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Anti-plane analysis of Francfort-Marigo model for quasi-static brittle fracture

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Summary

The study and prediction of crack behavior in materials have always been central to classical theories of solids. Since Griffith's 1921 seminal paper, all subsequent literature on brittle fracture has built on his work.
In this thesis, we focus on the extended variational setting of Francfort and Marigo, which aims at studying quasi-static brittle fracture by incorporating an interplay between elastic and fracture energies and a minimization over admissible crack evolutions. The assumptions and limitations of the model are clearly stated, as are the similarities and differences with Griffith's model.
Some existence results are needed for an appropriate formalization in the context of the calculus of variations. By focusing on two dimensional anti-plane cracks, we can avoid setting the problem in spaces of (special) functions of bounded variation or deformation, but this comes at the price of requiring some results on the fine properties of Sobolev functions and sets in metric space, as well as approximation results for the crack set. The existence of the quasi-static evolution is then obtained by discretizing the crack evolution and letting the time step tends to 0. We prove that Griffith's criteria can be obtained from this setting, as well as certain results on the regularity of the energy release rate and the stress intensity factor.

Contents

Uomini forti,
destini forti.
Uomini deboli,
destini deboli.

[Luciano Spalletti]

Introduction

Classical fracture mechanics was born almost entirely due to A. A. Griffith [2, 3] and, ever since, subsequent contributions to the engineering viewpoint have built on his works. Notable results were obtained by G. R. Irwin [7, 8], and J. R. Rice and G. P. Cherepanov [9, 11, 12, 13], expanding on Griffith's framework. The main ideas behind this framework were that the macroscopic cracks were the effect of microscopic defects in the atomic lattice, the actors involved could be represented by energy densities at each point, and the crack propagation was the result of an energy interplay between the growth of surface energy and the decrease of bulk energy. This last point in particular was quite the novelty in the literature at that time, and it proved useful for the next century or so.

Up until the end of the 20th century, mathematicians did not involve with matters of fracture, also due the to lack of appropriate tools, but with the advent of supercomputers, the engineering community felt the need for a formalization of Griffith's to do numerical simulations with it. Mathematical tools like measure theory and appropriate spaces of functions became widespread and well-researched, and eventually, the first variational model for fracture appeared in 1998 due to G. A. Francfort and J. J. Marigo [22], which dealt with a quasi-static crack evolution in a brittle material, similar to Griffith's setting. After just two years Francfort and Marigo, together with B. Bourdin [25], started to pave the way for the so-called *phase-field models*, essential in the numerical formulation.

This model, although seemingly powerful, lacked an existence result for its solutions for a few years. This turned out to be a challenging task, and the first result in that direction came from G. Dal Maso and R. Toader in 2002 [28], which analyzed a case of two dimensional anti-plane elasticity. Just one year later A. Chambolle [29] published an existence result for generalized two dimensional elasticity, and Francfort and C. J. Larsen [30] published an existence result for generalized anti-plane elasticity, both of which adopted the delicate framework of functions of bounded variations [24]. Up until now, a general existence result for three dimensional elasticity still seems formidable to obtain, and this has caused many mathematicians to start to look for alternative ways, like frameworks taken from continuum mechanics [32], from the engineering world [34], and more importantly, to look for local minima for the energies through artificial viscosity or gradient flows approaches [35, 36]. In addition to this, until recently there were no results regarding dynamic fracture, where kinetic effects are non-negligible and there might be transitions from linear to non-linear elastic behavior.

These difficulties arising from the mathematical formulation brought what felt like a separation between the mathematical and engineering worlds of fracture, which originally

worked together, but are now perceived as quite diverse. It must not come as a surprise, since the needed mathematical tools are quite difficult to work with and can intimidate many due to their apparent distance from the real world, while the engineering community is known to be concerned with more practical matters. In this work, our goal was in part to address this issue and show comparisons with the existing mathematical and engineering literature and results. In particular, we analyzed the Francfort-Marigo model for quasi-static brittle fracture and properly deduced the existing result of Dal Maso and Toader for two dimensional anti-plane elasticity and the behavior of solutions at the crack tip, all while making connections with classical fracture mechanics.

In Chapter 1 we briefly describe the classical theory of elastic fracture mechanics, starting from the reasoning behind the original energetic Griffith's criterion, derived from Inglis' computations of stress field around an elliptical hole in an infinite plate [1]. This global energetic argument is then localized following the works of Irwin, which described the strength of the singularity around a crack tip through a parameter called *Stress Intensity Factor* and described the different ways in which a crack can propagate. For a more modern approach, we described the general framework of the Rice-Cherepanov J-integral, which also incorporates non-linear elastic behaviors, and thus obtained the modern dissipative formulation of Griffith's criterion.

In Chapter 2 we described the Francfort-Marigo model for quasi-static brittle fracture, aiming at making clear its strengths against Griffith's criterion, as well as its weaknesses, and describing some more realistic models. Following an already-established idea in the calculus of variations, the crack evolution has to be in such a way that at every time the crack is a stationary point for the total energy and it satisfies an energy balance condition. We show how this model can accurately capture the phenomena of crack initiation and failure, as well as predict the crack path.

In Chapter 3 we introduce the necessary mathematical tools required for an appropriate formalization of the two dimensional anti-plane case. The concept of the capacity of a set is introduced to formalize the idea of sets being *invisible* to certain Sobolev spaces, like the Deny-Lions spaces used in the setting of the model. We recalled the notion of the Hausdorff dimension, which is a real-valued dimension for sets, as well as the Hausdorff distance between compact sets, essential to the description of the crack evolution. Moreover, we also linked the concept of harmonic conjugate for the solution to a harmonic problem with mixed boundary problems, which is useful in converting the boundary value problems into something more manageable.

In Chapter 4 we describe the two dimensional anti-plane case of Dal Maso and Toader, showing how the model is simplified under certain assumptions. We first prove the convergence of minimizers to the minimum problem under appropriate convergence of the crack set and the boundary loadings. Then we study the behavior in time of a general compact-valued non-decreasing function representing the crack evolution, and from this, we deduce the existence result following a time-discretization and a study of convergence of solutions to the discretized problem when the time-step goes to zero.

In Chapter 5 we study the behavior of solutions near the crack tip, to show how the dissipative formulation of Griffith's criterion and the existence of a Stress Intensity Factor measuring the strength of the singularity are both obtained through this variational

model. This requires a study of solutions to harmonic boundary-value problems in polygonal domains, which shows that certain angles can create singularities in the solution, whereby a singularity means a function that is in H^1 but not in H^2. This was done by treating the Laplacian as an operator from the space of strong solutions to L^2 and applying Fredholm's alternative theorem. Moreover, to show Griffith's criteria we straightened the crack with a diffeomorphism and localized the problem around the crack tip: after a few tedious computations, we deduce Irwin's formula, essential to the final result.

Chapter 1

Classical theory of elastic fracture mechanics

1.1 First studies and Griffith's approach

At the end of the nineteenth century, solid mechanics, intended as the study of deformations and motion of bodies subjected to external agents, was an already established tool to be applied in many real-life scenarios, for both construction and risk assessment of structures. However, the work done up to this point almost never included the study of bodies with simply connected but not connected geometries, that is bodies with holes, and in particular, in the studies of bodies with cracks, the resistance was studied with the classical criteria for brittle failure (Mohr-Coulomb, Tresca, Rankine, etc). This was because of two reasons: the appropriate mathematical tools were not yet fully developed; and the study of structures with holes or cracks, which were known to cause catastrophic effects, was mainly carried out with experiments.

These experiments though showed how a thoroughly polished material could withstand way more externally imposed stresses than a real material, with defects and micro-cracks, a result which was not availed by the state of the art of theoretical solid mechanics. This could mean that either classical resistance criteria were wrong and needed to account also for defects, or theoretical methods for computing stresses and strains in a real material were wrong by at least an order of magnitude.

Following the second option, in 1913 Inglis published a theoretical study [[1]] of stresses around an elliptical hole in an infinite plate, undergoing an externally imposed uniform vertical traction σ_0 (see Fig. 1.1). The ellipse has major and minor axis respectively equal to $2a$ and $2b$, with the minor axis parallel to the

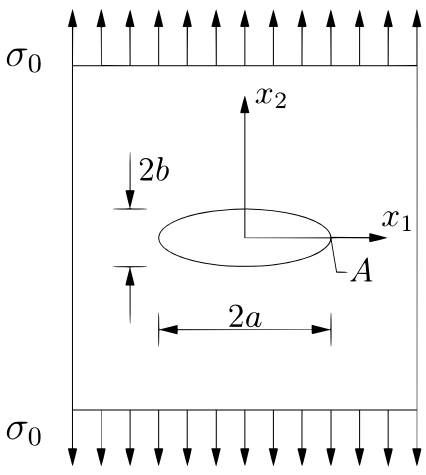

Figure 1.1: Inglis' setting

traction, while for simplicity the setting is that of a linear elastic, homogeneous, and isotropic material. To solve this problem, Inglis used a system of curvilinear coordinates around the ellipse and complex potentials reminiscent of conformal mappings to deduce a formula for the tangential stress at the rightmost part and the upmost part of the ellipse:

$$\sigma_{1,2}(x_1 = a, x_2 = 0) = \sigma_0\left(1 + 2\,\frac{a}{b}\right), \qquad \sigma_{2,1}(x_1 = 0, x_2 = b) = -\sigma_0.$$

As Inglis observed from the results, in the upper part there is always a tangential compression equal to the imposed stress σ_0, independently from the dimensions of the ellipse. On the other hand, in the rightmost part there is always tangential traction greater or equal to the imposed stress, and its value is dependent on the ratio $\frac{a}{b}$: when $\frac{a}{b} = 1$ there will be a circular hole, and the traction will be equal to $3\sigma_0$, that is with a Stress Concentration Factor (SCF) of 3. Since a and b can be independently varied, one can also consider extreme cases, in particular letting one of the dimensions approach 0: with $a \to 0$ the ellipse will approach a vertical crack of length $2b$, parallel to the imposed stress, with rightmost traction of σ_0, so SCF equal to 1.
When $b \to 0$ things get very interesting because $\frac{a}{b} \to \infty$, with a horizontal crack of length $2a$, perpendicular to the imposed stress. In this case, the traction at the rightmost part approaches ∞, which is something that will cause all existing resistance criteria to fail. It is also important to note that this result does not really depend on the value of the imposed stress, meaning that an arbitrarily small stress can still generate infinite stresses at the crack tips, and it did not take long to understand that another criterion was needed for bodies with cracks. In the same article, Inglis also did the calculations using the curvature of the hole, getting the analogous result that the smaller the curvature, the bigger the stress, and the stress approaches ∞ when the curvature is zero.
To be completely fair, Inglis did not dare to consider degenerate ellipses, as that was not his goal. It was not until 8 years later that Griffith, in his pioneering work [2], thought of using Inglis work to compute stresses at the tip of cracks. His interest in this computation arose after he conducted two experiments, in which he found that a thin iron rod with spiral cracks and a thin glass plate with a horizontal crack were way less resistant to vertical traction than their smooth counterparts.
Since, after the work of Inglis, it was suggested that stresses at the crack tip were always infinite, and so the crack would propagate for whatever imposed stress, Griffith thought of a different formulation for the criterion, one which could explain why real crack tend to have a critical imposed stress that makes them propagate. He thought of the imposed stress as energy given to the system, and so he formulated his energetic formulation: in the setting of brittle materials (which are easier to treat), he assumed that the solid would have surface energy which was to increase if the crack propagated, and this was to be compensated by a decrease in bulk energy and an overall decrease in total energy, thus meaning that a stable configuration was to be a minimum of the total energy. Defining E_t total energy, E_e elastic strain energy, E_p potential energy due to external loads, and E_f the new surface energy used to create new cracks, the relation would be:

$$E_t = E_e + E_p + E_f.$$

For two dimensional systems with a single crack, one could write every energy in terms of the extension ℓ of the crack, and the equilibrium of a configuration would be the same as the equilibrium with respect to a virtual extension of the crack:

$$\frac{\mathrm{d}E_t(\ell)}{\mathrm{d}\ell} = \frac{\mathrm{d}E_e(\ell)}{\mathrm{d}\ell} + \frac{\mathrm{d}E_p(\ell)}{\mathrm{d}\ell} + \frac{\mathrm{d}E_f(\ell)}{\mathrm{d}\ell} = 0. \tag{1.1}$$

Griffith considered an infinite plate with a crack of length 2ℓ, and his goal was to actually compute the energies as a function of ℓ and then substitute them in (1.1) to compute the critical imposed stress, which is the stress after which the crack propagates.
Using Clapeyron's Theorem for linear elastic materials, the potential energy for external loads can be expressed conveniently as

$$E_p(\ell) = -2E_e(\ell) \quad \Longrightarrow \quad E_t(\ell) = -E_e(\ell) + E_f(\ell).$$

In order to define the surface energy term, Griffith decided to make a simple assumption, that is that the surface energy is proportional to the dimension of the crack (in our simple case the length), with a proportionality constant of $2 \cdot 2\gamma$, so that

$$E_f(\ell) = 4\gamma\ell,$$

where γ is the surface energy for unit area, considered as a constant of the chosen material, and the premultiplied factor 4 is there because we are considering a crack expanding in two ways, both having two lips. The computation of the elastic strain energy in terms of the length of the crack has had a weird history - it was initially obtained by Griffith in [2], using Inglis relations for the stresses around an elliptical hole [1], only problem was that the formula was wrong because there was no continuity between the stress field at infinity and the imposed tractions. This was corrected by Griffith in [3], but without shedding light on his new calculations, which of course did not convince many contemporary engineers, and they were forced to come up with other ways to derive the formula. It was not until 1967 that [10] could finally use Inglis relations in the correct way, even though many others reached the same conclusions using alternative methods.

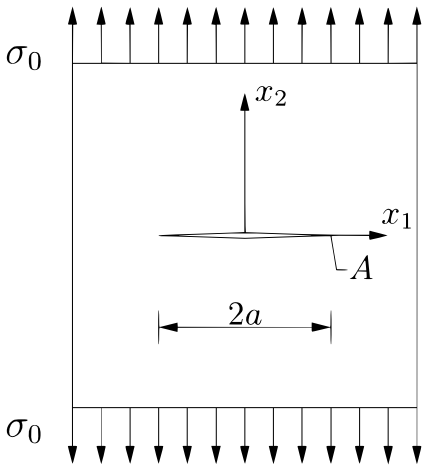

Figure 1.2: Griffith's setting

The right formula for the elastic strain energy thus was

$$E_e(\ell) = E_e^0 + \Delta E_e(\ell) = E_e^0 + \frac{\pi\sigma_0^2\ell^2}{E'},$$

where E_e^0 is the elastic strain energy of the uncracked specimen, trivially independent from the length ℓ. Moreover, the modified Young's modulus E' is such that $E' = E$ for plane stress and $E' = \frac{E}{1-\nu^2}$ for plane strain, with E being Young's modulus, and ν Poisson's

modulus. Substituting everything found up till now:

$$\begin{aligned} E_t(\ell) &= E_e(\ell) + E_p(\ell) + E_f(\ell) \\ &= E_e(\ell) - 2E_e(\ell) + E_f(\ell) \\ &= -E_e^0 - \Delta E_e(\ell) + E_f(\ell) \\ &= -E_e^0 - \frac{\pi\sigma_0^2\ell^2}{E'} + 4\gamma\ell. \end{aligned} \tag{1.2}$$

Substituting this in the crack equilibrium equation (1.1), the critical value of the imposed stress can finally be found:

$$\frac{\mathrm{d}E_t(\ell)}{\mathrm{d}\ell} = 0 \quad \Longrightarrow \quad 4\gamma = \frac{2\pi\sigma_{0c}^2\ell}{E'} \quad \Longrightarrow \quad \sigma_{0c} = \sqrt{\frac{2E'\gamma}{\pi\ell}}. \tag{1.3}$$

This result proposed a shift in the common perspective about cracks: a crack resistance to propagation is not that much about the stress field around the crack tip, which we have seen being always infinite, but more about the energy given to the system in a sense (in our case it is the imposed tension on the boundary). Simply put, when the given energy reaches a certain critical value, then and only then the crack propagates. This seems wonderful, and this model was regarded as the best model for fracture for some decades, but it has its obvious limitations: first of all, it works only for linear elastic brittle materials; moreover, the critical stress is inversely proportional to the length of the crack, meaning that for an undamaged material ($\ell = 0$), the critical stress is infinite, so the material will never crack. This is obviously not true based on common knowledge, which led Griffith to hypothesize the existence of micro-cracks and defects at the atomic level in every material. Last thing, for a given imposed tension σ_0, from (1.2) the critical length of the crack can be deduced

$$2\ell_c = \frac{4E'\gamma}{\pi\sigma_0^2},$$

so that if $\ell < \ell_c$ then the crack will not propagate: in other words, the body will be called *invisible to cracks*; if that is not the case, obviously the crack will propagate as predicted by the model.

What the formulae tell is that the total energy has a stationary point, and this point is a maximum or minimum of the energy (or neither) by looking at the sign of the second derivative in length. In the case study, there will be a maximum since the second derivative is negative, so this configuration is unstable, meaning that once the propagation has started, there is no way to tell in what configuration it will stop. Obreimoff [4] in 1930 showed a configuration with a positive second derivative, a stable one, meaning that by knowing the imposed stress one can always compute the end configuration of the crack. Summing up:

$$\frac{\mathrm{d}^2E_t(\ell)}{\mathrm{d}\ell^2} \quad \begin{cases} < 0 & \text{unstable;} \\ = 0 & \text{neutral;} \\ > 0 & \text{stable.} \end{cases} \tag{1.4}$$

Some years later, Irwin [7] decided to make (1.1) into an actual criterion for crack propagation:

$$\underbrace{-\frac{1}{2}\left(\frac{\mathrm{d}E_e}{\mathrm{d}\ell}+\frac{\mathrm{d}E_u}{\mathrm{d}\ell}\right)}_{=:G}=\underbrace{\frac{1}{2}\frac{\mathrm{d}E_f}{\mathrm{d}\ell}}_{=:G_c}. \tag{1.5}$$

The left term G is called *energy release rate (ERR)* and represents the elastic energy for unit length (unit surface in general) available for the crack to initiate propagation, and it was the same Irwin that chose the letter G for the symbol, as a tribute to Griffith. The right term G_c represents the energy needed for making the crack propagate for a unit length, and from Griffith's hypotheses, it is a constant of the chosen material. For the model analyzed, by substituting everything obtained:

$$G=-\frac{1}{2}\left(\frac{\mathrm{d}E_e}{\mathrm{d}\ell}+\frac{\mathrm{d}E_u}{\mathrm{d}\ell}\right)=\frac{1}{2}\frac{\mathrm{d}E_e}{\mathrm{d}\ell}=\frac{\pi\sigma_{0c}^2\ell}{E'}\quad\Longrightarrow\quad\frac{\pi\sigma_0^2\ell}{E'}=G_c. \tag{1.6}$$

By looking at this form of Griffith criterion, one can imagine that if $G<G_c$ then the propagation will not initiate, since the available energy is less than the needed energy, and this was also Griffith's idea, even though he did not say anything about the case $G>G_c$, which is something that can definitely happen, judging from (1.6).

1.2 Irwin's approach and modes of fracture propagation

The first who tried to actually change Griffith's perspective on fracture was Irwin in 1958 [8], which wanted to study the singular stress field around the crack tip.

By reading Griffith's work, he noticed that what Griffith called the *energy given* to the system, was in his framework an imposed traction on the boundary, that always generated infinite stresses at the crack tip. Irwin thought that a change in the imposed stress must bring some change in the stress field around the crack, which he knew had to be singular, but he hypothesized that the singularity had a kind of *strength* dependent on the imposed stress. In order to measure this strength of singularity, he used a combination of complex potentials, Airy stress functions, and eigenvalues problems, quite lengthy to follow. There will be presented here a simplified treatment, in an anti-plane shear case: let Ω be a set in $\mathbb{R}^2$, fixed on $\partial_D\Omega$, thus with Dirichlet condition $u=g$; it will have a crack K with length ℓ, whose upper and lower lips will be respectively K_+ and K_-. On $\partial_N\Omega=\partial\Omega\setminus(\partial_D\Omega\cup(\partial\Omega\cap K))$ there will be a Neumann condition, with an imposed traction T.

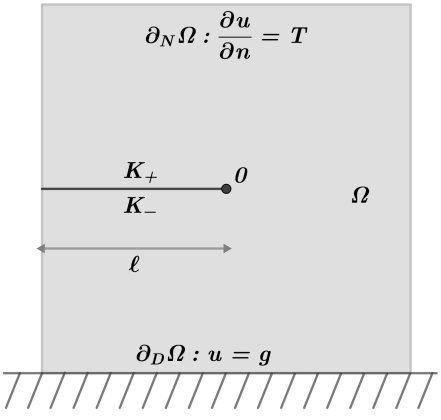

Figure 1.3: Irwin's setting

There is also the assumption of no bulk forces. Let us assume also a linear elastic, homogeneous, and isotropic body, so that its constitutive equation will be represented by Hooke's Law:

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}.$$

Given the anti-plane shear, it is more than reasonable to assume a displacement field $u = (u_1, u_2, u_3)$ in the form of:

$$u = (0, 0, u_3(x, y)),$$

and that the stress field will be so that only σ_{13} and σ_{23} are non-zero. Eventually, the constitutive equation will boil down to:

$$\sigma_{13} = 2\mu\varepsilon_{13}, \qquad \sigma_{23} = 2\mu\varepsilon_{23},$$

with $\varepsilon_{13} = \frac{1}{2}u_{3,1}$ and $\varepsilon_{23} = \frac{1}{2}u_{3,2}$. Given the absence of bulk forces, the equilibrium equation will be:

$$\begin{aligned} \nabla \cdot \sigma = 0 \quad &\implies \quad \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} = 0 \quad \implies \quad \frac{\partial \varepsilon_{31}}{\partial x_1} + \frac{\partial \varepsilon_{32}}{\partial x_2} = 0 \\ &\implies \quad \frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} = 0 \quad \implies \quad \Delta u_3 = 0. \end{aligned}$$

Following the assumption of the absence of traction on the lips of the crack, the boundary condition will be:

$$\sigma \cdot n = 0 \quad \text{on} \quad K^{\pm} \quad \implies \quad \frac{\partial u_3}{\partial y} = 0 \quad \text{on} \quad K^{\pm}.$$

In addition to these, there will also be adequate boundary conditions on $\partial_D\Omega$ and $\partial_N\Omega$, as previously described. Since the ultimate goal is to study the behavior of the material near the crack tip, let us focus on:

$$\begin{cases} \Delta u_3 = 0 & \forall x \in \Omega, \\ \lim\limits_{y \to 0^+} \dfrac{\partial u_3}{\partial y}(x, y) = 0 & \forall x < 0, \\ \lim\limits_{y \to 0^-} \dfrac{\partial u_3}{\partial y}(x, y) = 0 & \forall x < 0. \end{cases} \tag{1.7}$$

The objective of this treatment is not to study the general properties and regularity of solutions of this system, which requires the introduction and study of some Sobolev-like function spaces. For an adequate study, the reader is advised to consult the next chapters and [19]. Let us now look for solutions in polar coordinates centered at the crack tip of the form:

$$w(r, \theta) = r^{\alpha}\, \widehat{u}_3(\theta),$$

where α is the proposed *degree* of singularity. Since the displacement cannot be singular at the crack tip (nowhere in the body to be precise), this means that $\alpha > 0$. By writing

1.7 in polar coordinates, together with the boundary conditions at the crack lips, after a bit of rearrangement, the system will be:

$$\begin{cases} \widehat{u}_3'' + \alpha^2 \widehat{u}_3 = 0 & \forall \theta, \\ \widehat{u}_3'(\pm\pi) = 0. \end{cases} \tag{1.8}$$

The first equation is easily solvable and has the solution:

$$\widehat{u}_3(\theta) = A\cos(\alpha\theta) + B\sin(\alpha\theta).$$

By forcing the boundary conditions, and remembering the restriction $\alpha > 0$, the degree of singularity will be forced to be $\alpha = \frac{1}{2} + k,\ k \in \mathbb{N}$.
Since $\sigma_{i3} \propto \varepsilon_{i3} \propto \frac{\partial u_3}{\partial r} \propto r^{\alpha-1}$, and since for this model there is the need for a stress singularity at the origin, the only possible choice will be $\alpha = \frac{1}{2}$.

For a general problem involving more complex settings, Irwin found that:

$$\sigma_{ij} = \left(\frac{k}{\sqrt{r}}\right)\cos\frac{\theta}{2} + \sum_{n=0}^{\infty} A_n\, r^{\frac{n}{2}}\, g_{ij}^n(\theta),$$

where k is a constant (independent from r or θ, but possibly dependent on other variables) called *Stress Intensity Factor (SIF)*, representing the sought-after *strength* of the singularity. Moreover, Irwin was the first to provide a way to classify fractures, by pinpointing three different *modes*, in order to study each mode adequately. By looking at

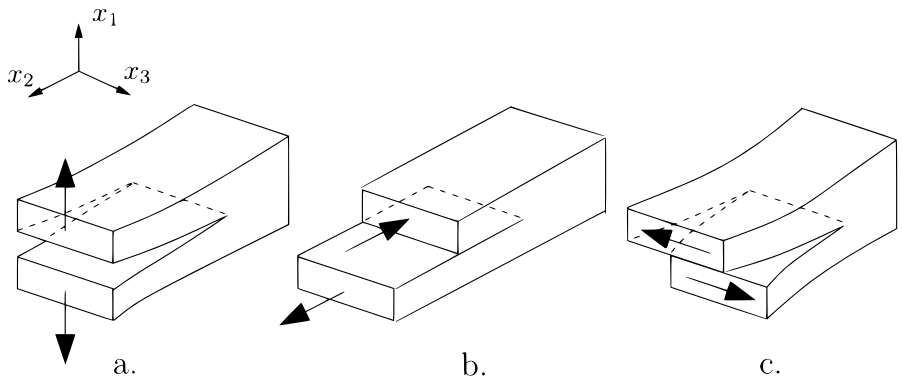

Figure 1.4: a. Mode I - opening, b. Mode II - sliding, c. Mode III - tearing

Fig. 1.4, let us consider a flat fracture on the plane $x_2 - x_3$, extended through all of x_3 (for visualization purposes, there is only a unit slice), propagating towards x_2. Irwin's three different modes are

a. Mode I - opening: $[\![u_1]\!] \neq 0$, $[\![u_2]\!] = 0$, $[\![u_3]\!] = 0$;

b. Mode II - sliding: $[\![u_1]\!] = 0$, $[\![u_2]\!] \neq 0$, $[\![u_3]\!] = 0$;

c. Mode III - tearing: $[\![u_1]\!] = 0$, $[\![u_2]\!] = 0$, $[\![u_3]\!] \neq 0$.

These three modes are not mutually exclusive, and a fracture can be a combination of more than one mode, in which case it is called *mixed mode*. The thing to keep in mind is that in whatever the case may be, the singularity for $r \to 0$ behaves like $r^{-\frac{1}{2}}$, and what actually changes from case to case is the constant k and the function $f_{ij}(\theta)$: the notation for the three modes is K_I for Mode I, K_{II} for Mode II and K_{III} for Mode III; for the stresses:

$$\sigma_{ij}^{I} = \frac{K_I}{\sqrt{2\pi r}} f_{ij}^{I}(\theta), \qquad \sigma_{ij}^{II} = \frac{K_{II}}{\sqrt{2\pi r}} f_{ij}^{II}(\theta), \qquad \sigma_{ij}^{III} = \frac{K_{III}}{\sqrt{2\pi r}} f_{ij}^{III}(\theta).$$

In the case of a mixed mode, due to the linearity of the material, the stress becomes:

$$\sigma_{ij} = \sigma_{ij}^{I} + \sigma_{ij}^{II} + \sigma_{ij}^{III} = \frac{1}{\sqrt{2\pi r}} \left(K_I f_{ij}^{I}(\theta) + K_{II} f_{ij}^{II}(\theta) + K_{III} f_{ij}^{III}(\theta) \right).$$

At this point, since the strength of the singularity is determined by the Stress Intensity Factor, taking from Griffith the idea that propagation starts due to an excess of stress, Irwin hypothesized the existence of a critical strength of singularity which causes the crack to propagate. He called this number K_c the tenacity to fracture, and the propagation criterion was thus established:

$$K = K_c.$$

The only thing to clarify is maybe that the criterion is given in terms of one number, but there are three different modes determining three different SIFs: there actually are three different tenacities, and the fracture occurs in the mode which is the first to satisfy the criterion. In general, one would expect $K_{Ic} \neq K_{IIc} \neq K_{IIIc}$, and usually $K_{Ic} < K_{IIc}, K_{IIIc}$, so in most of the cases the Mode I controls the crack propagation.

For the Griffith model in Fig. 1.2, there is clearly a Mode I, and in this and in some other simple cases, the SIF can be computed exactly. All the computations leading to it are quite lengthy though, so let us start with a result due to Westergaard in 1939 [5] for the stress in the x_1 direction computed on the line $x_2 = 0$ (see Fig. 1.2):

$$\sigma_{1,2}(x) = \frac{|x|\,\sigma_0}{\sqrt{x^2 - \ell^2}} \quad \text{for } |x| > \ell.$$

Considering just the right part ($x > \ell$), let us make the substitution $x = \ell + r$, thus obtaining:

$$\sigma_{1,2} = \frac{(r+\ell)\,\sigma_0}{\sqrt{r\,(r+2\ell)}} \qquad \Longrightarrow \qquad K_I = \lim_{r \to 0} \sigma_{1,2} \sqrt{2\pi r} = \sigma_0 \sqrt{\pi \ell}. \tag{1.9}$$

From (1.9) it is quite clear that Irwin's model is plagued by the same problem Griffith's model had: it cannot predict crack initiation. To see this, it suffices to write $\sigma_0 \sqrt{\pi \ell} = K_c$ and solve for ℓ: there will be the same factor $\sqrt{\ell}$ at the denominator as in (1.3).
Up to this point, there are two important descriptors of crack propagation: the Energy Release Rate G, representing the change in potential energy due to a unit length propagation; the Stress Intensity Factor K, representing the strength of the singularity of the

stress field near the crack tip. Based on their formulae, it looks like G is a global parameter, while K is a local parameter: however, there exists a relation between them, as easily obtainable from (1.6) and (1.9). By combining them:

$$G = \frac{K_I^2}{E'}.$$

This is valid in our particular geometry, and in general for whatever crack propagating in Mode I in a brittle linear elastic material. Irwin proved this, following a crack closure procedure, with the application of a compression stress field on the crack. In the same work [7], Irwin also gives the relation in the case of mixed mode, only valid though if the Mode II crack propagates on the same plane as the other two modes:

$$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}.$$

In the case of anti-plane fracture following a Mode III crack, the relation will be

$$G = \frac{K_{III}^2}{2\mu}. \tag{1.10}$$

We will derive this particular relation rigorously, in Chapter 5.

1.3 Dissipation analysis and J-integral

In both Griffith and Irwin formulations, it is not clear one of the key aspects of the phenomenon under study: the *irreversibility*. This is the inability of the crack to heal, that is to decrease its length, while some texts confuse it with the dissipative aspect of fracture: this is also important, but the concepts are not the same. To better understand this difference and their respective role, let us do a dissipation analysis of a body with a crack, in the spirit of continuum mechanics: there will be the assumption of negligible thermal effects (which requires the loading process to be quasi-static) so that the process is isothermal. The setting is the same already considered in Fig. 1.3, we just need to introduce the strain energy density as:

$$W(\varepsilon) \coloneqq \int_0^{\varepsilon} \sigma : \mathrm{d}\varepsilon.$$

The total dissipation Φ will be the difference between the power of applied forces and the stored elastic energy. Just as before we assume the absence of bulk forces, so the forces will all be applied at the boundary, hence:

$$\Phi = \int_{\partial\Omega} \frac{\partial u}{\partial t} \cdot (\sigma \cdot n)\ \mathrm{d}s - \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} W(\varepsilon(u))\, \mathrm{d}V. \tag{1.11}$$

Since the forces are applied at the boundary, and we also assume the crack lips to be traction free, then the applied forces are independent of the propagation of the crack. This means that the important contribution is the second integral in (1.11), in which the domain changes with time.

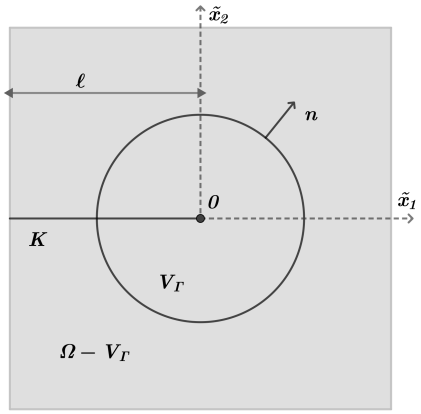

Figure 1.5: Integration around the crack tip

Let us define a set V_Γ with contour Γ, encapsulating the crack tip as in Fig. 1.5, and let us divide the integral in the contributions from V_Γ and $\Omega \setminus V_\Gamma$:

$$\int_\Omega W(\varepsilon(u))\,\mathrm{d}V = \int_{\Omega\setminus V_\Gamma} W(\varepsilon(u))\,\mathrm{d}V + \int_{V_\Gamma} W(\varepsilon(u))\,\mathrm{d}V.$$

Since V_Γ is moving together with the crack tip, we will simply have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\Gamma} W(\varepsilon(u))\,\mathrm{d}V &= \int_{V_\Gamma} \frac{\partial W}{\partial \varepsilon} : \frac{\partial \varepsilon(u)}{\partial t}\mathrm{d}V \\ &= \int_{V_\Gamma} \sigma : \frac{\partial \varepsilon}{\partial t}\mathrm{d}V.\end{aligned}$$

For the other integral we will use the formula for the material derivative of an integral with respect to a non-material volume moving in the x_1 direction, thus obtaining:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\setminus V_\Gamma} W(\varepsilon(u))\,\mathrm{d}V = \int_{\Omega\setminus V_\Gamma} \sigma : \frac{\partial \varepsilon}{\partial t}\mathrm{d}V - \int_\Gamma W(\varepsilon(u))\,\dot{\ell}\,n_1\,\mathrm{d}s, \tag{1.12}$$

where $n_1 = e_1 \cdot n$, with e_1 propagation direction of the volume and n the normal to said volume, pointing outward.
Let us now write an energy balance for the region $\Omega \setminus V_\Gamma$:

$$\begin{aligned}\int_{\Omega\setminus V_\Gamma} \sigma : \frac{\partial \varepsilon}{\partial t}\mathrm{d}V &= \int_{\partial(\Omega\setminus V_\Gamma)} \frac{\partial u}{\partial t}\cdot(\sigma\cdot n)\;\mathrm{d}s \\ &= \int_{\partial\Omega} \frac{\partial u}{\partial t}\cdot(\sigma\cdot n)\;\mathrm{d}s - \int_\Gamma \frac{\partial u}{\partial t}\cdot(\sigma\cdot n)\;\mathrm{d}s.\end{aligned} \tag{1.13}$$

By using (1.11), (1.12), (1.13) we will get to:

$$\Phi = \int_\Gamma \left(W(\varepsilon(u))\dot{\ell} n_1 + \frac{\partial u}{\partial t}\cdot(\sigma\cdot n)\right)\mathrm{d}s - \int_{V_\Gamma} \sigma : \frac{\partial \varepsilon}{\partial t}\mathrm{d}V.$$

If we statically observe the system, then the integrand of the second integral is independent of V_Γ, so it will vanish if we let the measure of V_Γ go to 0, summed up in $\Gamma \to 0$.

$$\Phi = \lim_{\Gamma\to 0} \int_\Gamma \left(W(\varepsilon(u))\dot{\ell} n_1 + \frac{\partial u}{\partial t}\cdot(\sigma\cdot n)\right)\mathrm{d}s. \tag{1.14}$$

The key point is that due to the movement of the crack tip, the singularity in u moves with velocity $\dot{\ell}\,e_1$. This is also reflected in the singularity in $\frac{\partial u}{\partial t}$, which has a singular contribution of the form $-\dot{\ell}\frac{\partial u}{\partial x_1}$. Substituting in (1.14) we will get:

$$\begin{cases}\Phi = J\cdot\dot{\ell}, \\ J = \lim\limits_{\Gamma\to 0} J_\Gamma = \lim\limits_{\Gamma\to 0} \int_\Gamma \left(W(\varepsilon(u)) n_1 - \dfrac{\partial u}{\partial x_1}\cdot(\sigma\cdot n)\right)\mathrm{d}s.\end{cases} \tag{1.15}$$

This integral J is called *Rice-Cherepanov J-integral*, and it was independently studied by Rice [12, 13] and Cherepanov [9, 11]. It can be written in another way, which generalizes it:

$$J = \lim_{\Gamma \to 0} \int_{\Gamma} \left[\Big(W\mathrm{I} - \nabla u^{\top} \mathrm{D}W \Big) n \right]^{\top} \tau \, \mathrm{d}s \tag{1.16}$$

where I is the identity matrix, n is the normal to the integration path, and τ is the tangent to the crack path at the crack tip, which in our case is $(0 \;\; 1)^{\top}$. The term $W\mathrm{I} - \nabla u^{\top} \mathrm{D}W$ is known in the literature as the *Eshelby* tensor.
From now on, we will follow the classic treatment by Rice, in which the curve Γ is oriented anticlockwise.

This integral, which we obtained with a fairly simple procedure, has some important properties, among which we have that it is always equal to 0 for contours without singularities. Let us call Γ^* a contour without singularities in its inner set B^*. Then we can apply the divergence theorem:

$$\begin{aligned} J_{\Gamma^*} = \int_{\Gamma^*} \left(W n_1 - \frac{\partial u}{\partial x_1} \cdot (\sigma \cdot n) \right) \mathrm{d}s &= \int_{\Gamma^*} \left(W e_1 - \frac{\partial u}{\partial x_1} \cdot \sigma \right) \cdot n \, \mathrm{d}s \\ &= \int_{B^*} \mathrm{div} \Big(W \, e_1 - \frac{\partial u}{\partial x_1} \cdot \sigma \Big) \, \mathrm{d}V. \end{aligned}$$

We can rewrite the integrand of the last integral in indicial notation to show that it is always equal to 0:

$$\begin{aligned} \Big[W\delta_{1j} - u_{k,1}\sigma_{kj} \Big]_{,j} &= W_{,1} - u_{k,1j}\sigma_{kj} - u_{k,1}\sigma_{kj,j} \\ &= \sigma_{mn}\varepsilon_{mn,1} - u_{k,1j}\sigma_{kj} \\ &= \sigma_{mn}u_{m,n1} - u_{m,n1}\sigma_{mn} \\ &= 0. \end{aligned}$$

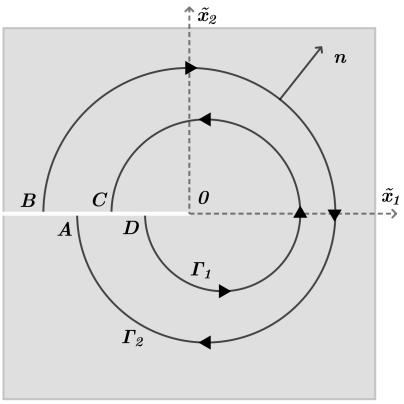

Figure 1.6: Path independence

Moreover, J_Γ is independent of the chosen path: let us consider the closed path $\Gamma = \Gamma_1 CB \Gamma_2 AD$ depicted in Fig. 1.6. Since it does not contain singularities, for what we have just shown, the total integral has to vanish $J_\Gamma = 0$. Due to linearity, we can write:

$$\underbrace{\int_{\Gamma_1} [\ldots] \, \mathrm{d}s}_{=: J_{\Gamma_1}} + \underbrace{\int_{C}^{B} [\ldots] \, \mathrm{d}s}_{=: J_{CB}} + \underbrace{\int_{\Gamma_2} [\ldots] \, \mathrm{d}s}_{=: -J_{\Gamma_2}} + \underbrace{\int_{A}^{D} [\ldots] \, \mathrm{d}s}_{=: -J_{AD}} = 0.$$

where $[\ldots] = W n_1 - \frac{\partial u}{\partial 1} \cdot (\sigma \cdot n)$, and the minus signs are there to preserve the anticlockwise orientation. The integrals J_{CB} and J_{AD} are equal to 0 since their normals have no component in the x_1 direction ($n_1 = 0$), and also because, as we said earlier, we are assuming the crack lips to be traction free ($\sigma \cdot n = 0$). We can thus conclude that $J_{\Gamma_1} = J_{\Gamma_2}$, proving the path independence in the case of no bulk forces and with parallel and flat crack lips. The last property is quite an astonishing result since it says that the J-integral is equal

to the Energy Release Rate G. By looking at the setting of Fig. 1.3, the potential energy will be:

$$E_e(\ell) + E_p(\ell) = \int_\Omega W \, \mathrm{d}V - \int_{\partial\Omega_T} T \cdot u \, \mathrm{d}s.$$

As in Irwin's study, let us suppose the absence of bulk forces, the independence of T from the crack length ℓ, and crack lips to be traction-free. Differentiating with respect to ℓ:

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\Big(E_e(\ell) + E_p(\ell)\Big) = \int_\Omega \frac{\mathrm{d}W}{\mathrm{d}\ell}\mathrm{d}V - \int_{\partial\Omega} T \cdot \frac{\mathrm{d}u}{\mathrm{d}\ell}\mathrm{d}s, \tag{1.17}$$

where we switched $\partial_N\Omega$ with $\partial\Omega$ because $\frac{\mathrm{d}u}{\mathrm{d}\ell} = 0$ on $\partial_D\Omega$. Moreover, since $T = 0$ on K, we can also switch $\partial\Omega$ with the contour Γ of the J-integral.
Let us now introduce a change of coordinates $\tilde{x}_1 - \tilde{x}_2$, attached to the crack tip, such that $\tilde{x}_1 = x_1 - \ell$. We will have:

$$\frac{\mathrm{d}}{\mathrm{d}\ell} = \frac{\partial}{\partial\ell} + \frac{\partial}{\partial x_1}\underbrace{\frac{\partial x_1}{\partial\ell}}_{=-1} = \frac{\partial}{\partial\ell} - \underbrace{\frac{\partial}{\partial x_1}}_{=\frac{\partial}{\partial\tilde{x}_1}} = \frac{\partial}{\partial\ell} - \frac{\partial}{\partial\tilde{x}_1}.$$

Substituting this in (1.17):

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\Big(E_e(\ell) + E_p(\ell)\Big) = \int_\Omega \left[\sigma : \nabla\Big(\frac{\partial u}{\partial\ell}\Big) - \frac{\partial W}{\partial x_1}\right]\mathrm{d}V - \int_\Gamma T \cdot \left[\frac{\partial u}{\partial\ell} - \frac{\partial u}{\partial x_1}\right]\mathrm{d}s. \tag{1.18}$$

Furthermore, if we treat $\frac{\partial u}{\partial\ell}$ as a kinematically admissible displacement field, we can write the principle of virtual work as follows:

$$\int_\Omega \left[\sigma : \nabla\Big(\frac{\partial u}{\partial\ell}\Big)\right]\mathrm{d}V = \int_\Gamma T \cdot \frac{\partial u}{\partial\ell}\mathrm{d}s.$$

Substituting this in (1.18) we will get:

$$-\frac{\mathrm{d}}{\mathrm{d}\ell}\Big(E_e(\ell) + E_p(\ell)\Big) = \int_\Omega \frac{\partial W}{\partial x_1}\mathrm{d}V - \int_\Gamma T \cdot \frac{\partial u}{\partial x_1}\mathrm{d}s,$$

where we can apply the divergence theorem to the first integral. From the boundary conditions we have $T = \sigma \cdot n$, so we will finally find:

$$\underbrace{-\frac{\mathrm{d}}{\mathrm{d}\ell}\Big(E_e(\ell) + E_p(\ell)\Big)}_{=:G} = \underbrace{\int_\Gamma \left[W n_1 - \frac{\partial u}{\partial x_1} \cdot (\sigma \cdot n)\right]\mathrm{d}s}_{=:J} \quad \Longrightarrow \quad J = G. \tag{1.19}$$

This means that we can compute the value of G by computing the J-integral, and this can be done more easily by conveniently choosing the contour Γ. Besides, we never made any assumptions on the linearity of the elastic equations, so it is valid also in non-linear settings, which at times can be considered part of plastic deformation theory.
Going back to (1.15), we can now write $\Phi = G \cdot \dot{\ell}$: in the formalization of analytical

thermodynamics, the term G represents the thermodynamic dual in dissipation to $\dot{\ell}$. It represents, in a sense, the force due to the singularity propagating, and since dimensionally it is energy dissipated per unit length, it makes sense to introduce a dissipation-based propagation criterion, equivalent to Griffith criterion thanks to previous results: it exists a critical dissipated energy G_c so that

$$\begin{cases} G < G_c & \Longrightarrow \quad \dot{\ell} = 0; \\ G = G_c & \Longrightarrow \quad \dot{\ell} \geq 0. \end{cases} \tag{1.20}$$

If $G > G_c$, the propagation is labeled *unstable* or *dynamic* (see [17]), and kinetic and thermal effects start to come into play. To remove this difficult case, we will restrict our study to a quasi-static crack evolution.
All these results, together with the irreversibility of crack length, are summed up in what is now known as the modern formulation of Griffith criterion for a quasi-static evolution of a crack in a brittle material:

$$\begin{cases} \dot{\ell} \geq 0; \\ (G_c - G) \geq 0; \\ (G_c - G)\, \dot{\ell} = 0. \end{cases} \tag{1.21}$$

The first equation is about the irreversibility of the process; the second equation is about the quasi-static nature of the process; the third equation is the true Griffith criterion. This treatment can be readily seen as an application of the Clausius-Duhem inequality for dissipation, with the introduction of a new thermodynamical variable, the crack length. This is the form of Griffith criterion which is usually used in theoretical works in solid mechanics, and also the one which modern theories of fracture in applied mathematics want to reconstruct, based on different principles from Griffith.

Chapter 2

Francfort-Marigo model for quasi-static brittle fracture

As we have said in Chapter 1, the Griffith model of linear elastic quasi-static brittle fracture, although powerful and widely used, has some obvious disadvantages, as discussed before.
First of all, as seen with the original Griffith model and subsequent classical models, if the length ℓ of the crack goes to 0, that is an entirely uncracked material, then the critical stress goes to $+\infty$, effectively rendering the criterion useless. This boils down to not having singularities in the stress field on the inside of the body, where the crack tips should be. In hindsight, this can be adjusted by permitting stress singularities on the boundary of the body, for example by having a non-convex shape admitting notches on the edge.
Moreover, as observed from 1.21, the Griffith criterion uses just the length of the crack, as a function of time, but it does not give indications on what path the crack will follow, nor if the path evolution will be smooth or it admits some *jumps*, that is isolated points in time where the crack length jumps instantaneously from length ℓ_1 to length ℓ_2. This is not even far from reality, because in many experiments a crack jump is observed, especially in cases near failure. Truth be told, in a pure quasi-static setting, a crack jump would be impossible to obtain, as one would always have $G < G_c$, a stable propagation. Of course, there have been some experimental results to try and predict the crack path, but they assume the path to be piecewise C^1 curves, a quite strong hypothesis.
This model, proposed in 1998 by G.A. Francfort and J.J. Marigo, was created with the intent of solving these issues contained within Griffith's approach, by setting the elastic brittle fracture problem in a variational setting. This model does not arise as a generalization of Griffith's criteria, but just as an independent model borrowing some hypothesis from Griffith. To be fair, no model is perfect as of now, and we will do our best to point out the strengths and weaknesses of this model, as we have done in Chapter 1 with Griffith's model.
The basic idea of the model is that the total energy is a sum of two terms: the bulk part of the body, that is the crack-free part has a certain energy; the cracks have another energy, possibly depending on the crack length. The evolution of the crack must be so

that at every time, the crack minimizes the total energy among all possible crack paths containing the crack paths at previous times. We will give now the details of the model, giving some proofs where possible, without going into the details of the existence and regularity of the eventual solutions of the model, as that, especially in the case of a general three dimensional domain, would require an extensive amount of theoretical framework, to begin with.

2.1 Model formulation

2.1.1 Domains and energies

The domain Ω will be a bounded connected open subset of $\mathbb{R}^N$, with $1 \leq N \leq 3$, with Lipschitz boundary. From a configurational viewpoint, it represents the crack-free reference configuration of the body, which is subjected to Dirichlet conditions on a relative open subset $\partial_D\Omega \subset \partial\Omega$, and to Neumann conditions on $\partial_N\Omega = \partial\Omega \setminus \overline{\partial_D\Omega}$.
The crack set will be a compact subset of the domain, $K \subset \overline{\Omega}$, which may possibly intersect $\partial_D\Omega$ and $\partial_N\Omega$. It need not be the image of a single curve, in the sense that the crack can also present some kind of branching in its path; moreover, it need not be connected, as we can have different crack paths starting at different points of the domain.
The variable that we will study at first, and on which we will apply the boundary conditions is the displacement field v. In the cracked configuration, the boundary conditions will slightly change, taking into account also the crack, to which we will impose a traction-free condition, which is a homogeneous Neumann condition. We need just to be cautious of the intersection between the crack and the boundary: specifically, on $\partial_D\Omega \cap K$, on $K \cap \Omega$, and on $\partial_N\Omega$ we will impose a traction-free condition, while on $\partial_D\Omega \setminus K$ we will impose a Dirichlet condition.
The traction-free nature of the crack has to be read as a no-contact condition between the lips of the crack, and in a quasi-static setting, this makes total sense. If the setting were to be of dynamic fracture, then one would have to also factor in some kind of contact between the lips, be it with or without friction. Of course, one can also see what happens in a quasi-static setting with a contact condition, but this would require a more appropriate formalization.
This would be it if this was not a variational model, but since it is, we need to define the set of admissible displacement fields, from where we can later extract the set of test functions. In the strong formulation, we denote the imposed Dirichlet loading as g, imposed on $\partial_D\Omega \setminus K$; in the variational formulation, we will then have a space of functions whose gradient is in $L^2(\Omega \setminus K; \mathbb{R}^N)$, which we will denote for now just $L^{1,2}(\Omega \setminus K)$, and we will leave the formalization details for the next chapter. We require the gradient to be at least square-integrable since, as we will see, the elastic energy density depends on the symmetrized gradient $\varepsilon(v) = \frac{1}{2}(\nabla v + \nabla^T v)$ of the displacement field v.
As we have thus seen, the set of admissible displacement fields depends obviously on the boundary condition U (we consider $\partial_D\Omega$ to be fixed), but it also depends on the crack set K, since in a sense it represents the current state of the body, and the more the fracture

propagates, the less control we have on the admissible displacements. This will be particularly clear when studying the failure of the materials undergoing cracking. Thus, the space of admissible displacements will be:

$$\mathcal{V}(g,K) \coloneqq \left\{ v \in L^{1,2}(\Omega \setminus K) : v = g \text{ on } \partial_D \Omega \setminus K \right\}. \tag{2.1}$$

As for the energies involved, the idea is that the total energy E is equal to:

$$E = E_b + E_f,$$

with E_b being the bulk energy, and E_f being the energy of the crack.
In order to define the bulk energy, we have to define the material we will be working with. For simplicity, we will consider only infinitesimal elasticity, even though a setting with finite elasticity should not be too challenging to pose in this variational setting.
We define the elastic energy density as a function of both space and the symmetrized gradient, $W = W(x, \varepsilon(v)(x))$, defined only on the crack-free part of the domain $\Omega \setminus K$. In general, since we need to integrate this density over the domain, and since for the appropriate minimization we need growth and coercitivity conditions, we require:

1. $W(x,\xi)$ is measurable in x and continuous in ξ for a.e. $x \in \Omega$, strictly convex and p-homogeneous in ξ;
2. $\alpha|\xi|^p \leq W(x,\xi) \leq \beta(|\xi|^p + 1)$ with $\alpha, \beta > 0$ for a.e. $x \in \Omega$.

In particular, linearized elasticity is always taken into consideration, where

$$W(x,\xi) = \frac{1}{2} A(x)\xi \cdot \xi \qquad \text{s.t. } A(x) = A^T(x), \;\; \alpha I \leq A(x) \leq \beta I \;\; \text{for a.e. } x \in \Omega.$$

This is the most used setting, especially in the setting of an isotropic homogeneous linear elastic material, as the constitutive equations are well known. To be general, we will try to deduce the result in this chapter for a general class of elastic densities, namely the ones which are p-homogeneous in ξ, and depend implicitly on x through a symmetric matrix, as in the linearized elasticity case.
Note that, if we keep K fixed, the loading on the boundary defines the displacement field which reaches elastic equilibrium by means of a minimization of the total elastic energy on $\Omega \setminus K$ among all admissible displacement fields. Then we can define the bulk energy:

$$E_b(g,K) \coloneqq \inf_{v \in \mathcal{V}(g,K)} \int_{\Omega \setminus K} W(x, \varepsilon(v)(x)) \, \mathrm{d}x. \tag{2.2}$$

As we will see, such an infimum exists due to the Lipschitzianity of the boundary and the restrictions on the set of possible cracks. Moreover, it is easy to see that, for a fixed external loading g, the bulk energy E_b is monotonically decreasing in K, meaning that if the crack grows, then the bulk energy will decrease, as a consequence of the integral over $\Omega \setminus K$. Also, in the setting of linearized elasticity, changing the external load g by a factor of c, results in the minimum of the bulk energy to be multiplied by a factor of c as well. This, due to the p-homogeneity of the linearized elastic density in ξ, translates into the

p-homogeneity of the bulk energy in g.
Of course, the bulk energy may also be generalized to more complex elastic materials, maybe also with a density depending on the second derivatives of the displacement fields. Following the energetic approach by Griffith, the total energy will also be due to the energy related to the crack. In Griffith's approach, the energy of the crack was simply proportional to its length. Here we are presented with a question that Griffith did not ponder: what kind of length are we to deal with? More precisely, what kind of cracks are we excluding with the choice of the proper measure of length? The normal choice would be to use the $N-1$ dimensional Hausdorff measure $\mathcal{H}^{N-1}$. Of course, this implies two major things, that is:

- compact subsets of $\overline{\Omega}$ with Hausdorff dimension strictly greater than $N-1$ have an infinite $\mathcal{H}^{N-1}$-measure, hence they cannot be created as they would give a crack with infinite energy;

- conversely, compact subsets of $\overline{\Omega}$ with Hausdorff dimension strictly lower than $N-1$ have zero $\mathcal{H}^{N-1}$-measure, hence they can be created without the expense of energy.

The first point solves itself, but the second may be troublesome: luckily, in most applications, we have $N=3$, and as we will see in the next chapter, sets of zero $\mathcal{H}^2$-measure also have zero capacity, meaning they are invisible to the Sobolev-type spaces we will use.
Back to the model, we also want that the energy required to create an infinitesimal crack is not the same at every point in the crack and that it may, in principle, change. This density, or toughness of the material, will be denoted by $k(x)$, and it is such that $k(x) \geq k_0 > 0$ so that energy has to be always used in order to create new cracks. This toughness in principle can also approach ∞, in points where the material is close to impossible to break. Hence the energy of the crack will be the integral of this toughness over the *length* of the crack:

$$E_f(K) = \int_K k(x)\,\mathrm{d}\mathcal{H}^{N-1}(x). \tag{2.3}$$

It is trivial to see that the energy of the crack increases with the crack, due to the integral over K.
Let us observe that as of this choice of model, the energy of the crack does not depend on the external loading g. This implies that the total energy may be written in two equivalent formulations:

$$\begin{aligned} E(g,K) &\coloneqq E_b(g,K) + E_f(K) \\ &= \min_{v\in\mathcal{V}(g,K)} \left\{ \int_{\Omega\setminus K} W(x,\varepsilon(v)(x))\,\mathrm{d}x \right\} + \int_K k(x)\,\mathrm{d}\mathcal{H}^{N-1}(x) \qquad (2.4) \\ &= \min_{v\in\mathcal{V}(g,K)} \left\{ \int_{\Omega\setminus K} W(x,\varepsilon(v)(x))\,\mathrm{d}x + \int_K k(x)\,\mathrm{d}\mathcal{H}^{N-1}(x) \right\}. \qquad (2.5) \end{aligned}$$

These two formulations are useful because we need to study the convergence of minimizers of the former, in order to study the convergence of minimizers for the latter.

2.1.2 The quasi-static evolution law

The formulas for the total energy given in the previous subsection are written not considering time as a variable. We will not change the expression for the total energy to account for time, but we will write the evolution of the crack by means of a quasi-static evolution which minimizes at every time the total energy. The idea, given a certain loading $g(t)$ applied to $\partial_D\Omega$ (which is fixed), and given an initial crack set K_0 (which may also be empty), is that at a given time t, the crack evolves such that $K(t) \subset \overline{\Omega}$ minimizes the total energy $E(g(t), K)$ among all admissible sets K containing all the cracks at previous times, $K(s) \subseteq K$ for $s < t$.
This is quite a simple idea, yet it contains some strong assumptions about the model. First of all the simple fact that the crack minimizes the energy at every time speaks for the quasi-static nature of the process: indeed this may not be true in all cases, for example when thermal effects are considered, or simply with external loading quite steep in time. It is important to observe that, while the 1.21 formulation of Griffith's criterion is quasi-static in the sense that the energy release rate is not allowed to become bigger than the critical value, in the Francfort-Marigo model the quasi-static nature is shifted in the time evolution dynamics of the system, and it does not express on the energy release rate. However, under some mild assumptions, we will prove that this model predicts entirely the Griffith criterion, as well as the more modern 1.21 formulation.
Second of all, this model involves a step of global minimization of the total energy, but it would be more realistic to study some local minimizers of the energy and their properties. Lastly, the hypothesis that the crack has to contain all the cracks at previous times, as in 1.21, expresses the irreversibility of a dissipative process, and the absence of crack healing. In some models, this assumption may fall, as in more complex materials the cracks can actually heal under some conditions. To see a recent real-life example of this, we refer to [40], where the scope is far from mathematical, but it provides a nice insight into the concept of self-healing materials being used in ancient times.
Now the precise formulation of this quasi-static process requires a time discretization, after which the time step goes to 0. Of course, one must be careful in formalizing this process, and that is not the scope of this chapter: right now we just give an idea of the process.
In the discretized setting, given $\delta > 0$, we define the time steps to be $t_i^\delta := i\delta$ for $0 \le i \le \lfloor \frac{t_f}{\delta} \rfloor$ where t_f is the final time, and we consequently define $g_i^\delta := g(t_i^\delta)$. Then the discretized crack evolution K_i^δ has to satisfy

$$K_{i-1}^\delta \subseteq K_i^\delta, \qquad E(g_i^\delta, K_i^\delta) \le E(g_i^\delta, K), \quad \text{for every } K \supseteq K_{i-1}^\delta. \tag{2.6}$$

It follows that the continuous crack evolution is constructed by taking the limit of the discrete crack evolution as $\delta \to 0$. Under some mild assumptions on the regularity of the loading $g(t)$, we will prove that this procedure makes sense. In fact, for a general class of loadings, we will see that the next result holds true.

Proposition 2.1.1. *A quasi-static evolution of a crack has to satisfy these conditions:*

1. *$K(t)$ is non-decreasing in t, and $K_0 \subseteq K(0)$;*

2. *$K(t)$ minimizes the energy among all possible crack such that*

$$E(g(t), K(t)) \leq E(g(t), K) \qquad \forall K \supseteq \bigcup_{s<t} K(s);$$

3. *The function $t \mapsto E(g(t), K(t))$ is absolutely continuous, and for a fixed loading $g(t)$, we have that $K(t)$ is a stationary point for the function $s \mapsto E(g(t), K(s))$, that is*

$$\frac{\mathrm{d}}{\mathrm{d}s} E(g(t), K(s))\Big|_{s=t} = 0 \qquad \text{for a.e. } t.$$

The first condition is quite straightforward, and we will show that if $g(0) = 0$ then $K(0) = K_0$. In fact, since we know that $K_0 \subseteq K(0)$, due to $E_f(K)$ being strictly increasing in K, we also know that $E_f(K_0) \leq E_f(K(0))$. Moreover, if there are no external loadings applied, then the total energy is only made up of the energy of the crack set: then, by the second condition we have:

$$E_f(K(0)) = E(0, K(0)) = E(g(0), K(0)) \leq E(g(0), K_0) = E(0, K_0) = E_f(K_0).$$

Thus we know that $E_f(K(0)) = E_f(K_0)$, and again due to $E_f(K)$ being strictly increasing in K, this implies that $K(0) = K_0$.
The second condition expresses the minimality of the energy in $K(t)$ with respect to all cracks bigger than the cracks at previous times. The third condition is a stability condition of the energy at time t with respect to all the crack evolution.
For simplicity we will examine in this chapter a special class of loadings, called Monotonically Increasing Loadings (MIL), such that the loading is of the form $g(t) = \varphi(t)g_0$ for $t \leq 0$, for a non-decreasing and non-negative function $\varphi(t)$, and for a certain function $g_0 \in H^1(\Omega)$. Under this class of loadings, the first condition implies that $K(t)$ is increasing and that $K(0) = K_0$ (as seen before) and we have:

Proposition 2.1.2. *Under a continuous evolution as in Proposition 2.1.1, if the loading is non-decreasing in time, then the third condition is equivalent to:*

$$E(g(t), K(t)) \leq E(g(t), K(s)) \quad \forall s \leq t. \tag{2.7}$$

The formal derivation of this reformulation requires a bit of work, but it can be formally deduced from the discretized crack evolution. Let us take two crack sets K, K' such that $K \subset K'$, and then $s \geq t$. Then, since $g(t)$ is a MIL, we know that the bulk energy E_b is p-homogeneous in g, and since E_b is also decreasing in K for a fixed loading, we find that:

$$\begin{aligned}
&E(g(s), K) - E(g(s), K') \\
&= E(g(t), K) - E(g(t), K) + E(g(s), K) - E(g(t), K') + E(g(t), K') - E(g(s), K') \\
&= E(g(t), K) - E(g(t), K') + (\varphi(s)^p - \varphi(t)^p)(E_b(g_0, K) - E_b(g_0, K')) \\
&\geq E(g(t), K) - E(g(t), K').
\end{aligned}$$

Hence if $E(g(t), K') \leq E(g(t), K)$, then also $E(g(s), K') \leq E(g(s), K) \quad \forall s \geq t$. Then if we take K_i to be a solution to the discretized evolution with discretized MIL loading U_i,

since $K_i \supset K_{i-1}$, then we know $E(g(t), K_i) \leq E(g(t), K_{i-1}) \quad \forall t \geq t_i$. Then taking $t = t_i$, it is clear that

$$E(g_i, K_i) \leq E(g_i, K_{i-1}) \leq E(g_i, K_{i-2}) \leq \cdots \implies E(g_i, K_i) \leq E(g_i, K_j) \qquad \forall j \leq i.$$

Now we can formally take the limit as the step δ goes to 0, and take $t_i = t$, so that $K_i = K(t)$ and $K_j = K(s)$ with $s \leq t$, thus obtaining precisely 2.7.

2.2 Properties of the Model

As explained at the beginning of the chapter, this simple yet powerful model permits us to compute some things that were inaccessible with Griffith's previous theory, as we will now see. All these results assume the existence of a well-behaved solution of the model, where, by a solution, we mean the evolution of the crack given the initial crack set (also empty) and the evolution of the external loadings in time. For the results in this section, we will assume the loadings to be MIL, and we will be especially careful in pointing out where this assumption is used.

2.2.1 Crack initiation and failure

One of the first key points of this model is the ability the predict the onset of cracks even in a crack-free material, contrary to Griffith's model, which predicts infinite imposed stresses.
We first define the set of cracks that eliminate the bulk energy at a given time t, the bulk-free cracks, simply as:

$$\mathcal{F}(g)(t) \coloneqq \{K : E_b(g(t), K) = 0\}.$$

This simply implies that for $K \in \mathcal{F}(g)(t)$ we have $E(g(t), K) = E_f(K)$. This set of bulk-free cracks in general depends on time, as the loading depends on time, but in the case of MIL loadings the formulation simplifies, as

$$E_b(g(t), K) = E_b(\varphi(t) g_0, K) = \varphi(t)^p E_b(g_0, K) = 0.$$

Then in this case the set does not depend on time, and we get

$$\mathcal{F}(g) \coloneqq \{K : E_b(g_0, K) = 0\}.$$

We will go with the convention that when we indicate time in $\mathcal{F}(g)$ then we are using a general loading.
This set is not empty, as whatever crack separating the body from the physical object providing the Dirichlet condition, effectively gives a homogeneous Dirichlet condition, and we have already seen that this makes the bulk energy equal to 0 because then $v = 0$ becomes an admissible displacement field. Then $\partial_D\Omega \in \mathcal{F}(g)(t)$ and also whatever K such that $K \supset \partial_D\Omega$. Moreover, it is easy to see that if $K \in \partial_D\Omega \in \mathcal{F}(g)(t)$ then also $K' \supset K$ is in the same set. Let us also note that in the case of MIL loadings, $K_0 = \varnothing$

can never be a bulk-free crack if $g_0 \neq 0$.
We also define the time at which the eventual crack onset takes place, also called *initiation time*, with t_i:

$$t_i \coloneqq \sup_{t \geq 0} \{t : K(t) = K_0\}.$$

Of course, this time may take any value in $[0, +\infty]$, as the crack may propagate right at the beginning or can also never propagate.

Proposition 2.2.1. *In the case of MIL loadings, the crack never propagates, that is $t_i = +\infty$, if and only if the initial crack is bulk-free, $K_0 \in \mathcal{F}(g)(0)$.*

Proof. If $K_0 \in \mathcal{F}(g)$, then since any eventual evolution is such that $K(t) \supset K_0$, then we also have $K(t) \in \mathcal{F}(g)$ hence $E_b(g(t), K(t)) = 0$ for $t > 0$. Since the loading is MIL, we know that $E(g(t), K(t) \leq E(g(t), K(0)) = E(g(t), K_0)$ for $t > 0$ which implies $E_f(K(t)) \leq E_f(K_0)$ for $t > 0$. However $K(t) \supset K_0$ and $E_f(K)$ is increasing in K, so $E_f(K(t)) \geq E_f(K_0)$ hence the two energies are equal. Since the fracture toughness is strictly positive, then it must be that $K(t) = K_0$ for $t \geq 0$, hence $t_i = +\infty$.
Conversely if $t_i = +\infty$, then obviously $K(t) = K_0$ for $t \geq 0$. Then using the second property of the cracks and the MIL loading we have:

$$\varphi(t)^p E_b(g_0, K_0) + E_f(K_0) \leq \varphi(t)^p E_b(g_0, K) + E_f(K) \qquad K \supseteq K_0.$$

Let us now take a particular crack K that is the union of the initial crack and a bulk-free crack $K^{'}$, such that $K = K_0 \cup K^{'}$. This is always possible since at this stage we do not require any connectedness, and in we are always sure we can take $K^{'} = \partial_D \Omega$. Since $K \supset K^{'}$, then K is also bulk-free. Thus, rewriting the previous formula yields:

$$\varphi(t)^p E_b(g_0, K_0) + E_f(K_0) \leq E_f(K).$$

This has to be true even if we let t approach $+\infty$, and since the energies are positive, the only solution is if $E_b(g_0, K_0) = 0$. □

Since in the case of MIL loadings, this result says that a necessary and sufficient condition for no crack initiation is that the initial crack is bulk-free, this also implies that if the initial crack is not bulk-free, or simply such that $E_b(g(0), K_0) \neq 0$, then the crack will start to propagate always in a finite time. In particular, if $K_0 = \varnothing$ then a crack will always form in a finite time.
Another point interesting to the literature is the eventual failure of the sample, which can intuitively be thought of as a crack that split the sample. Analogously to the crack initiation, let us define now the least possible bulk energy as

$$E_b^{\min} \coloneqq \inf_t \{E_b(g(t), K) : K \supset K_0\}.$$

Just as before, in the case of MIL loadings, we get a superfluous term $\varphi(t)^p$, and we can rewrite it as

$$E_b^{\min} \coloneqq \inf \{E_b(g_0, K) : K \supseteq K_0\}.$$

A most useful observation is that, if $\mathcal{F}(g) \neq \varnothing$, as we are supposing, then there exists a crack for which $E_b(g_0, K) = 0$, and due to the non-negativity of the energies, we would have $E_b^{\min} = 0$.
Then, again under the case of MIL loadings, we have:

Proposition 2.2.2. *In the case of MIL loading:*

- $\lim_{t\to\infty} E_b(g_0, K(t)) = E_b^{\min}$;
- *if the limit value is attained at a time $t_f < +\infty$, then $K(t) = K(t_f) \quad \forall t \geq t_f$.*

Proof. Let us take (K_n) as a minimizing sequence for the least possible bulk energy. Then, since the bulk energy is monotonically decreasing in K for fixed $g(t)$, then also $(K_n \cup K(t))$ is a minimizing sequence for every $t \geq 0$. The second condition of 2.1.1 tells us that

$$\begin{aligned}\varphi(t)^p E_b(g_0, K(t)) + E_f(K(t)) &\leq \varphi(t)^p E_b(g_0, K_n \cup K(t)) + E_f(K_n \cup K(t)) \\ &\leq \varphi(t)^p E_b(g_0, K_n) + E_f(K_n) + E_f(K(t)).\end{aligned}$$

Dividing now by $\varphi(t)^p$ and letting $n \to \infty$ we get

$$E_b(g_0, K(t)) \leq E_b^{\min} + \frac{1}{\varphi(t)^p}\left(E_f(K_{\min} + E_f(K(t))\right),$$

where $K_{\min}$ is a set such that $E_b^{\min} = E_b(g_0, K_{\min})$. Letting now $t \to \infty$ gives us half of the first result, $\lim_{t\to+\infty} E_b(g_0, K(t)) \leq E_b^{\min}$, where the other inequality is obtained thanks to the bulk energy being decreasing in K.
Now, if the limit is attained at a time $t_f < +\infty$, by the third condition in the monotone continuous evolution we can say that

$$\begin{aligned} E(g(t), K(t)) &\leq E(g(t), K(t_f)) \qquad \forall t_f \leq t, \\ \varphi(t)^p E_b(g_0, K(t)) + E_f(K(t)) &\leq \varphi(t)^p E_b(g_0, K(t_f)) + E_f(K(t_f)) \qquad \forall t_f \leq t, \\ \varphi(t)^p E_b^{\min} + E_f(K(t)) &\leq \varphi(t)^p E_b^{\min} + E_f(K(t_f)) \qquad \forall t_f \leq t, \\ E_f(K(t)) &\leq E_f(K(t_f)) \qquad \forall t_f \leq t. \end{aligned}$$

Now, since if $t \geq t_f$ then $K(t) \supseteq K(t_f)$, we deduce that $E_f(K(t) = E_f(K(t_f) \quad \forall t \geq t_f$, which lets us conclude that $K(t) = K(t_f) \quad \forall t \geq t_f$. □

What this tells us is that under monotone increasing loadings, the sample will reach (eventually in an infinite time) a state of least bulk energy, which, under our assumptions, is obtained by a bulk-free crack.
There is also a result that lets us combine the crack initiation and crack failure mechanisms.

Proposition 2.2.3. *If we have a strictly increasing MIL loading, and if $K_0 \notin \mathcal{F}(g)$, then*

$$0 \leq t_i \leq \varphi^{-1}\left(\sqrt[p]{\frac{\inf\left\{E_f(K \setminus K_0) : K \supseteq K_0,\ K \in \mathcal{F}(g)\right\}}{E_b(g_0, K_0)}}\right) \leq t_f \leq +\infty.$$

Proof. Let us take a bulk-free crack $K \in \mathcal{F}(g)$ such that $K \supset K_0$. If $t_i = 0$ the first inequality is trivial. Hence if $t_i > 0$ we know that $K(t) = K_0$ for every $0 \leq t < t_i$. From the second condition on continuous evolution, we can then say for such a time $t < t_i$:

$$\begin{aligned}
\varphi(t)^p E_b(g_0, K_0) + E_f(K_0) &\leq \varphi(t)^p E_b(g_0, K) + E_f(K), \\
\varphi(t)^p E_b(g_0, K_0) + E_f(K_0) &\leq E_f(K), \\
\varphi(t)^p E_b(g_0, K_0) &\leq E_f(K) - E_f(K_0) = E_f(K \setminus K_0), \\
\varphi(t)^p &\leq \frac{E_f(K \setminus K_0)}{E_b(g_0, K_0)} \qquad \forall K \in \mathcal{F}(g) : K \supset K_0.
\end{aligned}$$

Taking the inf over such set of K, then taking the p-th square root and taking the limit of $t \to t_i$, we obtain the first inequality, since $\varphi(t)$ is strictly increasing and thus invertible. For the second part, we assume $t_f < +\infty$, otherwise the statement would be trivial. In such a case, taking $t > t_f$ means that $K(t) = K(t_f)$. Keeping in mind that for a MIL loading we have $E_b(g_0, K(t_f)) = E_b^{\min} = 0$, the third law of continuous monotone evolution would yield for a time $t \geq t_f$:

$$\begin{aligned}
&\varphi(t)^p E_b(g_0, K(t)) + E_f(K(t)) \leq \varphi(t)^p E_b(g_0, K(s)) + E_f(K(s)) \qquad \forall s \leq t, \\
s = 0 \implies\ &\varphi(t)^p E_b(g_0, K(t_f)) + E_f(K(t_f)) \leq \varphi(t)^p E_b(g_0, K_0) + E_f(K_0), \\
&E_f(K(t_f)) \leq \varphi(t)^p E_b(g_0, K_0) + E_f(K_0), \\
&E_f(K(t_f) \setminus K_0) \leq \varphi(t)^p E_b(g_0, K_0).
\end{aligned}$$

Noting that $K(t_f) \in \mathcal{F}(g)$ and $K(t) \supset K_0$, we can then provide an upper bound for the inf taken before as:

$$\inf \left\{ E_f(K \setminus K_0) : K \supseteq K_0,\ K \in \mathcal{F}(g) \right\} \leq E_f(K(t_f) \setminus K_0) \leq \varphi(t)^p E_b(g_0, K_0).$$

Dividing by $E_b(g_0, K_0)$, taking the root and then the limit $t \to t_f$ just as before, we obtain the second part of the inequality. □

It is interesting to observe that in the case when $0 < t_i = t_f < +\infty$, the evolution can be brutal if $K_0 \neq K_{\min}$, in the sense that at time $t_i = t_f$, $K(t)$ goes instantaneously from K_0 to $K_{\min}$. This brutal behavior will be better studied in the next subsection.

2.2.2 Comparison with Griffith's criterion

Even though this model was not created with the idea of being an offspring of Griffith's theory, it can be considered an extension of it, as the dissipative formulation of Griffith's criterion can be obtained through this model.
Some work needs to be done though, because Griffith's model only concerned itself with the crack length and not the crack path. So first of all we will put this constraint in the model, that of a Predefined Crack Path (PCP). We assume the existence of regular simple arc Γ parameterized by its arc length ℓ via a path $\phi\colon [\ell_0, \ell_1] \to \Omega$, where ℓ_0 and ℓ_1 are respectively initial and final length. We identify the crack set K with the rectifiable curve $K(\ell(t)) = K_0 \cup \Gamma.$, of total length ℓ_1, and initial point x_0 and final point x_1. As we take

x_0 to be the endpoint of the initial crack set K_0, which we assume to have length ℓ_0, we will then define

$$K(\ell(t)) \coloneqq K_0 \cup \Gamma(\ell(t)) = K_0 \cup \{\phi(\mathbb{l}) : \ell_0 \leq \mathbb{l} \leq \ell(t)\}.$$

At the moment we do not concern ourselves with the question of the regularity of energies and length, as that is a question for the following chapters: for now, we will assume all the necessary regularity.
Let us briefly recall a certain notion of continuity used in a lot of results.

Definition 2.2.4. *A function $f\colon [a,b] \to \mathbb{R}$ is said to be absolutely continuous on $[a,b]$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \varepsilon,$$

for an arbitrary collection of mutually disjoint sub-intervals $\{[x_i, y_i] : i = 1, \dots, n\}$ of $[a,b]$ with $\sum_{i=1}^{n} |y_i - x_i| < \delta$.

A much more useful definition of absolute continuity is the following, which establishes an equivalence with the existence of an integrable derivative.

Definition 2.2.5. *We say that a function $f\colon [a,b] \to \mathbb{R}$ is absolutely continuous if and only if there exists an integrable function g on $[a,b]$ such that*

$$f(x) = f(a) + \int_a^x g(t)dt \qquad \forall x \in [a,b].$$

If such a g exists, then $f' = g$ a.e. in $[a,b]$.

We also give a definition of absolute continuity for curves in a general metric space: this will be useful in the next chapters.

Definition 2.2.6. *Let (X,d) be a metric space. We say that a curve $u\colon [a,b] \to X$ is absolutely continuous from $[a,b]$ to X if there exists a function $m \in L^1([a,b])$ such that:*

$$d\big(u(s), u(t)\big) \leq \int_s^t m(\tau)\,\mathrm{d}\tau \qquad \forall a \leq s \leq t \leq b.$$

What we want to study right now is the behavior of the function $t \mapsto \ell(t)$. Under the assumption of an absolute continuous evolution, we have the main theorem of this chapter:

Theorem 2.2.7. *If $\ell(t)$ is an absolutely continuous function of time, then the dissipative Griffith's criteria are satisfied. Namely:*

1. $\dot{\ell}(t) \geq 0$;
2. $\Big(k\big(x(\ell(t))\big) + \dfrac{\mathrm{d}E_b\big(g(t), K(\ell)\big)}{\mathrm{d}\ell}\Big|_{\ell=\ell(t)}\Big) \geq 0$;

3. $\Big(k\Big(x(\ell(t))\Big) + \dfrac{\mathrm{d}E_b(g(t),K(\ell)}{\mathrm{d}\ell}\Big|_{\ell=\ell(t)}\Big)\cdot\dot{\ell}(t) = 0.$

Proof. The first item is obvious from the definition.
For the second item, we use the second condition of continuous evolution, with a test crack of length $\ell(t)+\varepsilon$, such that $K(\ell(t)) \subset K(\ell(t)+\varepsilon)$:

$$E_b(g(t),K(\ell(t))) + E_f(K(\ell(t))) \leq E_b(g(t),K(\ell(t)+\varepsilon)) + E_f(K(\ell(t)+\varepsilon)).$$

By rearranging and dividing everything by ε, we get

$$\left(\frac{E_b(g(t),K(\ell(t)+\varepsilon)) - E_b(g(t),K(\ell(t)))}{\varepsilon}\right) + \left(\frac{E_f K(\ell(t)+\varepsilon)) - E_f(K(\ell(t)))}{\varepsilon}\right) \geq 0.$$

Taking now the limit of $\varepsilon \to 0^+$ we get:

$$\frac{\mathrm{d}E_b(g(t),K(\ell))}{\mathrm{d}\ell}\Big|_{\ell=\ell(t)} + \frac{\mathrm{d}E_f(K(\ell))}{\mathrm{d}\ell}\Big|_{\ell=\ell(t)} = \frac{\mathrm{d}E_b(g(t),K(\ell))}{\mathrm{d}\ell}\Big|_{\ell=\ell(t)} + k(x(\ell(t))) \geq 0.$$

For the third item, we consider the third condition from the continuous evolution:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Big(E_b\Big(g(t),K(\ell(s))\Big) + E_f\Big(K(\ell(s))\Big)\Big)\Big|_{s=t} &= 0,\\
\frac{\mathrm{d}E_b(g(t),K(\ell))}{\mathrm{d}\ell}\Big|_{\ell=\ell(t)}\cdot\frac{\mathrm{d}\ell(s)}{\mathrm{d}s}\Big|_{s=t} + \frac{\mathrm{d}E_f(K(\ell))}{\mathrm{d}\ell}\Big|_{\ell=\ell(t)}\cdot\frac{\mathrm{d}\ell(s)}{\mathrm{d}s}\Big|_{s=t} &= 0,\\
\Big(\frac{\mathrm{d}E_b(g(t),K(\ell))}{\mathrm{d}\ell}\Big|_{\ell=\ell(t)} + k\Big(x(\ell(t))\Big)\Big)\cdot\dot{\ell}(t) &= 0,
\end{aligned}$$

which is precisely what we wanted. □

Let us observe that we did not use the assumption of MIL loadings, so this result is quite general, and looking at the original dissipative formulation of Griffith's criteria (1.21), we may conclude that:

$$\begin{cases} G(t) = -\dfrac{\mathrm{d}E_b(g(t),K(\ell))}{\mathrm{d}\ell}\Big|_{\ell=\ell(t)}, \\ G_c(t) = k\Big(x(\ell(t))\Big). \end{cases} \tag{2.8}$$

If we drop altogether the hypothesis of continuity for $\ell(t)$, there may also be some jumps in the path, points in time where the length goes instantaneously between two different values.

Proposition 2.2.8. *If at a time t_0 we have $\ell^+(t_0) \coloneqq \lim\limits_{t\to t_0^-}\ell(t) \neq \lim\limits_{t\to t_0^+}\ell(t) =: \ell^-(t_0)$, then:*

$$-\big(E_b(g(t_0),K(\ell^+(t_0))) - E_b(g(t_0),K(\ell^-(t_0)))\big) \leq E_f(K(\ell^+(t_0))) - E_f(K(\ell^-(t_0))).$$

Moreover, if the loading is MIL, there is the equality.

Proof. We can apply the second condition of continuous evolution at time $t_0 - \varepsilon$, with test crack $K\big(\ell(t_0 + \varepsilon)\big)$, obtaining:

$$E_b(g(t_0-\varepsilon), K(\ell(t_0-\varepsilon))) + E_f(K(\ell(t_0-\varepsilon))) \leq E_b(g(t_0-\varepsilon), K(\ell(t_0+\varepsilon))) + E_f(K(\ell(t_0+\varepsilon))).$$

Rearranging and taking the limit as $\varepsilon \to 0$ we get precisely:

$$-\big(E_b(g(t_0), K(\ell^+(t_0))) - E_b(g(t_0), K(\ell^-(t_0)))\big) \leq E_f(K(\ell^+(t_0))) - E_f(K(\ell^-(t_0))).$$

We cannot use the third condition of the general continuous evolution since that derivative is not defined in t_0, but in case the loading is MIL we can use the third condition of the monotone continuous evolution, with $t = t_0 + \varepsilon$ and $s = t_0 - \varepsilon$. By taking the limit as before, we directly get:

$$E_b(g(t_0), K(\ell^+(t_0))) + E_f(K(\ell^+(t_0))) \leq E_b(g(t_0), K(\ell^-(t_0))) + E_f(K(\ell^-(t_0))).$$

Rearranging we get the opposite inequality, thus the equality. □

2.2.3 Progressive or brutal evolution

We now would like to study the assumptions under which the crack evolution is progressive (continuous) or brutal (discontinuous). This will permit us to overcome one of the limitations of the original Griffith's theory, which is the inability to predict what happens when $G > G_c$.
As in the previous subsection, we assume a Predefined Crack Path (PCP); moreover, we also assume throughout this subsection, that the loadings are MIL, such that the bulk and crack energies can be parameterized only in terms of the crack length ℓ, as follows:

$$\begin{cases} E_b(\ell) \coloneqq E_b(g_0, K(\ell)), \\ E_f(\ell) \coloneqq E_f(K(\ell)) - E_f(K_0). \end{cases}$$

We chose to subtract $E_f(K_0)$ from $E_f(\ell)$ such that the crack energy has a minimum in $\ell = 0$. Of course, since the bulk and crack energies are respectively monotonically decreasing and increasing with K, then their parameterized counterpart will be as well, but in ℓ. As before, we do not ask about the regularity of these parameterized energies, but we will be careful in stating the needed regularity for the next results, as this will be a key point for the distinction between progressive and brutal growth, as well as convexity. Since during crack propagation, the length always grows, then also the crack energy must grow. It then makes sense to try and re-parameterize the bulk energy in terms of the crack energy as follows:

$$\begin{cases} \lambda \coloneqq E_f(\ell), \\ \widehat{E}_b(\lambda) \coloneqq E_b(E_f^{-1}(\lambda)). \end{cases}$$

This is permitted since $E_f(\ell)$ is non-negative and strictly increasing in ℓ, and thus invertible at least locally: since ℓ can exhibits localized jumps, then it will be invertible on every interval without a jump. Moreover its inverse $E_f^{-1}(\lambda)$ is strictly increasing as well,

which means that $\widehat{E}_b(\lambda)$ is decreasing in λ. It is not strictly decreasing since ℓ can exhibit jumps.
With this change of variables, and keeping in mind that this time the crack path is fixed *a priori*, the monotone continuous evolution gets translated into:

Proposition 2.2.9. *Under the assumptions of Predefined Crack Path and Monotone Increasing Loadings, we get:*

1. $\lambda(t)$ *is increasing in* t *and* $\lambda(0) = 0$*;*
2. $\varphi(t)^p \widehat{E}_b(\lambda(t)) + \lambda(t) \leq \varphi(t)^p \widehat{E}_b(\lambda) + \lambda \qquad \forall \lambda \geq \lambda^-(t)$*;*
3. $\varphi(t)^p \widehat{E}_b(\lambda(t)) + \lambda(t) \leq \varphi(t)^p \widehat{E}_b(\lambda(s)) + \lambda(s) \qquad \forall s \leq t.$

Moreover, we can also translate the jump condition 2.2.8 in terms of this parameterization:

Proposition 2.2.10. *If at a time* t_0 *we have* $\lambda^+(t_0) \neq \lambda^-(t_0)$*, then:*

$$-\varphi(t_0)^p \Big(\widehat{E}_b(\lambda^+(t_0)) - \widehat{E}_b(\lambda^-(t_0)) \Big) = \lambda^+(t_0) - \lambda^-(t_0).$$

If $\widehat{E}_b$ is continuous in λ, there is a nice result that lets us link the progressive or brutal evolution with properties of convexity.

Proposition 2.2.11. *If* $\widehat{E}_b(\lambda)$ *is continuous and* $\varphi(0) = 0$*, then for each* t*,* $\lambda(t)$ *is one of the minimizers of* $\varphi(t)^p \widehat{E}_b(\lambda) + \lambda$ *on the interval* $[0, E_f(L)]$.

Since L is the total possible length of the fixed crack set, then $E_f(L)$ is the maximum possible value of the crack energy.

Proof. We will prove that $\lambda(t)$ is a solution of the fixed monotone evolution if and only if it is a minimizer to $\varphi(t)^p \widehat{E}_b(\lambda) + \lambda$ on the interval $[0, E_f(L)]$.
If $\lambda(t)$ is a minimizer, the minimum with $t = 0$ is obtained when $\lambda = 0$, and since the function to minimize grows with t, then also the minimum must increase with time, so the first condition of the fixed evolution is satisfied. Then, due to the minimality in $\lambda(t)$, the second and third conditions are trivially satisfied. Hence $\lambda(t)$ is also a solution.
Conversely, if $\lambda(t)$ is a solution, then the second condition of the fixed evolution is already half of the minimal inequality. We now need to extend the inequality to $\lambda \in [0, \lambda^-(t))$. Now, if $\lambda(t)$ is continuous on such interval, then by the third condition we can conclude. Conversely, λ might have a finite or countable number of isolated jump discontinuities, which we will label (t_n) with the corresponding jump $[\lambda_n^-, \lambda_n^+]$, such that the minimum $\lambda(t_n) \in [\lambda_n^-, \lambda_n^+]$. Let us note that if $t \in (t_{n-1}, t_n)$ for some n, then the opposite inequality is satisfied by the third condition. We can now write the third condition in the fixed evolution with $t = t_n + \varepsilon$, such that:

$$\varphi(t_n+\varepsilon)^p \widehat{E}_b(\lambda(t_n+\varepsilon)) + \lambda(t_n+\varepsilon) \leq \varphi(t_n+\varepsilon)^p \widehat{E}_b(\lambda(t)) + \lambda(t) \qquad \forall t \leq t_n + \varepsilon;$$
$$\lim_{\varepsilon \to 0^+} \implies \varphi(t_n)^p \widehat{E}_b(\lambda_n^+) + \lambda_n^+ \leq \varphi(t_n)^p \widehat{E}_b(\lambda) + \lambda \qquad \forall \lambda \leq \lambda_n^+,$$

which is the inequality for $\lambda = \lambda_n^+$. Then, rearranging 2.2.10 for t_p and using the second condition of the fixed evolution we get:

$$\varphi(t_n)^p \widehat{E}_b(\lambda_n^-) + \lambda_n^- = \varphi(t_n)^p \widehat{E}_b(\lambda_n^+) + \lambda_n^+ \leq \varphi(t_n)^p \widehat{E}_b(\lambda) + \lambda \qquad \forall \lambda \geq \lambda_n^-,$$

which proves the inequality for $\lambda = \lambda_n^-$. Now, since $\widehat{E}_b$ is decreasing in λ, we can conclude that $\forall \lambda \in (\lambda_n^-, \lambda_n^+)$:

$$\begin{aligned}
\varphi(t)^p \widehat{E}_b(\lambda_n^+) + \lambda_n^+ &\leq \varphi(t_n)^p \widehat{E}_b(\lambda_n^+) + \lambda_n^+ + (\varphi(t)^p - \varphi(t_n)^p)\widehat{E}_b(\lambda_n^+) \\
&\leq \varphi(t_n)^p \widehat{E}_b(\lambda_n^+) + \lambda_n^+ + (\varphi(t)^p - \varphi(t_n)^p)\widehat{E}_b(\lambda) \\
&\leq \varphi(t_n)^p \widehat{E}_b(\lambda) + \lambda + (\varphi(t)^p - \varphi(t_n)^p)\widehat{E}_b(\lambda) \\
&= \varphi(t)^p \widehat{E}_b(\lambda) + \lambda,
\end{aligned}$$

which means that the at every time t, $\lambda(t)$ is a minimizer. □

If the function $\widehat{E}_b(\lambda)$ has also a continuous derivative, then the minimizers will be among the stationary points, and if the function is also convex, the minimizers will be unique and equal to the unique stationary point of the derivative. From this observation follows this important result.

Theorem 2.2.12.

1. *If $\widehat{E}_b(\lambda)$ is a strictly convex and C^1 function, and $\varphi(t)$ is strictly increasing, then $\lambda(t)$ is given by:*

$$\lambda(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq t_i, \\ \widehat{E}_b'^{-1}\left(-\dfrac{1}{\varphi(t)^p}\right) & \text{if } t_i < t < t_L, \\ E_f(L) & \text{if } t_L \leq t, \end{cases} \tag{2.9}$$

   where the times are given by:

$$\begin{cases} t_i = \varphi^{-1}\left(\sqrt[p]{-\dfrac{1}{\widehat{E}_b'(0)}}\right), \\ t_L = \varphi^{-1}\left(\sqrt[p]{-\dfrac{1}{\widehat{E}_b'(E_f(L))}}\right). \end{cases}$$

2. *If $\widehat{E}_b(\lambda)$ is a concave and C^0 function, and $\varphi(t)$ is strictly increasing, then $\lambda(t)$ is given by:*

$$\lambda(t) = \begin{cases} 0 & \text{if } 0 \leq t < t_J, \\ E_f(L) & \text{if } t_J < t, \end{cases} \tag{2.10}$$

   where the time t_J is given by:

$$t_J = \varphi^{-1}\left(\sqrt[p]{-\frac{E_f(L)}{\widehat{E}_b(E_f(L)) - \widehat{E}_b(0)}}\right).$$

Proof. For the first part, if $\widehat{E}_b$ is strictly convex and C^1, then also $\varphi(t)^p\widehat{E}_b(\lambda)+\lambda$ has the same properties, hence it has a unique minimizer. Using 2.2.11, if the minimum is in $(0, E_f(L))$ then the minimum is found by setting the first derivative to 0:

$$\varphi(t)^p\widehat{E}_b'(\lambda)+1=0 \implies \lambda=\widehat{E}_b'^{\,-1}\left(-\frac{1}{\varphi(t)^p}\right).$$

In addition, due to the strict convexity, the minimum is at $\lambda=0$ if the derivative in that point is non-negative, $\varphi(t)^p\widehat{E}_b'(0)+1\geq 0$. However, since increasing t makes the minimum of the extended function move rightward, there will be a value t_i such that:

$$\varphi(t_i)^p\widehat{E}_b'(0)+1=0 \implies t_i=\varphi^{-1}\left(\sqrt[p]{-\frac{1}{\widehat{E}_b'(0)}}\right),$$

and for $t<t_i$ the minimum is in $\lambda=0$. Analogously, the minimum is at $\lambda=E_f(L)$ is the derivative in that point is non-positive, $\varphi(t)^p\widehat{E}_b'(E_f(L))+1\leq 0$. For the same reason as before, decreasing t makes the minimum of the extended function move leftward, so there exists a value t_L such that:

$$\varphi(t_L)^p\widehat{E}_b'(E_f(L))+1=0 \implies t_L=\varphi^{-1}\left(\sqrt[p]{-\frac{1}{\widehat{E}_b'(E_f(L))}}\right),$$

and for $t>t_L$ the minimum is in $\lambda=E_f(L)$.
For the second part, if $\widehat{E}_b$ is continuous and concave, then also $\varphi(t)^p\widehat{E}_b(\lambda)+\lambda$ has the same properties. This means that the minimum can only be either in $\lambda=0$ or in $\lambda=E_f(L)$: for small t it will surely be the first case, while for large t it will be the second. Since increasing t means that the minimum increases, there has to be a point t_J where the minimum jump between these two values. For this time t_J, the function has the same values in $\lambda=0$ and $\lambda=E_f(L)$, meaning that:

$$\varphi(t_J)^p\widehat{E}_b(0)+0=\varphi(t_J)^p\widehat{E}_b(E_f(L))+E_f(L) \implies t_J=\varphi^{-1}\left(\sqrt[p]{-\frac{E_f(L)}{\widehat{E}_b(E_f(L))-\widehat{E}_b(0)}}\right),$$

while for $t<t_J$ the minimum is $\lambda=0$ and for $t>t_J$ the minimum is $\lambda=E_f(L)$. □

What this theorem is telling us is conditions such that the growth is progressive or brutal.
If $\widehat{E}_b$ is convex but not strictly convex, it may have parts where it is linear: in those parts, there will be a brutal growth of the crack, and to generalize it further, if the function is not convex, the minimum points will coincide with the minimum points of the convex envelope. Hence, considering the convex envelope, the growth will be progressive on the strictly convex parts, and brutal in the linear parts, which in the original function translates to the concave parts. This means that the growth can be studied for an arbitrary continuous function.
What we discussed until now regarded just the speed of propagation, and the conditions

on the energies. We disregarded completely the study of the path of the crack by fixing a crack path, and that is fine with us since such a study would require at least some existence and regularity results on the solutions of this model. Nonetheless, even having fixed the crack path, we are forgetting about another important condition for the crack to propagate, which is inherently linked with energies and loadings: the displacement and stress fields in the domain.

This is evident in Griffith's and Irwin's formulation for a two dimensional domain, where the singularities of displacement and stress play an essential role. In Sect. 1.2, we derived that around the crack tip, the stress field has a $r^{-\frac{1}{2}}$-singularity, and the displacement field has a $r^{\frac{1}{2}}$-singularity. These kinds of singularities may arise not only on crack tips, but also on the non-smooth parts of the domain (the so-called *concave corners* of the domain), and also in points of the boundary experiencing a change in boundary conditions, like when the propagating crack intersects $\partial_D\Omega$. All of these conditions can be summarized in the next assumption: given $u_0 = u(g_0, K_0)$ the displacement field giving elastic equilibrium in $\Omega \setminus K_0$, and given (x_j) a finite collection of singular points for the displacement field inside Ω, we assume that u_0 is of the form

$$u_0(x) = \widehat{u}(x) + \sum_{j=1}^{n} r_j^{\alpha_j} v_j(\theta_j),$$

where $\widehat{u}$ is the singularity-free part of the field, (r_j, θ_j) are polar coordinates centered in the singular points, and α_j is the degree of singularity, assumed to be $0 < \alpha_p < 1$. Such a restriction on the degree of singularity is because the stresses have to be unbounded, as we already discussed in 1.2, which gives $\alpha_j < 1$, and also because due to the minimization the bulk energy has to be finite, which gives $\alpha_j > 0$. We also assume, just like in Griffith's model, we say that the fracture toughness is a constant, meaning that in a neighborhood of the singular points, $k(x) = \kappa_j$.

As a last assumption, we take a stronger condition than the Predefined Crack Path, namely that of Predefined Incremental Crack Path (PICP), such that for small times the crack propagates along $n+1$ rectifiable curves Γ_{ℓ_j} of finite length ℓ_j, where for $j \leq n$ the curves are located in a neighborhood of a singular point, while for $j = n+1$, the curve is away from such neighborhoods.

This assumption, which is relevant only for small times, is useful for studying the propagation speed at the initiation time. To do that, we need an expansion of the bulk energy in terms of the lengths of crack curves, which is provided by [18], given that $E_b(g_0, K_0) > 0$:

$$E_b\left(g_0, K_0 \cup \bigcup_{j=1}^{n+1} \Gamma_{\ell_j}\right) = E_b(g_0, K_0) - \sum_{j=1}^{n+1} \left\{ S_j \ell_j^{2\alpha_j} + o\left(\ell_j^{2\alpha_j}\right) \right\}, \tag{2.11}$$

where S_j is a factor depending on the shape of the defect and the strength of singularity, such that for $j \leq n$ we have $S_j > 0$, while for $j = n+1$, we have $S_j \geq 0$.

Proposition 2.2.13. *In the setting described above:*

1. *if at least one of the singularities has $\alpha_j < \frac{1}{2}$, the $t_i = 0$, the growth will be progressive, and under PICP assumption, the length corresponding to such singularities will behave like $t^{\frac{2}{1-2\alpha_j}}$;*

2. *if, under PICP assumption, every singularity has* $\alpha_j > \frac{1}{2}$*, then* $t_i \in (0, \infty)$ *and the growth will be brutal;*

3. *if, under PICP assumption, there is a singularity with* $\alpha = \frac{1}{2}$*, and all other singularities having* $\alpha_j < \frac{1}{2}$*, then* $t_i \in (0, \infty)$*;*

4. *if, under PICP assumption, there are no singular points (*$\alpha_j = 1$*), then either the crack will not propagate, or* $t_i \in (0, \infty)$ *and the growth will be brutal.*

This provides yet another result about the growth of the crack, proving again the superiority of this model to Griffith's model. We remind that in Irwin's theory, the displacement has $\alpha = \frac{1}{2}$, and assuming no other singularities, this predicts that the crack will have a non-zero finite initiation time, under ever-increasing loadings.

Chapter 3

Mathematical preliminaries

The results presented in the previous chapter seem to be quite precise and formal. Still, as we said multiple times, they lose all rigor in the absence of some results of existence for the continuous evolution given the initial crack and the loadings. This question though is no mere task, especially for an arbitrary body in $\mathbb{R}^3$, arbitrary loadings, and arbitrary constitutive law for the material. Since a complete study has not been done, we will focus on linearly elastic homogeneous isotropic material, with Lamé coefficients λ and μ. Moreover, we will restrict ourselves to a two dimensional anti-plane shear, to avoid using the spaces of special functions of bounded variations.
We can start now the computation of the bulk energy under these assumptions: for ease of calculation, we will use the reduced elasticity tensor, under the assumptions of symmetry. In general, if the displacement is of the form $u = (u_1, u_2, u_3)$, then the infinitesimal strain tensor ε and Cauchy stress tensor σ are two second-order tensors described by:

$$\varepsilon = \begin{bmatrix} \varepsilon_{x,x} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_{zz} \end{bmatrix} \qquad \sigma = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

where $\varepsilon_{i,i} = \frac{\partial u_i}{\partial x_i}$ and $\varepsilon_{i,j} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$. These two tensors are related through the elasticity tensor C, a fourth-order tensor with 81 entries, such that $\sigma = C\varepsilon$. At this point in general, for a linear elastic material, the elastic energy per unit volume turns out to be

$$\mathrm{d}W = \frac{1}{2}\sigma^\top \varepsilon \,\mathrm{d}x = \frac{1}{2}\varepsilon^\top C\varepsilon \,\mathrm{d}x. \tag{3.1}$$

It can be easily proved that ε and σ are symmetric tensors, hence they both have 6 independent entries: we can thus define two vectors, with these 6 respective entries, obtaining

$$\begin{aligned} \varepsilon_0 &\coloneqq \begin{bmatrix} \varepsilon_x & \varepsilon_y & \varepsilon_z & \gamma_{xy} & \gamma_{xz} & \gamma_{yz} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_3}{\partial x_3} & \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2} \end{bmatrix}, \end{aligned}$$

and also:

$$\sigma_0 \coloneqq \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z & \tau_{xy} & \tau_{xz} & \tau_{yz} \end{bmatrix}.$$

With these definitions, it follows that the elasticity tensor C can be reduced to a 6×6 tensor, therefore greatly reducing the number of entries. Under the further assumptions of isotropy and homogeneity, it can be further simplified: it can be proved that it depends on two parameters, the already mentioned Lamé coefficients λ and μ. Hence σ_0 and ε_0 are linked through the reduced elasticity tensor H, such that:

$$\sigma_0 = H\varepsilon_0 \qquad H \coloneqq \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}. \tag{3.2}$$

In our case, the reference configuration is an infinite cylinder $\Omega \times \mathbb{R}$, with $\Omega \subset \mathbb{R}^2$, so that for every $(x_1, x_2, y) \in \Omega \times \mathbb{R}$ the displacement field has the special form $\big(0, 0, u_3(x_1, x_2)\big)$, which yields a simpler form of the energy. Moreover, the cracks will be of the form $K \times \mathbb{R}$, where $K \in \mathcal{K}(\Omega)$, the sets of non-empty compact subsets of Ω. In this setting, the bulk and crack energies for the whole body will be infinite, as the length of the cylinder is infinite. Still, we can consider them as energies of a cylinder of unit length.
From these assumptions, we get:

$$\varepsilon_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_3}{\partial x_2} \end{bmatrix}.$$

Substituting in (3.1) we get that the elastic energy per unit volume is:

$$\mathrm{d}W = \frac{1}{2}\varepsilon_0^\top H \varepsilon_0 \,\mathrm{d}x = \frac{\mu}{2}\left[\left(\frac{\partial u_3}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_2}\right)^2\right]\mathrm{d}x = \frac{\mu}{2}|\nabla u_3|^2 \,\mathrm{d}x. \tag{3.3}$$

The bulk elastic energy for a given displacement will then be:

$$\mathcal{E}_b(u_3, K) \coloneqq \frac{\mu}{2}\int_{\Omega\setminus K} |\nabla u_3|^2 \,\mathrm{d}x,$$

where μ, already introduced as the second Lamé coefficient, is also known as the shear modulus of the material. The crack energy will be the area of the crack, but since we consider it with unit height, we can write

$$E_f \coloneqq \gamma\, \mathcal{H}^1(K),$$

where γ is the constant fracture toughness, borrowed from Griffith's work.
The set Ω will be a bounded connected open set, with a sufficiently regular boundary, such that $\partial\Omega$ can be divided in $\partial_D\Omega$, the Dirichlet part, and $\partial_N\Omega = \partial\Omega \setminus \overline{\partial_D\Omega}$, the Neumann part. On the Dirichlet part of the boundary, a displacement will be imposed, through a

sufficiently regular function g defined on $\Omega \setminus K$.
As we do not impose a displacement across the entire body, but just a boundary displacement, we will need to minimize the total energy across all displacement satisfying the boundary condition, as follows:

$$E_b(g,K) \coloneqq \min_{v\in\mathcal{V}(g,K)} \mathcal{E}_b(v,K) = \min_{v\in\mathcal{V}(g,K)} \left\{ \frac{\mu}{2} \int_{\Omega\setminus K} |\nabla v|^2 \,\mathrm{d}x \right\}.$$

The set on which we look for the minimizer $\mathcal{V}(g,K)$ has to be a space of functions with a square-integrable gradient on $\Omega \setminus K$: for now it would be too restrictive to assume $v \in H^1(\Omega \setminus K)$ since we do not need the global square integrability of the function, but it would also be too general to assume the structure of distributions since it would not be easy to work with due to the presence of non-regular distributions. Hence we choose to work with a subspace of the distributions with square-integrable gradient, that is the Deny-Lions space $L^{1,2}(\Omega \setminus K)$ of locally square-integrable functions with square-integrable gradient, so that

$$\mathcal{V}(g,K) = \left\{ v \in L^{1,2}(\Omega \setminus K) : v = g \text{ on } \partial_D\Omega \setminus K \right\}.$$

The value of the function at the boundary has to be taken carefully since its regularity is not clear. To do this, the concept of *quasi-everywhere* has to be introduced, basically the analog of almost-everywhere but for Sobolev spaces.
With all of this in mind, the total energy will be

$$\begin{aligned} E(g,K) &\coloneqq E_b(g,K) + E_f(K) \\ &= \min_{v\in\mathcal{V}(g,K)} \{\mathcal{E}_b(v,K)\} + E_f(K) \\ &= \min_{v\in\mathcal{V}(g,K)} \left\{ \frac{\mu}{2} \int_{\Omega\setminus K} |\nabla v|^2 \,\mathrm{d}x + \gamma\, \mathcal{H}^1(K) \right\}. \end{aligned}$$

In most of the next results, we will usually denote the admissible displacements with v and the minimum with u, if not mentioned otherwise.

In this chapter, we will present some results that are needed in Chapter 4 in order to formalize in a mathematically rigorous way the Francfort-Marigo model. Not every result presented will be used in this thesis, as this is also a really short and self-contained introduction to some of these concepts.

3.1 Capacity of a set and Deny-Lions spaces

First of all, we say that an open set $\Omega \subset \mathbb{R}^N$ has a *Lipschitz boundary*, if for some $a, r \in (0, +\infty)$, and for every $x_0 \in \partial\Omega$, there is an orthogonal reference system with origin precisely in x_0, a cylinder $C = C' \times (-a, a)$, with $C' = B_{N-1}(r)$ being the open ball in $\mathbb{R}^{N-1}$ with radius r, and a Lipschitz function $\phi\colon C' \to (-a, a)$ with $\phi(0) = 0$ such that

$$\partial\Omega \cap C = \{(x', \phi(x')) : x' \in C'\}; \tag{3.4}$$

$$\Omega \cap C = \{(x', x_N) : x' \in C', x_N > \phi(x')\}. \tag{3.5}$$

This definition simply states that each point in $\partial\Omega$ has a neighborhood in $\partial\Omega$ which is the graph of a Lipschitz function, whose graph divides Ω from its complementary. This definition could also be employed for every single point, in which case the set of points for which such a function exists is called the *Lipschitz part of the boundary*, and will be identified with $\partial_L\Omega$.
For appropriately describing the fine properties of functions in Sobolev spaces, the appropriate kind of measure of negligible sets is not the Lebesgue measure, as in $L^p(\Omega)$ spaces, but a notion of outer measure on Ω called *p-capacity*.

Definition 3.1.1. *Let Ω be a bounded open set in $\mathbb{R}^N$, and $A \subset \Omega$ an arbitrary subset. We define $\mathcal{W}(A;\Omega)$ to be:*

$$\mathcal{W}(A;\Omega) \coloneqq \{u \in W_0^{1,p}(\Omega) \cap C(\Omega) : A \subseteq \{u \geq 1\}\},$$

that is the space of functions in $W^{1,p}(\Omega)$, vanishing at the boundary of Ω, that are greater than 1 in a neighborhood of A. Then we can finally define the p-capacity as:

$$\mathrm{Cap}_p(A;\Omega) \coloneqq \inf_{u\in\mathcal{W}(A;\Omega)} \int_\Omega |\nabla u|^p \, \mathrm{d}x.$$

Sometimes another definition of capacity is used:

$$\mathrm{cap}_p(A;\Omega) \coloneqq \inf_{u\in\mathcal{W}(A;\Omega)} \int_\Omega (|u|^p + |\nabla u|^p) \, \mathrm{d}x;$$
$$\mathcal{W}(A;\Omega) \coloneqq \{u \in W_0^{1,p}(\Omega) \cap C(\Omega) : A \subseteq \{u \geq 1\}\}.$$

We will stick with the definition involving only the integral with the gradient because it only requires a p-integrable gradient, which is precisely what is needed in our treatment. Let us observe that, based on the definition just given, we can also restrict ourselves to functions u such that $0 \leq u \leq 1$. We will stick with the definition given, but in some results, we will use this equivalent formulation.
We are now going to present a list of results about capacity, and we refer to [38, 39] for a more delicate treatment.

Theorem 3.1.2. *The function $A \mapsto \mathrm{Cap}_p(A;\Omega)$ has the following properties:*

1. $\mathrm{Cap}_p(\varnothing;\Omega) = 0$;
2. *if $A_1 \subset A_2 \subset \Omega$, then $\mathrm{Cap}_p(A_1;\Omega) \leq \mathrm{Cap}_p(A_2;\Omega)$;*
3. *if $A \subset \Omega_1 \subset \Omega_2$, then $\mathrm{Cap}_p(A;\Omega_2) \leq \mathrm{Cap}_p(A;\Omega_1)$;*
4. *if K_1, K_2 are two compact subset of Ω, then*

$$\mathrm{Cap}_p(K_1 \cup K_2;\Omega) + \mathrm{Cap}_p(K_1 \cap K_2;\Omega) \leq \mathrm{Cap}_p(K_1;\Omega) + \mathrm{Cap}_p(K_2;\Omega),$$

   this is sometimes called Choquet*'s property;*
5. *if K_i are decreasing compacts such that $K = \bigcap K_i$, then $\mathrm{Cap}_p(K;\Omega) = \lim_{i\to\infty} \mathrm{Cap}_p(K_i;\Omega)$;*

6. *if A_i are increasing sets such that $A = \bigcup A_i$, then* $\operatorname{Cap}_p(A;\Omega) = \lim_{i\to\infty} \operatorname{Cap}_p(A_i;\Omega)$*;*

7. *if A_i are arbitrary sets such that $A = \bigcup A_i$, then* $\operatorname{Cap}_p(A;\Omega) \le \sum \operatorname{Cap}_p(A_i;\Omega)$.

Proof. 1. Clearly if $A = \varnothing$, then we can take $u = 0$ as a minimizer.

2. If $A_1 \subset A_2 \subset \Omega$, then $\mathcal{W}(A_2;\Omega) \subset \mathcal{W}(A_1;\Omega)$, hence the result.

3. Analogously, if $A \subset \Omega_1 \subset \Omega_2$, then $\mathcal{W}(A;\Omega_1) \subset \mathcal{W}(A;\Omega_2)$. Thus

$$\inf_{u\in\mathcal{W}(A;\Omega_2)} \int_{\Omega_2} |\nabla u|^p \,\mathrm{d}x \le \inf_{u\in\mathcal{W}(A;\Omega_1)} \int_{\Omega_2} |\nabla u|^p \,\mathrm{d}x = \inf_{u\in\mathcal{W}(A;\Omega_1)} \int_{\Omega_1} |\nabla u|^p \,\mathrm{d}x,$$

since if $u \in \mathcal{W}(A;\Omega_1)$, then it is extended to Ω_1 by setting it to 0.

4. Since first-order Sobolev spaces are closed under absolute values, if $u_1 \in \mathcal{W}(K_1;\Omega)$ and $u_2 \in \mathcal{W}(K_2;\Omega)$, then

$$\int_\Omega |\nabla \max(u_1,u_2)|^p \,\mathrm{d}x + \int_\Omega |\nabla \min(u_1,u_2)|^p \,\mathrm{d}x = \int_\Omega |\nabla u_1|^p \,\mathrm{d}x + \int_\Omega |\nabla u_2|^p \,\mathrm{d}x.$$

From the properties of the admissible sets we can easily deduce that:

$$\max(u_1,u_2) \in \mathcal{W}(K_1\cup K_2;\Omega), \qquad \min(u_1,u_2) \in \mathcal{W}(K_1\cap K_2;\Omega).$$

From these last two conclusions, it follows that:

$$\operatorname{Cap}_p(K_1\cup K_2;\Omega) + \operatorname{Cap}_p(K_1\cap K_2;\Omega) \le \int_\Omega |\nabla u_1|^p \,\mathrm{d}x + \int_\Omega |\nabla u_2|^p \,\mathrm{d}x.$$

Taking the infimum over all $u_1 \in \mathcal{W}(K_1;\Omega)$ and $u_2 \in \mathcal{W}(K_2;\Omega)$, we get the result.

5. Since $K_i \supset K$ for every i, we get that:

$$b = \lim_{i\to\infty} \operatorname{Cap}_p(K_i;\Omega) \ge \operatorname{Cap}_p(K;\Omega).$$

Due to the properties of the inf, $\forall \varepsilon > 0$ there exists $u \in \mathcal{W}(K;\Omega)$ such that

$$\int_\Omega |\nabla u|^p \,\mathrm{d}x < \operatorname{Cap}^p(K;\Omega) + \varepsilon.$$

For i large enough, the set K_i is sufficiently close to K, such that $K_i \subset \{u \ge 1-\varepsilon\}$. Then we can say

$$b = \lim_{i\to\infty} \operatorname{Cap}_p(K_i;\Omega) \le \operatorname{Cap}_p(\{u\ge 1-\varepsilon\};\Omega) \le (1-\varepsilon)^{-p} \int_\Omega |\nabla u|^p \,\mathrm{d}x.$$

Letting $\varepsilon \to 0$ yields the result.

For the last two points, we will need this lemma.

Lemma 3.1.3. *Suppose $E_1, \dots, E_k \subset \Omega$, and $(F_i)_{i=1}^{i=k}$ such that $F_i \subset E_i$ for every i, and $\mathrm{Cap}_p(\bigcup_{i=1}^k F_i; \Omega) < \infty$. Then*

$$\mathrm{Cap}_p\left(\bigcup_{i=1}^k E_i; \Omega\right) - \mathrm{Cap}_p\left(\bigcup_{i=1}^k F_i; \Omega\right) \leq \sum_{i=1}^k \Big(\mathrm{Cap}_p(E_i; \Omega) - \mathrm{Cap}_p(F_i; \Omega)\Big).$$

Proof. The key point here is that, for three compact subsets C, K, F of Ω such that $C \subset K$, it follows from previous results that

$$\begin{aligned}\mathrm{Cap}_p(K \cup F; \Omega) + \mathrm{Cap}_p(C; \Omega) &\leq \mathrm{Cap}_p(K \cup (C \cup F); \Omega) + \mathrm{Cap}_p(K \cap (C \cup F); \Omega) \\ &\leq \mathrm{Cap}_p(K; \Omega) + \mathrm{Cap}_p(C \cup F; \Omega).\end{aligned}$$

Rearranging we get:

$$\mathrm{Cap}_p(K \cup F; \Omega) - \mathrm{Cap}_p(C \cup F; \Omega) \leq \mathrm{Cap}_p(K; \Omega) - \mathrm{Cap}_p(C; \Omega).$$

The lemma then follows by iterating this procedure in the case when $K_i = E_i$ and $F_i = C_i$ are compact, and then it is easily extended to the case of open sets and then arbitrary sets. □

6. Thanks to the already proven monotonicity of capacity, we only need to prove the opposite inequality:

$$\mathrm{Cap}_p(A; \Omega) \leq \lim_{i \to \infty} \mathrm{Cap}_p(A_i; \Omega).$$

Let us fix $\varepsilon > 0$ and choose open sets U_i such that $A_i \subset U_i \subset \Omega$ and

$$\mathrm{Cap}_p(U_i; \Omega) \leq \mathrm{Cap}_p(A_i; \Omega) + \varepsilon 2^{-i}.$$

Due to the increasing nature of the sets, $\mathrm{Cap}_p(\bigcup_{i=1}^k A_i; \Omega) = \mathrm{Cap}_p(A_k; \Omega) < \infty$, and from the lemma it follows that:

$$\begin{aligned}\mathrm{Cap}_p\left(\bigcup_{i=1}^k U_i; \Omega\right) - \mathrm{Cap}_p\left(\bigcup_{i=1}^k A_i; \Omega\right) &\leq \sum_{i=1}^k \Big(\mathrm{Cap}_p(U_i; \Omega) - \mathrm{Cap}_p(A_i; \Omega)\Big) \\ &\leq \sum_{i=1}^k \varepsilon 2^{-i} \\ &< \varepsilon.\end{aligned}$$

From this it follows that if $K \subset \bigcup_{i=1}^\infty U_i$ is a compact, then $K \subset \bigcup_{i=1}^k U_i$, and:

$$\mathrm{Cap}_p\left(K; \Omega\right) \leq \mathrm{Cap}_p\left(\bigcup_{i=1}^k U_i; \Omega\right) \leq \mathrm{Cap}_p\left(\bigcup_{i=1}^k A_i; \Omega\right) + \varepsilon \leq \lim_{k \to \infty} \mathrm{Cap}_p\left(A_k; \Omega\right) + \varepsilon.$$

From this we can conclude:

$$\begin{aligned}\mathrm{Cap}_p\left(A; \Omega\right) &\leq \mathrm{Cap}_p\left(\bigcup_{i=1}^\infty U_i; \Omega\right) \\ &= \sup_{K \subset \bigcup_{i=1}^\infty U_i} \mathrm{Cap}_p\left(K; \Omega\right) \\ &\leq \lim_{k \to \infty} \mathrm{Cap}_p\left(A_k; \Omega\right) + \varepsilon.\end{aligned}$$

The inequality is then obtained by taking the limit of $\varepsilon \to 0$.

7. This is a consequence of the previous point. Let us observe that, by taking $F_i = \varnothing$, the lemma implies
$$\mathrm{Cap}_p\left(\bigcup_{i=1}^{k} A_i; \Omega\right) \leq \sum_{i=1}^{k} \mathrm{Cap}_p(A_i; \Omega).$$
The union is increasing, meaning that we can use the previous point to conclude, due to the capacity being a non-negative quantity. □

Note that properties 1), 2), and 7) are used to describe an outer measure, so it follows that *capacitable* set are defined with the usual Carathéodory construction. Moreover, if a set function satisfies properties 2), 5) and 6), it describes what is called a *Choquet* or *generalized capacity* relative to Ω. It can be proved that for this kind of generalized capacity, all Borel subsets A of Ω are capacitable.
This is sometimes called *variational capacity* due to its definition as an infimum, and in certain cases, it reminds us of the capacity of a capacitor with the walls being the boundaries of A and Ω. Besides, it was introduced as a means to study the properties of solutions of elliptic equations in non-linear potential theory. From a heuristical point of view, based on the definition, it seems likely an appropriate minimizer of that integral would have to be exactly equal to 1 on an open subset of the interior of A, and exactly equal to 0 on an open subset of $\Omega \setminus A$, in order to minimize the norm of the gradient. In fact, on a compact set, we can conclude

Proposition 3.1.4. *If $K \subset \Omega$ is a compact, then* $\mathrm{Cap}_p(K; \Omega) = \mathrm{Cap}_p(\partial K; \Omega)$.

The importance of this notion of measure comes from the fact that it is both used to define *invisible* (or *removable*) sets, such that a Sobolev space is isomorphic to the Sobolev space without the set, and to define fine properties of Sobolev functions and the concept of *quasi-everywhere* and *quasi-continuity*.
We define a set $E \subset \mathbb{R}^N$ to be of zero *p-capacity* ($\mathrm{Cap}_p(E) = 0$) if
$$\mathrm{Cap}_p(E \cap \Omega; \Omega) = 0 \quad \forall \Omega \subset \mathbb{R}^N,$$
and we say that a certain property holds *quasi-everywhere* if it holds except for a set of p-capacity 0. As a connection with the Lebesgue measure, we have:

Proposition 3.1.5. *If a set E is of zero p-capacity, then it is also of zero Lebesgue measure.*

Proof. Let us fix a small enough $\varepsilon > 0$. Since $\mathrm{Cap}_p(E \cap \Omega; \Omega) = 0 \quad \forall \Omega \subset \mathbb{R}^N$, we can choose a bounded open Ω such that there exists an open neighborhood $U \subset \Omega$ of $E \cap \Omega$, such that $\mathrm{Cap}_p(U; \Omega) < \varepsilon$. We can now choose a compact $K \subset U$, and a function $\varphi \in \mathcal{W}(A; \Omega)$ such that:
$$\int_\Omega |\nabla \varphi|^p \, \mathrm{d}x \leq \mathrm{Cap}_p(K; \Omega) + \varepsilon \leq 2\varepsilon.$$
Since Ω is bounded, and φ has zero trace at the boundary, we can use Poincare's inequality to deduce that:
$$|K| = \int_K \mathrm{d}x \leq \int_K |\varphi|^p \, \mathrm{d}x \leq \int_\Omega |\varphi|^p \, \mathrm{d}x \leq C \int_\Omega |\nabla \varphi|^p \, \mathrm{d}x \leq 2C\varepsilon.$$

We can now conclude thanks to the inner regularity of the Lebesgue measure, letting then $\varepsilon \to 0$. □

In general, the converse is not true, and this result only shows that the capacity is finer than the Lebesgue measure. Interestingly, there is a simple counter-example in the case $p > N$, because it turns out that every singleton has non-zero capacity. To prove this, we need Morrey's inequality, whose proof can be found in [38].

Lemma 3.1.6 (Morrey's inequality)**.** *Let Ω be a bounded connected open set with Lipschitz boundary, and let $v \in W^{1,p}(\Omega)$ with $p > N$ and $p < \infty$. Then there exists a constant $C = C(N, p, \Omega) > 0$ such that*

$$|v(z) - v(y)| \leq C|z - y|^{1-\frac{N}{p}} \|\nabla v\|_{L^p(\Omega)} \tag{3.6}$$

for almost every z and y in Ω.

Proposition 3.1.7. *If $p > N$, then $\mathrm{Cap}_p(\{x\}, \Omega) > 0$ for every $x \in \Omega$.*

Proof. Take $x \in \Omega$ and a function $v \in \mathcal{W}(\{x\}, \Omega)$. Then there exists a neighborhood of x where the $v \geq 1$. Without loss of generality, we can say that $v(y) \geq 1$ for every $y \in B_r(x)$, for a certain small enough r, such that

$$r < \frac{1}{5}\mathrm{dist}(\{x\}, \partial\Omega).$$

Let's take a truncation function $\eta \in C_0^\infty(B_{2r}(x))$, such that $0 \leq \eta \leq 1$, and $\eta = 1$ in $B_r(x)$, with a bounded gradient $|\nabla v| \leq \frac{2}{r}$.
By Lemma 3.1.6 applied to $\eta v \in W_0^{1,p}(\Omega)$, there exists a constant $C = C(N, p, \Omega) > 0$ such that

$$|(\eta v)(y) - (\eta v)(z)| \leq C|y - z|^{1-\frac{N}{p}} \|\nabla(\eta v)\|_{L^p(\Omega)}$$

for almost every $y, z \in \Omega$. Let us observe that $\|\nabla(\eta v)\|_{L^p(\Omega)} = \|\nabla(\eta v)\|_{L^p(B_{2r}(x))}$, since $\eta \equiv 0$ outside $B_{2r}(x)$. Let us now choose $y \in B_r(x)$ and $z \in B_{4r}(x) \setminus B_{2r}(x)$, such that

$$(\eta v)(y) \geq 1, \qquad (\eta v)(z) = 0, \qquad |y - z| \leq 5r.$$

Then we can write

$$\begin{aligned}
\int_{B_{2r}(x)} |\nabla(\eta v)|^p \,\mathrm{d}y &= \|\nabla(\eta v)\|^p_{L^p(\Omega)} \\
&\geq \frac{1}{C^p}|y - z|^{N-p} \left|(\eta v)(y) - (\eta v)(z)\right|^p \\
&\geq \frac{1}{C^p} 5^{N-p}\, r^{N-p}
\end{aligned}$$

We can also derive another bound on the integral, assuming $r < 1$:

$$\int_{B_{2r}(x)} |\nabla(\eta v)|^p \,\mathrm{d}y \leq 2^p \left(\int_{B_{2r}(x)} |v\nabla\eta|^p \,\mathrm{d}y + \int_{B_{2r}(x)} |\eta\nabla v|^p \,\mathrm{d}y \right)$$

$$
\begin{aligned}
&= 2^p \left(\int_{B_{2r}(x)} |v|^p \, |\nabla \eta|^p \, \mathrm{d}y + \int_{B_{2r}(x)} |\eta|^p \, |\nabla v|^p \, \mathrm{d}y \right) \\
&\leq 2^p \left(\frac{2^p}{r^p} \int_{B_{2r}(x)} |v|^p \, \mathrm{d}y + \int_{B_{2r}(x)} |\nabla v|^p \, \mathrm{d}y \right) \\
&\leq 2^p \left(\frac{2^p}{r^p} \int_{B_{2r}(x)} |v|^p \, \mathrm{d}y + 2^p \int_{B_{2r}(x)} |\nabla v|^p \, \mathrm{d}y \right) \\
&= \frac{4^p}{r^p} \left(\int_{B_{2r}(x)} |v|^p \, \mathrm{d}y + r^p \int_{B_{2r}(x)} |\nabla v|^p \, \mathrm{d}y \right) \\
&\leq \frac{4^p}{r^p} \left(\int_{B_{2r}(x)} |v|^p \, \mathrm{d}y + \int_{B_{2r}(x)} |\nabla v|^p \, \mathrm{d}y \right) \\
&\leq \frac{4^p}{r^p} \left(\int_{\Omega} |v|^p \, \mathrm{d}y + \int_{\Omega} |\nabla v|^p \, \mathrm{d}y \right) =: \frac{4^p}{r^p} \|v\|^p_{W^{1,p}(\Omega)}.
\end{aligned}
$$

These two together show that

$$
\|u\|^p_{W^{1,p}(\Omega)} \geq \frac{r^p}{4^p} \int_{B_{2r}(x)} |\nabla(\eta v)|^p \, \mathrm{d}y \geq \frac{r^p}{4^p} \frac{1}{C^p} 5^{N-p} \, r^{N-p} =: C_0 > 0
$$

for every $u \in \mathcal{W}(\{x\}, \Omega)$, hence $\mathrm{Cap}_p(\{x\}, \Omega) > 0$. □

What this tells us is that this concept of capacity is only useful when $p \leq N$, as is our case of study.
As for the *quasi-continuity*, intuitively it may be continuity except for a set of small capacity.

Definition 3.1.8. *We define a function* $u \colon A \to \overline{\mathbb{R}}$ *to be* quasi-continuous *if for every* $\varepsilon > 0$ *there exists an open set* $U_\varepsilon \subset A$ *such that* $\mathrm{Cap}_p(U_\varepsilon; \Omega) < \varepsilon$*, and* u *restricted to* $A \setminus U_\varepsilon$ *is continuous.*

This concept of *quasi-continuity* is especially useful with a new set of functions, more complicated than the usual Sobolev space. It is somewhat linked to our model since this space requires the gradient to be p-integrable, and the function to be locally p-integrable, since in the formula (2.2) the displacement at the boundary can theoretically be unbounded, but their gradient must be square-integrable. We will define this space by following the original treatment by Deny and Lions [6].
Let $\Omega \in \mathbb{R}^N$ be a connected open set, and let $\mathcal{D}_\Omega$ be the set of complex-valued compactly-supported test functions in Ω, along with the usual topology. As usual, its dual $\mathcal{D}'_\Omega$ will be the set of distributions on Ω, endowed with the dual topology. Let us now choose E as a topological vector space, locally convex, separable and complete, contained in $\mathcal{D}'_\Omega$ and endowed with a finer topology.

Definition 3.1.9. *We define a Beppo Levi space on* E*, denoted as* $BL(E)$*, to be the subspace of distributions* $T \in \mathcal{D}'_\Omega$ *such that* $\frac{\partial}{\partial x_i} T \in E \quad \forall i$*, endowed with the least fine topology such that the applications* $T \mapsto \frac{\partial}{\partial x_i} T$ *are continuous from* $BL(E)$ *to* E*.*

Upon further inspection, if $T \in BL(E)$, then also $T + c \in BL(E) \quad \forall c \in \mathbb{R}$, meaning that the application $T \mapsto \frac{\partial}{\partial x_i} T$ is continuous but not injective. If we could make it so that this application is injective, then since E is separable, also its image space would be as well. To do this, by a standard argument, we define the quotient of $BL(E)$ with respect to the kernel of the application, which in this case is simply the space of constants $\mathbb{R}$. By making this, we know that the quotient space is separable, hence:

Definition 3.1.10. *We define* $BL^\circ(E) = {}^{BL(E)}\!/_{\mathbb{R}}$ *as the separable topological vector space defined by this quotient.*

Proposition 3.1.11. *The space* $BL^\circ(E)$ *is complete.*

The most famous example of these spaces is when $E = L^p(\Omega)$, in which case it is endowed with the norm on the gradient:

$$\|\nabla T\|^p_{L^p(\Omega;\mathbb{R}^N)} = \sum_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} T \right\|^p_{L^p(\Omega)}.$$

In general $BL^\circ(L^p)$ is a Banach space, but when $p = 2$ it is a Hilbert space. In many applications, it is better to consider the T to be in a p-integrable type subspace of the distributions, since the original space may contain too many functions. In general:

Definition 3.1.12. *We define the Deny-Lions space*

$$L^{1,p}(\Omega) \coloneqq \left\{ v \in L^p_{\mathrm{loc}}(\Omega) : \nabla v \in L^p(\Omega;\mathbb{R}^N) \right\},$$

to be the space of locally p-integrable distributions with a p-integrable gradient, equipped with the norm $\|\nabla v\|_{L^p(\Omega;\mathbb{R}^N)}$.

Theorem 3.1.13. *The space* $L^{1,p}(\Omega)$ *is complete. In particular this implies that the set* $\left\{ \nabla v : v \in L^{1,p}(\Omega) \right\}$ *is closed in* $L^p(\Omega;\mathbb{R}^N)$.

There is an interesting connection between the space $L^{1,2}(\Omega)$ and the classic Sobolev space $H^1(\Omega)$, in the presence of a Lipschitz regularity for the boundary.

Proposition 3.1.14. *Let* u *be a function in* $L^{1,2}(\Omega)$*, and* x *be a point on the Lipschitz part of the boundary* $\partial_L\Omega$*. Then there exists a neighborhood* $I(x)$ *of* x*, such that* u *restricted to* $\Omega \cap I(x)$ *is in the Sobolev space,* $u|_{\Omega\cap I(x)} \in H^1(\Omega \cap I(x))$*. In particular, if the set* Ω *is bounded and with a Lipschitz boundary, we can take the neighborhood to be the whole* Ω*, from which follows that* $L^{1,2}(\Omega) = H^1(\Omega)$.

This is useful since in general, it is not clear how one can define traces at the boundary of a function in $L^{1,2}(\Omega)$, and in some cases, it is not even possible. This is due to the fact that functions in $L^{1,2}(\Omega)$ are only in $L^2_{\mathrm{loc}}(\Omega)$, which means that if $\Omega' \Subset \Omega$, then $u \in L^2(\Omega')$. On the other hand, this is not a problem in $H^1(\Omega)$, as the theory of traces is well-defined. This helps as 3.1.14 states that if $u \in L^{1,2}(\Omega)$ and $\partial_L\Omega \neq \varnothing$, then u is locally a function on $H^1(\Omega \cup \Gamma)$ with $\Gamma \subset \partial_L\Omega$. Then u has a well-defined trace everywhere on the relative interior of $\partial_L\Omega$.

Moreover, if Ω is connected, we can take Γ as before, and define the Deny-Lions space of functions vanishing *quasi-everywhere* on Γ. From now on we will identify *quasi-everywhere* with *q.e.*, hence:

$$L^{1,p}_{0,\Gamma}(\Omega) \coloneqq \{u \in L^{1,p}(\Omega) : u = 0 \text{ q.e. on } \Gamma\}.$$

In the case when $\Gamma = \partial_L\Omega = \partial\Omega$, we will simply write $L^{1,p}_0(\Omega)$.
From now on we are going to focus on the case $p = 2$. As before, we indicate with $(L^{1,2})^\circ(\Omega)$ the separable space that is the quotient between $L^{1,2}(\Omega)$ and the space of constants $\mathbb{R}$. For ease of notation, we will indicate $(L^{1,2})^\circ(\Omega)$ also with $L^{1,2}(\Omega)$.
One would question whether or not the space $L^{1,2}_{0,\Gamma}(\Omega)$ has some nice properties, first of all being Hilbertian. We have the following result, deriving from Theorem 3.1.13 and Proposition 3.1.14:

Corollary 3.1.15. *With the same notation as before, we have that the space $L^{1,2}_{0,\Gamma}(\Omega)$ is a Hilbert space endowed with the norm of the gradient $\|\nabla u\|_{L^2(\Omega;\mathbb{R}^N)}$. In addition, a bounded sequence admits a sub-sequence whose gradients converge weakly in $L^2(\Omega;\mathbb{R}^N)$.*

Proof. Since $\Gamma \subset \partial_L\Omega$, we iteratively construct a sequence of connected open sets (Ω_k) in such a way: Ω_1 is so that $\Omega_1 \subset \Omega$ and $(\partial\Omega_1 \cap \partial\Omega) \subset \Gamma$; then Ω_2 is such that $\Omega_1 \subset \Omega_2 \subset \Omega$ and $(\partial\Omega_1 \cap \partial\Omega) \subset (\partial\Omega_2 \cap \partial\Omega) \subset \Gamma$. Thus we can easily construct an increasing sequence such that they all have a Lipschitz boundary since $\partial\Omega_k \cap \partial\Omega \subset \Gamma \subset \partial_L\Omega$, and also $\Omega = \bigcup \Omega_k$ and $\Gamma = \bigcup(\partial\Omega_k \cap \partial\Omega)$.
Now let (v_n) be a Cauchy sequence in $L^{1,2}_{0,\Gamma}(\Omega)$. Thanks to Proposition 3.1.14 we have $L^{1,2}(\Omega_k) = H^1(\Omega_k)$ so the functions (v_n) restricted to Ω_k belong to $H^1(\Omega_k)$ and $v_n = 0$ q.e on $\partial\Omega_k \cap \partial\Omega$. By using a generalized Poincaré inequality for $L^{1,2}(\Omega_k)$ to control the norm, we can conclude that (v_n) is also a Cauchy sequence in $H^1(\Omega_k)$ which we know to be complete, hence $v_n \to v \in H^1(\Omega_k)$ strongly such that $v = 0$ q.e. on $\partial\Omega_k \cap \partial\Omega$. This is valid for every k, so we can take the union of the sets and construct a function v such that $v = 0$ q.e. on Γ which is in $H^1(\Omega) = L^{1,2}(\Omega)$, with $v_n \to v \in H^1(\Omega_k)\ \forall k$. Then also ∇v_n converges strongly in $L^2(\Omega_k;\mathbb{R}^N)$ due to the completeness of the space, and in particular it converges also in $L^2(\Omega;\mathbb{R}^N)$, hence $v_n \to v$ strongly in $L^{1,2}_{0,\Gamma}(\Omega)$. This means that every Cauchy sequence has a strong limit, so the space is complete.
For the second part, let us take (u_n) to be a bounded sequence in $L^{1,2}_{0,\Gamma}(\Omega)$. For what we have discussed in the first part, (u_n) is also a bounded sequence in $H^1(\Omega_k)\ \forall k$: so there exists a weakly convergent sub-sequence converging to a function $u \in L^{1,2}(\Omega)$. By Mazur's lemma there exists a convex combination of the functions u_n, converging to u strongly in $H^1(\Omega_k)$, and this, in turn, implies $u = 0$ q.e on $\partial\Omega_k \cap \partial\Omega$, hence $u \in L^{1,2}_{0,\Gamma}(\Omega)$. To conclude, we note that also (∇u_n) is bounded in $L^2(\Omega;\mathbb{R}^N)$, so $\nabla u_n \to \nabla u$ weakly in the same space. □

For more results on these kinds of spaces we refer the reader to the much more exhaustive treatment in [37].

The proof of this next result is quite long and requires some theory on maximal operators in Sobolev spaces: being outside the scope of this thesis, we will only state it

without proof. Besides, its importance in this treatment is confined to being able to use a quasi-continuous representative for function in a Deny-Lions or Sobolev space.

Theorem 3.1.16. *Let $u \in L^{1,p}(\Omega)$, $1 < p < \infty$, with $\Omega \subset \mathbb{R}^N$. Then outside of a set $E : \mathrm{Cap}_p(E) = 0$ there is a p-quasi-continuous representative $\tilde{u}$ defined* p-quasi-everywhere *on $\Omega \cup \partial_L\Omega$, defined by:*

$$\tilde{u}(x) \coloneqq \lim_{r\to 0^+} \fint_{B_r(x)\cap\Omega} u(y)\,\mathrm{d}y \qquad \forall x \in (\Omega \cup \partial_L\Omega) \setminus E.$$

Moreover the quasi-continuous representative $\tilde{u}$ satisfies:

$$\lim_{r\to 0^+} \fint_{B_r(x)\cap\Omega} |u(y) - \tilde{u}(x)|\,\mathrm{d}y = 0 \qquad \forall x \in (\Omega \cup \partial_L\Omega) \setminus E.$$

The extension of $\tilde{u}$ on the Lipschitz part of the boundary is a consequence of the well-known extension operators for Lipschitz domains in Sobolev spaces, and for Deny-Lions spaces it is a direct consequence of the equivalence with H^1 near $\partial_L\Omega$. From now on, we will always identify each function in $L^{1,2}$ with its quasi-continuous representative.
A nice link between functions defined *quasi-everywhere* and the usual Sobolev space is the next result, which links functions with zero trace on the boundary with a quasi-continuous function.

Theorem 3.1.17. *Let $u \in W^{1,p}(\Omega)$. Then $u \in W_0^{1,p}(\Omega)$ if and only if there exists a p-quasi-continuous function v in $\mathbb{R}^N$ such that $v = u$ a.e. in Ω and $v = 0$ q.e. on $\partial\Omega$.*

Note the difference between the *almost everywhere* inside the domain, and the *quasi everywhere* on the boundary.
As briefly introduced before, another important application of capacity is to understand when a set in *invisible* (or *removable*) for a Sobolev space, and we will see this will be useful in the existence results for discarding sets of Hausdorff dimension strictly lower than $N-1$. As a notation, if X and Y are two normed spaces, we will write $X = Y$ when there exists an isometric isomorphism between them.

Theorem 3.1.18. *Let E be a relatively closed subset of Ω. Then*

- *E has zero p-capacity if and only if $L_0^{1,p}(\Omega) = L_0^{1,p}(\Omega \setminus E)$;*
- *if E has zero p-capacity then $L^{1,p}(\Omega) = L^{1,p}(\Omega \setminus E)$.*

Proof. We will split the proof into two points.

- Suppose $\mathrm{Cap}_p(E) = 0$. Then it is also of zero Lebesgue measure. Now, since the inclusion $L_0^{1,p}(\Omega \setminus E) \subseteq L_0^{1,p}(\Omega)$ is trivial, we focus on the opposite inclusion. Let $u_j \in L_0^{1,p}(\Omega)$ with $0 \le u_j \le 1$, be a sequence such that $u_j = 1$ in a neighborhood of E, and such that $u_j \to 0$ in $L^{1,p}(\Omega)$. Let now $\varphi \in C_0^\infty(\Omega)$, then $(1-u_j)\varphi \in L_0^{1,p}(\Omega)$, but since it has compact support in $\Omega \setminus E$, then it also is in $L_0^{1,p}(\Omega \setminus E)$. Moreover, it converges to φ in $L^{1,p}(\Omega \setminus E)$, such that $\varphi \in L_0^{1,p}(\Omega \setminus E)$, from which follows the reverse inclusion.

Suppose now that $L_0^{1,p}(\Omega \setminus E) = L_0^{1,2}(\Omega)$. Let $K \subset E$ be a compact set. If we are able to prove that K is of p-capacity zero, then we are done, due to the outer continuity of capacity. Let us pick $\varphi \in C_0^\infty(\Omega)$ such that $\varphi = 1$ in a neighborhood of K. Due to the equivalence of spaces, we can take a sequence $\varphi_j \in C_0^\infty(\Omega \setminus E)$ such that $\varphi_j \to \varphi$ in $L^{1,p}(\Omega)$, and $\phi - \phi_j = 1$ in a neighborhood of K, since the support of the φ_j does not contain K. This means that $\varphi_j - \varphi \in L_0^{1,p}(\Omega)$, which implies

$$\mathrm{Cap}_p(E;\Omega) \leq \lim_{j\to\infty} \int_\Omega |\nabla\varphi - \nabla\varphi_j|^p \,\mathrm{d}x = 0.$$

The conclusion follows.

- As before, the inclusion $L^{1,p}(\Omega\setminus E) \subseteq L^{1,p}(\Omega)$ is trivial since E has also zero Lebesgue measure. For the other inclusion, let us take $u \in L^{1,p}(\Omega \setminus E)$: we need to show that u can be approximated in $L^{1,p}(\Omega \setminus E)$ by a sequence in $L^{1,p}(\Omega)$. Let us then take a sequence $(v_j) \subset L^{1,p}(\Omega)$, such that $0 \leq v_j \leq 1$, $v_j = 1$ in an open neighborhood U_j of E, and such that $v_j \to 0$ in $L^{1,p}(\Omega)$. Then it follows that the functions $u_j \coloneqq (1-v_j)u$ are in $L^{1,p}(\Omega \setminus E)$ since it is basically a truncation of u to $\Omega \setminus U_j \subset \Omega \setminus E$, and in $L^{1,p}(U_j)$ since $u_j = 0$ q.e. in U_j up to a set of capacity zero (E), and thus $u_j \in L^{1,p}(\Omega \setminus E) \cap L^{1,p}(U_j) = L^{1,p}\big((\Omega \setminus E) \cup U_j\big) = L^{1,p}(\Omega)$. Since $u_j \to u$ in $L^{1,p}(\Omega \setminus E)$ because U_j gets closer and closer to E, we can then conclude, having found the sequence. □

If Ω has a Lipschitz boundary, then this last result can be used to prove that the Deny-Lions space $L_0^{1,2}(\Omega)$ is isometrically isomorphic to $L^{1,2}(\Omega)$, only if the boundary is of zero p-capacity. Keep in mind that the assumption of the Lipschitz boundary is needed, because otherwise, as already discussed, we would not be able to define traces at the boundary.

Theorem 3.1.19. *$L^{1,p}(\Omega) = L_0^{1,p}(\Omega)$ if and only if $\partial\Omega$ has zero p-capacity.*

Proof. Suppose $\partial\Omega$ is of zero p-capacity. The inclusion $L_0^{1,p}(\Omega) \subseteq L^{1,p}(\Omega)$ is trivial. For the other inclusion, we can use the previous theorem to deduce that:

$$\begin{aligned} L^{1,p}(\Omega) &\subseteq L^{1,p}(\mathbb{R}^N \setminus \partial\Omega) = L^{1,p}(\mathbb{R}^N) \\ &= L_0^{1,p}(\mathbb{R}^N) = L_0^{1,p}(\mathbb{R}^N \setminus \partial\Omega) \subseteq L_0^{1,p}(\Omega). \end{aligned}$$

Suppose now that $L_0^{1,p}(\Omega) = L^{1,p}(\Omega)$. We can then say:

$$L_0^{1,p}(\mathbb{R}^N) = L^{1,p}(\mathbb{R}^N) \subseteq L^{1,p}(\Omega) = L_0^{1,p}(\Omega) \subseteq L_0^{1,p}(\mathbb{R}^N).$$

Thanks to this we have:

$$L_0^{1,p}(\mathbb{R}^N) = L_0^{1,p}(\Omega) \subseteq L_0^{1,p}(\mathbb{R}^N \setminus \partial\Omega).$$

Since the other inclusion $L_0^{1,p}(\mathbb{R}^N \setminus \partial\Omega) \subseteq L_0^{1,p}(\Omega)$ is trivial, we deduce that $L_0^{1,p}(\mathbb{R}^N) = L_0^{1,p}(\mathbb{R}^N \setminus \partial\Omega)$. Thanks to the previous theorem we can conclude that $\partial\Omega$ has zero p-capacity. □

As the last result of this section, we present a property of constant $L^{1,2}(\Omega)$ functions on two connected sets.

Proposition 3.1.20. *Let $u \in L^{1,2}(\Omega)$ with $\Omega \subset \mathbb{R}^2$ and C_1, C_2 two connected open subsets of $\Omega \cup \partial_L \Omega$ such that $\overline{C}_1 \cap \overline{C}_2 \neq \varnothing$. Then if u is constant q.e. on C_1 and C_2, it is also constant q.e. on $C_1 \cup C_2$.*

Proof. We assume C_1, C_2 to have more than one point each. We assume the constant values on each set to be different, such that $u = c_1$ q.e. on C_1 and $u = c_2$ q.e. on C_2 with $c_1 \neq c_2$. Since the intersection of the closure is non-empty, we can take

$$x \in (\overline{C_1} \cap \overline{C_2}) \subset \Omega \cup \partial_L \Omega,$$

and by using Proposition 3.1.14 we can assume $u \in H^1(B_r(x))$ for a certain $r > 0$, with r small enough so that $C_i \cap B_r(x) \neq \varnothing$. This implies $C_i \cap \partial B_\rho(x) \neq \varnothing \quad \forall \rho \in (0, r)$. Considering u as its quasi-continuous representative, we know that for every $\varrho \in (0, r)$, u takes two distinct values c_1 and c_2 in at least two points. Due to the *quasi-continuity*, since the minimizing function for the Dirichlet functional is linear when the boundary conditions are both Dirichlet, we can estimate the integral by constructing a linear function going around $\partial B_\varrho(x)$, attaining the values c_1 and c_2 in two points. We do not know anything about where these points will be positioned, but we can parameterize it with the angular distance θ between them. We will then obtain a function of θ, which we can further minimize, therefore obtaining:

$$\int_{\partial B_\rho(x)} |\nabla u|^2 \, \mathrm{d}\mathcal{H}^1 \geq \frac{(c_2 - c_1)^2}{\rho\,(2\pi - \theta)} + \frac{(c_2 - c_1)^2}{\rho\theta} \geq \frac{2\,(c_2 - c_1)^2}{\rho\pi}.$$

This implies that the integral over all $B_r(x)$ diverges, so $\nabla u \notin L^2(B_r(x); \mathbb{R}^2)$, hence $u \notin H^1(B_r(x))$, contradicting our assumption. In hindsight, the only way that integral could be non-divergent is if $c_1 = c_2$. □

This proposition is not straightforward due to the functions being defined *quasi-everywhere*, hence in principle, we could have a set with zero p-capacity *disconnecting* the sets, or maybe the interiors of the sets are disjoint, but the closures intersect in one point: as we have just proved, this is enough so that the function has the same constant values of C_1 and C_2.

3.2 Hausdorff measure and Hausdorff distance between compact sets

In Proposition 3.1.20 of the last section, we briefly used in the integral the so-called Hausdorff measure $\mathcal{H}^1$, which is the correct measure for dealing with integrals of surfaces embedded in a space of higher dimension. This notion of measure has actually some nice properties, and we will present a brief treatment.
Let $h(A)$ be a non-decreasing set function called *Gauge function*, such that if $A_1 \subset A_2$ then $h(A_1) \leq h(A_2)$. Moreover, $h(\varnothing) = 0$ and it is continuous in 0. Given a covering $\mathcal{A}$

of a certain set A, we define it to be a h_ε-covering if for all the sets in the covering (A_i), we have $h(A_i) \leq \varepsilon$: we will refer to these coverings with $\mathcal{A}_\varepsilon$.

Definition 3.2.1. *The set function defined by*

$$\Delta^h(E) = \sup_{\varepsilon \to 0} \Delta^h_\varepsilon(E) = \sup_{\varepsilon \to 0} \inf_{\mathcal{A}_\varepsilon} \sum_{A \in \mathcal{A}_\varepsilon} \alpha(h) h(A),$$

is an outer measure, whose restriction to all the sets E for which

$$\Delta^h(E) = \Delta^h(E \cap U) + \Delta^h(E \cap U^{\complement}) \quad \forall U \subset \mathbb{R}^N,$$

is a proper measure.

This definition is quite general, but for most of the purposes in the literature, it suffices to choose $h(A) = \big(diam(A)\big)^s$. With this choice of Gauge function, the measure thus defined will be the *s dimensional Hausdorff measure*, denoted by $\mathcal{H}^s(E)$. As we can see, the choice of s is quite important, since it is supposed to be the dimension of the set we want to measure. Moreover, one would expect this notion of measure to be equal to the Lebesgue measure when $s = N$. That is the case, though with the constant α equal to

$$\alpha(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\big(\frac{s}{2} + 1\big)}.$$

We can easily observe that, even though the Lebesgue measure is defined only for sets with integer dimension, in the definition of the Hausdorff dimension, the exponent s can take any value $s > 0$. In fact:

Proposition 3.2.2. *Let X be a metric space, and $E \subseteq X$ an arbitrary Borelian subset. It follows from the definition of the s dimensional Hausdorff measure that there exists at most one value s for which $\mathcal{H}^s(E) \in (0, \infty)$. We will call this value the* Hausdorff dimension *of the set E, defined by*

$$dim_{\mathcal{H}}(E) \coloneqq \inf\big\{s \in [0, \infty) : \mathcal{H}^s(E) = 0\big\}.$$

The importance of this measure comes from the fact that it is used to determine the fractal dimension of a set. For more results on this and Hausdorff measure in general we refer to [23, 33].
The concepts of Hausdorff measure and capacity are linked by the next proposition, which requires this next lemma to be proved. Let us keep in mind that with the classic Lebesgue measure, the measure of the ball $B(x, r)$ scales like r^N.

Lemma 3.2.3. *Let $x \in \Omega \subset \mathbb{R}^N$, and let $0 < r \leq 1$ be such that $B(x, r) \Subset \Omega$. Then there exists a constant $C = C(N, p)$ such that:*

$$\mathrm{Cap}_p(B(x, r); \Omega) \leq C r^{N-p}.$$

Proof. Let us take a small enough $\varepsilon > 0$ such that $B(x, r(1+\varepsilon)) \subset \Omega$. We next define a function $u(y)$ such that:

$$u(y) \coloneqq \begin{cases} 1 & y \in B(x,r), \\ \dfrac{1}{\varepsilon}\left(1+\varepsilon - \dfrac{\|y-x\|}{r}\right) & y \in B(x, r(1+\varepsilon)) \setminus B(x,r), \\ 0 & y \in \Omega \setminus B(x, r(1+\varepsilon)). \end{cases}$$

This function is such that $0 \leq u \leq 1$, it is $\frac{1}{\varepsilon r}$-Lipschitz and also $|\nabla u| \leq \frac{1}{\varepsilon r}$ a.e. in Ω. This means that $u \in H^{1,p}(\Omega)$. Moreover u has also zero trace at the boundary, hence $u \in H_0^{1,p}(\Omega)$, and thus $u \in \mathcal{W}(A;\Omega)$. Then

$$\begin{aligned} \operatorname{Cap}_p(B(x,r);\Omega) &\leq \int_\Omega |\nabla u(y)|^p \,\mathrm{d}y \leq \int_{B(x,r(1+\varepsilon))} |\nabla u(y)|^p \,\mathrm{d}y \\ &\leq \frac{1}{\varepsilon^p r^p} |B(x, r(1+\varepsilon))| \leq \frac{C_0}{\varepsilon^p r^p} (1+\varepsilon)^N r^N = C r^{N-p}. \end{aligned}$$

Hence the natural scaling for the p-capacity is r^{N-p}. □

Proposition 3.2.4. *Assume that $1 < p < N$. Then for any subset $E \subset \Omega$, there exists a constant $C = C(N,p)$ such that:*

$$\operatorname{Cap}_p(E;\Omega) \leq C\, \mathcal{H}^{N-p}(E).$$

Proof. Let $\mathcal{B}_\delta$ be the family of coverings of E of the type $\big(B(x_i,r_i)\big)_i$, such that $E \subset \bigcup_i B(x_i,r_i)$ and that $r_i \leq \delta$ for every i. Then the sub-additivity of the p-capacity and Lemma 3.2.3 imply:

$$\operatorname{Cap}_p(E;\Omega) \leq \sum_{i=1}^{\infty} \operatorname{Cap}_p(B(x_i,r_i);\Omega) \leq C \sum_{i=1}^{\infty} r_i^{N-p}.$$

Since this is true for all coverings, we can take the infimum across $\mathcal{B}_\delta$, obtaining precisely the definition of $\mathcal{H}_\delta^{N-p}(E)$. Then we conclude by:

$$\operatorname{Cap}_p(E;\Omega) \leq C\, \mathcal{H}_\delta^{N-p}(E) \leq C\, \mathcal{H}^{N-p}(E).$$

In particular this implies that if $\mathcal{H}^{N-p}(E) = 0$, then $\operatorname{Cap}_p(E;\Omega) = 0$. □

This result can also be extended to the case $p = N$, by using a Gauge function of the form

$$h(A) = \left(\log\left(\frac{1}{\operatorname{diam}(A)}\right)\right)^{1-N},$$

though we are not going to dwell on the much more difficult proof: we refer to the already mentioned books on the subject. Keep in mind that, as said in the previous section, the capacity only makes sense when $p \leq N$.
There is also another important concept, also studied initially by Hausdorff, which is a kind of distance between sets.

Let X be a metric space with metric $d(x, y)$, and $A, B \subseteq X$. Given $r > 0$, we define the open r-neighborhood of A as

$$N_r(A) \coloneqq \{y \in X : d(x, y) < r \text{ for some } x \in A\}.$$

In 1914 Hausdorff defined a distance between two sets as

$$\mathcal{D}(A, B) \coloneqq \inf \{r > 0 : A \subseteq N_r(B) \;\; B \subseteq N_r(A)\}.$$

This distance does not define a metric on the set of all subsets of X because, if we consider the simple case $X = \mathbb{R}$, then:

- $\mathcal{D}(\{0\}, [0, \infty)) = \infty$, so we restrict to bounded sets;
- $\mathcal{D}(\varnothing, \{0\}) = \infty$, so we restrict to non-empty sets;
- $\mathcal{D}((0,1), (0,1]) = 0$ even though the sets are not the same, so we restrict to closed sets.

Turns out this distance is actually a metric on the set $\mathcal{K}(\overline{\Omega})$ of all compact non-empty subsets of $\overline{\Omega}$, and for two compact non-empty sets $K_1, K_2 \subset \overline{\Omega}$ its formulation is equivalent to:

$$\mathcal{D}(K_1, K_2) \coloneqq \max \{ \sup_{x \in K_1} d(x, K_2), \; \sup_{y \in K_2} d(y, K_1)\},$$

with $d(x, \varnothing) = \operatorname{diam}(\Omega)$ and $\sup_{x \in \varnothing}[\ldots] = 0$.

Theorem 3.2.5. *Let $\Omega \subseteq X$ be a subset of a metric space, and let $\mathcal{K}(\overline{\Omega})$ be the collection of all compact non-empty subsets of $\overline{\Omega}$. Then if X is complete, then the Hausdorff distance $\mathcal{D}$ is a metric which renders $\mathcal{K}(\overline{\Omega})$ also complete. Moreover, if X is also compact, then $\mathcal{K}(\overline{\Omega})$ will also be compact, meaning that every sequence in $\mathcal{K}(\overline{\Omega})$ has a subsequence converging to a set $K \in \mathcal{K}(\overline{\Omega})$ in the Hausdorff metric.*

The Hausdorff metric and Hausdorff measure are related through various results, first of all, this lower semi-continuity result for connected sets. For this, we have to introduce a new space of compact sets, and some more that will be useful later on.
We define $\mathcal{K}^f(\overline{\Omega}) \coloneqq \{K \in \mathcal{K}(\overline{\Omega}) : \mathcal{H}^1(K) < \infty\}$ to be the set of compacts with Hausdorff dimension 1. We then define, given an integer $m \geq 1$, $\mathcal{K}_m(\overline{\Omega})$ to be the set of $K \in \mathcal{K}(\overline{\Omega})$ with at most m connected components. Consequently, we define $\mathcal{K}_m^f(\overline{\Omega}) \coloneqq \{K \in \mathcal{K}_m(\overline{\Omega}) : \mathcal{H}^1(K) < \infty\}$, the collection of compact sets with finite length and limited connected components. Also the set $\mathcal{K}_m^\lambda(\overline{\Omega}) \coloneqq \{K \in \mathcal{K}_m(\overline{\Omega}) : \mathcal{H}^1(K) < \lambda\}$ of compacts with limited length and limited connected components will be used.

Theorem 3.2.6. *Let (K_n) be a sequence in $\mathcal{K}_1(\overline{\Omega})$ converging to a set K in the Hausdorff metric. Then $K \in \mathcal{K}_1(\overline{\Omega})$ and $\forall U \subset \mathbb{R}^2$ open:*

$$\mathcal{H}^1(K \cap U) \leq \liminf_{n \to \infty} \mathcal{H}^1(K_n \cap U).$$

Remark 3.2.7. *If the sets are not connected, one can exhibit a counterexample. In* $\mathbb{R}$ *we can take:*

$$K_n = \bigcup_{i=0}^{2^n-1} \left[\frac{i}{2^n}, \frac{i+0.5}{2^n}\right].$$

Then $\mathcal{H}^1(K_n) = 0.5$*, but one can check that* $K_n \to [0,1]$ *in the Hausdorff metric. The problem here lies not in the set being disconnected, but in the fact that the number of connected components of the set is unbounded.*
In reality, this result can be extended to the case of at most m *connected components, since one can easily prove that there exists a sequence of each connected component converging to the connected components of* K*.*

Corollary 3.2.8. *Let* $m \geq 1$*, and* (K_n) *a sequence in* $\mathcal{K}_m(\overline{\Omega})$*, such that* $K_n \to K$ *in the Hausdorff metric. Then* $K \in \mathcal{K}_m(\overline{\Omega})$ *and* $\forall U \subset \mathbb{R}^2$ *open:*

$$\mathcal{H}^1(K \cap U) \leq \liminf_{n\to\infty} \mathcal{H}^1(K_n \cap U).$$

Proof. Let $K_n^1, \ldots, K_n^{k_n}$, with $k_n \leq m$, be the connected components of K_n. Then there exists $k \leq m$ such that, up to a sub-sequence, $k_n = k$ for every n. By the compactness of the Hausdorff space we know that $K_n^1 \to \widehat{K}^1, \ldots, K_n^k \to \widehat{K}^k$, and $\widehat{K}^i \in \mathcal{K}_1(\overline{\Omega})$.
Then, for every $x \in K$ there exists a sequence $x_n \to x$ with $x_n \in K_n$. Then there also exists a certain index j_n such that $x_n \in K_n^{j_n}$, with $j_n \leq k$. As before, up to a sub-sequence, we can take $j_n = j$, which translates to $x \in \widehat{K}^j$. From this, it follows that

$$K \subseteq \widehat{K}^1 \cup \cdots \cup \widehat{K}^k,$$

which implies that K as at most $k \leq m$ connected components. Then, from Theorem. 3.2.6 applied to each $\widehat{K}^i$, we deduce that

$$\mathcal{H}^1(\widehat{K}^i \cap U) \leq \liminf_{n\to\infty} \mathcal{H}^1(K_n^i \cap U).$$

It now follows that

$$\begin{aligned}
\mathcal{H}^1(K \cap U) &\leq \sum_{i=1}^{k} \mathcal{H}^1(\widehat{K}^i \cap U) \\
&\leq \sum_{i=0}^{k} \liminf_{n\to\infty} \mathcal{H}^1(K_n^i \cap U) \\
&= \liminf_{n\to\infty} \mathcal{H}^1(K_n \cap U)
\end{aligned}$$

which concludes the proof. □

Corollary 3.2.9. *Let* (H_n) *be a sequence in* $\mathcal{K}(\overline{\Omega})$ *converging to* H*. Let* $m \geq 1$ *and* (K_n) *a sequence in* $\mathcal{K}_m(\overline{\Omega})$ *converging to* K*. Then*

$$\mathcal{H}^1(K \setminus H) \leq \liminf_{n\to\infty} \mathcal{H}^1(K_n \setminus H_n).$$

Proof. Let $\varepsilon > 0$, and $N_\varepsilon(H)$ be the ε-neighborhood of H. Obviously $H_n \subset N_\varepsilon(H)$ for n large enough due to the convergence in the Hausdorff metric. This implies $K_n \setminus N_\varepsilon(H) \subset K_n \setminus H_n$. By applying Theorem 3.2.6 in the extended case of $m \geq 1$, with $U = \mathbb{R}^2 \setminus N_\varepsilon(H)$, since $K \cap U = K \cap (N_\varepsilon(H))^{\complement} = K \setminus N_\varepsilon(H)$ we have:

$$\mathcal{H}^1(K \setminus N_\varepsilon(H)) \leq \liminf_{n\to\infty} \mathcal{H}^1(K_n \setminus N_\varepsilon(H)) \leq \liminf_{n\to\infty} \mathcal{H}^1(K_n \setminus H_n).$$

Letting $\varepsilon \to 0$ we get the result. □

We will now prove three lemmas, used for the proof of an approximation result used in the next chapter.

Lemma 3.2.10. *Let $H \in \mathcal{K}_1(\overline{\Omega})$ and (H_n) a sequence in $\mathcal{K}_p(\overline{\Omega})$ converging to H. Then there exists a sequence (K_n) in $\mathcal{K}_1(\overline{\Omega})$ with $K_n \to H$, $H_n \subseteq K_n$ and $\mathcal{H}^1(K_n \setminus H_n) \to 0$.*

Proof. Let $H_n^1, \dots, H_n^{k_n}$ be all the connected components of H_n. Since $H_n \in \mathcal{K}_p(\overline{\Omega})$ we have $k_n \leq p$ for every n: this means that up to a sub-sequence (without renaming) there exists a constant $k \leq p$ such that every H_n has exactly k connected components $H_n^1, \dots, H_n^k$.
Due to the compactness of the space, every connected component converges to a compact and connected set $H_n^1 \to \widehat{H}^1, \dots, H_n^k \to \widehat{H}^k$, and obviously $H = \widehat{H}^1 \cup \dots \cup \widehat{H}^k$.
As $H \in \mathcal{K}_1(\overline{\Omega})$, it will be connected hence every set in $\widehat{H}^1, \dots, \widehat{H}^k$ will have at least a non-empty intersection with at least another set in the same family. Let us then fix a point in a non-empty intersection $x^{i,j} \in \widehat{H}^i \cap \widehat{H}^j$. By the convergence stated above, there will exists points $x_n^{i,j} \in H_n^i$ and $y_n^{i,j} \in H_n^j$ such that $x_n^{i,j} \to x^{i,j}$ and $y_n^{i,j} \to x^{i,j}$.
Since we have a Lipschitz boundary, there will be Lipschitz curves (hence $\mathcal{H}^1$-measurable) $X_n^{i,j}$ and $Y_n^{i,j}$ connecting respectively $x_n^{i,j}$ and $x_n^{i,j}$ to $x^{i,j}$. Due to the convergence, obviously $\mathcal{H}^1(X_n^{i,j}) \to 0$ and $\mathcal{H}^1(Y_n^{i,j}) \to 0$. Since these curves are compacts, we can then define a compact non-empty set as

$$K_n \coloneqq H_n \cup \bigcup_{i,j} X_n^{i,j} \cup \bigcup_{i,j} Y_n^{i,j}.$$

Thanks to the sub-additivity of the $\mathcal{H}^1$ measure, it is clear that $H_n \subseteq K_n$, $K_n \to H$ in the Hausdorff metric, and $\mathcal{H}^1(K_n \setminus H_n) \to 0$. To conclude, the set K_n is also connected since the connected components $H_n^1, \dots, H_n^k$ of H_n are connected to one another through $X_n^{i,j}$ and $Y_n^{i,j}$ which start respectively in $x_n^{i,j} \in H_n^i$ and $y_n^{i,j} \in H_n^j$ and intersect at the end in $x^{i,j}$. □

Lemma 3.2.11. *Let $K \in \mathcal{K}_1^f(\overline{\Omega})$ and $H \subseteq K$ be a non-empty compact with $p \geq 2$ connected components $H^1, \dots, H^p$. Then there exists a family of connected components of $K \setminus H$, connecting all the disjoint connected components of H.*

Proof. First of all $K \setminus H \neq \varnothing$, since this would imply $p = 1$. By hypothesis, K is connected and $K \setminus H$ is open in the K-subspace topology, so every connected component C of $K \setminus H$ is open as well in the K-subspace topology. On the contrary, each C is closed in the $(K \setminus H)$-subspace topology, so by definition $C = \overline{C} \cap (K \setminus H)$. If $C = \overline{C}$ then K would

contain a set which is both open and closed in the K-subspace topology, which then is a connected component. But K is connected, hence $C \neq \overline{C}$. Since $C = \overline{C} \cap (K \setminus H)$, then $\varnothing \neq \overline{C} \setminus C \subset H$, therefore $\overline{C} \cap H \neq \varnothing$ for every connected component C of $K \setminus H$.
Given the collection of connected components of H, we define the collection of connected components of $K \setminus H$ such that $\overline{C} \cap H^j \neq \varnothing$ as $C^j \coloneqq (C^{j,1}, \dots, C^{j,\ell_j})$. Let us now define a collection of sets $(\widehat{K}^1, \dots, \widehat{K}^p)$ such that

$$\widehat{K}^j \coloneqq H^j \cup \bigcup_{i=1}^{\ell_j} C^{j,i}.$$

We now prove that these sets are open. To do this we prove that $K \setminus \widehat{K}^j$ is closed, so we aim at proving that every converging sequence in $K \setminus \widehat{K}^j$ converges also to an element in $K \setminus \widehat{K}^j$, that is to an element not in $\widehat{K}^j$. Let us then pick a sequence $(x_n) \subset K \setminus \widehat{K}^j$ converging to $x \in K$.
If $x_n \in H \setminus H^j$ for $n > n_0$, then $x \in H \setminus H^j$ due to closure, so $x \notin C^j$ and $x \notin H^j$, which implies $x \notin \widehat{K}^j$.
If there is a connected component C_0 of $K \setminus H$ such that $\overline{C}_0 \cap H^j = \varnothing$, and if $x_n \in \overline{C}_0$ for $n > n_0$, then $x \in \overline{C}_0$ due to closure, and of course $x \notin \widehat{K}^j$ since $\overline{C}_0 \cap C^{j,i} = \varnothing \ \ \forall C^{j,i} \in C^j$.
In the remaining case, there is a sequence (C_n) of connected components such that $\overline{C}_n \cap H^j = \varnothing$, and such that, up to a sub-sequence, $x_n \in \overline{C}_n$. Since K is a connected compact, it has finite $\mathcal{H}^1$ measure, hence $\mathcal{H}^1(\overline{C}_n) \to 0$. This implies $\operatorname{dist}(x_n, H \setminus H^j) \to 0$ which gives $x \in H \setminus H^j$ due to closure. This implies as well $x \notin \widehat{K}^j$. Hence $\widehat{K}^j$ is always open in K.
Since every connected component of $K \setminus H$ has non-empty intersection with H, we have $K \subseteq \widehat{K}^1 \cup \dots \cup \widehat{K}^p$ since $\widehat{K}^j$ is open for every j, from which obviously follows $K = \widehat{K}^1 \cup \dots \cup \widehat{K}^p$. Since we know K to be connected, we know that for a family of indices, $\widehat{K}^i \cap \widehat{K}^j \neq \varnothing$. Obviously $H^i \cap H^j = \varnothing$, so there will be a family of connected components of $K \setminus H$ such that $H^i \cap C_{i,j} \neq \varnothing \neq H^j \cap C_{i,j}$, thus connecting the family H^j. □

Lemma 3.2.12. *Let $p > 0$ be an integer, and let (H_n) be a sequence in $\mathcal{K}_p^f(\overline{\Omega})$ converging to $H \in \mathcal{K}_p^f(\overline{\Omega})$. Let $K \in \mathcal{K}_1^f(\overline{\Omega})$ such that $H \subset K$. Then there exists a sequence (K_n) in $\mathcal{K}_1^f(\overline{\Omega})$ such that $K_n \to K$, $H_n \subseteq K_n$ and $\mathcal{H}^1(K_n \setminus H_n) \to \mathcal{H}^1(K \setminus H)$.*

Proof. We assume $H \neq \varnothing$, since in the opposite case we could take $K_n = K$. Let $H^1, \dots, H^k$ with $k \leq p$ be its connected components. If there is only one connected component, we can take $\widehat{K} \coloneqq H = H^1$. If $k \geq 2$, by Lemma 3.2.11 there exists a family of ℓ connected components $\{\Gamma_j\}$ of $K \setminus H$ such that the set

$$\widehat{K} \coloneqq H \cup \bigcup_{j=1}^{\ell} \overline{\Gamma}_j,$$

is connected. Given this set, we would like to construct a sequence of connected sets satisfying the Lemma stated above.
Let us fix a small enough $\varepsilon > 0$ such that $N_\varepsilon(H^i) \cap N_\varepsilon(H^j) = \varnothing$ for every $i \neq j$, and $\widetilde{H}_n^i \coloneqq \{x \in H_n : x \in N_\varepsilon(H^i)\}$. By definition $\widetilde{H}_n^i \in \mathcal{K}_p^f(\overline{\Omega})$ and in general it holds

$\bigcup \widetilde{H}_n^i \subseteq H_n$, but for n large enough there is enough separation between the sets $(\widetilde{H}_n^i)$ so $\bigcup \widetilde{H}_n^i = H_n$. Moreover, due to the assumption of ε, we have $\widetilde{H}_n^i \to H^i$ for $n \to \infty$.
By Lemma 3.2.10 applied to each H^i, there is a sequence $(\widehat{H}_n^i) \subset \mathcal{K}_1^f(\overline{\Omega})$ such that $\widehat{H}_n^i \to H^i$, $\widetilde{H}_n^i \subseteq \widehat{H}_n^i$ and $\mathcal{H}^1(\widehat{H}_n^i \setminus \widetilde{H}_n^i) \to 0$. We now need to find the appropriate sequence $\widehat{K}_n$. As before, if H has only a single connected component, then $\widehat{K}_n \coloneqq \widehat{H}_n^1$.
If $k \geq 2$, for every disjoint pair H^i and H^j and its connected component $\Gamma_{\sigma(i,j)}$ of $K \setminus H$, we can pick $x^i \in H^i \cap \Gamma_{\sigma(i,j)}$ and $y^j \in H^j \cap \Gamma_{\sigma(i,j)}$. Due to the Hausdorff convergence in $\widehat{H}_n^i$ stated above, there are points $x_n^i \in \widehat{H}_n^i$ and $y_n^j \in \widehat{H}_n^j$ such that $x_n^i \to x^i$ and $y_n^j \to y^j$ as $n \to \infty$. As in the proof of Lemma 3.2.10, since the domain has a Lipschitz boundary, we can take the Lipschitz curves X_n^i connecting x_n^i to x^i and Y_n^j connecting y_n^j to y^j. These will be $\mathcal{H}^1$-measurable and as $n \to \infty$ we will have $\mathcal{H}^1(X_n^i) \to 0$ and $\mathcal{H}^1(Y_n^j) \to 0$. We can now define

$$\widehat{K}_n \coloneqq \bigcup_{i=1}^{k} \widehat{H}_n^i \cup \bigcup_{j=1}^{\ell} \overline{\Gamma}_j \cup \bigcup_{j=1}^{\ell} X_n^j \cup \bigcup_{j=1}^{\ell} Y_n^j.$$

Since $\bigcup \widehat{H}_n^i \to \bigcup H^i = H$, it holds that $\widehat{K}_n \to \widehat{K}$ for $n \to \infty$. Moreover

$$\begin{aligned}
&\limsup_{n\to\infty} \mathcal{H}^1(\widehat{K}_n \setminus H_n) \\
&\leq \limsup_{n\to\infty} \left[\mathcal{H}^1\left(\bigcup_{i=1}^{k} \widehat{H}_n^i \setminus H_n \right) + \mathcal{H}^1\left(\bigcup_{j=1}^{\ell} \overline{\Gamma}_j \right) + \mathcal{H}^1\left(\bigcup_{j=1}^{\ell} X_n^j \right) + \mathcal{H}^1\left(\bigcup_{j=1}^{\ell} Y_n^j \right) \right] \\
&\leq \limsup_{n\to\infty} \mathcal{H}^1\left(\bigcup_{i=1}^{k} \widehat{H}_n^i \setminus H_n \right) + \mathcal{H}^1(\widehat{K} \setminus H) = \mathcal{H}^1(\widehat{K} \setminus H).
\end{aligned}$$

These sets $\widehat{K}_n$ will be compacts, since they are the union of compact sets, and also connected, but they still do not satisfy all the requests, since $\widehat{K}_n \nrightarrow K$. To solve this, we could add to $\widehat{K}_n$ the connected components of $K \setminus \widehat{K}$. These connected components are also connected components of $K \setminus H$, so as in the proof of Lemma 3.2.11 we can conclude that each connected component C is open in the K-subspace topology and $\overline{C} \cap H \neq \varnothing$. Moreover, K is separable, so these connected components $\{C_i\}$ are at most countable.
For every C_i we pick $z^i \in \overline{C}_i \cap H$. Due to the Hausdorff convergence, there will be points $z_n^i \in H_n$ such that $z_n^i \to z^i$. As before, due to the Lipschitz boundary, there will be a Lipschitz curve Z_n^i connecting z_n^i to z^i, and $\mathcal{H}^1(Z_n^i) \to 0$. We would like to control the sum of the $\mathcal{H}^1$ measure of these sets, being careful if they are countable since the sum could explode to ∞ for every n. To avoid this, we restrict the sum to a certain number of terms, which grows with n, such that at every step it is not infinite. We call this sequence (h_n) such that

$$\lim_{n\to\infty} \sum_{i=1}^{h_n} \mathcal{H}^1(Z_n^i) = 0.$$

If the connected components are finite, we can just take a constant number of terms in the sum, $h_n = h$. Finally, we can define

$$K_n \coloneqq \widehat{K}_n \cup \bigcup_{i=1}^{h_n} Z_n^i \cup \bigcup_{i=1}^{h_n} \overline{C}_i.$$

These sets are obviously compact and connected, they contain H_n from the definition of $\widehat{K}_n$, and now we also have $K_n \to K$. At this point

$$\mathcal{H}^1(K_n \setminus H_n) \leq \mathcal{H}^1(\widehat{K}_n \setminus H_n) + \sum_{i=1}^{h_n} \mathcal{H}^1(Z_n^i) + \sum_{i=1}^{h_n} \mathcal{H}^1(C_i).$$

Noting that the connected components (C_i) are disjoint, taking the lim sup we can write

$$\begin{aligned}\limsup_{n\to\infty} \mathcal{H}^1(K_n \setminus H_n) &\leq \limsup_{n\to\infty} \mathcal{H}^1(\widehat{K}_n \setminus H_n) + \mathcal{H}^1\left(\bigcup C_i\right) \\ &\leq \mathcal{H}^1(\widehat{K} \setminus H) + \mathcal{H}^1\left(\bigcup C_i\right) \\ &= \mathcal{H}^1(K \setminus H).\end{aligned}$$

From 3.2.9 we can then deduce

$$\limsup_{n\to\infty} \mathcal{H}^1(K_n \setminus H_n) \leq \mathcal{H}^1(K \setminus H) \leq \liminf_{n\to\infty} \mathcal{H}^1(K_n \setminus H_n),$$

from which we can conclude. □

Proposition 3.2.13. *Let $p, m > 0$ be integers, (H_n) be a sequence in $\mathcal{K}_p^f(\overline{\Omega})$ converging to $H \in \mathcal{K}_p^f(\overline{\Omega})$, and let $K \in \mathcal{K}_m^f(\overline{\Omega})$ such that $H \subseteq K$. Then there exists a sequence (K_n) in $\mathcal{K}_m^f(\overline{\Omega})$ such that $K_n \to K$, $H_n \subseteq K_n$ and $\mathcal{H}^1(K_n \setminus H_n) \to \mathcal{H}^1(K \setminus H)$.*

Proof. As usual let $K^1, \ldots, K^k$ with $k \leq m$ be the connected components of K. Similar to the proof of Lemma 3.2.11, we choose $\varepsilon > 0$ small enough such that the sets $N_\varepsilon(K^i)$ are disjoint. We then define

$$\widehat{H}_n^i \coloneqq \{x \in H_n : x \in N_\varepsilon(K^i)\}.$$

These sets will have at most the number of connected components of H_n, so $\widehat{H}_n^i \in \mathcal{K}_p^f(\overline{\Omega})$, and as before, for n large enough we have $H_n = \widehat{H}_n^1 \cup \cdots \cup \widehat{H}_n^i$. Moreover due to the separation of the ε-neighborhoods, we have $\widehat{H}_n^i \to H^i \coloneqq H \cap K^i$ as $n \to \infty$. We can now apply Lemma 3.2.12 to every connected component K^i, so for every i we will get $(K_n^i) \subset \mathcal{K}_1^f(\overline{\Omega})$ such that $K_n^i \to K^i$, $\widehat{H}_n^i \subseteq K_n^i$, and $\mathcal{H}^1(K_n^i \setminus \widehat{H}_n^i) \to \mathcal{H}^1(K^i \setminus H^i)$. To conclude we just need to take

$$K_n \coloneqq K_n^1 \cup \cdots \cup K_n^k,$$

which ticks all the boxes of the proposition. □

3.3 Harmonic conjugate for boundary value problems

It is known from complex analysis that a necessary condition for a function to be holomorphic on a domain is that the real and imaginary parts of the function have to satisfy the Cauchy-Riemann conditions in the domain. In particular, these imply that the real and

imaginary parts are harmonic, that is they satisfy the Laplace equation in the domain. One can also show that if a function u is harmonic and the domain is simply connected, then there exists a so-called *harmonic conjugate* v such that it is harmonic, and they are the real and imaginary part of a holomorphic function. The harmonic conjugate is not unique though, as is defined up to an additive function, and that u is the harmonic conjugate of v if and only if v is the harmonic conjugate of $-u$. Geometrically u and v have orthogonal trajectories, such that the gradients are perpendicular, there exists a rotation matrix R with $R \cdot e_1 = e_2$ and $R \cdot e_2 = -e_1$, such that $\nabla u = R \nabla v$. We are now going to study the properties of this harmonic conjugate in the setting of our model.
From now on, the boundary $\partial\Omega$ will be divided in $\partial_N\Omega$, fixed and relatively open, on which we will impose Neumann boundary conditions, and in $\partial_D\Omega = \partial\Omega \setminus \overline{\partial_N\Omega}$ fixed and relatively open as well, on which we will impose Dirichlet boundary conditions. Both $\partial_N\Omega$ and $\partial_D\Omega$ are supposed to have a finite number of connected components. We can observe that by the properties of the harmonic conjugate v of a harmonic function u, the Neumann condition on $\partial_N\Omega$ gets rotated in the direction tangential to the boundary, meaning that v will be constant along $\partial_N\Omega$. Since we impose the cracks to be traction-free, v will also be constant along K.
Given $K \in \mathcal{K}(\overline{\Omega})$ which will represent the cracks, we have this boundary value problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus K; \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial(\Omega \setminus K) \cap (K \cup \partial_N\Omega). \end{cases} \tag{3.7}$$

By a weak solution of 3.7 we mean a function $u \in L^{1,2}(\Omega \setminus K)$ satisfying:

$$\int_{\Omega\setminus K} \nabla u^\top \nabla z \, \mathrm{d}x = 0 \qquad \forall z \in L^{1,2}_{0,\partial_D\Omega\setminus K}(\Omega \setminus K). \tag{3.8}$$

As we can see, there is no boundary condition on $\partial_D\Omega$, so as of now, we do not have any constraint on the value of the function on the boundary, meaning that the solution will not be unique. If we want to prescribe a Dirichlet boundary condition, we can take:

$$g \in L^{1,2}(\Omega \setminus K), \quad u = g \quad \text{q.e. on } \partial_D\Omega. \tag{3.9}$$

With this boundary condition, the problem 3.8 is solvable, and since by Corollary 3.1.15 we know that $L^{1,2}_{0,\partial_D\Omega\setminus K}(\Omega \setminus K)$ is a Hilbert space, by the Babuška–Lax–Milgram Lemma we know that there exists a unique solution in the connected components of $\Omega \setminus K$ whose boundaries intersect $\partial_D\Omega \setminus K$, and in all the other connected components the solution will be defined up to an additive constant. As a consequence, since the solution is either unique or defined up to a constant, the gradient will be unique across all the domain and also square-integrable.
Due to this uniqueness the map $g \mapsto \nabla u$ is linear and satisfies a nice inequality: if we consider the function $u - g \in L^{1,2}(\Omega \setminus K)$, we notice it will also be in $L^{1,2}_{0,\partial_D\Omega\setminus K}(\Omega \setminus K)$ so

we can use it as a test function in 3.8, obtaining:

$$\begin{aligned}\|\nabla u\|^2_{L^2(\Omega\setminus K;\mathbb{R}^2)} &= \int_{\Omega\setminus K} \|\nabla u\|^2_{\mathbb{R}^2}\,\mathrm{d}x \\ &= \int_{\Omega\setminus K} \nabla u^\top \nabla g\,\mathrm{d}x \\ &\leq \|\nabla u\|_{L^2(\Omega\setminus K;\mathbb{R}^2)} \cdot \|\nabla g\|_{L^2(\Omega\setminus K;\mathbb{R}^2)}.\end{aligned}$$

Since $\|\nabla u\| \neq 0$ when $g \neq 0$, this implies:

$$\|\nabla u\|_{L^2(\Omega\setminus K;\mathbb{R}^2)} \leq \|\nabla g\|_{L^2(\Omega\setminus K;\mathbb{R}^2)}. \tag{3.10}$$

By the same procedure, one can take an arbitrary function in the set

$$\mathcal{V}(g,K) \coloneqq \left\{v \in L^{1,2}(\Omega\setminus K) : v = g \text{ on } \partial_D\Omega\setminus K\right\}.$$

As before, for whatever $v \in \mathcal{V}(g,K)$ we have $u - v \in L^{1,2}_{0,\partial_D\Omega\setminus K}(\Omega\setminus K)$ and we can use it as a test function in the weak formulation to obtain that the solution u is such that ∇u is a solution of:

$$\min_{v\in\mathcal{V}(g,K)} \int_{\Omega\setminus K} |\nabla u|^2\,\mathrm{d}x. \tag{3.11}$$

As a side note, if the boundary datum is in $H^1(\Omega\setminus K)\cap L^\infty(\Omega\setminus K)$, hence bounded, then one can use a version of the maximum principle to show that the solution u is also bounded and $u \in H^1(\Omega\setminus K)\cap L^\infty(\Omega\setminus K)$, therefore we can spare the use of the Deny-Lions spaces. We refer to [28] for further details.
Due to the uniqueness of the gradient in $\Omega\setminus K$, from now on we will extend the gradient to Ω by setting $\nabla u = 0$ on K.

Lemma 3.3.1. *Let (K_n) be a sequence in $\mathcal{K}(\overline{\Omega})$ converging to K, and let (u_n) be a sequence such that $u_n \in L^{1,2}_{0,\partial_D\Omega\setminus K_n}(\Omega\setminus K_n)$, and such that (∇u_n) is bounded in $L^2(\Omega;\mathbb{R}^2)$. Then there exists a sub-sequence (u_n) converging to $u \in L^{1,2}_{0,\partial_D\Omega\setminus K}(\Omega\setminus K)$ such that $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(U;\mathbb{R}^2)$ for every $U \Subset \Omega\setminus K$. Moreover, if $\mathcal{H}^1(K_n)\to\mathcal{H}^1(K)$, then the weak convergence is in $L^2(\Omega;\mathbb{R}^2)$.*

Proof. Let C be a connected component of $\Omega\setminus K$: we pick $x \in C$ and we choose a ε such that $0 < \varepsilon < \operatorname{dist}(x,\partial C)$. Then it is well-defined the set

$$N^\varepsilon \coloneqq \{x \in \mathbb{R}^2 : \operatorname{dist}(x,\partial_N\Omega\cup K) \leq \varepsilon\},$$

which is the closed ε-neighborhood of $\partial_N\Omega\cup K$: it follows that due to the Hausdorff convergence of $K_n \to K$, for n large enough we have $K_n \subset N^\varepsilon$. We denote by C^ε the connected component of $C\setminus N^\varepsilon$ such that $x \in C^\varepsilon$.
For every C we have two cases, either it touches the Dirichlet boundary or it does not.
If $\partial C\cap(\partial_D\Omega\setminus K) \neq \varnothing$, for ε small enough we will have $\partial C^\varepsilon\cap\partial_D\Omega \neq \varnothing$, so we will denote the relative interior of this intersection contained in ∂C^ε by Γ^ε. We know that $u_n = 0$ on $\partial_D\Omega\setminus K_n$, and since $K_n \subset N^\varepsilon$, we deduce $u_n = 0$ on Γ^ε, hence $u_n \in L^{1,2}_{0,\Gamma^\varepsilon}(C^\varepsilon)$.

We know that the sequence (∇u_n) is bounded, hence from Corollary 3.1.15 we know that $\exists u \in L^{1,2}_{0,\Gamma^\varepsilon}(C^\varepsilon)$ such that $\nabla u_n \rightharpoonup \nabla u$ in $L^2(C^\varepsilon;\mathbb{R}^2)$. Due to the arbitrariness of ε, and since by construction $C = \bigcup_\varepsilon C^\varepsilon$, we can extend the limit to $u \in L^{1,2}_{0,(\partial C \cap \partial_D\Omega)\setminus K}(C)$ such that $\nabla u_n \rightharpoonup \nabla u$ in $L^2(U;\mathbb{R}^2)$ for every open $U \Subset C$.
On the other hand, if $\partial C \cap (\partial_D\Omega \setminus K) = \varnothing$, if we consider the domain C^ε, the sequence (∇u_n) is still bounded, so there is a converging sub-sequence. Due to the compactness of the space $\{\nabla u : u \in L^{1,2}(C^\varepsilon)\}$, there exists $u \in L^{1,2}(C^\varepsilon)$ such that $\nabla u_n \rightharpoonup \nabla u$ in $L^2(C^\varepsilon;\mathbb{R}^2)$. As before, we extend the limit analogously to $u \in L^{1,2}(C)$ such that the gradients converge weakly for every open $U \Subset C$.
In any case, we have constructed $u \in L^{1,2}_{0,\partial_D\Omega\setminus K}$ such that $\nabla u_n \rightharpoonup \nabla u$ in $L^2(U;\mathbb{R}^2)$ for every open $U \Subset \Omega \setminus K$.
We now assume $\mathcal{H}^1(K_n) \to \mathcal{H}^1(K)$. We will use that, due to the absolute continuity of the integral, given a function $\psi \in L^2(\Omega;\mathbb{R}^2)$, $\forall \varepsilon > 0\ \exists \delta > 0$ and $\exists A \subset \mathbb{R}^2$ with $\mathcal{H}^2(A) < \delta$ such that

$$\|\psi\|^2_{L^2(A;\mathbb{R}^2)} = \int_A |\psi|^2 \,\mathrm{d}x < \varepsilon^2.$$

Let us now take a open $U \Subset \Omega \setminus K$ such that $\mathcal{H}^2\big((\Omega \setminus K) \setminus U\big) < \delta$. Since for n large enough we also have $U \Subset \Omega \setminus K_n$, and since the sets K_n converge in measure, we deduce $\mathcal{H}^2\big((\Omega \setminus K_n) \setminus U\big) < \delta$. Then

$$\begin{aligned}
\left|\int_{\Omega\setminus K_n} (\nabla u_n - \nabla u)^\top \psi \,\mathrm{d}x\right| &\le \left|\int_U (\nabla u_n - \nabla u)^\top \psi \,\mathrm{d}x\right| + \left|\int_{(\Omega\setminus K_n)\setminus U} (\nabla u_n - \nabla u)^\top \psi \,\mathrm{d}x\right| \\
&\le \left|\int_U (\nabla u_n - \nabla u)^\top \psi \,\mathrm{d}x\right| + \|\psi\| \cdot \|\nabla u_n - \nabla u\| \\
&\le \underbrace{\left|\int_U (\nabla u_n - \nabla u)^\top \psi \,\mathrm{d}x\right|}_{=0} + \varepsilon\Big(\|\nabla u_n\| + \|\nabla u\|\Big).
\end{aligned}$$

Since the sequence ∇u_n is bounded, by taking the lim sup of the left hand side we deduce:

$$\limsup_n \left|\int_{\Omega\setminus K_n} (\nabla u_n - \nabla u)^\top \psi \,\mathrm{d}x\right| \le C\varepsilon.$$

From the arbitrariness of ε, it follows that the gradients converge weakly on $\Omega \setminus K$. □

From the properties of a generic harmonic conjugate, we look for a result that guarantees these properties at least locally.

Theorem 3.3.2. *Let $K \in \mathcal{K}(\overline{\Omega})$ and u be a solution of (3.8), and let $U \subseteq \Omega$ be a Lipschitz open set. Then there exists $v \in H^1(U \cap \Omega)$ such that $\nabla v = R\,\nabla u$ a.e. on $U \cap \Omega$, and v is constant q.e. on the connected components of $\overline{U} \cap K$ and $\overline{U} \cap \partial_N\Omega$.*

Proof. Let $\phi \in C^\infty_c(U \cap \Omega)$ be a test function. Due to the compact support $\phi = 0$ on $\partial_D\Omega$, so we can use it as a test function in 3.8, and since $\nabla u = 0$ a.e. in K, we have:

$$\int_{U\cap\Omega} \nabla u^\top \nabla\phi \,\mathrm{d}x = \int_{\Omega\setminus K} \nabla u^\top \nabla\phi \,\mathrm{d}x = 0.$$

This implies $\nabla \cdot (\nabla u) = 0$ on the space of distributions over $U \cap \Omega$. This in turn implies also $\nabla \times (R\,\nabla u) = 0$ on the same space, because:

$$\begin{aligned}\nabla \times (R\,\nabla u) &= \frac{\partial}{\partial x_1}\Big(R\,\nabla u\Big)_2 - \frac{\partial}{\partial x_2}\Big(R\,\nabla u\Big)_1 \\ &= \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) u \\ &= \nabla \cdot (\nabla u) = 0.\end{aligned}$$

The domain $U \cap \Omega$ is simply connected and since we are assuming Ω to have a Lipschitz boundary, it will also have a Lipschitz boundary. Hence, since $\nabla \times \nabla f = 0$, there exists $v \in H^1(U \cap \Omega)$ such that $\nabla v = R\,\nabla u$ a.e. in $U \cap \Omega$.
Due to the traction-free nature of the lips of the cracks, we know that $\nabla v = 0$ a.e. in $U \cap K$, which implies that v is constant on each connected component C of $U \cap K$. We now need to extend this result to the domain $\overline{U} \cap K$, to have a compact as well.
Let us write K as $K = \cap_j K_j$ with $K_{j+1} \subset K_j \quad \forall j$ and $(K_j) \subset \mathcal{K}(\overline{\Omega})$, such that K is contained in the interior of K_j for every j.
We can write problem 3.8 restricted to the set $(U \cap \Omega) \setminus K$, since for every test function $z \in L^{1,2}_{0,\partial_D\Omega\setminus K}(\Omega \setminus K)$ can be restricted to $(U \cap \Omega) \setminus K$ by then setting $z = 0$ in $(\Omega \setminus U) \setminus K$:

$$\int_{(U\cap\Omega)\setminus K} \nabla u^\top \nabla z \,\mathrm{d}x = 0 \qquad \forall z \in L^{1,2}_{0,\partial(U\cap\Omega)\setminus K}((U \cap \Omega) \setminus K). \tag{3.12}$$

The solution u to this problem is in $L^{1,2}((U \cap \Omega) \setminus K)$, and since $K \subset K_j$, we also have $u \in L^{1,2}((U \cap \Omega) \setminus K_j)$. We can use this u as a Dirichlet boundary for another problem with solution u_j:

$$\begin{aligned}&u_j \in L^{1,2}(U \cap \Omega) \setminus K_j) \qquad u_j = u \ \text{ q.e. on } \partial(U \cap \Omega) \setminus K_j; \\ &\int_{(U\cap\Omega)\setminus K_j} \nabla u_j^\top \nabla z \,\mathrm{d}x = 0 \qquad \forall z \in L^{1,2}_{0,\partial(U\cap\Omega)\setminus K_j}((U \cap \Omega) \setminus K_j).\end{aligned} \tag{3.13}$$

Using a method used before, we can now choose $u_j - u$ as a test function in the problem above, to obtain that $\|\nabla u_j\| \leq \|\nabla u\| \leq C \quad \forall j$, hence the gradients are uniformly bounded. From Lemma 3.3.1, since due to the decreasing nature of (K_j) we have that $(U \cap \Omega) \setminus K_j \to (U \cap \Omega) \setminus K$ in the Hausdorff metric, we can deduce that there exists $u^* \in L^{1,2}((U \cap \Omega) \setminus K)$, with $u^* = u$ q.e. on $\partial(U \cap \Omega) \setminus K$, such that $\nabla u_j \rightharpoonup \nabla u^*$ in $L^2(U \cap \Omega; \mathbb{R}^2)$.
We can now also use $u_j - u^*$ as a test function in the same problem to obtain an integral relation between the gradients in $(U \cap \Omega) \setminus K_j$, which we can extend on $U \cap \Omega$ because the gradients are equal to 0 on K_j.
We now need to prove that $\nabla u^* = \nabla u$ a.e. in $(U \cap \Omega) \setminus K$. Since the gradients of the solutions are uniquely defined, we need to prove that u^* satisfies the same problem as u, Problem 3.12. If z is a test function such that $z \in L^{1,2}_{0,\partial(U\cap\Omega)\setminus K}((U \cap \Omega) \setminus K)$, by a truncation argument we know also $z \in L^{1,2}_{0,\partial(U\cap\Omega)\setminus K_j}((U \cap \Omega) \setminus K_j)$ since $K \subset K_j$ and since we can choose a z such that $z = 0$ on $(U \cap \Omega) \cap K_j$. Then we can use z also as a test

function on Problem 3.13, and to conclude, we just need to pass the limit for $j \to \infty$, obtaining Problem 3.12.
Now we know that for each j there is a function $v_j \in H^1(U \cap \Omega)$ such that $\nabla v = R\,\nabla u$ in $U \cap \Omega$. Let now K^0 be a connected component of $\overline{U} \cap K$. Then there will exist a connected component C of the interior of $U \cap K_j$ such that $K^0 \subset C \cup \partial_L C$ since $K \subset K_j$ and the Lipschitz part follows from the Lipschitzianity of the boundary of U and Ω. Due to u_j being the solution to 3.13, we know v_j will be constant q.e. on $C \cup \partial_L C$ since it is constant q.e. also on K^0.
Since a harmonic conjugate is defined up to a constant, we can assume:

$$⨍_{U\cap\Omega} v_j \,\mathrm{d}x = 0 \qquad \forall j.$$

From a previous result in the proof, we know that $\nabla v_j \to R\,\nabla u$ in $L^2(U \cap \Omega; \mathbb{R}^2)$: using now the Poincaré–Wirtinger inequality we know

$$\Big\| v_j - ⨍_{U\cap\Omega} v_j \,\mathrm{d}x \Big\|_{L^2(U\cap\Omega)} \leq C \|\nabla v_j\|_{L^2(U\cap\Omega;\mathbb{R}^2)},$$

hence (v_j) converges strongly also in $H^1(U\cap\Omega)$. Then there exists a function $v \in H^1(U\cap\Omega)$ satisfying $\nabla v = R\,\nabla u$ on $U \cap \Omega$. Moreover, since v_j is constant q.e. on K^0, then also v will be constant q.e. on K^0, hence v is constant on all the connected components of $\overline{U} \cap K$.
For the last part of the proof, we need to show that v is also constant on the connected components of $\overline{U} \cap \partial_N \Omega$. It is enough to prove that v is constant q.e. for every $V \subset U$ with Lipschitz boundary such that $V \cap \partial\Omega = V \cap \partial_N\Omega$ since we are only interested in what happens on the boundary. Let us now extend the vector field ∇u to a vector field ψ on V such that $\psi = \nabla u$ q.e. on $V \cap \overline{\Omega}$ and $\psi = 0$ q.e. on $V \setminus \Omega$. As before, we now know that $\nabla \cdot \psi = 0$ on the space of distributions over V. This implies $\nabla \times (R\psi) = 0$ over the same space. Just as before, this implies the existence of $z \in H^1(V)$ such that $\nabla z = R\psi$ a.e. in V. By construction $\nabla z = 0$ a.e. in $V \setminus \overline{\Omega}$, but due to the compactness of the space of gradients in the Deny-Lions space, we also know $\nabla z = 0$ a.e. in $V \setminus \Omega$. On the other hand, on $V \cap \Omega$ we know that $\nabla z = R\psi = R\,\nabla u = \nabla v$, then for the same reason $\nabla(z - v) = 0$ q.e. in $V \cap \overline{\Omega}$. Since $(V \cap \overline{\Omega}) \cap (V \setminus \Omega) = V \cap \partial\Omega = V \cap \partial_N\Omega$ then by combining the two previous result we know that $\nabla v = 0$ q.e. on $V \cap \partial_N\Omega$. □

Theorem 3.3.3. *Let $K \in \mathcal{K}(\overline{\Omega})$ be locally connected, and let $u \in L^{1,2}(\Omega \setminus K)$. Assume that for every point in $\overline{\Omega}$ there exists an open neighborhood U of that point and a function $v \in H^1(U \cap \Omega)$ satisfying the properties of a harmonic conjugate in $U \cap \Omega$. Then u is harmonic and is a solution of 3.8.*

Proof. To show that u is a solution of 3.8, it is enough to show that for every point in $\overline{\Omega}$ there exists an open neighborhood V such that u is a solution to 3.8 restricted to $V \cap \Omega$ since then it suffices to piece together the solutions.
Let U be the neighborhood given in the theorem. We assume $U \cap \Omega$ to have a Lipschitz boundary, and we know from the statement that v is constant q.e. on each connected component of $U \cap K$ and $U \cap \partial_N\Omega$. Let now V be an open neighborhood of a point such

that $V \Subset U$. Since K is a locally connected compact set, the connected components of $U \cap K$ are open in K. This implies that only a finite number of them intersect with $\overline{V} \cap K$. The same goes for $\overline{V} \cap \partial_N \Omega$. The connected components of $\overline{V} \cap K$ and $\overline{V} \cap \partial_N \Omega$ are respectively disjoint, but they can intersect between the two families, let us say m times. Since v is constant on all of them, using 3.1.20, we can show that there exist m compact sets $(\widehat{K}^i)$ such that

$$(\overline{V} \cap K) \cap (\overline{V} \cap \partial_N \Omega) = \overline{V} \cap (K \cup \partial_N \Omega) = \widehat{K}^1 \cup \cdots \cup \widehat{K}^m,$$

such that $v = c^i$ q.e. on $\widehat{K}^i$.
Let us now construct a sequence of functions $v_n \in C^\infty(\mathbb{R}^2)$ such that $v_n \to v$ in $H^1(V \cap \Omega)$ and $v_n = c^i$ in a neighborhood U_n^i of $\widehat{K}^i$ for each i, such that $U_n^i \cap U_n^j = \varnothing$ for $i \neq j$.
Let now $z \in L^{1,2}_{0,(V\cap\partial_D\Omega)\setminus K}((V \cap \Omega) \setminus K)$ be a test function of Problem 3.8 restricted to $V \cap \Omega$. Let also ϕ_n^i be a test function in U_n^i such that $\phi_n^i = 1$ in a neighborhood of $\widehat{K}^i$ contained in U_n^i. Let us now define $\psi_n = 1 - \sum_i \phi_n^i$, a function which is equal to 1 everywhere except for a neighborhood of $\widehat{K}^i$, where it is equal to 0. Then, since $U \cap \Omega$ has a Lipschitz boundary, by Proposition 3.1.14, the function $z\psi_n \in L^{1,2}_{0,(V\cap\partial_D\Omega)\setminus K}((V\cap\Omega)\setminus K)$ belongs also to $H^1((V \cap \Omega) \setminus K)$. By Theorem 3.1.17, since by multiplying by ψ_n we are forcing z to be 0 also around K, we deduce also that $z\psi_n \in H_0^1((V \cap \Omega) \setminus K)$.
Since v_n is constant in U_n^i around $\widehat{K}^i$, hence $\nabla v_n = 0$, we can say that the $\psi_n = 1$ if $\nabla v_n \neq 0$, thus:

$$\begin{aligned}\int_{(V\cap\Omega)\setminus K} \nabla v_n^\top R^\top \nabla z \,\mathrm{d}x &= \int_{(V\cap\Omega)\setminus K} \nabla v_n^\top R^\top \nabla(z\psi_n)\,\mathrm{d}x \\ &= -\int_{(V\cap\Omega)\setminus K} \nabla \cdot (R\,\nabla v_n)\, z\psi_n \,\mathrm{d}x = 0,\end{aligned}$$

since $\nabla \cdot (R\,\nabla v_n) = \nabla \times (\nabla v_n) = 0$ and for what just discussed $z\psi_n \in H_0^1((V \cap \Omega) \setminus K)$. Since, due to the properties of the rotation by 90 degrees, if $\nabla v = R\,\nabla u$, then also $\nabla u = -R\,\nabla v$. Passing to the limit for $n \to \infty$ we have:

$$\int_{(V\cap\Omega)\setminus K} \nabla u^\top \nabla z \,\mathrm{d}x = -\int_{(V\cap\Omega)\setminus K} \nabla v^\top R^\top \nabla z \,\mathrm{d}x = 0.$$

If z is a test function of the restricted problem, then this implies that u is a solution to such a problem. □

Chapter 4

Existence results in two dimensional anti-plane setting

Using the tools explained in the previous section, our goal is now to prove the existence of a continuous evolution satisfying the conditions of the Francfort-Marigo model. We will make some natural assumptions about the agents involved, in order to better formalize our setting, and also to state the full version of the model studied.
First of all, we are studying the system at time $t \in [0,1]$, under a time-dependent loading $g(t)$, such that $g \in \mathrm{AC}([0,1]; H^1(\Omega))$, with the usual Bochner notation. The use of absolutely continuous functions is useful since for compact intervals it implies uniform continuity, and also implies that the derivatives are integrable. At every time t, the total energy is minimized across all admissible displacements:

$$E(g(t), K(t)) \coloneqq \min_{v \in \mathcal{V}(g(t),K(t))} \left\{ \frac{\mu}{2} \int_{\Omega \setminus K(t)} |\nabla v|^2 \, \mathrm{d}x + \gamma \, \mathcal{H}^1(K(t)) \right\};$$
$$\mathcal{V}(g(t), K(t)) \coloneqq \left\{ v \in L^{1,2}(\Omega \setminus K(t)) : v = g(t) \text{ on } \partial_D \Omega \setminus K(t) \right\}.$$

While the evolution of $g(t)$ is fixed, we need to describe the evolution of the crack set K. We will consider as an evolution a compact-valued non-decreasing function $K \colon [0,1] \to \mathcal{K}^f_m(\overline{\Omega})$. This evolution has to satisfy these conditions, taken from 2.1.1:

- $K(t)$ minimizes the energy among all possible crack K such that $K \supseteq \bigcup_{s<t}(K(s))$;
- $K(t)$ is a stationary point for the function $s \mapsto E(g(t), K(s))$.

The result that we want to obtain is the following:

Theorem 4.0.1. *Let $g \in \mathrm{AC}([0,1]; H^1(\Omega))$ and $K_0 \in \mathcal{K}^f_m(\overline{\Omega})$. Then there exists a continuous evolution $K \colon [0,1] \to \mathcal{K}^f_m(\overline{\Omega})$ such that:*

1. *$K_0 \subseteq K(s) \subseteq K(t)$ for $0 \le s < t \le 1$;*
2. *$E(g(0), K(0)) \le E(g(0), K) \quad \forall K \in \mathcal{K}^f_m(\overline{\Omega}), K \supseteq K_0$;*

3. *for* $t \in (0,1]$, $E(g(t),K(t)) \leq E(g(t),K) \quad \forall K \in \mathcal{K}_m^f(\overline{\Omega}), K \supseteq \bigcup_{s<t} K(s)$;

4. *the function* $t \mapsto E(g(t),K(t))$ *is absolutely continuous on* $[0,1]$;

5. $\dfrac{\mathrm{d}}{\mathrm{d}s} E(g(t),K(s))\Big|_{s=t} = 0$ *for a.e.* $t \in [0,1]$.

Moreover, if every function $K\colon [0,1] \to \mathcal{K}_m^f(\overline{\Omega})$ *satisfies all these conditions, then it also satisfies:*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(g(t),K(t)) = \mu \int_{\Omega\setminus K(t)} \nabla u(t)^\top \nabla \dot{g}(t)\,\mathrm{d}x \quad \text{for a.e. } t \in [0,1],$$

where $u(t)$ *is a solution of the minimum problem defining* $E(g(t),K(t))$.

This result is basically the existence result for the continuous evolution, given an arbitrary loading, the equivalent of Proposition 2.1.1. We will also prove that under non-decreasing and non-negative loadings, the conditions on the previous theorem imply:

$$E(g(t),K(t)) \leq E(g(t),K(s)) \quad \forall s \leq t,$$

which is the third condition in the monotone continuous evolution 2.1.2.
This theorem will be proven by a time discretization process, where we will first derive the solutions for a discrete evolution, and then study the properties of these solutions as the time step approaches 0. To do this, we must first establish the proper convergence of the solution to the minimum problems defining $E(g,K)$ under appropriate sequences $(g_n)_n$ and $(K_n)_n$. Then we will study the properties of a general continuous evolution, which is simply a compact-valued increasing function. At last, we will tackle the discretization process and its difficulties.

4.1 Convergence of minimizers

As discussed, the first thing is to study the convergence of the minimums of the problems defining the total energy under converging sequences. To do this we need some results on the convergence of solutions of boundary value problems under domain perturbation, which we will just present without giving proofs. These will be used for an important lemma, which tells us that if $v_n = 0$ *quasi-everywhere* in K_n, then this property is stable under limits.
First of all, we say that an open set $\Omega \subseteq B \subset \mathbb{R}^N$ has the (r,C)-capacity condition if:

$$\forall x \in \partial\Omega \qquad \frac{\mathrm{Cap}_2\left(\Omega^{\complement} \cap B_r(x); B_{2r}(x)\right)}{\mathrm{Cap}_2\left(B_r(x); B_{2r}(x)\right)} \geq C.$$

Now, if $r < 1$, we define the following family of open subsets of B:

$$\mathcal{O}_{r,C}(B) \coloneqq \left\{\Omega \subseteq B : \forall r_0 \in (0,r)\ \ \Omega \text{ has the } (r_0,C)\text{-capacity condition}\right\}.$$

We now define the convergence in the Hausdorff complementary topology, simply the notion distance defined on the complement of the sets:

$$K_n \xrightarrow{H^{\complement}} K \iff K_n^{\complement} \xrightarrow{H} K^{\complement}.$$

Now, for an arbitrary $f \in L^2(B)$, we can define as u_Ω the solution to the Dirichlet problem:

$$-\Delta u_\Omega = f \qquad u_\Omega \in H_0^1(\Omega).$$

The main result is the following:

Theorem 4.1.1. *Let $B \in \mathbb{R}^N$ be an arbitrary open set. Let (Ω_n) be a sequence in $\mathcal{O}_{r,C}(B)$, such that $\Omega_n \xrightarrow{H^{\complement}} \Omega$ for a certain open Ω. Then $u_{\Omega_n} \to u_\Omega$ strongly in $H_0^1(B)$.*

The reader can find the proof of this theorem in [21]. Moreover, if $B \subset \mathbb{R}^2$, from the previous theorem we can deduce a result from [20]:

Corollary 4.1.2. *Let $B \in \mathbb{R}^2$ be an arbitrary open set, and $\varkappa$ a positive integer. Define the set*

$$\mathcal{O}_\varkappa := \{\Omega \subseteq B : \text{ number of connected components of } B \setminus \Omega \text{ is } \leq \varkappa\}.$$

Then it follows that the set $\mathcal{O}_\varkappa$ is compact in the $H^{\complement}$-topology and the application $\mathcal{O}_\varkappa \ni \Omega \mapsto u_\Omega \in H_0^1(B)$ is continuous, meaning that if a sequence of sets converges in the Hausdorff complementary topology, then the solutions to the respective boundary value problems converge strongly in $H_0^1(B)$.

This last result will play a key role in this section.
We will now define another notion of convergence, which can be used both for sets and operators. This notion was introduced in [14], and is thus called Mosco convergence. His original definition goes as follows:

Definition 4.1.3. *Let $(Z_n)_n$ be a sequence of closed convex subsets of a reflexive Banach space X. Let us define two sets:*

$$S\{(Z_n)_n\} := \{v \in X : \exists (v_n)_n, v_n \in Z_n, v_n \to v\};$$
$$W\{(Z_n)_n\} := \{v \in X : \exists (v_k)_k, v_k \in Z_{n_k}, v_k \rightharpoonup v\}.$$

We say that the sequence converges in the sense of Mosco to a closed convex subset Z, denoted by $Z_n \xrightarrow{M} Z$ or $M - \lim Z_n = Z$ if:

$$S\{(Z_n)_n\} = W\{(Z_n)_n\} = Z \implies Z_n \xrightarrow{M} Z.$$

What this tells us is that if every weakly convergent sequence with $v_n \in Z_n$ converges also strongly, then we can define the Mosco limit set. From the definition, it follows that is $Z_n \xrightarrow{M} Z \neq \varnothing$, then $Z_n \neq \varnothing$ for every $n \geq n_0$. Moreover, even if $Z_n \neq \varnothing$ for every n, this does not imply $Z \neq \varnothing$.
From this general definition, one can deduce another equivalent one, which gives more characterization to the Mosco limit set:

Definition 4.1.4. *A sequence $(Z_n)_n$ of closed convex subsets of a reflexive Banach space X converges in the sense of Mosco to a closed convex subset Z if:*

1. *for every $v \in Z$ there exists a sequence $(v_n)_n$, $v_n \in Z_n$ for every n, such that $v_n \to v$;*
2. *if $(n_k)_k$ is an increasing sequence of positive integers, and $v_{n_k} \in Z_{n_k}$ for every k, then wherever $v_{n_k} \rightharpoonup v \in X$, this implies $v \in Z$.*

This notion of convergence, just as the more famous Γ-convergence, can also be written in terms of functionals on X. The definition is quite similar to the one of the Γ-convergence, and the M-convergence, as it is sometimes referred to, is also phrased as *weak Γ-liminf and strong Γ-limsup.*

Definition 4.1.5. *Let $(F_n)_n$ be a sequence of functionals on X. This sequence converges in the sense of Mosco to another functional F if:*

1. *for every sequence $(v_n)_n \subset X$ such that $v_n \rightharpoonup v \in X$, then*
$$F(v) \leq \liminf_{n\to\infty} F_n(v_n);$$
2. *for every $v \in X$ there exists a sequence $(v_n)_n \subset X$ with $v_n \to v$, such that*
$$\limsup_{n\to\infty} F_n(v_n) \leq F(v).$$

This notion of convergence of sets has proven to be very useful in studying boundary value problems under domain perturbation. For a Dirichlet problem with purely Dirichlet boundary conditions, in [15] it was proven that:

Theorem 4.1.6. *Let $(Z_n)_n, Z$ be a sequence of non-empty closed convex subsets of a reflective Banach space X. Then the following are equivalent:*

- *the sequence $(Z_n)_n$ converges in the sense of Mosco to Z;*
- *$u_{Z_n} \to u_Z$ strongly in X, with u_Ω being the solution to the Dirichlet problem, as outlined above.*

Lemma 4.1.7. *Let (K_n) be a sequence in $\mathcal{K}_1(\overline{\Omega})$ of connected compacts such that $K_n \to K$ in the Hausdorff metric. Let (v_n) be a sequence in $H^1(\Omega)$ such that $v_n \rightharpoonup v$ in $H^1(\Omega)$. Let us also assume that $v_n = 0$ q.e. on K_n for every n. Then it follows that the limit function v is such that $v = 0$ q.e. on the limit set K.*

Proof. Since we know from the assumption of the model that Ω is bounded, then also $\overline{\Omega}$ is bounded. We can then take a ball B centered in the origin, with a big enough radius such that $\overline{\Omega} \subset B$. We know that if $v_n, v \in H^1(\Omega)$ such that $v_n \rightharpoonup v$ in $H^1(\Omega)$, then, since Ω has a Lipschitz boundary, we can extend these functions to $v_n, v \in H^1_0(B)$ such that $v_n \rightharpoonup v$ in $H^1(B)$. Moreover, trivially if we have $A \Subset B$, then we can extend a function $u \in H^1_0(A)$ to a function $u \in H^1_0(B)$ by setting $u = 0$ in the difference set $\overline{B} \setminus A$. Then, by applying Theorem 3.1.17, we can say:

$$H^1_0(A) = \left\{u \in H^1(A) : u = 0 \text{ q.e. on } \partial A\right\} = \left\{u \in H^1(B) : u = 0 \text{ q.e. on } \overline{B} \setminus A\right\}. \quad (4.1)$$

Looking now at the sequence of sets $\left((B \setminus K_n)^{\complement}\right)_n$, it always has a finite number of connected components, hence from Corollary 4.1.2, since obviously

$$B \setminus K_n \xrightarrow{H^{\complement}} B \setminus K,$$

we can deduce that, given $f \in L^2(B)$, we have $u_{B\setminus K_n} \to u_{B\setminus K}$ strongly in $H^1_0(B)$, with $u_{B\setminus K_n}$ and $u_{B\setminus K}$ being the solutions to the Dirichlet problems:

$$\begin{aligned} u_{B\setminus K_n} \in H^1_0(B \setminus K_n) \qquad & \Delta u_{B\setminus K_n} = f \text{ in } B \setminus K_n; \\ u_{B\setminus K} \in H^1_0(B \setminus K) \qquad & \Delta u_{B\setminus K} = f \text{ in } B \setminus K. \end{aligned}$$

Thanks to the equivalence theorem 4.1.6, we now know that $H^1_0(B \setminus K_n) \xrightarrow{M} H^1_0(B \setminus K)$ in the space $H^1_0(B)$. Since $v_n = 0$ q.e. on K_n, from (4.1) we know $v_n \in H^1_0(B \setminus K_n)$ since $v = 0$ q.e. on $\overline{B} \setminus (B \setminus K_n) \supset K_n$. Thanks to the second point of the definition 4.1.4 of Mosco convergence, since $v_n \in H^1_0(B \setminus K_n)$ and $v_n \rightharpoonup v$ in $H^1(B)$ from the hypotheses of the theorem, we deduce that $v \in H^1_0(B \setminus K)$. We can conclude that $v = 0$ q.e. on K. □

Theorem 4.1.8. *Let $m \geq 1$ and $\lambda \geq 0$, and let (K_n) be a sequence in $\mathcal{K}^\lambda_m(\overline{\Omega})$, such that $K_n \to K$ in the Hausdorff metric. Let (g_n) be a sequence in $H^1(\Omega)$ such that $g_n \to g$ in $H^1(\Omega)$. For every n let u_n be a solution (not necessarily unique since in certain parts of the domain it can be defined up to a constant) of the problem:*

$$\min_{v \in \mathcal{V}(g_n, K_n)} \int_{\Omega\setminus K_n} |\nabla v|^2 \, \mathrm{d}x, \quad \mathcal{V}(g_n, K_n) = \left\{ v \in L^{1,2}(\Omega \setminus K_n) : v = g_n \text{ q.e. on } \partial_D\Omega \setminus K_n \right\},$$

and analogously let u be a solution of:

$$\min_{v \in \mathcal{V}(g, K)} \int_{\Omega\setminus K} |\nabla v|^2 \, \mathrm{d}x, \quad \mathcal{V}(g, K) = \left\{ v \in L^{1,2}(\Omega \setminus K) : v = g \text{ q.e. on } \partial_D\Omega \setminus K \right\}.$$

Then it follows that $\nabla u_n \to \nabla u$ in $L^2(\Omega; \mathbb{R}^2)$.

Proof. Since the minimum problems are equivalent to (3.8), then we know that $\|\nabla u_n\| \leq \|\nabla g_n\| < +\infty$ for every n, and $\|\nabla u\| \leq \|\nabla g\| < +\infty$, meaning that (∇u_n) is a bounded sequence in $L^2(\Omega; \mathbb{R}^2)$. Then from 3.3.1 we know that there exists $u^* \in L^{1,2}(\Omega \setminus K)$ such that $u^* = g$ q.e. on $\partial_D\Omega \setminus K$, and also $\nabla u_n \rightharpoonup \nabla u^*$ in $L^2(U; \mathbb{R}^2)$ with $U \Subset \Omega \setminus K$. Moreover, since the sets K_n have a finite number of connected components and finite $\mathcal{H}^1$-measure, and due to the lower semicontinuity of the $\mathcal{H}^1$-measure:

$$\mathcal{H}^1(K) \leq \liminf_{n\to\infty} \mathcal{H}^1(K_n) \leq \lambda \implies K \in \mathcal{K}^\lambda_m(\overline{\Omega}).$$

It also follows that $\mathcal{H}^1(K_n) \to \mathcal{H}^1(K)$, which tells us that $\nabla u_n \rightharpoonup \nabla u^*$ in $L^2(\Omega; \mathbb{R}^2)$. What we want now to prove is that $\nabla u^* = \nabla u$ a.e. in $\Omega \setminus K$, and to do this, due to the uniqueness of the gradients, it is enough to show that u^* is a solution of 3.8. Instead of proving this directly, we will use Theorem 3.3.3 by showing that there exists a harmonic conjugate. Thanks to a localization argument, we can take for every $x \in \overline{\Omega}$ a neighborhood U, and we will prove that there exists $v \in H^1(U \cap \Omega)$ such that $\nabla v = R\,\nabla u^*$ in $U \cap \Omega$,

and v is constant on the connected components of $U \cap K$ and $U \cap \partial_N \Omega$.
Let us choose $x \in \overline{\Omega}$ and an open neighborhood U, and let us take a set $V \subseteq \overline{\Omega}$ such that $U \Subset V$, and let us define $\delta \coloneqq \operatorname{dist}(U, \partial V)$. We now want to estimate the number of connected components C of $\overline{V} \cap K_n$, such that $C \cap (U \cap K_n) \neq \varnothing$. If $C \cap \partial V \neq \varnothing$, then C connects points in U with points in ∂V, hence $\mathcal{H}^1(C) \geq \delta$. But then we know that $\mathcal{H}^1(\overline{V} \cap K_n) \leq \mathcal{H}^1(K_n) \leq \lambda$, hence the cardinality of this subset of the connected components is $\leq \frac{\lambda}{\delta}$. On the contrary, if $C \cap \partial V = \varnothing$, then C is a connected component of K_n, hence the cardinality in this case is $\leq m$. Summing up, we have that $\#C \leq m + \frac{\lambda}{\delta}$.
Let $K_n^1, \dots, K_n^{k_n}$ be the connected components of $\overline{V} \cap K_n$ intersecting $U \cap K_n$. For what we just said, $k_n \leq m + \frac{\lambda}{\delta}$, hence, following a procedure outlined in the previous chapter, up to a sub-sequence we can take $k_n = k$ for every n, such that $K_n^1 \to \widehat{K}^1, \dots, K_n^k \to \widehat{K}^k$, with $\widehat{K}^i \in \mathcal{K}_1(\overline{\Omega})$. As in the proof of Corollary 3.2.8, we can deduce that

$$U \cap K \subset \widehat{K}^1 \cup \ldots \cup \widehat{K}^k.$$

Let us take now a harmonic conjugate v_n of u_n in $V \cap \Omega$, such that $\nabla v_n = R \nabla u_n$. Since a harmonic conjugate is defined up to a constant, we can assume that:

$$⨍_{V \cap \Omega} v \, \mathrm{d}x = 0.$$

From what shown before we can assert that $\nabla v_n \rightharpoonup R \nabla u^*$ in $L^2(V \cap \Omega; \mathbb{R}^2)$. Moreover, by the Poincaré-Wirtinger inequality, we can also say that there exists a v such that $v_n \rightharpoonup v$ with $\nabla v = R \nabla u^*$ in $H^1(V \cap \Omega)$.
Now, if $\widehat{K}^i$ has just one point, then of course $v = c^i$. If it has more than one point, then we know that $v_n = c_n^i$ on K_n^i, hence $v_n - c_n^i$ is bounded in $H^1(V \cap \Omega)$. Since v_n is bounded in $H^1(V \cap \Omega)$, since v_n converges weakly, then c_n has to be bounded. Then it has a limit c^i, such that $v_n - c_n^i \rightharpoonup v - c^i$. But then we can apply 4.1.7, since $v_n - c_n^i = 0$ q.e. on K_n^i, it follows that $v - c^i = 0$ q.e. on $\widehat{K}^i$. To finish this first part, by Proposition 3.1.20, if $\widehat{K}^i \cap \widehat{K}^j \neq \varnothing$, it follows that v is constant in $\widehat{K}^i \cup \widehat{K}^j$, such that v is constant on $U \cap K$.
For the part about $U \cap \partial_N \Omega$, we know that v_n is constant in $\partial_N \Omega$: then v_n is also constant in $V \cap \partial_N \Omega$. We know that $v_n \rightharpoonup v$, then by Mazur's lemma there exists a convex combination of the elements in the sequence such that, up to a sub-sequence, $v_n \to v$. This implies that v is constant in $V \cap \partial_N \Omega$, hence also in $U \cap \partial_N \Omega$.
Therefore, v is a harmonic conjugate for u^*, and this, in turn, implies that u^* is a solution of 3.8. To conclude, it is enough to remember that $\nabla u_n \rightharpoonup \nabla u$, and $\nabla g_n \to \nabla g$, such that

$$\int |\nabla u_n|^2 \, \mathrm{d}x = \int \nabla u_n^\top \nabla g_n \, \mathrm{d}x \to \int \nabla u^\top \nabla g \, \mathrm{d}x = \int |\nabla u|^2 \, \mathrm{d}x,$$
$$\|\nabla u_n\| \to \|\nabla u\|.$$

From this it follows that $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega; \mathbb{R}^2)$. □

From what we said in the previous proof, we can easily deduce an extension of this result to the whole energy:

Corollary 4.1.9. *Let $m \geq 1$ and $\lambda \geq 0$, and let (K_n) be a sequence in $\mathcal{K}_m^\lambda(\overline{\Omega})$, such that $K_n \to K$ in the Hausdorff metric. Let (g_n) be a sequence in $H^1(\Omega)$ such that $g_n \to g$*

in $H^1(\Omega)$. Let u_n and u be the solutions to the minimum problems defining $E(g_n, K_n)$ and $E(g, K)$ respectively. Since from previous result we know that $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega;\mathbb{R}^2)$ then we can deduce that $E(g_n, K_n) \to E(g, K)$.

Proof. Since $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega;\mathbb{R}^2)$, we know that $E_b(g_n, K_n) \to E_b(g, K)$ since the gradients are assumed to be zero on the crack sets. Moreover, since the sets K_n have a finite number of connected components and finite $\mathcal{H}^1$-measure, due to the lower semi-continuity of the $\mathcal{H}^1$-measure we also know that $K \in \mathcal{K}^\lambda_m(\overline{\Omega})$. It follows that $\mathcal{H}^1(K_n) \to \mathcal{H}^1(K)$, which implies that $E_f(K_n) \to E_f(K)$. □

4.2 Behavior in time of admissible crack sets and loadings

Up till now, we have not really studied the behavior in time of the admissible evolution. We will consider thus a non-decreasing function $K\colon [0,1] \to \mathcal{K}(\overline{\Omega})$. We are already assuming the evolution to be non-decreasing: it makes sense since we are excluding crack healing. Most of these results will be the extension of properties of real-valued monotone functions to compact-valued functions. To prove the first result about continuity we will need the following lemma.

Lemma 4.2.1. *Let K_1, K_2 be to non-decreasing compact-valued functions defined on $[0,1]$, such that:*

$$K_1(s) \subseteq K_2(t), \qquad K_2(s) \subseteq K_1(t), \qquad s \leq t. \tag{4.2}$$

Let us define $\Theta \coloneqq \{t \in [0,1] : K_1(t) = K_2(t)\}$. Then the set $[0,1] \setminus \Theta$ is at most countable.

Proof. Let us define a function $f_i\colon \overline{\Omega} \times [0,1] \to \mathbb{R}$ such that $f_i(x,t) = \operatorname{dist}\big(x, K_i(t)\big)$, with the convention $\operatorname{dist}(x, \varnothing) = \operatorname{diam}(\Omega)$. Now, it is known that, for a fixed time t, the function $x \mapsto f_i(x,t)$ is Lipschitz continuous with Lipschitz constant 1. Moreover, since $K_i(t)$ is non-decreasing, for a fixed x the function $t \mapsto f_i(x,t)$ is non-increasing.
Let us now take a countable set D which is dense in $\overline{\Omega}$. For every $x \in D$, by Luzin's theorem there exists a countable set $N_x \subset [0,1]$ such that $f_i(x,\cdot)\big|_{[0,1]\setminus N_x}$ is continuous.
By (4.2) we can write

$$f_1(x,s) \geq f_2(x,t), \qquad f_2(x,s) \geq f_1(x,t), \qquad \forall x \in \overline{\Omega} \quad \forall s,t \in [0,1] : s \leq t.$$

By the continuity stated above, it follows that $\forall x \in D$, $\forall t \in [0,1] \setminus N_x$, then $f_1(x,t) = f_2(x,t)$. We now define $N = \bigcup_{x\in D} N_x$, which is at most countable, since it is a countable union of sets that are at most countable. Let us then take $t \in [0,1] \setminus N$, we will have $\forall x \in D$ that $f_1(x,t) = f_2(x,t)$. Due to the density of D, this implies that $\forall t \in [0,1] \setminus N$, $f_1(x,t) = f_2(x,t)$ for every $x \in \overline{\Omega}$. Looking at the definition of f_i, this implies that $K_1(t) = K_2(t)$ $\forall t \in [0,1] \setminus N =: \Theta$. Then $[0,1] \setminus \Theta = N$, which is at most countable. □

Theorem 4.2.2. *Let K be a non-decreasing compact-valued function defined on* [0,1]. *We can then define the analogous of the one-sided limits for real-valued functions as:*

$$K^-\colon (0,1] \to \mathcal{K}(\overline{\Omega}) \qquad K^-(t) = \overline{\bigcup_{s<t} K(s)} \quad 0 < t \leq 1;$$
$$K^+\colon [0,1) \to \mathcal{K}(\overline{\Omega}) \qquad K^+(t) = \bigcap_{s>t} K(s) \quad 0 \leq t < 1.$$

Obviously for every $t \in (0,1)$ we have $K^-(t) \subseteq K(t) \subseteq K^+(t)$. Let us define the set $\Theta := \{t \in (0,1) : K^-(t) = K^+(t)\}$. Then the set $[0,1] \setminus \Theta$ is at most countable and for every $t \in \Theta$ and for every sequence $(t_n) \subset [0,1]$ such that $t_n \to t$, we have $K(t_n) \to K(t)$ in the Hausdorff metric.

Proof. From the definition of $K^-(t)$ and $K^+(t)$, it is clear that they are both non-decreasing. Moreover if $s < t$, we have $K^-(s) \subseteq K^+(t)$ and $K^+(s) \subseteq K^-(t)$, hence from previous lemma, defining the same Θ, we have that $[0,1] \setminus \Theta$ is at most countable. Let us choose $t \in \Theta$, and a sequence $(t_n) \subset [0,1]$ such that $t_n \to t$. Then due to the compactness of the Hausdorff space, the sequence $(K(t_n))$ converges to K^*.
Let us choose now $s_1, s_2 \in [0,1]$ such that $s_1 < t < s_2$ with $t \in \Theta$. Since $t_n \to t$, for n large enough we have $s_1 < t_n < s_2$, which implies $K(s_1) \subseteq K(t_n) \subseteq K(s_2)$. Taking now the limit as $n \to \infty$, since K^* is closed, we get $K(s_1) \subseteq K^* \subseteq K(s_2)$. Taking also the limits $s_1 \to t^-$ and $s_2 \to t^+$, we get $K^-(t) \subseteq K^* \subseteq K^+(t)$. Recall now that $t \in \Theta$, hence $K^-(t) = K(t) = K^+(t)$, from which follows that $K^* = K(t)$. □

The next result is the equivalent of Helly's selection theorem for real-valued monotone functions, which states that a uniformly bounded sequence of real-valued monotone functions admits a convergent sub-sequence.

Theorem 4.2.3. *Let (K_n) be a sequence of compact-valued non-decreasing functions. Then there exists a compact-valued non-decreasing function K, such that, up to sub-sequences, $K_n(t) \to K(t)$ for every $t \in [0,1]$.*

Proof. Let D be a countable dense subset of [0,1]. Fixing $t \in D$, due to the compactness of the Hausdorff space, the sequence $\big(K_n(t)\big)_n$ admits a sub-sequence converging to K_t^*. For every n we have that $K_n(t) \subseteq K_n(s)$ if $t < s$, hence taking the limit we get $K_t^* \subseteq K_s^*$. We can then define $K\colon D \to \mathcal{K}(\overline{\Omega})$, where $K(t) = K_t^*$ for every $t \in D$ from the previous limits. We can now define an analogous of the functions $K^-(t)$ and $K^+(t)$, except only for the set D:

$$K_D^-\colon (0,1] \cap D \to \mathcal{K}(\overline{\Omega}), \qquad K^-(t) = \overline{\bigcup_{\substack{s<t \\ s\in D}} K(s)} \quad t \in (0,1] \cap D;$$
$$K_D^+\colon [0,1) \cap D \to \mathcal{K}(\overline{\Omega}), \qquad K^+(t) = \bigcap_{\substack{s>t \\ s\in D}} K(s) \quad t \in [0,1) \cap D.$$

Since D is dense in [0,1], these operators can be extended to $K_D^-\colon (0,1] \to \mathcal{K}(\overline{\Omega})$ and to $K_D^+\colon [0,1) \to \mathcal{K}(\overline{\Omega})$. Let us define the set $\Theta = \{t \in [0,1] : K_D^-(t) = K_D^+(t)\}$: since

obviously if $s < t$ we have $K_D^-(s) \subseteq K_D^+(t)$ and $K_D^+(s) \subseteq K_D^-(t)$, from 4.2.1 we deduce that the set $[0,1] \setminus \Theta$ is at most countable.
For every $t \in D$ we now have $K_D^-(t) \subseteq K(t) \subseteq K_D^+(t)$, and for every $t \in \Theta$ we have $K_D^-(t) = K_D^+(t)$, we deduce that for every $t \in \Theta \cap D$ we have $K_D^-(t) = K(t) = K_D^+(t)$. Instead for $t \in \Theta \setminus D$ we can define $K(t) \coloneqq K_D^-(t) = K_D^+(t)$, but we need to make sure that $K_n(t) \to K(t)$ for $t \in \Theta \setminus D$. Let us choose $s_1, s_2 \in D$ and $t \in \Theta \setminus D$ such that $s_1 < t < s_2$: we will then have $K_n(s_1) \subseteq K_n(t) \subseteq K_n(s_2)$. Taking the limit as $n \to \infty$, due the compactness of the Hausdorff space we deduce that $K_n(t) \to K^*$, hence $K(s_1) \subseteq K^* \subseteq K(s_2)$. Taking now the limits as $s_1 \to t^-$ and $s_2 \to t^+$, since K^* is closed we deduce $K_D^-(t) \subseteq K^* \subseteq K_D^+(t)$. Since $t \in \Theta$ we have $K_D^-(t) = K_D^+(t)$, hence $K(t) = K^*$. It remains to check the convergence in $[0,1] \setminus (\Theta \cup D)$. Since we know that it is at most countable, with the same method as the beginning of the proof, thanks to the compactness of the Hausdorff space, we can find a sub-sequence (K_n) and a non-decreasing limit $K \colon [0,1] \setminus (\Theta \cup D) \to \mathcal{K}(\overline{\Omega})$ such that $K_n(t) \to K(t)$ for every $t \in [0,1] \setminus (\Theta \cup D)$. □

We will now introduce the time dependence also in the loading. We need to have that for each t the function $g(t) \in H^1(\Omega)$, since it seems natural that the function g would be in some Bochner-type space. We chose $g \in \mathrm{AC}\big([0,1]; H^1(\Omega)\big)$, since it implies:

$$\begin{cases} \dot g \in L^1\big([0,1]; H^1(\Omega)\big), \\ \nabla g \in \mathrm{AC}([0,1]; L^2(\Omega;\mathbb{R}^2)), \\ \nabla \dot g \in L^1\big([0,1]; L^2(\Omega;\mathbb{R}^2)\big); \end{cases} \implies \begin{cases} g(t) \in H^1(\Omega), \\ \dot g(t) \in H^1(\Omega), \\ \nabla g(t) \in L^2(\Omega;\mathbb{R}^2), \\ \nabla \dot g(t) \in L^2(\Omega;\mathbb{R}^2); \end{cases} \implies \begin{cases} \|\dot g\|_{H^1(\Omega)} \in L^1([0,1]), \\ \|\nabla g\|_{L^2(\Omega;\mathbb{R}^2)} \in \mathrm{AC}([0,1]), \\ \|\nabla \dot g\|_{L^2(\Omega;\mathbb{R}^2)} \in L^1([0,1]). \end{cases}$$

We will keep these in mind for the next proofs. We will now prove three lemmas which will be used in the proof of the last result of this section.

Lemma 4.2.4. *Let $K \in \mathcal{K}^f(\overline{\Omega})$ and define $F \colon H^1(\Omega) \to \mathbb{R}$ such that $F(g) \coloneqq E(g,K)$ for every $g \in H^1(\Omega)$. Then F is continuously Gateaux-differentiable and*

$$\mathrm{d}F(g)h = \mu \int_{\Omega\setminus K} \nabla u_g^\top \nabla h \,\mathrm{d}x \qquad \forall g, h \in H^1(\Omega),$$

where u_g is the solution to the minimum problem defining $E(g,K)$.

Proof. Choose an $\varepsilon > 0$. Due to the linearity of the application $g \mapsto \nabla u_g$ proved in Section 3.3, we can consider $g + \varepsilon h$ as a loading, obtaining

$$\nabla u_{g+\varepsilon h} = \nabla u_g + \nabla u_{\varepsilon h} = \nabla u_g + \varepsilon \nabla u_h.$$

We can now write the first variation of the F as

$$\begin{aligned} F(g+\varepsilon h) - F(g) &= \frac{\mu}{2} \int_{\Omega\setminus K} |\nabla u_g + \varepsilon \nabla u_h|^2 \,\mathrm{d}x - \frac{\mu}{2} \int_{\Omega\setminus K} |\nabla u_g|^2 \,\mathrm{d}x \\ &= \mu\varepsilon \int_{\Omega\setminus K} \nabla u_g^\top \nabla u_h \,\mathrm{d}x + \varepsilon^2 \frac{\mu}{2} \int_{\Omega\setminus K} |\nabla u_h|^2 \,\mathrm{d}x \\ &= \mu\varepsilon \int_{\Omega\setminus K} \nabla u_g^\top \nabla h \,\mathrm{d}x + \varepsilon^2 \frac{\mu}{2} \int_{\Omega\setminus K} |\nabla u_h|^2 \,\mathrm{d}x, \end{aligned}$$

where the last equality follows from (3.8), taking $z = u_h - h \in L^{1,2}_{0,\partial_D\Omega\setminus K}(\Omega\setminus K)$. Dividing by ε and taking the limit as $\varepsilon \to 0$, we get that

$$\mathrm{d}F(g)h = \lim_{\varepsilon\to 0} \frac{F(g+\varepsilon h) - F(g)}{\varepsilon} = \mu \int_{\Omega\setminus K} \nabla u_g^\top \nabla h \,\mathrm{d}x.$$

Since this limit exists for every $h \in H^1(\Omega)$, we can then conclude that F is Gateaux-differentiable at g. Moreover, we know that the application $g \mapsto \nabla u_g$ is also bounded, since $\|\nabla u_g\| \le \|g\|$, hence this implies that it is also continuous. This implies that $\mathrm{d}F(g)h$ is also continuous, hence F is a C^1 function. □

Lemma 4.2.5. *Let $m \ge 1$ and $\lambda \ge 0$. Let $K\colon [0,1] \to \mathcal{K}^\lambda_m(\overline{\Omega})$ be a non-decreasing compact-valued function. We define the function in two variables $F\colon H^1(\Omega)\times[0,1]\to\mathbb{R}$ as $F(g,t) \coloneqq E\big(g, K(t)\big)$. Then the differential $\mathrm{d}_g F$ with respect to g is continuous at every point $(g,t) \in H^1(\Omega)\times[0,1]$ such that if $t_n \to t$ then $K(t_n) \to K(t)$ in the Hausdorff metric.*

Proof. From previous lemma we know that, keeping $K(t)$ fixed, we have that

$$\mathrm{d}_g F(g,t)h = \mu \int_{\Omega\setminus K(t)} \nabla u_g^\top \nabla h \,\mathrm{d}x,$$

and that $\mathrm{d}_g F(g,t)$ is continuous in g. We need to check whether this is continuous in t as well. If $t_n \to t$, from 4.1.8, taking $K_n = K(t_n)$ and $K = K(t)$, $g_n = g$ for every n, we deduce that $\nabla u_{g,n} \to \nabla u_g$ strongly in $L^2(\Omega;\mathbb{R}^2)$.
We can then conclude that

$$\mathrm{d}_g F(g,t_n)h = \mu \int_{\Omega\setminus K(t_n)} \nabla u_{g,n}^\top \nabla h \,\mathrm{d}x \to \mu \int_{\Omega\setminus K(t)} \nabla u_g^\top \nabla h \,\mathrm{d}x = \mathrm{d}_g F(g,t)h.$$

Then we know that it is also continuous in t, whenever $\exists(t_n)$ with $t_n \to t$ such that $K(t_n) \to K(t)$. □

Lemma 4.2.6. *Let X be a Hilbert space, and let $F\colon X\times[0,1]\to\mathbb{R}$ be a function such that $F(\cdot,t)\in C^1(X)$ for every $t\in[0,1]$. Fixing $t_0\in[0,1]$ and $g\in \mathrm{AC}\big([0,1];X\big)$, we define $\psi(t) \coloneqq F(g(t),t)$ and $\psi_0(t) \coloneqq F(g(t_0),t)$. We assume that t_0 is a differentiable point for ψ and g, and also a Lebesgue point for $\dot g$. We also assume that the first differential $\mathrm{d}_g F$ is continuous as $(g(t_0),t_0)$. Then ψ_0 is differentiable at t_0, and*

$$\dot\psi_0(t_0) = \dot\psi(t_0) - \mathrm{d}_g F(g(t_0),t_0)\dot g(t_0).$$

Proof. Let us keep in mind that $\psi_0(t_0) = F(g(t_0),t_0) = \psi(t_0)$. Then we can write:

$$\begin{aligned}
\psi_0(t) - \psi_0(t_0) &= F(g(t_0),t) - \psi(t_0) + \psi(t) - F(g(t),t)\\
&= F(g(t_0),t) - F(g(t),t) + \psi(t) - \psi(t_0)\\
&= F(g(s),t)\Big|_{s=t}^{s=t_0} + \psi(t) - \psi(t_0)\\
&= \int_t^{t_0} \mathrm{d}_g F(g(s),t)\dot g(s)\,\mathrm{d}s + \psi(t) - \psi(t_0).
\end{aligned}$$

Dividing now by $t - t_0$ we get

$$\frac{\psi_0(t) - \psi_0(t_0)}{t - t_0} = -\fint_{t_0}^{t} \mathrm{d}_g F(g(s), t)\dot{g}(s)\,\mathrm{d}s + \frac{\psi(t) - \psi(t_0)}{t - t_0}.$$

Taking now the limit as $t \to t_0$, due to the assumptions on t_0 and on the continuity of $\mathrm{d}_g F$, we can then deduce

$$\dot{\psi}_0(t_0) = \lim_{t \to t_0} \frac{\psi_0(t) - \psi_0(t_0)}{t - t_0} = -\mathrm{d}_g F(g(t_0), t_0)\dot{g}(t_0) + \dot{\psi}(t_0).$$

The quantities on the right side are finite, hence ψ_0 is differentiable at t_0. □

We will now prove the last result of this section, which proves an equivalence used in the next section to study the stability of the continuous evolution.

Theorem 4.2.7. *Let $m \geq 1$, and $g \in \mathrm{AC}\big([0,1], H^1(\Omega)\big)$ an absolutely continuous loading, and $K\colon [0,1] \to \mathcal{K}^f_m(\overline{\Omega})$ be a non-decreasing compact-valued function. We also assume that the function $t \mapsto E(g(t), K(t))$ is in $\mathrm{AC}\big([0,1]\big)$. Then the following are equivalent:*

1. $\dfrac{\mathrm{d}}{\mathrm{d}s} E(g(t), K(s))\Big|_{s=t} = 0$ *for a.e.* $t \in [0,1]$;
2. $\dfrac{\mathrm{d}}{\mathrm{d}t} E(g(t), K(t)) = \mu \left\langle \nabla u(t), \nabla \dot{g}(t) \right\rangle_{L^2(\Omega;\mathbb{R}^2)}$ *for a.e.* $t \in [0,1]$;

Here $u(t)$ is a solution to the minimum problem defining $E(g(t), K(t))$.

Proof. We define a function $F\colon H^1(\Omega) \times [0,1] \to \mathbb{R}$ as $F(g,t) \coloneqq E(g, K(t))$. By 4.2.1 we know that, defining $\Theta = \big\{t \in [0,1] : K^-(t) = K^+(t)\big\}$, then the set $[0,1] \setminus \Theta$ is at most countable, hence of zero Lebesgue measure. Moreover, for every $t \in \Theta$, there exists $(t_n) \subset [0,1]$ with $t_n \to t$, such that $K(t_n) \to K(t)$ in the Hausdorff metric. We can now apply Lemma 4.2.5 to deduce that $\mathrm{d}_g F$ is continuous in (g,t) for every $g \in H^1(\Omega)$ in for a.e. $t \in [0,1]$ (since Θ has zero measure).
From Lemma 4.2.4, we know that, fixing $K(t)$, then for every $g, h \in H^1(\Omega$ we have

$$\mathrm{d}_g F(g,t)h = \mu \int_{\Omega \setminus K(t)} \nabla u_g^\top \nabla h \,\mathrm{d}x \qquad \forall g, h \in H^1(\Omega), \text{ for a.e. } t \in [0,1].$$

But, for what was discussed before about absolutely continuous functions, we know that both $g(t)$ and $\dot{g}(t)$ are in $H^1(\Omega)$, hence we take $g = g(t)$ and $h = \dot{g}(t)$, such that:

$$\mathrm{d}_g F(g(t), t)\dot{g}(t) = \mu \int_{\Omega \setminus K(t)} \nabla u_{g(t)}^\top \nabla \dot{g}(t)\,\mathrm{d}x = \mu \left\langle \nabla u(t), \nabla \dot{g}(t) \right\rangle_{L^2(\Omega;\mathbb{R}^2)} \qquad \text{for a.e. } t \in [0,1].$$

Then, from 4.2.6, if we define

$$\psi(t) = F(g(t), t) = E(g(t), K(t)), \qquad \psi_0(t) = F(g(t_0), t) = E(g(t_0), K(t)),$$

we can then deduce:

$$\frac{\mathrm{d}}{\mathrm{d}s} E(g(t_0), K(s))\Big|_{s=t_0} = -\mu \left\langle \nabla u(t_0), \nabla \dot{g}(t_0) \right\rangle_{L^2(\Omega;\mathbb{R}^2)} + \frac{\mathrm{d}}{\mathrm{d}t} E(g(t), K(t))\Big|_{t=t_0}.$$

This will be valid for a.e. $t \in [0,1]$, hence the theorem. □

4.3 Existence for irreversible quasi-static evolution

As outlined in the introduction to the previous chapter, the existence result for the continuous evolution of the crack set will be obtained through a time discretization. For clarity we will repeat the statement of the result we wish to prove, with a slight change at point 3):

Theorem 4.3.1. *Let $g \in \mathrm{AC}([0,1]; H^1(\Omega))$ and $K_0 \in \mathcal{K}^f_m(\overline{\Omega})$. Then there exists a continuous evolution $K\colon [0,1] \to \mathcal{K}^f_m(\overline{\Omega})$ such that:*

1. *$K_0 \subseteq K(s) \subseteq K(t)$ for $0 \le s < t \le 1$;*
2. *$E(g(0), K(0)) \le E(g(0), K) \quad \forall K \in \mathcal{K}^f_m(\overline{\Omega}), K \supseteq K_0$;*
3. *for $t \in (0,1]$, $E(g(t), K(t)) \le E(g(t), K) \quad \forall K \in \mathcal{K}^f_m(\overline{\Omega}), K \supseteq K(t)$;*
4. *the function $t \mapsto E(g(t), K(t))$ is absolutely continuous on $[0,1]$;*
5. *$\dfrac{\mathrm{d}}{\mathrm{d}s} E(g(t), K(s))\Big|_{s=t} = 0$ for a.e. $t \in [0,1]$.*

Moreover, if every function $K\colon [0,1] \to \mathcal{K}^f_m(\overline{\Omega})$ satisfies all these conditions, then it also satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} E(g(t), K(t)) = \mu \int_{\Omega \setminus K(t)} \nabla u(t)^\top \nabla \dot{g}(t)\, \mathrm{d}x \quad \text{for a.e. } t \in [0,1],$$

where $u(t)$ is a solution of the minimum problem defining $E(g(t), K(t))$.

The main idea is to compute the minimum not instantaneously, as in the quasi-static setting, but at every successive time step, and then look for a convergence of some kind as the time step goes to 0. Let us introduce then some notation: let us take $\delta > 0$, and define N_δ as the biggest integer such that $\delta N_\delta \le 1$. Then we define the discretized times as $t^\delta_i = i\delta$, and the discretized loadings as $g^\delta_i = g(t^\delta_i)$, for every $0 \le i \le N_\delta$. We will define K^δ_i inductively, as the minimum of the evolution restricted to the discretized times and loadings, as:

$$K^\delta_i = \operatorname*{argmin}_K \left\{ E\left(g^\delta_i, K\right) : K \in \mathcal{K}^f_m(\overline{\Omega}), K \supseteq K^\delta_{i-1} \right\}, \qquad K^\delta_{-1} = K_0. \tag{4.3}$$

The first thing we need to make sure of is that this restricted evolution problem actually has a solution.

Lemma 4.3.2. *The minimum problem 4.3 has a solution.*

Proof. We know already that $K^\delta_{-1} = K_0 \in \mathcal{K}^f_m(\overline{\Omega})$. We assume by induction that $K^\delta_{i-1} \in \mathcal{K}^f_m(\overline{\Omega})$ is a minimum for $E(g^\delta_{i-1}, K)$. Then, since K^δ_{i-1} is admissible for the minimum of $E(g^\delta_i, K)$, there will be $\lambda > 0$ such that $E(g^\delta_i, K^\delta_{i-1}) < \lambda$. Let us take (K_n) as a minimizing sequence for this problem since we are sure that such a sequence always exists. Due to the upper bound on the energy that we just deduced, we can assume $K_n \in \mathcal{K}^\lambda_m(\overline{\Omega})$ for n sufficiently large. Since $K_n \supseteq K^\delta_{i-1}$ for every n, then due to the compactness of the

Hausdorff space, there exists $K^* \supseteq K_{i-1}^\delta$ such that $K_n \to K^*$ in the Hausdorff metric. If we take now u_n as the minimum of the problem defining $E(g_i^\delta, K_n)$, by Theorem 4.1.8 we can deduce that $\nabla u_n \to \nabla u^*$ strongly in $L^2(\Omega;\mathbb{R}^2)$, where u^* is the minimum of the problem defining $E(g_i^\delta, K^*)$. Moreover, from Corollary 3.2.8, we deduce that $K^* \in \mathcal{K}_m(\overline{\Omega})$, and also that

$$\mathcal{H}^1(K^*) \leq \liminf_{n\to\infty} \mathcal{H}^1(K_n) \leq \lambda \quad \Longrightarrow \quad K^* \in \mathcal{K}_m^\lambda(\overline{\Omega}) \subset \mathcal{K}_m^f(\overline{\Omega}). \tag{4.4}$$

Since $\nabla u_n \to \nabla u^*$ strongly, then $\|\nabla u_n\| \to \|\nabla u^*\|$, which, together with (4.4), implies that

$$E(g_i^\delta, K^*) \leq \liminf_{n\to\infty} E(g_i^\delta, K_n).$$

Since (K_n) is a minimizing sequence, it then follows that K^* is a solution to the minimum problem. □

Once we know that the restricted evolution has a solution, we can define some step function analogous to the restricted problems, namely we define g_δ, K_δ and u_δ as step function on [0,1] by setting;

$$g_\delta(t) \coloneqq g_i^\delta, \qquad K_\delta(t) \coloneqq K_i^\delta, \qquad u_\delta(t) \coloneqq u_i^\delta, \qquad t \in \left[t_i^\delta, t_{i+1}^\delta\right)$$

where u_i^δ is a solution to the minimum problem defining $E(g_i^\delta, K_i^\delta)$.
We will now give an estimate of the energy at a time step with respect to a previous time step, which will be useful later in obtaining an upper bound for the norms of the step functions defined above.

Lemma 4.3.3. *There exists a positive function $\rho(\delta)$, with $\rho(\delta) \to 0^+$ as $\delta \to 0^+$, such that:*

$$\frac{\mu}{2}\left\|\nabla u_j^\delta\right\|^2 + \gamma\,\mathcal{H}^1\left(K_j^\delta\right) \leq \frac{\mu}{2}\left\|\nabla u_i^\delta\right\|^2 + \gamma\,\mathcal{H}^1\left(K_i^\delta\right) + \mu\int_{t_i^\delta}^{t_j^\delta}\left\langle \nabla u_\delta(t), \nabla \dot{g}(t)\right\rangle \mathrm{d}t + \rho(\delta),$$

for each $0 \leq i < j \leq N_\delta$.

Proof. Just to remember the framework we are in, we know that

$$g_i^\delta \in H^1(\Omega), \qquad K_i^\delta \in \mathcal{K}_m^f(\overline{\Omega}), \qquad u_i^\delta \in L^{1,2}(\Omega \setminus K_i^\delta) \quad u_i^\delta = g_i^\delta \text{ q.e. on } \partial_D\Omega \setminus K_i^\delta.$$

Let us take now an integer r such that $i \leq r < j$. By the absolute continuity of g, we know that

$$g_{r+1}^\delta - g_r^\delta = \int_{t_r^\delta}^{t_{r+1}^\delta} \dot{g}(t)\,\mathrm{d}t \quad \Longrightarrow \quad \nabla g_{r+1}^\delta - \nabla g_r^\delta = \int_{t_r^\delta}^{t_{r+1}^\delta} \nabla\dot{g}(t)\,\mathrm{d}t,$$

where the first and the second integrals are Bochner integrals, respectively on $H^1(\Omega)$ and on $L^2(\Omega;\mathbb{R}^2)$.
Let us look now at the function $u_r^\delta + g_{r+1}^\delta - g_r^\delta$: this is a sum of functions in $L^{1,2}(\Omega \setminus K_r^\delta)$

and in $H^1(\Omega)$, hence it is in $L^{1,2}(\Omega\setminus K_r^\delta)$. Moreover $u_r^\delta+g_{r+1}^\delta-g_r^\delta=g_{r+1}^\delta$ q.e. on $\partial_D\Omega\setminus K_r^\delta$, hence it is an admissible function for the minimum defining $E\left(g_{r+1}^\delta,K_r^\delta\right)$, hence:

$$E\left(g_{r+1}^\delta,K_r^\delta\right)\leq\frac{\mu}{2}\left\|\nabla u_r^\delta+\nabla g_{r+1}^\delta-\nabla g_r^\delta\right\|^2+\gamma\,\mathcal{H}^1\left(K_r^\delta\right).$$

Now, since u_{r+1}^δ is a minimum for the problem defining $E\left(g_{r+1}^\delta,K_{r+1}^\delta\right)$, and $K_{r+1}^\delta\subseteq K_r^\delta$, we have that:

$$\frac{\mu}{2}\left\|\nabla u_{r+1}^\delta\right\|^2+\gamma\,\mathcal{H}^1\left(K_{r+1}^\delta\right)=E\left(g_{r+1}^\delta,K_{r+1}^\delta\right)\leq E\left(g_{r+1}^\delta,K_r^\delta\right).$$

Combining these two relations we get:

$$\begin{aligned}
&\frac{\mu}{2}\left\|\nabla u_{r+1}^\delta\right\|^2+\gamma\,\mathcal{H}^1\left(K_{r+1}^\delta\right)\\
\leq&\frac{\mu}{2}\left\|\nabla u_r^\delta+\nabla g_{r+1}^\delta-\nabla g_r^\delta\right\|^2+\gamma\,\mathcal{H}^1\left(K_r^\delta\right)\\
\leq&\frac{\mu}{2}\left\|\nabla u_r^\delta\right\|^2+\mu\left\langle\nabla u_r^\delta,\nabla g_{r+1}^\delta-\nabla g_r^\delta\right\rangle+\frac{\mu}{2}\left\|\nabla g_{r+1}^\delta-\nabla g_r^\delta\right\|^2+\gamma\,\mathcal{H}^1\left(K_r^\delta\right)\\
\leq&\frac{\mu}{2}\left\|\nabla u_r^\delta\right\|^2+\mu\int_{t_r^\delta}^{t_{r+1}^\delta}\left\langle\nabla u_r^\delta,\nabla\dot g(t)\right\rangle\,\mathrm{d}t+\frac{\mu}{2}\left(\int_{t_r^\delta}^{t_{r+1}^\delta}\|\nabla\dot g(t)\|\,\mathrm{d}t\right)^2+\gamma\,\mathcal{H}^1\left(K_r^\delta\right).
\end{aligned}$$

We know that $u_r^\delta=u_\delta(t)$ for $t\in[t_r^\delta,t_{r+1}^\delta)$, and also that:

$$\int_{t_r^\delta}^{t_{r+1}^\delta}\|\nabla\dot g(t)\|\,\mathrm{d}t\leq\max_{r=0,1,\dots,N_\delta-1}\int_{t_r^\delta}^{t_{r+1}^\delta}\|\nabla\dot g(t)\|\,\mathrm{d}t=:\sigma(\delta).$$

Due to the absolute continuity of the integral, if $\delta\to0$, then also $\sigma(\delta)\to0$. Hence we get:

$$\begin{aligned}
&\frac{\mu}{2}\left\|\nabla u_{r+1}^\delta\right\|^2+\gamma\,\mathcal{H}^1\left(K_{r+1}^\delta\right)\\
\leq&\frac{\mu}{2}\left\|\nabla u_r^\delta\right\|^2+\gamma\,\mathcal{H}^1\left(K_r^\delta\right)+\mu\int_{t_r^\delta}^{t_{r+1}^\delta}\left\langle\nabla u_\delta(t),\nabla\dot g(t)\right\rangle\,\mathrm{d}t+\frac{\mu}{2}\sigma(\delta)\int_{t_r^\delta}^{t_{r+1}^\delta}\|\nabla\dot g(t)\|\,\mathrm{d}t.
\end{aligned}$$

Iterating this inequality for $i\leq r<j$ we get:

$$\begin{aligned}
&\frac{\mu}{2}\left\|\nabla u_j^\delta\right\|^2+\gamma\,\mathcal{H}^1\left(K_j^\delta\right)\\
\leq&\frac{\mu}{2}\left\|\nabla u_i^\delta\right\|^2+\gamma\,\mathcal{H}^1\left(K_i^\delta\right)+\mu\int_{t_i^\delta}^{t_j^\delta}\left\langle\nabla u_\delta(t),\nabla\dot g(t)\right\rangle\,\mathrm{d}t+\frac{\mu}{2}\sigma(\delta)\int_{t_i^\delta}^{t_j^\delta}\|\nabla\dot g(t)\|\,\mathrm{d}t\\
\leq&\frac{\mu}{2}\left\|\nabla u_i^\delta\right\|^2+\gamma\,\mathcal{H}^1\left(K_i^\delta\right)+\mu\int_{t_i^\delta}^{t_j^\delta}\left\langle\nabla u_\delta(t),\nabla\dot g(t)\right\rangle\,\mathrm{d}t+\frac{\mu}{2}\sigma(\delta)\int_0^1\|\nabla\dot g(t)\|\,\mathrm{d}t,
\end{aligned}$$

which is the relation we are looking for, where $\rho(\delta)=\frac{\mu}{2}\sigma(\delta)\int_0^1\|\nabla\dot g(t)\|\,\mathrm{d}t$. □

Lemma 4.3.4. *There exists a positive constant $\lambda \in \mathbb{R}$, depending on g and K_0, such that:*

$$\left\|\nabla u_i^\delta\right\| \leq \lambda \qquad \mathcal{H}^1\left(K_i^\delta\right) \leq \lambda, \tag{4.5}$$

for every $\delta > 0$ and every $i = 0, 1, \ldots, N_\delta$.

Proof. Since $g_i^\delta \in H^1(\Omega) \subset L^{1,2}(\Omega \setminus K_i^\delta)$, we have that $v = g_i^\delta$ is an admissible function for the minimum problem defining $E(g_i^\delta, K_i^\delta)$, whose minimum is u_i^δ. Then by the equivalence to 3.8 we know that $\|\nabla u_i^\delta\| \leq \|\nabla g_i^\delta\|$ for every i, hence $\|\nabla u_\delta(t)\| \leq \|\nabla g_\delta(t)\|$ for every t. Since $g \in \mathrm{AC}\left([0,1]; H^1(\Omega)\right)$, we know that $\|\nabla \dot{g}(t)\| \in L^1\left([0,1]\right)$, and that $\nabla g(t) \in L^2(\Omega; \mathbb{R}^2)$, hence there exists a positive constant C_1 depending only on g such that $\|\nabla g(t)\| \leq C_1$ for every t. Then we can conclude with the first upper bound, since if $t = t_i^\delta$, then we have

$$\left\|\nabla u_i^\delta\right\| \leq \left\|\nabla g_i^\delta\right\| = \left\|\nabla g(t_i^\delta)\right\| \leq C_1.$$

Looking now at (4.3) for $i = 0$, we can say that since K_0^δ is the minimum, and since $\|\nabla u_0\| \leq \|\nabla g(0)\|$, we know:

$$E\left(g(0), K_0^\delta\right) \leq E\left(g(0), K_0\right) \implies \frac{\mu}{2}\left\|\nabla u_0^\delta\right\|^2 + \gamma\,\mathcal{H}^1\left(K_0^\delta\right) \leq \frac{\mu}{2}\left\|\nabla g(0)\right\|^2 + \gamma\,\mathcal{H}^1\left(K_0\right).$$

Using now Lemma 4.3.3 we can write:

$$\begin{aligned}\frac{\mu}{2}\left\|\nabla u_i^\delta\right\|^2 + \gamma\,\mathcal{H}^1\left(K_i^\delta\right) &\leq \frac{\mu}{2}\left\|\nabla u_0^\delta\right\|^2 + \gamma\,\mathcal{H}^1\left(K_0^\delta\right) + \mu\int_{t_0^\delta}^{t_i^\delta}\left\langle\nabla u_\delta(t), \nabla\dot{g}(t)\right\rangle \mathrm{d}t + \rho\left(\delta\right) \\ &\leq \frac{\mu}{2}\left\|\nabla g(0)\right\|^2 + \gamma\,\mathcal{H}^1\left(K_0\right) + \mu\int_0^{t_i^\delta}\left\langle\nabla u_\delta(t), \nabla\dot{g}(t)\right\rangle \mathrm{d}t + \rho\left(\delta\right),\end{aligned}$$

From this it follows that there exists a positive constant C_2 depending on g, K_0 and δ such that

$$\mathcal{H}^1\left(K_i^\delta\right) \leq \frac{1}{\gamma}\left(\frac{\mu}{2}\left\|\nabla g(0)\right\|^2 + \gamma\,\mathcal{H}^1\left(K_0\right) + \mu\int_0^{t_i^\delta}\left\langle\nabla u_\delta(t), \nabla\dot{g}(t)\right\rangle \mathrm{d}t + \rho\left(\delta\right)\right) \leq C_2.$$

To conclude we may take $\lambda = \max\left\{C_1, C_2\right\}$. □

One of the main results of this section, and one that is going to be used frequently in the next results, is the following:

Theorem 4.3.5. *Take λ as the constant of Lemma 4.3.4. Then there exists a non-decreasing compact valued function $K\colon [0,1] \to \mathcal{K}_m^\lambda(\overline{\Omega})$ such that $K_\delta(t) \to K(t)$ in the Hausdorff metric as $\delta \to 0$, along a sequence independent of t.*

Proof. From Theorem 4.2.3, we know that there exists a sub-sequence independent of t, and a non-decreasing compact valued function $K\colon [0,1] \to \mathcal{K}(\overline{\Omega})$ such that for every $t \in [0,1]$, $K_\delta(t) \to K(t)$ as $\delta \to 0$. By Lemma 4.3.4 we know that $\mathcal{H}^1\left(K_i^\delta\right) \leq \lambda$, hence for every $t \in [0,1]$ we have $\mathcal{H}^1\left(K_\delta(t)\right) \leq \lambda$. Then we can conclude by Corollary 3.2.8 that $K(t) \in \mathcal{K}_m(\overline{\Omega})$ and also that:

$$\mathcal{H}^1\left(K(t)\right) \leq \liminf_{n\to\infty} \mathcal{H}^1\left(K_\delta(t)\right) \leq \lambda.$$

Thus $K(t) \in \mathcal{K}_m^\lambda(\overline{\Omega})$. □

Note that we just proved point 1) of Theorem 4.3.1. From now on, we will use the fact that the limit is a non-decreasing compact-valued function, and the limit will be always taken along the sub-sequence independent of t.

Lemma 4.3.6. *For every $t \in [0,1]$ we have that $\nabla u_\delta(t) \to \nabla u(t)$ in $L^2(\Omega;\mathbb{R}^2)$, where $u(t)$ is the solution to the minimum defining $E(g(t), K(t))$.*

Proof. Let us fix a $t \in [0,1]$, and let $u_\delta(t)$ be the solution to the minimum defining $E(g_\delta(t), K_\delta(t))$. Since we know that $g_\delta(t) \to g(t)$ in $H^1(\Omega)$ and $K_\delta(t) \to K(t)$ in the Hausdorff metric, by Theorem 4.1.8 we can conclude that $\nabla u_\delta(t) \to \nabla u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$ for every $t \in [0,1]$. □

We will now prove points 2) and 3) the 4.3.1.

Theorem 4.3.7. *For every $t \in [0,1]$, we have:*

$$E\left(g(0), K(0)\right) \leq E\left(g(0), K\right), \qquad \forall K \in \mathcal{K}_m^f(\overline{\Omega}), K \supseteq K_0; \tag{4.6}$$

$$E\left(g(t), K(t)\right) \leq E\left(g(t), K\right), \qquad \forall K \in \mathcal{K}_m^f(\overline{\Omega}), K \supseteq K(t). \tag{4.7}$$

Proof. For the first point let us fix $K \in \mathcal{K}_m^f(\overline{\Omega}) : K \supseteq K_0$. By the minimality of 4.3, we have:

$$E\left(g_\delta(0), K_\delta(0)\right) \leq E\left(g_\delta(0), K\right) = E\left(g(0), K\right),$$

$$\frac{\mu}{2}\left\|\nabla u_\delta(0)\right\|^2 + \gamma\,\mathcal{H}^1\left(K_\delta(0)\right) \leq E\left(g(0), K\right).$$

From Corollary 3.2.8 we deduce that:

$$\begin{aligned}\mathcal{H}^1\left(K(0)\right) &\leq \liminf_{\delta\to 0} \mathcal{H}^1\left(K_\delta(0)\right)\\ &\leq \liminf_{\delta\to 0} \frac{1}{\gamma}\left(E\left(g(0), K\right) - \frac{\mu}{2}\left\|\nabla u_\delta(0)\right\|^2\right)\\ &\leq \frac{1}{\gamma}\left(E\left(g(0), K\right) - \frac{\mu}{2}\left\|\nabla u(0)\right\|^2\right),\end{aligned}$$

hence the conclusion follows:

$$E\left(g(0), K(0)\right) = \frac{\mu}{2}\left\|\nabla u(0)\right\|^2 + \gamma\,\mathcal{H}^1\left(K(0)\right) \leq E\left(g(0), K\right),$$

for whatever $K \in \mathcal{K}_m^f(\overline{\Omega}) : K \supseteq K_0$.
For the second point let us fix $t \in [0,1]$, and $K \in \mathcal{K}_m^f(\overline{\Omega}) : K \supseteq K(t)$. Since $K_\delta(t) \to K(t)$ in the Hausdorff metric, from Proposition 3.2.13, we can deduce that there exists a sequence $(K_\delta) \subset \mathcal{K}_m^f(\overline{\Omega})$ with $K_\delta \to K$ in the Hausdorff metric, such that

$$K_\delta(t) \subseteq K_\delta, \qquad \mathcal{H}^1(K_\delta \setminus K_\delta(t)) \to \mathcal{H}^1(K \setminus K(t)) \quad \delta \to 0.$$

Since K has finite $\mathcal{H}^1$-measure, then $\mathcal{H}^1(K_\delta)$ will be bounded for δ small enough.
Let us define now v_δ as the solution to the minimum problem defining $E\left(g_\delta(t), K_\delta\right)$, and v a solution to the minimum problem defining $E\left(g(t), K\right)$. Since $g_\delta(t) \to g(t)$ in $H^1(\Omega)$ and $K_\delta \to K$ in the Hausdorff metric, from 4.1.8 we know that $\nabla v_\delta \to \nabla v$ strongly in $L^2(\Omega;\mathbb{R}^2)$. Now, due to $K_\delta(t)$ being a solution to 4.3, we know

$$\begin{aligned} E\left(g_\delta(t), K_\delta(t)\right) &\leq E\left(g_\delta(t), K_\delta\right), \\ \frac{\mu}{2}\left\|\nabla u_\delta(t)\right\|^2 + \gamma\,\mathcal{H}^1\left(K_\delta(t)\right) &\leq \frac{\mu}{2}\left\|\nabla v_\delta\right\|^2 + \gamma\,\mathcal{H}^1\left(K_\delta\right), \\ \frac{\mu}{2}\left\|\nabla u_\delta(t)\right\|^2 &\leq \frac{\mu}{2}\left\|\nabla v_\delta\right\|^2 + \gamma\,\mathcal{H}^1\left(K_\delta \setminus K_\delta(t)\right). \end{aligned}$$

Taking now the limit as $\delta \to 0$, we easily get

$$\frac{\mu}{2}\left\|\nabla u(t)\right\|^2 \leq \frac{\mu}{2}\left\|\nabla v\right\|^2 + \gamma\,\mathcal{H}^1\left(K \setminus K(t)\right) \implies E\left(g(t), K(t)\right) \leq E\left(g(t), K\right),$$

for whatever $K \in \mathcal{K}_m^f(\overline{\Omega}) : K \supseteq K(t)$. □

To show that also 4) and 5) are satisfied, we need the following estimate, a consequence of Lemma 4.3.3:

Lemma 4.3.8. *For every $0 \leq s < t \leq 1$ we have:*

$$\frac{\mu}{2}\left\|\nabla u(t)\right\|^2 + \gamma\,\mathcal{H}^1\left(K(t)\right) \leq \frac{\mu}{2}\left\|\nabla u(s)\right\|^2 + \gamma\,\mathcal{H}^1\left(K(s)\right) + \mu\int_s^t \left\langle \nabla u(\tau), \nabla \dot{g}(\tau)\right\rangle \mathrm{d}\tau, \quad (4.8)$$

Proof. Fix some $0 \leq s < t \leq 1$, and choose δ small enough so that there exist $i \neq j$ such that $s \in [t_i^\delta, t_{i+1}^\delta)$ and $t \in [t_j^\delta, t_{j+1}^\delta)$. Define now $s_\delta \coloneqq t_i^\delta$ and $t_\delta \coloneqq t_j^\delta$. Obviously $s_\delta \neq t_\delta$, and by Lemma 4.3.3 we get:

$$\frac{\mu}{2}\left\|\nabla u_\delta(t)\right\|^2 + \gamma\,\mathcal{H}^1\left(K_\delta(t) \setminus K_\delta(s)\right) \leq \frac{\mu}{2}\left\|\nabla u_\delta(s)\right\|^2 + \mu\int_{s_\delta}^{t_\delta} \left\langle \nabla u_\delta(\tau), \nabla \dot{g}(\tau)\right\rangle \mathrm{d}\tau + \rho\left(\delta\right).$$

From Corollary 3.2.9 we know that

$$\mathcal{H}^1\left(K(t) \setminus K(s)\right) \leq \liminf_{\delta\to 0} \mathcal{H}^1\left(K_\delta(t) \setminus K_\delta(s)\right),$$

hence we can write

$$\begin{aligned} &\mathcal{H}^1\left(K(t) \setminus K(s)\right) \\ &\leq \liminf_{\delta\to 0} \mathcal{H}^1\left(K_\delta(t) \setminus K_\delta(s)\right) \\ &\leq \liminf_{\delta\to 0} \frac{1}{\gamma}\left[\frac{\mu}{2}\left\|\nabla u_\delta(s)\right\|^2 - \frac{\mu}{2}\left\|\nabla u_\delta(t)\right\|^2 + \mu\int_{s_\delta}^{t_\delta} \left\langle \nabla u_\delta(\tau), \nabla \dot{g}(\tau)\right\rangle \mathrm{d}\tau + \rho\left(\delta\right)\right]. \end{aligned}$$

Obviously $\rho(\delta) \to 0$ as $\delta \to 0$, and from Lemma 4.3.6 we know that $\nabla u_\delta(t) \to \nabla u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$ for every $t \in [0,1]$, and from Lemma 4.3.4 we get that $\|\nabla u_\delta(\tau)\| \leq \lambda$

for every $\tau \in [0,1]$. We just need to check if the integral converges: if $\delta \to 0$ then $s_\delta \to s$ and $t_\delta \to t$. We can then use the dominated convergence theorem by writing:

$$\left| \int_{s_\delta}^{t_\delta} \left\langle \nabla u_\delta(\tau), \nabla \dot{g}(\tau) \right\rangle \mathrm{d}\tau \right| \leq \int_s^t \mathbb{1}_{[s_\delta, t_\delta]} \left| \left\langle \nabla u_\delta(\tau), \nabla \dot{g}(\tau) \right\rangle \right| \mathrm{d}\tau.$$

Since

$$\mathbb{1}_{[s_\delta, t_\delta]} \left\langle \nabla u_\delta(\tau), \nabla \dot{g}(\tau) \right\rangle \longrightarrow \mathbb{1}_{[s,t]} \left\langle \nabla u(\tau), \nabla \dot{g}(\tau) \right\rangle \qquad \delta \to 0,$$

and

$$\mathbb{1}_{[s_\delta, t_\delta]} \left\langle \nabla u_\delta(\tau), \nabla \dot{g}(\tau) \right\rangle \leq \mathbb{1}_{[s_\delta, t_\delta]} \left\| \nabla u_\delta(\tau) \right\| \left\| \nabla \dot{g}(\tau) \right\| \in L^1(\Omega) \qquad \forall \tau \in [0,1],$$

we can then conclude:

$$\begin{aligned}
&\mathcal{H}^1 \left(K(t) \setminus K(s) \right) \\
&\leq \liminf_{\delta \to 0} \frac{1}{\gamma} \left[\frac{\mu}{2} \left\| \nabla u_\delta(s) \right\|^2 - \frac{\mu}{2} \left\| \nabla u_\delta(t) \right\|^2 + \mu \int_{s_\delta}^{t_\delta} \left\langle \nabla u_\delta(\tau), \nabla \dot{g}(\tau) \right\rangle \mathrm{d}\tau + \rho(\delta) \right] \\
&\leq \frac{1}{\gamma} \left(\frac{\mu}{2} \left\| \nabla u(s) \right\|^2 - \frac{\mu}{2} \left\| \nabla u(t) \right\|^2 + \mu \int_s^t \left\langle \nabla u(\tau), \nabla \dot{g}(\tau) \right\rangle \mathrm{d}\tau \right).
\end{aligned}$$

The conclusion now follows. □

Now we can conclude the proof of Theorem 4.3.1. We have to recall the definition of absolutely continuous function given in 2.2.4, 2.2.5, 2.2.6: these are all equivalent, but we will use 2.2.6.

Theorem 4.3.9. *The function $t \mapsto E(g(t), K(t))$ is in $\mathrm{AC}([0,1])$ and:*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(g(t), K(t)) = \mu \left\langle \nabla u(t), \nabla \dot{g}(t) \right\rangle_{L^2(\Omega; \mathbb{R}^2)} \qquad \text{for a.e. } t \in [0,1]. \tag{4.9}$$

Moreover:

$$\frac{\mathrm{d}}{\mathrm{d}s} E(g(t), K(s)) \Big|_{s=t} = 0 \qquad \text{for a.e. } t \in [0,1]. \tag{4.10}$$

Proof. Fix $0 \leq s < t \leq 1$, then from Lemma 4.3.8 we know that:

$$E(g(t), K(t)) - E(g(s), K(s)) \leq \mu \int_s^t \left\langle \nabla u(\tau), \nabla \dot{g}(\tau) \right\rangle \mathrm{d}\tau. \tag{4.11}$$

From Lemma 4.3.4 there exists a constant C such that $\|\nabla u(\tau)\| \leq C$ for every $\tau \in [0,1]$, hence:

$$\begin{aligned}
\left| E(g(t), K(t)) - E(g(s), K(s)) \right| &\leq \left| \mu \int_s^t \left\langle \nabla u(\tau), \nabla \dot{g}(\tau) \right\rangle \mathrm{d}\tau \right| \\
&\leq \mu \int_s^t \left| \left\langle \nabla u(\tau), \nabla \dot{g}(\tau) \right\rangle \right| \mathrm{d}\tau \\
&\leq \mu \int_s^t \left\| \nabla u(\tau) \right\| \left\| \nabla \dot{g}(\tau) \right\| \mathrm{d}\tau \\
&\leq \mu C \int_s^t \left\| \nabla \dot{g}(\tau) \right\| \mathrm{d}\tau.
\end{aligned}$$

Since $\|\nabla\dot{g}\| \in L^1([0,1])$, this is equivalent to the definition of an absolutely continuous function given in 2.2.6.
From Theorem 4.3.7 we know that

$$E\big(g(s),K(s)\big) \le E\big(g(s),K(t)\big), \tag{4.12}$$

since $K(t) \supseteq K(s)$, and from Lemma 4.2.4 with $K = K(t)$, and $h = \dot{g}(\tau)$ and integrating with respect to τ from s to t, we have

$$\begin{aligned} \mathrm{d}_g E\big(g(\tau),K(t)\big)\dot{g}(\tau) &= \mu\langle\nabla u_t(\tau),\nabla\dot{g}(\tau)\rangle, \\ E\big(g(t),K(t)\big) - E\big(g(s),K(t)\big) &= \mu\int_s^t \langle\nabla u_t(\tau),\nabla\dot{g}(\tau)\rangle\,\mathrm{d}\tau, \end{aligned} \tag{4.13}$$

where $u_t(\tau)$ is the solution to the minimum problem defining $E\big(g(\tau),K(t)\big)$. Combining (4.12) and (4.13) we get:

$$\begin{aligned} E\big(g(t),K(t)\big) - E\big(g(s),K(s)\big) &\ge E\big(g(t),K(t)\big) - E\big(g(s),K(t)\big) \\ &= \mu\int_s^t \langle\nabla u_t(\tau),\nabla\dot{g}(\tau)\rangle\,\mathrm{d}\tau. \end{aligned} \tag{4.14}$$

Notice that (4.11) and (4.14) are quite similar, but they differ in $\nabla u(\tau)$ and $\nabla u_t(\tau)$. Let us observe though that from Theorem 4.1.8 we know that $\nabla u_t(\tau) \to \nabla u(\tau)$ as $\tau \to t$, since $g(\tau) \to g(t)$ strongly in $H^1(\Omega)$ as $\tau \to t$.
From (4.11), we can divide by $t - s$ and take the limit as $s \to t^-$, and we will obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}E\left(g(t),K(t)\right) \le \mu\lim_{s\to t^-}\fint_s^t \langle\nabla u(\tau),\nabla\dot{g}(\tau)\rangle\,\mathrm{d}\tau = \mu\langle\nabla u(t),\nabla\dot{g}(t)\rangle.$$

Analogously from (4.14):

$$\frac{\mathrm{d}}{\mathrm{d}t}E\left(g(t),K(t)\right) \ge \mu\lim_{s\to t^-}\fint_s^t \langle\nabla u_t(\tau),\nabla\dot{g}(\tau)\rangle\,\mathrm{d}\tau = \mu\langle\nabla u(t),\nabla\dot{g}(t)\rangle.$$

Combining these last two relations we get (4.9).
To conclude, we just need to apply the equivalence of Theorem 4.2.7 to obtain (4.10). □

Hence the existence result 4.3.1 is proved. As we can see the assumptions taken are quite natural, like the absolute continuity of the loading, which in most applications could also be taken Lipschitz or even continuously differentiable. Even if the assumption on the finite number of connected components of the crack set may seem quite technical, it feels quite reasonable given the observed structure of a crack propagating through a real material. Moreover, if this hypothesis is dropped, the convergence of $K_\delta(t) \to K(t)$ in the Hausdorff metric does not imply the convergence of the solutions to the respective minimum problems. If the number of holes or cracks in the domain is allowed to be infinite, the problem is known in the literature as *Neumann sieve* (see [31]).
One can see that the third point of 4.3.1 has a different formulation from 4.0.1, but actually, it is a consequence:

Theorem 4.3.10. *Let $K\colon [0,1] \to \mathcal{K}^f_m(\overline{\Omega})$ a non-decreasing compact-valued function satisfying the hypothesis of Theorem 4.3.1. Then:*

$$E\left(g(t), K(t)\right) \leq E\left(g(t), K\right) \qquad \forall K \in \mathcal{K}^f_m(\overline{\Omega}) : K \supseteq \bigcup_{s<t} K(s) \qquad \forall t \in (0,1]. \tag{4.15}$$

Proof. Fix $t \in (0,1]$ and an arbitrary $K \in \mathcal{K}^f_m(\overline{\Omega})$ such that $K \supseteq \bigcup_{s<t} K(s)$.
If $0 \leq s < t$ then $K \supseteq K(s)$, hence from third point of 4.3.1 we deduce:

$$E\left(g(s), K(s)\right) \leq E\left(g(s), K\right).$$

We know that the functions $t \mapsto E\left(g(t), K(t)\right)$ and $t \mapsto E\left(g(t), K\right)$ are absolutely continuous on a compact interval, hence they are continuous: taking then the limit as $s \to t^-$ we get:

$$E\left(g(t), K(t)\right) \leq E\left(g(t), K\right),$$

for whatever $K \in \mathcal{K}^f_m(\overline{\Omega})$ such that $K \supseteq \bigcup_{s<t} K(s)$. □

Observation 4.3.11. *We can easily prove another proposition of Francfort and Marigo, namely that if $g(0) = 0$ then $K(0) = K_0$.*
If $g(0) = 0$, then by the second point in 4.3.1 we have:

$$\begin{aligned}
E\left(0, K(0)\right) \leq E\left(0, K\right) &\qquad \forall K \in \mathcal{K}^f_m(\overline{\Omega}) : K \supseteq K_0,\\
\mathcal{H}^1\big(K(0) \setminus K\big) \leq 0 &\qquad \forall K \in \mathcal{K}^f_m(\overline{\Omega}) : K \supseteq K_0,\\
\max_K \mathcal{H}^1\big(K(0) \setminus K\big) \leq 0 &\qquad \forall K \in \mathcal{K}^f_m(\overline{\Omega}) : K \supseteq K_0,\\
\mathcal{H}^1\big(K(0) \setminus K_0\big) \leq 0 \implies \mathcal{H}^1\big(K(0) \setminus K_0\big) = 0.&
\end{aligned}$$

This implies that $K(0) = K_0$ up to sets of zero $\mathcal{H}^1$-measure, which we have already seen to be removable *for this model.*

In their model, Francfort and Marigo focused on a special class of loadings, called MIL: Monotone Increasing Loadings. They formally asserted that under MIL loadings, Proposition 2.1.1 implied Proposition 2.1.2. We will prove this result rigorously.

Proposition 4.3.12. *Let $K\colon [0,1] \to \mathcal{K}^f_m(\overline{\Omega})$ a non-decreasing compact-valued function satisfying the hypothesis of Theorem 4.3.1, and let g be an absolutely continuous function on $[0,1]$ with values in $H^1(\Omega)$ with the particular form $g(t) = \varphi(t)g_0$, where $\varphi \in \mathrm{AC}([0,1])$ non-decreasing and non-negative, and $g_0 \in H^1(\Omega)$ a fixed function. Then:*

$$E\left(g(t), K(t)\right) \leq E\left(g(t), K(s)\right) \qquad \forall 0 \leq s < t \leq 1. \tag{4.16}$$

Proof. Fix s, t such that $0 \leq s < t \leq 1$. Due to the structure of the model, we have

$$E\left(g(\tau), K(\tau)\right) = E\left(\varphi(\tau)g_0, K(\tau)\right) = \frac{\mu}{2}\varphi(\tau)^2 \left\|\nabla v(\tau)\right\|^2 + \gamma\, \mathcal{H}^1\left(K(\tau)\right),$$

where $v(\tau)$ is the solution to the minimum problem defining $E\left(g_0, K(\tau)\right)$. Then $u(\tau) = \varphi(\tau)v(\tau)$ where $u(\tau)$ is the solution to the minimum problem defining $E\left(\varphi(\tau)g_0, K(\tau)\right)$; moreover $\dot{g}(\tau) = \dot{\varphi}(\tau)g_0$. From Theorem 4.3.1 we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E(g(t), K(t)) = \mu \int_{\Omega \setminus K(t)} \nabla u(t)^\top \nabla \dot{g}(t)\, \mathrm{d}x \quad \text{for a.e. } t \in [0,1].$$

Integrating in time from s to t we get:

$$E\left(g(t),K(t)\right)-E\left(g(s),K(s)\right)=\mu\int_s^t\Big\langle\nabla u(\tau),\nabla\dot g(\tau)\Big\rangle\,\mathrm{d}\tau$$

From this, adding and subtracting $E\left(g(s),K(s)\right)$ we get

$$\begin{aligned}
&E\left(g(t),K(t)\right)-E\left(g(t),K(s)\right)\\
=&\,\mu\int_s^t\Big\langle\nabla u(\tau),\nabla\dot g(\tau)\Big\rangle\,\mathrm{d}\tau+E\left(g(s),K(s)\right)-E\left(g(t),K(s)\right)\\
=&\,\mu\int_s^t\Big\langle\nabla v(\tau),\nabla g_0\Big\rangle\varphi(\tau)\dot\varphi(\tau)\,\mathrm{d}\tau+\frac{\mu}{2}\left(\varphi(s)^2-\varphi(t)^2\right)\|\nabla v(s)\|^2.
\end{aligned}$$

Now, since $v(\tau)$ is such that $v(\tau)=g_0$ q.e. on $\partial_D\Omega\setminus K(\tau)$, we can take in (3.8) $z=v(\tau)-g_0$, thus:

$$\Big\langle\nabla v(\tau),\nabla g_0\Big\rangle_{L^2(\Omega;\mathbb{R}^2)}=\Big\|\nabla v(\tau)\Big\|^2_{L^2(\Omega;\mathbb{R}^2)}.$$

Due to the monotonicity of K, if $\tau\geq s$ then $K(\tau)\supseteq K(s)$, thus $\Omega\setminus K(\tau)\subseteq\Omega\setminus K(s)$. If $v(s)\in L^{1,2}(\Omega\setminus K(s))$, we can then truncate it on $\Omega\setminus K(\tau)$, therefore obtaining $v(s)\in L^{1,2}(\Omega\setminus K(\tau))$ and $v(s)=g_0$ q.e. on $\partial_D\Omega\setminus K(\tau)$, hence $v(s)$ is an admissible function for the minimum problem defining $E\left(g_0,K(\tau)\right)$. Due to the minimality of $v(\tau)$ we get $\|\nabla v(\tau)\|^2\leq\|\nabla v(s)\|^2$. Combining these results we get:

$$\begin{aligned}
&E\left(g(t),K(t)\right)-E\left(g(t),K(s)\right)\\
=&\,\mu\int_s^t\Big\langle\nabla v(\tau),\nabla g_0\Big\rangle\varphi(\tau)\dot\varphi(\tau)\,\mathrm{d}\tau+\frac{\mu}{2}\left(\varphi(s)^2-\varphi(t)^2\right)\|\nabla v(s)\|^2\\
=&\,\mu\int_s^t\Big\|\nabla v(\tau)\Big\|^2\varphi(\tau)\dot\varphi(\tau)\,\mathrm{d}\tau+\frac{\mu}{2}\left(\varphi(s)^2-\varphi(t)^2\right)\|\nabla v(s)\|^2\\
\leq&\,\mu\Big\|\nabla v(s)\Big\|^2\int_s^t\varphi(\tau)\dot\varphi(\tau)\,\mathrm{d}\tau+\frac{\mu}{2}\left(\varphi(s)^2-\varphi(t)^2\right)\|\nabla v(s)\|^2\\
=&\,\frac{\mu}{2}\Big\|\nabla v(s)\Big\|^2\left[2\int_s^t\varphi(\tau)\dot\varphi(\tau)\,\mathrm{d}\tau+\varphi(s)^2-\varphi(t)^2\right]\\
=&\,\frac{\mu}{2}\Big\|\nabla v(s)\Big\|^2\left[\int_s^t\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\varphi(\tau)^2\Big)\,\mathrm{d}\tau+\varphi(s)^2-\varphi(t)^2\right]\\
=&\,0,
\end{aligned}$$

from which the conclusion follows. □

A nice corollary of Theorem 4.3.1 admits the existence of a left-continuous and a right-continuous solution:

Corollary 4.3.13. *Let $K\colon[0,1]\to\mathcal{K}^f_m(\overline{\Omega})$ a non-decreasing compact-valued function satisfying the hypothesis of Theorem 4.3.1. Then:*

$$E\left(g(t),K(t)\right)=E\left(g(t),K^-(t)\right)\quad 0<t\leq 1,\tag{4.17}$$

$$E\left(g(t),K(t)\right)=E\left(g(t),K^+(t)\right)\quad 0\leq t<1.\tag{4.18}$$

Proof. For the first part, fix a $t \in (0,1]$, and a succession (t_n) such that $t_n \to t^-$. Then we have $K(t_n) \to K^-(t)$ in the Hausdorff metric, and $g(t_n) \to g(t)$ in $H^1(\Omega)$ due to the absolute continuity. From Theorem 4.1.8 then we have that $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega;\mathbb{R}^2)$, where u_n, u^- are the solutions to the minimum problems defining respectively $E\left(g(t_n), K(t_n)\right)$ and $E\left(g(t), K^-(t)\right)$. Moreover, from Corollary 3.2.8 we also know that $\mathcal{H}^1(K(t_n)) \to \mathcal{H}^1(K^-(t))$, hence we conclude that:

$$E\left(g(t_n), K(t_n)\right) \to E\left(g(t), K^-(t)\right) \qquad t_n \to t^-.$$

From Theorem 4.3.1 we know that the function $s \mapsto E\left(g(s), K(s)\right)$ is absolutely continuous on a compact interval, hence it is continuous, hence by the uniqueness of the limit we get:

$$E\left(g(t), K(t)\right) = E\left(g(t), K^-(t)\right),$$

for every $t \in (0,1]$.
The second part follows analogously. □

From this result, we deduce that the problem has a left-continuous and a right-continuous solution, given by:

$$t \mapsto K_{\rightarrow}(t) \coloneqq \begin{cases} K(0) & t = 0, \\ K^-(t) & 0 < \quad t \leq 1; \end{cases} \qquad t \mapsto K_{\leftarrow}(t) \coloneqq \begin{cases} K^+(t) & 0 \leq \quad t < 1, \\ K(1) & t = 1; \end{cases}$$

These are solutions to Theorem 4.3.1, since from Corollary 4.3.13 we have that the properties of 4.3.1 are invariant under changes from K to K^+ or K^-.

Chapter 5

Study of solutions near the crack tip

At this point we know that under some natural assumptions on the agents involved, the proposed model always admits a solution, being a compact-valued non-decreasing function representing the evolution of the crack set.
The strength of this model, as anticipated, is found not only in this existence result, but also in the fact that it summarizes in a way the classical theory of brittle fracture. That is why the objective of this chapter is three-fold:

1. first we want to study the behavior of the displacements near the crack tip, in order to prove the existence of the Stress Intensity Factor and a certain dependence on the square root of the distance from the crack tip;

2. then study the Energy Release Rate under propagation of the crack, to prove Irwin's relation between the Stress Intensity Factor and the Energy Release Rate in our case of anti-plane shear;

3. reconstruct Griffith's criteria in its dissipative formulation.

The tools that we will use in this chapter, especially in the first section, will be quite different from the ones used up until now, and due to the amount of necessary introduction, we will often omit the proofs of some results, for the sake of simplicity.

5.1 Regularity of displacements near the crack tip

To study this problem, we can avoid the propagation, and thus avoid considering the energy of the fracture, which will be considered constant and thus will not contribute. Effectively this is done by freezing the system in time and considering only the local behavior of displacements. We will assume the crack to be a curve of class C^2, in order that it always exists a diffeomorphism that, in a sufficiently small neighborhood of the crack tip, allows us to consider a straight crack. We will not give details for this diffeomorphism,

which can be constructed explicitly: we refer to [16] for further details.
We can take a neighborhood of the crack tip U, and then consider the truncated problem through a function $\eta \in \mathcal{D}(\overline{U})$, in order to avoid having to consider non-homogeneous boundary conditions. Under this setting, we consider U to be a polygonal boundary, of the type in the right figure in Figure 5.1. For clarity, we will be studying the general case in which the boundary has a concave corner, and then deduce the result for a crack.

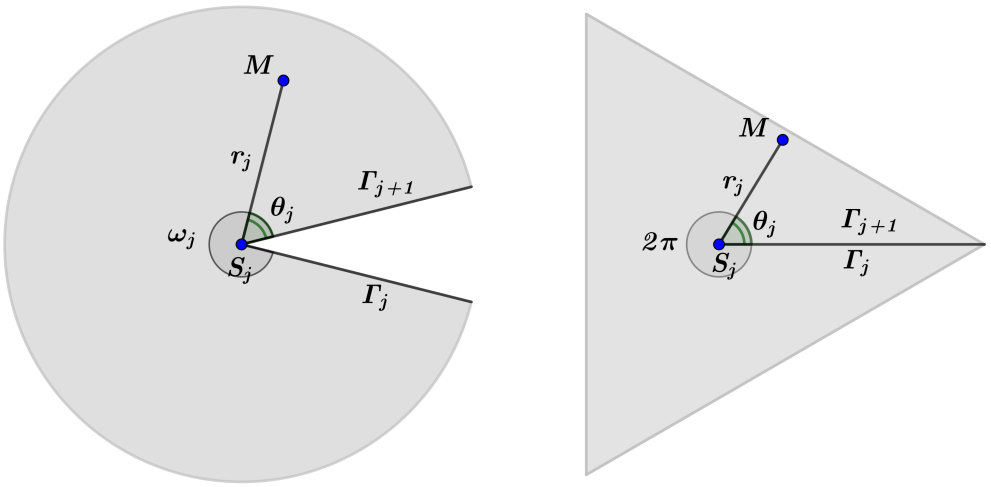

Figure 5.1: Polygonal boundary being considered

Now for some notation, we define each vertex as S_j, with the canonical orientation, and then each edge as Γ_j, connecting S_j to S_{j+1}. We define for each vertex the angle pointing into the domain as ω_j, and also a system of polar coordinates (r_j, θ_j) where taken a point $M \in U$, we get $r_j = M - S_j$ and θ_j equal to the angle between Γ_{j+1} and $M - S_j$, as in Figure 5.1: obviously $\theta_j \in [0, \omega_j]$. Given a vertex and its polar coordinates, we can define a truncation function $\eta_j \in \mathcal{D}(\overline{U})$, function only of r_j, such that $\eta_j \equiv 1$ near S_j, and it decreases until $\eta_j \to 0$ near $\overline{\Gamma}_k$, for $k \neq j, j+1$. Moreover, these functions can be made so that they have disjoint support.
We will now divide the edges between the Dirichlet and Neumann edges, dividing the indices of the edges in a disjoint union of two sets, as $\{1, 2, \dots, N\} = \mathcal{D} \sqcup \mathcal{N}$, with the obvious choice of notation. We will also divide the vertices in an analogous way, dividing them as follows:

$$\begin{aligned}
\mathcal{D}^2 &\coloneqq \{j : j \in \mathcal{D}, j+1 \in \mathcal{D}\};\\
\mathcal{N}^2 &\coloneqq \{j : j \in \mathcal{N}, j+1 \in \mathcal{N}\};\\
\mathcal{M} &\coloneqq \{j : j \in \mathcal{D}, j+1 \in \mathcal{N} \text{ or } j \in \mathcal{N}, j+1 \in \mathcal{D}\}.
\end{aligned}$$

Up until now, we have omitted the trace operator γ on the formulation we have given, but now we will have to define a trace operator γ_k for each of the edges. Summing up,

the problem we wish to study is the following:

$$\begin{cases} \Delta u = 0 \text{ on } U, \\ \gamma_k u = 0 \text{ on } \Gamma_k, k \in \mathcal{D}, \\ \gamma_k \dfrac{\partial u}{\partial n_k} = 0 \text{ on } \Gamma_k, k \in \mathcal{N}. \end{cases}$$

Define then the space of weak solutions $V \coloneqq \{u \in H^1(U) : \gamma_k u = 0 \ \ \forall k \in \mathcal{D}\}$. From the usual theory on boundary value problems, it is known that there exists always a solution $u \in V$ to this problem. Moreover, it can be proven that, is $u \in V$ is a solution, and $\varphi \in \mathcal{D}(\overline{U})$ such that its support only intersects the edges away from the vertices, then $\varphi u \in H^2(U)$, meaning that the solution is regular up to the corners.
Let us now define the space of strong solutions

$$V^2 \coloneqq \left\{ u \in H^2(U) : \gamma_k u = 0 \ \ \forall k \in \mathcal{D}, \ \ \gamma_k \frac{\partial u}{\partial n_k} = 0 \ \ \forall k \in \mathcal{N} \right\}.$$

Consider the Laplacian as an operator from V^2 to $L^2(U)$. What we want to do is prove that the image of this operator is a closed subset of $L^2(U)$, meaning that for an arbitrary polygonal boundary, we are not sure that there exists $v \in V^2$ such that $\Delta v = 0$. In order to make the operator surjective, we will need to add to the domain a space of a certain dimension, and this will be done with the use of the theory of Fredholm operators, which we will briefly introduce now without proofs: we refer to Section 3.3 of [26].

Definition 5.1.1. *Let X and Y be Banach spaces, and $T\colon X \to Y$ be a bounded linear operator. We say that T is a Fredholm operator if it satisfies these three conditions:*

- $\operatorname{Ker} T$ *has finite dimension;*
- $\operatorname{Im} T$ *is closed in Y;*
- $\operatorname{Coker} T$ *has finite dimension.*

If T is a Fredholm operator we can define its index as

$$\begin{aligned} \operatorname{ind} T &= \dim(\operatorname{Ker} T) - \dim(\operatorname{Coker} T) \\ &= \dim(\operatorname{Ker} T) - \operatorname{Codim}(\operatorname{Im} T). \end{aligned}$$

Proposition 5.1.2. *$T\colon X \to Y$ is a Fredholm operator if and only if there exists a bounded linear operator $R\colon Y \to X$ such that both $RT - I_X$ and $TR - I_Y$ are compact operators.*

As a standard notation, we define $\mathcal{B}(X, Y)$ to be the set of bounded linear operators between X and Y. We also now introduce $\operatorname{Fred}(X, Y)$ to be the set of Fredholm operators between X and Y.

Proposition 5.1.3. $\operatorname{Fred}(X, Y)$ *is an open subset of $\mathcal{B}(X, Y)$, and the index is a continuous function on* $\operatorname{Fred}(X, Y)$.

A property that is used frequently in the study of kernel operators is the following:

Proposition 5.1.4. *If $K \in \mathcal{B}(X,X)$ is compact, then $I+K \in \operatorname{Fred}(X,X)$.*

As a last property, the space of Fredholm operators is closed under composition.

Proposition 5.1.5. *Let $T \in \operatorname{Fred}(X,Y)$ and $S \in \operatorname{Fred}(Y,Z)$. Then $ST \in \operatorname{Fred}(X,Z)$ and $\operatorname{ind} ST = \operatorname{ind} S + \operatorname{ind} T$.*

We will use the Fredholm alternative theory to solve our problems, which in our case states that T is a second-order elliptic operator in a Hilbert space H, then the elliptic problem $L\,u = 0$ has a non-trivial weak solution. For further details, we refer to Section 6.2.3 of [38].
The first two properties of Fredholm operators are usually quite easy to prove, and for elliptic problems, they involve finding a certain inequality, for which we require the following lemma.

Lemma 5.1.6. *For each $u \in V^2$ we have*

$$\|\Delta u\|^2 = \|\partial_{11} u\|^2 + \|\partial_{22} u\|^2 + 2\|\partial_{12} u\|^2.$$

Proof. By straightforward calculation, we get:

$$\|\Delta u\|^2 = \|\partial_{11} u\|^2 + \|\partial_{22} u\|^2 + 2\int_U \partial_{11} u \partial_{22} u \,\mathrm{d}x.$$

Let us look at the last term on the right-hand side: we would like to prove that

$$\int_U \partial_{11} u \partial_{22} u \,\mathrm{d}x = \int_U (\partial_{12} u)^2 \,\mathrm{d}x \qquad \forall u \in V^2.$$

Thanks to density results, we can prove this for $u \in V^2 \cap H^3(U)$. Integrating by parts twice we get:

$$\begin{aligned}
&\int_U \partial_{11} u \partial_{22} u \,\mathrm{d}x \\
&= \sum_j \int_{\Gamma_j} \gamma_j(\partial_1 u)\gamma_j(\partial_{22} u) n_1 d\sigma - \int_U \partial_1 u \partial_{122} u \,\mathrm{d}x \\
&= \sum_j \int_{\Gamma_j} \gamma_j(\partial_1 u)\gamma_j(\partial_{22} u) n_1 d\sigma - \sum_j \int_{\Gamma_j} \gamma_j(\partial_{12} u)\gamma_j(\partial_1 u) n_2 d\sigma + \int_U (\partial_{12} u)^2 \,\mathrm{d}x,
\end{aligned}$$

which brings to:

$$\begin{aligned}
\int_U \partial_{11} u \partial_{22} u \,\mathrm{d}x - \int_U (\partial_{12} u)^2 \,\mathrm{d}x &= \sum_j \int_{\Gamma_j} \gamma_j(\partial_1 u)\Big[\gamma_j(\partial_{22} u) n_1 - \gamma_j(\partial_{12} u) n_2\Big] d\sigma \\
&= \sum_j \int_{\Gamma_j} \gamma_j(\partial_1 u) d[\gamma_j(\partial_2 u)],
\end{aligned}$$

which is well defined since $u \in H^3(U)$. Looking at the boundary conditions, they imply $\gamma_j(\nabla u \cdot \mu_j) = 0$, where $\mu_j = \tau_j$ is $j \in \mathcal{D}$, and $\mu_j = n_j$ if $j \in \mathcal{N}$. Now, if μ_j is parallel to the x axis, then $\gamma_j(\partial_1 u) = 0$ and the conclusion follows. If μ_j is not parallel to the x axis, we define α_j and β_j as the components of μ_j with respect to two axes: it follows that $\beta_j \neq 0$. It follows that $\gamma_j(\partial_2 u) = -\frac{\alpha_j}{\beta_j}\gamma_j(\partial_1 u)$, from which we deduce:

$$\begin{aligned}\int_{\Gamma_j} \gamma_j(\partial_1 u) d[\gamma_j(\partial_2 u)] &= -\frac{\alpha_j}{\beta_j}\int_{\Gamma_j} \gamma_j(\partial_1 u) d[\gamma_j(\partial_1 u)] \\ &= -\frac{\alpha_j}{2\beta_j}\Big[\big(\gamma_j(\partial_1 u)\big)^2(S_j) - \big(\gamma_j(\partial_1 u)\big)^2(S_{j-1})\Big].\end{aligned}$$

Again, this is well defined since if $u \in H^3(U)$ then $\partial_1 u \in C(\overline{U})$.
From the trace theorems we know that $(\gamma_j \nabla u)(S_j) = (\gamma_{j+1}\nabla u)(S_j)$. Moreover, we also know that $(\gamma_j \nabla u)(S_j)$ is perpendicular both to μ_j and μ_{j+1}. Now, if $\mu_j \nparallel \mu_{j+1}$ then obviously $(\gamma_j \nabla u)(S_j) = 0$, from which the conclusion follows. If $\mu_j \parallel \mu_{j+1}$ then:

$$-\frac{\alpha_j}{\beta_j}\Big[\big(\gamma_j(\partial_1 u)\big)^2(S_j) - \big(\gamma_j(\partial_1 u)\big)^2(S_{j-1})\Big] = \left[\frac{\alpha_{j+1}}{\beta_{j+1}} - \frac{\alpha_j}{\beta_j}\right](\gamma_j \partial_1 u)^2(S_j) = 0,$$

from which the conclusion follows. □

Theorem 5.1.7. *Let U be a bounded polygonal open set, and let the set $\mathcal{D}$ of Dirichlet edges be non-empty. Then there exists a constant $C(U)$ such that*

$$\|u\|_{H^2} \leq C(U)\left(\|\Delta u\|_{L^2} + \|u\|_{L^2}\right).$$

Proof. We will look for a bound for every derivative. For the second derivative, from the previous lemma we have:

$$|u|_2^2 = \|\partial_{11} u\|^2 + \|\partial_{22} u\|^2 + 2\|\partial_{12} u\|^2 = \|\Delta u\|^2.$$

For the first derivative, they are equivalent to the gradient, hence we can use Poincare inequality to obtain:

$$\begin{aligned}\|\nabla u\|^2 = \int_U |\nabla u|^2 \,\mathrm{d}x &= -\int_U u\Delta u \,\mathrm{d}x + \sum_j \int_{\Gamma_j} \gamma_j u \gamma_j \frac{\partial u}{\partial n_j} d\sigma = -\int_U u\Delta u \,\mathrm{d}x \\ &\leq \|u\|\|\Delta u\| \leq C(U)\|\nabla u\|\|\Delta u\|.\end{aligned}$$

This of course implies:

$$\|\nabla u\| \leq C(U)\|\Delta u\|.$$

We can now conclude by:

$$\|u\|_{H^2} \leq (1 + C(U))\|\Delta u\|_{L^2} + \|u\|_{L^2} \leq C(U)\left(\|\Delta u\|_{L^2} + \|u\|_{L^2}\right),$$

where we just renamed every constant with $C(U)$. □

Thanks to this theorem, since V^2 is compactly embedded in L^2, we can prove the semi-Fredholm property of Δ, because it follows that the operator has a finite dimensional kernel and has a closed image, as shown in Section 2.3 of [19].
What we are missing is a result about the dimension of the Coker of Δ, or equivalently, a result about the co-dimension of the image of Δ, which has to be finite. In order to do this, we will study the orthogonal to the image of Δ, which we will call N, and we will prove that its dimension is finite, by calculating it explicitly. This will prove that, given $f \in L^2(\Omega)$, then there exists a weak solution to $\Delta u = f$, or in other words, the only solution to $\Delta u = 0$ is the trivial solution $u \equiv 0$. To do this, we need to define a new space of functions:

$$D(\Delta, L^2(U)) \coloneqq \{v \in L^2(U) : \Delta v \in L^2(U)\}.$$

With this space, we will prove an equivalence relation between N and $D(\Delta, L^2(U))$.

Proposition 5.1.8. *If $v \in N$, then $v \in D(\Delta, L^2(U))$ and is the solution to the following adjoint problem:*

$$\begin{cases} \Delta v = 0 & \text{in } U; \\ \gamma_j v = 0 & \text{per } j \in \mathcal{D}; \\ \gamma_j \dfrac{\partial v}{\partial n_j} = 0 & \text{per } j \in \mathcal{N}. \end{cases}$$

This proposition makes sense since it can be proved that for every $v \in H^2$, the mapping $v \mapsto \{\gamma_j v, \gamma_j \frac{\partial v}{\partial n_j}\}$ has a unique continuous extension as an operator from $D(\Delta, L^2(U))$ to $H^{-\frac{1}{2}}(\Gamma_j) \times H^{-\frac{3}{2}}(\Gamma_j)$.

Proof. By definition of N, if $v \in N$, then

$$v \in L^2(U), \qquad \int_U v \Delta u \, \mathrm{d}x = 0 \quad \forall u \in V^2.$$

This is true also for all $u \in \mathcal{D}(U)$, hence we can integrate by parts to obtain that v is harmonic, with of course $\nabla v \in L^2(U)$. It follows then $v \in D(\Delta, L^2(U))$.
We now need to deduce the boundary conditions. Given $\varphi_j \in \mathcal{D}(\Gamma_j)$ for $j \in \mathcal{D}$ and $\psi_j \in \mathcal{D}(\Gamma_j)$ for $j \in \mathcal{N}$, from the trace embeddings we know that there exists $u \in H^2(U)$ such that:

$$\gamma_j u = \varphi_j \qquad \gamma_j \frac{\partial u}{\partial n_j} = 0 \qquad j \in \mathcal{N};$$

$$\gamma_j u = 0 \qquad \gamma_j \frac{\partial u}{\partial n_j} = \psi_j \qquad j \in \mathcal{D}.$$

Using Green's identity, adjusted for polygonal boundaries, we can write:

$$\int_U u \Delta v \, \mathrm{d}x - \int_U v \Delta u \, \mathrm{d}x = \sum_j \left[\left\langle \gamma_j u, \gamma_j \frac{\partial v}{\partial n_j} \right\rangle - \left\langle \gamma_j v, \gamma_j \frac{\partial u}{\partial n_j} \right\rangle \right].$$

Since v is harmonic, and due to the properties of the chosen u, we get

$$\sum_{j\in\mathcal{N}} \left\langle \varphi_j, \gamma_j \frac{\partial v}{\partial n_j} \right\rangle - \sum_{j\in\mathcal{D}} \langle \gamma_j v, \psi_j \rangle = 0.$$

Due to the arbitrariness of φ_j and ψ_j, we deduce the given boundary conditions. □

To prove the converse of this proposition, we need to first introduce some new notation, and prove an orthogonality result which will be useful in more results.
We define two subsets of $\mathcal{M}$, namely:

$$\mathcal{M}' \coloneqq \left\{ j \in \mathcal{M} : j \in \mathcal{N}, j+1 \in \mathcal{D}, \omega_j = \frac{\pi}{2} \text{ or } \omega_j = \frac{3\pi}{2} \right\};$$
$$\mathcal{M}'' \coloneqq \left\{ j \in \mathcal{M} : j \in \mathcal{D}, j+1 \in \mathcal{N}, \omega_j = \frac{\pi}{2} \text{ or } \omega_j = \frac{3\pi}{2} \right\}.$$

These sets may very well be empty, as they depend on the angles of the polygon.

Lemma 5.1.9. *If $v \in N$, then v has to satisfy these three orthogonality conditions:*

$$\int_U v\Delta(\eta_j)\,\mathrm{d}x = 0 \qquad \forall j \in \mathcal{N},$$
$$\int_U v\Delta(y_j\eta_j)\,\mathrm{d}x = 0 \qquad \forall j \in \mathcal{M}',$$
$$\int_U v\Delta(x_j\eta_j)\,\mathrm{d}x = 0 \qquad \forall j \in \mathcal{M}'',$$

where $x_j = r_j\cos\theta_j$ and $y_j = r_j\sin\theta_j$.

Proof. Since $v \in N$, then we know that $v \in D(\Delta, L^2(U))$, $\Delta v = 0$ and it satisfies boundary conditions. Moreover, $\eta_j, y_j\eta_j, x_j\eta_j \in \mathcal{D}(U)$ hence they are differentiable and we can use Green's formula.
For the first formula, we get:

$$\begin{aligned}\int_U v\Delta(\eta_j)\,\mathrm{d}x &= \int_U \eta_j\Delta v\,\mathrm{d}x + \sum_k \left\langle \gamma_k v, \gamma_k \frac{\partial \eta_j}{\partial n_k} \right\rangle - \sum_k \left\langle \gamma_k\eta_j, \gamma_k \frac{\partial v}{\partial n_k} \right\rangle \\ &= \sum_{k\in\mathcal{N}} \left\langle \gamma_k v, \gamma_k \frac{\partial \eta_j}{\partial n_k} \right\rangle - \sum_{k\in\mathcal{D}} \left\langle \gamma_k\eta_j, \gamma_k \frac{\partial v}{\partial n_k} \right\rangle \\ &= 0.\end{aligned}$$

where the last equality is due to the supports of η_j being disjoint, and due to η_j depending only on r_j. In particular, the second term is equal to zero since in the first case $j \in \mathcal{N}^2$, hence if $k \in \mathcal{D}$ then $\gamma_k\eta_j = 0$. This is the part that needs to be modified for the three different cases: for example, if $j \in \mathcal{M}'$, then we need to check for $j = k+1 \in \mathcal{D}$, but this we obviously have $y_j\eta_j = 0$ due to the presence of y_j. The same goes for $j \in \mathcal{M}''$, hence the conclusion. □

Proposition 5.1.10. *If $v \in D(\Delta, L^2(U))$, solves the adjoint boundary value problem and satisfies the orthogonality conditions in 5.1.9, then $v \in N$.*

Proof. Since we want to prove $v \in N$, we will prove

$$\int_U v\,\Delta u\,\mathrm{d}x = 0 \qquad \forall u \in V^2.$$

By density we can consider $u \in V^2 \cap H^4(U)$, since this implies $u \in C^2(\overline{U})$. Then it is well defined the function:

$$w \coloneqq u - \sum_{j\in\mathcal{N}^2} u(S_j)\cdot\eta_j - \sum_{j\in\mathcal{M}'} \partial_{y_j}u(S_j)\cdot y_j\eta_j - \sum_{j\in\mathcal{M}''} \partial_{x_j}u(S_j)\cdot x_j\eta_j.$$

Taking the Laplacian, multiplying by v and integrating we get:

$$\Delta w = \Delta u - \sum_{j\in\mathcal{N}^2} u(S_j)\cdot\Delta(\eta_j) - \sum_{j\in\mathcal{M}'} \partial_{y_j}u(S_j)\cdot\Delta(y_j\eta_j) - \sum_{j\in\mathcal{M}''} \partial_{x_j}u(S_j)\cdot\Delta(x_j\eta_j)$$

$$v\Delta w = v\Delta u - \sum_{j\in\mathcal{N}^2} u(S_j)\cdot v\Delta(\eta_j) - \sum_{j\in\mathcal{M}'} \partial_{y_j}u(S_j)\cdot v\Delta(y_j\eta_j) - \sum_{j\in\mathcal{M}''} \partial_{x_j}u(S_j)\cdot v\Delta(x_j\eta_j),$$

where the last equality is due to the orthogonality conditions of 5.1.9. This obviously implies

$$\int_U v\,\Delta w\,\mathrm{d}x = \int_U v\,\Delta u\,\mathrm{d}x.$$

We observe that $u(S_j) = 0$ for all $j \in \mathcal{D}^2 \cup \mathcal{M}$: since this excludes $j \in \mathcal{N}^2$, due to the nature of the support of η_j we can conclude that $w(S_j) = 0$ for all j.
Looking now at ∇w, we know for a fact that $\nabla u^\top \mu_j = 0$ for all j, with the same μ_j introduced a while ago, i.e. $\mu_j = n_j$ if $j \in \mathcal{N}$, while $\mu_j = \tau_j$ if $j \in \mathcal{D}$. In S_j we have $\nabla u(S_j) \perp \mu_j, \mu_{j+1}$. If $\mu_j \nparallel \mu_{j+1}$ then $\nabla u(S_j) = 0$. On the other hand, $\mu_j \parallel \mu_{j+1}$ only when $j \in \mathcal{M}'$ or $j \in \mathcal{M}''$: if $j \in \mathcal{M}'$, then $\gamma_{j+1}u = 0$ on Γ_{j+1}, hence $\partial_{x_j}u(S_j) = 0$, implying $\nabla w(S_j) = 0$. Same goes for $j \in \mathcal{M}''$: hence $\nabla w(S_j) = 0$ for all j. Since we know that

$$\gamma_j w \in H^{\frac{3}{2}}(\Gamma_j), \qquad \gamma_j \frac{\partial w}{\partial n_j} \in H^{\frac{1}{2}}(\Gamma_j), \qquad \forall j,$$

and $u \in H^2(U) \subset C^1(\overline{U})$, we then deduce that

$$\gamma_j w = 0, \qquad \gamma_j \frac{\partial w}{\partial n_j} = 0, \qquad \forall j.$$

We can apply Green's formula to obtain:

$$\int_U v\Delta w\,\mathrm{d}x = \int_U w\Delta v\,\mathrm{d}x + \sum_j \left\langle \gamma_j v, \gamma_j \frac{\partial w}{\partial n_j} \right\rangle - \sum_j \left\langle \gamma_j w, \gamma_j \frac{\partial v}{\partial n_j} \right\rangle = 0,$$

where the last equality follows from the properties of w, hence the conclusion. □

The next proposition will give us a result about the smoothness of v far from the corners.

Proposition 5.1.11. *If $v \in N$, then $v \in C^\infty(\overline{U} \setminus I)$, where I is a neighborhood of all S_j.*

Proof. From previous proofs we know that v is harmonic, hence it is smooth in every open set contained in U. We just need to prove the smoothness along the edges, away from corners. We can do this independently for every edge, so let us choose a j, and change coordinates so that $\Gamma_j \subset \{x_2 = 0\}$ and $U \subset \{x_2 \geq 0\}$. Define now a rectangle $R \subset U$ such that $R = (a,b) \times (0,c)$, such that $\Gamma_0 = (a,b) \times \{0\}$ is such that $\Gamma_0 \subset \Gamma_j$.
Since $v \in D(\Delta, L^2(U))$, this implies $v \in D(\Delta, L^2(R))$. Moreover, since $v \in N$ is a solution to the adjoint problem, we also know $\gamma_0 v = 0$ on $H^{\frac{1}{2}}(a,b)$, where γ_0 is the trace operator on Γ_0.
Now, if $j \in \mathcal{D}$, we define an odd reflection through Γ_0. If $w \in L^2(R)$ we define the Laplacian of the odd reflection as

$$\Delta W = 2(\gamma_0 w) \otimes \delta(x_2) \qquad \text{in } (a,b) \times (-c,c),$$

which is defined for $w \in \mathcal{D}(\overline{R})$. Due to the density of $\mathcal{D}(\overline{R})$ in $D(\Delta, L^2(R))$ and the continuity of the trace operator, this equality is also true for $w \in D(\Delta, L^2(R))$.
We can apply this identity to $v \in N$ since this implies $v \in D(\Delta, L^2(R))$. We also know $\gamma_0 v = 0$, hence $\Delta V = 0$, i.e. it is harmonic in $(a,b) \times (-c,c)$. This implies that V is smooth in the reflected rectangle, and by restriction, is smooth on the rectangle with its lower boundary.
If $j \in \mathcal{N}$, the result works as well, we just need to use the even reflection. □

The key point is to try and expand v as a function depending on the polar coordinates centered on the corners. Let us in fact take a $\rho_j > 0$ such that the support of η_j is contained in $D_j \coloneqq U \cap B(S_j, \rho_j)$, it does not intersect other D_i for $i \neq j$, and it only intersects the edges at Γ_j and Γ_{j+1}.
Since we proved that a function $v \in N$ solves the adjoint problem, and is harmonic up to the corners, we can write the Laplacian in polar coordinates with origin at S_j:

$$\frac{\partial^2 v}{\partial r_j^2} + \frac{1}{r_j}\frac{\partial v}{\partial r_j} + \frac{1}{r_j^2}\frac{\partial^2 v}{\partial \theta_j^2} = 0 \qquad \theta_j \in (0, \omega_j), \quad r_j \in (0, \rho_j). \tag{5.1}$$

We also know that it must fulfill the following boundary conditions, which are well-defined since v is smooth away from corners:

$$\theta_j = 0 \implies \begin{cases} v = 0 & \text{if } j+1 \in \mathcal{D}, \\ \frac{\partial v}{\partial \theta_j} = 0 & \text{if } j+1 \in \mathcal{N}; \end{cases} \qquad \theta_j = \omega_j \implies \begin{cases} v = 0 & \text{if } j \in \mathcal{D}, \\ \frac{\partial v}{\partial \theta_j} = 0 & \text{if } j \in \mathcal{N}; \end{cases}$$

Due to the dependence on the second derivative with respect to θ_j in (5.1), we will need to study the eigenvalues and eigenfunctions of $\varphi \mapsto -\varphi''$, with appropriate boundary conditions on 0 and ω_j, depending on the corner type. These results are well known, but we will need to introduce some notation.
We define an unbounded operator as follows:

$$\Upsilon_j \colon D(\Upsilon_j) \to \mathcal{L}_j = L^2(0, \omega_j) \qquad \Upsilon_j \varphi = -\varphi'' \quad \forall \varphi \in D(\Upsilon_j),$$

where $D(\Upsilon_j)$ is the domain of Υ_j, which depends on the boundary condition as follows:

$$\begin{aligned}
\text{if } j \in \mathcal{D}^2 \qquad & D(\Upsilon_j) \coloneqq \left\{\varphi \in H^2(0,\omega_j) : \varphi(0) = 0, \varphi(\omega_j) = 0\right\};\\
\text{if } j \in \mathcal{N}^2 \qquad & D(\Upsilon_j) \coloneqq \left\{\varphi \in H^2(0,\omega_j) : \varphi'(0) = 0, \varphi'(\omega_j) = 0\right\};\\
\text{if } j \in \mathcal{N},\ j+1 \in \mathcal{D} \qquad & D(\Upsilon_j) \coloneqq \left\{\varphi \in H^2(0,\omega_j) : \varphi(0) = 0, \varphi'(\omega_j) = 0\right\};\\
\text{if } j \in \mathcal{D},\ j+1 \in \mathcal{N} \qquad & D(\Upsilon_j) \coloneqq \left\{\varphi \in H^2(0,\omega_j) : \varphi'(0) = 0, \varphi(\omega_j) = 0\right\}.
\end{aligned}$$

It can be proved that Υ_j is non-negative, self-adjoint, and with discrete spectrum, hence the eigenvalues and eigenfunctions are well-defined. We will denote the eigenfunctions with $\varphi_{j,m}$, $m \geq 1$, and the eigenvalues in increasing order with $\lambda_{j,m}^2$, $m \geq 1$. We have:

$$\begin{aligned}
&j \in \mathcal{D}^2 \quad m \geq 1 \quad \varphi_{j,m}(\theta_j) = \sqrt{\frac{2}{\omega_j}} \sin \frac{m\pi\theta_j}{\omega_j} \quad \lambda_{j,m} = \frac{m\pi}{\omega_j};\\
&j \in \mathcal{N}^2 \quad \begin{cases} m = 1 \quad \varphi_{j,1}(\theta_j) = \sqrt{\frac{1}{\omega_j}} \quad \lambda_{j,1} = 0, \\ m \geq 2 \quad \varphi_{j,m}(\theta_j) = \sqrt{\frac{2}{\omega_j}} \cos \frac{(m-1)\pi\theta_j}{\omega_j} \quad \lambda_{j,m} = \frac{(m-1)\pi}{\omega_j}; \end{cases}\\
&j \in \mathcal{N},\ j+1 \in \mathcal{D} \quad m \geq 1 \quad \varphi_{j,m}(\theta_j) = \sqrt{\frac{2}{\omega_j}} \sin \frac{(m-\frac{1}{2})\pi\theta_j}{\omega_j} \quad \lambda_{j,m} = \frac{(m-\frac{1}{2})\pi}{\omega_j};\\
&j \in \mathcal{D},\ j+1 \in \mathcal{N} \quad m \geq 1 \quad \varphi_{j,m}(\theta_j) = \sqrt{\frac{2}{\omega_j}} \sin \frac{(m-\frac{1}{2})\pi(\omega_j - \theta_j)}{\omega_j} \quad \lambda_{j,m} = \frac{(m-\frac{1}{2})\pi}{\omega_j}.
\end{aligned}$$

If we fix $r_j > 0$, we can write v in a compact way as $v\left(r_j e^{i\theta_j}\right)$, which is obviously in $D(\Upsilon_j)$ for $r_j \in (0, \rho_j)$. We can thus rewrite (5.1) as:

$$\frac{\partial^2 v}{\partial r_j^2} + \frac{1}{r_j}\frac{\partial v}{\partial r_j} - \frac{1}{r_j^2}\Upsilon_j v = 0 \qquad r_j \in (0, \rho_j), \tag{5.2}$$

where we consider $v \in C^\infty((0,\rho_j); D(\Upsilon_j))$. Due to this smoothness, we can now expand it in a series with the eigenfunctions.

Proposition 5.1.12. *Assume that $v \in C^\infty((0,\rho_j); D(\Upsilon_j))$ is a solution of (5.2), and assume that $v \in L^2(D_j)$, where $D_j = U \cap B(S_j, \rho_j)$. Then*

$$v\left(r_j e^{i\theta_j}\right) = v_{1,j}(r_j)\varphi_{j,1}(\theta_j) + \sum_{m\geq 2} \alpha_{j,m} r_j^{\lambda_{j,m}} \varphi_{j,m}(\theta_j) + \sum_{m\,:\,\lambda_{j,m}\in(0,1)} \beta_{j,m} r_j^{-\lambda_{j,m}} \varphi_{j,m}(\theta_j), \tag{5.3}$$

where

$$v_{1,j}(r_j) = \begin{cases} \alpha_{j,1} r_j^{\lambda_{j,1}} & \lambda_{j,1} > 0, \\ \alpha_{j,1} + \beta_{j,1} \log r_j & \lambda_{j,1} = 0, \end{cases}$$

and $\alpha_{j,m}, \beta_{j,m} \in \mathbb{R}$, such that $|\alpha_{j,m}| < L\rho_j^{-\lambda_{j,m}}$ when $\lambda_{j,m} > 1$, and $f_{j,m}(\alpha_{j,m}, \beta_{j,m}) < 0$ for a certain function $f_{j,m}$ depending on ρ_j and $\lambda_{j,m}$ when $\lambda_{j,m} \in (0,1)$.

Proof. Since $(\varphi_{j,m})_m$ is an orthonormal basis for $\mathcal{L}_j$, if we fix $r_j \in (0, \rho_j)$ we can write:

$$v\left(r_j e^{i\theta_j}\right) = \sum_{m\geq 1} v_m(r_j)\varphi_{j,m}(\theta_j), \qquad v_m(r_j) = \int_0^{\omega_j} v\left(r_j e^{i\theta_j}\right)\, \varphi_{j,m}(\theta_j)\, \mathrm{d}\theta_j.$$

Since $v \in C^\infty\left((0,\rho_j); D(\Upsilon_j)\right)$, we can now write:

$$v_m^{''}(r_j) + \frac{1}{r_j} v_m^{'}(r_j) - \lambda_{j,m}^2 \frac{1}{r_j^2} v_m(r_j) = 0 \qquad r_j \in (0,\rho_j),$$

whose solutions are parameterized in a sense by $\lambda_{j,m}$. We know it to be positive, but the critical case is when $\lambda_{j,m} = 0$. In any case, we have:

$$\begin{cases} v_m(r_j) = \alpha_{j,m} r_j^{\lambda_{j,m}} + \beta_{j,m} r_j^{-\lambda_{j,m}} & \lambda_{j,m} > 0; \\ v_m(r_j) = \alpha_{j,m} + \beta_{j,m} \log r_j & \lambda_{j,m} = 0. \end{cases} \tag{5.4}$$

Let us observe that $\lambda_{j,m} = 0$ only when $j \in \mathcal{N}^2$ and $m = 1$.
At this point, since $v \in L^2(D_J)$, we can bound $v_m(r_j)$ by:

$$v_m^2(r_j) = \left(\int_0^{\omega_j} v\left(r_j e^{i\theta_j}\right)\varphi_{j,m}(\theta_j)\,\mathrm{d}\theta_j\right)^2 \leq \int_0^{\omega_j} \left|v\left(re^{i\theta_j}\right)\right|^2 \underbrace{|\varphi_{j,m}(\theta_j)|^2}_{=1}\,\mathrm{d}\theta_j.$$

From this, we can deduce a bound on the L^2 norm of v_m:

$$\int_0^{\rho_j} v_m^2(r_j) r_j\,\mathrm{d}r_j \leq \int_0^{\rho_j}\int_0^{\omega_j} \left|v\left(re^{i\theta_j}\right)\right|^2 r_j\,\mathrm{d}r_j\,\mathrm{d}\theta_j \leq \|v\|^2 < \infty.$$

Knowing that the norm has to be finite, we can substitute for $v_m(r_j)$ obtained in (5.4). When $\lambda_{j,m} = 0$, $v_m(r_j)$ does not depend on $\lambda_{j,m}$, meaning that the integral will always be finite, as can be checked with a quick calculation. When $\lambda_{j,m} > 0$, we have to write:

$$\int_0^{\rho_j} v_m^2(r_j) r_j\,\mathrm{d}r_j = \alpha_{j,m}^2 \underbrace{\int_0^{\rho_j} r_j^{1+2\lambda_{j,m}}\,\mathrm{d}r_j}_{<\infty \text{ if } \lambda_{j,m}>-1} + 2\alpha_{j,m}\beta_{j,m} \underbrace{\int_0^{\rho_j} r_j\,\mathrm{d}r_j}_{<\infty\ \forall\lambda_{j,m}} + \beta_{j,m}^2 \underbrace{\int_0^{\rho_j} r_j^{1-2\lambda_{j,m}}\,\mathrm{d}r_j}_{<\infty \text{ if } \lambda_{j,m}<1}.$$

From this it follows that if $\lambda_{j,m} \in (0,1)$, then the integral is surely finite; however, if $\lambda_{j,m} \geq 1$, then the last integral goes to ∞, hence in this case we require $\beta_{j,m} = 0$.
When $\lambda_{j,m} > 1$, we can just write:

$$|\alpha_{j,m}|^2 \frac{\rho_j^{2+2\lambda_{j,m}}}{2+2\lambda_{j,m}} < \|v\|^2 \implies \alpha_{j,m} < \underbrace{\frac{\|v\|\sqrt{2+2\lambda_{j,m}}}{\rho_j}}_{=:L} \rho_j^{-\lambda_{j,m}}.$$

When $\lambda_{j,m} \in (0,1)$, we write:

$$\alpha_{j,m}^2 \frac{\rho_j^{2+2\lambda_{j,m}}}{2+2\lambda_{j,m}} + \alpha_{j,m}\beta_{j,m}\rho_j^2 + \beta_{j,m}^2 \frac{\rho_j^{2-2\lambda_{j,m}}}{2-2\lambda_{j,m}} - \|v\|^2 < 0,$$

hence the conclusion. □

Let us observe that the case in which $\lambda_{j,m} = 0$ appears to be quite troublesome, due to the presence of the logarithm: this problem solves itself when $v \in N$.
We know that $\lambda_{j,m} = 0$ only when $j \in \mathcal{N}^2$ and $m = 1$, in which case $\varphi_{j,1}(\theta_j) = \sqrt{\frac{1}{\omega_j}}$. We remember now that if $v \in N$ and $j \in \mathcal{N}^2$, we can use the first orthogonality condition of 5.1.9, keeping in mind that η_j is a radial function.

$$\begin{aligned}
0 &= \int_U v\,\Delta\eta_j\,\mathrm{d}x \\
&= \int_0^{\rho_j}\int_0^{\omega_j} v\left(r_j e^{i\theta_j}\right)\Delta\left(\eta_j(r_j)\right) r_j\,\mathrm{d}r_j\,\mathrm{d}\theta_j \\
&= \left(\int_0^{\rho_j} v_1(r_j)\Delta\left(\eta_j(r_j)\right) r_j\,\mathrm{d}r_j\right)\left(\int_0^{\omega_j}\varphi_{j,1}(\theta_j)\,\mathrm{d}\theta_j\right) \\
&\quad + \sum_{m\geq 2}\left(\int_0^{\rho_j} v_m(r_j)\Delta\left(\eta_j(r_j)\right) r_j\,\mathrm{d}r_j\right)\underbrace{\left(\int_0^{\omega_j}\varphi_{j,m}(\theta_j)\,\mathrm{d}\theta_j\right)}_{=0}.
\end{aligned}$$

The integral $\int_0^{\omega_j}\varphi_{j,m}(\theta_j)\,\mathrm{d}\theta_j$ is equal to 0 when $m \geq 2$ because, if $j \in \mathcal{N}^2$, we can substitute and deduce:

$$\int_0^{\omega_j}\varphi_{j,m}(\theta_j)\,\mathrm{d}\theta_j \propto \int_0^{\omega_j}\cos\frac{(m-1)\pi}{\omega_j}\theta_j\,\mathrm{d}\theta_j \propto \int_0^{(m-1)\pi}\cos\psi\,\mathrm{d}\psi = 0.$$

We then get:

$$\begin{aligned}
0 &= \int_0^{\rho_j} v_1(r_j)\Delta\left(\eta_j(r_j)\right) r_j\,\mathrm{d}r_j \int_0^{\omega_j}\varphi_{j,1}(\theta_j)\,\mathrm{d}\theta_j \\
&= \int_0^{\rho_j}\left(\alpha_{j,1}+\beta_{j,1}\log r_j\right)\left(\eta_j''(r_j)+\frac{1}{r_j}\eta_j'(r_j)\right) r_j\,\mathrm{d}r_j \int_0^{\omega_j}\sqrt{\frac{1}{\omega_j}}\,\mathrm{d}\theta_j \\
&= \alpha_{j,1}\sqrt{w_j}\int_0^{\rho_j}\left(r_j\eta_j''(r_j)+\eta_j'(r_j)\right)\mathrm{d}r_j + \beta_{j,1}\sqrt{\omega_j}\int_0^{\rho_j}\left(r_j\log(r_j)\eta_j''(r_j)+\log(r_j)\eta_j'(r_j)\right)\mathrm{d}r_j \\
&= \alpha_{j,1}\sqrt{\omega_j}\left(r_j\eta_j'(r_j)\right)_0^{\rho_j} + \beta_{j,1}\sqrt{\omega_1}\left(r_j\log(r_j)\eta_j'(r_j)-\eta_j(r_j)\right)_0^{\rho_j} \\
&= \alpha_{j,1}\sqrt{\omega_j}\Big(\underbrace{\rho_j\eta_j'(\rho_j)}_{=0}\Big) + \beta_{j,1}\sqrt{\omega_1}\Big(\underbrace{\rho_j\log(\rho_j)\eta_j'(\rho_j)-\eta_j(\rho_j)}_{=0}+\underbrace{\eta_j(0)}_{=1}\Big) \\
&= \beta_{j,1}\sqrt{\omega_j},
\end{aligned}$$

from which follows that $\beta_{j,1} = 0$ for all $j \in \mathcal{N}^2$, as we wanted.
In order to explicitly compute the dimension of N, from the local estimates on the behavior of v near each corner, we have to deduce the global behavior.

Lemma 5.1.13. *For all j and for all $\lambda_{j,m} \in (0,1)$, there exists $z_{j,m} \in N$ such that*

$$z_{j,m} - \eta_j(r_j)\,r_j^{-\lambda_{j,m}}\,\varphi_{j,m}(\theta_j) \;\in\; H^1(U). \tag{5.5}$$

Proof. Let us define $u_{j,m} \coloneqq \eta_j(r_j)\,r_j^{-\lambda_{j,m}}\,\varphi_{j,m}(\theta_j)$. Due to the construction given in previous proofs, we know that $\Delta u_{j,m} = 0$ in $\mathcal{D}(\overline{D_j})$, hence also in $\mathcal{D}(\overline{U})$, and it satisfies the

boundary conditions.
Moreover, as recalled at the beginning of this section, there always exists $v \in H^1(U)$ variational solution of the boundary value problem, such that

$$v \in N, \qquad \int_U \nabla u^\top \nabla v \, \mathrm{d}x = 0 \quad \forall u \in V.$$

Let us define then $z_{j,m} = u_{j,m} + v$. This is from what we have discussed, $z_{j,m} \in D(\Delta, L^2(U))$ and it solves the adjoint problem, simply because it is a linear sum of functions solving the adjoint problem. In order to prove that $z_{j,m} \in N$, we still need to prove the orthogonality conditions of 5.1.9.
If $k \in \mathcal{N}^2$, since $v \in H^1$ and $\eta_k \in \mathcal{D}(\overline{U})$ we can use integration by parts to deduce:

$$\begin{aligned}
\int_U z_{j,m} \Delta \eta_k \, \mathrm{d}x &= \int_U u_{j,m} \Delta \eta_k \, \mathrm{d}x + \int_U v \Delta \eta_k \, \mathrm{d}x \\
&= \int_U u_{j,m} \Delta \eta_k \, \mathrm{d}x - \int_U \nabla v^\top \nabla \eta_k \, \mathrm{d}x + \sum_j \int_{\Gamma_j} \gamma_j v \, \gamma_j \frac{\partial \eta_k}{\partial n_j} \mathrm{d}\sigma \\
&= \int_U u_{j,m} \Delta \eta_k \, \mathrm{d}x + \sum_{j \in \mathcal{N}^2} \int_{\Gamma_j} \gamma_j v \, \gamma_j \frac{\partial \eta_k}{\partial n_j} \mathrm{d}\sigma \\
&= \int_U u_{j,m} \Delta \eta_k \, \mathrm{d}x, \qquad (5.6)
\end{aligned}$$

where the last equality is due to η_k being a radial function. In the integral that is left, if $j \neq k$ then it vanishes since η_j and η_k have disjoint support. If $j = k$ then we have:

$$\begin{aligned}
\int_U u_{j,m} \Delta \eta_j \, \mathrm{d}x &= \int_{D_j} \eta_j(r_j) \, r_j^{-\lambda_{j,m}} \, \varphi_{j,m}(\theta_j) \Delta \eta_j \, \mathrm{d}x \\
&= \int_0^{\rho_j} \eta_j(r_j) \, r_j^{1-\lambda_{j,m}} \Delta \eta_j(r_j) \, \mathrm{d}r_j \underbrace{\int_0^{\omega_j} \varphi_{j,m}(\theta_j) \, \mathrm{d}\theta_j}_{=0} \\
&= 0,
\end{aligned}$$

because if $\lambda_{j,m} \neq 0$, then the eigenfunction, as computed previously, has zero average.
If $k \in \mathcal{M}'$, then it means $j \in \mathcal{N}$, $j+1 \in \mathcal{D}$ and ω_j is either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$. Up to (5.6), the procedure follows analogously. Then as before, if $j \neq k$ the integral is automatically zero. If $j = k$ we are left with:

$$\int_U u_{j,m} \Delta \left(y_k \eta_k \right) \, \mathrm{d}x = \int_0^{\rho_j} \int_0^{\omega_j} \eta_j(r_j) r_j^{1-\lambda_{j,m}} \varphi_{j,m}(\theta_j) \Delta \Big(r_j \sin\left(\theta_j\right) \eta_j(r_j) \Big) \, \mathrm{d}r_j \, \mathrm{d}\theta_j.$$

Let us take a closer look at the Laplacian in the integral: by writing it in polar coordinates, we can deduce:

$$\begin{aligned}
\Delta \Big(r_j \sin\left(\theta_j\right) \eta_j(r_j) \Big) &= \sin(\theta_j) \frac{\partial^2 r_j \eta_j(r_j)}{\partial r_j^2} + \sin(\theta_j) \frac{1}{r_j} \frac{\partial r_j \eta_j(r_j)}{\partial r_j} + \frac{1}{r_j^2} r_j \eta_j(r_j) \frac{\partial^2 \sin(\theta_j)}{\partial \theta_j^2} \\
&= \sin(\theta_j) \, f_1(r_j) + \sin(\theta_j) \, f_2(r_j) - \sin(\theta_j) \, f_3(r_j) \\
&= \sin(\theta_j) \, f(r_j).
\end{aligned}$$

Substituting this in the integral we get:

$$\int_U u_{j,m}\Delta\left(y_k\eta_k\right)\,\mathrm{d}x \propto \int_0^{\rho_j}\eta_j(r_j)r_j^{1-\lambda_{j,m}}f(r_j)\,\mathrm{d}r_j\int_0^{\omega_j}\sin(\theta_j)\,\sin\left(\left(m-\frac{1}{2}\right)\frac{\pi}{\omega_j}\theta_j\right)\,\mathrm{d}\theta_j.$$

Since $j\in\mathcal{M}^{'}$, we can either have $\omega_j=\frac{\pi}{2}$ or $\omega_j=\frac{3\pi}{2}$: from this, we can substitute these two values and get:

$$\omega_j=\frac{\pi}{2}\implies\int_0^{\frac{\pi}{2}}\sin(\theta_j)\,\sin\left((2m-1)\theta_j\right)\mathrm{d}\theta_j=0\qquad\forall m\neq 1;$$
$$\omega_j=\frac{3\pi}{2}\implies\int_0^{\frac{3\pi}{2}}\sin(\theta_j)\,\sin\left(\frac{2m-1}{3}\theta_j\right)\mathrm{d}\theta_j=0\qquad\forall m\neq 2.$$

So the critical cases are when $(\omega_j,m)=(\frac{\pi}{2},1)$ and $(\omega_j,m)=(\frac{3\pi}{2},2)$: in these cases we have:

$$(\omega_j,m)=(\frac{\pi}{2},1)\implies\lambda_{j,m}=\left(m-\frac{1}{2}\right)\frac{\pi}{\omega_j}=\frac{1}{2}\frac{\pi}{\pi}2=1;$$
$$(\omega_j,m)=(\frac{3\pi}{2},2)\implies\lambda_{j,m}=\left(m-\frac{1}{2}\right)\frac{\pi}{\omega_j}=\frac{3}{2}\frac{\pi}{\pi}\frac{2}{3}=1.$$

In any case this is not an acceptable value of $\lambda_{j,m}$, hence the conclusion.
The case $j\in\mathcal{M}^{''}$ follows analogously. □

Theorem 5.1.14. *The dimension of N is equal to:*

$$\dim N=\chi=\sum_j\#\{m:\lambda_{j,m}\in(0,1)\}.$$

Proof. By combining (5.3) and (5.5) we deduce:

$$v\left(r_je^{i\theta_j}\right)-v_1(r_j)\varphi_{j,1}(\theta_j)-\sum_{m\geq 2}\alpha_{j,m}r_j^{\lambda_{j,m}}\varphi_{j,m}(\theta_j)-\sum_{m\,:\,\lambda_{j,m}\in(0,1)}\beta_{j,m}z_{j,m}\;\in\;H^1(D_j).$$

Let us now check that:

$$w_j\coloneqq v_1(r_j)\varphi_{j,1}(\theta_j)+\sum_{m\geq 2}\alpha_{j,m}r_j^{\lambda_{j,m}}\varphi_{j,m}(\theta_j)\;\in\;H^1\left(D\left(\rho_j^{'}\right)\right),$$

where $D(\rho_j^{'})=U\cap B(S_j,\rho_j^{'})$, with $\rho_j^{'}<\rho_j$. It is obvious that $w_j\in L^2(D(\rho_j^{'}))$, but we need to check that its gradient is in $L^2(D(\rho_j^{'});\mathbb{R}^2)$. Writing the gradient in polar coordinates we get:

$$\nabla w_j=\frac{\partial w_j}{\partial r_j}e_{r_j}+\frac{1}{r_j}\frac{\partial w_j}{\partial\theta_j}e_{\theta_j}\implies\begin{cases}\dfrac{\partial w_j}{\partial r_j}=\displaystyle\sum_{m\geq 1}\alpha_{j,m}r_j^{\lambda_{j,m}-1}\lambda_{j,m}\varphi_{j,m}(\theta_j);\\[2ex]\dfrac{1}{r_j}\dfrac{\partial w_j}{\partial\theta_j}=\displaystyle\sum_{m\geq 1}\alpha_{j,m}r_j^{\lambda_{j,m}-1}\varphi_{j,m}^{'}(\theta_j).\end{cases}$$

By defining $h_{j,m} = \max\{1, \lambda_{j,m}\}$, together with the bounds for $|\alpha_{j,m}|$, we can deduce:

$$\begin{aligned}\|\nabla w_j\| &\leq c \sum_{m\geq 1} |\alpha_{j,m}| h_{j,m} r_j^{\lambda_{j,m}-1} \\ &\leq cL \sum_{m\geq 1} h_{j,m} \frac{r_j^{\lambda_{j,m}-1}}{\rho_j^{\lambda_{j,m}}},\end{aligned}$$

which is in $L^2\big(D(\rho_j^{'})\big)$ for $\rho_j^{'} < \rho_j$. So what we just proved is that:

$$v\left(r_j e^{i\theta_j}\right) - \sum_{m\,:\,\lambda_{j,m}\in(0,1)} \beta_{j,m} z_{j,m} \;\in\; H^1\left(D\left(\rho_j^{'}\right)\right).$$

Since we know that v is smooth far from the corners, globally we have:

$$w \coloneqq v - \sum_j \sum_{m\,:\,\lambda_{j,m}\in(0,1)} \beta_{j,m} z_{j,m} \;\in\; H^1(U).$$

By hypothesis on v and 5.1.13 we know that $w \in N \cap H^1(U)$. Due to the compact and disjoint supports of $(\eta_j)_j$, and due to v being harmonic away from corners, we also easily get $\Delta w = 0$. Moreover, due to the properties of $z_{j,m}$, w also satisfies boundary conditions. This, together with the orthogonality condition implied by w being in N, implies

$$\int_U \nabla w^\top \nabla v \,\mathrm{d}x = 0 \qquad \forall v \in V,$$

and thus $w \equiv 0$. Thus we have just shown that:

$$v = \sum_j \sum_{m\,:\,\lambda_{j,m}\in(0,1)} \beta_{j,m} z_{j,m}. \tag{5.7}$$

This means that v is a linear combination of the functions $z_{j,m}$, which are linearly independent due to their support. □

For a general polygonal domain, we can study the contribution of each corner type by studying the eigenvalues in each case, and checking whether they are in (0,1).

- $j \in \mathcal{N}^2$: for $m = 1$ we have $\lambda_{j,1} = 0$, not acceptable; for $m = 2$ we have $\lambda_{j,2} = \frac{\pi}{\omega_j}$, which is less than 1 when $\omega_j > \pi$; for $m \geq 3$ we have $\lambda_{j,m} \geq \frac{2\pi}{\omega_j} \geq 1$, not acceptable.
- $j \in \mathcal{D}^2$: for $m = 1$ we have $\lambda_{j,1} = \frac{\pi}{\omega_j}$, which is less than 1 when $\omega_j > \pi$; for $m \geq 2$ we have $\lambda_{j,m} \geq \frac{2\pi}{\omega_j} \geq 1$, not acceptable.
- $j \in \mathcal{M}$: for $m = 1$ we have $\lambda_{j,1} = \frac{\pi}{2\omega_j}$, which is less than 1 when $\omega_j > \frac{\pi}{2}$; for $m = 2$ we have $\lambda_{j,2} = \frac{3\pi}{2\omega_j}$ which is less than 1 when $\omega_j > \frac{3\pi}{2}$; for $m \geq 3$ we have $\lambda_{j,m} \geq \frac{5\pi}{2\omega_j} \geq \frac{5}{4} > 1$, which is not acceptable.

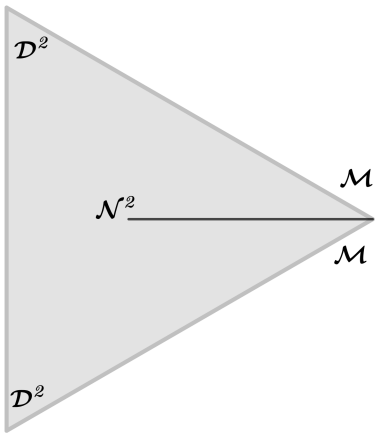

Figure 5.2: Corner types

For the specific choice of our domain, we can see from Figure 5.2 the corner types. It follows that the only contribution from the dimension of N comes from the corner around the crack tip, hence $\dim N = 1$. This also explains why we could restrict our problem to a neighborhood of the crack tip.
What we proved is that basically V^2 is not a big enough space to be sure that there exists a non-trivial solution to $\Delta u = 0$ in a polygonal boundary. We have to augment the space by *adding* a space of the same dimension of N to V^2 because, by the Fredholm alternative theorem, we know that if there exists a non-trivial solution, then the space of such solutions has the same dimension of the orthogonal to the image of Δ. To define this space, we need to define its basis. For a general polygonal boundary, for each corner, and for each eigenvalue, define:

$$S_{j,m}(r_j, \theta_j) \coloneqq \eta_j(r_j)\, r_j^{\lambda_{j,m}}\, \varphi_{j,m}(\theta_j). \tag{5.8}$$

For all that we have discussed, the set of $S_{j,m}$ has the same dimension of N, and every function is the variational solution to the adjoint problem. Moreover, they are linearly independent in V, due to the disjoint supports for different j, and the orthogonality of $\varphi_{j,m}$ for the same j. The final result of this section is the following.

Theorem 5.1.15. *Assume U is a bounded polygonal open subset of $\mathbb{R}^2$. Then there exists a unique variational solution*

$$u \in V, \qquad \int_U \nabla u^\top \nabla v \,\mathrm{d}x = 0 \quad \forall v \in V,$$

and there exist constants $c_{j,m}$ such that:

$$u - \sum_j \sum_{m\,:\,\lambda_{j,m}\in(0,1)} c_{j,m} S_{j,m} \in H^2(U).$$

Proof. Since $S_{j,m} \in H^1(U) \setminus H^2(U)$, we have that $S_{j,m} \notin V^2$. This implies that $\Delta S_{j,m}$ does not belong to the image of V^2 through Δ, which we will denote with R. Since we know that these functions $(S_{j,m})$ are linearly independent in V and they have the same cardinality as the dimension of N, which is the orthogonal of R in $L^2(U)$, we can conclude that each function in $L^2(U)$ can be written as a combination of a function in R and a function $\Delta S_{j,m}$. The conclusion comes from the uniqueness of the variational solution u: basically, if we *remove* the badly-behaved function $S_{j,m}$ from a neighborhood of a corner, the corresponding solution is exactly in V^2, a subspace of $H^2(U)$. □

This result may be extended to a non-homogeneous problem with $f \in L^2(U)$, $g_j \in H^{\frac{3}{2}}(U)$ for $j \in \mathcal{D}$, and $h_j \in H^{\frac{1}{2}}(U)$ for $j \in \mathcal{N}$, but some compatibility conditions for

adjacent edges have to be introduced. Moreover, it can also be extended to curvilinear polygons, which are an extended concept of polygonal domain in which edges are assumed to be arbitrary $C^{1,1}$ curves, meeting in a finite number of corners. The proofs, in either case, require extensive generalizations, but the theorem holds without further assumptions. Just for closure, for our problem, since the dimension of N is just one and $\omega = 2\pi$, we have that in a bounded neighborhood U of the crack tip, there exists a constant κ such that:

$$u - \frac{k}{\mu}\sqrt{r}\cos\frac{\theta}{2} \in H^2(U \setminus K) \cap H^{1,\infty}(U \setminus K), \tag{5.9}$$

where the $H^{1,\infty}$ comes from the fact that both functions have gradients that are bounded on compact sets. We have also avoided the truncation function η_j, since we are assuming to be on a small enough neighborhood of the tip, such that $\eta_j \equiv 1$.
We could extend this theorem to all of Ω, though assuming that $\partial\Omega$ has the necessary regularity to use all of the theory thus written. We would need to be careful though, cause the intersection between $\partial\Omega$ and K may generate additional terms in the global behavior: due to the disjoint support of η_j though, this does not change the local behavior near each corner.
To conclude, (5.9) reminds us of the formula that Irwin derived: in particular, κ now plays the role of the Stress Intensity Factor, which in our two dimensional anti-plane shear is translated to the SIF of the third mode of fracture.

5.2 Derivation of Griffith's criteria

In order to derive Griffith's criteria from this variational model, we have to study the moving nature of the singularity at the crack tip, which we have studied in the previous section. In order to apply the results there given, we will assume the crack to be moving in the same leftward fashion. We will consider the same straight crack, because, for a general crack represented by a C^2 curve, it is always possible to define a diffeomorphism that rectifies the crack in a neighborhood of the tip, therefore making it possible to apply our result.
For simplicity, we will introduce some easy notation. We assume that the crack tip at the beginning is in the origin, the initial crack set has length ℓ_0, and the crack propagation is parameterized by its straight length $\ell(t)$: since we want to study its behavior with respect to the moving of the tip, we will assume the length to be strictly increasing in the time frame we are analyzing, keeping in mind that in any case, the length is at most non-decreasing. For this reason, for $|s - s_0| < \delta$ with $\delta > 0$ small enough, we define:

$$K_{\ell(s_0)} \coloneqq \Big\{x \in \overline{\Omega} : 0 \le x_1 \le \ell(s_0),\ x_2 = 0\Big\};$$
$$K_{\ell(s)} \coloneqq \Big\{x \in \overline{\Omega} : \ell(s_0) - \ell(s) \le x_1 \le \ell(s_0),\ x_2 = 0\Big\}.$$

Define the sets $\Omega_{\ell(s_0)} \coloneqq \Omega \setminus K_{\ell(s_0)}$ and $\Omega_{\ell(s)} \coloneqq \Omega \setminus K_{\ell(s)}$, and then further define:

$$H_{\ell(s_0)} \coloneqq \left\{ w \in H^1(\Omega_{\ell(s_0)}) : w = 0 \text{ on } \partial\Omega \setminus K_{\ell(s_0)} \right\};$$
$$H_{\ell(s)} \coloneqq \left\{ w \in H^1(\Omega_{\ell(s)}) : w = 0 \text{ on } \partial\Omega \setminus K_{\ell(s)} \right\}.$$

Our goal is to study the rate of change of the energy with respect to an increase in crack length: since the fracture energy depends linearly on the length, its derivative will simply be equal to γ, meaning that we can concentrate on the bulk energy, effectively rendering the problem similar to the study of a Dirichlet problem. Since we are interested in the local behavior, we are once again not interested in global boundary conditions, and thus we will restrict ourselves to the local behavior by choosing a bounded open neighborhood N of the crack tip with Lipschitz boundary, where only Dirichlet boundary conditions are imposed, and defining the problem as follows:

$$E(g, K, N) \coloneqq \min_{v \in \mathcal{V}(g,K,N)} \left\{ \frac{\mu}{2} \int_{(\Omega \cap N) \setminus K} |\nabla v|^2 \, \mathrm{d}x + \gamma \, \mathcal{H}^1(K \cap (\overline{N} \cap \overline{\Omega})) \right\},$$
$$\mathcal{V}(g, K, N) \coloneqq \left\{ v \in H^1((\Omega \cap N) \setminus K) : v = g \text{ q.e. on } \partial(N \cap \Omega) \setminus K \right\},$$

where $g \in H^1(\partial(N \cap \Omega) \setminus K)$. Later, when we will actually apply this restricted problem, we will not use the same g as the global problem, but just the restriction of the global displacement solution to the set $\partial(N \cap \Omega) \setminus K$. For ease of notation, we will re-define $\Omega_{\ell(s_0)} = (\Omega \cap N) \setminus K_{\ell(s_0)}$ and $\Omega_{\ell(s)} = (\Omega \cap N) \setminus K_{\ell(s)}$.
Remembering (3.8), with the notation just defined, we introduce these two weak formulations:

$$u_{\ell(s_0)} : \begin{cases} u_{\ell(s_0)} \in H_{\ell(s_0)}, \\ \displaystyle\int_{\Omega_{\ell(s_0)}} \nabla u_{\ell(s_0)}^{\top}(x) \, \nabla w(x) \, \mathrm{d}x = 0 \quad \forall w \in H_{\ell(s_0)}; \end{cases}$$
$$u_{\ell(s)} : \begin{cases} u_{\ell(s)} \in H_{\ell(s)}, \\ \displaystyle\int_{\Omega_{\ell(s)}} \nabla u_{\ell(s)}^{\top}(x) \, \nabla w(x) \, \mathrm{d}x = 0 \quad \forall w \in H_{\ell(s)}. \end{cases}$$

In order to capture the propagation of the tip, since we are interested in vector fields that modify only the crack set, we take a smooth vector field V with compact support in Ω, such that $V(x) = (V_1(x) \quad 0)$, with $V_1 \equiv -1$ in a neighborhood of the origin. Define then $F_{\ell(s)}(x) \coloneqq I(x) + (\ell(s) - \ell(s_0))V(x)$: if $s \in (s_0 - \delta, s_0 + \delta)$ for $\delta > 0$ small enough, then $F_{\ell(s)}$ is a small smooth perturbation of the identity, hence it is invertible, with smooth inverse. As one can easily observe, due to the compact support of V we have that $\Omega_{\ell(s)} = F_{\ell(s)}(\Omega_{\ell(s_0)})$, and this prompts us to also define $U_{\ell(s)} = u_{\ell(s)} \circ F_{\ell(s)}$ and $W = w \circ F_{\ell(s)}$ for every $w \in H_{\ell(s)}$, which now are functions defined on $\Omega_{\ell(s_0)}$.

We can now rewrite the weak formulation for $u_{\ell(s)}$ as:

$$\begin{aligned}
&\int_{\Omega_{\ell(s)}} \nabla u_{\ell(s)}^{\top}(x)\nabla w(x)\,\mathrm{d}x \\
&= \int_{\Omega_{\ell(s_0)}} \left(\nabla F_{\ell(s)}^{-1}(F_{\ell(s)}(x))\,\nabla U_{\ell(s)}(x)\right)^{\top} \left(\nabla F_{\ell(s)}^{-1}(F_{\ell(s)}(x))\,\nabla W(x)\right) \det\left(\nabla F_{\ell(s)}\right)(x)\,\mathrm{d}x \\
&= \int_{\Omega_{\ell(s_0)}} \nabla U_{\ell(s)}^{\top}(x) \underbrace{\left[\nabla F_{\ell(s)}^{-\top}(F_{\ell(s)}(x))\,\nabla F_{\ell(s)}^{-1}(F_{\ell(s)}(x))\,\det\left(\nabla F_{\ell(s)}\right)(x)\right]}_{=:A(x,\ell(s))} \nabla W(x)\,\mathrm{d}x. \quad (5.10)
\end{aligned}$$

So we define the new parameterized operator A as:

$$A(x,\ell(s)) = \nabla F_{\ell(s)}^{-\top}(F_{\ell(s)}(x))\,\nabla F_{\ell(s)}^{-1}(F_{\ell(s)}(x))\,\det\left(\nabla F_{\ell(s)}\right)(x).$$

Since $F_{\ell(s)}$ is a smooth invertible function with smooth inverse, we have that the function $\ell(s) \mapsto A(x,\ell(s))$ is continuously differentiable, and moreover by computation we have that

$$\frac{\partial A(x,\ell)}{\partial \ell}\Big|_{\ell=\ell(s_0)} = -\nabla V^{\top}(x) - \nabla V(x) + I\mathrm{div}(V)(x),$$

which we know to be bounded for a.e. $x \in \Omega_{\ell(s_0)}$ thanks the properties of V.
We define now an operator $T\colon \mathbb{R} \to H_{\ell(s_0)}$ such that $T(\ell) = U_\ell$. It is proved in [19] that this operator is continuously differentiable, hence we can define

$$\dot{U}(\ell(s_0)) \coloneqq \frac{\mathrm{d}T(\ell)}{\mathrm{d}\ell}\Big|_{\ell=\ell(s_0)}.$$

Now, keeping in mind that $U_{\ell(s_0)} = u_{\ell(s_0)} \circ F_{\ell(s_0)} = u_{\ell(s_0)} \circ I = u_{\ell(s_0)}$, we can write:

$$\begin{aligned}
U_{\ell(s)} &= u_{\ell(s_0)} + \big(\ell(s) - \ell(s_0)\big)\frac{\mathrm{d}T}{\mathrm{d}\ell}\Big|_{\ell=\ell(s_0)} + \mathcal{O}\left(\big(\ell(s) - \ell(s_0)\big)^2\right) \\
&= u_{\ell(s_0)} + \big(\ell(s) - \ell(s_0)\big)\dot{U}(\ell(s_0)) + \mathcal{O}\left(\big(\ell(s) - \ell(s_0)\big)^2\right).
\end{aligned}$$

Thanks to this, we can actually write:

$$\begin{aligned}
A(\cdot,\ell(s)) &= I + \big(\ell(s) - \ell(s_0)\big)\frac{\partial A(\cdot,\ell)}{\partial \ell}\Big|_{\ell=\ell(s_0)} + \mathcal{O}\left(\big(\ell(s) - \ell(s_0)\big)^2\right) \\
&= I + \big(\ell(s) - \ell(s_0)\big)\left(-\nabla V^{\top} - \nabla V + I\mathrm{div}(V)\right) + \mathcal{O}\left(\big(\ell(s) - \ell(s_0)\big)^2\right).
\end{aligned}$$

We can now substitute in (5.10), and by calling for brevity $\mathcal{o}\Big(\big(\ell(s)-\ell(s_0)\big)^2\Big) =: \mathcal{o}$ we get:

$$
\begin{aligned}
&\int_{\Omega_{\ell(s)}} \nabla u_{\ell(s)}^\top \nabla w \,\mathrm{d}x \\
&= \int_{\Omega_{\ell(s_0)}} \nabla U_{\ell(s)}^\top A(x,\ell(s))\,\nabla W \,\mathrm{d}x \\
&= \int_{\Omega_{\ell(s_0)}} \Big(\nabla u_{\ell(s_0)}^\top + \big(\ell(s)-\ell(s_0)\big)\nabla \dot{U}^\top(\ell(s_0)) + \mathcal{o}\Big)\Big(I + \big(\ell(s)-\ell(s_0)\big)\frac{\partial A(x,\ell)}{\partial \ell}\Big|_{\ell=\ell(s_0)} + \mathcal{o}\Big)\nabla W \,\mathrm{d}x \\
&= \int_{\Omega_{\ell(s_0)}} \nabla u_{\ell(s_0)}^\top \nabla W dx + \big(\ell(s)-\ell(s_0)\big)\int_{\Omega_{\ell(s_0)}} \nabla u_{\ell(s_0)}^\top \frac{\partial A(x,\ell)}{\partial \ell}\Big|_{\ell=\ell(s_0)} \nabla W \,\mathrm{d}x \\
&+\big(\ell(s)-\ell(s_0)\big)\int_{\Omega_{\ell(s_0)}} \nabla \dot{U}^\top(\ell(s_0))\,\nabla W \,\mathrm{d}x + \mathcal{o}.
\end{aligned}
$$

Due to the weak formulations of $u_{\ell(s)}$ and $u_{\ell(s_0)}$ we have:

$$
0 = \big(\ell(s)-\ell(s_0)\big)\left(\int_{\Omega_{\ell(s_0)}} \nabla u_{\ell(s_0)}^\top \frac{\partial A(x,\ell)}{\partial \ell}\Big|_{\ell=\ell(s_0)} \nabla W \,\mathrm{d}x + \int_{\Omega_{\ell(s_0)}} \nabla \dot{U}^\top(\ell(s_0))\,\nabla W \,\mathrm{d}x\right) + \mathcal{o}.
$$

Dividing by $\big(\ell(s)-\ell(s_0)\big)$ we get:

$$
0 = \int_{\Omega_{\ell(s_0)}} \nabla u_{\ell(s_0)}^\top \frac{\partial A(x,\ell)}{\partial \ell}\Big|_{\ell=\ell(s_0)} \nabla W \,\mathrm{d}x + \int_{\Omega_{\ell(s_0)}} \nabla \dot{U}^\top(\ell(s_0))\,\nabla W \,\mathrm{d}x + \mathcal{o}\Big(\ell(s)-\ell(s_0)\Big),
$$

and taking the limit as $s \to s_0^+$ we get:

$$
0 = \int_{\Omega_{\ell(s_0)}} \nabla u_{\ell(s_0)}^\top \frac{\partial A(x,\ell)}{\partial \ell}\Big|_{\ell=\ell(s_0)} \nabla W \,\mathrm{d}x + \int_{\Omega_{\ell(s_0)}} \nabla \dot{U}^\top(\ell(s_0))\,\nabla W \,\mathrm{d}x.
$$

This result has to be read as an integration by parts of the weak problem, since:

$$
\int_{\Omega_{\ell(s_0)}} \nabla \dot{U}^\top(\ell(s_0))\,\nabla W \,\mathrm{d}x = -\int_{\Omega_{\ell(s_0)}} \nabla u_{\ell(s_0)}^\top \frac{\partial A(x,\ell)}{\partial \ell}\Big|_{\ell=\ell(s_0)} \nabla W \,\mathrm{d}x. \tag{5.11}
$$

Keeping in mind that this result hold for every $W \in H_{\ell(s_0)}$ such that $W = w \circ F_{\ell(s)}$ with $w \in H_{\ell(s)}$, we can apply it with $w = u_{\ell(s)}$. We can therefore write:

$$
\begin{aligned}
E_b(g, K_{\ell(s)}, N) &= \frac{\mu}{2}\int_{\Omega_{\ell(s)}} |\nabla u_{\ell(s)}|^2 \,\mathrm{d}x \\
&= \frac{\mu}{2}\int_{\Omega_{\ell(s)}} \nabla u_{\ell(s)}^\top \nabla u_{\ell(s)} \,\mathrm{d}x \\
&= \frac{\mu}{2}\int_{\Omega_{\ell(s_0)}} \nabla U_{\ell(s)}^\top A(x,\ell(s)) \nabla U_{\ell(s)} \,\mathrm{d}x.
\end{aligned}
$$

Since we know E_b to be absolutely continuous with respect to K, we can now derive with respect to the crack length without having to worry about the domain dependence:

$$\frac{1}{\mu}\frac{\partial E_b(g,K_\ell,N)}{\partial \ell}\Big|_{\ell=\ell(s_0)} = \int_{\Omega_{\ell(s_0)}} \left[\nabla \dot{U}^\top(\ell(s_0))\, A(x,\ell(s_0))\, \nabla u_{\ell(s_0)} + \frac{1}{2}\nabla u_{\ell(s_0)}^\top \frac{\partial A(x,\ell)}{\partial \ell}\Big|_{\ell=\ell(s_0)} \nabla u_{\ell(s_0)}\right] \mathrm{d}x.$$

Thanks to $A(x,\ell(s_0)) = I$ and (5.11), we can easily deduce:

$$\frac{1}{\mu}\frac{\partial E_b(g,K_\ell,N)}{\partial \ell}\Big|_{\ell=\ell(s_0)} = \int_{\Omega_{\ell(s_0)}} \nabla u_{\ell(s_0)}^\top \Big(\nabla V - \frac{1}{2}\mathrm{div}(V) I\Big)\nabla u_{\ell(s_0)}\, \mathrm{d}x. \tag{5.12}$$

We recall from the previous section, that $u_{\ell(s_0)}$ has a particular behavior at the crack tip. More precisely:

$$u_{\ell(s_0)} = R + \frac{\kappa}{\mu} S, \qquad S = \sqrt{r}\cos\frac{\theta}{2},$$

with $R \in H^2(\Omega_{\ell(s_0)}) \cap H^{1,\infty}(\Omega_{\ell(s_0)})$ and $S \in H^1(\Omega_{\ell(s_0)}) \setminus H^2(\Omega_{\ell(s_0)})$. In order to do away with this behavior, the usual way is to integrate on $\Omega_\varepsilon = \Omega_{\ell(s_0)} \setminus B_\varepsilon$ with ε small enough such that $B_\varepsilon \subset N$, and then letting $\varepsilon \to 0$, hoping that the singular terms vanish.

$$\frac{1}{\mu}\frac{\partial E_b(g,K_\ell,N)}{\partial \ell}\Big|_{\ell=\ell(s_0)} = \lim_{\varepsilon\to 0^+} \int_{\Omega_{\ell(s_0)}\setminus B_\varepsilon} \nabla u_{\ell(s_0)}^\top \Big(\nabla V - \frac{1}{2}\mathrm{div}(V) I\Big)\nabla u_{\ell(s_0)}\, \mathrm{d}x.$$

By writing this in components we can greatly reduce this computation, since $V_2(x) \equiv 0$. For now, we will avoid writing the limit at every step, keeping it in mind for later usage:

$$\begin{aligned}
&= \int_{\Omega_{\ell(s_0)}\setminus B_\varepsilon} \Big\{\partial_1 V_1\, (\partial_1 u_{\ell(s_0)})^2 + \partial_2 V_1\, \partial_1 u_{\ell(s_0)}\, \partial_2 u_{\ell(s_0)} - \frac{1}{2}\,\partial_1 V_1 \Big[(\partial_1 u_{\ell(s_0)})^2 + (\partial_2 u_{\ell(s_0)})^2\Big]\Big\}\, \mathrm{d}x \\
&= -\int_{\Omega_{\ell(s_0)}\setminus B_\varepsilon} V_1\, \partial_1 u_{\ell(s_0)} \Big(\partial_{1,1} u_{\ell(s_0)} + \partial_{2,2} u_{\ell(s_0)}\Big)\, \mathrm{d}x \\
&\qquad + \int_{\partial B_\varepsilon} V_1 \Big[\cos(\theta)\frac{(\partial_1 u_{\ell(s_0)})^2 - (\partial_2 u_{\ell(s_0)})^2}{2} + \sin(\theta)\partial_1 u_{\ell(s_0)}\partial_2 u_{\ell(s_0)}\Big]\, \mathrm{d}\sigma.
\end{aligned}$$

The volume term vanishes since $\partial_{1,1} u_{\ell(s_0)} + \partial_{2,2} u_{\ell(s_0)} = \Delta u_{\ell(s_0)}$, and we know that away from corners $\Delta u_{\ell(s_0)} = 0$. Now we choose ε small enough such that $V_1 \equiv -1$, and we are left with:

$$-\frac{1}{\mu}\frac{\partial E_b(g,K_\ell)}{\partial \ell}\Big|_{\ell=\ell(s_0)} = \lim_{\varepsilon\to 0^+} \int_{\partial B_\varepsilon} \Big[\cos(\theta)\frac{(\partial_1 u_{\ell(s_0)})^2 - (\partial_2 u_{\ell(s_0)})^2}{2} + \sin(\theta)\partial_1 u_{\ell(s_0)}\partial_2 u_{\ell(s_0)}\Big]\, \mathrm{d}\sigma.$$

There are two things we now need to do: firstly, we can now substitute $u_{\ell(s_0)} = R + \frac{\kappa}{\mu}S$, in the formula, which, due to the presence of quadratic terms, will be equal to the sum

of three terms, respectively an integral with only R terms, one with only S terms, and one with R and S terms; secondly, since we are integrating on a circle of radius ε, we can write everything in polar coordinates. The right-hand side is now:

$$\begin{aligned}
&= \lim_{\varepsilon\to 0^+} \int_{\partial B_\varepsilon} \left[\cos(\theta) \frac{(\partial_1 u_{\ell(s_0)})^2 - (\partial_2 u_{\ell(s_0)})^2}{2} + \sin(\theta) \partial_1 u_{\ell(s_0)} \partial_2 u_{\ell(s_0)} \right] \mathrm{d}\sigma \\
&= \lim_{\varepsilon\to 0^+} \int_{\partial B_\varepsilon} \left[\cos(\theta) \frac{(\partial_1 R)^2 - (\partial_2 R)^2}{2} + \sin(\theta) \partial_1 R \partial_2 R \right] \mathrm{d}\sigma \\
&\qquad + \frac{\kappa}{\mu} \lim_{\varepsilon\to 0^+} \int_{\partial B_\varepsilon} \left[\cos(\theta) \Big(\partial_1 R\, \partial_1 S - \partial_2 R\, \partial_2 S \Big) + \sin(\theta) \Big(\partial_1 R\, \partial_2 S + \partial_1 S\, \partial_2 R \Big) \right] \mathrm{d}\sigma \\
&\qquad\qquad + \frac{\kappa^2}{\mu^2} \lim_{\varepsilon\to 0^+} \int_{\partial B_\varepsilon} \left[\cos(\theta) \frac{(\partial_1 S)^2 - (\partial_2 S)^2}{2} + \sin(\theta) \partial_1 S \partial_2 S \right] \mathrm{d}\sigma \\
&= \lim_{\varepsilon\to 0^+} \left(R_\varepsilon + \frac{\kappa}{\mu} R S_\varepsilon + \frac{\kappa^2}{\mu^2} S_\varepsilon \right).
\end{aligned}$$

The short end of this computation will be that the terms involving the regular contribution to the displacement will go to 0 as $\varepsilon \to 0$, and to prove this we will need some inequalities. For RS_ε, due to the singular nature of the derivatives of S, we know that

$$|\partial_1 S| \leq \varepsilon^{-\frac{1}{2}}, \qquad |\partial_2 S| \leq \varepsilon^{-\frac{1}{2}}, \qquad \text{on } \partial B_\varepsilon.$$

This allows us to write:

$$\begin{aligned}
|RS_\varepsilon| &\leq \int_{\partial B_\varepsilon} \left[|\cos(\theta)| \Big(|\partial_1 R|\, |\partial_1 S| + |\partial_2 R|\, |\partial_2 S| \Big) + |\sin(\theta)| \Big(|\partial_1 R|\, |\partial_2 S| + |\partial_1 S|\, |\partial_2 R| \Big) \right] \mathrm{d}\sigma \\
&\leq \varepsilon^{-\frac{1}{2}} \int_{\partial B_\varepsilon} 2 \Big(|\partial_1 R| + |\partial_2 R| \Big) \mathrm{d}\sigma \\
&\leq 2\varepsilon^{-\frac{1}{2}} \int_{\partial B_\varepsilon} |\nabla R| \, \mathrm{d}\sigma \\
&\leq 2\varepsilon^{-\frac{1}{2}} \, \|\nabla R\|_{L^2(\partial B_\varepsilon)} \, |\partial B_\varepsilon|^{\frac{1}{2}} \\
&= C_1 \|\nabla R\|_{L^2(\partial B_\varepsilon)}.
\end{aligned}$$

For R_ε the estimate is simpler:

$$\begin{aligned}
|R_\varepsilon| &\leq \int_{\partial B_\varepsilon} \left[|\cos(\theta)| \frac{1}{2} \Big(|\partial_1 R|^2 + |\partial_2 R|^2 \Big) + |\sin(\theta)|\, |\partial_1 R|\, |\partial_2 R| \right] \mathrm{d}\sigma \\
&\leq \int_{\partial B_\varepsilon} \frac{1}{2} |\nabla R|^2 \, \mathrm{d}\sigma + \int_{\partial B_\varepsilon} |\partial_1 R|\, |\partial_2 R| \, \mathrm{d}\sigma \\
&\leq C_2 \int_{\partial B_\varepsilon} |\nabla R|^2 \, \mathrm{d}\sigma \\
&= C_2 \|\nabla R\|^2_{L^2(\partial B_\varepsilon)}.
\end{aligned}$$

In both cases, we can conclude on R_ε and RS_ε if we prove that $\|\nabla R\|_{L^2(\partial B_\varepsilon)} \to 0$ as $\varepsilon \to 0$. From $R \in H^2$ it follows that $\nabla R \in H^1$, hence the trace at the boundary ∂B_ε is

well-defined, and thus we can use the trace inequality to write:

$$\|\nabla R\|^2_{L^2(\partial B_\varepsilon)} \leq C\|\nabla R\|^2_{L^2(B_\varepsilon)} = C\int_{B_\varepsilon} |\nabla R(x)|^2 \,\mathrm{d}x = C\int_{B_1} |\nabla R(x)|^2 \mathbb{1}_{B_\varepsilon}(x)\,\mathrm{d}x.$$

The last integral goes to 0 due to dominated convergence, because it obviously converges pointwise to 0, and the integrand is also bounded by the square of the norm of the gradient, which we know to be in $L^{1,\infty}(B_1) \subset L^1(B_1)$.
Now we can just focus on the last term S_ε:

$$\begin{aligned} S_\varepsilon &= \int_{\partial B_\varepsilon} \left[\cos(\theta)\frac{(\partial_1 S)^2 - (\partial_2 S)^2}{2} + \sin(\theta)\partial_1 S \partial_2 S\right] \mathrm{d}\sigma \\ &= \int_0^{2\pi} \left[\cos(\theta)\frac{(\partial_1 S|_{r=\varepsilon})^2 - (\partial_2 S|_{r=\varepsilon})^2}{2} + \sin(\theta)\partial_1 S|_{r=\varepsilon}\,\partial_2 S|_{r=\varepsilon}\right]\varepsilon\,\mathrm{d}\theta. \end{aligned}$$

Writing the derivatives in polar coordinates we get:

$$\begin{aligned} \frac{\partial\sqrt{r}\cos(\frac{\theta}{2})}{\partial x}\Big|_{r=\varepsilon} &= \frac{1}{2\sqrt{\varepsilon}}\Big(\cos(\theta)\,\cos\Big(\frac{\theta}{2}\Big) + \sin(\theta)\,\sin\Big(\frac{\theta}{2}\Big)\Big); \\ \frac{\partial\sqrt{r}\cos(\frac{\theta}{2})}{\partial y}\Big|_{r=\varepsilon} &= \frac{1}{2\sqrt{\varepsilon}}\Big(\sin(\theta)\,\cos\Big(\frac{\theta}{2}\Big) - \cos(\theta)\,\sin\Big(\frac{\theta}{2}\Big)\Big). \end{aligned}$$

Substituting all of these derivatives, squaring, and summing, we are left at the end with a surprising result: the integrand does not depend on ε, which is something that we hoped for, but most importantly, it does not even depend on θ!

$$\begin{aligned} -\frac{\partial E_b(g, K_\ell, N)}{\partial \ell}\Big|_{\ell=\ell(s_0)} &= \mu\frac{\kappa^2}{\mu^2}\lim_{\varepsilon\to 0^+}\int_0^{2\pi}\frac{1}{8\varepsilon}\Big(\cos^2(\theta)+\sin^2(\theta)\Big)\varepsilon\,\mathrm{d}\theta \\ &= \frac{\kappa^2}{\mu}\lim_{\varepsilon\to 0^+}\int_0^{2\pi}\frac{1}{8\varepsilon}\varepsilon\,\mathrm{d}\theta \\ &= \frac{\pi}{2}\frac{\kappa^2}{2\mu}. \end{aligned}$$

This is a most interesting result: in Griffith's theory, what we have just computed is the Energy Release Rate in terms of the Stress Intensity Factor. In particular, under an easy change of variable, this can be brought up to be precisely the Irwin relation for the third mode (1.10): it suffices to define $\kappa_1 = \kappa\sqrt{\frac{\pi}{2}}$, which also transforms (5.9) into

$$u_{\ell(s_0)} = u^R_{\ell(s_0)} + \frac{\kappa}{\mu}\sqrt{\frac{2r}{\pi}}\cos\left(\frac{\theta}{2}\right).$$

We conclude this long streak of computations by writing the analogous formula including also the fracture energy:

$$\frac{\partial E(g, K_\ell, N)}{\partial \ell}\Big|_{\ell=\ell(s_0)} = -\frac{\pi}{2}\frac{\kappa^2}{2\mu} + \gamma. \tag{5.13}$$

This is the formula that we will be using for deriving Griffith's criteria: in fact, this is precisely equal to $G - G_c$.
Let us take now the continuous evolution $K\colon [0,1] \to \mathcal{K}^f_m(\overline{\Omega})$: our goal is now to study the propagation by parameterizing an already existing evolution with respect to the crack length. Let us take $0 \le t_0 < t_1 \le 1$, which is the time interval that we will be analyzing. Given a certain initial crack K_0 corresponding to the crack set at time $t = t_0$, we assume that the successive evolution can be described by a simple arc $\Gamma \subset \overline{\Omega}$, parameterized by arc length by a C^2 path $\phi\colon [\ell_0, \ell_1] \to \overline{\Omega}$, where ℓ_0 and ℓ_1 are the initial and final length respectively, and there is a non-decreasing length function $\ell\colon [t_0, t_1] \to [\ell_0, \ell_1]$ such that:

$$K(t) = K(t_0) \cup \Gamma(\ell(t)) \qquad \Gamma(\ell) \coloneqq \Big\{\phi(\mathbb{l}) : \ell_0 \le \mathbb{l} \le \ell\Big\}.$$

In order to prove Griffith's criteria, we first need a preliminary result connecting the global solution of $E(g, K)$ to the local solution of $E(g, K, N)$.

Lemma 5.2.1. *Let $m \ge 1$, and $H \in \mathcal{K}^f_m(\overline{\Omega})$ with $h \le m$ connected components. Let g be a function in $H^1(\Omega)$, and u be the solution to the minimum problem defining $E(g, H)$. Take a bounded open subset N of $\overline{\Omega}$ with Lipschitz boundary such that $H \cap \overline{N} \neq \varnothing$, meaning that this set is a neighborhood of the crack tip. Suppose also that there are $q \le h \le m$ connected components of H that intersect $\overline{N}$. Now, if we have*

$$E(g, H) \le E(g, K) \qquad \forall K \in \mathcal{K}^f_m(\overline{\Omega}),\ K \supseteq H,$$

then we also have

$$E(u, H, N) \le E(u, K, N) \qquad \forall K \in \mathcal{K}^f_{q+m-h}(\overline{N}),\ K \supseteq H \cap \overline{N}.$$

Proof. Let us take an arbitrary $K \in \mathcal{K}^f_{q+m-h}(\overline{N})$ such that $K \supseteq H \cap \overline{N}$, and define v as the solution to the minimum problem defining $E(u, K, N)$. Since v is defined on $N \setminus K$, we can extend it by taking u on the rest of the domain. We then define:

$$w \coloneqq \begin{cases} v & \text{on } \overline{N} \setminus K, \\ u & \text{on } (\overline{\Omega} \setminus \overline{N}) \setminus H. \end{cases}$$

Due to the formulation of $E(u, K, N)$ we have that $v = u$ q.e. on $\partial N \setminus K$, meaning that the function w does not have jumps across $\partial N \setminus K$, and hence $H^1(\Omega \setminus (H \cup K))$. Moreover, since $u = g$ q.e. on $\partial_D \Omega \setminus H$, then we also have $w = g$ q.e. on $\partial_D \Omega \setminus (H \cup K)$, meaning that w is an admissible function for the minimum problem defining $E(g, H \cup K)$, and thus:

$$\begin{aligned} E(g, H \cup K) &\le \frac{\mu}{2} \int_{\Omega \setminus (H \cup K)} |\nabla w|^2 \,\mathrm{d}x + \gamma\, \mathcal{H}^1(H \cup K) \qquad (5.14) \\ &\le \frac{\mu}{2} \int_{N \setminus K} |\nabla v|^2 \,\mathrm{d}x + \gamma\, \mathcal{H}^1(K \cap \overline{N}) + \frac{\mu}{2} \int_{(\Omega \setminus N) \setminus H} |\nabla u|^2 \,\mathrm{d}x + \gamma\, \mathcal{H}^1(H \setminus \overline{N}). \end{aligned}$$

Due to the minimality of u for the problem defining $E(g,H)$, we have:

$$E(g,H) = \frac{\mu}{2}\int_{\Omega\setminus H}|\nabla u|^2\,\mathrm{d}x + \gamma\,\mathcal{H}^1(H) \tag{5.15}$$
$$= \frac{\mu}{2}\int_{N\setminus H}|\nabla u|^2\,\mathrm{d}x + \gamma\,\mathcal{H}^1(H\cap\overline{N}) + \frac{\mu}{2}\int_{(\Omega\setminus N)\setminus H}|\nabla u|^2\,\mathrm{d}x + \gamma\,\mathcal{H}^1(H\setminus\overline{N}).$$

Our goal now is to apply the second condition of the continuous evolution to the sets H and $H\cup K$: in order to do this, since we obviously have $H\cup K\supseteq K$, we just need to prove that $H\cup K\in\mathcal{K}^f_m(\overline{\Omega})$. We know from the hypotheses that the number of connected components of H intersecting $\overline{N}$ is $q\le h\le m$. Since H has at $q\le m$ connected components, it follows that the number of connected components of H not intersecting $\overline{N}$ is $h-q$. Then every connected component of $H\cup K$ intersecting $\overline{N}$ contains also a connected component of K because $\overline{N}\cap K\neq\varnothing$. Since the number of connected components of K is at most $q+m-h$, it follows that the total number of connected components of $H\cup K$ is at most $(h-q)+(q+m-h)=m$. Then it follows that $H\cup K$ is an admissible competitor for $E(g,H)$. From:

$$E(g,H)\le E(g,H\cup K),$$

using (5.14) and (5.15) we can deduce:

$$\frac{\mu}{2}\int_{N\setminus H}|\nabla u|^2\,\mathrm{d}x + \gamma\,\mathcal{H}^1(H\cap\overline{N}) + \frac{\mu}{2}\int_{(\Omega\setminus N)\setminus H}|\nabla u|^2\,\mathrm{d}x + \gamma\,\mathcal{H}^1(H\setminus\overline{N})$$
$$\le \frac{\mu}{2}\int_{N\setminus K}|\nabla v|^2\,\mathrm{d}x + \gamma\,\mathcal{H}^1(K\cap\overline{N}) + \frac{\mu}{2}\int_{(\Omega\setminus N)\setminus H}|\nabla u|^2\,\mathrm{d}x + \gamma\,\mathcal{H}^1(H\setminus\overline{N}).$$

After a little simplification, we are left with:

$$\frac{\mu}{2}\int_{N\setminus H}|\nabla u|^2\,\mathrm{d}x + \gamma\,\mathcal{H}^1(H\cap\overline{N}) \le \frac{\mu}{2}\int_{N\setminus K}|\nabla v|^2\,\mathrm{d}x + \gamma\,\mathcal{H}^1(K\cap\overline{N}).$$

We know that v is the minimum for the problem defining $E(u,K,N)$, but it is also obvious that u is admissible for the minimum problem defining $E(u,H,N)$. We thus write:

$$E(u,H,N) \le \frac{\mu}{2}\int_{N\setminus H}|\nabla u|^2\,\mathrm{d}x + \gamma\,\mathcal{H}^1(H\cap\overline{N})$$
$$\le \frac{\mu}{2}\int_{N\setminus K}|\nabla v|^2\,\mathrm{d}x + \gamma\,\mathcal{H}^1(K\cap\overline{N}) = E(u,K,N).$$

We then conclude by the arbitrariness of K. □

Theorem 5.2.2. *Let $m\ge 1$, and let $K\colon[0,1]\to\mathcal{K}^f_m(\overline{\Omega})$ be a continuous evolution satisfying the hypothesis for Theorem 4.0.1. Let $g\in AC\big([0,1],H^1(\Omega)\big)$, and let u be the solution to the minimum problem defining $E(g(t),K(t))$. Let $0\le t_0<t_1\le 1$ and let*

$K(t) = K(t_0) \cup \Gamma(\ell(t))$ with the same notation as before. Then we have:

$$\begin{cases} \dot{\ell}(t) \geq 0 & \text{for a.e. } t \in (0,1), \\ \gamma - \dfrac{\pi}{2}\dfrac{\kappa^2}{2\mu} \geq 0 & \text{for every } t \in (0,1), \\ \left(\gamma - \dfrac{\pi}{2}\dfrac{\kappa^2}{2\mu}\right)\dot{\ell}(t) = 0 & \text{for a.e. } t \in (0,1). \end{cases} \tag{5.16}$$

Proof. Take $t \in (t_0, t_1)$ such that ℓ is continuous at t and such that the derivative $\dot{\ell}(t_0)$ exists (this is possible since the derivative is defined almost everywhere).
Define a ball B as the neighborhood N of the local minimum, centered on the crack tip $\phi(\ell(t))$, such that $\overline{B} \subset \Omega$, and with a small enough radius such that $\overline{B} \cap K(t_0) = \varnothing$. We can assume, for a small enough radius, that:

$$B \cap \Gamma(\ell_1) = \left\{\phi(\ell) : \ell_0^B < \ell < \ell_1^B\right\}, \qquad \ell_0 < \ell_0^B < \ell(t) < \ell_1^B < \ell_1.$$

Again, for a small enough radius, we may also assume that $\Gamma(\ell_1)$ intersects ∂B only at $\phi(\ell_0^B)$ and $\phi(\ell_1^B)$, in particular not tangentially.
All of these assumptions imply the possibility of a local parameterization inside $\overline{B}$, namely:

$$\overline{B} \cap K(s) = \overline{B} \cap \Gamma(\ell(s)) = \left\{\phi(\ell) : \ell_0^B \leq \ell \leq \ell(s)\right\}, \qquad \ell_0^B \leq \ell(s) \leq \ell_1^B, \tag{5.17}$$

which remains valid for $s = t$, or for every s sufficiently close to t if t is a continuity point for ℓ.
Since we are considering just a simple branch, $m = 1$, and from Theorem 4.0.1 we have that:

$$E(g(t), K(t)) \leq E(g(t), K), \qquad \forall K \in \mathcal{K}_1^f(\overline{\Omega}),\ K \supseteq K(t).$$

Thanks to this, from Lemma 5.2.1 applied to B, we deduce:

$$E(u(t), K(t), B) \leq E(u(t), K, B), \qquad \forall K \in \mathcal{K}_1^f(\overline{B}),\ K \supseteq K(t) \cap \overline{B}.$$

This, together with the local re-parameterization (5.17), implies:

$$E(u(t), \Gamma(\ell(t)), B) \leq E(u(t), \Gamma(\ell), B), \qquad \ell(t) \leq \ell \leq \ell_1^B.$$

Due to the absolute continuity of the energy and formula (5.13), this implies:

$$\frac{\partial E(u(t), \Gamma(\ell), B)}{\partial \ell}\Big|_{\ell=\ell(t)} \geq 0 \quad \Longrightarrow \quad \gamma - \frac{\pi}{2}\frac{\kappa^2}{2\mu} \geq 0.$$

Since the first condition of (5.16) is true for a.e. $t \in (0,1)$ given the assumptions of the theorem, we just need to prove the third condition.
From Theorem 4.0.1, we know that

$$\frac{\partial E(g(t), K(s))}{\partial s}\Big|_{s=t} = 0, \qquad \text{for a.e. } t \in (0,1). \tag{5.18}$$

Let us choose a $t \in (0,1)$ such that this derivative exists, and also that $\dot{\ell}(t)$ exists. We can then take s close enough to t such that, due to (5.17) and to the properties of the integral and $\mathcal{H}^1$-measure, we have:

$$E\Big(g(t), K(s)\Big) \leq E\Big(u(t), \Gamma\big(\ell(s)\big), B\Big) + E\Big(u(t), K(t_0) \cup \Gamma(\ell_0^B), \overline{B}^{\complement}\Big). \tag{5.19}$$

This is in general an inequality, but observe that it trivially becomes an equality for $s = t$. We know from 4.0.1 that the functions $s \mapsto E\big(g(t), K(s)\big)$ and $s \mapsto E\big(u(t), \Gamma(\ell(s)), B\big)$ are absolutely continuous on (0,1), meaning that they have an integrable derivative almost everywhere. We chose t precisely such that these derivatives exist at $s = t$, which is something that we can do for a.e. $t \in (0,1)$. Thus, since the last term in the (5.19) does not depend on s, we can derive and thanks to (5.18), we deduce:

$$\begin{aligned}
0 &= \frac{\partial E(g(t), K(s))}{\partial s}\Big|_{s=t} && (5.20)\\
&= \frac{\partial E(u(t), \Gamma(\ell(s)), B)}{\partial s}\Big|_{s=t} && (5.21)\\
&= \frac{\partial E(g(t), \Gamma(\ell), B)}{\partial \ell}\Big|_{\ell=\ell(t)} \cdot \frac{\mathrm{d}\ell(s)}{\mathrm{d}s}\Big|_{s=t} && (5.22)\\
&= \left(\gamma - \frac{\pi}{2}\frac{\kappa^2}{2\mu}\right)\dot{\ell}(t). && (5.23)
\end{aligned}$$

We can then conclude by noting that this is true for a.e. $t \in (0,1)$. □

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