Polytechnic University of Turin

Master degree
in Engineering Mathematics

Master thesis

Redundancy of finite frames

with a discussion on Gabor frames

Supervisor
prof. Patrik Wahlberg

Candidate
Felix Ackermann

signature

signature

Academic year 2021-2022

Summary

Frames represent an important theoretical concept for signal processing. A frame is a collection of vectors satisfying the frame inequality, i.e. a relaxed form of Parseval's identity for which the vectors involved do not need to form an orthonormal basis. Frames are used in order to provide a redundant representation of a vector (signal) in terms of coefficients associated to the frame's elements.

The goal of this thesis is to quantify redundancy for frames. Therefore we present a combinatorial- and an analytical redundancy measure. Both yield information on the maximal number of spanning sets and the minimal number of linearly independent sets one can partition the frame into, though in general the two measures are not equivalent.
For the combinatorial measure we show a proof of the *Rado-Horn* theorem and two results basing upon it. We then examine a well-known example: the *Fourier frame*. Instead the analytical measure is characterized in terms of properties of the redundancy function. It is known that these two measures coincide for an equal norm Parseval frame, but when relaxing equal norm, this does not hold anymore. Exploiting the properties of the redundancy function, we show that the two redundancy measures still coincide if we consider Parseval frames with some additional property.
Finally, we examine *Gabor frames*. The frame elements are modulated translates of a window function. We characterize the frame both in general terms and with respect to its redundancy.

Contents

1 Finite frames on a Hilbert space

A separable complex Hilbert space H is a complete metric space where the norm squared on it is induced by a scalar product $< \cdot, \cdot >$. For a given vector $x \in H$, this means

$$\|x\|_H^2 = < x, x > \tag{1.1}$$

where the scalar product is a bilinear form satisfying

$$x, y \in H \;\; \Rightarrow \;\; \alpha x + \beta y \in H \;\; \forall \alpha, \beta \in \mathbb{C} \tag{1.2}$$

$$< x, \alpha y > = \overline{\alpha} < x, y > \quad . \tag{1.3}$$

The vector $x \in H$ can be expressed with respect to an orthonormal basis $\{e_j\}_{k=1}^{\infty} \subset H$, which yields

$$x = \sum_{k=1}^{\infty} \alpha_k e_k \quad \text{where} \quad \alpha_k = < x, e_k > \quad . \tag{1.4}$$

Moreover, the Parseval identity holds, that is

$$\|x\|_H^2 = \sum_{k=1}^{\infty} |\alpha_k|^2 \quad . \tag{1.5}$$

As commonly done, we denote strict subsets with the symbol $\subsetneq$. We briefly recall for a space what it means to be separable. Note that every finite dimensional metric space is separable (see [9], p.72).

Definition 1.1. A metric space E is separable if there exists a subset $D \subset E$ that is countable and dense.

We come to the definition of a frame. Simply put, a frame yields a different representation of a vector $x \in H$ than that of eq. (1.4). This will be clearer when we define the operator associated to a frame.

Definition 1.2. A sequence $\Phi = \{\varphi_n\}_{n=1}^{\infty}$ of elements of a Hilbert space H is called a frame if there are constants $A, B > 0$ such that

$$A \|x\|_H^2 \;\; \leq \;\; \sum_{k=1}^{\infty} |< x, \varphi_k >|^2 \;\; \leq \;\; B \|x\|_H^2 \tag{1.6}$$

where A,B are the lower and upper frame bounds.

If A can be chosen to be equal to B, the frame is called a tight frame. Moreover, when $A = B = 1$, then it is referred to as a Parseval frame. Intuitively, a Parseval frame is a frame with no preferred direction. In fact, we then have

$$\sum_{k=1}^{\infty} |\alpha_k|^2 = \sum_{k=1}^{\infty} |< x, \varphi_k >|^2 \quad , \tag{1.7}$$

so none of the vectors in Φ has, relatively seen, more weight with respect to the other frame vectors. Lastly, if $\|\varphi_j\|_H = c$ for all $j = 1, 2, \ldots$, then the frame is called equal norm.

For the purpose of the thesis, we consider a Hilbert space of finite dimension n (dim $H = n$), denoted by H^n. For notational simplicity, we drop the index H^n from the Hilbert space norm. We will consider vectors $x \in \mathbb{C}^n$ with norm

$$\|x\|_2 = \left(\sum_{k=1}^{n} |x_j|^2 \right)^{\frac{1}{2}} \quad . \tag{1.8}$$

As we shall see by Prop. 3, considering $x \in H^n$ or $x \in \mathbb{C}^n$ is basically the same. Definitions, theorems etc. are stated in general form for H^n. Plus, we limit the discussion to frames Φ of finite size N with $N \geq n$. As we will see later in Sec. 1.3, this assumption is necessary since if $N < n$, then span $\Phi \subsetneq H^n$, so there exists $x \in H^n \setminus \{0\}$ such that $\sum_{k=1}^{N} |< x, \varphi_k >|^2 = 0$. Therefore we can assume the frame to be given in the form

$$\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_N\} \, . \tag{1.9}$$

The frame concept is linked to that of a Riesz basis. As we will see, a Riesz basis automatically satisfies the frame inequality given by eq. (1.6).

Definition 1.3. Let $\{\varphi_i\}_{i=1}^n \subset H^n$ be a sequence of vectors, T an invertible linear operator on H^n and $\{e_i\}_{i=1}^n \subset H^n$ an orthonormal basis. A Riesz basis $\{\varphi_i\}_{i=1}^n$ satisfies

$$\varphi_j = Te_j \ , j = 1, \ldots, n \quad .$$

By the above definition, there is a one-to-one correspondance between a Riesz basis and an orthonormal basis. Using Def. 1.3, we may now rewrite eq. (1.4), giving

$$\begin{aligned} x &= \sum_{j=1}^{n} < x, e_j > e_j \\ &= \sum_{j=1}^{n} < x, T^{-1}\varphi_j > T^{-1}\varphi_j \quad . \end{aligned} \tag{1.10}$$

There is an equivalent definition for a Riesz basis. It goes as follows.

Definition 1.4. A sequence of vectors $\{\varphi_i\}_{i=1}^n \subset H^n$ is a Riesz basis if there are constants λ_1, $\lambda_2 > 0$ so that for all sequencies of scalars $\{a_i\}_{i=1}^n$ we have

$$\lambda_1 \sum_{k=1}^n |a_i|^2 \le \left\| \sum_{k=1}^n a_i \varphi_i \right\|^2 \le \lambda_2 \sum_{k=1}^n |a_i|^2 \quad . \tag{1.11}$$

In finite dimension n, every basis is a Riesz basis. Differently, in infinite dimension, this is not true: given the orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ for H, the sequence $\{e_j/j\}_{j\in\mathbb{N}}$ does not allow for a suitable lower bound λ_1 as choosing $a_k = 0$ for some $k \in \mathbb{N}$ and $a_j = 0$ for $j \neq k$, $j \in \mathbb{N}$, by the above definition we have (see [8] for this example)

$$\frac{1}{k^2} < e_k, e_k > = \frac{1}{k^2} \;\; \to \;\; 0 \;\; \text{for } k \to \infty \quad . \tag{1.12}$$

We now present some definitions and results from Hilbert space theory and operator theory we need throughout this chapter (see [4]). Until now we have relied on [1].

For demonstrations, the *Cauchy-Schwarz* and *Triangle* inequality are often used. The first states, given x, $y \in H$

$$|< x, y >| \le \|x\| \, \|y\| \quad , \tag{1.13}$$

equality holding if and only if $x = cy$ for some constant c, while by the Triangle inequality we mean, given any $x, y \in H$

$$\|x + y\| \le \|x\| + \|y\| \, . \tag{1.14}$$

Definition 1.5. If S, T are positive operators on H^n, we write $S \le T$ if $T - S \ge 0$.

Definition 1.6. An operator $P : H^n \to H^n$ is called a projection if $P^2 = P$. If P is also self-adjoint it is an orthogonal projection.

Often one wants to project onto a linear subspace $W \subsetneq H^n$. This may be done using an orthonormal basis $\{e_j\}_{j=1}^m$ of W (dim $W = m < n$). We define P as

$$Px = \sum_{k=1}^m < x, e_k > e_k \quad , \tag{1.15}$$

thereby P is an orthogonal projection. Indeed, given $y \in W$, by the antilinearity of the scalar product, we have

$$\begin{aligned} < Px, y > &= < \sum_{k=1}^{m} < x, e_k > e_k, y > \\ &= \sum_{k=1}^{m} < x, e_k >< e_k, y > \\ &= \sum_{k=1}^{m} < x, e_k > \overline{< y, e_k >} \\ &= \sum_{k=1}^{m} < x, \sum_{k=1}^{m} < y, e_k > e_k > \; = \; < x, Py > \quad , \end{aligned} \tag{1.16}$$

so P is self-adjoint and thus an orthogonal projection. Relative to the frame context, an orthogonal projection maps a frame to a frame. This is the content of the next proposition.

Proposition 1. *Let $\{\varphi_i\}_{i=1}^{N}$ be a frame for H^n with frame bounds A, B and let P be an orthogonal projection on H^k ($k \leq n$), $P : H^n \to H^k$ being onto and $H^k = P(H^n)$. Then $\{P\varphi_i\}_{i=1}^{N}$ is a frame for H^k with frame bounds A,B. In particular, an orthogonal projection of an orthonormal basis (or a Parseval frame) is a Parseval frame for span P.*

Proof. Given any $x \in P(H^n)$, it holds that $Px = x$. By the fact that P is self-adjoint, we have

$$\sum_{k=1}^{N} |< x, P\varphi_k >|^2 = \sum_{k=1}^{N} |< Px, \varphi_k >|^2 = \sum_{k=1}^{N} |< x, \varphi_k >|^2 \quad ,$$

so P does not alter the frame inequality and thus the frame bounds. □

Proposition 2. *If S, $T : H^n \to H^k$ are operators satisfying*

$$< Tx, y > = 0 \;\; \forall x \in H^n, y \in H^k \quad ,$$

then $T = 0$. Hence if

$$< Tx, y > = < Sx, y > \;\; \forall x, y \in H^n \quad ,$$

then $S = T$.

Proof. Take $x \in H^n$. By $y = Tx$ we obtain

$$0 = < Tx, Tx > = \|Tx\|^2$$

and so by the norm property $Tx = 0$ for all $x \in H^n$, thus $T = 0$. □

Later (see Prop. 5 and Sec. 1.2) we show that two frames which are isomorphic share some properties with one another. Therefore we now define both when two generic Hilbert spaces and when two frames are isomorphic.

Definition 1.7. A linear operator $T : H^n \to H^k$ is a Hilbert space isomorphism is for every $x, y \in H^n$ we have

$$< Tx, Ty >_{H^k} = < x, y >_{H^n} \ .$$

Two Hilbert spaces H^n and H^m are isomorphic if there is a Hilbert space isomorphism $T : H^n \to H^m$.

Definition 1.8. Two frames $\{\varphi_i\}_{i=1}^N$, $\{\psi_i\}_{i=1}^N$ for H^n are isomorphic if there exists a bounded, invertible operator $T : H^n \to H^n$ so that $T\varphi_i = \psi_i$ for all $1 \leq i \leq N$.

Considering $x \in H^n$ or $x \in \mathbb{C}^n$, $\mathbb{C}^n$ being a Hilbert space, is basically the same as the two are isomorphic.

Proposition 3. *Every two n-dimensional Hilbert spaces are Hilbert space isomorphic. Thus, any n-dimensional Hilbert space is isomorphic to* $\mathbb{C}^n$.

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis for H_1^n. Denote by $\{g_i\}_{i=1}^n$ an orthonormal basis for H_2^n obtained with an operator $T : H_1^n \to H_2^n$, $Te_i = g_i$, $1 \leq i \leq n$. Then, by Def. 1.7 T is a Hilbert space isomorphism, thus H_1^n isomorphic to H_2^n. $\square$

Next we state a corollary we use in the next section (see [9], p.167). Before we define the spectrum $\sigma(T)$ of an operator $T : H \to H$. It is the set

$$\sigma(T) = \{\lambda \in \mathbb{C} : Tx = \lambda x, x \in H\} \quad . \tag{1.17}$$

Corollary 1.1. *Let* $T : H \to H$ *be a linear, self-adjoint operator such that* $\sigma(T) = \{0\}$. *Then* $T = 0$.

1.1 Frame Operator

In order to read a vector $x \in H^n$ by a frame we need to determine its frame operator $S = TT^*$, where T^* and T are respectively the analysis and synthesis operator of a frame (see [1] for the part following). The former maps $x \in H^n$ as $x \mapsto \{< x, \varphi_i >\}_{i=1}^N$, the latter rebuilds a vector from a sequence of coefficients: T acts as $Te_j = \varphi_j$ $(j = 1, \ldots, N)$ with $\{e_i\}_{i=1}^N \subset \mathbb{C}^N$ being the standard basis from now onwards. The before said is equal to the operators acting like

$$T^* : H^n \to \mathbb{C}^N,\ x \mapsto T^*x = \{< x, \varphi_i >\}_{i=1}^N = \{\alpha_i\}_{i=1}^N \quad , \tag{1.18}$$

$$T : \mathbb{C}^N \to H^n,\ \{\alpha_i\}_{i=1}^N \mapsto T\alpha = \sum_{k=1}^N \alpha_k \varphi_k \quad . \tag{1.19}$$

Note that T^* is the adjoint of T, meaning it is the unique operator satisfying

$$< Tx, y > = < x, T^*y > \quad \forall x, y \in \mathbb{C}^N \quad . \tag{1.20}$$

One can rewrite eq. (1.6) using the definition of T^*, now in finite dimension ($x \in H^n$). This gives

$$A \|x\|^2 \leq \|T^*x\|_2^2 \leq B \|x\|^2 \quad \forall x \in H^n \quad , \tag{1.21}$$

as

$$\|T^*x\|_2^2 = \sum_{k=1}^N |< x, \varphi_k >|^2 \quad . \tag{1.22}$$

The theorem following characterizes a frame in terms of its analysis and synthesis operator (see [4])

Theorem 1.1. *Let $\{\varphi_i\}_{i=1}^N$ be a family of vectors in a Hilbert space H^n. Then the following are equivalent:*

1. *$\{\varphi_i\}_{i=1}^N$ is a frame for H.*
2. *The operator T is bounded, linear and onto.*
3. *The operator T^* is bounded, linear and one-to-one.*

Moreover, $\{\varphi_i\}_{i=1}^N$ is a Parseval frame if and only if $T^*T = \mathbb{I}$ if and only if T is an isometry.

As T^* is a linear operator, we may write T^* in matrix form, which gives

$$T^* = \begin{pmatrix} - & \overline{\varphi_1} & - \\ - & \overline{\varphi_2} & - \\ \cdots & \cdots & \cdots \\ - & \overline{\varphi_N} & - \end{pmatrix} \quad . \tag{1.23}$$

Then for T^*x we have

$$T^*x = \sum_{k=1}^N < x, \varphi_k > e_k \quad . \tag{1.24}$$

As a consequence of eq. (1.23) above, T has in its matrix representation the frame vectors as columns, that is

$$T = (\varphi_1|\varphi_2|\dots|\varphi_N) \quad . \tag{1.25}$$

Note that as both T and T^* are linear operators, the composition $S = TT^*$ is again a linear operator. Taking x,$y \in H^n$ and α, $\beta \in \mathbb{C}$, we have

$$\begin{aligned} TT^*(\alpha x + \beta y) &= T(\alpha T^* x + \beta T^* y) \\ &= \alpha S x + \beta S y \quad . \end{aligned} \tag{1.26}$$

We may now read a vector $x \in H^n$ by the frame operator S as defined above. Using eq. (1.24) and the linearity property of T, this yields

$$\begin{aligned} Sx &= TT^* x \\ &= T\left(\sum_{k=1}^{N} < x, \varphi_k > e_k\right) \\ &= \sum_{k=1}^{N} < x, \varphi_k > \varphi_k \quad . \end{aligned} \tag{1.27}$$

Building the scalar product $< \cdot, x >$, again by linearity we have

$$\begin{aligned} < Sx, x > &= \ < \sum_{k=1}^{N} < x, \varphi_k > \varphi_k, x > \\ &= \sum_{k=1}^{N} |< x, \varphi_k >|^2 \\ &= \ < x, S^* x > \\ &= \ < x, \sum_{k=1}^{N} < x, \varphi_k > \varphi_k > \quad . \end{aligned} \tag{1.28}$$

We observe that by the frame inequality (eq. (1.6)) S is strictly positive, meaning $< Sx, x > > 0 \ \forall x \neq 0$, and self-adjoint ($S^* = S$), which results from eq. (1.28). Plus, replacing $\sum_{k=1}^{N} |< x, \varphi_k >|^2$ in eq. 1.6 by eq. (1.28), we have that $\{\varphi_i\}_{i=1}^{N}$ is a frame with respective bounds A, B if and only if

$$< Ax, x > \ \leq \ < Sx, x > \ \leq \ < Bx, x > \ \ \forall x \in H^n \quad . \tag{1.29}$$

Moreover, S is bounded. Its norm $\|S\|$ is given by

$$\begin{aligned} \|S\| &= \sup_{\|x\|=1, \|y\|=1} |< TT^* x, y >| \\ &= \sup_{\|x\|=1, \|y\|=1} |< T^* x, T^* y >| \quad . \end{aligned} \tag{1.30}$$

The right-hand side of eq. (1.30) is maximal when T^*x is colinear with T^*y. As follows,

$$\|S\| = \sup_{\|x\|=1} \|T^*x\|^2 \quad , \tag{1.31}$$

and because T^* is bounded by Theorem 1.1, S is bounded as well.
Lastly, S is invertible. In finite dimension, this is equivalent to S being one-to-one. For proving the latter we need to show that for $x \in H^n$

$$Sx = 0 \;\Rightarrow\; x = 0 \quad . \tag{1.32}$$

We prove by contradiction: take $x \in H^n$, $x \neq 0$ and suppose $Sx = 0$. As $x \neq 0$, the frame inequality (eq. (1.29)) gives

$$< Sx, x > = 0 \not\geq A\,\|x\|^2 \; > 0 \;\forall x \in H^n \quad , \tag{1.33}$$

thus the frame inequality would not hold for the frame operator S. Therefore we necessarily have $Sx \neq 0$, so S is one-to-one and therefore invertible. This will be used later, together with a result of [4], p. 12: it states that if $\tilde{T}$ is a nonnegative, invertible and diagonalizable operator, then its powers $\tilde{T}^\alpha$ with $\alpha \in \mathbb{R}$, $\alpha \geq 0$, are well defined. They add that, if an operator $\tilde{T}$ is self-adjoint, then it is also diagonalizable. These two facts will be used some lines below. The part following bases on [4], p. 23.

The characterization of frame types can be rephrased in terms of the eigenvalues λ_j, $j = 1, \ldots, n$ of the frame operator S. Generally one has that the sum of the eigenvalues is equal to the squared sum of the frame vectors φ_j, $j = 1, \ldots, N$, meaning

$$\sum_{k=1}^{n} \lambda_k = \sum_{k=1}^{N} \|\varphi_k\|^2 \quad . \tag{1.34}$$

Equation (1.34) can be obtained starting from the eigenvalue equation for the frame operator S

$$S\,x_j = \lambda_j x_j \tag{1.35}$$

where $x_j \in \mathbb{C}^n$ is an eigenvector corresponding to the eigenvalue λ_j. As S is strictly positive, we have that $\lambda_j > 0\,\forall j$, therefore $\det(S) \neq 0$ and $\mathrm{rank}(S) = n$, giving finally n eigenvalues and n eigenvectors x_j which are orthogonal to one another. Writing the last equation in matrix form yields

$$SX = X\Lambda \quad , \tag{1.36}$$

taking X as $X = [x_1|x_2|\ldots|x_n]$ and Λ as $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. X is invertible as its columns, namely the eigenvectors x_j, are linearly independent of one another. Left multiplying by X^{-1} and taking the trace of the resulting equation one obtains

$$\mathrm{tr}[X^{-1}SX] = \sum_{k=1}^{n} \lambda_k \tag{1.37}$$

which, after applying the trace property $\mathrm{tr}[AB] = \mathrm{tr}[BA]$, becomes

$$\begin{aligned} \mathrm{tr}[S] &= \mathrm{tr}[T^*T] \\ &= \mathrm{tr}[\left(\varphi_1^T, \varphi_2^T, \dots, \varphi_N^T\right)(\varphi_1|\varphi_2|\dots|\varphi_N)] \\ &= \sum_{k=1}^{N} < \varphi_k, \varphi_k > \\ &= \sum_{k=1}^{N} \|\varphi_k\|^2 \quad . \end{aligned} \tag{1.38}$$

This proves eq. (1.34). When a frame is equal norm ($\|\varphi_k\| = c > 0\ \forall k = 1, \dots, N$), then

$$\sum_{k=1}^{N} \|\varphi_k\|^2 = N \cdot c^2 \quad . \tag{1.39}$$

Being tight means

$$\begin{aligned} < Sx, x > &= \sum_{k=1}^{N} |< x, \varphi_k >|^2 \\ &= A\|x\|^2 \\ &= < Ax, x > \end{aligned} \tag{1.40}$$

and thus

$$< (S - A \cdot \mathbb{I}_n)\, x, x > \; = \; 0 \ \forall x \in H^n \ . \tag{1.41}$$

We claim $S = A \cdot \mathbb{I}_n$. Suppose $S - A \cdot \mathbb{I}_n \neq 0$. The operator $\tilde{S} = (S - A \cdot \mathbb{I}_n)$ is nonnegative (a) and diagonalizable (b). Fact (a) follows by the frame inequality (eq. 1.29), giving

$$< (S - A \cdot \mathbb{I}_n)\, x, x > \geq 0 \ \forall x \in H^n \quad ,$$

while fact (b) by $\tilde{S}$ being self-adjoint. Indeed, rewriting eq. (1.29) we get

$$< Sx, x > - < Ax, x > \; = \; < x, (S - A)x > \quad \forall x \in H^n \quad ,$$

so $(S - A \cdot \mathbb{I}_n)$ is diagonalizable. We prove by contradiction. By (a), for every eigenvalue λ of $\tilde{S}$ we have $\lambda \geq 0$. If there exists $\lambda > 0$ for some $\lambda \in \sigma(\tilde{S})$, then, being $y \in H^n$ the corresponding eigenvector, it follows that

$$\tilde{S}y = \lambda y \quad .$$

This implies

$$< \tilde{S}y, y > = < \lambda y, y > = \lambda \|y\|^2 > 0 \quad ,$$

contradicting eq. (1.41), so $\sigma(\tilde{S}) = \{0\}$. Next, by Cor. 1.1 it follows that $\tilde{S} = 0$, so $S = A \cdot \mathbb{I}_n$, meaning S is a multiple of the identity matrix. Thus for a tight frame we have n eigenvalues λ_j where each $\lambda_j = A$, whereby

$$\sum_{k=1}^{n} \lambda_k = n \cdot A \quad . \tag{1.42}$$

If then the frame is also Parseval ($A = 1$), it follows that

$$\begin{aligned} S = \mathbb{I} \quad \text{and} \quad \lambda_k = 1 \ \forall k = 1, \dots, n \\ \text{so} \quad \sum_{k=1}^{n} \lambda_k = n = \sum_{k=1}^{N} \|\varphi_k\|^2 \quad . \end{aligned} \tag{1.43}$$

Thus, eq. (1.43) shows that the eigenvalues of S may not be necessarily distinct. Adding the equal norm property on top of the Parseval property, by eq. (1.43) and eq. (1.39), eq. (1.34) gives

$$n = N \cdot c^2 \quad . \tag{1.44}$$

Instead, if we keep the tightness (eq. (1.42)), but drop Parseval and impose equal norm (eq. (1.39)), then one obtains the condition $n \cdot A = N \cdot c^2$. Thereby $A = Nc^2/n$ and the frame condition (eq. 1.6) becomes

$$\sum_{k=1}^{N} |< x, \varphi_k >|^2 = \frac{N}{n} c^2 \|x\|^2 \quad . \tag{1.45}$$

1.2 Dual Frames

Reconstruction of a vector $x \in H^n$ can occur by the formula $x = SS^{-1}x$, and using eq. (1.27) one obtains

$$\begin{aligned} x &= SS^{-1}x \\ &= \sum_{k=1}^{N} < S^{-1}x, \varphi_k > \varphi_k \\ &= \sum_{k=1}^{N} < x, S^{-1}\varphi_k > \varphi_k \end{aligned} \tag{1.46}$$

where $\{< S^{-1}x, \varphi_i >\}_{i=1}^{N}$ are called frame coefficients. Eq. (1.46) states that in order to reconstruct a vector $x \in H^n$ we need to find a right inverse of

S. In general, as we have more vectors in our frame Φ with respect to an orthonormal basis, that is the basis is *overcomplete*, neither the right inverse is unique nor the expansion coefficients will be. A generic right inverse takes a special name: it is referred to as a dual frame (see [4]). This will be more clear by Prop. 4 below. Now follows the definition of a dual frame.

Definition 1.9. Let $\Phi = \{\varphi_i\}_{i=1}^N$ be a frame for H^n. Then a frame $\{\psi_i\}_{i=1}^N$ is called a dual frame for Φ if

$$\sum_{k=1}^{N} < x, \psi_k > \varphi_k = x \ \forall x \in H^n \quad .$$

By this definition, in eq. 1.46 $\{S^{-1}\varphi_i\}_{i=1}^N$ is dual to $\{\varphi_i\}_{i=1}^N$, in particular it is referred to as the canonical dual. The canonical dual frame is unique and it leads to the smallest expansion in l^2-norm, that is the vector $\{\alpha_i{'}\}_{i=1}^N$ solution of the following optimization problem (see [6], pp. 145-151 for this fact and Example 1 following):

$$\min_{\alpha \in l^2} \|\alpha\| \ \text{ s.t. } T\alpha = x \ . \tag{1.47}$$

By the next proposition, the reconstruction problem in eq. (1.46) may be simplified: not necessarily one has to determine S and invert it.

Proposition 4. *Let $\Phi = \{\varphi_i\}_{i=1}^N$ and $\Psi = \{\psi_i\}_{i=1}^N$ with analysis operator T_1^*, T_2^* respectively. Then the following are equivalent:*

1. *Ψ is a dual frame for Φ,*
2. *$T_1 T_2^* = \mathbb{I}$.*

Proof. Take $x \in H^n$, then by Eq. 1.18 and Eq. 1.19 for all $x \in H^n$

$$x = T_1 T_2^* x = T_1 \left(\{< x, \psi_k >\}_{k=1}^N \right) = \sum_{k=1}^{N} < x, \psi_k > \varphi_k \quad .$$

Reading the equation from the left to the right corresponds to the implication (2) $\Rightarrow$ (1). Instead, one has the implication (1) $\Rightarrow$ (2) when reading the equations leftwards. □

Considering now the canonical dual frame, by its definition we have $T_2^* = (S^{-1}T_1)^*$, so

$$\begin{aligned} T_1 T_2^* &= T_1 T_1^* \left(S^{-1}\right)^* \\ &= S S^{-1} = \mathbb{I} \end{aligned} \tag{1.48}$$

using again S being self-adjoint. Moreover, using eq. (1.27) we have

$$\begin{aligned} x &= S^{-1/2}S^{1/2}x \\ &= S^{-1/2}\left(S\left(S^{-1/2}x\right)\right) \\ &= S^{-1/2}\left(\sum_{k=1}^{N} < S^{-1/2}x, \varphi_k > \varphi_k\right) \\ &= \sum_{k=1}^{N} < x, S^{-1/2}\varphi_k > S^{-1/2}\varphi_k \quad . \end{aligned} \tag{1.49}$$

Note that $\tilde{\Phi} = \left\{S^{-1/2}\varphi_n\right\}_{n=1}^{N}$ is Parseval. Indeed, as

$$\tilde{T} = S^{-1/2}T \quad \text{and} \quad \tilde{T}^* = \left(S^{-1/2}T\right)^* = T^*S^{-1/2} \tag{1.50}$$

with T, T^* being as before respectively the analysis and synthesis operator of the frame Φ , we have

$$\begin{aligned} \tilde{S} &= \tilde{T}\tilde{T}^* \\ &= S^{-1/2}TT^*S^{-1/2} \\ &= S^{-1/2}SS^{-1/2} = \mathbb{I} \quad . \end{aligned} \tag{1.51}$$

Inserting now in the frame inequality for the frame operator $\tilde{S}$ yields

$$< \tilde{S}x, x > \; = \; < x, x > \; = \|x\|^2 \quad . \tag{1.52}$$

allowing for the choice $A = B = 1$. Therefore $\tilde{\Phi}$ is Parseval. Moreover, as $S^{-1/2}$ is an invertible and bounded operator, then by Def. 1.8 $\Phi = \{\varphi_i\}_{i=1}^{N}$ and $\tilde{\Phi} = \left\{S^{-1/2}\varphi_i\right\}_{i=1}^{N}$ are isomorphic. In pratice, this means that partitions of Φ and $\tilde{\Phi}$ share the same spanning and linear independence properties (see [18] p. 58, Theorem 2.4). Thus defining $E = \{\varphi_i : i \in I\}$ for some index set $I \in [1, N]$, it holds that

E is spanning/ lin. indipendent $\Leftrightarrow S^{-1/2}(E)$ is spanning/ lin. indipendent .

When two frames are isomorphic, then their analysis and synthesis operators share some properties.

Proposition 5. *Let* $\Phi = \{\varphi_i\}_{i=1}^{N}$ *and* $\Psi = \{\psi_i\}_{i=1}^{N}$ *be frames for* H^n *with analysis operator* T_1 *and* T_2 *respectively. The following are equivalent:*

1. Φ *and* Ψ *are isomorphic,*

2. *rank* $T_1 =$ *rank* T_2,

3. *ker* $T_1^* =$ *ker* T_2^*.

We now show the reconstruction process at an example.

Example 1. *In* $\mathbb{R}$, *take* $\Phi = \frac{1}{5}\begin{pmatrix}1 & 2\end{pmatrix}$ *and* $x = 1$, *so* $n = 1$, $N = 2$.

As $T_1 = \frac{1}{5}\begin{pmatrix}1 & 2\end{pmatrix}$, the adjoint T_1^* is just its transpose, that is $T_1^* = \frac{1}{5}\begin{pmatrix}1 & 2\end{pmatrix}^T$. By Prop. 4, the adjoint operator $T_2^* = \begin{pmatrix}\tilde{\varphi_1} & \tilde{\varphi_2}\end{pmatrix}^T$ of the dual frame Ψ must satisfy

$$\frac{1}{5}\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}\tilde{\varphi_1} \\ \tilde{\varphi_2}\end{pmatrix} = 1 \quad . \tag{1.53}$$

Therefore we may choose $\tilde{\varphi_1} = 1$ and $\tilde{\varphi_2} = 2$. We note that the choice of the dual frame is not unique. For $\alpha = \begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix}^T$, we have ($x = 1$)

$$\alpha = T_2^* x = \begin{pmatrix}1 \\ 2\end{pmatrix} \quad .$$

Adding β such that $T_1\beta = 0$ to the solution α does not alter the reconstructed vector x. Thus we are considering a general solution $\bar{\alpha}$ of the form

$$\bar{\alpha} = \alpha + \beta \quad . \tag{1.54}$$

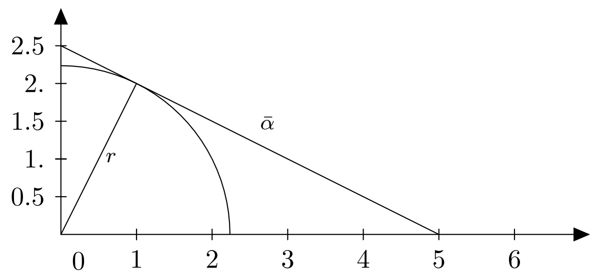
The vector $\beta = \begin{pmatrix}\beta_1 & \beta_2\end{pmatrix}^T$ we look for satisfies

$$\beta_1 + 2\beta_2 = 0 \quad ,$$

allowing for the choice $\beta_1 = 2$ and $\beta_2 = -1$ and any multiple $\gamma \in \mathbb{R} \setminus \{0\}$ of the two. Inserting the expressions obtained in eq. (1.54), for $\overline{\alpha}$ we have

$$\overline{\alpha} = \begin{pmatrix}1 \\ 2\end{pmatrix} x + \begin{pmatrix}2 \\ -1\end{pmatrix}\gamma \quad . \tag{1.55}$$

Consider now $x = 1$. The parameter γ in eq. (1.55) describes the set of points along the line having co-vertex $\alpha_{10} = (5, 0)$ and vertex $\alpha_{01} = (0, 5/2)$, obtained respectively using $\gamma = 2$ and $\gamma = -1/2$. Now the set of points (α_1, α_2) giving the same l^2 norm lie on a circle around the origin with radius r satisfying $\alpha_1^2 + \alpha_2^2 = r^2$. As $\alpha_1 = 1$ and $\alpha_2 = 2$, it follows $r^2 = 5$.
Plotting the line and the circle, their intersection gives the point $\alpha = (1, 2)^T$, meaning the coefficients α_1, α_2 minimizing $\|\alpha\|_2$ in eq. (1.47) are unique.



1.3 Spanning and independence properties

Central to frames are the questions concerning the number of spanning sets and that of linear independent sets one can partition the frame into. Linear independence of a set is expressed as follows: for an index set I, considering a set of vectors $\{\varphi_j : j \in I\}$, these are linearly independent if

$$\sum_{k \in I} c_k \varphi_k = 0 \Rightarrow c_k = 0 \ \forall k \tag{1.56}$$

with $c_k \in \mathbb{C}$. Next, a frame contains a basis. That is the content of the next proposition. This section bases entirely on [3].

Proposition 6. *Let $\Phi = \{\varphi_i\}_{i=1}^N \subset H^n$ be a collection of vectors. Then Φ is a frame if and only span $\Phi = H^n$.*

Proof. By eq. (1.29) and Prop. 2, Φ is a frame with frame operator S, comparing the operators we have $A \cdot \mathbb{I} \leq S$ for some $A > 0$. If rank (S) was such that rank $(S) < n$, then there exists a vector $x \in H^n$ ($\|x\| = 1$) with $Sx = 0$. But this contradicts the frame condition (eq. 1.6), so necessarily we have rank $(S) = n$ and span $\Phi = H^n$.

Now consider the backwards direction. Suppose Φ is not a frame, therefore the frame inequality does not hold (eq. (1.6)), and following, the middle term can become arbitrary small if $\|x\|$ is fixed. This means that there exists a sequence $\{x_k\}_{k=1}^\infty \in H^n$ with fixed norm (e.g. $\|x_k\| = 1$) satisfying

$$\sum_{i=1}^{N} |< x_k, \varphi_i >|^2 \leq \frac{1}{k} \quad \text{for all } k = 1, 2, \ldots, \quad . \tag{1.57}$$

As we are in finite dimension, there exists a subsequence $\left\{x_{k_j}\right\} \to x \in H^n$,

$x \neq 0$, which for simplicity we denote by $\{x_k\}_{k=1}^{\infty}$. Building the difference

$$\begin{aligned}
\sum_{i=1}^{N} |< x, \varphi_i >|^2 &= \sum_{i=1}^{N} |< x - x_k + x_k, \varphi_i >|^2 \\
&= \sum_{i=1}^{N} < x - x_k + x_k, \varphi_i > \overline{< x - x_k + x_k, \varphi_i >} \\
&= \sum_{i=1}^{N} \big(|< x - x_k, \varphi_i >|^2 + |< x_k, \varphi_i >|^2 + \\
&\quad + < x_k, \varphi_i > \overline{< x - x_k, \varphi_i >} + < x - x_k, \varphi_i > \overline{< x_k, \varphi_i >} \big) .
\end{aligned} \tag{1.58}$$

We recall Young's inequality (see [9], p.92), given any a, $b \in \mathbb{C}$

$$|a|\,|b| \leq \frac{1}{2} \left(|a|^2 + |b|^2 \right) \quad . \tag{1.59}$$

Then for the third and fourth term in eq. (1.58) it holds that

$$\begin{aligned}
|x_k, \varphi_i >|\,|\overline{< x - x_k, \varphi_i >}| &\leq \frac{1}{2} \left(|< x_k, \varphi_i >|^2 + |< x - x_k, \varphi_i >|^2 \right) \\
|< x - x_k, \varphi_i >|\,|\overline{< x_k, \varphi_i >}| &\leq \frac{1}{2} \left(|< x - x_k, \varphi_i >|^2 + |< x_k, \varphi_i >|^2 \right) \quad .
\end{aligned}$$

With the two relations obtained, using eq. (1.57), eq. (1.58) becomes

$$\begin{aligned}
\sum_{i=1}^{N} |< x, \varphi_i >|^2 &\leq 2 \left(\sum_{i=1}^{N} |< x - x_k, \varphi_i >|^2 + |< x_k, \varphi_i >|^2 \right) \\
&\leq 2 \left(\frac{1}{k} + \sum_{i=1}^{N} |< x - x_k, \varphi_i >|^2 \right) \quad .
\end{aligned}$$

Finally, by the Cauchy-Schwarz inequality (eq. (1.13))

$$\begin{aligned}
\sum_{i=1}^{N} |< x, \varphi_i >|^2 &\leq 2 \left(\frac{1}{k} + \sum_{i=1}^{N} \|x - x_k\|^2 \|\varphi_i\|^2 \right) \\
&= 2 \left(\frac{1}{k} + \|x - x_k\|^2 \sum_{i=1}^{N} \|\varphi_i\|^2 \right) \quad .
\end{aligned}$$

Letting $k \to \infty$, the right-hand side of the equation goes to zero, so

$$\sum_{i=1}^{N} |< x, \varphi_i >|^2 = 0 \quad ,$$

and $x \perp \varphi_i$ for all $i = 1, \ldots, N$, implying that Φ can not span H^n. $\square$

Next we characterize a frame by linear independence.

Proposition 7. *Let $\Phi = \{\varphi_i\}_{i=1}^N \subset H^n$ be a collection of vectors. Then Φ is linearly independent if and only if Φ is a Riesz basis for its span.*

Proof. For the forward direction, if Φ is linearly independent, then $N \leq n$ and $x \in \text{span } \Phi$ has a unique representation by a sequence of scalars $\{a_i\}_{i=1}^N$ in the form $x = \sum_{k=1}^N a_k \varphi_k$. If $x = 0$, then necessarily we have $a_k = 0$ for all $k = 1, \ldots, N$. Differently, when $x \neq 0$ by the norm property it follows that

$$\|x\|^2 = < x, x > = \left\| \sum_{k=1}^N a_k \varphi_k \right\|^2 \; > \; 0$$

and $\sum_{k=1}^N |a_k|^2 > 0$ as $a_k \neq 0$ for some k. Next, by the linearity property of the scalar product and following the Cauchy-Schwarz inequality (eq. (1.13)) we get

$$\begin{aligned} < x, x > &\leq \sum_{j=1}^N \sum_{k=1}^N |< a_j \varphi_j, a_k \varphi_k >| \\ &\leq \sum_{j=1}^N \sum_{k=1}^N \|a_j \varphi_j\| \, \|a_k \varphi_k\| = \left(\sum_{k=1}^N |a_k| \, \|\varphi_k\| \right)^2 \end{aligned}$$

We define $d := \max_{j \in \{1,\ldots,N\}} \|\varphi_j\|$. Moreover, as H^n is finite dimensional, any two norms $\|\cdot\|_p$, $\|\cdot\|_q$ with $p, q \in [1, \infty]$ are equivalent (see Cor. 2.17 in [19], p.43). Thereby it follows that

$$\begin{aligned} < x, x > \; &\leq \left(d \sum_{k=1}^N |a_k| \right)^2 \\ &= d^2 \|a\|_1^2 \leq \lambda_2 \|a\|_2^2 = \lambda_2 \sum_{k=1}^N |a_k|^2 \end{aligned}$$

for some $\lambda_2 > 0$. In order to find a lower bound, let T_a be the linear operator having as matrix representation

$$T_a = (\varphi_1 | \varphi_2 | \ldots | \varphi_N) \quad .$$

Then $x = T_a a = \sum_{k=1}^N a_k \varphi_k$. Plus, as T_a's columns are linearly independent, there exists a left inverse T_a^{-1}, so $a = T_a^{-1} x$. It then follows

$$\begin{aligned} < a, a > = \sum_{k=1}^N |a_k|^2 &= \; < T_a^{-1} x, T_a^{-1} x > \\ &= \left\| T_a^{-1} x \right\|^2 \leq \left\| T_a^{-1} \right\|^2 \|x\|^2 \quad , \end{aligned}$$

where $\|T_a^{-1}\| > 0$ is a generic matrix norm (see [20], p.33 for some examples). In particular, this will be a finite number as we are in finite dimension. Reordering and using the above expression for x then gives a lower Riesz basis bound $\lambda_1 := \|T_a^{-1}\|^{-2}$ resulting from

$$\left\|T_a^{-1}\right\|^{-2} \sum_{k=1}^{N} |a_k|^2 \leq \left\| \sum_{k=1}^{N} a_k \varphi_k \right\|^2 \quad .$$

For the backwards direction, suppose Φ is a Riesz basis. Let $\{a_i\}_{i=1}^N$ be a sequence of scalars. If Φ was linearly dependent, there exists $x = \sum_{k=1}^N a_k \varphi_k = 0$ with $a_j \neq 0$ for some j, $j \in \{1, \ldots, N\}$. Thus $\sum_{k=1}^N |a_k|^2 > 0$, contradicting eq. (1.11). Therefore Φ must be linearly independent. □

We now make some general considerations about partitioning a collection of vectors Φ into spanning and linear independent sets. Suppose one finds a partition $\{A_1, \ldots, A_K\}$ of Φ into minimal spanning sets ($K \in \mathbb{N}$), meaning each spanning set counts n elements, so $K \cdot n = N$. As there could remain a partition element A_{K+1} which is not spanning, the partition now being $\{A_1, \ldots, A_K, A_{K+1}\}$, generally we have

$$K \leq \lfloor N/n \rfloor \quad \text{spanning sets} \quad . \tag{1.60}$$

Concerning linear independence, we know that a minimal spanning set is in particular a linear independent set. Plus, if $|A_{K+1}| = 1$, then it can be partitioned into one linearly independent set, namely that composed by the vector itself. Differently, if $|A_{K+1}| > 1$, then therein we can find at least one linear independent set, namely that composed by a single vector. Thus we get

$$K \geq \lceil N/n \rceil \ \text{linearly independent sets} \quad . \tag{1.61}$$

In this chapter, we prove a version of the *Rado-Horn* theorem, upon which several other results build. We first introduce two concepts and a theorem we need. The first notion is that of K-ordering of dimensions. A partition $\{I_1, \ldots, I_K\}$ of an index set $[1, N]$ is said to maximize the k-ordering of dimensions if, given any other partition $\left\{\tilde{I}_k\right\}_{k=1}^K$ of $[1, N]$ satisfying

$$\text{dim span } \{\varphi_j : j \in I_k\} \leq \text{dim span } \left\{\varphi_j : j \in \tilde{I}_k\right\} \ , 1 \leq k \leq K \tag{1.62}$$

then

$$\text{dim span } \{\varphi_j : j \in I_k\} = \text{dim span } \left\{\varphi_j : j \in \tilde{I}_k\right\} \ , 1 \leq k \leq K \quad . \tag{1.63}$$

Next we introduce a chain of dependency. The idea is that, given a partition $\{I_k\}_{k=1}^2$ of $[1, N]$, a chain of dependency 'records' the indices i_k associated to vectors φ_{i_k} $(k = 1, \ldots, N)$ that may be moved between the two sets

$$A_1 = \{\varphi_j : j \in I_1\} \text{ and } A_2 = \{\varphi_j : j \in I_2\} \quad . \tag{1.64}$$

Necessary condition for a vector $\varphi_{i_k} \in A_1$ (respectively $\varphi_{i_k} \in A_2$) to be moved to A_2 (A_1) is that A_1 (A_2) is linearly dependent and that φ_{i_k} is in the span of A_2 (A_1). We show a brief example to explain the concept before coming to the formal definition.

Example 2. *In $\mathbb{R}^3$, take $\varphi_1 = \varphi_5 = (1, 0, 0)^T$, $\varphi_2 = \varphi_6 = (0, 1, 0)^T$, $\varphi_3 = \varphi_7 = (0, 0, 1)^T$ and $\varphi_4 = (1, 1, 1)^T$.*

Given the partition $I_1 = \{1, 2, 3, 4\}$, $I_2 = \{5, 6, 7\}$, we note that A_1 is linearly dependent. Therefore we take 4 and move it to I_2, thereby A_2 becomes linear dependent as $I_2 = \{4, 5, 6, 7\}$, I_1 being $I_1 = \{1, 2, 3\}$. One can now move either φ_5, φ_6 or φ_7 from A_2 to A_1 as by linear combinations of the other vectors plus φ_4 one obtains the chosen vector. Iterating this procedure, one gets a chain of dependency of length 7. This procedure is shown below:

$$\begin{aligned}
I_1 = \{1, 2, 3, 4\} &\text{ and } I_2 = \{5, 6, 7\} \quad \text{(Start)} \\
I_1 = \{1, 2, 3\} &\xrightarrow{4} I_2 = \{4, 5, 6, 7\} \\
I_1 = \{1, 2, 3, 5\} &\xleftarrow{5} I_2 = \{4, 6, 7\} \\
I_1 = \{2, 3, 5\} &\xrightarrow{1} I_2 = \{1, 4, 6, 7\} \\
I_1 = \{2, 3, 5, 6\} &\xleftarrow{6} I_2 = \{1, 4, 7\} \\
I_1 = \{3, 5, 6\} &\xrightarrow{2} I_2 = \{1, 2, 4, 7\} \\
I_1 = \{3, 5, 6, 7\} &\xleftarrow{7} I_2 = \{1, 2, 4\} \\
I_1 = \{5, 6, 7\} &\xrightarrow{3} I_2 = \{1, 2, 3, 4\}
\end{aligned}$$

giving the chain of dependency $\{4, 5, 1, 6, 2, 7, 3\}$. Note that in general a chain of dependency might not contain all vector indices, i.e. its length M being shorter than N. This is equivalent to saying that not all indexes in I_1 at start end up in I_2 when the chain of dependency terminates.

Definition 1.10. A chain of dependency $\{i_1, i_2, \ldots, i_M\} \subset I_1 \cup I_2$ of length M has the following properties:

1. i_k will be an element of I_1 for odd indices k, and an element of I_2 for even indices k,

2. $\varphi_{i_1} \in \text{span}\{\varphi_j : j \in I_1 \setminus \{i_1\}\}$, and $\varphi_{i_1} \in \text{span}\{\varphi_j : j \in I_2\}$,

3. for odd k, $1 < k \leq M$:

$$\varphi_{i_k} \in \text{span}\{\varphi_j : j \in (I_1 \cup \{i_2, i_4, \ldots, i_{k-1}\}) \setminus \{i_1, i_3, \ldots, i_k\}\} \ , \text{ and}$$
$$\varphi_{i_k} \in \text{span}\{\varphi_j : j \in (I_2 \cup \{i_1, i_3, \ldots, i_{k-2}\}) \setminus \{i_2, i_4, \ldots, i_{k-1}\}\} \ ,$$

4. for even k, $1 < k \leq M$:

$$\varphi_{i_k} \in \text{span}\{\varphi_j : j \in (I_2 \cup \{i_1, i_3, \ldots, i_{k-1}\}) \setminus \{i_2, i_4 \ldots, i_k\}\} \ , \text{ and}$$
$$\varphi_{i_k} \in \text{span}\{\varphi_j : j \in (I_1 \cup \{i_2, i_4, \ldots, i_{k-2}\}) \setminus \{i_1, i_3, \ldots, i_{k-1}\}\} .$$

Condition 1 describes the structure of a chain of dependency: it is a sequence of distinct indices, alternatingly chosen from I_1 and I_2. Next, condition 2 poses a condition for an index i_1 to be moved to I_2. Condition 3 and 4 just extend condition 2 for $k > 1$, respectively for odd k (condition 3) and even k (condition 4.), by the fact that one has to take into account the vectors that have already been moved. Both in condition 3 and 4 the first line concerns the set from which a vector φ_{i_k} is moved, the second the set to which φ_{i_k} is moved.
Def. 1.10 has been slightly adjusted with respect to the original formulation in [3]: it seems plausible that the authors meant i_k instead of i_{k-2} in the first line of condition 4 as otherwise $k-2$ would give 0 for $k = 2$, but there is no index i_0 in the chain of dependency. This adaptation makes also sense when we compare condition 4 to condition 2.

The two concepts just introduced are related to one another: a partition maximizing the K-ordering of dimensions will allow for a chain of dependency of maximum length. The latter has been observed in Ex. 2. Concerning the fact that $I_1 = \{1, 2, 3, 4\}$ and $I_2 = \{5, 6, 7\}$ maximize the K-ordering of dimensions, there does not exist a partition $\tilde{A}_1$, $\tilde{A}_2$ whose elements span counts more dimensions as, recalling that we are in $\mathbb{R}^3$, both I_1 and I_2 are maximal in the sense that

$$\dim \text{span } \{\varphi_j : j \in I_1\} = 3 = \dim \text{span } \{\varphi_j : j \in I_2\} \quad .$$

Differently, if one had started with $\tilde{I}_1 = \{1, 2, 4, 5, 7\}$ and $\tilde{I}_2 = \{3, 6\}$, then we would have obtained a shorter chain of dependency. Indeed, we have

$$\begin{aligned}\tilde{I}_1 = \{1,2,4,5,7\} &\text{ and } \tilde{I}_2 = \{3,6\} \quad \text{(Start)}\\ \tilde{I}_1 = \{1,4,5,7\} &\overset{2}{\rightarrow} \tilde{I}_2 = \{2,3,6\}\\ \tilde{I}_1 = \{1,4,5,6,7\} &\overset{6}{\leftarrow} \tilde{I}_2 = \{2,3\}\\ \tilde{I}_1 = \{1,4,5,6\} &\overset{7}{\rightarrow} \tilde{I}_2 = \{2,3,7\}\\ \tilde{I}_1 = \{1,3,4,5,6\} &\overset{3}{\leftarrow} \tilde{I}_2 = \{2,7\} \quad . \end{aligned} \tag{1.65}$$

Now, one could choose either $1, 4$ or 5, but none of φ_1, φ_4 or φ_5 lies in span $\{\varphi_2, \varphi_7\}$. As a consequence, we may say that a partition allowing for a longer chain of dependency is a partition for which the linearly independent vectors are more equally distributed between the two partition elements: for $I_1 = \{1,2,3,4\}$ and $I_2 = \{5,6,7\}$, A_1 and A_2 each contain 3 linearly independent vectors. Instead, for the choice $I_1 = \{1,2,4,5,7\}$ and $I_2 = \{3,6\}$ A_1 has three linearly independent vectors, whereas A_2 just two.
In fact, more is true: a partition I_1, I_2 maximizing the 2-ordering of dimensions leads to A_1, A_2 generating the same subspace S. That is the content of the next Lemma, which for brevity we state without proving it (see [3]).

Lemma 1.2. *Let $\Phi = \{\varphi_i\}_{i=1}^N \subset H^n$, and suppose that Φ cannot be partitioned into two linearly independent sets. Let $\{I_1, I_2\}$ be a partition of $[1, N]$ which maximizes the 2-ordering of dimensions. Let J be the union of all chains of dependencies of Φ based on the partition $\{I_1, I_2\}$. Let $J_1 = J \cap I_1$ and $J_2 = J \cap I_2$ and $S = \text{span } \{\varphi_i : i \in J\}$. Then*

$$S = \text{span } \{\varphi_i : i \in J_k\} \ , \ k = 1, 2 \quad .$$

The hypothesis that when partitioning Φ one obtains at least one linearly dependent set implies that Φ has some redundancy within it. Consider the following example.

Example 3. *In $\mathbb{R}^2$, take $\Phi = \{e_1, e_1, e_1, e_2\}$ ($n = 2$, $N = 4$).*

Omitting trivial partitions and those obtained by switching the indexes of A_1 and A_2, there remain two cases to be considered:

1. $A_1 = \{e_1, e_1, e_1\}$ and $A_2 = \{e_2\}$,

2. $A_1 = \{e_1, e_1\}$ and $A_2 = \{e_1, e_2\}$.

Both in case 1 and 2 we have at least two occurences of e_1 in A_1, meaning that A_1 is linearly dependent. The redundancy within Φ is represented by the fact that dim span $\{\varphi_j : j \in \Phi\} = 2$, while Φ has length four.
Lastly we state the theorem we need for the proof of the Rado-Horn theorem.

Theorem 1.3. *Let $\Phi = \{\varphi_i\}_{i=1}^N \subset H^n$ be a collection of vectors. If for every non-empty $J \subset [1, N]$*

$$\frac{|J|}{dim\ span\ \{\varphi_j : j \in J\}} \leq 2$$

then Φ can be partitioned into two linear independent sets.

Proof. We show the converse implication of the theorem: suppose Φ cannot be partitioned into two linearly independent sets. We start with a partition I_1, I_2 maximizing the 2-ordering of dimensions. By assumption, at least one of the two sets is linearly dependent, e.g. the set $\{\varphi_j : j \in I_1\}$. Take J as the union of all possible chain of dependencies on the partition of I_1, I_2. Let J_1, J_2 be such that

$$J_1 = J \cap I_1 \quad \text{and} \quad J_2 = J \cap I_2 \quad .$$

This means J_1 (respectively J_2) contains only those indices of I_1 (I_2) that may be chosen in some chain of dependency. Then, as $J_1 \cap J_2 = \varnothing$, we may write J as $J = J_1 \cup J_2$ leading to $|J| = |J_1| + |J_2|$. By Lemma 1.2, the sets $\{\varphi_j : j \in J_k\}$ with $k = 1, 2$ span the same subspace $S = \text{span}\ \{\varphi_i : i \in J\}$. Then, as $\{\varphi_i : i \in J_1\}$ is not linearly independent, we have $|J_1| > \dim S$, and as $|J_2| \geq \dim S$

$$\begin{aligned} |J| &= |J_1| + |J_2| \\ &> \dim S + \dim S = 2 \dim \text{span}\ \{\varphi_j : j \in J\} \quad , \end{aligned} \tag{1.66}$$

so by reordering one obtains

$$\frac{|J|}{\dim \text{span}\ \{\varphi_i : i \in J\}} > 2 \quad ,$$

that is the converse equation of that in the theorem. □

Now follows the Rado-Horn theorem. For the purpose of the thesis, we show a reduced proof in the case of two linearly independent sets. This contains also the main idea upon which the general proof works.

Theorem 1.4. *Let $\Phi = \{\varphi_i\}_{i=1}^N \subset H^n$. Then there exists a partition $\{I_1, \ldots, I_K\}$ of $[1, N]$ where each subset $\{\varphi_j : j \in I_k\}$ is linearly independent for every k, $1 \leq k \leq K$, if and only if the following inequality holds:*

$$\frac{|J|}{dim\ span\{\varphi_i : i \in J\}} \leq K \quad \forall J \subset [1, N], J \neq \varnothing \quad . \tag{1.67}$$

Proof. We start with the forward direction. Suppose $\{I_1, I_2, \ldots, I_K\}$ partitions $[1, N]$ so that each of the sets $\{\varphi_i : i \in I_k\}$, $k = 1, \ldots, K$ is linearly independent. Take J as a non-empty subset of $[1, N]$, $J \neq \{1, \ldots, N\}$ and let $J_k = J \cap I_k$ for each $1 \leq k \leq K$. Then as $J_i \cap J_k = \varnothing$ for each $i \neq k$, $i, k \in \{1, \ldots, K\}$, we may write $J = \cup_{k=1}^{K} J_k$. Next

$$\begin{aligned} |J| = \sum_{k=1}^{K} |J_k| &= \sum_{k=1}^{K} \dim \operatorname{span} \{\varphi_i : i \in J_k\} \\ &\leq K \cdot \dim \operatorname{span} \{\varphi_i : i \in J\} \end{aligned} \tag{1.68}$$

so by division one gets eq. (1.67). Concerning the existence of the partition $\{I_1, \ldots, I_K\}$, as we are dealing with finite frames, the biggest K we can get when partitioning is N, meaning every vector index forms a partition element for itself.
The proof of the backwards direction for $K = 2$ is Theorem 1.3. □

We now show without proofs two results based on the Rado-Horn theorem, both of which we will use in Section 2: the first for equal norm Parseval frames, the second is under a more general hypothesis and involves the lower frame bound.

Theorem 1.5. *Let $\Phi = \{\varphi_i\}_{i=1}^{N}$ be an equal norm Parseval frame for H^n, and let $K = \lceil N/n \rceil$. There exists a partition $\{I_k\}_{k=1}^{K}$ of $[1, N]$ such that*

1. *$\{\varphi_j : j \in I_k\}$ is linearly independent for $1 \leq k \leq K$, and*

2. *$\{\varphi_j : j \in I_k\}$ spans H^n for $1 \leq k \leq K - 1$.*

Theorem 1.6. *Let $\Phi = \{\varphi_i\}_{i=1}^{N}$ be a frame for H^n with lower frame bound $A \geq 1$ satisfying $\|\varphi_j\| \leq 1$ for all $j = 1, \ldots, N$ and set $K = \lfloor A \rfloor$. Then there exists a partition $\{I_k\}_{k=1}^{K}$ of $[1, N]$ so that*

$$\textit{span } \{\varphi_i : i \in I_k\} = H^n \textit{ for all } k = 1, 2, \ldots, K \quad .$$

1.4 Full-spark frames

Full-spark frames are a special type of frames. Formally the spark of a frame Φ is defined as follows (see [7] for this section).

Definition 1.11. Let $\Phi = \{\varphi_i\}_{i=1}^{N} \subset H^n$ be a frame for H^n. Then the spark of Φ is the cardinality of the smallest linearly dependent subset of the frame. The frame is full-spark if every $m-$element subset of the frame is linearly independent ($m \in \{1, \ldots, n\}$).

Full-spark frames have, among its properties, that every $n \times n$ submatrix of the matrix associated to the frame operator S is invertible. Therefore they are said to be *maximally robust to erasures* as the erasure of any up to $N - n$ expansion coefficients α_j $(j = 1, \dots, N)$ allows for perfect reconstruction of the vector $x \in H^n$.
We now give an example for a full-spark frame: the Discrete Fourier Transform (DFT) Matrix, often referred to as *Fourier frame*. Recalling that one obtains the discrete Fourier transform $\hat{x}$ of a vector $x = (x_0, x_1, \dots, x_{N-1}) \in \mathbb{C}^N$ by

$$\hat{x}(w) = \sum_{n=0}^{N-1} x_n e^{-i\omega n} \tag{1.69}$$

where $\omega \in \mathbb{R}$ is the physical frequency. Sampling in frequency, that is by defining ω_k as $\omega_k = \frac{2\pi}{N}k$ $(k = 0, 1, \dots, N-1)$ leads to the sequence $\hat{x} = (\hat{x}_0, \hat{x}_1, \dots, \hat{x}_{N-1})$ where the $\hat{x}_k$ are respectively defined as

$$\begin{aligned} \hat{x}_k &= \sum_{n=0}^{N-1} x_n e^{-i\frac{2\pi}{N}kn} \\ &= \sum_{n=0}^{N-1} x_n w_N^{kn} \end{aligned} \tag{1.70}$$

where $w_N = e^{-i\frac{2\pi}{N}}$. Writing in matrix form gives

$$\begin{pmatrix} \hat{x}_0 \\ \hat{x}_1 \\ \dots \\ \hat{x}_{N-1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w_N & w_N^2 & \dots & w_N^{N-1} \\ 1 & w_N^2 & w_N^4 & \dots & w_N^{2(N-1)} \\ \dots & \dots & & & \dots \\ 1 & w_N^{N-1} & w_N^{2(N-1)} & \dots & w_N^{(N-1)^2} \end{pmatrix}}_{=:\mathbb{W_N}^*} \begin{pmatrix} x_0 \\ x_1 \\ \dots \\ x_{N-1} \end{pmatrix} . \tag{1.71}$$

where $\mathbb{W}_N^*$ is the adjoint of the Fourier matrix $\mathbb{W}_N$. In compact form, the above equation may be rewritten as

$$\hat{x} = \mathbb{W}_N^* \, x \tag{1.72}$$

Recalling eq. (1.25), a frame has as vectors the complex-conjugates of the synthesis operator's rows: thus $\mathbb{W}_N^*$'s rows yield the vectors for the Fourier frame. Defining x^j to be the componentwise power by a factor $j \in \mathbb{N}$ of the elements of $x \in \mathbb{C}^N$, the collection of vectors

$$\Phi = \{\varphi_0, \varphi_1, \dots, \varphi_N\} \quad \text{with} \quad \varphi_k = \left(w_N^0, w_N^{-k}, \dots, w_N^{-k(N-1)}\right)^T \tag{1.73}$$

is a frame by Prop. 6 as its elements form an orthogonal basis. Indeed, we have that

$$\begin{aligned}
< \varphi_j, \varphi_k > &= < \begin{pmatrix} 1 \\ w_N^{-j} \\ \dots \\ w_N^{-j(N-1)} \end{pmatrix}, \begin{pmatrix} 1 \\ w_N^{-k} \\ \dots \\ w_N^{-k(N-1)} \end{pmatrix} > \\
&= \sum_{n=0}^{N-1} w_N^{(k-j)n} = \begin{cases} N & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases} \quad , \qquad (1.74)
\end{aligned}$$

as by the geometric sum for $k \neq j$, $k - j \in \mathbb{Z}$

$$\begin{aligned}
\sum_{n=0}^{N-1} w_N^{(k-j)n} &= \frac{1 - w_N^{(k-j)N}}{1 - w_N^{k-j}} \\
&= \frac{1 - [\cos\ (2\pi(k-j)) - i \cdot \sin\ (2\pi(k-j))]}{1 - w_N^{k-j}} = 0 \quad . \qquad (1.75)
\end{aligned}$$

Plus, Φ is equal norm as

$$\begin{aligned}
\|\varphi_j\|^2 &= \overline{\varphi_j}^T \varphi_j \\
&= \sum_{k=0}^{N} \left(w_N^{-j}\right)^k \overline{\left(w_N^{-j}\right)^k} \\
&= \sum_{k=0}^{N} w_N^{-jk} w_N^{jk} = N \quad \forall j = 0, \dots, N-1 \quad . \qquad (1.76)
\end{aligned}$$

Finally, Φ is a tight frame as, given the inversion formula for the DTFT

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} \hat{x}_k w_N^{-kn} \quad , \qquad (1.77)$$

then computing $< x, x >= \|x\|_2^2$ we get

$$\begin{aligned}
\|x\|_2^2 &= < \begin{pmatrix} x_0 \\ \dots \\ x_{N-1} \end{pmatrix}, \begin{pmatrix} x_0 \\ \dots \\ x_{N-1} \end{pmatrix} > \\
&= \sum_{n=0}^{N-1} \frac{1}{N^2} \left| \sum_{k=0}^{N-1} \hat{x}_k w_N^{kn} \right|^2 \\
&= \sum_{n=0}^{N-1} \frac{1}{N^2} \left(\hat{x}_0 w_N^0 + \dots + \hat{x}_{N-1} w_N^{N-1}\right) \\
&\qquad \left(\overline{\hat{x}_0} w_N^0 + \dots + \overline{\hat{x}_{N-1}} w_N^{-(N-1)}\right) \quad .
\end{aligned}$$

We recall eq. (1.74): any cross term with $k \neq j$, $k - j \in \mathbb{Z}$ gives

$$\hat{x}_k \overline{\hat{x}_j} \sum_{n=0}^{N-1} w_N^{(k-j)n} = 0 \quad .$$

Thus

$$\begin{aligned} \|x\|_2^2 &= \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} |\hat{x}_k|^2 \\ &= \frac{1}{N^2} \cdot \sum_{n=0}^{N-1} \|\hat{x}\|_2^2 = \frac{1}{N} \|\hat{x}\|_2^2 \quad . \end{aligned} \tag{1.78}$$

Thus $N \|x\|_2^2 = \|\hat{x}\|_2^2$, so the frame is N-tight (compare with eq. (1.6).

2 Redundancy of frames

Spanning and independence properties may be condensed in the concept of redundancy of a frame, specified by a lower redundancy R^- and an upper redundancy R^+. The former is defined by

$$R^- = \max_{\cup_{k=1}^L A_k \subseteq \Phi} L \tag{2.1}$$

where $A_k \subset \Phi$ are pairwise disjoint subsets and each A_k is spanning, that is span $A_k = H^n$. Differently, the upper redundancy R^+ is taken as

$$R^+ = \min_{\cup_{k=1}^L A_k = \Phi} L, \tag{2.2}$$

the $A_k \subset H^n$ being linearly independent (see eq. (1.56) with $I = \{j : \varphi_j \in A_k\}$). These definitions have combinatorial nature: one has to identify a partition $(A_k)_{k=1}^L$ which minimizes or maximizes L. As we shall see, these bounds may be very far from one another. Let us start with two examples.

Example 4. *In $\mathbb{R}^n$, take $\Phi = (e_1, e_2/\sqrt{2}, e_2/\sqrt{2}, \ldots)$ where $e_j/\sqrt{j}$ is repeated j times, $j = 1, \ldots, n$ and $N = n(n+1)/2$.*

Example 5. *In $\mathbb{R}^n$, take $\Phi = (e_1, e_2, e_2, \ldots)$ where each e_j is repeated j times, $j = 1, \ldots, n$ and $N = n(n+1)/2$.*

Examining the linear independence property for example 5, we might determine R^+ by reasoning as follows: minimizing L means finding the partition with the least possibile number of subsets A_k. Choosing the A_k as $A_1 = \{e_1, e_2 \ldots, e_n\}$, $A_2 = \{e_2, e_3, \ldots, e_n\}$, we obtain n linearly independent subsets. Plus, moving any vector into another set makes that set linear dependent. Therefore $R^+ = n$. In example 4 the same reasoning holds, although one might observe that the constant $\sqrt{j}$ is an additional factor which might restrict choices when building the A_k's. But that constant does not play a role because the linear independence is due to the fact that each A_k contains vectors φ_j that are mutually orthogonal to one another.
Concerning the spanning property, for both example 4 and 5 there exists only one subset generating $\mathbb{R}^n$, thereby $L = 1 = R^-$. This is because φ_1 is orthogonal to all φ_i, $i \neq 1$.

Coming back to the examples, Theorem 1.5 is neither applicable to example 4 nor to example 5 as both are not equal norm Parseval frames, example 4 being at least Parseval. One interesting experiment is to relax the conditions for the theorem, requesting only Parseval without equal norm. Thus the estimate provided by the Rado-Horn theorem on example 4 would yield

$$\begin{aligned} R^- &= \lfloor N/n \rfloor = \lfloor (n+1)/2 \rfloor \\ R^+ &= \lceil N/n \rceil = \lceil (n+1)/2 \rceil \quad , \end{aligned} \tag{2.3}$$

differing from $R^+ = n$ and $R^- = 1$. Therefore, the Rado-Horn theorem does not hold for a Parseval frame which is not equal norm. Moreover, they are at the furthest distance from uniform redundancy N/n. Clearly, this is an extreme case as vectors are concentrated around e_n: we have n repetitions of $e_n/\sqrt{n}$ but just one of e_1.

Comparing example 4 with example 5, one sees that the Parseval property prevents concentration of the 'energy' of the φ_i. Put differently, the l^2 norm of the projections of the orthonormal basis $\{e_n\}_{j=1}^n$ on the φ_i is equal to 1 for example 4, but ranges from 1 to n, depending on e_j, for example 5. In formulae respectively:

$$\sum_{k=1}^{N} |< e_j, \varphi_k >|^2 = 1 \ \forall j \quad \text{for ex. 4} \tag{2.4}$$

$$\sum_{k=1}^{N} |< e_1, \varphi_k >|^2 = 1 \ \text{ but } \sum_{k=1}^{N} |< e_n, \varphi_k >|^2 = n \quad \text{for ex. 5} \tag{2.5}$$

A different measure of redundancy is that proposed by *Bodmann et al.* [2]. It is given by the sum of the normalized projections on the φ_i, that is

$$R_\Phi(x) = \sum_{k=1}^{N} \|\varphi_k\|^{-2} |< x, \varphi_k >|^2 \quad . \tag{2.6}$$

The lower redundancy R^- and upper redundancy R^+ are then respectively defined as

$$R_\Phi^- = \min_{x \in \mathbb{S}} R_\Phi(x) \quad \text{and} \quad R_\Phi^+ = \max_{x \in \mathbb{S}} R_\Phi(x) \tag{2.7}$$

where $\mathbb{S}$ is the unit sphere given by $\mathbb{S} = \{x : \|x\| = 1\}$. Using this redundancy measure on example 4, we observe that by taking x as the normalized all-ones vector, that is $x = \sum_{j=1}^n e_j/\sqrt{n}$, and observing that each e_j ($j \in \{1, \ldots, n\}$) gives j terms,we get

$$\begin{aligned} R_\Phi\left(\sum_{j=1}^{n} e_j/\sqrt{n}\right) &= \left|< \frac{e_1}{\sqrt{n}}, e_1 >\right|^2 + \frac{2}{1/2}\left|< \frac{e_2}{\sqrt{n}}, \frac{e_2}{\sqrt{2}} >\right|^2 + \ldots \\ &= \frac{1}{n} + \frac{2}{n} + \ldots + \frac{n-1}{n} + \frac{n}{n} \\ &= \frac{1}{n}\left(n(n+1)/2\right) = \frac{n+1}{2} \quad . \end{aligned} \tag{2.8}$$

On the other hand

$$R_\Phi(e_1) = |< e_1, e_1 >|^2 = 1$$
$$R_\Phi(e_2) = \frac{2}{1/2}\left|< e_2, \frac{e_2}{\sqrt{2}} >\right|^2 = 2$$

so it is convenient to 'concentrate' the energy of the vector x in the last component as we have n normalized projections. That following

$$R_\Phi(e_n) = \frac{n}{1/n}\left|< e_n, \frac{e_n}{\sqrt{n}} >\right|^2 = n \quad . \tag{2.9}$$

Overall, $R_\Phi^- = 1$ as e_1 realizes the minimum, respectively $R_\Phi^+ = n$ as e_n realizes the maximum, so the result of the first redundancy measure (equations (2.1) and (2.2)) coincides with that of the second (eq. 2.7)

2.1 Analytical redundancy measure

We now characterize the analytical redundancy measure outlining the main properties presented by *Casazza et al.* [2] in form of a theorem. Before that we state a lemma that we need when proving one of the properties.

Lemma 2.1. *Let $\Phi = \{\varphi_n\}_{n=1}^N$ be an equal norm frame for a Hilbert space H^n, having frame bounds A, B. Set $c = \|\varphi_j\|^2$ for all $j = 1, \ldots, N$. Then*

$$R_\Phi^- = \frac{A}{c} \quad \textit{and} \quad R_\Phi^+ = \frac{B}{c} \quad .$$

Proof. Using the definition

$$R_\Phi\,(x) = \sum_{k=1}^{N} \frac{1}{\|\varphi_k\|^2}\,|< x, \varphi_k >|^2$$
$$= \frac{1}{c}\sum_{k=1}^{N}|< x, \varphi_k >|^2 \quad .$$

Next, by the frame condition (eq. (1.6)) for $x \in \mathbb{S}$ we have

$$\sum_{k=1}^{N}|< x, \varphi_k >|^2 \geq A \qquad \text{and} \qquad \sum_{k=1}^{N}|< x, \varphi_k >|^2 \leq B \quad ,$$

whereby the conclusion directly follows. □

Now we come to the theorem.

Theorem 2.2. *Let $\Phi = \{\varphi_i\}_{i=1}^N$ be a frame for an n dimensional real or complex Hilbert space H^n ($N \geq n$) and S the frame operator associated to Φ. Then R_Φ has the following properties.*

P1 *If Φ is an equal norm Parseval frame, then $R_\Phi^- = R_\Phi^+ = \frac{N}{n}$.*

P2 *The two are equivalent:*

(i) $R_\Phi^- = R_\Phi^+$.

(ii) the normalized sequence $\{\varphi_i / \|\varphi_i\|\}_{i=1}^N$ is a tight frame.

Moreover, the following two are also equivalent:

(i) $R_\Phi^- = R_\Phi^+ = 1$.

(ii) Φ is an orthogonal collection of vectors.

P3 *The following inequality holds for the upper and lower redundancy R_Φ^+, R_Φ^-: $0 < R_\Phi^- \leq R_\Phi^+ < \infty$.*

P4 *Let $\{e_i\}_{i=1}^n$ be an orthonormal basis and $\Phi' = \{\varphi_i'\}_{i=1}^N$ be a frame. Then:*

$$R^{\pm}_{\Phi \cup (e_i)_{i=1}^n} = R_\Phi^{\pm} + 1 \quad . \tag{2.10}$$

Moreover:

$$\begin{aligned} R^-_{\Phi \cup \Phi'} &\geq R^-_\Phi + R^-_\Phi \\ R^+_{\Phi \cup \Phi'} &\leq R^+_\Phi + R^+_\Phi \quad . \end{aligned}$$

P5 *Let U be a unitary operator, $\{c_n\}_{n=1}^N$ be a sequence of scalars. Then:*

$$\begin{aligned} R^{\pm}_{U(\Phi)} &= R^{\pm}_\Phi \\ R^{\pm}_{(c_i\varphi_i)_{i=1}^N} &= R^{\pm}_\Phi \quad . \end{aligned}$$

P6 *Φ contains at least $\lfloor R_\Phi^- \rfloor$ disjoint spanning sets. In particular, any set of $\lfloor R_\Phi^- \rfloor - 1$ vectors can be deleted yet leave a frame.*

P7 *If Φ does not contain any zero vectors, then it can be partitioned into $\lceil R_\Phi^+ \rceil$ linearly independent sets.*

We briefly comment the properties and then provide a formal proof of each one. P1 states that equal norm Parseval frames have uniform redundancy.

Proof. (P1) As Φ is an equal norm frame, we have that

$$R_\phi(x) = \frac{1}{\|\varphi_1\|^2} \sum_{k=1}^{N} |< x, \varphi_k >|^2 \quad .$$

By definition of lower and upper redundancy, taking $x \in \mathbb{S}$, one has that the sum in the above equation must be greater or equal to A, and lower or equal to B. We recall that for an equal norm Parseval frame $A = B = 1$, so finally $\sum_{k=1}^{N} |< x, \varphi_k >|^2 = 1$. On the other hand $n = N \cdot c^2$ (see eq. (1.44)) where $c = \|\varphi_j\|$, $j = 1, \dots, N$, which leads to

$$\begin{aligned} R_\Phi(x) &= \|\varphi_1\|^{-2} \\ &= c^{-2} = \frac{N}{n} \end{aligned}$$

giving the conclusion. □

The second property is the so-called *Nyquist* property, which characterizes the redundancy measure of tight frames.

Proof. (P2) We start with the forward direction. Normalizing a frame Φ means mapping $\{\varphi_i\}_{i=1}^{N} \mapsto \{\varphi_i / \|\varphi_i\|\}_{i=1}^{N}$. Thereby the frame condition (eq. (1.6)) for $\tilde{\Phi} = \{\varphi_i / \|\varphi_i\|\}_{i=1}^{N}$ becomes, considering $x \in \mathbb{S}$

$$A \leq \underbrace{\sum_{k=1}^{N} \frac{1}{\|\varphi_k\|^2} |< x, \varphi_k >|^2}_{=R_\Phi} \leq B \quad .$$

Now, if $R_\Phi^+ = R_\Phi^-$, then $R_\Phi(x)$ does not depend on $x \in \mathbb{S}$. Thus one can choose $A = B$ in the frame condition of $\tilde{\Phi} = \{\varphi_i / \|\varphi_i\|\}$, so $\tilde{\Phi}$ is tight. Note that the restriction to $x \in \mathbb{S}$ does not change the fact whether a frame is tight or not. Indeed, taken $y = \alpha \cdot x$ with $x \in H^n$, $\alpha \in \mathbb{C}$, we have that

$$\|y\|^2 = |\alpha|^2 \|x\|^2 = \sum_{k=1}^{N} |< y, \varphi_k >|^2 = |\alpha|^2 \sum_{k=1}^{N} |< x, \varphi_k >|^2 \quad .$$

For the backwards direction, if $\tilde{\Phi}$ is tight, then by the tightness property and after reordering we have

$$\begin{aligned} A\|x\|^2 &= \sum_{k=1}^{N} \left| < x, \frac{\varphi_k}{\|\varphi_k\|} > \right|^2 \\ &= \sum_{k=1}^{N} \frac{1}{\|\varphi_k\|^2} |< x, \varphi_k >|^2 \\ &= R_\Phi(x) \ \ \forall x \in H^n \quad . \end{aligned} \tag{2.11}$$

Therefore $R_\Phi^+ = R_\Phi^-$.

Now consider the moreover part. Starting with the forward direction, let us assume that Φ was not orthogonal, meaning there exists φ_j such that $< \varphi_i, \varphi_j > \neq 0$ for some $j,\, i \in \{1, \dots, N\}$. Then taking $x = \varphi_j / \|\varphi_j\|$ we have that

$$\begin{aligned} R_\Phi \left(\varphi_j / \|\varphi_j\|\right) &= \frac{1}{\|\varphi_j\|^2} \left| < \frac{\varphi_j}{\|\varphi_j\|}, \varphi_j > \right|^2 + \sum_{k \neq j} \frac{1}{\|\varphi_k\|^2} \left| < \frac{\varphi_j}{\|\varphi_j\|}, \varphi_k > \right|^2 \\ &= \underbrace{\frac{1}{\|\varphi_j\|^2} \left(\frac{1}{\|\varphi_j\|} \|\varphi_j\|^2 \right)^2}_{=1} + \underbrace{\sum_{k \neq j} \frac{1}{\|\varphi_k\|^2} \left| < \frac{\varphi_j}{\|\varphi_j\|}, \varphi_k > \right|^2}_{\neq 0} \end{aligned}$$

leading to a contradiction as $R_\Phi(x) \leq R_\Phi^+ = 1\ \forall x \in H^n$. For the backwards direction, as Φ is an orthogonal collection of vectors, we can normalize each frame vector in Φ, so $\tilde{\Phi} = \{\tilde{\varphi}_i\}_{i=1}^N = \{\varphi_i / \|\varphi_i\|\}_{i=1}^N$ is an orthonormal basis. We may then use Parseval's identity (eq. (1.5)), giving for all $x \in \mathbb{S}$

$$1 = \sum_{k=1}^{n} |< x, \tilde{\varphi}_k >|^2 = \sum_{k=1}^{n} \frac{1}{\|\varphi_k\|^2} |< x, \varphi_j >|^2 = R_\Phi(x) \quad ,$$

implying $R_\Phi^+ = R_\Phi^-$. □

P3 is an inequality for R_Φ^+, R_Φ^- providing an upper and lower bound and can be well observed in example 5. Here R_Φ^+, R_Φ^- were at the furthest distance of uniform redundancy N/n, namely $R_\Phi^- = 1$ and $R_\phi^+ = n$.

Proof. (P3) The equation follows directly from the frame inequality. □

The fourth property concerns the effect of a merge of two frames on the redundancy measure. We have that the lower (upper) redundancy measure R_Φ^- (R_Φ^+) is superadditive (subadditive). Moreover, the merge of any frame Φ with an orthonormal basis $\{e_j\}_{j=1}^n$ alters the lower and upper redundancy measure respectively by exactly 1.

Proof. (P4) Demonstrations are straightforward using the definition of R_Φ, linearity arguments and the fact that for any $x \in \mathbb{S}$ we have that

$$\sum_{k=1}^{n} |< x, e_k >|^2 = 1 \quad .$$

□

P5 considers invariance of the lower and upper redundancy measure by linear combinations of the frame vectors and by applying a unitary operator.

Proof. (P5) The second equation follows directly by the linearity property of the scalar product and the homogeneity of the norm. Concerning the first, by taking the adjoint of U we have

$$R^{\pm}_{U(\Phi)} = R^{\pm}_{\Phi}\left(U^{*}x\right) \quad .$$

As $\|U^{*}x\| = \|x\| = 1$, we have $R^{\pm}_{\Phi}\left(U^{*}x\right) = R^{\pm}_{\Phi}\left(x\right)$ and thereby the claim. $\square$

(P6) and *(P7)* state that by R_{Φ} one can identify both the minimum number of spanning sets one can partition the frame into, and the maximum number of linearly independent sets, thus characterize the redundancy of the frame.

Proof. (P6) We denote by $\tilde{S}$ the frame operator associated to the unit norm frame $\tilde{\Phi} = \{\varphi_i / \|\varphi_i\|\}_{i=1}^{N}$. Then, by Lemma 2.1 $A = R^{-}_{\tilde{\Phi}}$ is the lower frame bound of $\tilde{\Phi}$. Suppose $A \geq 1$. Thus, using Theorem 1.6 we have that $\tilde{\Phi}$ can be partitioned into $\left\lfloor R^{-}_{\tilde{\Phi}} \right\rfloor$ spanning sets.
Note that the normalization $\{\varphi_i\}_{i=1}^{N} \mapsto \{\varphi_i / \|\varphi_i\|\}_{i=1}^{N}$ does not alter the spanning properties of a frame: in fact, normalization is also part of the *Gram-Schmidt* orthogonalization procedure (see [4], p. 34). Generally, as Φ is a spanning set for H^n, we can write any $x \in H^n$ in the form (see [18], p. 42)

$$x = a_1\varphi_1 + \ldots + a_N\varphi_N$$

with $\{a_i\}_{i=1}^{N}$ being a sequence of scalars, not necessarily unique. Now if we consider the normalized frame, this does not alter its spanning properties, as rewriting the above equation

$$\begin{aligned} x &= \underbrace{a_1\|\varphi_1\|}_{=:c_1}\frac{\varphi_1}{\|\varphi_1\|} + \ldots + \underbrace{a_N\|\varphi_N\|}_{=:c_N}\frac{\varphi_N}{\|\varphi_N\|} \\ &= c_1\frac{\varphi_1}{\|\varphi_1\|} + \ldots + c_N\frac{\varphi_N}{\|\varphi_N\|} \quad , \end{aligned}$$

we obtain a new representation with respect to $\tilde{\Phi}$ and coefficients $\{c_i\}_{i=1}^{N}$. Thus Φ can be partitioned into the same number of spanning sets $\tilde{\Phi}$ can be partitioned into. $\square$

This proof for property (P6) works as long as $A \geq 1$. That is probably the reason why the authors *Bodmann et al.* [2] add 'at least' when stating the property. We now see an example in which $\left\lfloor R^{-}_{\Phi} \right\rfloor = 0$.

Example 6. *Let $0 < \varepsilon < 1$ and $\{e_i\}_{i=1}^n \subset \mathbb{R}^n$ be the standard basis. Consider $\Phi = \{\varphi_i\}_{i=1}^n$*

$$\varphi_j = \begin{cases} e_1 & , \quad i = 1 \\ \sqrt{1-\varepsilon^2}\, e_1 + \varepsilon e_i & , \quad i \neq 1 \end{cases}$$

The frame's vectors are concentrated around e_1. In fact,

$$R_\Phi\left(e_1\right) = 1 + \sum_{k=2}^{n} \left|< e_1, \sqrt{1-\varepsilon^2}\, e_1 >\right|^2 = 1 + (n-1)(1-\varepsilon^2)$$

but for $j \neq 1$ we have

$$R_\Phi\left(e_j\right) = \left|< e_j, \varepsilon e_j >\right|^2 = \varepsilon^2 \quad .$$

Thus $R_\Phi^- = \varepsilon^2$, so $\left\lfloor R_\phi^- \right\rfloor = 0$ but there exists a partition into spanning sets - the frame itself is a spanning set.

Proof. (P7) As done when demonstrating property (P6), we denote by $\tilde{S}$ the frame operator associated to the unit norm frame $\tilde{\Phi} = \{\varphi_i / \|\varphi_i\|\}_{i=1}^N$. We prove that $\tilde{\Phi}$ can be partitioned into $\left\lceil R_{\tilde{\Phi}}^+ \right\rceil$ linear independent sets. This is sufficient as $c_i = \|\varphi_i\|$ is a candidate for each $i \in \{1, \ldots, N\}$ when checking eq. 1.56. Thus,

$$\sum_{k=1}^{N} c_k \frac{\varphi_k}{\|\varphi_k\|} = 0 \quad \text{if and only if } c_k = 0 \ \forall k = 1, \ldots, N$$

is equivalent to

$$\sum_{k=1}^{N} c_k \varphi_k = 0 \quad \text{if and only if } c_k = 0 \ \forall k = 1, \ldots, N \quad .$$

As $\tilde{\Phi}$ is unit norm, we can use Lemma 2.1, so $B = R_{\tilde{\Phi}}^+$. The frame inequality (eq. (1.29)) then becomes

$$< \tilde{S}x, x > \leq R_{\tilde{\Phi}}^+ \|x\|^2 \quad .$$

As both $\tilde{S}$ and $R_{\tilde{\Phi}}^+ \mathbb{I}$ are positive operators, then using Def. 1.5 the frame inequality (eq. (1.29)) gives the relation

$$\tilde{S} \leq R_{\tilde{\Phi}}^+ \mathbb{I} \quad ,$$

so

$$\frac{\mathbb{I}}{R_{\tilde{\Phi}}^+} \leq \tilde{S}^{-1} \quad .$$

Similarly, by Lemma 2.1 $A = R^-_{\tilde{\Phi}}$ and one obtains

$$\tilde{S}^{-1} \leq \frac{\mathbb{I}}{R^-_{\tilde{\Phi}}} \quad .$$

Putting the equations together we get

$$\frac{\mathbb{I}}{R^+_{\tilde{\Phi}}} \leq \tilde{S}^{-1} \leq \frac{\mathbb{I}}{R^-_{\tilde{\Phi}}} \quad ,$$

which gives a modified frame inequality, namely (compare with eq. (1.29))

$$\frac{1}{R^+_{\tilde{\Phi}}} \|x\|^2 \ \leq < \tilde{S}^{-1}x, x > \leq \frac{1}{R^-_{\tilde{\Phi}}} \|x\|^2 \quad .$$

Hence, testing the above equation with the unit norm vector $x = \varphi_i / \|\varphi_i\|$ ($i \in \{1, \ldots, N\}$), then using the fact that $\tilde{S}^{-1}$ is self-adjoint it follows that

$$\frac{1}{R^+_{\tilde{\Phi}}} \leq \left\| \tilde{S}^{-1/2} \left(\varphi_i / \|\varphi_i\|\right) \right\|^2 \leq \frac{1}{R^-_{\tilde{\Phi}}} \ \ \forall i = 1, \ldots, N \quad . \tag{2.12}$$

Let $I \subsetneq \{1, 2, \ldots, N\}$, then taking the sum over $i \in I$ we have

$$\frac{|I|}{R^+_{\tilde{\Phi}}} \leq \sum_{i \in I} \left\| \tilde{S}^{-1/2} \left(\varphi_i / \|\varphi_i\|\right) \right\|^2 \leq \frac{|I|}{R^-_{\tilde{\Phi}}} \quad . \tag{2.13}$$

We now take the corresponding Parseval frame to $\tilde{\Phi}$. In order to do so, we need to apply $\tilde{S}^{-1/2}$ on each frame vector (recall eq. (1.50)). Let P be the orthonormal projection of span $\left\{\tilde{S}^{-1/2}(\varphi_i / \|\varphi_i\|)\right\}_{i=1}^N$ onto span $\left\{\tilde{S}^{-1/2}\varphi_i / \|\varphi_i\|\right\}_{i \in I}$. Then by Prop. 1 the set $\left\{P\tilde{S}^{-1/2}(\varphi_i / \|\varphi_i\|)\right\}_{i=1}^N$ is a Parseval frame for span P. We now use eq. (1.43): therein we had dim span $\Phi = n$, whereas here it holds that dim span $\left\{P\tilde{S}^{-1/2} \left(\varphi_i / \|\varphi_i\|\right)\right\}_{i=1}^N \neq n$. Thus the equivalent of eq. (1.43) will be given by

$$\text{dim span } \left\{\tilde{S}^{-1/2}\varphi_i / \|\varphi_i\|\right\}_{i \in I} = \sum_{i=1}^N \left\| P\tilde{S}^{-1/2} \left(\varphi_i / \|\varphi_i\|\right) \right\|^2 \quad .$$

Moreover, using eq. (2.13)

$$\begin{aligned} \sum_{i=1}^N \left\| P\tilde{S}^{-1/2} \left(\varphi_i / \|\varphi_i\|\right) \right\|_2^2 &\geq \sum_{i \in I} \left\| P\tilde{S}^{-1/2} \left(\varphi_i / \|\varphi_i\|\right) \right\|_2^2 \\ &= \sum_{i \in I} \left\| \tilde{S}^{-1/2} \left(\varphi_i / \|\varphi_i\|\right) \right\|_2^2 \geq \frac{|I|}{R^+_{\tilde{\Phi}}} \quad . \end{aligned}$$

Combining the last two equations obtained, by reordering we get

$$\frac{|I|}{\dim \operatorname{span} \left\{ \tilde{S}^{-1/2} \left(\varphi_i / \left\| \varphi_i \right\| \right) \right\}} \leq R^{+}_{\tilde{\Phi}} \quad . \tag{2.14}$$

Now the K in the Rado-Horn condition must be a positive integer, but $R^{+}_{\tilde{\Phi}}$ might not necessarily be. If we round towards 0, we would find a subset I contradicting the Rado-Horn condition. Thus $K = \left\lceil R^{+}_{\tilde{\Phi}} \right\rceil$, so $\tilde{\Phi}$ and thereby also Φ can be partitioned into $\left\lceil R^{+}_{\tilde{\Phi}} \right\rceil$ linearly independent sets. □

2.2 Redundancy measures - comparison

How do the two redundancy measures relate to one another? For the upper redundancy it is straightforward by property P7: $R^{+} = \left\lfloor R^{+}_{\Phi} \right\rfloor$.

For the lower redundancy, suppose to consider the optimal partition for R^{-}, that is the one maximizing the number of subsets spanning H^n. We indicate the partition with $\{A_1, \ldots, A_L\}$[1]. Defining the reduced frame $\tilde{\Phi}$ as

$$\tilde{\Phi} = \{A_1, A_2, \ldots, A_L\} \quad , \tag{2.15}$$

$\tilde{\Phi}$ still is a frame by proposition 6 as each of the A_j, $j = 1, 2, \ldots, L$ spans H^n, thus $R^{-} = L$. Now by the moreover part of property P4 we have:

$$\begin{aligned} R^{-}_{\tilde{\Phi}} &= \min_{x \in \mathbb{S}} R_{A_1 \cup A_2 \cup \ldots \cup A_L}(x) \\ &\geq R^{-}_{A_1} + R^{-}_{A_2} + \ldots + R^{-}_{A_L} \quad . \end{aligned} \tag{2.16}$$

Plus, by property P6: $R^{-} \geq \left\lfloor R^{-}_{\Phi} \right\rfloor$. Therefore, combining the two equations gives

$$\begin{aligned} R^{-} \geq \left\lfloor R^{-}_{\Phi} \right\rfloor \geq \left\lfloor R^{-}_{\tilde{\Phi}} \right\rfloor & \\ \geq R^{-}_{A_1} + R^{-}_{A_2} + \ldots + R^{-}_{A_L} & \quad . \end{aligned} \tag{2.17}$$

Then, reordering the above equation gives

$$R^{-} - R^{-}_{A_1} - \ldots - R^{-}_{A_L} \geq R^{-} - \left\lfloor R^{-}_{\Phi} \right\rfloor \tag{2.18}$$

so by property P2, if each A_j ($j = 1, \ldots, L$) is an orthogonal collection of vectors, then $1 = R^{+}_{A_j} = R^{-}_{A_j}$. Using $R^{-} = L$ we get

$$L - L \geq L - \left\lfloor R^{-}_{\Phi} \right\rfloor \quad . \tag{2.19}$$

[1] generally we can have $\cup_{j=1}^{L} A_j \subsetneq \Phi$, meaning that not all vectors in Φ are part of the partition

This means that for frames being orthogonal collection of vectors the two redundancy measures yield the same result ($R^- = \lfloor R^-_\Phi \rfloor$). This applies to the Fourier frame discussed in Sec. 1.4 (see eq. (1.73)). Differently, if Φ is a Riesz basis, then $R^- = 1$ by Prop. 7 but $R^-_\Phi \neq 1$ in general as we could observe by Ex. 6. Eq. (2.17) becomes

$$1 - \lfloor R^-_\Phi \rfloor \geq 0 \quad , \tag{2.20}$$

with $R^-_\Phi \leq 1$ as $R^+_\Phi = 1$ by property P7. This means that the two redundancy measures do generally not coincide ($R^- \neq \lfloor R^-_\Phi \rfloor$).

Now let us consider Parseval frames. The frame condition gives Parseval's identity, that is (compare with eq. (1.5))

$$\|x\|^2 = \sum_{k=1}^{N} |< x, \varphi_k >|^2 =: \sum_{k=1}^{N} |\alpha_k|^2 \quad . \tag{2.21}$$

Though, as one could observe in Example 4, this does not imply that the frame is an orthonormal sequence. When demonstrating property P6 in the previous subsection (see Sec. 2.1), we showed that if $\tilde{A} \geq 1$, $\tilde{A}$ being the lower frame bound of the normalized frame $\tilde{\Phi}$, then $\left\lfloor R^-_{\tilde{\Phi}} \right\rfloor = R^- = \lfloor R^-_\Phi \rfloor$. Now the difference for Parseval frames is that $A = 1$, so we do not need to normalize the frame in order to satisfy conditions for Theorem 1.6. So we ask, does $R^- = \lfloor R^-_\Phi \rfloor$ hold? As we shall see by Theorem 2.3 we propose, under certain conditions the answer is affirmative.

Theorem 2.3. *Let $\Phi = \{\varphi_i\}_{i=1}^N$ be a Parseval frame satisfying $\|\varphi_j\| \leq 1$ for all $j = 1, \ldots, N$ and $I \subset \{1, \ldots, N\}$ be such that $\{\varphi_j\}_{j \in I}$ is an orthogonal spanning set for H^n. Then Φ can be partitioned into one spanning set, so $R^- = 1$. Moreover, $R^- = R^-_\Phi$.*

Proof. The first fact directly follows by Theorem 1.6 and the fact that Φ is Parseval ($A = B = 1$). Concerning the moreover part, by hypothesis, $\tilde{\Phi} = \{\varphi_j : j \in I\}$ is an orthogonal spanning subset within Φ, thus by property P2: $R^-_{\tilde{\Phi}} = 1 = R^+_{\tilde{\Phi}}$. Using property P4 we get

$$\begin{aligned} R^-_\Phi &= \min_{x \in \mathbb{S}} R_{\tilde{\Phi} \cup \left(\Phi \setminus \tilde{\Phi}\right)}(x) \\ &\geq R^-_{\tilde{\Phi}} + R^-_{\Phi \setminus \tilde{\Phi}} = 1 + R^-_{\Phi \setminus \tilde{\Phi}} \quad . \end{aligned} \tag{2.22}$$

As $R^-_\Phi \leq R^-$ by property P6, it follows that

$$1 = R^- \geq R^-_\Phi \geq 1 + R^-_{\Phi \setminus \tilde{\Phi}} \quad , \tag{2.23}$$

so necessarily $R^-_{\Phi \setminus \tilde{\Phi}} = 0$ and $R^-_\Phi = 1$. $\square$

One can apply Theorem 2.3 on Ex. 4 discussed in Sec. 2. The subset $\tilde{\Phi} \subset \Phi$

$$\tilde{\Phi} = \left\{ e_1, e_2/\sqrt{2}, e_3/\sqrt{3}, \ldots, e_n/\sqrt{n} \right\}$$

still is a frame as it is spanning. Moreover, $\|\tilde{\varphi}_j\| \leq 1$ for all $j = 1, \ldots, N$ and it holds that the $\tilde{\varphi}_j$ are mutually orthogonal, so $< \tilde{\varphi}_i, \tilde{\varphi}_j >= 0$ for $i \neq j$. Indeed, we observed that the two redundancy measures did coincide (see p. 28).

Finally, let us consider equal norm Parseval frames. By property P1 of Theorem 2.2 we have $R_\Phi^- = R_\Phi^+ = N/n$. By the properties P6 and P7 of the same theorem we get

$$R^- \geq \left\lfloor R_\Phi^- \right\rfloor = \lfloor N/n \rfloor \quad \text{and} \quad R^+ = \left\lceil R_\Phi^+ \right\rceil = \lceil N/n \rceil \quad , \tag{2.24}$$

which is close to the statement of Theorem 1.5. Remembering eq. (1.60) we get $R^- = R_\Phi^-$ - the result of the theorem.

3 Discrete Gabor Frames

Gabor Frames denote a class of frames named after D. Gabor. Their peculiarity is that we have modulated time-frequency shifts of a window function ψ as frame vectors. Before defining a Gabor system, we first introduce the translation and modulation operator. For this section, we mostly rely on [13].

Given $x \in \mathbb{C}^N$, the translation operator $\mathbb{T}_k$ shifts its components by k, $k \in \{0, \dots, N-1\}$. In formula this is equal to:

$$\mathbb{T}_k : \mathbb{C}^N \to \mathbb{C}^N, x \mapsto \mathbb{T}_k x = x_{n-k} \bmod N \quad . \tag{3.1}$$

Instead, the modulation operator $\mathbb{M}_l$ modulates the sequence $x \in \mathbb{C}^N$ by a harmonic of order l. Differently than in Sec. 1.4, we now take $w_N = e^{2\pi i/N}$, so M_l is defined to be

$$\mathbb{M}_l : \mathbb{C}^N \to \mathbb{C}^N, x \mapsto \mathbb{M}_l x = \left(w_N^0 x_0, w_N^l x_1, \dots, w_N^{l(N-1)} x_{N-1} \right) \quad , \tag{3.2}$$

meaning that each component x_j is multiplicated with w_N^{jl}. Translation operators are commonly referred to as time-shift operators, modulation operators as frequency-shift operators. Indeed, taking the Fourier transform of $\mathbb{M}_l x$, then for the k-th component we have

$$\begin{aligned} \left(\widehat{\mathbb{M}_l x} \right)_k &= \sum_{n=0}^{N-1} \left(e^{2\pi i l n/N} x_n \right) e^{-2\pi i k n/N} \\ &= \sum_{n=0}^{N-1} x_n e^{2\pi i (k-l) n/N} \\ &= \hat{x}_{k-l} = \left(T_l \hat{x} \right)_k \quad . \end{aligned} \tag{3.3}$$

This means that a modulation in time equals a translation in frequency. Vice versa, a translation in time causes a modulation in frequency. Indeed, translating x by k, then taking the l-th component of the resulting DFT gives

$$\begin{aligned} \left(\widehat{T_k x} \right)_l &= \sum_{n=0}^{N-1} x_{n-k} e^{-2\pi i l n/N} \\ &= \sum_{\tilde{n}=0}^{N-1} x_{\tilde{n}} e^{-2\pi i l (\tilde{n}+k)/N} = \left(\mathbb{M}_{-k} \hat{x} \right)_l \end{aligned} \tag{3.4}$$

with $\tilde{n} = n - k$. We define now the composition of the two operators introduced above. This is

$$\pi : \mathbb{C}^N \to \mathbb{C}^{N \times N}, \;\; x \mapsto \pi(k, l)\, x = \left(\mathbb{M}_l \mathbb{T}_k \right) x \quad . \tag{3.5}$$

Applying π to some $y \in \mathbb{C}^N$, the effect is that of a time-frequency shift. We show an example in $\mathbb{C}^4$ with the operators $\mathbb{T}_2$, $\mathbb{M}_3$ and $\pi(2,3)$. They are respectively given by

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & w_4^{-3} & 0 & 0 \\ 0 & 0 & w_4^{-6} & 0 \\ 0 & 0 & 0 & w_4^{-9} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & w_4^{-3} \\ w_4^{-6} & 0 & 0 & 0 \\ 0 & w_4^{-9} & 0 & 0 \end{pmatrix}. \tag{3.6}$$

We note that supp $\mathbb{T}_2 =$ supp $\mathbb{M}_3\mathbb{T}_2$ where by supp $\cdot$ we mean the set containing the indices of the nonzero entries of the argument, e.g. for $x \in H^n$ or $A = (a_{ij})_{i,j=1}^N \in \mathbb{C}^{N\times N}$

$$\begin{aligned} \text{supp } x &= \{n : x_n \neq 0\} \quad , \\ \text{supp } A &= \{(i,j) : a_{ij} \neq 0 \text{ and } i,j \in \{1,\ldots,N\}\} \quad . \end{aligned}$$

Now follows the definition of a Gabor system. We denote with $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ the indices of $\mathbb{C}^N$, meaning we consider an index set of length N with integer numbers as indices. In this thesis, we limit the discussion to cyclic groups of the form $\mathbb{Z}_N = \{0,1,\ldots,N-1\} \times \{0,1,\ldots,N-1\}$.

Definition 3.1. Let $\psi \in \mathbb{C}^N \setminus \{0\}$ and $\Lambda \subset \mathbb{Z}_N \times \mathbb{Z}_N$, then the Gabor system generated is

$$\{\psi, \Lambda\} = \{\pi(k,l)\,\psi\}_{(k,l)\in\Lambda} \tag{3.7}$$

where ψ is being called window function and Λ is a chosen lattice. If the Gabor system spans $\mathbb{C}^N$, then it is a Gabor frame.

The possible choices of a particular lattice are limited by the group structure of Λ, e.g. if $(a,b) \in \{0,1,\ldots,\lfloor N/2\rfloor\} \times \{0,1,\ldots,\lfloor N/2\rfloor\}$, then also $(2a,2b) \in \Lambda$. In general, there exists several ways to pick Λ, i.e. hexagonal lattices (see [11]).

A Gabor system is related to the *short-time Fourier transform* (STFT), or equivalently *windowed Fourier transform*. Now follows its definition.

Definition 3.2. Let $x \in \mathbb{C}^N$. Then its short-time Fourier transform (STFT) $V_\psi : \mathbb{C}^N \to \mathbb{C}^{N\times N}$ with respect to the window function $\psi \in \mathbb{C}^N \setminus \{0\}$ is given by

$$\begin{aligned} V_\psi : \mathbb{C}^N \to \mathbb{C}^{N\times N},\ V_\psi x(k,l) &= \langle x, \pi(k,l)\,\psi \rangle \\ &= \sum_{n=0}^{N-1} x_n \overline{\psi_{n-k}} w_N^{-ln} \quad . \end{aligned} \tag{3.8}$$

Simply put, applying V_ψ on $x \in \mathbb{C}^n$ returns the coefficient corresponding to an element $\pi(k,l)\psi$ of a Gabor system, $(k,l) \in \Lambda$. Note that V_ψ maps from $\mathbb{C}^N$ to the Hilbert-Schmidt space of linear operators on $\mathbb{C}^N$ (see also [9], p.169). The inner product on it, given two matrices A, B, is [2]

$$< A, B >= \sum_{n=0}^{N-1} \sum_{\tilde{n}=0}^{N-1} < Ae_n, e_{\tilde{n}} >< Be_n, e_{\tilde{n}} > \quad . \tag{3.9}$$

We are now able to characterize the set of time-frequency shift operators $\{\pi(k,l)\}_{k,l\in\Lambda}$.

Proposition 8. *The set of normalized time-frequency shift operators*

$$\left\{ \frac{1}{\sqrt{N}} \pi(k,l) \right\}_{k,l\in\Lambda}$$

is an orthonormal basis for the Hilbert-Schmidt space of linear operators on $\mathbb{C}^N$.

Proof. Denoting with λ_{ij} the entries of the matrix Λ associated to the operator $\pi(k,l)$ (respectively $\gamma_{i,j}$ for the matrix Γ associated to $\pi\left(\tilde{k},\tilde{l}\right)$), we have

$$\left\langle \pi(k,l), \pi\left(\tilde{k},\tilde{l}\right) \right\rangle = \sum_{n=0}^{N-1} \sum_{\tilde{n}=0}^{N-1} \lambda_{\tilde{n}n} \gamma_{\tilde{n}n} \quad .$$

If $k \neq \tilde{k}$, then as mentioned before $\pi(k,l)$ and $\pi\left(\tilde{k},l\right)$ have disjoint support. Hence

$$< \pi(k,l), \pi\left(\tilde{k},l\right) > = 0 \quad .$$

Plus,

$$\frac{1}{N} < \pi(k,l), \pi\left(k,\tilde{l}\right) > = \frac{1}{N} \sum_{n=0}^{N-1} w_N^{-ln} w_N^{\tilde{l}n}$$

as the matrix composition reduces to the vector inner product. Then, using the expression for the geometric sum (see eq. (1.74)), we have

$$\frac{1}{N} < \pi(k,l), \pi\left(k,\tilde{l}\right) > = \delta_{\tilde{l}-l} \quad .$$

completing the proof. □

[2] Sometimes we find the notation $A : B$ instead of $< A, B >$ (see for example [10]): it emphasizes the contraction between the indices of A and those of B. In Einstein's notation, this is equal to $A : B = A_{ij} B_{ij}$.

Similarly to Fourier frames, if we consider the full Gabor system $\{\psi, \Lambda\}$ with $\Lambda = \mathbb{Z}_N \times \mathbb{Z}_N$, then it is a tight frame. As we shall see in the following, if we take $\Lambda \subsetneq \{0, 1, \ldots, N-1\} \times \{0, 1, \ldots, N-1\}$ then $\{\psi, \Lambda\}$ might still be a frame, but will generally not be tight. Now follows the proof of the tightness property.

Eq. 3.7 characterizes the elements of a Gabor system. We consider the full Gabor system with $\Lambda = \{0, 1, \ldots, N-1\} \times \{0, 1, \ldots, N-1\}$. Consider an element $\pi(k, l)\psi$ with $(k, l) \in \Lambda$. Then

$$\begin{aligned}
\pi(k,l)\psi &= \left((\mathbb{M}_l \mathbb{T}_k)\, \psi \right)^{\top\top} && (3.10)\\
&= \left(\psi^\top \mathbb{T}_k^\top \mathbb{M}_l \right)^\top \\
&= \left((\mathbb{T}_k \psi)^\top \mathbb{M}_l \right)^\top \quad . && (3.11)
\end{aligned}$$

Remembering eq. (3.2), eq. (3.10) means we are multiplying componentwise the vector $(\mathbb{T}_k\psi)^T$ with $\left(1, e^{2\pi i l/N}, \ldots, e^{2\pi i l(N-1)/N}\right)^T$, the result of the operation being a vector. This reminds us of the Fourier matrix $\mathbb{W}_N$ defined by eq. (1.73): its columns are formed by the vectors

$$\varphi_l = \left(w_N^0, w_N^l, \ldots, w_N^{l(N-1)} \right)^T = \left(1, e^{2\pi i l/N}, \ldots, e^{2\pi i l(N-1)/N}\right)^T \tag{3.12}$$

with $l = 0, 1, \ldots, N-1$. As one might imagine, we can express the synthesis operator for $\{\psi, \Lambda\}$ by means of $\mathbb{W}_N$. Recalling eq. (1.25), T's columns give the frame vectors. In compact form, T may be written as (see [14])

$$T = (\mathbb{D}_0 \mathbb{W}_N | \mathbb{D}_1 \mathbb{W}_N | \ldots | \mathbb{D}_{N-1} \mathbb{W}_N) \tag{3.13}$$

where

$$\mathbb{D}_j = \text{diag } (\psi_j, \psi_{j+1}, \ldots, \psi_{N-1}, \psi_0, \ldots, \psi_{j-1}) \;\;,\;\; j \in \{0, 1, \ldots, N-1\} \quad .$$

We briefly check its consistency. Recall that $\mathbb{W}_N = (\varphi_0 | \varphi_1 | \ldots | \varphi_{N-1})$.

Example 7. *In $\mathbb{C}^3$, take $\psi = (\psi_0, \psi_1, \psi_2)$ and* $\mathbb{W}_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & w_3 & w_3^2 \\ 1 & w_3 & w_3^4 \end{pmatrix}$.

We start writing the matrices in the set $\{\pi(0, l)\psi\}_{l=0}^2$:

$$\begin{aligned}
\pi\left((0,0)\right) &= \mathbb{M}_0 \mathbb{T}_0 = \text{diag } (1, 1, 1) \quad , \\
\pi\left((0,1)\right) &= \mathbb{M}_1 \mathbb{T}_0 = \text{diag } \left(1, w_3, w_3^2\right) \quad , \\
\pi\left((0,2)\right) &= \mathbb{M}_2 \mathbb{T}_0 = \text{diag } \left(1, w_3^2, w_3^4\right) \quad ,
\end{aligned}$$

so

$$\pi\left((0,0)\right)\psi = \begin{pmatrix}\psi_0 & \psi_1 & \psi_2\end{pmatrix}^\top \quad , \tag{3.14}$$
$$\pi\left((0,1)\right)\psi = \begin{pmatrix}\psi_0 & w_3\psi_1 & w_3^2\psi_2\end{pmatrix}^\top \quad , \tag{3.15}$$
$$\pi\left((0,2)\right)\psi = \begin{pmatrix}\psi_0 & w_3^2\psi_1 & w_3^4\psi_2\end{pmatrix}^\top \quad . \tag{3.16}$$

Similarly, by composing $\mathbb{D}_0\mathbb{W}_3$ we obtain

$$\mathbb{D}_0\mathbb{W}_3 = \mathrm{diag}\ \left(\psi_0\psi_1\psi_2\right)\begin{pmatrix}1 & 1 & 1\\ 1 & w_3^1 & w_3^2\\ 1 & w_3^2 & w_3^4\end{pmatrix} = \begin{pmatrix}\psi_0 & \psi_0 & \psi_0\\ \psi_1 & w_3\psi_1 & w_3^2\psi_1\\ \psi_2 & w_3^2\psi_2 & w_3^4\psi_2\end{pmatrix} \quad ,$$

meaning the columns in $\mathbb{D}_0\mathbb{W}_3$ coincide with eq. (3.14-3.16), which motivates eq. (3.13).

Let T^* be the analysis operator associated to $\{\psi, \Lambda\}$. By eq. (3.13), T^* follows as

$$T^* = \begin{pmatrix}\left(\mathbb{D}_0\mathbb{W}_N\right)^*\\ \left(\mathbb{D}_1\mathbb{W}_N\right)^*\\ \dots\\ \left(\mathbb{D}_{N-1}\mathbb{W}_N\right)^*\end{pmatrix} \quad . \tag{3.17}$$

Combining eq. (3.13) and (3.17) for S we get

$$\begin{aligned} S &= TT^* \\ &= \mathbb{D}_0\mathbb{W}_N\left(\mathbb{D}_0\mathbb{W}_N\right)^* + \mathbb{D}_1\mathbb{W}_N\left(\mathbb{D}_1\mathbb{W}_N\right)^* + \ldots + \mathbb{D}_{N-1}\mathbb{W}_N\left(\mathbb{D}_{N-1}\mathbb{W}_N\right)^* \quad . \end{aligned} \tag{3.18}$$

Accounting for the fact that

$$\begin{aligned} \mathbb{W}_N\mathbb{W}_N^* &= \left(\varphi_0|\varphi_1|\ldots|\varphi_{N-1}\right)\begin{pmatrix}\overline{\varphi_0}\\ \overline{\varphi_1}\\ \dots\\ \overline{\varphi_{N-1}}\end{pmatrix} \\ &= N\cdot\mathbb{I} \quad , \end{aligned} \tag{3.19}$$

eq. (3.18) becomes

$$S = N\left(\mathbb{D}_0\mathbb{D}_0^* + \mathbb{D}_1\mathbb{D}_1^* + \ldots + \mathbb{D}_{N-1}\mathbb{D}_{N-1}^*\right) \quad . \tag{3.20}$$

The matrix S is diagonal as it is the result of sum of diagonal matrices. Next, considering a component $(S)_{jj}$ ($j \in \{0, 1, \ldots, N-1\}$), each term in the above sum contributes differently to each component on the diagonal, e.g. $(\mathbb{D}_k\mathbb{D}_k^*)_{jj}$ gives $|\psi_{j-k}|^2$. Thus for eq. (3.20) we get

$$S = \left(N \cdot \sum_{k=0}^{N-1} |\psi_k|^2 \right) \mathbb{I} = N \, \|\psi\|_2^2 \, \mathbb{I} \quad , \tag{3.21}$$

so by eq. (1.42) and the before discussion $\{\psi, \Lambda\}$ is $N \|\psi\|^2$ tight. If then $\|\psi\|^2 = 1/N$, the frame is Parseval (compare with eq. (1.43)). Plus, as we shall see in the following, $\{\psi, \Lambda\}$ is equal norm. Indeed, $\mathbb{T}_k$ is a unitary operator, thus

$$\|\mathbb{M}_l \mathbb{T}_k \psi\|_2^2 = \|\mathbb{M}_l \psi\|_2^2 \tag{3.22}$$

so

$$\begin{aligned} \|\mathbb{M}_l \psi\|_2^2 &= < \mathbb{M}_l \psi, \mathbb{M}_l \psi > \\ &= \sum_{n=0}^{N-1} w_N^{ln} \psi_n w_N^{-ln} \overline{\psi}_n \\ &= \sum_{n=0}^{N-1} |\psi_n|^2 \quad , \end{aligned} \tag{3.23}$$

and by Parseval's equality (eq. (1.5)) we get

$$\|\psi\|_2^2 = \|\mathbb{M}_l \psi\|_2^2 \quad . \tag{3.24}$$

Finally

$$\|\mathbb{M}_l \mathbb{T}_k \psi\|_2^2 = \|\psi\|_2^2 \quad . \tag{3.25}$$

Until now we have not considered the impact of choosing a particular $\psi \in \mathbb{C}^N$: different window functions have different localization properties. Roughly speaking, we have good localization in time (respectively in frequency) when $|\psi_k|^2$ ($|\hat{\psi}_k|$) is large for a few k, meaning the energy $\|\psi\|_2^2$ is carried by a few components of ψ ($\hat{\psi}$). Generally, good localization in time results in poorer localization in frequency and vice versa. We see this at an example.

Example 8. *In $\mathbb{R}^N$, take $\psi = \delta_{N/2}$ (N even).*

Taking the DFT of ψ, we get

$$\begin{aligned} \hat{\psi}_k &= \sum_{n=0}^{N-1} \delta_{N/2}(n) \, e^{-2\pi i k n/N} \\ &= e^{-\pi i k} = \begin{cases} 1 & \text{for } k = 0, 2, \dots \\ -1 & \text{for } k = 1, 3, \dots \end{cases} \end{aligned} \tag{3.26}$$

meaning $\hat{\psi}$ is not very localized. By *Heisenberg's Uncertainity principle*, there is a certain limit to having both good localization in time and frequency. This may be quantified by Prop. 9 below. Before that we define $\|\cdot\|_0$: by $\|x\|_0$, $x \in H^n$, we mean the cardinality of the set supp x, that is

$$\|x\|_0 = |\text{supp } x| \quad , \tag{3.27}$$

which intuitively may be rewritten as

$$\|x\|_0 = \|\mathbf{1}_{\text{supp } x}\|_2^2 \quad . \tag{3.28}$$

Proposition 9. *Let $x \in \mathbb{C}^N \setminus \{0\}$, then $\|x\|_0 \|\hat{x}\|_0 \geq N$.*

Proof. Take $x \in \mathbb{C}^N \setminus \{0\}$, then by eq. 1.70 we have

$$\hat{x}_k = \sum_{n=0}^{N-1} x_n w_N^{-kn} \quad .$$

As $\left|w_N^{-kn}\right| = 1$, it follows that $|\hat{x}_k| \leq \sum_{n=0}^{N-1} |x_n|$. This holds also for $\max \hat{x}_k$, $k \in \{0, 1, \ldots, N-1\}$, so multiplying by N gives

$$N \|\hat{x}\|_\infty^2 \leq N \left(\sum_{k=0}^{N-1} |x_k| \right)^2 \quad . \tag{3.29}$$

Using Cauchy-Schwarz's inequality (eq. (1.13)), then employing Parseval's equality (eq. (1.5)) we get

$$\begin{aligned} \sum_{k=0}^{N-1} |x_k| &= \sum_{k=0}^{N-1} \left| x_k \left(\mathbf{1}_{\text{supp } x}\right)_k \right| \\ &\leq \|x\|_2 \|\mathbf{1}_{\text{supp } x}\|_2 = \|\mathbf{1}_{\text{supp } x}\|_2 \left(\sum_{k=0}^{N-1} |x_k|^2 \right)^{1/2} \quad . \end{aligned} \tag{3.30}$$

Accounting for eq. (3.28)) and eq. (3.30), eq. (3.29) becomes

$$N \|\hat{x}\|_\infty^2 \leq N \|x\|_0 \sum_{k=0}^{N-1} |x_k|^2 \quad . \tag{3.31}$$

Employing the fact that the Fourier frame is an N-tight frame (see eq. (1.78)), the above equation then becomes

$$N \|\hat{x}\|_\infty^2 \leq \|x\|_0 \sum_{k=0}^{N-1} |\hat{x}_k|^2 \quad . \tag{3.32}$$

Then

$$\sum_{k=0}^{N-1} |\hat{x}_k|^2 = \sum_{k=0}^{N-1} \left|\hat{x}_k \left(\mathbf{1}_{\text{supp } \hat{x}}\right)_k\right|^2 \leq \|\hat{x}\|_\infty^2 \sum_{k=0}^{N-1} \left|\left(\mathbf{1}_{\text{supp } \hat{x}}\right)_k\right|^2 = \|\hat{x}\|_\infty^2 \|\hat{x}\|_0$$

whereby for eq. (3.32) it follows that

$$N \|\hat{x}\|_\infty^2 \leq \|x\|_0 \|\hat{x}\|_0 \|\hat{x}\|_\infty^2 \quad ,$$

so finally dividing by $\|\hat{x}\|_\infty^2$ we get the result in the proposition. □

3.1 Frame redundancy

In this section we show that there exists full spark Gabor frames, i.e. Gabor frames for which every N-element subset of the frame is linearly independent[3] (see Def. 1.11). Recall that as in the previous section we limit the discussion to $\Lambda = \mathbb{Z}_N \times \mathbb{Z}_N$. For this section we rely mostly on [14], except for a few cases we will mention. If $\{\psi, \Lambda\}$ is full-spark, then its upper redundancy is $R^+ = N^2/N$ - we divide its cardinality by the maximum number of elements which form a linearly independent set (recall eq. (2.2)). Plus, as any N linearly independent vectors in $\mathbb{C}^N$ are spanning, it follows that $R^- = R^+$.

Showing that $\{\psi, \Lambda\}$ is full-spark by means of the analytical redundancy measure is rather short. We can use Lemma 2.1: $\{\psi, \Lambda\}$ is both equal norm (see eq. (3.22), (3.23)) and tight (eq. (3.21)), so $A = B = N\|\psi\|^2$ and $c = \|\psi\|^2$. Thereby we get $R_\Phi^- = R_\Phi^+ = N\|\psi\|^2 / \|\psi\|^2$ linearly independent/ spanning sets.

As mentioned, we want to proof that any N element subset of $\{\psi, \Lambda\}$ is linearly independent. We denote with $G = G(\psi)$ the matrix associated to the synthesis operator T of $\{\psi, \Lambda\}$. By a minor of order N we mean the determinant of the matrix A having as columns any N vectors of G. We write it as

$$G\begin{pmatrix} 0 & 1 & \dots & N-1 \\ j_1 & j_2 & \dots & j_N \end{pmatrix} = \det A \tag{3.33}$$

where $j_1, j_2, \dots, j_N$ are the indices corresponding to the columns chosen and $0, 1 \dots N-1$ means we are considering all lines of G in the submatrix A.

[3]equivalently other authors (e.g. in [14]) indicate that a collection of vectors is linear position or has the *Haar* property

We briefly recall some facts concerning determinants of a matrix. Generally, given $A \in \mathbb{C}^{N\times N}$, $\det(A)$ may be expanded with respect to the elements of a row i, $i \in \{1,\ldots,N\}$, so

$$\det A = \sum_{j=1}^{N} (-1)^{i+j}\, a_{ij}\, |A_{ij}| \tag{3.34}$$

where each cofactor[4] $|A_{ij}|$ denotes the determinant of the matrix obtained after deleting the i-th row and j-th column, $j \in \{1,\ldots,N\}$. Thus $|A_{ij}|$ is the sum of $(N-1)!$ terms, and the right-hand side of eq. (3.34) is the sum of $N(N-1)! = N!$ terms with their respective sign (see [16], pp. 38-39). In terms of minors, the above equation may be rewritten as

$$\det A = (-1)^{i+1}\, a_{i1} A \begin{pmatrix} 1 & \ldots & i-1 & i+1 & \ldots & N \\ 2 & 3 & \ldots & \ldots & \ldots & N \end{pmatrix} + \ldots + \\ + (-1)^{i+N}\, a_{iN} A \begin{pmatrix} 1 & \ldots & i-1 & i+1 & \ldots & N \\ 1 & 2 & \ldots & \ldots & \ldots & N-1 \end{pmatrix} \,. \tag{3.35}$$

In compact form, each minor $|A_{ij}|$ may be written as (see [16], p.29-30,38)

$$|A_{ij}| = \sum_{l} \pm a_{i,l_1} a_{i,l_2} \ldots a_{i,l_{N-1}} \tag{3.36}$$

where the sum runs over all $(N-1)!$ permutations $l = (l_1,\ldots,l_{N-1})$ of the indexes $1,\ldots,j-1,j+1,\ldots,N$: within a permutation each column can occur once and once only. The sign attached to each element in the sum depends on the parity of l: it is negative (positive) when the number of inversions it takes to re-establish the natural order $1\ldots N$ of indices is odd (even). Consider for example (1243) and (1423): in the first (second) case we need one (two) interchange(s), so the sign is negative (positive).

Each minor can be expanded further: take $|A_{11}|$ and expand $m-1$ times. In eq. 3.35 there will then occur a term of the form

$$a_{11}a_{22}\ldots a_{mm} \sum_{l} a_{m+1,l_1} a_{m+2,l_2} \ldots a_{N,l_{N-m}} \tag{3.37}$$

where $l = (l_1,\ldots,l_{N-m})$ runs over $(N-m)!$ permutations of $[m+1,N]$. The aggregate of terms multiplying the minor $A\begin{pmatrix} m+1 & \ldots & N \\ m+1 & \ldots & N \end{pmatrix}$ is itself a

[4] also being referred to as a signed minor

minor - the complementary minor (see [16], pp. 76-77). So in the expression for $\det A$ (eq. (3.35)) there will occur a term of the form

$$A\begin{pmatrix}1 & \dots & m\\ 1 & \dots & m\end{pmatrix} A\begin{pmatrix}m+1 & \dots & N\\ m+1 & \dots & N\end{pmatrix} \quad ,$$

so we can express $\det A$ as a sum of products between minors. Now, one could further partition each of the minors in eq. (3.1), writing them respectively as a product of a minor with its complementary minor (see [16], p. 81-82). Thus, setting a partition $s = (s_1, \dots, s_m)$ of column indices where each s_j is of cardinality p_j ($|s_j| = p_j$), so $\sum_{k=1}^{m} p_j = N$, then $\det A$ may be written in the form (see [14])[5]

$$\det A = \sum_t (-1)^{\mu(t,s)} A\begin{pmatrix}t_1\\ s_1\end{pmatrix} A\begin{pmatrix}t_2\\ s_2\end{pmatrix} \dots A\begin{pmatrix}t_m\\ s_m\end{pmatrix} \tag{3.38}$$

where t runs through all partitions of row indices into subsets of size (p_1, p_2, ..., p_m) and $\mu(t, s)$ denotes the sign factor associated to the respective monomial. We leave it in general form as the specific sign is not important to our further discussion, but mention that for a minor $A\binom{t_k}{s_k}$ its sign is given by (see [16], p. 77)

$$(-1)^{\sum_{i \in t_k} t_{k_i} + s_{k_i}}$$

where t_{k_i} (s_{k_i}) means the i-th element within the index set t_k (s_k). Note that when building minors the rows in the sets $t_1 \dots t_m$ do not need to be consecutive rows. Instead, concerning the columns, no column can enter in multiple minors (see [16], p. 81-82).
We show the method at an example. Take $\mathbb{D}_0\mathbb{W}_3$ as in Ex. 7.

We choose $s_1 = \{1, 2\}$ and $s_2 = \{0\}$, thus $p_1 = 2$ and $p_2 = 1$. Feasible row partitions t are:

1. $t_1 = \{0, 1\}$ and $t_2 = \{2\}$,
2. $t_1 = \{0, 2\}$ and $t_2 = \{1\}$,
3. $t_1 = \{1, 2\}$ and $t_2 = \{0\}$.

For option 1. the product of minors in eq. (3.38) is

$$\begin{aligned} A\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix} A\begin{pmatrix}2\\ 0\end{pmatrix} &= \det\begin{pmatrix}\psi_0 & \psi_0\\ w_3\psi_1 & w_3^2\psi_1\end{pmatrix}\psi_2 \\ &= \psi_0\psi_1\psi_2\left(w_3^2 - w_3\right) \quad . \end{aligned} \tag{3.39}$$

[5] the author refers to [16] p. 81-82 when stating the result: as the notations are less handy in the original reference, we prefere stating the result as in [14]

We observe that det $\mathbb{D}_0\mathbb{W}_3$ is a homogeneous polynomial of degree 3 in the variables ψ_0, ψ_1, ψ_2. This is because $\mathbb{D}_0\mathbb{W}_3$ is a full-matrix having in each line i as elements multiples of ψ_i. However, as we shall see below, if $i \neq 0$, then det $\mathbb{D}_i\mathbb{W}_3$ will still be a homogeneous polynomial, but each line i will have as elements multiples of $\psi_{\tilde{i}}$ with $i \neq \tilde{i}$.

Example 9. *In $\mathbb{C}^3$, take $\mathbb{D}_1 = diag\ (\psi_1, \psi_2, \psi_0)$ and* $\mathbb{W}_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & w_3 & w_3^2 \\ 1 & w_3^2 & w_3^4 \end{pmatrix}$, *so*

$$\mathbb{D}_1\mathbb{W}_3 = \begin{pmatrix} \psi_1 & \psi_1 & \psi_1 \\ \psi_2 & w_3\psi_2 & w_3^2\psi_2 \\ \psi_0 & w_3^2\psi_0 & w_3^4\psi_0 \end{pmatrix} .$$

The above consideration allows for the following generalization: for a $p \times p$ submatrix A of G, det A is a homogeneous polynomial of degree p in the variables ψ_0, ψ_1, $\ldots$, ψ_{N-1}. Thus, defining the multiindex $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{N-1})$ where the element α_i denotes the power factor associated to ψ_i, $i \in \{0, 1, \ldots, N-1\}$, we may write det A as

$$\det\ (A) = \sum_{\substack{\alpha \in \mathbb{Z}_+^N, \\ |\alpha| = p}} a_\alpha \psi_0^{\alpha_0} \psi_1^{\alpha_1} \ldots \psi_{N-1}^{\alpha_{N-1}}. \tag{3.40}$$

where a_α is the respective coefficient associated to each monomial. We now characterize a full spark Gabor system. It is straightforward by Def. 1.11 and Prop. 10.

Proposition 10. *The Gabor system $\{\psi, \mathbb{Z}_N \times \mathbb{Z}_N\}$ is full spark if and only if every minor of $\{\psi, \Lambda\}$ of order N is nonzero.*

The question whether there actually exists a window function $\psi \in \mathbb{C}^N$ satisfying Prop. 10 can be answered affirmatively for N prime (see [14]). For N even, numerical results for $N = 4, 6$ indicate a positive answer (see [15]). In [12], Malikiosis proves the general fact for arbitrary finite N. We now state and prove the result for N prime, though we write the result as reported in [13]. Before doing that we briefly state a lemma we need for the proof.

Lemma 3.1. *If N is prime then every minor of the discrete Fourier matrix $\mathbb{W}_N$ is nonzero.*

Lemma 3.1 is of practical relevance: consider a minor $A\binom{t}{s}$ in a block $\mathbb{D}_i\mathbb{W}_N$ in G, t (s) being a set of rows (columns) and $i \in \{0, 1, \ldots, N-1\}$.

By the structure of G (recall eq. (3.13))

$$A\binom{t}{s} = \det\left(\mathbb{D}_i\binom{t}{s}\right)\det\underbrace{\left(W_N\binom{t}{s}\right)}_{\neq 0}\quad, \tag{3.41}$$

so if N is even, $A\binom{t}{s}$ being nonzero depends just on the minor $\mathbb{D}_i\binom{t}{s}$. This fact will be used below in the proof.

Theorem 3.2. *If $\Lambda = \mathbb{Z}_N \times \mathbb{Z}_N$ with N prime, then there exixts $\psi \in \mathbb{C}^N$ such that $\{\psi, \Lambda\}$ is a full spark Gabor frame. Moreover, we can choose the vector ψ to be unit norm.*

Proof. Let us consider an $N \times N$ submatrix A of G. Denote by $l = (l_0, l_1, \ldots, l_{N-1})$ the N-ple indicating the number of columns chosen from each block $\mathbb{D}_i\mathbb{W}_N$ in G that exist in A, $i \in \{0, 1, \ldots, N-1\}$. Let $s = (s_1, s_2, \ldots, s_m\}$ be the collection of column indices of A such that if $l_{j_k} > 0$, then s_k is the set of l_{j_k} column indices which correspond to those columns taken from the block $\mathbb{D}_{j_k}\mathbb{W}_N$ that exist in A. Now, by eq. (3.38) it follows that

$$\det A = \sum_t (-1)^{\mu(t,s)} A\binom{t_1}{s_1} A\binom{t_2}{s_2} \ldots A\binom{t_m}{s_m} \quad . \tag{3.42}$$

where t runs through all partitions of row indices into subsets of size ($|s_1|$, $|s_2|$, ..., $|s_m|$) and $\mu(t, s)$ denotes the sign factor associated to the respective monomial. We proceed as follows:

1. As we observed before discussing eq. (3.40), every monomium in the above sum is of degree N in the variables ψ_0, ψ_1, ..., ψ_{N-1}. We choose a polynomial p_A by identifying a diagonal such that the product of the elements on the diagonal are a multiple of p_A. Adopting the notation already used in eq. (3.40) we write

$$p_A = \psi_0^{\alpha_0}\psi_1^{\alpha_1} \ldots \psi_{N-1}^{\alpha_{N-1}}$$

where $\alpha = (\alpha_0, \ldots, \alpha_{N-1})$ is some multiindex satisfying $\sum_{k=0}^{N-1} \alpha_k = N$.

2. We determine a corresponding partition t: the constraint for every partition element is that the elements lying on the chosen diagonal (see point 1.) must lie on the main diagonal of the submatrix $A\binom{t_i}{s_i}$, $i \in \{1, \ldots, m\}$.

Now recalling eq. (3.41), every minor $A\begin{pmatrix} t_k \\ s_k \end{pmatrix}$ is the product of some variables ψ_i with a minor of the Fourier matrix W_N, $k \in \{1, \dots, m\}$ and $i \in \{0, 1, \dots, N-1\}$). Then we may write

$$A\begin{pmatrix} t_1 \\ s_1 \end{pmatrix} A\begin{pmatrix} t_2 \\ s_2 \end{pmatrix} \dots A\begin{pmatrix} t_m \\ s_m \end{pmatrix} = p_A \cdot c$$

where $c \in \mathbb{C}$ is the coefficient resulting from the product of the different minors of W_N. As N is prime, by Lemma 3.1 every minor of W_N is nonzero, so $c \neq 0$. Moreover, it can be argued that there is just one monomial in eq. (3.42) which is a multiple of the chosen p_A (see [14], p. 725 for the details), so the sign of the monomial can be neglected. Thereby, ψ or equivalently supp ψ can be chosen such that $p_A \neq 0$ (e.g. the trivial choice $\psi = \mathbf{1}$), so finally $\det A \neq 0$. This fact holds for any $N \times N$ submatrix A of G with N prime, so by Prop. 10 G is full-spark. □

We now show the result of Theorem 3.42 at an example.

Example 10. *In $\mathbb{C}^3$, consider*

$$\mathbb{D}_0\mathbb{W}_3 = \begin{pmatrix} \psi_0 & \psi_0 & \psi_0 \\ \psi_1 & w_3\psi_1 & w_3^2\psi_1 \\ \psi_2 & w_3^2\psi_2 & w_3^4\psi_2 \end{pmatrix} \quad \textit{and} \quad \mathbb{D}_1\mathbb{W}_3 = \begin{pmatrix} \psi_1 & \psi_1 & \psi_1 \\ \psi_2 & w_3\psi_2 & w_3^2\psi_2 \\ \psi_0 & w_3^2\psi_0 & w_3^4\psi_0 \end{pmatrix}$$

. Take then $l = (1, 2, 0)$ and $s_1 = \{2\}$, $s_2 = \{1, 2\}$, so

$$A = \begin{pmatrix} \psi_0 & \psi_1 & \psi_1 \\ w_3^2\psi_1 & w_3\psi_2 & w_3^2\psi_2 \\ w_3^4\psi_2 & w_3^2\psi_0 & w_3^4\psi_0 \end{pmatrix} \quad .$$

We choose the polynomial $p_A = \psi_1\psi_2^2$ and identify a partition of lines (t_1, t_2) satisfying the constraint posed by point 2 in the demonstration of Theorem 3.42: thereby $t_1 = \{2\}$ and $t_2 = \{0, 1\}$. Then the product of the minors in eq. (3.42) is

$$A\begin{pmatrix} 2 \\ 0 \end{pmatrix} A\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = \psi_1\psi_2^2 \det\left(w_3^4\right) \det\begin{pmatrix} 1 & 1 \\ w_3 & w_3^2 \end{pmatrix} = \psi_1\psi_2^2 w_3^5\left(w_3 - 1\right) \neq 0$$

as

$$w_3 = e^{-2\pi i/3} = \cos\left(\frac{2}{3}\pi\right) - \mathrm{i}\sin\left(\frac{2}{3}\pi\right) \neq \{0, 1\} \quad ,$$

so for $\psi \in \mathbb{C}^3$ satisfying ψ_1, $\psi_2 \neq 0$ the above monomial occuring in eq. (3.42) is different from 0.

References

[1] P.G. Casazza: The art of frame theory, `https://arxiv.org/abs/math/9910168` (1999)

[2] B.G. Bodmann, P.G. Casazza, G. Kutyniok: A quantitative notion of redundancy for finite frames, *Appl. Comp. Harmonic Anal. 30* pp. 348-362 (2011)

[3] P.G. Casazza, D. Speegle: Spanning and Independence Properties of Finite Frames, *Finite Frames* pp. 109-139 (2013)

[4] P.G. Casazza, R.G. Lynch: A brief introduction to Hilbert space frame theory and its applications (2015)

[5] G. E. Pfander: Gabor frames in finite dimensions, *Finite Frames* pp. 193-239 (2013)

[6] J. Kovacevic, Goyal V. K and M. Vetterli: Fourier and Wavelet Signal Processing, `https://fourierandwavelets.org` (2013)

[7] B. Alexeev, Jameson Cahill and D.G. Mixon: Full Spark Frames, *Journal of Fourier Analysis and Applications 18*, pp. 1167-1194 (2012)

[8] C. Heil: Linear Independence of Finite Gabor Systems, *Harmonic Analysis and Applications* pp. 171-206 (2006)

[9] H. Brezis: Functional Analysis, Sobolev Spaces and Partial Differential Equations (2011)

[10] S. di Stefano: Course notes for Continuum Mechanics (2017), Polytechnic University of Torino

[11] M. Faulhuber: Master Thesis on Geometry and Gabor Frames (2014), University of Vienna

[12] R.-D. Malikiosis: A note on Gabor frames in finite dimension, *Applied and Computational Harmonic Analysis 38* pp. 318-330 (2015)

[13] G.E. Pfander: Gabor frames in finite dimension, *Finite Frames* pp. 193–239 (2013)

[14] J. Lawrence, G.E. Pfander and D. Walnut: Linear independence of Gabor systems in finite dimensional vector spaces, *The Journal of Fourier Analysis and Applications 11*, pp. 715-726 (2005)

[15] H.B. Dang, K. Blanchfield, I. Bengtsson and D.M. Appleby: Linear Dependencies in Weyl-Heisenberg Orbits, *Quantum Information Processing 12*, pp. 3449-3475 (2013)

[16] A.C. Aitken: Determinants and Matrices (1944)

[17] T. Muir: A treatise on the Theory of Determinants (1933)

[18] S. Roman: Advanced Linear Algebra (2005)

[19] B.P. Rynne, M.A. Youngson: Linear Functional Analysis (2008)

[20] G. Monegato: Metodi e algoritmi per il calcolo numerico (2008)