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Structural models for the evaluation of eigen-properties in damaged spatial arches: a critical appraisal

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Abstract The paper considers the eigen-properties of spatial arch structures in both undamaged and damaged configurations. In the literature, several models have been proposed for modelling damage, mainly in beam-like structures, based on the concepts of equivalent elastic weakened section, for concentrated damage, or stiffness reduction influence area, for distributed damage. In damage identification procedures, it is needed to numerically represent the actual structure in which a reliable damage model has to be assumed. An unsuitable damage representation may lead to wrong or misleading results that, although consistent with the assumed model, could be not representative of the actual unhealthy condition of the structure. In this paper, focusing the attention on spatial Timoshenko arches, the eigen-properties of the undamaged and the damaged structure are exactly evaluated, consistent with an assumed continuous model, and compared to the corresponding results provided by several detailed finite element simulations based either on mono- or three-dimensional models. In the assumed continuous model, the damage is represented by considering an appropriate reduction, in a certain zone, in the arch cross section. The comparison with the widely adopted spring equivalent model is also made. A comparative parametric study is developed showing the role of damage on the natural frequencies of vibration of spatial arches and allowing to point out some ambiguous results that may lead to false inverse problem solutions.

Keywords Arch · Damage · Structural models · Natural frequencies · Cracked arch

1 Introduction

The early detection of the presence of damage is a key step on the health monitoring of structures since it can guarantee the operational condition and could prevent catastrophic collapse. The damage identification procedures are generally based on the investigation, by means of non-destructive tests, of the symptoms shown by the damaged structures according to an assumed diagnosis [1–3]. Since the presence of damage modifies, with respect to the healthy condition, both the static and the dynamic response of a structure, damage identification problems are generally based on the comparison of the responses of undamaged and damaged structures in terms of either static or dynamic behaviour. A great part of the papers, present in the literature,

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deals with damage identification taking into account the variation of dynamic characteristics such as natural frequencies and mode shapes [4–13], between undamaged and damaged structures, others consider static quantities [14–16]. Therefore, following this damage identification approach, it is important to have available correct structural models both for the undamaged and the damaged structure in order to correctly match the altered measured response to the results provided by the numerical simulations. An unsuitable representation of the damage may lead to wrong or misleading results that, although consistent with the assumed model, could not be representative of the actual unhealthy condition of the investigated structure. Several studies have been performed with reference to the direct and inverse problems of beam-like structures in the presence of single [17–23] or multiple damage [24–27]. On the contrary, very few applications relative to the evaluation of influence of damage and its detection in arch structures have been conducted so far [4,5,14,28].

Pau et al. [5] studied a damage identification problem, referred to a parabolic arch with a single notch, which was based on the knowledge of the natural frequencies, obtained experimentally, either of the undamaged or the damaged arch. Greco and Pau [14] solved the inverse problem of single damage identification on a parabolic arch by means of experimental static deflection measurements. Cerri and Ruta [4] studied the direct and the inverse problem related to a damaged doubly hinged plane circular arch by means of measured natural frequencies. Viola et al. [29] investigated the in-plane linear dynamic behaviour of multi-stepped and multi-damaged circular arches (modelled as massless elastic rotational hinges) under different boundary conditions.

All the above-mentioned studies refer to damaged plane arches, and the concentrated structural weakness due to damage has always been modelled by means of the well-known equivalent elastic hinge damage model. However, this simplified damage modelling strategy cannot be directly extended for the evaluation of the damage influence on spatial arches in which both the in-plane and out-of-plane behaviours have to be considered.

This paper is focused on the structural model of damage in spatial arches, aimed at evaluating the eigen-properties, as general parameters to be used in a dynamic damage identification procedure. The considered continuous model assumes that the damage is related to a cross section reduction in a small portion of the structural element. This assumption has been already presented by some authors in the literature [30], but at the authors knowledge, it has never been applied to curved beams. In the paper, other damage models, adopted in the literature for beam-like structures, are considered and compared.

The approach here proposed is consistent with a distributed parameter Timoshenko curved beam, for which the exact equation of motion governing the in-plane and out-of-plane free vibration has been solved by means of the Williams and Wittrick algorithm [31] according to a procedure, proposed by the same authors, for the undamaged spatial arch in a previous paper [32]. Namely the exact mode shapes and the corresponding frequencies, either in the undamaged or damaged arch, have been evaluated and compared. Besides the distributed parameter model, the reduced section model has been adopted in a discretized beam-based finite element simulation, provided by the commercial software Sap 2000, and in a detailed solid-element-based finite element discretization by using the FEM software ADINA. This latter detailed and computational demanding model should denote the most accurate representation of the actual damaged structure. The capability of the widely adopted model of the equivalent elastic hinge is also examined with reference to a FEM discretization.

For each structural model, the variations between the first natural frequencies of vibration in the damaged and undamaged arches are evaluated as a function of the location and intensity of damage. These variations can be considered as the fundamental data needed for the assessment of the frequency-based damage identification procedure.

The performed comparisons, since based on different damage modelling approaches, have allowed to highlight that in some cases, both stiffness and mass reductions can play an important role in the evaluation of the eigen-properties of the damaged structure. In fact, in the existing literature, the presence of damage has always been treated as a reduction in the stiffness of the structure, therefore leading to reduced values of natural frequencies compared to the undamaged configuration. Nevertheless, the analysis of the obtained results has allowed distinguishing some ranges of damage intensity and location in which the values of some frequencies of vibration, obtained by the numerical simulations, are increased by the presence of damage. This apparently unusual behaviour is related to the role of the mass reduction in the representation of damage and may not be highlighted by some damage models that, if wrongly used, could provide misleading results or false positive in frequency-based damage identification procedures.

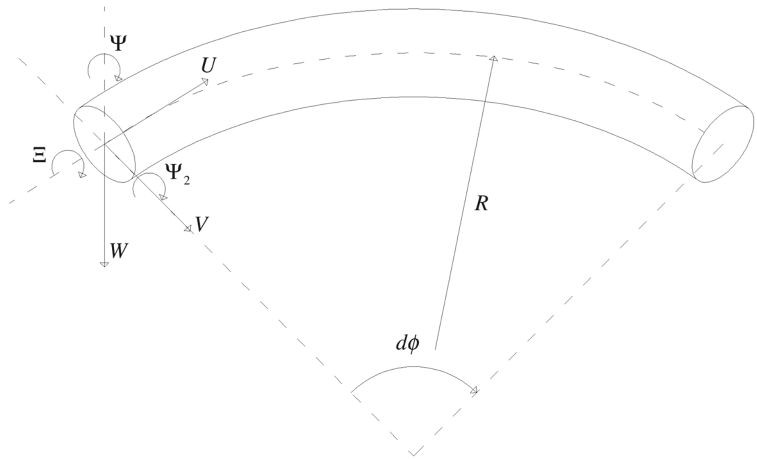

Fig. 1 Arch element

2 The continuum structural model and the equations of motion

The considered model is represented by a small length of a curved Timoshenko beam element which subtends an angle $d\phi$ at the centre of a circle of radius R; the cross section is assumed to be constant and of circular shape with radius r. The material properties, Young's modulus E, Poisson's ratio υ and density per unit length ρ, are uniformly distributed.

The degrees of freedom in the space, with reference to both the in-plane and the out-of-plane motions, of an arch element are reported in Fig. 1. By considering the equilibrium of the arch element, in its undeformed state, and the linear elastic constitutive law, the differential equations of motion governing the in-plane and out-of-plane free vibration of the considered Timoshenko arch can be derived [33–35]. The assumption that, in the undeformed configuration, the arch is contained in a principal plane of the cross section guarantees that the in-plane and out-of-plane vibrations of the planar arch are governed by uncoupled differential equations [33]. These equations have been written in a useful dimensionless form by the present authors, in a previous paper [32], for generic cross sections with a symmetry axis. With reference to the kinematic parameters U and V shown in Fig. 1, the uncoupled equations governing the in-plane motion of the arch are identical for both the parameters. For example, with reference to the tangential displacement U, the differential equation of in-plane motions, obtained by particularizing the equations given in [32] for circular cross sections, is:

$$\begin{aligned}&\left\{D^6+\left[2\left(1+\gamma^2\right)+\bar{\upsilon}\gamma^2\right]D^4+\left[1+\left(1-\bar{\upsilon}-\lambda^2\right)\gamma^2+\left(1+2\bar{\upsilon}\right)\gamma^4\right]D^2\right.\\&\quad\left.+\left[\left(1-2\lambda^2\right)\gamma^2-\left(1-2\bar{\upsilon}\right)\gamma^4\right]\right\}U=0\end{aligned}\tag{1}$$

and the same equation holds for the radial displacement V.

The symbol D^n denotes the nth derivation with respect to polar coordinate φ, and the following dimensionless parameters have been introduced:

$$\lambda^2=\left(\frac{2R}{r}\right)^2\qquad\gamma^2=\frac{\rho\omega^2R^2}{E}\qquad\bar{\upsilon}=\frac{E}{0.9G}\tag{2}$$

With reference to the kinematic parameters W and Ξ, shown in Fig. 1, the uncoupled equations governing the out-of-plane motion of the arch are identical for both the parameters. For example, with reference to the transversal displacement W, the differential equation of out-of-plane motions is:

$$\begin{aligned}&\left\{\mu D^6+\left(2\mu+2\gamma^2+\mu\left(1+\bar{\upsilon}\right)\gamma^2\right)D^4\right.\\&\quad+\left[\mu\left(1-2\gamma^2-\lambda^2\gamma^2\right)-\left(1-2\gamma^2-\mu\bar{\upsilon}\gamma^2\right)\gamma^2+\left(2\gamma^2+2\mu\right)\bar{\upsilon}\gamma^2\right]D^2\\&\quad\left.+\left[\lambda^2-2\lambda^2\gamma^2+\mu\bar{\upsilon}-2\mu\bar{\upsilon}\gamma^2+2\gamma^4\bar{\upsilon}-\bar{\upsilon}\gamma^2\right]\gamma^2\right\}W=0\end{aligned}\tag{3}$$

and the same equation holds for the torsional rotation Ξ.

In reference [32], the procedure for obtaining the exact solution of Eqs. (1) and (3), in terms of frequencies and modes of vibration, by means of an application of the Wittrick and William algorithm is reported.

3 Evaluation of the natural frequencies and modes of vibration

The knowledge of the solution of the equations of motion described in [32] allows the evaluation of the exact dynamic stiffness matrix $\mathbf{K}(\omega)$ of the arch which relates the vector $\mathbf{P}$ of the nodal static parameters at the end of the element, to the corresponding vector of kinematic parameters $\mathbf{d}$ through the equation:

$$\mathbf{P} = \mathbf{K}(\omega)\mathbf{d} \tag{4}$$

Once the dynamic stiffness matrix of the structure is assembled, the natural frequencies can be calculated by means of a very effective method based on the Wittrick and Williams Algorithm [31].

The algorithm allows to evaluate the number of frequencies of vibration which are lower than a trial value and, therefore, by means of an iteration procedure, to converge to any required frequency. In the present study, the frequencies of vibration have been calculated implementing the above-mentioned procedure with a Matlab code.

The correspondent ith mode of vibration ($i = 1, 2, 3\ldots$) can be evaluated calculating for each kinematic parameter X_i the following expression:

$$X_i(\phi) = \left(\sum_{m=1}^{6} A_m e^{\bar{\beta}_m \phi}\right)_i \tag{5}$$

where the constants A_m and $\bar{\beta}_m$ for each mode i are calculated for in-plane and out-of-plane behaviours when the boundary conditions for the single arch element are assigned [32].

4 Damage models

Most of the models presented in the literature aim to describe the effect of concentrated damage due to cracks. However, depending on material properties and loading conditions, in many cases damage can be related to a reduction in stiffness in a certain zone of the structure due to excessive loadings or degradation phenomenon, and in other cases, the local damage of the element can also involve local mass variation that could influence the overall dynamic behaviour of the structure. With the aim to experimentally simulate a concentrated damage condition, several experimental investigations of damaged structures have been performed on structural specimens in which the damage has been introduced by realizing a section notch of an element of the structure that, rigorously, determines both a stiffness and mass reductions.

The existing structural damage models generally address the damage effects in the structural response by expressing the damage in terms of an intensity parameter, an extension length and its position in the considered structure [36–38]. However, most of the adopted models avoid considering the damage extension by assuming the damage as concentrated in a zero length cross section (cracked section) so reducing the number of parameters to be identified in the inverse problem. These so-called *crack* models aim at describing the local variation of the flexural stiffness of the beam in the vicinity of the crack. Christides and Barr [17] proposed to consider a stiffness reduction due to the presence of a crack with an exponential variation law, not restricted to a local influence. Sinha et al. [39] assumed a stiffness reduction with a local effect governed by a triangular variation. Cerri and Vestroni [6,7] proposed a constant stiffness reduction, due to a concentrated crack, limited to an effective length around the crack. Bilello [40] modelled locally the effect of a concentrated crack by means of an ineffective area delimited by a linear reduction in its height starting from the cracked section. Chondros et al. [18] modelled the crack as a continuous flexibility by using the displacement field in the vicinity of the crack obtained through fracture mechanics methods. The consistent continuous model proposed by Chondros et al. is a generalization of the Christides and Barr cracked beam theory that is based on experimental determination of the exponent of the stress field. Other authors presented models of beams with transverse cracks showing that cracked structural members can be represented by a consistent static flexibility matrix [19] and evaluating alternative expressions by including coupling terms of the flexibility influence coefficients [41].

However, in the damage identification problems, the effect of concentrated damage on the flexural stiffness close to the crack is generally treated, in the literature, in a macroscopic way by means of the idea of an equivalent rotational spring connecting two adjacent segments of the beam [15,20,42–45] whose elasticity is related to the damage level. This very simple model, whose calibration is based on fracture mechanics concepts, [8,18,40,43] is able to capture the slope discontinuity at the cross sections where the cracks occur. In several researches relative to beam-like structures, Caddemi et al. described the effect of several cracks by means of the Dirac delta function introducing discontinuities on the stiffness distribution [46–48]. Caddemi and Morassi proved that this formulation can be re-conducted to the above-mentioned equivalent spring model [27]. Another simple approach already used in the literature consists in the use of a reduced stiffness element for the description of a diffuse damage [30].

However, the greatest part of the literature deals with straight Euler–Bernoulli beams and very few on cracked curved members [4,5,14,28,29]. A great number of scientific papers have been published evaluating stress intensity factors for cracked beams either with reference to I-beams, under both bending moment and axial load [49,50], or T-beams subjected to a bending moment, shear forces and torsion [51]. Also the case of stress intensity factors for a curved cracked beam of small to moderate curvature estimating the strain energy release rate based on elementary beam theory has been studied [52].

Independently on the structural damage model assumed, it is worth noticing that an unsuitable damage representation may lead to wrong or misleading results that, although consistent with the assumed model, could be not representative of the actual health condition of the structure. In the following, different damage models are applied to a circular arch structure with the aim to compare the results obtained by considering different damage scenario. After a brief description of the assumed models, a parametric study is performed.

4.1 Equivalent spring model

According to this model, it is assumed that the presence of a crack influences a portion of the length of the considered structure by inducing an increase $\Delta\theta$ in the relative rotation between the cross sections delimiting the damaged zone (θ^{D}) with respect to the same relative rotation for the undamaged structure (θ^{U}). Therefore:

$$\theta^{\mathrm{D}} = \theta^{\mathrm{U}} + \Delta\theta \tag{6}$$

Expressing the relative rotations as a function of the bending moment M, one gets:

$$\theta^{\mathrm{D}} = \frac{ML_{\mathrm{D}}}{EI^{\mathrm{D}}} \theta^{\mathrm{U}} = \frac{ML_{\mathrm{D}}}{EI^{\mathrm{U}}} \tag{7}$$

where L_{D} assumes the meaning of length of the damaged zone and EI^{U} and EI^{D} are, respectively, the bending stiffness of the undamaged and damaged cross sections.

By setting:

$$\delta = \frac{EI^{\mathrm{U}} - EI^{\mathrm{D}}}{EI^{\mathrm{U}}} \tag{8}$$

The increase in the relative rotation $\Delta\theta$ can be written as:

$$\Delta\theta = \frac{ML_{\mathrm{D}}}{EI^{\mathrm{U}}} \frac{\delta}{1-\delta} \tag{9}$$

The presence of the damaged zone can be therefore modelled by means of a concentrated rotational spring having the following rigidity [6]:

$$k_{=} \frac{M}{\Delta\theta} = \frac{EI^{\mathrm{U}}}{L_{\mathrm{D}}} \frac{1-\delta}{\delta} \tag{10}$$

In the case of circular arches of radius R, the length of the damaged zone can be expressed as [4]:

$$L_{\mathrm{D}} = \phi_{\mathrm{d}} R \tag{11}$$

where ϕ_{d} is the opening angle of the damaged zone.

Other studies [15,16,26] model the presence of a crack at a certain abscissa x_0 introducing a discontinuity in the moment of inertia of the cross section with respect to the undamaged value I_0 by means of the Dirac function $\delta(x - x_0)$ as follows:

$$I(x) = I_0 \left[1 - \alpha\delta(x - x_0)\right] \tag{12}$$

where the parameter α is related to damage intensity.

It can be shown that also this hypothesis leads to a relative rotation between the cross sections which is modelled introducing a concentrated rotational spring of suitable rigidity.

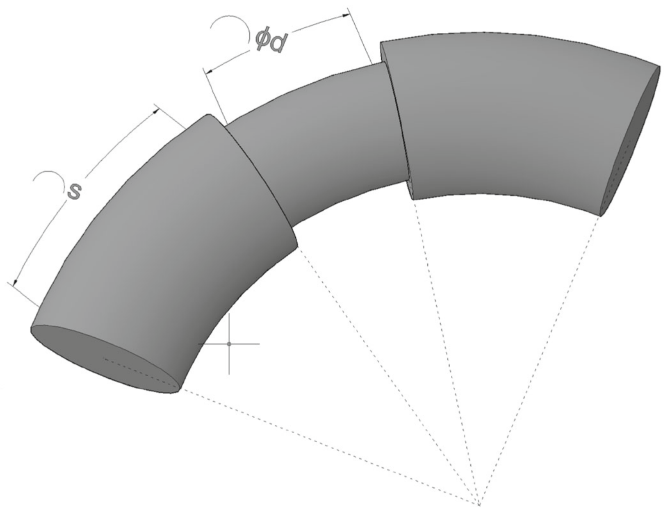

Fig. 2 Damaged element

4.2 The cross section reduction model of structural damage

In the cross section reduction model, the damage is simply taken into account introducing an element of reduced cross section with respect to the undamaged arch; this latter model can be preferable when the damage has a nonzero extension and in those cases in which both the in- and out-of-plane responses have to be investigated. In the case under investigation, the damaged element has a cross section of the same shape of the undamaged one but smaller sizes.

Three damage parameters, as highlighted in Fig. 2, are considered in the paper; they are:

- s: position of the damaged element, defined as the polar coordinate of the left reduced cross section;
- ϕ_{d}: extension of the damaged zone;
- β_{d}: intensity parameter of the damage, defined as the ratio between the diameters of the undamaged and damaged cross sections (therefore, $\beta_{\mathrm{d}} = 1$ refers to undamaged section).

In the parametric studies reported in the following, different damage scenarios relative to a circular arch with circular cross section are considered. The more general case of different cross section can be addressed introducing instead of β_{d} two different parameters $\beta_{\mathrm{d}1}$ and $\beta_{\mathrm{d}2}$ related to the principal axis of the cross section.

When the extension of damage is close to zero, no crack closure phenomenon is considered; hence, linear behaviour of the damaged curved beam is always assumed.

5 Parametric study on the frequencies of vibration of the damaged arch

Dynamic damage identification problems are generally based on the investigation of the effect of damage in terms of variation of the natural frequencies of vibration, as global response parameters are easy to identify both in the undamaged and damaged structure. The present study is focused on the influence of damage on spatial circular arch structures. Aiming at providing a numerical investigation by considering different damage models, circular arches of circular cross sections are investigated. The considered arches are assumed clamped at both ends, with opening angle $\Phi = 180°$, structural parameter $\lambda = 500$ and unitary radius. The extensions of the damaged zone has been set to $\phi_{\mathrm{d}} = \Phi/64$, being Φ the opening angle of the arch. The material has been considered to be steel, characterized by a Poisson's ratio $\upsilon = 0.3$ and a Young modulus $E = 210{,}000$ Mpa. Two different intensities of the damage have been considered, namely a strong damage, in which the radius of the damaged cross section is equal to 50 % of the undamaged one, characterized by $\beta_{\mathrm{d}} = 2$, and a weak damage with $\beta_{\mathrm{d}} = 1.1$ for which the radius of the cross section reduces of about 9 %.

The position s of the damage may vary in the entire arch; however, due to the symmetry of the problem, the damage has been considered only in half arch and therefore in the range [0–$\Phi/2$]. In the following figures, the differences in the first four vibration frequencies of the undamaged and the damaged arch are reported as a function of the damage position s. In particular, the ordinates report the difference $\Delta\gamma_i^2$ between the square

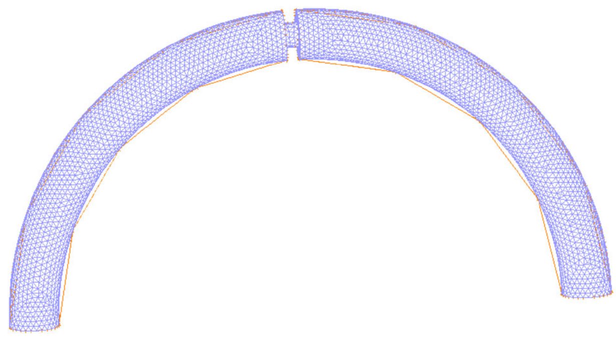

Fig. 3 Three-dimensional FE model and the damage representation

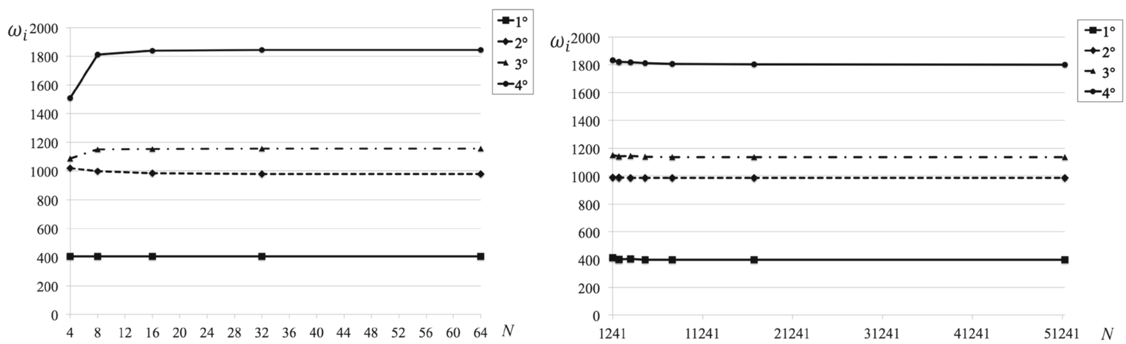

Fig. 4 Convergence study of the mono-dimensional FE model

of the ith frequency parameter of the undamaged arch (γ_i^2) and the correspondent parameter for the damaged arch ($\gamma_{\mathrm{D}i}^2$) normalized with respect to the undamaged one.

For each natural frequency, the correspondent mode of vibration has been calculated, for the undamaged arch, by means of Eq. (6) and has been plotted in the relative figure, thus allowing to recognize the in-plane or the out-of-plane character.

The study has been conducted with reference to three different structural models. Beside the exact mono-dimensional Timoshenko distributed parameter model, described in the previous paragraphs, two different finite elements (FE) approximations have been taken into account. These latter are represented, respectively, by a mono-dimensional and a three-dimensional FE discretization of the investigated arch.

In the mono-dimensional FE model, the undamaged arch has been studied by means of the commercial software SAP2000, considering 64 Timoshenko beam elements. The damage has been modelled assuming the element corresponding to the damaged zone with a reduced cross section, coherently with the three-dimensional model. Hence, in order to study the influence of the position s of a single damage on the frequencies of vibration, 32 different models have been considered.

The three-dimensional refined FE model, either undamaged or damaged, has been obtained by means of the software ADINA considering 11 nodes linear elastic brick elements, Fig. 3. The mesh has been automatically generated with a maximum extension, in all the directions, of 0.02 m which corresponds to 1/100 of the diameter of the arch. The damage has been modelled by considering the reduced cross section for the assumed extension, Fig. 3.

Aiming at evaluating the accuracy associated with the meshes adopted for the considered FE models, a convergence study for the investigated frequencies has been performed. Figure 4 reports the results of the convergence test for all the considered frequencies and for both finite element models. It can be observed how the considered meshes guarantee an accurate evaluation of the sought frequencies.

Figures 5, 6, 7 and 8 show the modes of vibration of the undamaged arch and the normalized frequency parameters $\Delta\gamma_i^2/\gamma_i^2$, as a function of the damage position s, calculated with respect to all the three considered models for the strongly damaged arch characterized by $\beta_{\mathrm{d}} = 2$.

The figures show a very good agreement between the results obtained by the proposed continuum model, solved exactly through the Wittrick and Williams algorithm, and those provided by SAP2000 based on 32

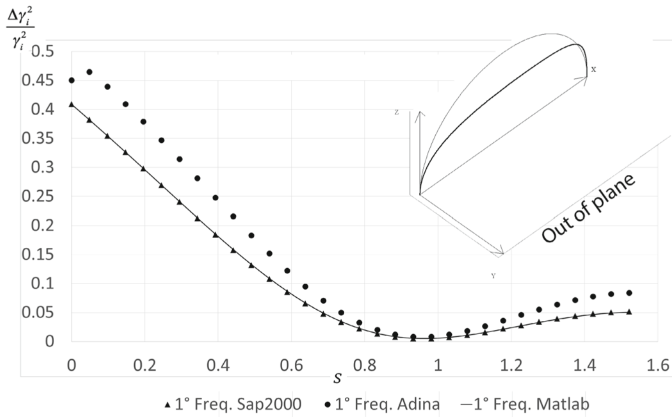

Fig. 5 First mode of vibration of the undamaged arch and variation of natural frequency for a damage with $\beta_{\mathrm{d}} = 2$

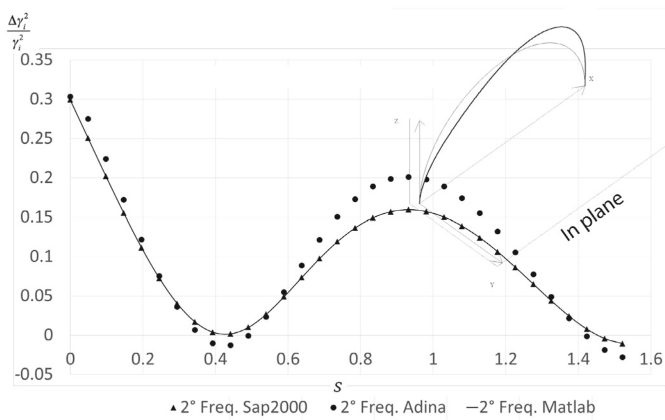

Fig. 6 Second mode of vibration of the undamaged arch and variation of natural frequency for a damage with $\beta_{\mathrm{d}} = 2$

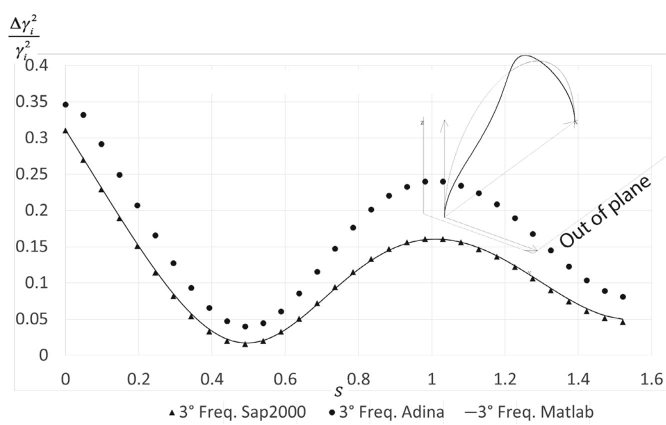

Fig. 7 Third mode of vibration of the undamaged arch and variation of natural frequency for a damage with $\beta_{\mathrm{d}} = 2$

beam elements. On the contrary, some differences with the three-dimensional detailed ADINA model can be recognized. By the observation of the figures, it can easily be noticed that when the damage is located at the curvature peaks of the undamaged arch, the differences $\Delta\gamma_i^2$reach a relative maximum, while if damage is located at a node of the bending curvature, differences $\Delta\gamma_i^2$ approach to zero, as already observed in the literature [4,5]. These differences appear further amplified in the results provided by the three-dimensional

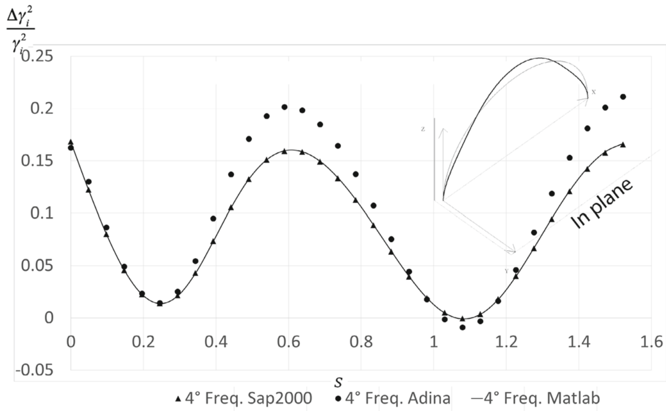

Fig. 8 Fourth mode of vibration of the undamaged arch and variation of natural frequency for a damage with $\beta_d = 2$

detailed models implemented in the FE software ADINA that, rigorously speaking, should represent the most fidelity model, being based on solid elements.

Figures 9, 10, 11 and 12 show the normalized frequency parameters $\Delta\gamma_i^2/\gamma_i^2$, as a function of the damage position s, calculated with respect to all the three cited models for the considered weakly damaged arch, characterized by $\beta_d = 1.1$.

Once again a good correspondence between the results has been obtained. In particular, the values $\Delta\gamma_i^2$ are almost identical for the continuum and the mono-dimensional models for the first three frequencies of vibration, while some differences arise in the fourth which, in the undamaged arch, corresponds to an in-plane extensional mode. This difference is due to the incapacity of the FE model, based on straight beam element, to correctly predict the frequencies associated with modes dominated by axial deformation.

5.1 Physical interpretation of the obtained results

The observation of the previous figures allows to evaluate some ranges in the damage position s in which the difference $\Delta\gamma_i^2$ takes negative values, and therefore, the ith frequency parameter of the undamaged arch is lower than the correspondent parameter for the damaged arch. This result that may appear as unusual denotes ranges of damage positions in which the damaged structure possesses a natural frequency higher than the undamaged one. For the strongly damaged arch, this phenomenon regards only the second natural frequency, in a range of damage positions close to the middle of the arch (Fig. 6), but for the weak damage intensity, it can be observed in all the first four natural frequencies (Figs. 9, 10, 11, 12).

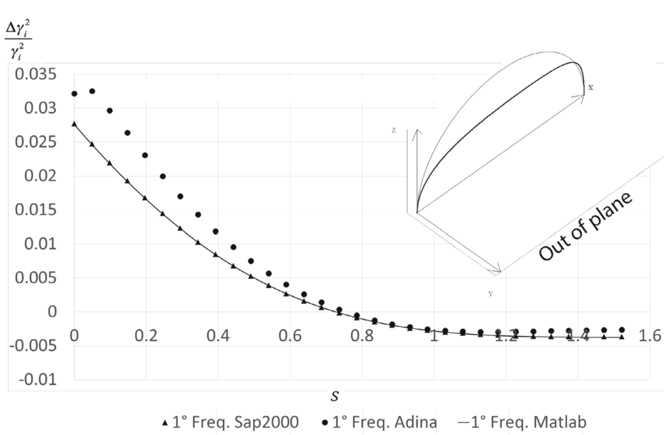

Fig. 9 First mode of vibration of the undamaged arch and variation of natural frequency for a damage with $\beta_d = 1.1$

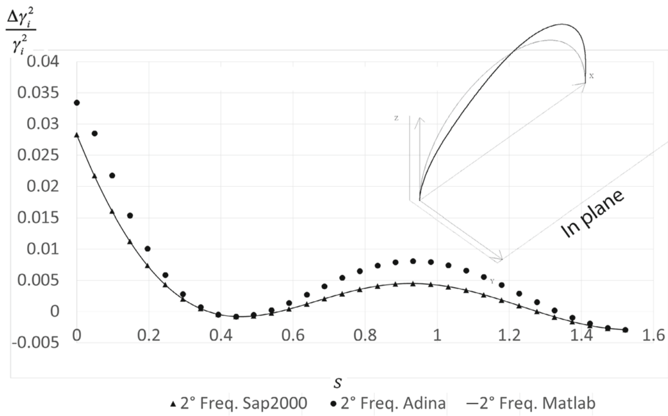

Fig. 10 Second mode of vibration of the undamaged arch and variation of natural frequency for a damage with $\beta_d = 1.1$

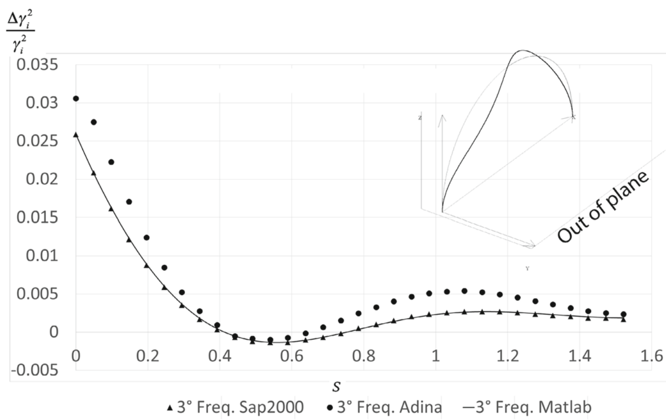

Fig. 11 Third mode of vibration of the undamaged arch and variation of natural frequency for a damage with $\beta_d = 1.1$

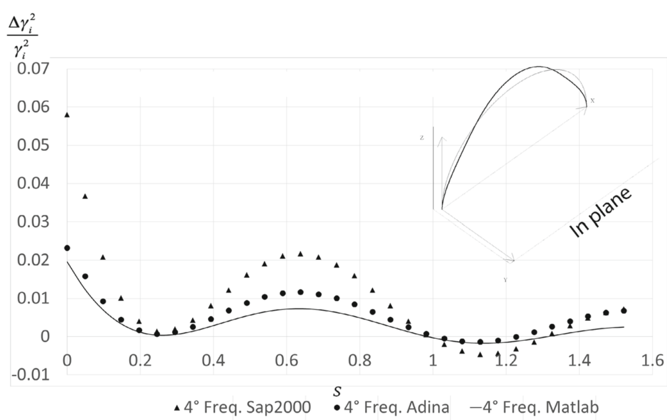

Fig. 12 Fourth mode of vibration of the undamaged arch and variation of natural frequency for a damage with $\beta_d = 1.1$

Either in the strongly damaged arch or in the case of weak damage, the phenomenon mainly appears when the damage is located in proximity of the middle section. In order to further investigate on the influence of the intensity of the damage on this unexpected behaviour, in Fig. 13 the first four normalized frequency differences $\Delta\gamma_i^2$ have been calculated, for the exact continuous model and for the ADINA model. The results are plotted for an arch damaged in the middle by assuming increasing values of the intensity of the damage.

As it can be observed, particularly for small damage intensities, the curves corresponding to the first two frequencies exhibit negative value. Furthermore, both the investigations present a value of damage intensity

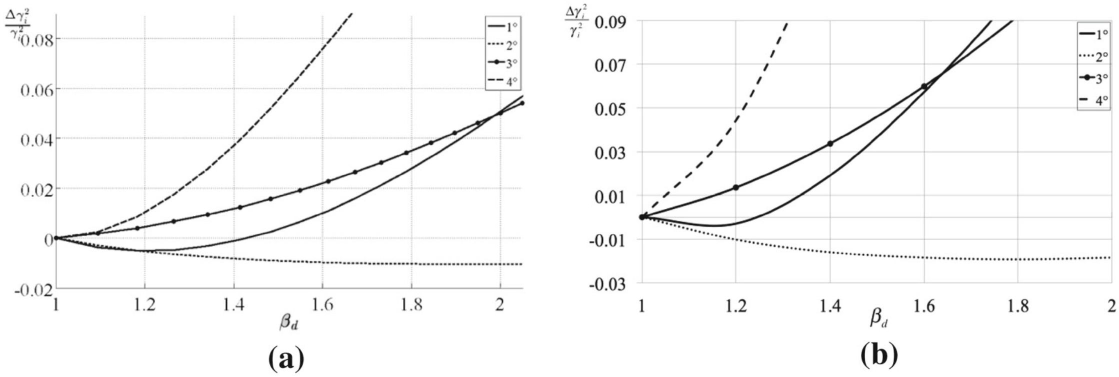

Fig. 13 Normalized frequency differences $\Delta\gamma_i^2$, obtained for the exact continuous model (**a**) and for the ADINA model (**b**), for an arch damaged in the middle with increasing damage intensity

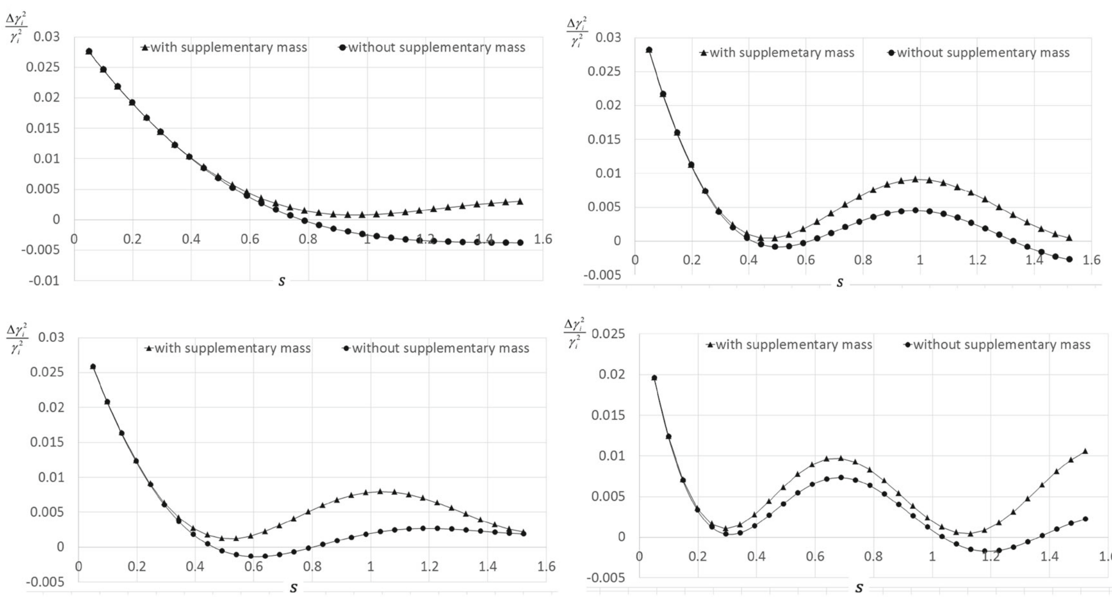

Fig. 14 Variation of the first four natural frequencies with damage position with and without supplementary mass

corresponding to zero variation of the fundamental frequency; this value identify particular cases in which the direct models show a condition in which damaged and the undamaged structure share the same fundamental frequency.

In order to justify this unexpected behaviour, it is worth pointing out that the considered model of the damage implies a reduction not only in the stiffness, but also in the total mass of the structure. It is worth noticing that this aspect is generally neglected in the equivalent elastic hinge model.

With the aim to confirm the justification of this unexpected behaviour, a further investigation has been performed in which the mass reduction has been balanced by means of supplementary concentrated masses uniformly applied in the damage zones. This has been obtained by increasing the mass density of the damaged zone in order to provide the correspondent total mass of the undamaged arch element. The considered structure has therefore a reduced stiffness but the same mass of the undamaged arch.

In Fig. 14, that refers to the weakly damaged continuous arch model characterized by $\beta_\mathrm{d} = 1.1$, the first four normalized frequency parameters $\Delta\gamma_i^2 / \gamma_i^2$ are reported as a function of the damage position, with and without considering the contribution of the supplementary concentrated mass in the damage abscissa s. As it can be seen, in all the four considered natural frequencies, the presence of the supplementary mass eliminates the zone characterized by frequency reduction since all the curves have now positive values.

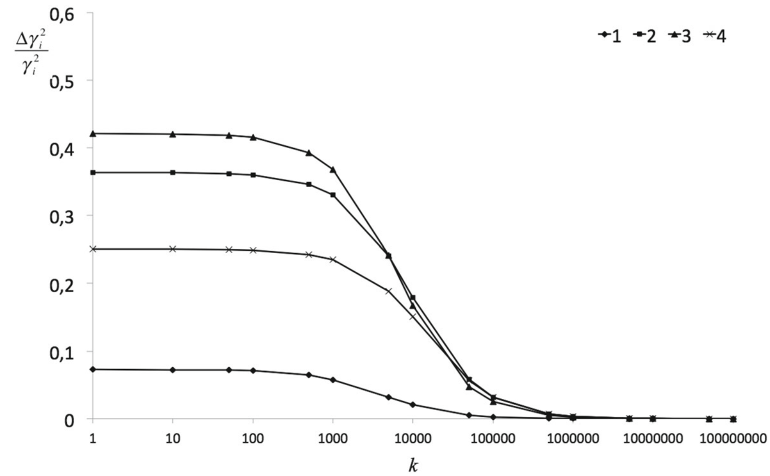

Fig. 15 Variation of natural frequency with respect to the stiffness of the rotational springs equivalent to the damage

5.2 Comparison with the concentrated rotational spring damage model

With the aim to compare the damage models previously considered to the well-known equivalent elastic hinge model, in this section two identical rotational springs, respectively, in the plane of the arch and normal to it, have been considered with the same stiffness; in this case, this assumption is justified by the assumed circular cross section. The influence of the stiffness of the springs on the variation of the first four natural frequencies, with respect to the undamaged values, has been reported in Fig. 15. The study has been conducted by means of the SAP discretization, considering two linear links of the same stiffness, that allow only flexural relative rotations in-plane and out-of-plane of the arch, but inhibit any further relative displacement. In this example, the damage has been located at a position $s = \pi/4$.

As it can be seen, the increase in the stiffness of the springs produces a reduction in the frequency variation. Besides, as expected, high values of k correspond to zero variation, and therefore, the damaged arch has the same natural frequencies of the undamaged one.

In order to compare the damage model considered in the present study with the widely adopted equivalent spring model, reference will be made to the spring stiffness proposed in Eqs. (11) and (12).

The following figures show the comparison between the damage model presented in the paper with supplementary mass and the equivalent spring model proposed by Cerri and Vestroni. In particular, Figs. 16, 17, 18 and 19 refer to an arch of unitary radius and $\Phi = 180°$ opening angle, clamped at both ends and affected by a damage with intensity $\beta_{\mathrm{d}} = 1.1$, length $\varphi_{\mathrm{d}} = \Phi/64$ located at variable positions. The stiffness of the equivalent rotational spring making use of Eqs. (11) and (12) therefore turns out to be $k = 488{,}727.27$ kN/m. It is worth noticing that in both models, the total mass is the same.

The comparison shows the variation of the first four frequencies of vibration when the damage position varies along the arch. The results related to the equivalent spring model have been obtained by means of a SAP discretization which divides the arch in 64 Timoshenko beam elements.

The distributed parameter model with the added mass provides results in great agreement with those obtained by the equivalent spring model. This behaviour is obviously predictable since the spring model does not imply a loss of mass.

6 Conclusions

This study is focused on the assessment of some existing numerical methods for evaluating frequencies and modes of vibration of spatial Timoshenko arches in the presence of damage.

The effects of the damage parameters, i.e. location, extension and intensity, on the natural frequencies of vibration of spatial arches have been evaluated through an extensive numerical investigation. The performed parametric study compares a continuum model, presented in the paper, to two different finite elements approximations, mono- and three dimensional, showing a very good correspondence.

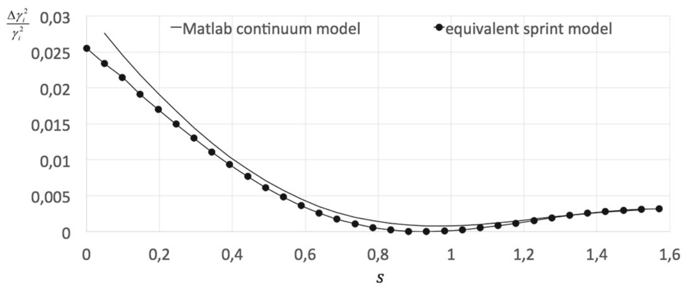

Fig. 16 Variation of the first natural frequency with damage position for the considered model and the equivalent spring one

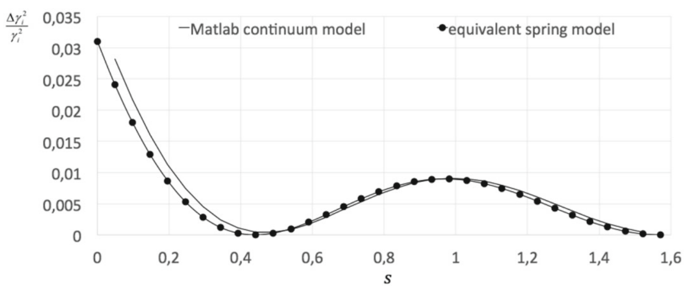

Fig. 17 Variation of the second natural frequency with damage position for the considered model and the equivalent spring one

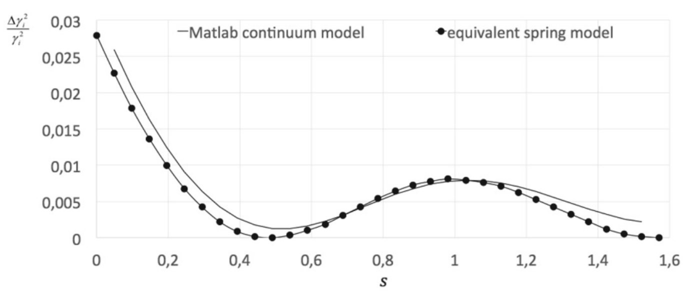

Fig. 18 Variation of the third natural frequency with damage position for the considered model and the equivalent spring one

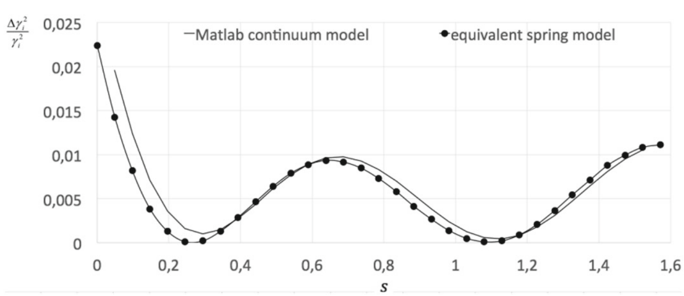

Fig. 19 Variation of the fourth natural frequency with damage position for the considered model and the equivalent spring one

The eigen-properties of the investigated structures have been evaluated according to an approach in which the presence of damage has been modelled by introducing a reduction in the dimension of the cross section of the damaged arch zone consistent with both the level of damage and its extension. Furthermore, the results have been compared to those obtained by the well-known equivalent spring model in which, independently on the damage extension, the effect of the concentrated damage is represented by an elastic hinge.

The variation of natural frequencies for undamaged and damaged arches obtained with reference to the considered structural models allows to point out some peculiar and unusual behaviour. Namely, some results denote apparently unexpected result according to which the damaged structure seems to be stiffer than the undamaged one, thus contradicting the well-known Rayleigh's theorem. In the paper, it has been highlighted that when modelling a damage, both the stiffness and mass reductions can influence the frequency variations. Consequently, a wrong modelling can lead to misleading assessments, particularly when frequency measurements are the key ingredients of a damage identification procedure.

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