on, we drop the dependency on ξ, t for simplicity of notation. Hence:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\hat{f}_i - \hat{g}_i\right) = \chi \sum_j P_{ij}\left(\hat{f}_j - \hat{g}_j\right) - \chi\left(\hat{f}_i - \hat{g}_i\right) + \\ \mu\left[\hat{f}_i\left(p\xi\right)\hat{f}_i\left(q\xi\right) - \hat{g}_i\left(p\xi\right)\hat{g}_i\left(q\xi\right)\right] - \mu\left(\varrho_i^f \hat{f}_i - \varrho_i^g \hat{g}_i\right) \end{aligned} \tag{3.16}$$

Applying the absolute value to both members, we obtain

$$\left|\frac{\partial}{\partial t}\left(\hat{f}_i - \hat{g}_i\right)\right| \leq \chi \sum_j P_{ij} \left|\hat{f}_j - \hat{g}_j\right| + \chi \left|\hat{f}_i - \hat{g}_i\right| + \mu \left|\hat{f}_i (p\xi)\, \hat{f}_i (q\xi) - \hat{g}_i (p\xi)\, \hat{g}_i (q\xi)\right| + \mu \left|\varrho_i^f \hat{f}_i - \varrho_i^g \hat{g}_i\right| \tag{3.17}$$

and using the inequality $|\partial_t (\cdot)| \geq \partial_t |(\cdot)|$ it becomes

$$\frac{\partial}{\partial t}\left|\hat{f}_i - \hat{g}_i\right| \leq \chi \sum_j P_{ij} \left|\hat{f}_j - \hat{g}_j\right| + \chi \left|\hat{f}_i - \hat{g}_i\right| + \mu \left|\hat{f}_i (p\xi)\, \hat{f}_i (q\xi) - \hat{g}_i (p\xi)\, \hat{g}_i (q\xi)\right| + \mu \left|\varrho_i^f \hat{f}_i - \varrho_i^g \hat{g}_i\right|. \tag{3.18}$$

If we multiply everything by $\left|\hat{f}_i - \hat{g}_i\right|$ (and remembering $P_{ij} \leq 1$) we get

$$\frac{1}{2}\frac{\partial}{\partial t}\left|\hat{f}_i - \hat{g}_i\right|^2 \leq \chi \sum_j \left|\hat{f}_j - \hat{g}_j\right|\left|\hat{f}_i - \hat{g}_i\right| + \chi \left|\hat{f}_i - \hat{g}_i\right|^2 + \mu \left|\hat{f}_i (p\xi)\, \hat{f}_i (q\xi) - \hat{g}_i (p\xi)\, \hat{g}_i (q\xi)\right|\left|\hat{f}_i - \hat{g}_i\right| + \mu \left|\varrho_i^f \hat{f}_i - \varrho_i^g \hat{g}_i\right|\left|\hat{f}_i - \hat{g}_i\right|. \tag{3.19}$$

Summing on every node, it results in

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\sum_i \left|\hat{f}_i - \hat{g}_i\right|^2\right) \leq \chi \sum_j \left|\hat{f}_j - \hat{g}_j\right| \sum_i \left|\hat{f}_i - \hat{g}_i\right| + \chi \sum_i \left|\hat{f}_i - \hat{g}_i\right|^2 + \mu \sum_i \left[\left|\hat{f}_i (p\xi)\, \hat{f}_i (q\xi) - \hat{g}_i (p\xi)\, \hat{g}_i (q\xi)\right|\left|\hat{f}_i - \hat{g}_i\right|\right] + \mu \sum_i \left(\left|\varrho_i^f \hat{f}_i - \varrho_i^g \hat{g}_i\right|\left|\hat{f}_i - \hat{g}_i\right|\right) \tag{3.20}$$

However, investigating the second term, we obtain

$$\begin{aligned}\left|\hat{f}_i (p\xi)\, \hat{f}_i (q\xi) - \hat{g}_i (p\xi)\, \hat{g}_i (q\xi)\right| &= \left|\hat{f}_i (p\xi)\, \hat{f}_i (q\xi) - \hat{f}_i (p\xi)\, \hat{g}_i (q\xi) + \hat{f}_i (p\xi)\, \hat{g}_i (q\xi) - \hat{g}_i (p\xi)\, \hat{g}_i (q\xi)\right| \\ &\leq \left|\hat{f}_i (p\xi)\right|\left|\hat{f}_i (q\xi) - \hat{g}_i (q\xi)\right| + \left|\hat{g}_i (q\xi)\right|\left|\hat{f}_i (p\xi) - \hat{g}_i (p\xi)\right|.\end{aligned} \tag{3.21}$$

Since both $|\hat{g}_i (q\xi)|$ and $\left|\hat{f}_i (p\xi)\right|$ are less or equal than 1, we get

$$\left|\hat{f}_i (p\xi)\, \hat{f}_i (q\xi) - \hat{g}_i (p\xi)\, \hat{g}_i (q\xi)\right| \leq \left|\hat{f}_i (q\xi) - \hat{g}_i (q\xi)\right| + \left|\hat{f}_i (p\xi) - \hat{g}_i (p\xi)\right| \tag{3.22}$$

Moreover,

$$\begin{aligned}\left|\varrho_i^f \hat{f}_i - \varrho_i^g \hat{g}_i\right| &= \left|\varrho_i^f \hat{f}_i - \varrho_i^f \hat{g}_i + \varrho_i^f \hat{g}_i - \varrho_i^g \hat{g}_i\right| \\ &\leq \left|\varrho_i^f\right| \left|\hat{f}_i - \hat{g}_i\right| + \left|\hat{g}_i\right| \left|\varrho_i^f - \varrho_i^g\right| \\ &\leq \left|\hat{f}_i - \hat{g}_i\right| + \left|\hat{g}_i\right| \left|\varrho_i^f - \varrho_i^g\right| .\end{aligned} \tag{3.23}$$

Combining Equations (3.22) and (3.23) we get

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\left(\sum_i \left|\hat{f}_i - \hat{g}_i\right|^2\right) \leq \chi \sum_j \left|\hat{f}_j - \hat{g}_j\right| \sum_i \left|\hat{f}_i - \hat{g}_i\right| + \chi \sum_i \left|\hat{f}_i - \hat{g}_i\right|^2 + \\ \mu \sum_i \left[\left|\hat{f}_i (q\xi) - \hat{g}_i (q\xi)\right| + \left|\hat{f}_i (p\xi) - \hat{g}_i (p\xi)\right| \left|\hat{f}_i - \hat{g}_i\right|\right] \\ + \mu \sum_i \left(\left|\hat{f}_i - \hat{g}_i\right|^2 + \left|\hat{g}_i\right| \left|\varrho_i^f - \varrho_i^g\right| \left|\hat{f}_i - \hat{g}_i\right|\right)\end{aligned} \tag{3.24}$$

However, using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\sum_j \left|\hat{f}_j - \hat{g}_j\right| \sum_i \left|\hat{f}_i - \hat{g}_i\right| &= \left(\sum_i \left|\hat{f}_i - \hat{g}_i\right|\right)^2 \\ &= \left(\sum_i \left|\hat{f}_i - \hat{g}_i\right| |1|\right)^2 \\ &\leq \left(\sum_{i=1}^N \left|\hat{f}_i - \hat{g}_i\right|^2\right) \left(\sum_{i=1}^N 1^2\right) \\ &= N \sum_i \left|\hat{f}_i - \hat{g}_i\right|^2\end{aligned} \tag{3.25}$$

Therefore, combining these considerations, Equation (3.24) reads

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\left(\sum_i \left|\hat{f}_i - \hat{g}_i\right|^2\right) \leq \left[\chi (N+1) + \mu\right] \sum_i \left|\hat{f}_i - \hat{g}_i\right|^2 + \mu \sum_i \left|\hat{f}_i (q\xi) - \hat{g}_i (q\xi)\right| \left|\hat{f}_i - \hat{g}_i\right| \\ + \mu \sum_i \left|\hat{f}_i (p\xi) - \hat{g}_i (p\xi)\right| \left|\hat{f}_i - \hat{g}_i\right| + \mu \sum_i \left(\left|\hat{g}_i\right| \left|\varrho_i^f - \varrho_i^g\right| \left|\hat{f}_i - \hat{g}_i\right|\right)\end{aligned} \tag{3.26}$$

that, integrated in dξ, becomes

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\left(\sum_i\int_{\mathbb{R}}\left|\hat{f}_i-\hat{g}_i\right|^2\,\mathrm{d}\xi\right)\leq\left[\chi\left(N+1\right)+\mu\right]&\sum_i\int_{\mathbb{R}}\left|\hat{f}_i-\hat{g}_i\right|^2\,\mathrm{d}\xi\\&+\mu\sum_i\int_{\mathbb{R}}\left|\hat{f}_i\left(q\xi\right)-\hat{g}_i\left(q\xi\right)\right|\left|\hat{f}_i-\hat{g}_i\right|\,\mathrm{d}\xi\\&+\mu\sum_i\int_{\mathbb{R}}\left|\hat{f}_i\left(p\xi\right)-\hat{g}_i\left(p\xi\right)\right|\left|\hat{f}_i-\hat{g}_i\right|\,\mathrm{d}\xi\\&+\mu\sum_i\int_{\mathbb{R}}\left|\hat{g}_i\right|\left|\varrho_i^f-\varrho_i^g\right|\left|\hat{f}_i-\hat{g}_i\right|\,\mathrm{d}\xi\end{aligned}\tag{3.27}$$

Using the Cauchy-Schwarz inequality for integrals, we get

$$\begin{aligned}&\int_{\mathbb{R}}\left|\hat{f}_i\left(q\xi\right)-\hat{g}_i\left(q\xi\right)\right|\left|\hat{f}_i\left(\xi\right)-\hat{g}_i\left(\xi\right)\right|\,\mathrm{d}\xi\\&=\int_{\mathbb{R}}\left|\left(\hat{f}_i-\hat{g}_i\right)\left(q\xi\right)\right|\left|\left(\hat{f}_i-\hat{g}_i\right)\left(\xi\right)\right|\,\mathrm{d}\xi\\&\leq\left(\int_{\mathbb{R}}\left|\left(\hat{f}_i-\hat{g}_i\right)\left(q\xi\right)\right|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\left|\left(\hat{f}_i-\hat{g}_i\right)\left(\xi\right)\right|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}\end{aligned}\tag{3.28}$$

Applying a change of variable, we obtain that

$$\int_{\mathbb{R}}\left|\left(\hat{f}_i-\hat{g}_i\right)\left(q\xi\right)\right|^2\,\mathrm{d}\xi=\frac{1}{|q|}\int_{\mathbb{R}}\left|\left(\hat{f}_i-\hat{g}_i\right)\left(\xi\right)\right|^2\,\mathrm{d}\xi.\tag{3.29}$$

This means that

$$\begin{aligned}&\int_{\mathbb{R}}\left|\hat{f}_i\left(q\xi\right)-\hat{g}_i\left(q\xi\right)\right|\left|\hat{f}_i\left(\xi\right)-\hat{g}_i\left(\xi\right)\right|\,\mathrm{d}\xi\\&\leq\left(\int_{\mathbb{R}}\left|\left(\hat{f}_i-\hat{g}_i\right)\left(q\xi\right)\right|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\left|\left(\hat{f}_i-\hat{g}_i\right)\left(\xi\right)\right|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}\\&=\left(\frac{1}{|q|}\int_{\mathbb{R}}\left|\left(\hat{f}_i-\hat{g}_i\right)\left(\xi\right)\right|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\left|\left(\hat{f}_i-\hat{g}_i\right)\left(\xi\right)\right|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}\\&=\frac{1}{\sqrt{|q|}}\int_{\mathbb{R}}\left|\left(\hat{f}_i-\hat{g}_i\right)\left(\xi\right)\right|^2\,\mathrm{d}\xi.\end{aligned}\tag{3.30}$$

Analogously,

$$\int_{\mathbb{R}}\left|\hat{f}_i\left(p\xi\right)-\hat{g}_i\left(p\xi\right)\right|\left|\hat{f}_i\left(\xi\right)-\hat{g}_i\left(\xi\right)\right|\,\mathrm{d}\xi\leq\frac{1}{\sqrt{|p|}}\int_{\mathbb{R}}\left|\left(\hat{f}_i-\hat{g}_i\right)\left(\xi\right)\right|^2\,\mathrm{d}\xi.\tag{3.31}$$

Therefore

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\sum_i \int_{\mathbb{R}} \left|\hat{f}_i - \hat{g}_i\right|^2 \mathrm{d}\xi\right) \leq \left[\chi\left(N+1\right) + \mu\left(1 + \frac{1}{\sqrt{|q|}} + \frac{1}{\sqrt{|p|}}\right)\right] \sum_i \int_{\mathbb{R}} \left|\hat{f}_i - \hat{g}_i\right|^2 \mathrm{d}\xi$$
$$+ \mu \sum_i \int_{\mathbb{R}} |\hat{g}_i| \left|\varrho_i^f - \varrho_i^g\right| \left|\hat{f}_i - \hat{g}_i\right| \mathrm{d}\xi \tag{3.32}$$

Let us now focus on the last term. Knowing that $ab \leq \frac{1}{2}\left(a^2 + b^2\right)$ and identifying $\left|\hat{f}_i - \hat{g}_i\right|$ as a and $|\hat{g}_i|$ as b, we get

$$\sum_i \int_{\mathbb{R}} \left|\varrho_i^f - \varrho_i^g\right| \left|\hat{f}_i - \hat{g}_i\right| |\hat{g}_i| \, \mathrm{d}\xi \leq \frac{1}{2} \sum_i \int_{\mathbb{R}} \left|\varrho_i^f - \varrho_i^g\right| \left|\hat{f}_i - \hat{g}_i\right|^2 \mathrm{d}\xi + \frac{1}{2} \sum_i \int_{\mathbb{R}} \left|\varrho_i^f - \varrho_i^g\right| |\hat{g}_i|^2 \, \mathrm{d}\xi. \tag{3.33}$$

Here we can estimate from above the term $\left|\varrho_i^f - \varrho_i^g\right|$ in two different ways, While in the first case it is sufficient to recall that $\varrho_i^f \leq 1$ and estimate this quantity with 2, the second case requires more attention. First of all, we introduce the vectors $\underline{\varrho}^f$ and $\underline{\varrho}^g$ defined as $\left(\underline{\varrho}^f\right)_i = \varrho_i^f$ and $\left(\underline{\varrho}^g\right)_i = \varrho_i^g$.
Choosing $\varphi(v) = 1$ in (2.51) we get

$$\frac{\partial}{\partial t} \int_{\mathbb{R}_+} f_i(v, t) \, \mathrm{d}v = \chi \int_{\mathbb{R}_+} \left[\sum_j P_{ij} f_j(v, t) - f_i(v, t)\right] \mathrm{d}v \tag{3.34}$$

that, recalling the Definition (3.5), becomes

$$\frac{\partial}{\partial t}\varrho_i^f = \chi\left[\sum_j P_{ij}\varrho_j^f - \varrho_i^f\right]. \tag{3.35}$$

and in this vector notation it reads

$$\frac{\partial}{\partial t}\underline{\varrho}^f = \chi\left(P - I\right)\underline{\varrho}^f = A\underline{\varrho}^f. \tag{3.36}$$

where $A := \chi(P - I) \in \mathbb{R}^{N\times N}$. By introducing the exponential matrix

$$e^A := \sum_{k=0}^{+\infty} \frac{A^k}{k!}, \tag{3.37}$$

we obtain that $\underline{\varrho}^f(t) = \underline{\varrho}^f(0)\, e^{At} = \varrho_0^f e^{At}$ and analogously $\underline{\varrho}^g(t) = \varrho_0^g e^{At}$. As a consequence, we deduce that

$$\left|\varrho_i^f - \varrho_i^g\right| \leq \sum_i \left|\varrho_i^f - \varrho_i^g\right| = \left\|\underline{\varrho}^f(t) - \underline{\varrho}^g(t)\right\|_1 \leq \left\|e^{At}\right\| \left\|\varrho_0^f - \varrho_0^g\right\|_1. \tag{3.38}$$

where $\left\|e^{At}\right\|$ is the matrix norm of e^{At}.
Let us now consider the eigenvalues of the matrix A. Since P is left-stochastic, its eigenvalues have absolute value not greater than 1, with one eigenvalue equal to 1 with multiplicity 1. Moreover, the identity matrix I has eigenvalues all equal to 1, that means that 0 is a simple eigenvalue of A and all the other eigenvalues have negative real part. This leads to $\left\|e^{At}\right\| \leq 1$, resulting in

$$\left|\varrho_i^f - \varrho_i^g\right| \leq \left\|\varrho_0^f - \varrho_0^g\right\|_1 \qquad \forall i. \tag{3.39}$$

Hence, we get

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\left(\sum_i \int_{\mathbb{R}} \left|\hat{f}_i - \hat{g}_i\right|^2 \mathrm{d}\xi\right) &\leq \left[\chi\left(N+1\right) + \mu\left(1 + \frac{1}{\sqrt{|q|}} + \frac{1}{\sqrt{|p|}}\right)\right] \sum_i \int_{\mathbb{R}} \left|\hat{f}_i - \hat{g}_i\right|^2 \mathrm{d}\xi \\
&\quad + \mu \sum_i \int_{\mathbb{R}} \left|\hat{f}_i - \hat{g}_i\right|^2 \mathrm{d}\xi + \frac{1}{2}\mu\left\|\varrho_0^f - \varrho_0^g\right\|_1 \sum_i \int_{\mathbb{R}} |\hat{g}_i|^2 \mathrm{d}\xi \\
&= \left[\chi\left(N+1\right) + \mu\left(2 + \frac{1}{\sqrt{|q|}} + \frac{1}{\sqrt{|p|}}\right)\right] \sum_i \int_{\mathbb{R}} \left|\hat{f}_i - \hat{g}_i\right|^2 \mathrm{d}\xi \\
&\quad + \frac{1}{2}\mu\left\|\varrho_0^f - \varrho_0^g\right\|_1 \sum_i \left\|\hat{g}_i\right\|_2^2
\end{aligned} \tag{3.40}$$

Using the result of the previous section - namely Theorem 3.1 - this inequality becomes

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\left(\sum_i \int_{\mathbb{R}} \left|\hat{f}_i - \hat{g}_i\right|^2 \mathrm{d}\xi\right) &\leq \left[\chi\left(N+1\right) + \mu\left(2 + \frac{1}{\sqrt{|q|}} + \frac{1}{\sqrt{|p|}}\right)\right] \sum_i \int_{\mathbb{R}} \left|\hat{f}_i - \hat{g}_i\right|^2 \mathrm{d}\xi + \frac{1}{2}\mu\left\|\varrho_0^f\right. \\
&\quad \left. - \varrho_0^g\right\|_1 \sum_i \left\|\hat{g}_i\left(\cdot, 0\right)\right\|_2^2 \exp\left\{2\left[\chi\left(N+1\right) + \mu\left(\frac{1}{\sqrt{|p|}} + 1\right)\right] t\right\}
\end{aligned} \tag{3.41}$$

Integrating in time from 0 to t we get

$$\begin{aligned}
\sum_i \int_{\mathbb{R}} \left|\hat{f}_i - \hat{g}_i\right|^2 (t)\, \mathrm{d}\xi &\leq \sum_i \int_{\mathbb{R}} \left|\hat{f}_i - \hat{g}_i\right|^2 (0)\, \mathrm{d}\xi \\
&\quad + 2t\left[\chi\left(N+1\right) + \mu\left(2 + \frac{1}{\sqrt{|q|}} + \frac{1}{\sqrt{|p|}}\right)\right] \sum_i \int_{\mathbb{R}} \left|\hat{f}_i - \hat{g}_i\right|^2 \mathrm{d}\xi + \mu\left\|\varrho_0^f\right. \\
&\quad \left. - \varrho_0^g\right\|_1 \sum_i \left\|\hat{g}_i\left(\cdot, 0\right)\right\|_2^2 \int_0^t \exp\left\{2\left[\chi\left(N+1\right) + \mu\left(\frac{1}{\sqrt{|p|}} + 1\right)\right] t\right\} \mathrm{d}t
\end{aligned} \tag{3.42}$$

that, using Gronwall's Lemma, becomes

$$
\begin{aligned}
&\sum_i \int_{\mathbb{R}} \left| \hat{f}_i - \hat{g}_i \right|^2 (t) \, \mathrm{d}\xi \\
&\leq \Bigg(\mu \left\| \varrho_0^f - \varrho_0^g \right\|_1 \sum_i \left\| \hat{g}_i (\cdot, 0) \right\|_2^2 \\
&\qquad \cdot \frac{1}{2 \left[\chi (N+1) + \mu \left(\frac{1}{\sqrt{|p|}} + 1 \right) \right]} \exp \left\{ 2 \left[\chi (N+1) + \mu \left(\frac{1}{\sqrt{|p|}} + 1 \right) \right] t \right\} \\
&\quad + \sum_i \int_{\mathbb{R}} \left| \hat{f}_i - \hat{g}_i \right|^2 (0) \, \mathrm{d}\xi \Bigg) \exp \left\{ 2 \left[\chi (N+1) + \mu \left(2 + \frac{1}{\sqrt{|p|}} + \frac{1}{\sqrt{|q|}} \right) \right] t \right\}
\end{aligned}
\tag{3.43}
$$

Hence, if f and g have the same initial conditions their distance is null for every time, proving that this solution is unique, provided it exists. If the two initial conditions differ slightly, the difference between the two solutions remains small at every time, therefore the continuous dependence on the initial data is proven.

□

Chapter 4

Study of the asymptotic behaviour of the solution

The aim of this chapter is to understand when an equilibrium can be reached. In Section 4.1 a new notation is introduced and some hypotheses are made, then in Section 4.2 the metric d_s is adapted to the new context. The last Section focuses on the asymptotic behaviour of the solutions under the new hypotheses, leading to some considerations on the role of the parameters.

4.1 Notation and hypotheses

Since the importance of what happens in a node can be thought as proportional to the number of elements in it - the density ϱ_i^f-, it is natural to write the density function f_i as $f_i = \varrho_i^f F_i$, hence $\hat{f}_i = \varrho_i^f \hat{F}_i$. Let us suppose the number of elements is sufficiently large.

Theorem 4.1. *If $\varrho_i^f(0) > 0$ for a fixed i, then $\varrho_i^f(t) > 0$ for every t.*

Proof. Choosing $\varphi(v) = 1$ in 2.51 we get

$$\frac{\partial}{\partial t} \int_{\mathbb{R}_+} f_i(v, t)\, \mathrm{d}v = \chi \int_{\mathbb{R}_+} \left[\sum_j P_{ij} f_j(v, t) - f_i(v, t) \right] \mathrm{d}v \tag{4.1}$$

that, recalling the Definition (3.5), becomes

$$\frac{\partial}{\partial t} \varrho_i^f = \chi \left[\sum_j P_{ij} \varrho_j^f - \varrho_i^f \right]. \tag{4.2}$$

Since ϱ_j^f is non-negative for every j by definition, it holds that

$$\frac{\partial}{\partial t}\varrho_i^f = \chi\left[\sum_j P_{ij}\varrho_j^f - \varrho_i^f\right] \geq -\chi\varrho_i^f \tag{4.3}$$

and, as a result, that

$$\varrho_i^f\left(t\right) \geq \varrho_i^f\left(0\right)e^{-\chi t} > 0 \qquad \forall t > 0. \tag{4.4}$$

□

Furthermore, in [6] it is shown that, for every i, ϱ_i^f tends to a strictly positive value. In view of the fact that we are studying the asymptotic behaviour, it is reasonable to assume that no node is empty at time zero (and consequently at every time $t > 0$). For the same reason, it can be assumed that the social contact rate varies from node to node, introducing a specific μ_i for each node. These considerations being done, Equation (2.51) in its Fourier form becomes

$$\frac{\partial}{\partial t}\varrho_i^f\hat{F}_i\left(\xi,\,t\right) = \chi\left[\sum_j P_{ij}\varrho_j^f\hat{F}_j\left(\xi,\,t\right) - \varrho_i^f\hat{F}_i\left(\xi,\,t\right)\right] + \left(\varrho_i^f\right)^2\mu_i\left[\hat{F}_i\left(p\xi,\,t\right)\hat{F}_i\left(q\xi,\,t\right) - \hat{F}_i\left(\xi,\,t\right)\right]. \tag{4.5}$$

Since the whole model focuses on the comparison between the two processes their effects should be comparable in each node, hence the two terms should be of the same order in ϱ_i^f. Therefore, it makes sense to assume

$$\mu_i\left(t\right) = \frac{\mu_0}{\varrho_i^f\left(t\right)}$$

with μ_0 constant, leading to

$$\frac{\partial}{\partial t}\varrho_i^f\hat{F}_i\left(\xi,\,t\right) = \chi\left[\sum_j P_{ij}\varrho_j^f\hat{F}_j\left(\xi,\,t\right) - \varrho_i^f\hat{F}_i\left(\xi,\,t\right)\right] + \varrho_i^f\mu_0\left[\hat{F}_i\left(p\xi,\,t\right)\hat{F}_i\left(q\xi,\,t\right) - \hat{F}_i\left(\xi,\,t\right)\right], \tag{4.6}$$

the Boltzmann-type equation in each node with the new hypotheses and notation.

4.2 Adaptation of the metric

The density plays a central role also in adapting the metric d_s: it can be considered as the weight each node has when computing the distance. This chapter will focus on studying the asymptotic behaviour of solutions with the same initial elements' distribution across the nodes. If v is the viral charge and the nodes represent the different cities, this means that, at time zero, the inhabitants of each city are the same number in both scenarios (i.e. the two solutions). Due to the fact that the adjacency matrix P_{ij} is the same for both functions, the distribution (and the density in each node) will coincide at every time. This means that $\varrho_i^f = \varrho_i^g = \varrho_i$, and with the notation introduced in the previous Section we can write $\hat{f}_i = \varrho_i \hat{F}_i$ and $\hat{g}_i = \varrho_i \hat{G}_i$.

The adapted metric reads

$$\sum_i \varrho_i d_s\left(F_i, G_i\right) = \sum_i \varrho_i \sup_{\xi \in \mathbb{R} \setminus \{0\}} \frac{\left|\hat{F}_i - \hat{G}_i\right|}{|\xi|^s}. \tag{4.7}$$

This is a metric because the weights are non-negative by definition and non-zero at every time by hypothesis, and since the weights are less than or equal to 1 and the number of nodes is finite, the finiteness comes from the finiteness of d_s discussed in Section 2.2.

4.3 Asymptotic trend of the solutions

The result presented in this Section is an adaptation of Theorem 2.3 with the above described changes in the norm and in the notation.

Theorem 4.2. *If $p^s + q^s < 1$, two solutions with the same initial distribution of elements across the nodes tend to each other, in particular it holds that*

$$\sum_i \varrho_i(t)\, d_s\left(F_i(t), G_i(t)\right) \leq \sum_i \varrho_i(0)\, d_s\left(F_i(0), G_i(0)\right) \exp\left\{\mu_0\left(p^s + q^s - 1\right)t\right\}. \tag{4.8}$$

Proof. Since f and g solve Equation (4.6) and using the new notation, we can write

$$\begin{aligned}\frac{\partial}{\partial t}\varrho_i\left(\hat{F}_i - \hat{G}_i\right) = \chi &\left[\sum_j P_{ij}\varrho_j\left(\hat{F}_j - \hat{G}_j\right) - \varrho_i\left(\hat{F}_i - \hat{G}_i\right)\right] \\ &+ \mu_0\varrho_i\left[\hat{F}_i(p\xi)\hat{F}_i(q\xi) - \hat{G}_i(p\xi)\hat{G}_i(q\xi)\right] - \mu_0\varrho_i\left(\hat{F}_i - \hat{G}_i\right)\end{aligned} \tag{4.9}$$

that can be reordered in the following way:

$$\begin{aligned}\frac{\partial}{\partial t}\varrho_i\left(\hat{F}_i - \hat{G}_i\right) + (\chi + \mu_0)\varrho_i\left(\hat{F}_i - \hat{G}_i\right) = \chi &\left[\sum_j P_{ij}\varrho_j\left(\hat{F}_j - \hat{G}_j\right)\right] \\ &+ \mu_0\varrho_i\left[\hat{F}_i(p\xi)\hat{F}_i(q\xi) - \hat{G}_i(p\xi)\hat{G}_i(q\xi)\right].\end{aligned} \tag{4.10}$$

Multiplying by $e^{(\chi+\mu_o)t}$ it becomes

$$\frac{\partial}{\partial t}\left[e^{(\chi+\mu_o)t}\varrho_i\left(\hat{F}_i-\hat{G}_i\right)\right]=e^{(\chi+\mu_o)t}\chi\left[\sum_j P_{ij}\varrho_j\left(\hat{F}_j-\hat{G}_j\right)\right] \\ +e^{(\chi+\mu_o)t}\mu_0\varrho_i\left[\hat{F}_i\left(p\xi\right)\hat{F}_i\left(q\xi\right)-\hat{G}_i\left(p\xi\right)\hat{G}_i\left(q\xi\right)\right]. \tag{4.11}$$

Considering its absolute value and fixing $s>0$ we get

$$\frac{\left|\frac{\partial}{\partial t}\left[e^{(\chi+\mu_o)t}\varrho_i\left(\hat{F}_i-\hat{G}_i\right)\right]\right|}{|\xi|^s} \\ =\frac{\left|e^{(\chi+\mu_o)t}\chi\left[\sum_j P_{ij}\varrho_j\left(\hat{F}_j-\hat{G}_j\right)\right]+e^{(\chi+\mu_o)t}\mu_0\varrho_i\left[\hat{F}_i\left(p\xi\right)\hat{F}_i\left(q\xi\right)-\hat{G}_i\left(p\xi\right)\hat{G}_i\left(q\xi\right)\right]\right|}{|\xi|^s} \tag{4.12}$$

that using the inequality $|\partial_t\left(\cdot\right)|\geq\partial_t\left|\left(\cdot\right)\right|$ becomes

$$\frac{\partial}{\partial t}\left[e^{(\chi+\mu_0)t}\varrho_i\frac{\left|\hat{F}_i-\hat{G}_i\right|}{|\xi|^s}\right]\leq\chi e^{(\chi+\mu_0)t}\sum_j P_{ij}\varrho_j\frac{\left|\hat{F}_j-\hat{G}_j\right|}{|\xi|^s} \\ +\mu_0 e^{(\chi+\mu_0)t}\varrho_i\frac{\left|\hat{F}_i\left(p\xi\right)\hat{F}_i\left(q\xi\right)-\hat{G}_i\left(p\xi\right)\hat{G}_i\left(q\xi\right)\right|}{|\xi|^s}. \tag{4.13}$$

Due to the fact that the right term is by definition less than its *supremum*, we get

$$\frac{\partial}{\partial t}\left[e^{(\chi+\mu_0)t}\varrho_i\frac{\left|\hat{F}_i-\hat{G}_i\right|}{|\xi|^s}\right]\leq\chi e^{(\chi+\mu_0)t}\sup_{\xi\in\mathbb{R}\setminus\{0\}}\sum_j P_{ij}\varrho_j\frac{\left|\hat{F}_j-\hat{G}_j\right|}{|\xi|^s} \\ +\mu_0 e^{(\chi+\mu_0)t}\sup_{\xi\in\mathbb{R}\setminus\{0\}}\varrho_i\frac{\left|\hat{F}_i\left(p\xi\right)\hat{F}_i\left(q\xi\right)-\hat{G}_i\left(p\xi\right)\hat{G}_i\left(q\xi\right)\right|}{|\xi|^s}. \tag{4.14}$$

We need now to investigate the last term. First of all, it is true that

$$\begin{aligned}
&\varrho_i \frac{\left|\hat{F}_i(p\xi)\hat{F}_i(q\xi) - \hat{G}_i(p\xi)\hat{G}_i(q\xi)\right|}{|\xi|^s} \\
&= \varrho_i \frac{\left|\hat{F}_i(p\xi)\hat{F}_i(q\xi) - \hat{F}_i(p\xi)\hat{G}_i(q\xi) + \hat{F}_i(p\xi)\hat{G}_i(q\xi) - \hat{G}_i(p\xi)\hat{G}_i(q\xi)\right|}{|\xi|^s} \\
&\leq \varrho_i \left|\hat{F}_i(p\xi)\right| \frac{\left|\hat{F}_i(q\xi) - \hat{G}_i(q\xi)\right|}{|\xi|^s} + \varrho_i \left|\hat{G}_i(q\xi)\right| \frac{\left|\hat{F}_i(p\xi) - \hat{G}_i(p\xi)\right|}{|\xi|^s} \\
&\leq \varrho_i \frac{\left|\hat{F}_i(q\xi) - \hat{G}_i(q\xi)\right|}{|\xi|^s} + \varrho_i \frac{\left|\hat{F}_i(p\xi) - \hat{G}_i(p\xi)\right|}{|\xi|^s}
\end{aligned} \tag{4.15}$$

and, since it holds for every ξ, it holds also for its *supremum*, leading us to

$$\begin{aligned}
\sup_{\xi \in \mathbb{R}\setminus\{0\}} \varrho_i \frac{\left|\hat{F}_i(p\xi)\hat{F}_i(q\xi) - \hat{G}_i(p\xi)\hat{G}_i(q\xi)\right|}{|\xi|^s} &\leq \sup_{\xi \in \mathbb{R}\setminus\{0\}} \varrho_i \frac{\left|\hat{F}_i(q\xi) - \hat{G}_i(q\xi)\right|}{|\xi|^s} \\
&\quad + \sup_{\xi \in \mathbb{R}\setminus\{0\}} \varrho_i \frac{\left|\hat{F}_i(p\xi) - \hat{G}_i(p\xi)\right|}{|\xi|^s} \\
&= \sup_{q\xi \in \mathbb{R}\setminus\{0\}} \varrho_i \frac{\left|\hat{F}_i(q\xi) - \hat{G}_i(q\xi)\right|}{|q\xi|^s} q^s \\
&\quad + \sup_{p\xi \in \mathbb{R}\setminus\{0\}} \varrho_i \frac{\left|\hat{F}_i(p\xi) - \hat{G}_i(p\xi)\right|}{|p\xi|^s} p^s \\
&\leq (p^s + q^s) \sup_{\xi \in \mathbb{R}\setminus\{0\}} \varrho_i \frac{\left|\hat{F}_i(\xi) - \hat{G}_i(\xi)\right|}{|\xi|^s}.
\end{aligned} \tag{4.16}$$

Hence, exploiting this result, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\left[e^{(\chi+\mu_0)t} \varrho_i \frac{\left|\hat{F}_i - \hat{G}_i\right|}{|\xi|^s}\right] &\leq \chi e^{(\chi+\mu_0)t} \sup_{\xi \in \mathbb{R}\setminus\{0\}} \sum_j P_{ij} \varrho_j \frac{\left|\hat{F}_j - \hat{G}_j\right|}{|\xi|^s} \\
&\quad + \mu_0 e^{(\chi+\mu_0)t} (p^s + q^s) \sup_{\xi \in \mathbb{R}\setminus\{0\}} \varrho_i \frac{\left|\hat{F}_i - \hat{G}_i\right|}{|\xi|^s}.
\end{aligned} \tag{4.17}$$

that, integrated in time between 0 and t, becomes

$$e^{(\chi+\mu_0)t}\varrho_i\frac{\left|\hat{F}_i-\hat{G}_i\right|}{|\xi|^s} \leq \left(e^{(\chi+\mu_0)t}\varrho_i\frac{\left|\hat{F}_i-\hat{G}_i\right|}{|\xi|^s}\right|_{t=0} + \int_0^t \left[\chi e^{(\chi+\mu_0)\tau} \sup_{\xi\in\mathbb{R}\setminus\{0\}} \sum_j P_{ij}\varrho_j\frac{\left|\hat{F}_j-\hat{G}_j\right|}{|\xi|^s} + \mu_0 e^{(\chi+\mu_0)\tau}\left(p^s+q^s\right) \sup_{\xi\in\mathbb{R}\setminus\{0\}} \varrho_i\frac{\left|\hat{F}_i-\hat{G}_i\right|}{|\xi|^s}\right] \mathrm{d}\tau. \tag{4.18}$$

Since it holds for every ξ, it holds also for its *supremum*, giving

$$\begin{aligned} e^{(\chi+\mu_0)t} \sup_{\xi\in\mathbb{R}\setminus\{0\}} \varrho_i\frac{\left|\hat{F}_i-\hat{G}_i\right|}{|\xi|^s} \leq & \left(e^{(\chi+\mu_0)t} \sup_{\xi\in\mathbb{R}\setminus\{0\}} \varrho_i\frac{\left|\hat{F}_i-\hat{G}_i\right|}{|\xi|^s}\right|_{t=0} \\ & + \int_0^t \chi\sum_j P_{ij}e^{(\chi+\mu_0)\tau} \sup_{\xi\in\mathbb{R}\setminus\{0\}} \varrho_j\frac{\left|\hat{F}_j-\hat{G}_j\right|}{|\xi|^s}\,\mathrm{d}\tau \\ & + \int_0^t \mu_0 e^{(\chi+\mu_0)\tau}\left(p^s+q^s\right) \sup_{\xi\in\mathbb{R}\setminus\{0\}} \varrho_i\frac{\left|\hat{F}_i-\hat{G}_i\right|}{|\xi|^s}\,\mathrm{d}\tau. \end{aligned} \tag{4.19}$$

Summing on every node we obtain

$$\begin{aligned} e^{(\chi+\mu_0)t}\sum_i \sup_{\xi\in\mathbb{R}\setminus\{0\}} \varrho_i\frac{\left|\hat{F}_i-\hat{G}_i\right|}{|\xi|^s} \leq & \left(e^{(\chi+\mu_0)t}\sum_i \sup_{\xi\in\mathbb{R}\setminus\{0\}} \varrho_i\frac{\left|\hat{F}_i-\hat{G}_i\right|}{|\xi|^s}\right|_{t=0} \\ & + \int_0^t \chi\sum_j\sum_i P_{ij}\, e^{(\chi+\mu_0)\tau} \sup_{\xi\in\mathbb{R}\setminus\{0\}} \varrho_j\frac{\left|\hat{F}_j-\hat{G}_j\right|}{|\xi|^s}\,\mathrm{d}\tau \\ & + \int_0^t \mu_0\left(p^s+q^s\right)e^{(\chi+\mu_0)\tau}\sum_i \sup_{\xi\in\mathbb{R}\setminus\{0\}} \varrho_i\frac{\left|\hat{F}_i-\hat{G}_i\right|}{|\xi|^s}\,\mathrm{d}\tau \end{aligned} \tag{4.20}$$

and since P is left stochastic then $\sum_i P_{ij} = 1$, and the previous inequality becomes

$$\begin{aligned}
e^{(\chi+\mu_0)t} \sum_i \sup_{\xi \in \mathbb{R}\setminus\{0\}} \varrho_i \frac{\left|\hat{F}_i - \hat{G}_i\right|}{|\xi|^s} \leq & \left(e^{(\chi+\mu_0)t} \sum_i \sup_{\xi\in\mathbb{R}\setminus\{0\}} \varrho_i \frac{\left|\hat{F}_i - \hat{G}_i\right|}{|\xi|^s} \right|_{t=0} \\
& + \int_0^t \chi e^{(\chi+\mu_0)\tau} \sum_j \sup_{\xi\in\mathbb{R}\setminus\{0\}} \varrho_j \frac{\left|\hat{F}_j - \hat{G}_j\right|}{|\xi|^s} \,\mathrm{d}\tau \\
& + \int_0^t \mu_0 \left(p^s + q^s\right) e^{(\chi+\mu_0)\tau} \sum_i \sup_{\xi\in\mathbb{R}\setminus\{0\}} \varrho_i \frac{\left|\hat{F}_i - \hat{G}_i\right|}{|\xi|^s} \,\mathrm{d}\tau \\
= & \left(e^{(\chi+\mu_0)t} \sum_i \sup_{\xi\in\mathbb{R}\setminus\{0\}} \varrho_i \frac{\left|\hat{F}_i - \hat{G}_i\right|}{|\xi|^s} \right|_{t=0} \\
& + \int_0^t \left[\chi + \mu_0 \left(p^s + q^s\right)\right] e^{(\chi+\mu_0)\tau} \sum_i \sup_{\xi\in\mathbb{R}\setminus\{0\}} \varrho_i \frac{\left|\hat{F}_i - \hat{G}_i\right|}{|\xi|^s} \,\mathrm{d}\tau
\end{aligned} \tag{4.21}$$

Using Gronwall's Lemma we get

$$\begin{aligned}
e^{(\chi+\mu_0)t} \sum_i \varrho_i \sup_{\xi \in \mathbb{R}\setminus\{0\}} \frac{\left|\hat{F}_i - \hat{G}_i\right|}{|\xi|^s} \leq & \left(\sum_i \varrho_i \sup_{\xi\in\mathbb{R}\setminus\{0\}} \frac{\left|\hat{F}_i - \hat{G}_i\right|}{|\xi|^s} \right|_{t=0} \exp\left\{ \int_0^t [\chi \right. \\
& \left. + \mu_0 \left(p^s + q^s\right)\right] \,\mathrm{d}\tau \Bigg\} \\
= & \left(\sum_i \varrho_i \sup_{\xi\in\mathbb{R}\setminus\{0\}} \frac{\left|\hat{F}_i - \hat{G}_i\right|}{|\xi|^s} \right|_{t=0} \exp\left\{ \left[\chi + \mu_0 \left(p^s + q^s\right)\right] t \right\}
\end{aligned} \tag{4.22}$$

that leads to

$$\begin{aligned}
\sum_i \varrho_i \sup_{\xi \in \mathbb{R}\setminus\{0\}} \frac{\left|\hat{F}_i - \hat{G}_i\right|}{|\xi|^s} \leq & \left(\sup_{\xi\in\mathbb{R}\setminus\{0\}} \sum_i \varrho_i \frac{\left|\hat{F}_i - \hat{G}_i\right|}{|\xi|^s} \right|_{t=0} \exp\left\{ \left[\chi + \mu_0 \left(p^s + q^s\right)\right] t - \mu_0 t - \chi t \right\} \\
= & \left(\sum_i \varrho_i \sup_{\xi\in\mathbb{R}\setminus\{0\}} \frac{\left|\hat{F}_i - \hat{G}_i\right|}{|\xi|^s} \right|_{t=0} \exp\left\{ \mu_0 \left(p^s + q^s - 1\right) t \right\}
\end{aligned} \tag{4.23}$$

that, recalling Equation (4.7), gives the thesis. □

It is interesting to underline that this result does not depend on the number of nodes nor on the mobility rate, but only on the interaction between individuals.

Corollary 4.3. *Let $p^s + q^s < 1$ and let us fix a given initial distribution of elements across the nodes. Then, if an equilibrium distribution exists, it is unique.*

Proof. If an equilibrium distribution (of elements with given states across the nodes) f exists, this is an equilibrium solution of (4.6). If another equilibrium solution g with the same initial distribution of elements across the nodes but different states exists and the parameters satisfy $p^s + q^s < 1$, then Theorem 4.2 holds and, consequently, the distance between f and g decreases in time. On the other hand, f and g are equilibria: their distance cannot change in time, leading to a contradiction. □

Another noteworthy consideration is that the sufficient condition for the uniqueness of the equilibrium is the same of Theorem 2.3 (and discussed in Section 2.3.3).

Chapter 5

Conclusion

In this work, we studied the Boltzmann-type equation on a graph structure as proposed in [6]. We first proved the uniqueness and the continuous dependence on the initial data of a solution, provided it exists, by extending the L^2-norm to the graph structure. This result is general, since it is obtained without the introduction of further hypotheses.

Moreover, the last chapter focuses on the asymptotic behaviour of a solution. It is shown that, if the initial distribution of the mass and its evolution are fixed, then, supposing there is an equilibrium distribution, the uniqueness of the equilibrium solution depends on the sign of a quantity based on the parameters that describe the microscopic interaction between two elements. This is shown to be the same quantity that states the uniqueness of the equilibrium in a Boltzmann-type equation without the graph structure (as proved in [8]).

Further research can be performed by weakening the hypotheses in the last section: it can be interesting to investigate the difference between two solutions with slightly different initial distributions in a sensitivity analysis.

In addition, in this work such as in [6], the act of moving from a node to another is modeled by a Markov-type process with an adjacency matrix P_{ij} that is constant in time. Although it does not play a central role in the study and in Chapters 3 and 4 only its left-stochasticity is used, the fact that it is constant in time is fundamental to prove that the ϱ_i^f are strictly positive at every time and allows us to use the densities as weights in the adaptation of the Fourier-based metric. It can be interesting to investigate a case where this matrix depends on the time, maybe where it is periodic (i.e. to describe periodic migrations) or where a node is temporarily disconnected (i.e. if the nodes represent some cities and it is impossible to reach or leave one of them).

Bibliography

[1] M. Bisi, J. A. Carrillo, and G. Toscani. "Decay rates in probability metrics towards homogeneous cooling states for the inelastic Maxwell model". In: *J. Stat. Phys.* 124 (2006), pp. 625–653.

[2] J. A. Carrillo, G. Toscani, et al. "Contractive probability metrics and asymptotic behavior of dissipative kinetic equations". In: *Riv. Mat. Univ. Parma* 6.7 (2007), pp. 75–198.

[3] B. Düring and M. Wolfram. "Opinion dynamics: inhomogeneous Boltzmann-type equations modelling opinion leadership and political segregation". In: *Proc. R. Soc. A* 471.2182 (2015), p. 20150345.

[4] G. Gabetta, G. Toscani, and B. Wennberg. "Metrics for probability distributions and the trend to equilibrium for solutions of the Boltzmann equation". In: *J. Stat. Phys.* 81 (1995), pp. 901–934.

[5] K. Kanazawa et al. "Derivation of the Boltzmann equation for financial Brownian motion: Direct observation of the collective motion of high-frequency traders". In: *Phys. Rev. Lett.* 120.13 (2018), p. 138301.

[6] N. Loy and A. Tosin. "A viral load-based model for epidemic spread on spatial networks". In: *Math. Biosci. Eng* 18.5 (2021), pp. 5635–5663.

[7] N. Loy and A. Tosin. "Boltzmann-type equations for multi-agent systems with label switching". In: *Kinet. Relat. Models* 14.5 (2021), pp. 867–894.

[8] L. Pareschi and G. Toscani. *Interacting multiagent systems: kinetic equations and Monte Carlo methods*. OUP Oxford, 2013.

[9] A. Pulvirenti and G. Toscani. "Asymptotic properties of the inelastic Kac model". In: *J. Stat. Phys.* 114 (2004), pp. 1453–1480.