Analysis of masonry arches and vaults

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Summary

A growing interest in the preservation of historic structures has created a need for methods for the analysis of load-bearing unreinforced masonry structures, such as arches, vaults, and buttresses. Although the plasticity methods, first applied to medieval structures in detail by Heyman, provide a useful and intuitive approach to the understanding of the behaviour of masonry arches and vaults, their usefulness in performing actual assessments of such structures has limitations. The constitutive laws of the materials used in masonry structures are not always amenable to accurate treatment by the rigid–plastic simplification, and the complexity of many vaulted masonry structures makes the application of these methods difficult. Moreover, empirical studies have shown that these structures may be subject to three-dimensional effects that are not entirely addressed by the application of plastic or elastic analysis in two dimensions. Progress has been made recently in the development of constitutive laws for ancient masonry structures and in the application of these to the analysis of unreinforced masonry structural systems. Various formulations of three-dimensional finite element analysis, including discrete element methods, and plasticity methods have also proven useful.

Key words: masonry; arch; vault; plasticity; finite element; analysis

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Introduction and scope

A growing interest in the stabilization, preservation, and repair of heritage and historic structures has created a need for accurate and efficient methods for the analysis of load-bearing unreinforced masonry structures. The principal components of these structures, whether medieval church buildings or nineteenth-century bridges, are masonry arches, vaults, and buttresses. Although these structures have endured for hundreds of years and despite the fact that they are one of the earliest types of structures to be subjected to scientific structural analysis, they are often considered difficult to analyse or to assess accurately. This attitude is due only in part to the usual uncertainties in assessing built structures: the presence of unknown or uncertain material properties, the difficulty of obtaining measurements, and the difficulty of assessing the conditions or construction without the ability to inspect the interior of the structure. The complexities of analysis are also due to the complexity of the system of which these elements are a part: a medieval cathedral is a convoluted assembly of piers, vaults, arches, and buttresses that all work together, and for which the load paths are not always obvious.

This paper reviews, from a very personal point of view, recent developments in the analysis of masonry arches and vaults. I have chosen to limit the scope of this paper in two ways. First, this review, with only occasional exceptions, is limited to papers published in refereed journals and to doctoral dissertations. Without having time to review critically every paper that has come across my desk, I find it necessary to confine the scope of this discussion to work that has already been peer-reviewed. Second, beyond the initial discussion of some of the papers that opened up this area of research, I am confining this review primarily to papers that have appeared in print from 1990 onwards. Certain earlier developments will be mentioned, but because the scope of this journal is a review of recent progress, this limitation is appropriate. Finally, it is inevitable that I have overlooked significant papers that lie within the scope that I have outlined. Where this has happened, I apologize to these authors for the incompleteness of my research, and invite them to correspond with me, so that we may attempt to set the record straight.

The initial papers on the topic of masonry arch and vault analysis in the late twentieth century were motivated by their authors' interest in Gothic architecture, and in explaining, using the vocabulary

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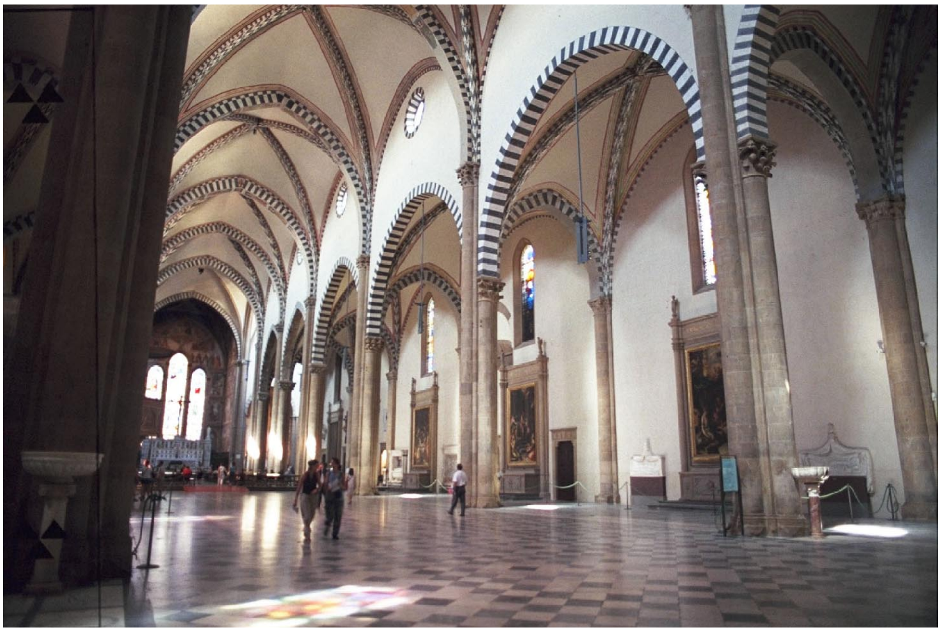

Fig. 1 A complex system of medieval vaulted masonry, Santa Maria Novella, Florence, Italy

of a modern engineer, the functioning of the elements of a Gothic cathedral: arches, roof vaults, piers, and buttresses. The principal papers, which are discussed in the next section, are disarming in their simplicity and clarity, and extraordinary in their ability to put an ancient art into the context of the most modern type of limit analysis, referencing directly the plastic theory of structures.

Within a decade, the combined effect of an increasing interest in and public awareness of the preservation of engineering heritage, an explosive growth in road and rail transportation demands, and the understanding of the very large number of masonry arch bridges in the existing road and rail infrastructure created a very real and practical need for assessment and repair methods for masonry arch bridges. This has resulted in the adoption of some of the literature meant to improve understanding of the arch as used in medieval construction to the development of tools for the assessment of the present population of masonry arch bridges.

Vaults come in various forms, including the barrel vault, which is an arch made wide enough to form a roof vault, groined vaults, composed of barrel vaults intersecting at right angles, and rib vaults, characteristic of gothic architecture. Fig. 1 shows a system of Gothic rib vaults, and Fig. 2 shows a typical masonry arch bridge. Although a masonry arch bridge is superficially an instance of a barrel vault, substantial differences exist between the masonry arch as applied to a bridge and the same principles applied to vaulting of a medieval building. The loads on a masonry bridge are dynamic and concentrated, and the principle of the arch is less effective in resisting the loads due to a heavy axle than a distributed gravity load. The primary load on a masonry roof structure is simply the self-weight of the material. Thus, Heyman's[1] postulate, written with a clear reference to medieval construction, 'If, on striking the centering for a flying buttress, that buttress stands for five minutes, then it will stand for 500 years' cannot be generalized to any part of a bridge structure. In the masonry bridge, the concentrated axle loads are compensated for by using much thicker voussoirs than in any sort of building, and by adding a large mass of fill to prestress and stabilize the vault. The presence of the fill adds complexities to the masonry arch bridge that are not present in the building structure. The spandrel walls that retain the fill on the sides of the bridge function as gravity retaining walls: soil pressures due to the fill and to loads superimposed on the fill cause transverse stresses in the spandrel walls and in the barrel vault of the arch that are potentially quite destabilizing. The springing for an arch bridge is very solid, and rarely more than 1–2 m above the ground line for the bridge, whereas the spring lines for roof vaults may be up to 20–30 m above the ground, and supported by complex networks of piers and buttresses. Long-term movements of the pier tops may be inevitable, and are of course very important for the overall stability of the structure, whereas this is hardly an issue when considering masonry bridges.

The different demands of the bridge and building structures are reflected in the very different characters of the papers written on the two topics. Papers focused on the analysis of masonry bridges tend to be practice oriented, use significant empirical and experimental evidence, discuss details of assessment methods, and offer some practical solutions to assessment and repair problems. Ultimate strength tests on masonry arch bridges conducted during the 1980s, and well

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Fig. 2 A typical masonry arch bridge, Griffith Bridge, Dublin, Ireland

summarized by Page[2], are often cited in the arch bridge literature. These bridge tests are often used to calibrate proposed models and to test results of analysis efforts, and are usually designated by the location of the bridge: Bridgemill, Bargower, and Preston are noted below.

Papers focused on building vaults tend to be more general, more conjectural, and have less opportunity to introduce positive confirming evidence (although several redundant masonry bridges have been available for destructive testing, one hardly expects to see testing of redundant cathedrals). However, the analysis of masonry building vaults has introduced more complex constitutive laws, including time-dependent effects, and more comprehensive three-dimensional analysis into the discussion of the analysis of masonry arches and vaults.

Without attempting an exhaustive review of the literature prior to 1990, I would like to begin this review with a brief discussion of a few classic papers that motivated much of the further research on this topic.

Three classic papers

Pippard, Trantner & Chitty[3] present the results of a combined experimental and analytical investigation of model masonry arches. The authors acknowledge the usual design procedure of the time, motivated by the understanding that the mortar joints are very weak in tension, of confining the thrust line to the middle third of the arch to ensure that compressive stresses only are present. However, it is noted that larger eccentricities produce a condition similar to the development of a plastic hinge, and this property is exploited in the analysis of arches. In this work, the arch under dead load is generally considered to be a two-hinged arch, and the horizontal thrust is chosen as the redundant quantity and solved for by Castigliano's theorem. For a two-hinged arch, the influence line of concentrated loading on a voussoir is developed by noting that, under the influence of a significant concentrated load, the arch will develop a third hinge and become statically determinate. The results of careful experiments are compared with the results of the analysis. The experiments also led to the observation that, in general, an arch produces initial settlement at the supports of the arch that tend to put the structure into a two-hinged configuration. The combination of experimental observations and rigorous analysis, in describing a structure for which approximate analysis methods were widespread, was very influential. Also of far-reaching importance were the departures from complete reliance on elastic analysis, and the use of observations to support the idea that the structure can support loads producing high eccentricities of the thrust.

These ideas were later refined and developed into an important tool for masonry arch analysis. Heyman[1], who was also influential in the development of plastic design methods for steel structures, applied the same principles to unreinforced masonry structures. The following simplifying assumptions are introduced into this analysis:

- the masonry units are infinitely rigid;
- the masonry units are infinitely strong;
- the masonry units do not slide at the joints;
- the joints transmit no tension.

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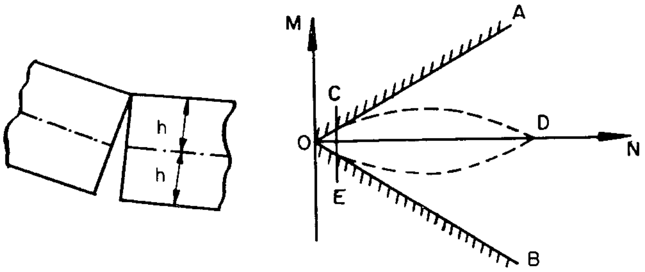

Fig. 3 Heyman's yield locus, and the hinging condition at a joint in a masonry arch [1]. Reprinted from *International Journal of Solids and Structures*, Heyman, J. The Stone Skeleton. pp. 249–279, copyright 1966, with permission from Elsevier Science

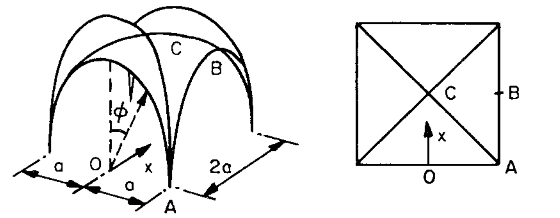

Fig. 4 A simplified vault, as analysed by Heyman[1]. Reprinted from *International Journal of Solids and Structures*, Heyman, J. The Stone Skeleton. pp. 249–279, copyright 1966, with permission from Elsevier Science

The assumptions are justified in their application to medieval stone construction on the grounds that the compressive stresses in the masonry are very low, and the deformations of ashlar masonry are very small. Under these assumptions, one can consider a yield locus (Fig. 3) in the axial force N and the bending moment M with an associated flow rule in the coordinates representing the axial and rotational displacements. The consequence of these assumptions, then, is that the bounding theorems of plasticity are directly applicable to the determination of a collapse load for a masonry arch and vault.

The theory of engineering plasticity incorporates two theorems: a lower bound, equilibrium, or safe theorem; and an upper bound, mechanism, or unsafe theorem. In the usual statement of these theorems, the loading is considered proportional, and parameterized by a load factor. The lower bound theorem states that if, for a given load factor, a statically admissible distribution of internal forces can be found that everywhere satisfies the yield condition, then the load factor is a lower bound on the collapse load. According to the upper bound theorem, a load factor for which the virtual work of the loads is equal to the virtual work of the internal forces as the structure works through a kinematically admissible mechanism – then the load factor is an upper bound on the collapse load factor. A uniqueness theorem shows that if both conditions are satisfied by a given load factor, then this must be the collapse load factor. Horne[4] provides a summary of these notions. For an arch, these notions are further simplified by the understanding that the yield condition can be represented by a thrust line superimposed on a drawing of the arch structure, and that the yield condition is represented by the thrust line being everywhere inscribed within the arch. The mechanism condition is also particularly attractive for analysis, because it is not particularly hard to impose mechanisms on a given arch and assess the virtual work. The paper gives further examples of calculation of the equilibrium condition for various configurations of buttresses in Gothic cathedrals. The discussion of vaults relies solely on equilibrium or lower bound analysis, as it is very difficult to attempt to work out the kinematics of a vault collapse mechanism. However, using the membrane equations of equilibrium, Heyman is able to deduce the stability of a number of three-dimensional vault configurations from the equilibrium condition. An example of the simplified vaults considered by Heyman is shown in Fig. 4.

Livesley[5] notes that the equilibrium and mechanism formulations of limit analysis, as applied by Heyman to the masonry arch and vault, are amenable to solution by linear programming, where the objective function is the maximization of the collapse load factor, subject to constraints furnished by the first-order equilibrium equations. He reduces the stresses in the joint of a voussoir to resultants at the intrados and the extrados. Heyman's assumptions can be recovered by constraining both normal forces to unbounded non-negative values. Under the four assumptions given above, he is able to reach the solution of collapse loads and collapse modes by application of a linear programming algorithm. Livesley is the first to attempt to relax the assumption that the blocks do not slide at the joints, but notes the computational difficulties that this poses owing to the non-associated flow rule for a joint subject to Coulomb friction.

Rigid–plastic analysis

Many authors, following the basic assumption set of the Heyman and the Livesley papers discussed in the preceding section, have elaborated the theories set out in these papers. Harvey[6], while embracing the rigid–plastic theory, discusses the apparent limitations of its implementation: uncertainties in the material properties and second order effects, especially those due to the deformations of the mortar. He adopts a highly simplified equilibrium approach, allowing the uncertainties of the analysis to be reflected in the use of a zone of thrust, which is notionally a volume in which the actual thrust line of the structure could be constructed. Smith, Harvey & Vardy[7] note that, in the context of equilibrium or lower bound analysis, it is generally a simple matter to locate the first three

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hinges, and the structure can then be analysed as a statically determinate structure to determine the load at which the fourth hinge forms. They implement the zone of thrust concept outlined above, note the loss of effective cross-section of the arch ring due to presumed tension cracking, and further discuss the sensitivity of arch structures to the spreading of the abutments. Many of the concepts outlined in these two papers have been implemented in a widely used computer program for the equilibrium analysis of arch bridges[8].

Gilbert & Melbourne[9], like Livesley[5], pose the problem of masonry arch stability as the mathematical procedure of maximizing the applied load subject to the constraints posed by restrictions on interpenetration of adjacent blocks. They adopt Heyman's assumptions, with the exception that they permit sliding between adjacent blocks, incorporating friction into the equilibrium equations under the assumption of normality, which Drucker[10] noted produces an upper bound solution. In this implementation, the program is linearized and the system is solved by the simplex algorithm. This solution is implemented to produce very realistic failures of multi-ring brick arches, including the stiffening effect of spandrel walls.

Boothby & Brown[11] pose the problem of arch stability as a mathematical program, under Heyman's assumptions. In this case, the objective function is to minimize the potential energy (the system is shown to be conservative) of the voussoirs in an arch subject to kinematic constraints. They are able to identify the upper and lower bound theorems with fundamental stability criteria, refuting Vilnay & Cheung's[12] assertion that a separate stability analysis is required in rigid–plastic analysis of masonry arches.

The attractiveness of the mechanism method for masonry arch analysis is not carried through to masonry domes and vaults. The mechanisms appear generally too complex to be visualized and for the kinematics to be formulated. Oppenheim, Gunaratnam & Allen[13] did consider both equilibrium and mechanism analysis of axisymmetrically loaded masonry domes, by considering the kinematics of a lune, or wedge incorporating the apex, of the dome, but this work has not been extended to other vault shapes, or arbitrary loading. O'Dwyer[14] presents the equilibrium solution of simplified masonry vaults by considering equilibrium on a lattice of bars incorporated within the thickness of the masonry. Beginning from an initial lattice based on qualitative analysis of the load paths, he applies linear programming, in the manner of Livesley[5] without the refinement of allowing sliding between blocks, to the solution of the maximum load factor for a concentrated load applied to the vault. As is typical for linear programming problems, a number of the bar forces are non-basic variables in the solution, that is, equal to zero. As a result, the remaining non-zero bar forces give some sense of the load paths for the resistance of a concentrated load applied to the vault.

Smars[15], in a recent doctoral dissertation, applies both standard and non-standard plasticity (associated and non-associated flow rule) to the analysis of Gothic masonry vaults in Brabant, a province of Belgium. He conducts an exhaustive review of the merits of using various formulations of non-standard plasticity, and the characteristics of the vaulting in the buildings under study. His analysis of vaults is a computerized implementation of the scheme from Wolfe[16], discussed below in the section entitled Finite Element Analysis.

Lucchesi *et al.*[17] present a strictly mathematical approach to the rigid–plastic analysis of masonry vaults, and present a closed-form example solution to a specific instance of a vault consisting of a quadrant of a torus.

COMMENTARY ON RIGID-PLASTIC ANALYSIS

The rigid–plastic analysis is a compelling treatment of the stability of masonry arches and vaults, and has advanced modern understanding of ancient construction techniques considerably. It is particularly useful in its appeal to physical intuition; the equilibrium method addresses the way that structures can redistribute internal forces to resist external loading, and the mechanism method appeals to the understanding of the way that a structure is expected to collapse. However, the assumptions made in invoking rigid–plastic analysis limit the ability to address apparent phenomena in the behaviour of actual arch and vault structures. Although the assumption of rigidity is fundamentally justified for the units in ashlar masonry, the mortars used in ancient masonry construction are deformable and have been found to exhibit significant deformation over time. Masonry units do fracture, and differences in properties between facing and core masonry present legitimate analytical difficulties. Neither the equilibrium method nor the mechanism method lend themselves comfortably to application to the analysis of structures with complex three-dimensional geometry, such as cathedral vaults or skew bridges. The limitations of rigid–plastic analysis can be addressed either by further investigations in the framework of engineering plasticity, or by computer tools for the investigation of complex geometry and constitutive laws. Both of these approaches have been initiated in the study of the analysis of masonry arches and vaults.

Classical analysis methods

Rigid–plastic analysis has not been universally embraced as the preferred means for the analysis of

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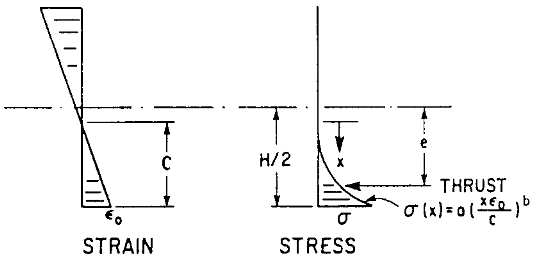

Fig. 5 Stress distribution at a cracked mortar joint, according to McNeely, Archer & Smith [19]. Courtesy of National Research Council, Canada

masonry arches and vaults. Further work on two-dimensional methods of masonry arch bridge assessment, and two- and occasionally three-dimensional analysis of masonry vaults has been pursued through elastic analysis.

Bridle & Hughes[18] present a method for masonry arch analysis, specifically adapted to bridge structures, invoking Castigliano's theorem to solve for redundant quantities. Since they propose to account for the loss of cross-section when cracking develops (presumably for any tension within the arch ring), the arch ring has to be divided into elements and solved iteratively under increments of loading. They note that a very large portion of the arch volume has become ineffective at the onset of failure, and caution the possibility of buckling as a result of the substantially reduced stiffness. They are able to compare their deflection results favourably with the test at Bridgemill, but note that the choice of modulus of elasticity is somewhat arbitrary.

McNeely, Archer & Smith[19] introduce a combination of effects into a classical analysis of masonry arch bridges in general, with reference to a particular case study located in Ottawa, Canada. In addition to the loss of stiffness considered by Bridle & Hughes, they also introduce a realistic non-linear stress–strain law for aged lime mortar which, owing to the effect of water seepage through the mortar, is often effectively a confined wet sand material. The parabolic, stiffening stress–strain law for mortar is converted to a moment–curvature relationship for cracked and uncracked sections, (Fig. 5) and the reduced stiffness of the mortar joint is used to reduce the stiffness of the masonry cross-section in proportion to the relative volumes of mortar and masonry units. The authors also incorporate second-order effects by updating the geometry of the cross-section at each equilibrium iteration. A significant difference in the ultimate load, both for a prototype bridge geometry, and for an actual bridge requiring assessment, is found for each of the refinements introduced: incorporation of second-order effects and modifications of stiffness due to the compliance of the mortar joints.

The work of Robert Mark[20,21], although most of the published work dates from the 1980s, has to be mentioned in the context of the analysis of masonry vaults. Mark conducted a very large series of experiments on two-dimensional scale models of the cross-sections of Gothic churches, using transmission photoelasticity techniques. This technique assumes a strictly linearly elastic material behaviour, with no allowance for reduced stiffness due to cracking. Most of the inferences are made as the result of the appearance of fictitious tensile stresses at certain points of the models. Although the quantitative conclusions are few – the technique is not amenable to comparison with sophisticated analytical results – conclusions about the competence of certain types of medieval construction, the effect of wind loads on Gothic buildings, and the success of particularly daring structures, such as the choir of Bourges cathedral, have contributed much to the dialogue between engineers and architectural historians on the topic of medieval construction.

Constitutive laws

Recent research on general constitutive laws for unreinforced masonry to be implemented into finite element codes is applicable to specific instances of masonry arches and vaults. It is beyond the scope of this paper to review all of these developments, but two in particular merit our attention: the development of homogenization procedures for unreinforced masonry and the implementation of discrete element methods. In the homogenization methods, the periodicity of masonry is recognized, and the properties of the two materials, units and mortar, are incorporated into a single element. The general features of this element are a general elastic–plastic constitutive law, with significantly lower strength in tension than in compression. This can addressed by a cohesive model[22], provided with a compression cap, or with a Drucker–Prager model[23], also provided with a compression cap.

Constitutive laws applicable to two-dimensional masonry arch analysis have been considered by Taylor & Mallinder[24,25] and by Mallinder[26]. In the earlier paper by Taylor & Mallinder[24], a simplified parabolic stress–strain law is developed for unreinforced masonry and, using limit states principles in much the same way as they are applied to reinforced concrete beam-columns, an interaction diagram is constructed, based on the assumption of ultimate strain distributions in the cross-section at various eccentricities. The later paper[25] implements this scheme, plus further empirical evidence, into the analysis of a masonry arch. By parameterizing the masonry stress–strain law to a parabolic shape:

$$\frac{\sigma}{\sigma_m} = \frac{1}{k-1}\left[k\left(\frac{\varepsilon}{\varepsilon_m}\right) - \left(\frac{\varepsilon}{\varepsilon_m}\right)^k\right]$$

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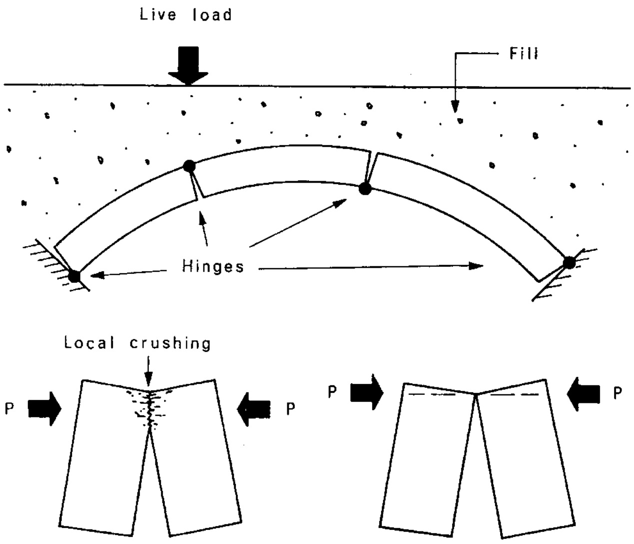

Fig. 6 Taylor & Mallinder's brittle hinge [25]. Courtesy of Institution of Structural Engineers, London

where σ_m and ε_m are the peak stress and the strain associated with this stress, and parameter $k = 0$ represents an infinitely strong, fully rigid material, while an infinite value of k represents a linearly elastic material. Practical values of k are found to be approximately[3–8]. This analysis can be immediately adapted to upper and lower bound analysis of arches, by imposing a lower bound condition that combinations of axial force and moment must be in the safe region of an interaction diagram generated from the stress–strain law, and the calculation of virtual work in the mechanism condition must take account of the shift in hinge location from the intrados or the extrados of the arch. Fig. 6 illustrates some of the main ideas of this paper.

Different approaches to the deformation characteristics of masonry arch assemblies, for masonry arches jointed with lime masonry, is proposed by Boothby[27] and by Rosson, Søyland & Boothby[28]. Motivated by experimental results showing that lime mortar exhibits a hardening elastic-plastic response with a very small elastic range, Boothby develops an approximate piecewise linear yield surface for the mortar, and determines collapse loads for masonry arches under this failure criterion. He notes that the post-yielding behaviour of most systems is hardening, and that visible deformations will precede the failure of an arch jointed with lime mortar. Rosson, Søyland & Boothby note that the joints allow the arch to shake down under a pattern of repeated loading. After the initial permanent deformations under loading, the structure becomes effectively elastic.

Finite element implementations

The simple graphical or semi-graphical method proposed by Heyman and his followers for arch structures becomes unwieldy, at best, for three-dimensional structures such as vaults. Although it is possible to apply strictly graphical lower bound methods to the analysis of vaults, as shown in Fig. 7, which comes from an early-twentieth-century textbook on graphic statics[16], this is probably too tedious for any modern engineer to accept as a basis for analysis or design, although it is strikingly similar to the computer-based method used by Smars[15]. Heyman, in the analysis of vaults, appeals directly to equations of equilibrium from the membrane theory of shells to find equilibrium surfaces contained within the body of the vault structure. Although this is certainly a useful procedure for proving the generality of lower bound analysis, in application, the attempt to find membrane surfaces in equilibrium, contained within the vault, is probably too reliant on heuristics for widespread application to problems such as analysis of rib vaults, or skew arch bridges. It is more straightforward, given the widespread distribution of general-purpose finite element computer programs incorporating three-dimensional analysis and various refined non-linear constitutive laws, to turn to these procedures for the solution of complex three-dimensional analysis problems. Most of the experiments with finite element analysis from the 1990s pre-date the wide distribution of three-dimensional general purpose programs that can be adapted to the analysis of masonry structures. In large part, these papers relate to the development of purpose-built finite element routines for the two-dimensional solution of masonry arch assessment problems.

Choo, Coutie & Gong[29] account for the loss of stiffness in cracked or crushed masonry by adopting a no-tension, trapezoidal stress–strain law for the masonry. Using beam elements with stiffness parameters at each end, adjusted for the state of nodal axial force and bending moment, they construct an iterative, incremental finite element routine, similar to the stiffness method routines[18,19] described earlier in this paper. They introduce the further refinement of considering the distribution of axle loads through the fill, and arrive at a satisfactory replication of the bridge tests at Bridgemill, Bargower and Preston.

Loo & Yang[30] consider the combined nonlinear stress–strain properties of mortar and brick masonry units to develop a simplified approach to the failure of masonry in cracking or crushing, and the post-failure behaviour. They settle on a von Mises failure criterion in compression, with a parabolic stress–strain law and a complete loss of normal and shear stiffness in the principal stress direction after reaching a failure strain. They propose a Coulomb–Mohr failure criterion, with a tension cut-off in tension and shear, and strain-softening behaviour after tension failure, primarily to achieve numerical stability. They incorporate this constitutive law into a nonlinear two-dimensional finite element scheme. The fill is modelled as dead

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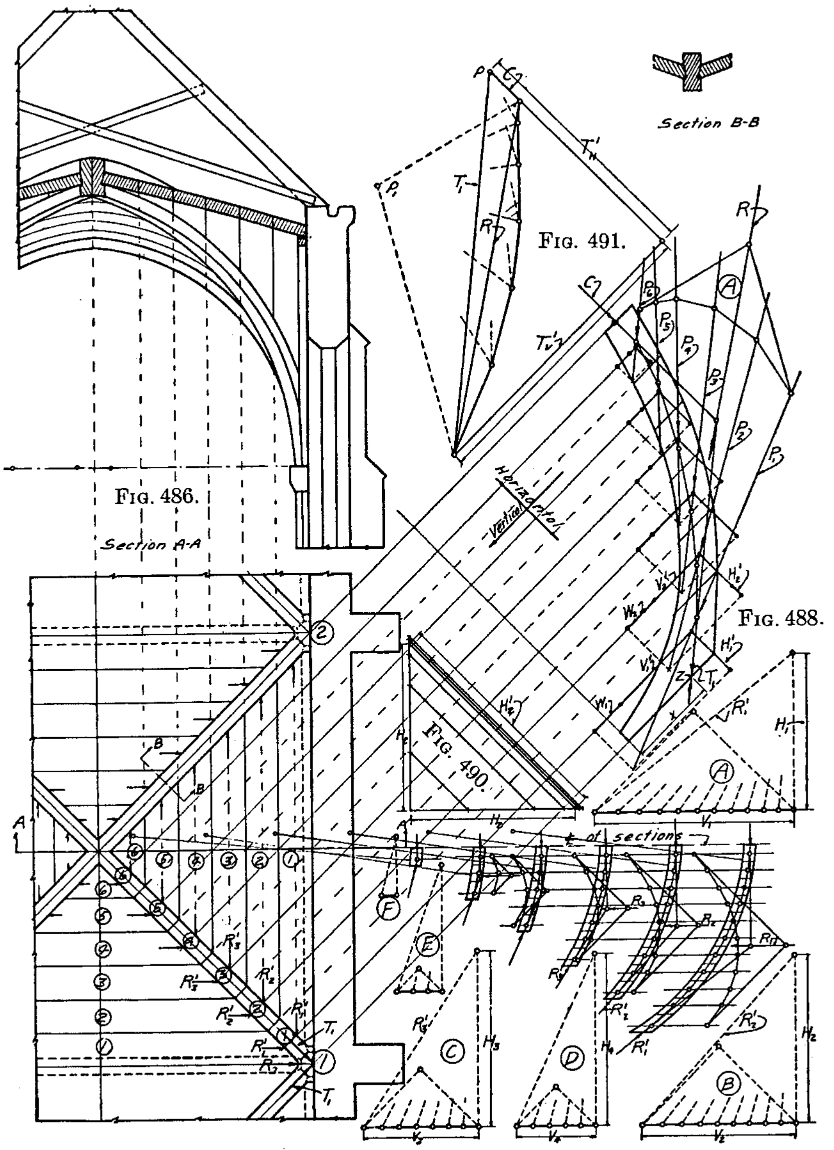

Fig. 487. Fig. 489.

Fig. 7 Wolfe's graphical analysis of a square-bayed rib vault, by the equilibrium method [16]

weight applied to the back of the arch barrel. Their results are not compared with any bridge tests or case studies of actual bridges, but they find arch bridges to be very sensitive to support movements, and that significant cracking precedes failure of arch bridges.

Ng, Fairfield & Sibbald[31] apply a commercial two-dimensional nonlinear finite element code to the analysis of some of the arch bridges in the ultimate strength testing program from the previous decade. The masonry is modelled as a von Mises material, with differing tensile and compressive strength, while a Coulomb–Mohr failure criterion is applied to the fill. Plane stress conditions were assumed for the masonry, and plane strain for the fill. The authors vary a number of the parameters of interest: masonry compressive and tensile strength; masonry modulus of elasticity; and load dispersal angle through the fill, and capture the collapse loads of bridges within the range of parameters. They discover that the collapse load is particularly sensitive to the tensile and compressive strength of the masonry, and insensitive to the elastic modulus of the masonry.

Molins & Roca[32] discuss some of the shortcomings of stiffness-based finite element approaches, as used by the authors described above: the use of a perfectly brittle constitutive law, as appropriate for masonry in tension, and in compression beyond the failure strain, leads to numerical instability, and the prescribed displacement interpolation causes damage to be spread far beyond the initial location, in contradiction to experimental observations, in which damage is very localized. To overcome these problems, they propose the use of a flexibility model, implemented into a finite element code. Using this method, they model a prototype arch bridge, the Bridgemill bridge from the UK testing project, and a complex three-dimensional choir vault. They are able to reproduce localized damage that does not spread far beyond the location of the original cracking.

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Fanning, Boothby & Roberts[33] create routine three-dimensional models of masonry arch bridges using a commercial finite element code. The masonry is modelled by a solid element developed specifically for reinforced concrete–by reducing the reinforcement ratio, and inputting appropriate tensile and compressive strengths, this element can simulate the cracking and crushing encountered in a masonry bridge. The fill is modelled as a nearly cohesionless Drucker–Prager material, and contact elements are used at the interface between masonry and fill. The results are found to match very well to the results of field tests under service loads. Moreover, the development of longitudinal cracks in nearly all the models, and in the prototype structures as well, shows the importance of incorporating transverse effects into masonry arch bridge analysis.

An emerging trend in the implementation of finite element analysis for assessment of masonry arches is the recent trend towards the use of discrete element methods. In these methods, individual voussoirs, or combinations of voussoirs can be discretized into finite elements joined by contact surfaces, which are themselves nonlinear finite elements. At each iteration, the contact surfaces are used to detect interpenetration of adjacent elements, and to update the stiffness at the interface to resist interpenetration in the following iteration. The discrete element method has been incorporated into specific computer programs, while earlier computer programs have incorporated contact surface elements, using many of the features of the discrete element method. This method seems particularly applicable to masonry arch and vault analysis because adjacent voussoirs are not strongly physically joined by the jointing material, but resist interpenetration and sliding across the contact surface. Similarly, the fill in an arch bridge, or the surcharge (material placed behind the vault to add weight and stiffness) in a vaulted structure share similar contact properties with the arch or vault.

Toi & Yoshida[34] propose a particular type of discrete element model, in which the rigid body modes of each of the blocks are incorporated into a single rigid element, whose elastic and interface properties are discretized as springs connecting the blocks. They apply this method to a number of two-dimensional problems, including an arch investigated by Livesley.

Gebara & Pan[35] apply a discrete element analysis specifically developed for masonry arches, and look at a number of two-dimensional problems by this method: support settlement; concentrated load applied to an arch; collapse of an arch due to shaking of the base. Their solutions show an unusual preponderance of sliding failures.

Although much of the work described in this section has been rendered obsolete by the availability of more powerful general purpose routines, many of the lessons learned in the development of these programs can be applied to the implementation of three-dimensional analysis. Both the compressive strength and tensile strength of the masonry and the failure criterion are of great significance in capturing the behaviour of masonry arches and vaults in service conditions and especially at loads close to failure. The deformability of older mortars, and their influence on the overall deformability of the masonry warrant particular attention. Failure modes other than four hinges forming for arches, or yield lines for vaults must be considered, including local punching of the arch barrel, buckling of the arch barrel, due to geometric changes, shortening of the arch barrel, and loss of effective cross-section of the arch barrel. As three-dimensional analysis schemes are implemented, many other failure modes will be noted, especially for arch bridges: failures due to transverse bending of the arch barrel, or due to sliding and overturning of the spandrel walls, or a combination of these effects.

Empirical and case studies

A number of available studies attempt to correlate the results of rigid–plastic analysis with the results of testing of experimental specimens constructed under controlled conditions. Royles & Hendry[36] constructed 24 model arch bridges, as plain masonry arch vaults, vaults with fill, vaults with fill and spandrel masonry, and vaults with fill, spandrel masonry and wingwalls. The structures were loaded with line loads that simulated, as closely as practicable, the effect of a two-dimensional analysis. They found high variability among nominally identical specimens, and a very large restraining and strengthening effect due to the presence of fill and spandrel walls. They found fair agreement between the results of a mechanism analysis and the actual collapse loads for specimens with a plain vault and for specimens with fill. The effect of the fill was only taken into account as additional dead load and as a means of dispersing the load applied to the arch.

Melbourne & Gilbert[37] and Melbourne, Gilbert & Wagstaff[38] looked at multi-ring brick arch bridges, and multi-span brick arch bridges. These structures were built with spandrel walls and fill, and loaded with a transverse line load–all structures in these studies failed by rigid body mechanisms. However a rigid block analysis, described above[9], underestimated the actual collapse load of all the specimens. Substantial slip between adjacent rings was observed in the multi-ring arches. The multi-span arches failed at substantially lower loads than equivalent single-span arches.

Hughes & Pritchard[39] conducted *in situ* measurements of dead and live load stresses in an arch bridge. They compared these measurements with the

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predictions made by the tapered beam element model described by Bridle & Hughes[18]. They found that the dead load stresses were suggestive of the development of abutment spreading subsequent to the initial construction of the bridge and the development of hinges at the abutments. Live load stresses could be predicted reasonably well, except in cases where the stresses were measured close to a crack in the masonry. Espion *et al.*[40] report dead load flat-jack measurements at the abutments of a monumental 72-m span arch in Luxembourg, and find reasonable agreement with the results of an elastic analysis conducted by the original designer of the bridge.

Conclusions

The rigid block theory should be accepted as primarily a means to understanding the fundamental behaviour of masonry arches and, by extension, vaults. However, the actual analysis of these structures does require more sophistication. The arguments that that elastic analysis makes too many questionable assumptions, that uniqueness is irrelevant, and that the approximate results found from plastic analysis are more relevant to a real understanding of the structures have validity. However, it should be noted that the analysis methods for dealing with uncertainty, and for dealing with variable loading environments are very well established, both in the context of elasticity and plasticity.

Two-dimensional finite element analysis should no longer be considered a legitimate means of research investigation of any structures other than wall arches. Three-dimensional programs are so widely distributed, and have such convenient interfaces, that it no longer makes sense to reduce what is known to be a complex three-dimensional system to two dimensions. For implementation in practice, on the other hand, two-dimensional analysis makes very good sense, as steel and concrete bridges are analysed as two-dimensional structures, and the complications and uncertainties in parameter estimation make three-dimensional analysis unfit for mainstream structural engineering practice.

The properties of the mortar have to be incorporated into any further analysis effort in masonry arches and vaults, either through a homogenization procedure, by the use of interface elements, or by simply considering the units to be relatively undeformable, and concentrating the deformations in the mortar joints, where appropriate. This applies particularly to the analysis of any old or ancient structures in which lime mortars, or high-lime-content mortars are used. The papers that took the trouble of considering the effects of deformations in the mortar have invariably found these effects to be significant, arguably of greater significance than effects of the strength or stiffness of the units. It has been shown that it is not necessary to abandon a limit states approach in order to introduce this refinement, but again, it is clear that a direct application of Heyman's assumptions, invaluable as these analyses are in interpreting arch and vault behaviour, does not result in realistic assessments of any real structure.

Possible areas for authentic progress include the development of a graphical method equivalent in power and scope to thrust line analysis for two-dimensional structures, such as walls subject to in-plane loading and buttress systems, or three-dimensional structures such as vaults. Means of approaching and analysing skew arches, which again are complex three-dimensional systems, have not been developed to the extent warranted by the large incidence of these structures. Hodgson[41] has applied mechanism analysis to these structures, but he concludes that more comprehensive constitutive models are required when the hinges are inclined relative to the abutments.

The application and validity of the discrete element method is currently being investigated in many quarters for its applicability to the analysis of vaulted masonry structures. It is reasonable to inquire whether the increased sophistication of these methods will yield results that are useful in the assessment of such structures.

Although rigid–plastic analysis does not admit movements of the springings of a roof vault at the top of the piers, this phenomenon is observable in nearly every medieval building at the top of the nave piers, due to either inadequate buttressing, or simply to the long-term loads imposed on the pier tops. This effect is a cause of great concern in the preservation of roof vaults. The explanation of this condition requires a detailed look at the properties of the materials used in these buildings, and the development of constitutive laws that replicate these actions and their overall effect on the structure.

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