POLITECNICO DI TORINO

Master's degree course
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New results on the a posteriori error analysis for Virtual Element Methods

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Summary

This thesis fits in the Adaptive Virtual Element Methods theory. It investigates the stabilization-free a posteriori error analysis in polygonal meshes in 2d. The first novelty brought by this work stands in the extension of Beirão da Veiga et al. [2021] to the cases of triangular meshes with hanging nodes and polygons of higher degree. Assuming that any chain of recursively created hanging nodes is uniformly bounded, stabilization-free upper and lower bounds for energy error are presented. The main difference with respect to the case of polynomials of degree one is that, given two triangles sharing an edge, the refinement of one of them brings some points to be both hanging nodes and proper nodes for the other triangle. On one hand because of this a re-definition of the hanging nodes is necessary, on the other hand it simplifies a lot the proof of the Scaled Poincaré inequality.

The second topic studied in this thesis is the extension of the analysis to the case of quadrangles. The main challenge here is the definition itself of the refinement. In this text the refinement consists in tracing the edges connecting the midpoints of two opposite edges of the quadrangles. In this way a quadrangle is reduced to four quadrangles. Also the space of polynomials of degree one has to be changed. Indeed, a polynomial of degree one is not uniquely determined by the value at the four vertices of the quadrangle. We then introduced a new functional space that contains the polynomials of degree one. Finally the enhanced version of this functional space of this Virtual Element and the stabilization-free a posteriori error analysis have been discussed.

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Introduction

In order to describe the world and its phenomena, an essential concept is necessary: the derivative. Derivative is the mathematical way to describe the evolution in time or in space. By combining derivatives and some constraints observed, which are described as equations, the partial differential equations (PDE) arise. Most of the time we can just discuss some properties of the solution of an PDE, but finding the explicit solution can be analytically impossible. Numerical Analysis tries to solve this problem with the Finite Element Method (FEM). This method is based on finding an approximation of the real solution; it gets more precise as the degree of accuracy grows. FEM considers a discretization of the domain made by finite elements, defined by a triple which consists in the 'geometrical shape' E of the element forming the partition, a space of approximation functions living in E and a set of degrees of freedom. The main theme of the thesis focuses on the Virtual Element Method (VEM), a type of FEM, which has been introduced less than ten years ago.

The first chapter of the thesis consists in an introduction of the VEM, starting from the first paper Beirão da Veiga et al. [2013]. The peculiarity of the VEM is the fact that the functional space defined on each element E concerns only the values of the functions at the boundary of E and a condition on the Laplacian of the functions. It means that we do not require to know the functions in the interior of E.

From the presentation of the VEM, the thesis focuses on the stabilization-free a posteriori error analysis, with the purpose to extend the work by Beirão da Veiga et al. [2021] from a degree of accuracy 1 to a general degree k.

In the last chapter a re-definition of the VEM has been proposed in order to adapt it to the case of quadrangles. The functional spaces introduced here become helpful when refining a discretization made by quadrangles, without reducing to triangles, is needed.

Chapter 1

Virtual element methods

In this chapter the **Virtual element methods** are presented. The main reference that will guide us to the description of this 'new' type of finite elements is the work *Basic principles of virtual elements methods* by Beirão da Veiga et al. [2013]. The main difference with respect to their work is that we here present the more general symmetric elliptic PDE, instead of the classical Poisson problem. This choice has been taken because it will be the one used in the second chapter and it does not add too many differences in the presentation of the virtual elements.

1.1 The continuous problem

The first step into the world of the virtual elements starts from the very simple and well known problem: the bi-dimensional symmetric elliptic problem with vanishing Dirichlet boundary conditions. This classical problem arises from several engineering and physical applications, such as Newtonian gravity, hydrodynamics, electrostatics, diffusion problems etc ....

Given $\Omega \subset \mathbb{R}^2$ a polygonal domain, our problem can be written as

$$\begin{cases} -\nabla \cdot (A \nabla u) + cu = f & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

where $A \in L^\infty(\Omega)^{2\times 2}$ is symmetric and uniformly positive definite in Ω, $c \in L^\infty(\Omega)$ and positive in Ω, $f \in L^2(\Omega)$. The variational formulation of the previous problem reads as we want to

$$\begin{cases} \text{find } u \in \mathbb{V} := H_0^1(\Omega) & \text{such that} \\ \mathcal{B}(u,v) = (f,v), & \forall\, v \in \mathbb{V}, \end{cases} \tag{1.2}$$

where $(\cdot,\cdot)$ is the scalar product in $L^2(\Omega)$ and $\mathcal{B}(u,v) := a(u,v) + m(u,v)$ is a bilinear form where

$$a(u,v) = (A\nabla u, \nabla v) \qquad\qquad m(u,v) = (cu,v).$$

Denoting by $|\cdot|_1$ the norm in $\mathbb{V}$ such that $|v|_1^2 = (\nabla v, \nabla v)$, it can be easily proved that the variational formulation of (1.1) has a unique solution. Indeed, the bilinear form $\mathcal{B}$, is continuous and coercive:

$$\mathcal{B}(u,v) \leq (||A||_\infty + ||c||_\infty)|v|_1|u|_1, \qquad \mathcal{B}(v,v) \geq \beta |v|_1^2, \tag{1.3}$$

where $\beta > 0$ exists because of the hypothesis on A and c.

1.2 The discrete problem and the assumptions needed

For the continuous problem it is guaranteed the existence of a unique solution. We want now to introduce a discretization such that the discrete solution we will found is 'close' to the continuous solution. The core of this section is to define the properties a discretization has to have in order to be 'well-built'. As done in Beirão da Veiga et al. [2013], we here recall these properties and in the next sections we will verify if they hold in the case of in the virtual elements.

Let us introduce a decomposition $\mathcal{T}$ of Ω into elements E. As usual, h will denote the maximum of the diameters of the elements in $\mathcal{T}$. A decomposition is 'well-built' if the following assumptions are satisfied.

Assumption 1.2.1. For every h, the decomposition $\mathcal{T}$ is made of simple polygons. The bilinear forms $a(\cdot,\cdot)$, $m(\cdot,\cdot)$ and the norm $|\cdot|_1$ can be split over the elements of the discretization, i.e.

$$a(u,v) = \sum_{E\in\mathcal{T}} a^E(u,v), \qquad m(u,v) = \sum_{E\in\mathcal{T}} m^E(u,v),$$

$$|v|_1^2 = \sum_{E\in\mathcal{T}} |v|_{1,E}^2, \qquad \forall\ v,\ u \in \mathbb{V}.$$

Given the space $H^1(\mathcal{T}) := \prod_{E\in\mathcal{T}} H^1(E)$, we can also define the H^1-seminorm as

$$|v|_{1,\mathcal{T}} := \left(\sum_{E\in\mathcal{T}} |\nabla v|_{0,E}^2 \right)^{1/2}, \qquad \forall\ v \in \mathbb{V}.$$

Assumption 1.2.2. For every h, we assume to have:

- a space $\mathbb{V}_\mathcal{T} \subset \mathbb{V}$;
- a symmetric bilinear form $\mathcal{B}_\mathcal{T} : \mathbb{V}_\mathcal{T} \times \mathbb{V}_\mathcal{T} \to \mathbb{R}$, such that

  $$\mathcal{B}_\mathcal{T}(u_h,v_h) = a_\mathcal{T}(u_h,v_h) + m_\mathcal{T}(u_h,v_h) = \sum_{E\in\mathcal{T}} \mathcal{B}_h^E(u_h,v_h),$$

  and

  $$a_\mathcal{T}(u_h,v_h) = \sum_{E\in\mathcal{T}} a_h^E(u_h,v_h), \qquad m_\mathcal{T}(u_h,v_h) = \sum_{E\in\mathcal{T}_h} m_h^E(u_h,v_h);$$

  $\forall\ u_h,\ v_h\ \in \mathbb{V}_\mathcal{T}$, where $\mathcal{B}_h^E(\cdot,\cdot)$, $a_h^E(\cdot,\cdot)$ and $m_h^E(\cdot,\cdot)$ are bilinear forms on $\mathbb{V}_\mathcal{T}|_E \times \mathbb{V}_\mathcal{T}|_E \to \mathbb{R}$;

- an element $f_h \in \mathbb{V}'_{\mathcal{T}}$.

These assumptions allow us to define the following problem

$$\begin{cases} \text{find } u_h \in \mathbb{V}_{\mathcal{T}} \text{ such that} \\ \mathcal{B}_{\mathcal{T}}(u_h, v_h) = \langle f_h, v_h \rangle, \ \forall\, v_h \in \mathbb{V}_{\mathcal{T}}. \end{cases} \tag{1.4}$$

We now want to define some assumptions such that this system has got a unique solution u_h which is close enough to the solution u of (1.2). If $k \geq 1$ is the degree of accuracy we want that the following inequality is valid:

$$|u - u_h|_1 \lesssim h^k |u|_{k+1,\Omega}. \tag{1.5}$$

Assumption 1.2.3. There exists an integer $k \geq 1$ such that for all h and $E \in \mathcal{T}_h$ the space of polynomials of degree k is a subset of $\mathbb{V}_{\mathcal{T}}|_E$. Moreover,

- *k-Consistency*: For all $p \in \mathbb{P}_k(E)$ and for all $v_h \in \mathbb{V}_{\mathcal{T}}|_E$,

$$\mathcal{B}_h^E(p, v_h) = \mathcal{B}^E(p, v_h). \tag{1.6}$$

- *Stability*: There exist two constants $\alpha_* > 0$ e $\alpha^* > 0$, independent from h and E, such that

$$\forall\, v_h \in \mathbb{V}_{\mathcal{T}}|_E, \qquad \alpha_* \mathcal{B}^E(v_h, v_h) \leq \mathcal{B}_h^E(v_h, v_h) \leq \alpha^* \mathcal{B}^E(v_h, v_h). \tag{1.7}$$

This last property implies the continuity of $\mathcal{B}_h^E$, indeed

$$\begin{aligned} \mathcal{B}_h^E(u, v) &\leq \left(\mathcal{B}_h^E(u, u)\right)^{\frac{1}{2}} \left(\mathcal{B}_h^E(v, v)\right)^{\frac{1}{2}} \\ &\leq \alpha^* \left(\mathcal{B}^E(u, u)\right)^{\frac{1}{2}} \left(\mathcal{B}^E(v, v)\right)^{\frac{1}{2}} \\ &= \beta^* |u|_{1,E} |v|_{1,E}, \ \text{ for all } u, v \in \mathbb{V}_{\mathcal{T}}|_E; \end{aligned} \tag{1.8}$$

where $\beta^* := \alpha^*(||A||_{\infty,E} + ||c||_{\infty,E})$.

Theorem 1.2.1. *Under the Assumptions 1.2.2 and 1.2.3, the discrete problem: Find $u_h \in \mathbb{V}_h$ such that*

$$\mathcal{B}_{\mathcal{T}}(u_h, v_h) = \langle f_h, v_h \rangle \quad \forall v_h \in \mathbb{V}_{\mathcal{T}}, \tag{1.9}$$

has a unique solution u_h. Moreover, for every approximation $u_I \in \mathbb{V}_{\mathcal{T}}$ of u for every approximation u_π of u that is piecewise in $\mathbb{P}_k$, we have

$$|u - u_h|_1 \leq C(|u - u_I|_1 + |u - u_\pi|_{1,\mathcal{T}} + ||f - f_h||_{\mathbb{V}'_{\mathcal{T}}}),$$

where C is a constant and for any h, $||f - f_h||_{\mathbb{V}'_{\mathcal{T}}}$ is the smallest constant such that

$$(f, v_h) - \langle f_h, v_h \rangle \leq ||f - f_h||_{\mathbb{V}'_{\mathcal{T}}} |v_h|_1, \qquad \forall\, v_h \in \mathbb{V}_{\mathcal{T}}. \tag{1.10}$$

Proof. From (1.3) and (1.7), the symmetric bilinear form $\mathcal{B}_{\mathcal{T}}$ is continuous and coercive so that the solution of (1.9) exists and it is unique.

Now, setting $\delta_h := u_h - u_I$, we have, from the continuity and the coecivity of $\mathcal{B}$, and using *Stabiliy* (1.7) property of $\mathcal{B}_{\mathcal{T}}$

$$\begin{aligned}
\beta\alpha_*|\delta_h|_1^2 &\leq \alpha_*\mathcal{B}(\delta_h,\delta_h) \leq \mathcal{B}_{\mathcal{T}}(\delta_h,\delta_h) \\
&= \mathcal{B}_{\mathcal{T}}(u_h,\delta_h) - \mathcal{B}_{\mathcal{T}}(u_I,\delta_h) \\
&= \mathcal{B}_{\mathcal{T}}(u_h,\delta_h) - \mathcal{B}_{\mathcal{T}}(u_I,\delta_h) + \mathcal{B}_{\mathcal{T}}(u_\pi,\delta_h) - \mathcal{B}_{\mathcal{T}}(u_\pi,\delta_h) \\
&= \mathcal{B}_{\mathcal{T}}(u_h,\delta_h) - \sum_{E\in\mathcal{T}} \left(\mathcal{B}_h^E(u_I - u_\pi,\delta_h) + \mathcal{B}_h^E(u_\pi,\delta_h)\right).
\end{aligned}$$

Using now (1.9) and the *k-Consistency* assumption (1.6), since $u_\pi|_E \in \mathbb{P}_k(E)$,

$$\begin{aligned}
\beta\alpha_*|\delta_h|_1^2 &\leq \langle f_h,\delta_h\rangle - \sum_{E\in\mathcal{T}} \left(\mathcal{B}_h^E(u_I - u_\pi,\delta_h) + \mathcal{B}^E(u_\pi,\delta_h)\right) \\
&= \langle f_h,\delta_h\rangle - \sum_{E\in\mathcal{T}} \left(\mathcal{B}_h^E(u_I - u_\pi,\delta_h) + \mathcal{B}^E(u_\pi - u,\delta_h) + \mathcal{B}^E(u,\delta_h)\right) \\
&= \langle f_h,\delta_h\rangle - \mathcal{B}(u,\delta_h) - \sum_{E\in\mathcal{T}} \left(\mathcal{B}_h^E(u_I - u_\pi,\delta_h) + \mathcal{B}^E(u_\pi - u,\delta_h)\right) \\
&= \langle f_h,\delta_h\rangle - (f,\delta_h) - \sum_{E\in\mathcal{T}} \left(\mathcal{B}_h^E(u_I - u_\pi,\delta_h) + \mathcal{B}^E(u_\pi - u,\delta_h)\right).
\end{aligned}$$

Applying now the inequality (1.8) and the definition of $||f - f_h||_{\mathbb{V}_h'}$ (1.10)

$$\beta\alpha_*|\delta_h|_1^2 \leq ||f - f_h||_{\mathbb{V}_{\mathcal{T}}'}|\delta_h|_1 + \sum_{E\in\mathcal{T}} \beta^*|u_I - u_\pi|_{1,E}|\delta_h|_{1,E} + \left(||A||_\infty + ||c||_\infty\right)|u_\pi - u|_1|\delta_h|_1.$$

Finally, by using the triangular inequality,

$$\begin{aligned}
|u - u_h|_1 &\leq |u - u_I|_1 + |u_h - u_I|_1 \\
&\leq C\left(||f - f_h||_{\mathbb{V}_{\mathcal{T}}'} + |u_I - u_\pi|_{1,\mathcal{T}} + |u - u_\pi|_{1,\mathcal{T}}\right)
\end{aligned}$$

Here, the only difference with respect to the proof shown in Beirão da Veiga et al. [2013] is that, expectedly, the constant C is a not only a combination of the stability constants α_* and α^*, but also of the coercive constant β and the data of the problem $||A||_\infty$ and $||c||_\infty$. □

1.3 The discretization: introduction of the virtual elements

In this section we want to introduce the main concept of this work, the virtual element methods, showing that with this definition the assumptions described in the previous section are satisfied.

First of all, we briefly recall the definition of **finite element**, due to Ciarlet (1975).

Definition 1.3.1. A finite element in $\mathbb{R}^d$ is a triple $(E, \mathbb{V}_E, \mathcal{L}_E)$, where:

- E is a non-empty compact and connected set in $\mathbb{R}^d$, such that $E = \overline{E}$ and the boundary ∂E is Lipschitz-continuous.
- $\mathbb{V}_E$ is a linear space of functions defined in E.
- $\mathcal{L}_E$, set of degrees of freedom, is the set collecting linear forms $\ell_j : \mathbb{V}_E \to \mathbb{R}^d$, which is unisolvent for E.

Following this definition, we want to define all the elements of the triple for a virtual element.

As before, let $\mathcal{T}$ be a conforming partition of $\overline{\Omega}$ made of a finite number of simple polygons E. We denote as h_E the diameter of the element in E, $\boldsymbol{x}_E$ the barycenter of E, e one of the n edges of E and $\mathcal{E}_E$ the set of all the edges of E.

We are now ready to define the linear spaces of functions in E. Firstly, fixing $k \geq 1$, we can define a space containing functions living in the boundary of E:

$$\mathbb{V}_{\partial E,k} := \{v \in C^0(\partial E) : v|_e \in \mathbb{P}_k(e), \forall e \subset \partial E\}. \tag{1.11}$$

A function $v \in \mathbb{V}_{\partial E,k}$ is a polynomial of degree k for each edge of the element E. The dimension of this space would so be $(k+1)n$, but v has also to be continuous in ∂E and for this reason the values at the n vertices are uniquely defined. The dimension of $\mathbb{V}_{\partial E,k}$ is then $(k+1)n - n = kn$.

It seems clear that the previous space is not enough to the describe an element. Indeed, by now it is not known anything about how the function in the 'middle' of E is made. We now give the definition of the following space,

$$\mathbb{V}_{E,k} := \{v \in H^1(E) : v|_{\partial E} \in \mathbb{V}_{\partial E,k}, \Delta v|_E \in \mathbb{P}_{k-2}(E)\}, \tag{1.12}$$

recalling that $\mathbb{P}_{-1}(E) = \{0\}$.

For clarity's sake, we here describe the spaces for $k = 1$ and $k = 2$.

$\mathbb{V}_{E,1}$ is the set of functions that are polynomials of degree 1 on the edges of E, determined by the n vertices and harmonic functions in E.

$\mathbb{V}_{E,2}$ is the set of functions v that are continuous on ∂E and polynomials of degree ≤ 2 defined by $2n$ nodes. Inside E the functions are defined by a constant c such that $\Delta v = c$. v is so uniquely determined by $2n + 1$ conditions.

In general, a function v_h in the space $\mathbb{V}_{E,k}$ is fully determined by a function g living in $\mathbb{V}_{\partial E,k}$ and a polynomial $q_{k-2} \in \mathbb{P}_{k-2}(E)$ such that $v_h|_{\partial E} = g$ and $\Delta v_h = q_{k-2}$. For these reasons

$$N_E := dim(\mathbb{V}_{E,k}) = nk + \frac{k(k-1)}{2}, \tag{1.13}$$

since the second term corresponds to the dimension of the space of polynomials with a degree $\leq k - 2$.

It becomes now clear that $nk + \frac{k(k-1)}{2}$ degrees of freedom $\mathcal{L}_{E,k}$ need to be defined. In particular, we choose:

- $\mathcal{V}_{E,k}$: set of the values of v_h at the vertices of E;
- $\mathcal{E}_{E,k}$: set of the values of v_h at the $k-1$ equi-spaced internal points of each edge of ∂E;
- $\mathcal{P}_{E,k}$: set of the moments $\frac{1}{|E|}\int_E m(\boldsymbol{x})v_h(\boldsymbol{x})d\boldsymbol{x}\ \forall m \in \mathcal{M}_{k-2}(E)$;

where the set $\mathcal{M}_{k-2}(E)$ is defined as

$$\mathcal{M}_{k-2}(E) = \left\{ \left(\frac{\boldsymbol{x} - \boldsymbol{x}_E}{h_E} \right)^s, |s| \leq k-2 \right\}. \tag{1.14}$$

In (1.14) s is a multi-index and $|s| = s_1 + s_2$ with $\boldsymbol{x}^s := x_1^{s_1} x_2^{s_2}$. The dimension of $\mathcal{P}_{E,k}$ is so $\frac{k(k-1)}{2}$. Before discussing if this choice of the degrees of freedom is properly defined, we show in Figure 1.1 two examples of virtual elements with their degrees of freedom. In the figure shown, E is a quadrangle ($n = 4$) and the case (a) is the one of $\mathbb{V}_{E,1}$, whose dimension is 4 and the degrees of freedom are the values of v_h at the vertices. In the case (b) $\mathbb{V}_{E,2}$ is used and, by (1.13), the dimension is 9 and degrees of freedom are the values of the function at the 4 vertices, the 4 midpoints of each edge and the value of mean of the function on E.

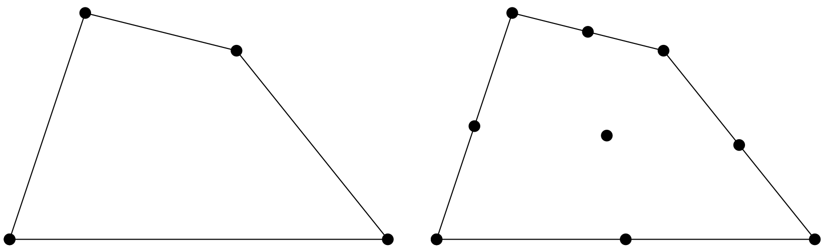


Figure 1.1: This figure shows two virtual elements where the dots represents the degrees of freedom. In (a) the space of functions is $\mathbb{V}_{E,1}$, while in (b) $\mathbb{V}_{E,2}$.

The last step that needs to be verified is if the set of the degrees of freedom is unisolvent for $\mathbb{V}_{E,k}$.

Proposition 1.3.1. *Let E be a simple polygon with n edges, and let the space $\mathbb{V}_{E,k}$ be defined as in (1.12). The degrees of freedom $\mathcal{V}_{E,k}$ plus $\mathcal{E}_{E,k}$ plus $\mathcal{P}_{E,k}$ are unisolvent for $\mathbb{V}_{E,k}$.*

Proof. We divide this proof into 3 steps, where the first two are just a recall of what is known for the basic finite elements. Given $v_h \in \mathbb{V}_{E,k}$, we want to prove that if $\ell_j(v_h) = 0$, $\forall \ell_j \in \mathcal{L}_{E,k}$, then $v_h = 0$ in E.

1. If $v_h(\boldsymbol{x}_i) = 0$, where $\boldsymbol{x}_i$ is a vertex or one of the $k-1$ equi-spaced internal points of each edge of E, then $v_h = 0$ on ∂E. Indeed: let e be one of the edges of E, by definition (1.11), $v_h|_e$ is a polynomial of degree k. Because of $v_h|_e(\boldsymbol{x_i}) = 0, i = 1 \ldots k+1$, then $v_h|_e = 0$. Repeating the same procedure for each edge, we obtain $v_h = 0$ on ∂E

2. If $\ell_j(v_h) = 0$, $\forall \ell_j \in \mathcal{P}_{E,k}$, then $P^E_{k-2} v_h = 0$ in E, where $P^E_{k-2} : L^2 \to \mathbb{P}_{k-2}(E)$ is the L^2-projection onto $\mathbb{P}_{k-2}(E)$. A polynomial of degree $\leq k-2$ that has got $\frac{k(k-1)}{2}$ moments vanishing is null in E.

3. If $v_h = 0$ on ∂E and $P^E_{k-2} v_h = 0$ in E, then $v_h = 0$ in E. By the definition of (1.12), we just need to prove that $\Delta v_h = 0$ in E. To this end, we define an auxiliary problem. For every $q \in \mathbb{P}_{k-2}(E)$, we find $w \in H^1_0(E)$, such that

$$(\nabla w, \nabla v)_{0,E} = (q, v)_{0,E}, \qquad \forall v \in H^1_0(E);$$

which can be re-written as

$$-\Delta w = q \text{ in } E, \qquad w = 0 \text{ on } \partial E,$$

or, formally, $w = -\Delta^{-1}_{0,E} q$. We consider now a map $R : \mathbb{P}_{k-2}(E) \to \mathbb{P}_{k-2}(E)$, such that

$$R(q) := P^E_{k-2}(\Delta^{-1}_{0,E} q) = P^E_{k-2}(w).$$

. R is an isomorphism, indeed, if $q \in \mathbb{P}_{k-2}(E)$, then

$$(R(q), q)_{0,E} = (P^E_{k-2}(\Delta^{-1}_{0,E} q), q)_{0,E} = (P^E_{k-2} w, q)_{0,E} = (w, q)_{0,E} = (\nabla w, \nabla w)_{0,E}.$$

Since $w \in H^1_0(E)$,

$$\{R(q) = 0\} \Leftrightarrow \{w = 0\}.$$

We notice that if $v_h = 0$ on ∂E, as in our case,

$$P^E_{k-2} v_h = P^E_{k-2}(-\Delta^{-1}_{0,E}(-\Delta v_h)) = R(-\Delta v_h).$$

By step 2, we know that $P^E_{k-2} v_h = 0$, which implies $R(-\Delta v_h) = 0$, hence $\Delta v_h = 0$ in E.

□

Remark 1. Given $v_h \in \mathbb{V}_{E,k}$ and $p \in \mathbb{P}_k(E)$, the degrees of freedom chosen allow us to compute the value of $a^E(v, p)$. Indeed, using the Green formula,

$$a^E(v_h, p) = \int_E A|_E \, \nabla v_h \cdot \nabla p = -\int_E A|_E \, \Delta p \, v_h + \int_{\partial E} A|_{\partial E} \, \frac{\partial p}{\partial n} \, v_h,$$

which implies that it can be computed without knowing the value of v_h in the interior of E. Indeed, the first integral can be computed using the values of the moments of v (because of $\Delta p \in \mathbb{P}_{k-2}(E)$), while the second integral considers the value of v_h on the boundary of E.

1.4 Construction of the discretization

In the previous section we have introduced the local spaces that we will use for the discretization. In this section we firstly define the spaces on the domain Ω, then the bilinear form $\mathcal{B}_{\mathcal{T}}$ and finally the right-hand term f_h. In the last paragraph we will discuss if the discretization is 'well-built', meaning if given the solution u of the continuous problem (1.2) is close enough to the discrete solution u_h of (1.4). In particular, we will show that the inequality (1.5) holds.

1.4.1 Construction of $\mathbb{V}_{\mathcal{T}}$

We are now ready to describe the space of virtual elements on the whole Ω. Given a $k \geq 1$, we have

$$\mathbb{V}_{\mathcal{T}} := \{v \in \mathbb{V} : v|_E \in \mathbb{V}|_{E,k}, \ \forall E \in \mathcal{T}\}.$$

Defining N^V, N^E and N^P, respectively, to be the number of internal vertices, of internal edges and of elements in $\mathcal{T}$; we can compute the dimension of $\mathbb{V}_{\mathcal{T}}$. By the description of $\mathbb{V}_{E,k}$, we have one degree of freedom for each vertex, $k-1$ for each internal points of the edge and those that can allow to have a polynomial of degree $k-2$ for each element. So that

$$N^{tot} := dim(\mathbb{V}_{\mathcal{T}}) = N^V + N^E(k-1) + N^P \frac{k(k-1)}{2}. \tag{1.15}$$

We remark the fact that we have considered only the internal vertices and edges because of the boundary Dirichlet conditions at $\partial\Omega$.

1.4.2 Construction of $\mathcal{B}_{\mathcal{T}}$

We are now ready to define the so called *Nabla operator*, that allows us to construct the form $\mathcal{B}_{\mathcal{T}}$. Let the operator $\Pi_E^{\nabla} : \mathbb{V}_{E,k} \to \mathbb{P}_k(E) \subset \mathbb{V}_{E,k}$ be the projector that guarantees the following system to have a unique solution:

$$\begin{cases} \int_E \nabla \Pi_E^{\nabla} v \cdot \nabla q = \int_E \nabla v \cdot \nabla q & \forall q \ \in \mathbb{P}_k(E) \\ \\ \overline{\Pi_E^{\nabla} v} = \overline{v}, & \forall v \ \in \mathbb{V}_{E,k}, \end{cases} \tag{1.16}$$

where, if φ is a smooth function and $\{V_i\}_{i=1\dots n}$ the set of vertices of E, we define

$$\overline{\varphi} := \frac{1}{n} \sum_{i=1}^{n} \varphi\left(V_i\right).$$

If $q \in \mathbb{P}_k(E)$, the previous system has got as solution

$$\Pi_E^{\nabla} q = q. \tag{1.17}$$

In the following, we need a projector defined on the entire discretization. In order to achieve this, we firstly need to define the space of functions that are polynomials on each element E, but we require them to be just L^2 on Ω. We so have that

$$\mathbb{W}_{\mathcal{T}}^k := \{w \in L^2(\Omega) : w|_E \in \mathbb{P}_k(E)\}.$$

This definition allows us to define the projector $\Pi_{\mathcal{T}}^{\nabla} : \mathbb{V}_{\mathcal{T}} \to \mathbb{W}_{\mathcal{T}}^k$, that restricts to Π_E^{∇} on each E. Moreover, because of it will be used in the second chapter, from the definition of $\mathbb{W}_{\mathcal{T}}^k$ we can here also define the space $\mathbb{V}_{\mathcal{T}}^0$ as

$$\mathbb{V}_{\mathcal{T}}^0 := \mathbb{V}_{\mathcal{T}} \cap \mathbb{W}_{\mathcal{T}}^k.$$

Remark 2. From the first condition of the system (1.16), we have that $\forall q \in \mathbb{P}_k(E)$,

$$a^E\left(\Pi_E^{\nabla} v, q\right) = a^E(v, q).$$

This implies that, $\forall v \in \mathbb{V}_{E,k}$

$$\mathcal{B}^E\left(\Pi_E^{\nabla} v, q\right) - \mathcal{B}^E(v, q) = c_E \int_E \left(\Pi_E^{\nabla} v - v\right) q = 0, \tag{1.18}$$

where the last integral is equal to 0, using the *enhanced* space definition, that will be discussed in the next section. By now, in order not to interrupt the flow, we take for valid this result.

Thanks to the projection operator, we can define the bilinear form $\mathcal{B}_h^E$ for which both the *k-Consistency* and the *Stability* assumptions are valid. In particular, we consider

$$a_h^E(v, w) = \int_E \left(A_E \nabla \Pi_E^{\nabla} v\right) \cdot \left(\nabla \Pi_E^{\nabla} w\right),$$
$$m_h^E(v, w) = \int_E \left(c_E \nabla \Pi_E^{\nabla} v\right) \cdot \left(\Pi_E^{\nabla} w\right),$$

where $A_E := A|_E$, $c_E := c|_E$ and $\mathcal{B}_h^E := a_h^E(u, v) + m_h^E(u, v)$. For such forms, for sure the *k-Consistency* condition (1.6) holds, but in general the *Stability* condition (1.7) does not. We need to define a symmetric bilinear form $s_E : \mathbb{V}_{E,k} \times \mathbb{V}_{E,k} \to \mathbb{R}$, such that

$$c_s |v|_{1,E}^2 \leq s_E(v, v) \leq C_s |v|_{1,E}^2, \qquad \forall v \in \mathbb{V}_{E,k}/\mathbb{R}, \tag{1.19}$$

where c_s and C_s are positive constants independent of E and h_E. Then set

$$\mathcal{B}_h^E(v, w) := \mathcal{B}^E(\Pi_E^{\nabla} v, \Pi_E^{\nabla} w) + s_E(v - \Pi_E^{\nabla} v, w - \Pi_E^{\nabla} w), \qquad \forall v,\ w\ \in \mathbb{V}_{E,k}, \tag{1.20}$$

which imitates the the Pythagoras theorem. Indeed,

$$\mathcal{B}^E(v, w) = \mathcal{B}^E(\Pi_E^{\nabla} v, \Pi_E^{\nabla} w) + \mathcal{B}^E(v - \Pi_E^{\nabla} v, w - \Pi_E^{\nabla} w), \qquad \forall v,\ w\ \in \mathbb{V}_{E,k}. \tag{1.21}$$

This definition satisfies *k-Consistency*, because of (1.17) and (1.18), indeed, $\forall p \in \mathbb{P}_k(E)$,

$$\mathcal{B}_h^E(p, v) = \mathcal{B}^E(\Pi_E^{\nabla} p, \Pi_E^{\nabla} v) + s_E(p - \Pi_E^{\nabla} p, w - \Pi_E^{\nabla} v) = \mathcal{B}^E(p, \Pi_E^{\nabla} v) = \mathcal{B}^E(p, v).$$

We need now to prove that also the *Stability* holds.

Theorem 1.4.1. *The bilinear form $\mathcal{B}_h^E(\cdot,\cdot)$, as defined in (1.20), satisfies the Stability property.*

Proof. For all $v \in \mathbb{V}_{E,k}$, by the definition of $\mathcal{B}^E(\cdot,\cdot)$ (1.3), $\mathcal{B}_h^E(\cdot,\cdot)$ (1.20), the stability bilinear form (1.19) and the orthogonality (1.21):

$$\begin{aligned}
\mathcal{B}_h^E(v,v) &= \mathcal{B}^E(\Pi_E^\nabla v, \Pi_E^\nabla v) + s_E(v-\Pi_E^\nabla v, v-\Pi_E^\nabla v) \\
&\le \mathcal{B}^E(\Pi_E^\nabla v, \Pi_E^\nabla v) + C_s|v-\Pi_E^\nabla v|_{1,E}^2 \\
&\le (||A_E||_\infty + ||c_E||_\infty)\,|\Pi_E^\nabla v|_{1,E}^2 + C_s|v-\Pi_E^\nabla v|_{1,E}^2 \\
&\le \max\{||A_E||_\infty + ||c_E||_\infty, C_s\}\left(|\Pi_E^\nabla v|_{1,E}^2 + |v-\Pi_E^\nabla v|_{1,E}^2\right) \\
&= \frac{\max\{||A_E||_\infty + ||c_E||_\infty, C_s\}}{\beta}\beta|v|_{1,E}^2 \le \alpha^*\mathcal{B}^E(v,v).
\end{aligned}$$

Similarity, $v \in \mathbb{V}_{E,k}$,

$$\begin{aligned}
\mathcal{B}_h^E(v,v) &\ge \mathcal{B}^E(\Pi_E^\nabla v, \Pi_E^\nabla v) + c_s|v-\Pi_E^\nabla v|_{1,E}^2 \\
&\ge \beta|\Pi_E^\nabla v|_{E,1}^2 + c_s|v-\Pi_E^\nabla v|_{1,E}^2 \\
&\ge \min\{\beta, c_s\}|v|_{1,E}^2.
\end{aligned}$$

□

As already discussed in the previous sections, from the *Stability* property the $\mathcal{B}_h^E$ results continuous and coercive on $\mathbb{V}_{E,k} \times \mathbb{V}_{E,k}$.

We can now extend the definition of the bilinear form to the entire discretization. We define $\mathcal{B}_\mathcal{T}(\cdot,\cdot) : \mathbb{V}_\mathcal{T} \times \mathbb{V}_\mathcal{T} \to \mathbb{R}$ as

$$\mathcal{B}_\mathcal{T}(v_h, w_h) := \sum_{E\in\mathcal{T}} \mathcal{B}_h^E(v_h, w_h),$$

which results continuous and coercive on $\mathbb{V}_\mathcal{T}$ guaranteeing the existence and uniqueness of the solution u_h in (1.9).

1.4.3 Choice of $S_\mathcal{T}$

Following Beirão da Veiga et al. [2013], the choice of s_E, in general, depends on the problem. Here we present the simplest form of s_E, that we will use in the next chapter.

We can choose a canonical basis $\varphi_1, \dots, \varphi_{N_E}$ such that

$$\ell_i(\varphi_j) = \delta_{i,j}, \qquad\qquad i,j = 1,\dots,N_E$$

where ℓ_i, as defined previously, are the local degrees of freedom.

We can define S_E as

$$S_E(\varphi_i,\varphi_j) := s_E\left(\varphi_i - \Pi_E^\nabla\varphi_i, \varphi_j - \Pi_E^\nabla\varphi_j\right) := \sum_{r=1}^{N_E} \ell_r\left(\varphi_i - \Pi_E^\nabla\varphi_i\right)\ell_r\left(\varphi_j - \Pi_E^\nabla\varphi_j\right).$$

In the following, we will use this definition of the stability form in the case of $\mathbb{V}_{E,1}$ and $\mathbb{V}_{E,2}$. In the case of $\mathbb{V}_{E,1}$, the degrees of freedom are the values of the functions at the vertices of E. In the case of $\mathbb{V}_{E,2}$, the degrees of freedom are the values of the functions at the vertices, at the midpoints of each edge and value the moments. So that we can define the stability form as:

$$s_E(v,w) := \sum_{i=1}^{N_E-dim(\mathcal{P}_{E,k})} v(\boldsymbol{x}_i)w(\boldsymbol{x}_i) + \sum_{i=1}^{dim(\mathcal{P}_{E,k})} \ell_i^m(v)\ell_i^m(w), \tag{1.22}$$

where $\{\boldsymbol{x}_i\}_{i=1\ldots\{N_E-dim(\mathcal{P}_{E,k})\}}$ are the nodes of E at the vertices or at the midpoints and $\{\ell_i^m\}_{i=1\ldots dim(\mathcal{P}_{E,k})}$ are the degrees of freedom related to the moments. We recall that in the case of $\mathbb{V}_{E,1}$ there is not the second part of (1.22), while in $\mathbb{V}_{E,2}$ there is only the mean of v on E.

The extension on Ω of $S_{\mathcal{T}}$ is

$$S_{\mathcal{T}}(v_h,w_h) := \sum_{E\in\mathcal{T}} S_E(v_h,w_h) \qquad\qquad \forall\ v_h,\ w_h \in \mathbb{V}_{\mathcal{T}}.$$

A stabilization constant $\gamma > 0$ is then added, bringing the definition of $\mathcal{B}_{\mathcal{T}}$ to

$$\mathcal{B}_{\mathcal{T}}(v_h,w_h) = a_{\mathcal{T}}(v_h,w_h) + m_{\mathcal{T}}(v_h,w_h) + \gamma S_{\mathcal{T}}(v_h,w_h), \qquad \forall\ v_h,\ w_h \in \mathbb{V}_{\mathcal{T}}. \tag{1.23}$$

1.4.4 Construction of the right-hand side

Given $k \geq 2$ and recalling the L^2-projection $P_{K-2}^E : L^2(E) \to \mathbb{P}_{k-2}(E)$, we define f_h as

$$f_h = P_{k-2}^E f, \quad \text{on each element } E \text{ in } \mathcal{T}.$$

Consequently, we have

$$\langle f_h, v_h\rangle = \sum_{E\in\mathcal{T}_h} \int_E f_h v_h := \sum_{E\in\mathcal{T}} \int_E \left(P_{k-2}^E f\right) v_h = \sum_{E\in\mathcal{T}} \int_E f \left(P_{k-2}^E v_h\right).$$

We here remark that the last integral can be computed by using the known degrees of freedom of each element. Indeed, the $P_{k-2}^E v_h$ is a polynomial of degree $k-2$ and can by computed by knowing the moments of v_h. This choice of f_h is closer to f, when h goes to 0. Indeed,

$$\langle f_h, v_h\rangle - (f, v_h) = \sum_{E\in\mathcal{T}} \int_E \left(P_{k-2}^E f - f\right)(v_h - P_0^E(v_h)),$$

using the Cauchy- Schwarz inequality, and the error estimates for the projector operators

$$\begin{aligned}\langle f_h, v_h\rangle - (f, v_h) &\leq \sum_{E\in\mathcal{T}} h_E^{k-1}|f|_{k-1,E} h_E |v_h|_{1,E} \\ &\leq Ch^k \left(\sum_{E\in\mathcal{T}} |f|_{k-1,E}^2\right)^{\frac{1}{2}} |v_h|_1.\end{aligned}$$

This implies, from (1.2.1), that we have an estimation on $||f - f_h||_{\mathbb{V}'_{\mathcal{T}}}$,

$$||f - f_h||_{\mathbb{V}'_{\mathcal{T}}} \leq Ch^k \left(\sum_{E \in \mathcal{T}} |f|^2_{k-1,E} \right)^{\frac{1}{2}}. \tag{1.24}$$

1.4.5 The error of the discretization

In order to discuss the error of the discretization we firstly need to remark that, given a smooth function w, the Assumption (1.2.1) does not imply the existence of $w_\pi \in \mathbb{P}_k(E)$ sufficiently close to w. For this reason, as done in Beirão da Veiga et al. [2013], we use a new assumption.

Assumption 1.4.1. There exists a $\delta > 0$ such that, for all h, each element E in $\mathcal{T}_h$ is **star-shaped** with respect to a ball of radius $\geq \delta h_E$.

Thanks to this assumption, according to the Scott-Dupont theory (See Brenner and Scott [2008]), we have the two following results.

Proposition 1.4.2. *If the assumption (1.4.1) is satisfied, there exists a constant C, depending only on k and γ such that for every s with $1 \leq s \leq k+1$ and for every $w \in H^s(E)$, there exists a $w_\pi \in \mathbb{P}_k(E)$ such that*

$$||w - w_\pi||_{0,E} + h_E|w - w_\pi|_{1,E} \leq Ch_E^s|w|_{s,E}. \tag{1.25}$$

Proposition 1.4.3. *If the assumption (1.4.1) is satisfied, there exists a constant C, depending only on k and γ such that for every s with $1 \leq s \leq k+1$, for every h_e, for all $E \in \mathcal{T}_h$ and for every $w \in H^s(E)$, there exists a $w_I \in \mathbb{V}^{k,E}$ such that*

$$||w - w_I||_{0,E} + h_E|w - w_I|_{1,E} \leq Ch_E^s|w|_{s,E}. \tag{1.26}$$

Because of the construction of the bilinear form $\mathcal{B}_{\mathcal{T}}$, we have that it is continuous and coercive and so the Theorem 1.2.1 is valid. Then, if u is the solution of (1.2) and u_h the solution of (1.4), we have

$$|u - u_h|_1 \lesssim |u - u_I|_1 + |u - u_\pi|_{h,1} + ||f - f_h||_{\mathbb{V}'_h},$$

that, by using (1.24), (1.25) and (1.26), becomes

$$|u - u_h|_1 \lesssim h^k|u|_{k+1,\Omega},$$

with $h = \max_{E \in \mathcal{T}_h}\{h_E\}$. The inequality we wanted, (1.5), holds.

1.5 The *enhanced* definition of VEM

In order to define a more advanced and more operative definition of the virtual element space, we here present the idea of Ahmad et al. [2013] of a new definition, known as *enhanced* space. The importance of this definition is so evident that we already needed it to prove the *k-Consistency* of $\mathcal{B}_h^E$ (see **Remark** 2).

The idea is to define a new space for the VEM such that the degrees of freedom are the same as before, but in which the moments of $\Pi_E^k v$ and v coincides, $\forall\, v \in \mathbb{W}_{E,k}$.

The way in which in Ahmad et al. [2013] the new space is defined consists in two stps.

- **The enlargement.** Firstly, we consider the space

$$\tilde{\mathbb{V}}_{E,k} := \{v : v|_{\partial E} \in \mathbb{V}_{\partial E,k} \;\text{ and }\; \Delta v \in \mathbb{P}_k(E)\}.$$

- **The restriction.** Secondly, we restrict $\tilde{\mathbb{V}}_{E,k}$ to the space where the moments of degree $k-1$ and k of v and $\Pi_E^\nabla v$ coincide. So,

$$\mathbb{W}_{E,k} := \{v \in \tilde{\mathbb{V}}_{E,k} : (v - \Pi_E^\nabla v, q)_E = 0,\; \text{where } q \in \mathcal{M}_{k-1}^*(E) \cup \mathcal{M}_k^*(E)\},$$

  where the set $\mathcal{M}_k^*(E)$ is defined as

$$\mathcal{M}_k^*(E) = \left\{ \left(\frac{\boldsymbol{x} - \boldsymbol{x}_E}{h_E} \right)^s, |s| = k \right\}.$$

Proposition 1.5.1. *The dimension of* $\tilde{\mathbb{V}}_{E,k}$ *is*

$$\dim(\tilde{\mathbb{V}}_{E,k}) = kn + \frac{(k+1)(k+2)}{2},$$

the degrees of freedom of $\tilde{\mathbb{V}}_{E,k}$ *are the same as those of* $\mathbb{V}_{E,k}$*, but the moments considered are up to order* k*.*

Proof. We can follow the same idea for the dimension of $\mathbb{V}_{E,k}$, but now the polynomial $\Delta v \in \mathbb{P}_k(E)$, not more in $\mathbb{P}_{k-2}(E)$. □

Proposition 1.5.2. *The dimension of* $\mathbb{W}_{E,k}$ *is*

$$\dim(\mathbb{W}_{E,k}) = kn + \frac{k(k-1)}{2}.$$

As the degrees of freedom of $\mathbb{W}_{E,k}$ *we take the same of* $\mathbb{V}_{E,k}$*.*

Proof. The dimension of $\mathcal{M}_{k-1}^*(E) \cup \mathcal{M}_k^*(E)$ is equal to $2k+1$ (number of polynomials of degree $k-1$ or k). For sure, we have that

$$\dim(\mathbb{V}_{E,k}) \geq kn + \frac{(k+1)(k+2)}{2} - (2k+1) = kn + \frac{k(k-1)}{2}. \tag{1.27}$$

By definition of Π_E^∇ (1.16) a function $v \in \mathbb{W}_{E,k}$ that vanishes on ∂E and with moments up to order $k-2$ are zeros, $\Pi_E^\nabla v = 0$. Since v is in $\mathbb{W}_{E,k}$, all the moments up to order k are zeros, which implies v is identically zero.

This implies that the dimension of $\mathbb{W}_{E,k}$ is $kn + k(k-1)/n$ and that the same choice of the degrees of freedom is unisolvent on $\mathbb{W}_{E,k}$. □

In order not to add to many notations, in the following, we will use the same notation $\mathbb{V}_{E,k}$ for the *enhanced* space. We will underline when some properties depend on the definition of the VEM space.

Chapter 2

Stabilization-free a posteriori error analysis with the space of polynomials of degree 2

Being inspired by the work *Adaptive vem: stabilization-free a posteriori error analysis* by Beirão da Veiga et al. [2021], in this chapter we present the stabilization-free a posteriori analysis using the space of polynomials of degree 2. With respect to the mentioned paper, we will highlight the main differences brought by the space $\mathbb{V}_{E,2}$. The analysis here carried out will take into consideration the problem (1.1), already widely discussed in the first section.

As done in Beirão da Veiga et al. [2021], this analysis will be done in 2 dimensions and considering as elements of triangulation E only the triangles. These choices have been made for two reasons: on one hand a lot of results with VEM in three dimensions are not valid and on the other hand to the author it is not known a uniqueness and acknowledged way to define the refinement of elements with more then three edges, if not reducing to triangles. An enlargement to the case of quadrangles is proposed in the next chapter.

2.1 Preliminary setting

Before going directly to the discussion of the a posteriori error analysis, we here briefly recall some properties of the Virtual Elements in the case examined. Given a triangulation $\mathcal{T}$ on Ω made of $N_{\mathcal{T}}$ elements E. With the definition discussed in the first chapter we have the local spaces

$$\mathbb{V}_{\partial E,2} := \{v \in C^0(\partial E) : v|_e \in \mathbb{P}_2(e), \forall e \subset \partial E\},$$
$$\mathbb{V}_{E,2} := \{v \in H^1(E) : v|_{\partial E} \in \mathbb{V}_{\partial E,2}, \Delta v|_E = c, \text{where } c \in \mathbb{R}\};$$

and spaces defined on Ω

$$\begin{aligned}
&\mathbb{V}_{\mathcal{T},2} := \{v \in \mathbb{V} : v|_E \in \mathbb{V}_{E,2},\ \forall E \in \mathcal{T}_h\},\\
&\mathbb{W}^2_{\mathcal{T}} := \{w \in L^2(\Omega) : w|_E \in \mathbb{P}_2(E)\},\\
&\mathbb{V}^0_{\mathcal{T}} := \mathbb{V}_{\mathcal{T},2} \cap \mathbb{W}^2_{\mathcal{T}}.
\end{aligned}$$

In the case examined, the degrees of freedom of $\mathbb{V}_{E,2}$ are 7, showed in Figure 2.1. The 3 value of the functions of $\mathbb{V}_{E,2}$ at the vertices, the 3 values at the midpoinst and the mean of the function on E.

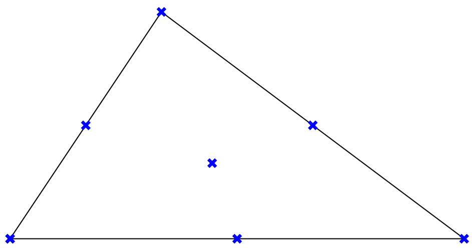


Figure 2.1: The figure represents with blue crosses the degrees of freedom of the element $\mathbb{V}_{E,2}$, in the case E is a triangle.

Following what written in the first chapter, from (1.22), the stabilization form then becomes,

$$s_E(v,w) := \sum_{i=1}^{6} v(\boldsymbol{x}_i)w(\boldsymbol{x}_i) + \ell^m(v)\ell^m(w), \qquad \forall v,\ w \in \mathbb{V}_{E,2} \tag{2.1}$$

where $\{\boldsymbol{x}_i\}_{i=1,\dots,6}$ are the vertices or the midpoints of E and ℓ^m is the degree of freedom related to the mean. Since, following Beirão da Veiga et al. [2017], the stabilization term in the a posteriori error analysis can be simplified by dropping the contribution of the degrees of freedom linked to the moments, we will consider the (2.1) without the term with ℓ^m. This choice of the stabilization form does not affect the stability and the convergence properties.

Because of some basic definitions will be fundamental in the following discussion, we here recall them here. In the literature the new nodes that occur after the refinement of an element are called *hanging nodes*. We here remark that in the case of $\mathbb{P}_2(E)$ after the first refinement one node will coincide with a proper node. In general, we notice that if the degree of the polynomials is even, then some proper nodes of the triangulation are midpoints of some edges. When a refinement occurs, these midpoints become also hanging nodes. For clarity's sake, we here redefine the sets of nodes.

Definition 2.1.1. Given a triangulation $\mathcal{T}$, we have the followings.

- A **node of the triangulation** is a vertex of some triangle or the midpoint of some edge of a triangle. The set of these points will be referred as $\mathcal{N}$.
- A **proper node** is a vertex or a midpoint of each triangle containing it. The set of the proper nodes will be called $\mathcal{P}$.

- A **hanging node** is a node of the triangulation which is not a proper node. We will use the letter $\mathcal{H}$ for referring to the set of these points.

We can also 'restrict' these definitions to an element of the triangulation $E \in \mathcal{T}$. In particular we have the following sets.

Definition 2.1.2. Let E be an element in $\mathcal{T}$. We can define

$\mathcal{N}_E$: subset of $\mathcal{N}$, with the nodes sitting on ∂E,
$\mathcal{P}_E$: set of the proper nodes of E,
$\mathcal{H}_E$: set of the hanging nodes of E.

An important definition, introduced in Beirão da Veiga et al. [2021], is the *Global index of a node*, that we here recall.

Definition 2.1.3 (Global index of a node)**.** The global index λ of a node $\boldsymbol{x} \in \mathcal{N}$ is recursively defined as follows:

- If $\boldsymbol{x}$ is a proper node, then set $\lambda(\boldsymbol{x}) := 0$;

- If $\boldsymbol{x}$ is a hanging node, with $\boldsymbol{x'}, \boldsymbol{x''} \in \mathcal{B}(\boldsymbol{x})$, then set $\lambda(\boldsymbol{x}) := \max\{\lambda(\boldsymbol{x'}), \lambda(\boldsymbol{x''})\} + 1$, where $\mathcal{B}(\boldsymbol{x})$ indicates the set of the endpoints $\{\boldsymbol{x'},\ \boldsymbol{x''}\}$ of the edge containing the node $\boldsymbol{x}$ as midpoint.

Moreover, we define the largest global index in $\mathcal{T}$ as

$$\Lambda_{\mathcal{T}} := \max_{\boldsymbol{x} \in \mathcal{N}} \lambda(\boldsymbol{x})$$

and we require it has got a limit.

Assumption 2.1.1. Given a triangulation $\mathcal{T}$, there exists a constant $\Lambda \geq 1$ such that

$$\Lambda_{\mathcal{T}} \leq \Lambda.$$

Remark 3. The **Assumption** 2.1.1, as discussed in Beirão da Veiga et al. [2021], has some implications on the number of the hanging nodes. Given an element E of the triangulation we have that, for each half side of the triangle, the maximum number of hanging nodes is $2^{\Lambda} - 2$, according to definition of hanging node previously given. The number of nodes is so $|\mathcal{N}_E| \leq 6 \cdot 2^{\Lambda}$. Moreover each refinement of E would introduce 2 degrees of freedoms, the moments of each 'new' triangle. This implies that we can bound the dimension of $\mathbb{V}_{E,2}$ by $6 \cdot 2^{\Lambda} + 2\Lambda$.

An example of the display of the global indexes is shown in Fig. 2.2. In particular, we notice that after 6 refinements $\Lambda_{\mathcal{T}}$ does not blow-up and, in this case, it remains equal to 2.

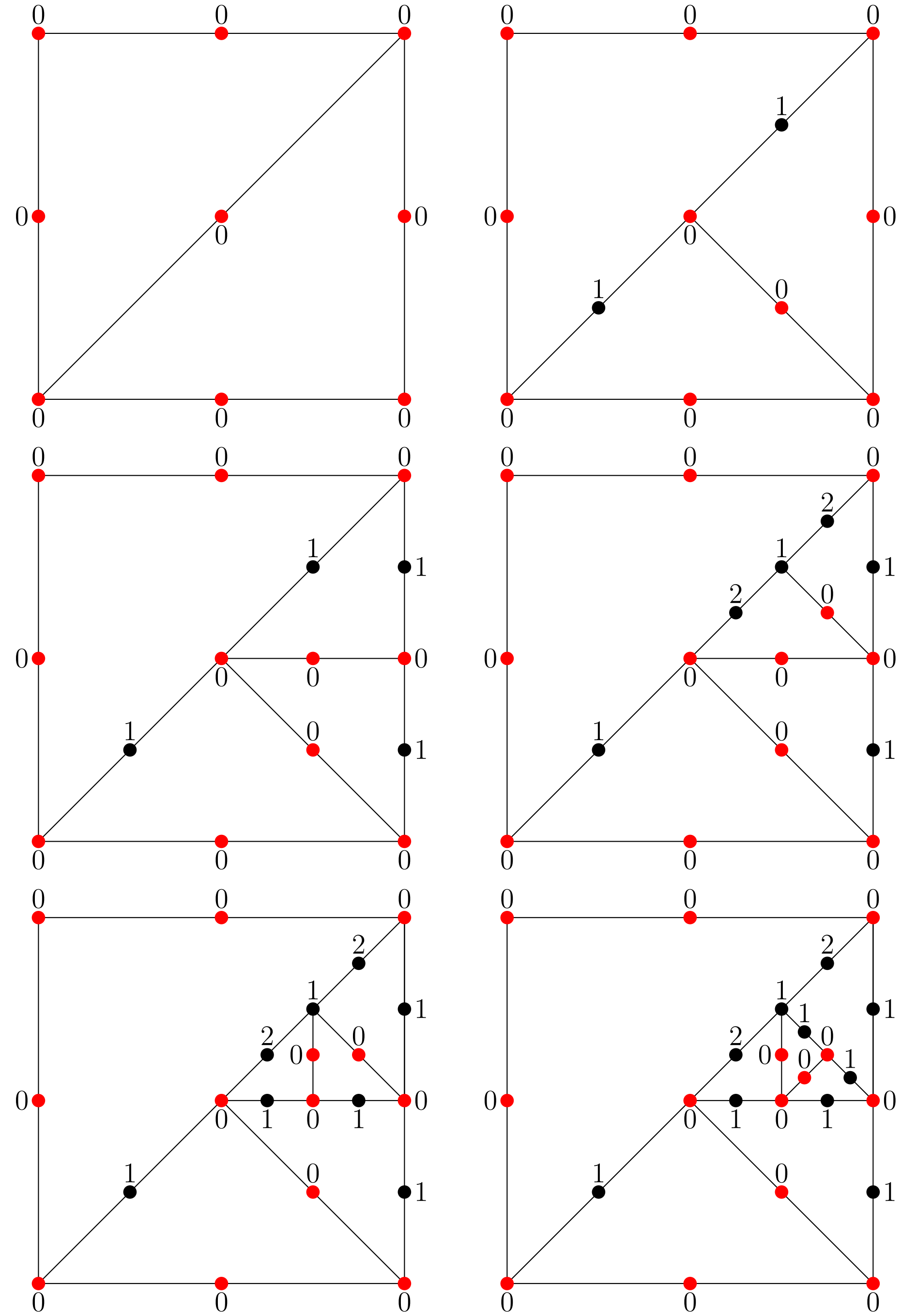


Figure 2.2: The figure shows the global indexes of the nodes that occur after 6 refinements. The proper nodes are represented in red, the hanging nodes in black.

2.2 The Poincaré inequality in $\mathbb{V}_{E,2}$

In Beirão da Veiga et al. [2021] a great effort has been carried out in order to prove the validity of a Scaled Poincaré Inequality on the whole triangulation. The inequality was necessary because in the case of $\mathbb{V}_{E,1}$ an element E can have all the vertices that are not proper node, as shown in Figure 2.3. For this reason it could happen that a function $v \in \mathbb{V}_{\mathcal{T},2}$ such that $v(\boldsymbol{x}) = 0$, for all $\boldsymbol{x}$ that are proper nodes, might not be zero in any point of E. On the other hand, this does not happen in the case of $\mathbb{V}_{E,2}$. Indeed, the new edge that arises after a refinement splits an element in two new triangles. As a result the node in the middle of the new edge cannot be an hanging node, as shown in Figure 2.4.

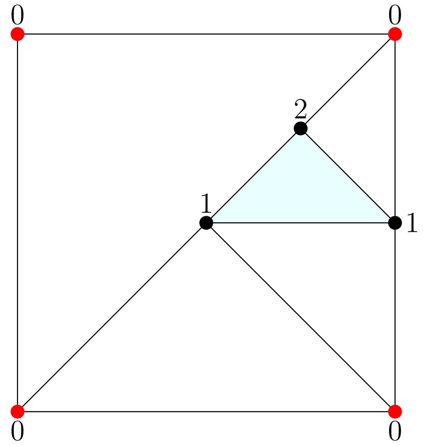


Figure 2.3: Example of triangulation using the space $\mathbb{V}_{E,1}$. We see that the triangle in light blue has all the vertices that are not proper nodes. The Poincaré inequality cannot be applied directly to this element. The numbers on the nodes are the global indexes, as described in Beirão da Veiga et al. [2021].

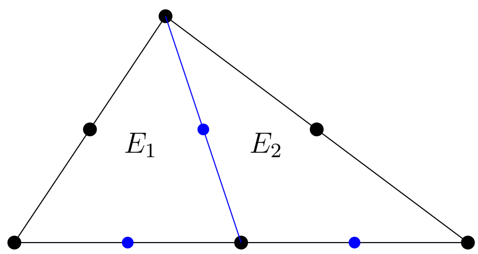


Figure 2.4: The figure shows the two elements that arise when a new edge (the blue one) is traced. In particular we notice that the midpoint of the blue edge is a proper node for both the elements E_1 and E_2.

We now recall the classical Poincaré inequality that now can be applied on each element E.

Proposition 2.2.1 (Poincaré inequality)**.** *Given an element E, there exists a constant $C_\Lambda > 0$ depending on Λ, such that*

$$h_E^{-2}||v||_{0,E}^2 \leq C_\Lambda |v|_{1,E}^2,$$

$\forall v \in \mathbb{V}_\mathcal{T}$ *such that* $v(\boldsymbol{x}) = 0$, $\forall \boldsymbol{x} \in \mathcal{P}$.

By summing on each element of the triangulation $\mathcal{T}$ we obtain

$$\sum_{E\in\mathcal{T}} h_E^{-2}||v||_{0,E}^2 \leq C_\Lambda |v|_1^2,$$

which is the same inequality proved in Beirão da Veiga et al. [2021] in the case of $\mathbb{V}_{E,1}$.

Remark 4. In the case of the space $\mathbb{V}_{E,k}$, with $k \geq 2$, the edge that arises with the refinement contains nodes that are not hanging nodes for the two new elements. The only case in which the Poincaré inequality cannot be applied directly on each element is the case with $k = 1$, as shown in Beirão da Veiga et al. [2021].

2.3 Other preparatory results

We now want to discuss some properties of the space $\mathbb{V}_\mathcal{T}^0$. This space will be essential in the following proof. A function v in $\mathbb{V}_\mathcal{T}^0$ is a polynomial of degree 2 in each element E of the triangulation and it is uniquely defined by the six values of the function at the vertices and at the midpoints. Let $\boldsymbol{z_{new}}$ be the hanging node that occurs after a refinement. This point is also a midpoint on the edge with endpoints $\boldsymbol{z'}$ and $\boldsymbol{z''}$. Because of $\boldsymbol{z_{new}}$ is not a proper node, at least one between $\boldsymbol{z'}$ and $\boldsymbol{z''}$ is a midpoint. We fix $\boldsymbol{z''}$ in the middle of the edge with end points $\boldsymbol{z'}$ and $\boldsymbol{z'''}$. There are two possible situations. The first one is showed in Fig. 2.5.

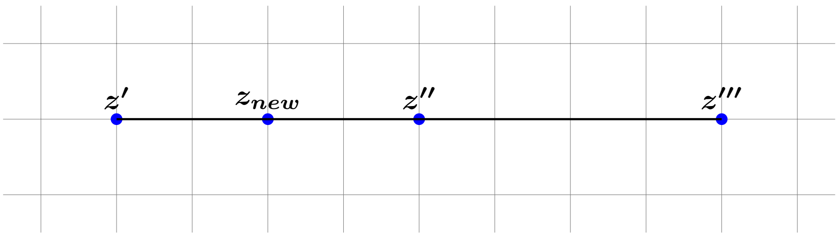


Figure 2.5: This figure shows the position of the nodes $\boldsymbol{z}_{new}$, $\boldsymbol{z'}$, $\boldsymbol{z''}$ and $\boldsymbol{z'''}$.

In this case, we can summarize the relations among these nodes as:

$$\boldsymbol{z'} = \boldsymbol{z_{new}} - \Delta\boldsymbol{z}, \qquad \boldsymbol{z''} = \boldsymbol{z_{new}} + \Delta\boldsymbol{z}, \qquad \boldsymbol{z'''} = \boldsymbol{z_{new}} + 3\Delta\boldsymbol{z},$$

where $\Delta\boldsymbol{z} = \frac{1}{4}(\boldsymbol{z'''} - \boldsymbol{z'})$. It is now possible to find a polynomial of degree 2 interpolating $v(\boldsymbol{z'})$, $v(\boldsymbol{z''})$ and $v(\boldsymbol{z'''})$. For this purpose it is convenient to write the polynomial as:

$$\Pi(\boldsymbol{x}) = a\left(\frac{\boldsymbol{x} - \boldsymbol{z_{new}}}{\Delta\boldsymbol{z}}\right)^2 + b\left(\frac{\boldsymbol{x} - \boldsymbol{z_{new}}}{\Delta\boldsymbol{z}}\right) + c. \tag{2.2}$$

We so have

$$\begin{aligned}
v(\boldsymbol{z'}) &= a\left(\frac{-\Delta \boldsymbol{z}}{\Delta \boldsymbol{z}}\right)^2 + b\left(\frac{-\Delta \boldsymbol{z}}{\Delta \boldsymbol{z}}\right) + c = a - b + c,\\
v(\boldsymbol{z''}) &= a\left(\frac{\Delta \boldsymbol{z}}{\Delta \boldsymbol{z}}\right)^2 + b\left(\frac{\Delta \boldsymbol{z}}{\Delta \boldsymbol{z}}\right) + c = a + b + c,\\
v(\boldsymbol{z'''}) &= a\left(\frac{3\Delta \boldsymbol{z}}{\Delta \boldsymbol{z}}\right)^2 + b\left(\frac{3\Delta \boldsymbol{z}}{\Delta \boldsymbol{z}}\right) + c = 9a + 3b + c.
\end{aligned}$$

The coefficients obtained are

$$\begin{aligned}
a &= \frac{v(\boldsymbol{z'''}) - 2v(\boldsymbol{z''}) + v(\boldsymbol{z'})}{8},\\
b &= \frac{v(\boldsymbol{z''}) - v(\boldsymbol{z'})}{2},\\
c &= \frac{-v(\boldsymbol{z'''}) + 6v(\boldsymbol{z''}) + 3v(\boldsymbol{z'})}{8}.
\end{aligned}$$

Posing the passage by $\boldsymbol{z_{new}}$, the polynomial obtained is

$$\Pi(\boldsymbol{z_{new}}) = \frac{-v(\boldsymbol{z'''}) + 6v(\boldsymbol{z''}) + 3v(\boldsymbol{z'})}{8}. \tag{2.3}$$

The other possible situation is showed in Fig. 2.6, with the relations,

$$\boldsymbol{z'} = \boldsymbol{z_{new}} + \Delta \boldsymbol{z}, \qquad \boldsymbol{z''} = \boldsymbol{z_{new}} - \Delta \boldsymbol{z}, \qquad \boldsymbol{z'''} = \boldsymbol{z_{new}} - 3\Delta \boldsymbol{z},$$

by the same steps we obtain the same formula (2.3).

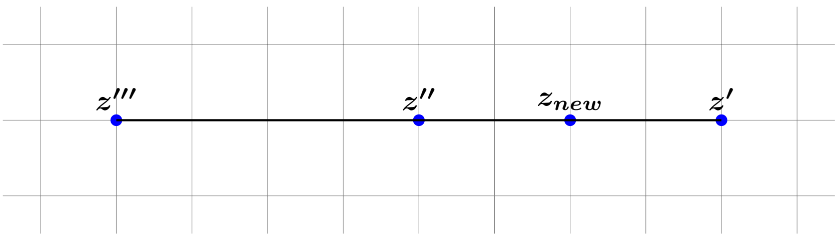


Figure 2.6: This figure shows the position of the nodes $\boldsymbol{z}_{new}$, $\boldsymbol{z'}$, $\boldsymbol{z''}$ and $\boldsymbol{z'''}$.

We can now define a basis for this space

$$\forall \boldsymbol{x} \in \mathcal{P}: \qquad \psi_{\boldsymbol{x}} \in \mathbb{V}_{\mathcal{T}}^0 \text{ satisfies } \psi_{\boldsymbol{x}}(\boldsymbol{z}) = \begin{cases} 1 \text{ if } \boldsymbol{z} = \boldsymbol{x},\\ 0 \text{ if } \boldsymbol{z} \in \mathcal{P} \setminus \{\boldsymbol{x}\}. \end{cases}$$

We can define the Lagrange interpolation operator

$$\begin{aligned}
&\mathcal{I}_{\mathcal{T}}^0 : \mathbb{V}_{\mathcal{T}} \to \mathbb{V}_{\mathcal{T}}^0 \text{ such that}\\
&\mathcal{I}_{\mathcal{T}}^0(\xi) = \sum_{\boldsymbol{x} \in \mathcal{P}} v(\boldsymbol{x}) \psi_{\boldsymbol{x}}(\xi).
\end{aligned}$$

We will also need the Clément quasi-interpolation operators. We will denote as $\overline{\mathcal{I}}^0_{\mathcal{T}}$ the classical Clément operator on $\mathbb{V}^0_{\mathcal{T}}$ and as $\overline{\mathcal{I}}_{\mathcal{T}}$ the classical Clément operator on $\mathbb{V}_{\mathcal{T}}$.

We here recall the following **Lemma**, as proved in Beirão da Veiga et al. [2021].

Lemma 2.3.1 (Clément interpolation estimate)**.** *It holds the following inequality*

$$\sum_{E\in\mathcal{T}} h_E^{-2}||v-\overline{\mathcal{I}}^0_{\mathcal{T}}v||^2_{0,E} \lesssim |v|^2_1, \qquad \forall v \ \in \mathbb{V},$$

where the hidden constant depends only on Λ*.*

Proof. Let define $v_{\mathcal{T}} = \overline{\mathcal{I}}_{\mathcal{T}}v$. We notice that

$$v-\overline{\mathcal{I}}^0_{\mathcal{T}}v = (v-v_{\mathcal{T}}) + \left(v_{\mathcal{T}}-\overline{\mathcal{I}}^0_{\mathcal{T}}v_{\mathcal{T}}\right) + \left(\overline{\mathcal{I}}^0_{\mathcal{T}}v_{\mathcal{T}}-\overline{\mathcal{I}}^0_{\mathcal{T}}v\right)$$

and we recall that $\overline{\mathcal{I}}^0_{\mathcal{T}}$ is locally stable in L^2, then

$$\begin{aligned}\sum_{E\in\mathcal{T}} h_E^{-2}||v-\overline{\mathcal{I}}^0_{\mathcal{T}}v||^2_{0,E} &\lesssim \sum_{E\in\mathcal{T}} h_E^{-2}||v-v_{\mathcal{T}}||^2_{0,E} + \sum_{E\in\mathcal{T}} h_E^{-2}||\overline{\mathcal{I}}^0_{\mathcal{T}}v_{\mathcal{T}}-v_{\mathcal{T}}||^2_{0,E}\\ &\lesssim |v|^2_1 + \sum_{E\in\mathcal{T}} h_E^{-2}||\overline{\mathcal{I}}^0_{\mathcal{T}}v_{\mathcal{T}}-v_{\mathcal{T}}||^2_{0,E}.\end{aligned}$$

We just now to prove that

$$\sum_{E\in\mathcal{T}} h_E^{-2}||\overline{\mathcal{I}}^0_{\mathcal{T}}v_{\mathcal{T}}-v_{\mathcal{T}}||^2_{0,E} \lesssim |v_{\mathcal{T}}|^2_1, \tag{2.4}$$

and then concluding with the stability of $\overline{\mathcal{I}}_{\mathcal{T}}$. The inequality (2.4) can be proved writing

$$v_{\mathcal{T}}-\overline{\mathcal{I}}^0_{\mathcal{T}}v_{\mathcal{T}} = v_{\mathcal{T}}-\mathcal{I}^0_{\mathcal{T}}v_{\mathcal{T}} + \overline{\mathcal{I}}^0_{\mathcal{T}}\left(\mathcal{I}^0_{\mathcal{T}}v_{\mathcal{T}}-v_{\mathcal{T}}\right),$$

because $\overline{\mathcal{I}}^0_{\mathcal{T}}$ is invariant in $\mathbb{V}^0_{\mathcal{T}}$. We can use again the stability of $\overline{\mathcal{I}}^0_{\mathcal{T}}$ and we have

$$\begin{aligned}\sum_{E\in\mathcal{T}} h_E^{-2}\left|\left|\left(v_{\mathcal{T}}-\mathcal{I}^0_{\mathcal{T}}v_{\mathcal{T}}\right)+\overline{\mathcal{I}}^0_{\mathcal{T}}\left(\mathcal{I}^0_{\mathcal{T}}v_{\mathcal{T}}-v_{\mathcal{T}}\right)\right|\right|^2_{0,E} &\lesssim \sum_{E\in\mathcal{T}} h_E^{-2}\left|\left|v_{\mathcal{T}}-\mathcal{I}^0_{\mathcal{T}}v_{\mathcal{T}}\right|\right|^2_{0,E}\\ &\lesssim C_\Lambda\,|v_{\mathcal{T}}|^2_1,\end{aligned}$$

where in the last inequality the Poincaré inequality (**Proposition** 2.2.1) has been used. □

2.4 A posteriori error analysis

In order to discuss the a posteriori error control, we firstly have to define the internal residual over E as

$$r_{\mathcal{T}}(E;v,\mathcal{D}) := f_E - c_E\Pi^\nabla_E v + \nabla\cdot\left(A_E\nabla\Pi^\nabla_E v\right), \qquad \forall v\in\mathbb{V}_{\mathcal{T},2}, \tag{2.5}$$

where $\mathcal{D} = (A, c, f)$ denotes the set of piecewise constant data.

Analogously, given two elements E_1 and E_2 in $\mathcal{T}$ and let e be the edge shared by the two elements. We can define the jump over e as

$$j_{\mathcal{T}}(e; v, \mathcal{D}) := [[A_E \nabla \Pi^{\nabla}_{\mathcal{T}} v]]_e = (A_{E_1} \nabla \Pi^{\nabla}_{E_1} v|_{E_1}) \cdot \boldsymbol{n_1} + (A_{E_2} \nabla \Pi^{\nabla}_{E_2} v|_{E_2}) \cdot \boldsymbol{n_2}, \tag{2.6}$$

where $\boldsymbol{n_i}$ denotes the unit vector to e pointing outward with respect to E_i. If $e \in \partial\Omega$ we set $j_{\mathcal{T}}(e; v, \mathcal{D}) = 0$. We then define the local residual estimator associated with E,

$$\eta^2_{\mathcal{T}}(E; v, \mathcal{D}) := h_E^2 ||r_{\mathcal{T}}(E; v, \mathcal{D})||^2_{0,E} + \frac{1}{2} \sum_{e \in \mathcal{E}_E} h_E ||j_{\mathcal{T}}(e; v, \mathcal{D})||^2_{0,e} \tag{2.7}$$

and the global residual as

$$\eta^2(v, \mathcal{D}) := \sum_{E \in \mathcal{T}} \eta^2_{\mathcal{T}}(E; v, \mathcal{D}).$$

In the following we present an upper and lower bounds for the energy norms. For the upper bound we will follow the proof showed in Cangiani et al. [2017].

Proposition 2.4.1 (Upper bound)**.** *There exists a constant C_{apost} depending only on Λ and $\mathcal{D}$, such that*

$$|u - u_{\mathcal{T}}|_1 \le C_{apost} \left(\eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D}) + S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}) \right).$$

Proof. Let $v \in H^1_0(\Omega)$ and $v_{\mathcal{T}} = \overline{\mathcal{I}}^0_{\mathcal{T}} v \in \mathbb{V}^0_{\mathcal{T}}$, where $\overline{\mathcal{I}}^0_{\mathcal{T}}$ is the Clément quasi-interpolation. We have

$$\mathcal{B}(u - u_{\mathcal{T}}, v) = ((f, v - v_{\mathcal{T}})_\Omega - \mathcal{B}(u_{\mathcal{T}}, v - v_{\mathcal{T}})) + \mathcal{B}(u - u_{\mathcal{T}}, v_{\mathcal{T}}) =: I + II.$$

The first term can be estimated as follows.

$$\begin{aligned} I = &\sum_{E \in \mathcal{T}} \{(f, v - v_{\mathcal{T}})_E - (A_E \nabla \Pi^{\nabla}_E u_{\mathcal{T}}, \nabla(v - v_{\mathcal{T}}))_E - c_E(\Pi^{\nabla}_E u_{\mathcal{T}}, v - v_{\mathcal{T}})_E\} \\ &+ \sum_{E \in \mathcal{T}} \{(A_E \nabla(\Pi^{\nabla}_E u_{\mathcal{T}} - u_{\mathcal{T}}), \nabla(v - v_{\mathcal{T}}))_E + c_E(\Pi^{\nabla}_E u_{\mathcal{T}} - u_{\mathcal{T}}, v - v_{\mathcal{T}})_E\} =: I_1 + I_2. \end{aligned}$$

Then, integrating by parts

$$\begin{aligned} |I_1| &\le \sum_{E \in \mathcal{T}} \left| (f, v - v_{\mathcal{T}})_E - (A_E \nabla \Pi^{\nabla}_E u_{\mathcal{T}}, \nabla(v - v_{\mathcal{T}}))_E - c_E(\Pi^{\nabla}_E u_{\mathcal{T}}, v - v_{\mathcal{T}})_E \right| \\ &= \sum_{E \in \mathcal{T}} \left| (f + \nabla \cdot (A_E \nabla \Pi^{\nabla}_E u_{\mathcal{T}}) - c_E \Pi^{\nabla}_E u_{\mathcal{T}}, v - v_{\mathcal{T}})_E - (A_E \nabla \Pi^{\nabla}_E u_{\mathcal{T}}, v - v_{\mathcal{T}})_{\partial E} \right| \\ &= \sum_{E \in \mathcal{T}} h_E^2 ||r_{\mathcal{T}}||_{0,E} h_E^{-2} ||v - v_{\mathcal{T}}||_{0,E} + \frac{1}{2} \sum_{e \in \mathcal{E}_E} h_E ||j_{\mathcal{T}}(e; v, \mathcal{D})||_{0,e} h_E^{-1} ||v - v_{\mathcal{T}}||_{0,e} \\ &\lesssim \eta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D}) |v|_1, \end{aligned}$$

where in the last passage it has been used the Cauchy-Schwarz inequality and the **Lemma** 2.3.1. While for the second part

$$
\begin{aligned}
|I_2| &\leq \sum_{E\in\mathcal{T}} \left|(A_E\nabla(\Pi_E^\nabla u_\mathcal{T} - u_\mathcal{T}), \nabla(v - v_\mathcal{T}))_E\right| + \sum_{E\in\mathcal{T}} \left|c_E(\Pi_E^\nabla u_\mathcal{T} - u_\mathcal{T}, v - v_\mathcal{T})_E\right| \\
&\lesssim \sum_{E\in\mathcal{T}} h_E \left(h_E^{-1}||\nabla(\Pi_E^\nabla u_\mathcal{T} - u_\mathcal{T})||_{0,E} + ||\Pi_E^\nabla u_\mathcal{T} - u_\mathcal{T}||_{0,E}\right) h_E^{-1}||v - v_\mathcal{T}||_{0,E} \\
&\lesssim \left(\sum_{E\in\mathcal{T}} ||\nabla(\Pi_E^\nabla u_\mathcal{T} - u_\mathcal{T})||_{0,E}^2 + h_E||\Pi_E^\nabla u_\mathcal{T} - u_\mathcal{T}||_{0,E}^2\right)^{1/2} \left(\sum_{E\in\mathcal{T}} h_E^{-2}||v - v_\mathcal{T}||_{0,E}^2\right)^{1/2} \\
&\lesssim \left(\sum_{E\in\mathcal{T}} ||\nabla(\Pi_E^\nabla u_\mathcal{T} - u_\mathcal{T})||_{0,E}^2 + h_E||\Pi_E^\nabla u_\mathcal{T} - u_\mathcal{T}||_{0,E}^2\right)^{1/2} |v_\mathcal{T}|_1,
\end{aligned}
$$

where we used again **Lemma** 2.3.1, concluding

$$
|I_2| \lesssim S_\mathcal{T}(u_\mathcal{T}, u_\mathcal{T})^{1/2}\ |v_\mathcal{T}|_1.
$$

For the term II we firstly apply (1.18)

$$
\mathcal{B}(u - u_\mathcal{T}, v_\mathcal{T}) = \sum_{E\in\mathcal{T}} c_E \int_E (\Pi_E^\nabla u_\mathcal{T} - u_\mathcal{T})v_\mathcal{T},
$$

because $v_\mathcal{T} \in \mathbb{V}_\mathcal{T}^0$. By the definition of Π_E^∇ and the scaled Poincaré inequality $||\Pi_E^\nabla u_\mathcal{T} - u_\mathcal{T}||_{0,E} \lesssim h_E|\Pi_E^\nabla u_\mathcal{T} - u_\mathcal{T}|_{1,E}$, $\forall E \in \mathcal{T}$ we have

$$
\mathcal{B}(u - u_\mathcal{T}, v_\mathcal{T}) \lesssim h_E S_\mathcal{T}(u_\mathcal{T}, u_\mathcal{T})^{1/2}\ |v_\mathcal{T}|.
$$

Taking now $v = u - u_\mathcal{T}$, we end the proof by using the coercivity of $\mathcal{B}(\cdot,\cdot)$. □

We report here the **Proposition** showed in Cangiani et al. [2017] concerning the local lower bound.

Proposition 2.4.2 (Local lower bound)**.** *There holds*

$$
\eta^2(E; u_\mathcal{T}, \mathcal{D}) \lesssim \sum_{E'\in\omega_E} \left(|u - u_\mathcal{T}|_{1,E'}^2 + S_{E'}(u_\mathcal{T}, u_\mathcal{T})\right),
$$

where $\omega_E := \{E' : |\partial E \cap \partial E'| \neq 0\}$. *The hidden constant does not depend on* γ, h, u *and* $u_\mathcal{T}$.

Corollary 2.4.3 (Global lower bound)**.** *There exists a constant* $c_{apost} > 0$, *depending on* Λ, *but independent of* $u, \mathcal{T}, u_\mathcal{T}$ *and* γ *such that*

$$
c_{apost}\eta^2(u_\mathcal{T}, \mathcal{D}) \leq |u - u_\mathcal{T}|_1^2 + S_\mathcal{T}(u_\mathcal{T}, u_\mathcal{T}).
$$

2.5 Bound of the stabilization term by the residual

Following Beirão da Veiga et al. [2021], in this section we introduce the bound for the stabilization term by the residual in the case of $\mathbb{V}_{E,2}$.

As first step we need to discuss the interpolation error in $\mathbb{V}^0_{\mathcal{T}}$, $\mathcal{I}^0_{\mathcal{T}}$, and $\mathbb{V}_{\mathcal{T},2}$, $\mathcal{I}_{\mathcal{T}}$. We notice that, by the triangle inequality, $\forall v \in \mathbb{V}_{\mathcal{T},2}$

$$|v - \mathcal{I}^0_{\mathcal{T}} v|_1 = |v - \mathcal{I}^0_{\mathcal{T}} v|_{1,\mathcal{T}} \leq |v - \mathcal{I}_{\mathcal{T}} v|_{1,\mathcal{T}} + |\mathcal{I}_{\mathcal{T}} v - \mathcal{I}^0_{\mathcal{T}} v|_{1,\mathcal{T}}.$$

If we show that

$$|\mathcal{I}_{\mathcal{T}} v - \mathcal{I}^0_{\mathcal{T}} v|_{1,\mathcal{T}} \lesssim |v - \mathcal{I}_{\mathcal{T}} v|_{1,\mathcal{T}}, \tag{2.8}$$

then we can conclude that

$$|v - \mathcal{I}^0_{\mathcal{T}} v|_1 \lesssim |v - \mathcal{I}_{\mathcal{T}} v|_{1,\mathcal{T}}. \tag{2.9}$$

In order to prove (2.8), we need to write the function of the *hierarchical detail of* v. In section 2.3, we have already discussed how it is possible to build the polynomial of degree 2 interpolating three nodes. Thanks to this, we can write the following function.

Definition 2.5.1 (Hierarchical detail of v)**.** To each function $v \in \mathbb{V}_{E,2}$ we associate a vector $d(v) = \{d(v;\boldsymbol{z})\}_{\boldsymbol{z}\in\mathcal{N}_E}$ that collects the following values, so called *hierarchical details of* v

$$d(v;\boldsymbol{z}) = \begin{cases} v(\boldsymbol{z}) & \text{if } \boldsymbol{z} \in \mathcal{P}_E, \\ v(\boldsymbol{z}) + \dfrac{1}{8} v(\boldsymbol{z}''') - \dfrac{3}{4} v(\boldsymbol{z}'') - \dfrac{3}{8} v(\boldsymbol{z}') & \text{if } \boldsymbol{z} \in \mathcal{H}_E, \end{cases} \tag{2.10}$$

where $\boldsymbol{z}'$, $\boldsymbol{z}''$ and $\boldsymbol{z}'''$, are defined as follows. $\boldsymbol{z}''$ is the midpoint of the edge with endpoints $\boldsymbol{z}'$ and $\boldsymbol{z}'''$ and $\boldsymbol{z}$ is the midpoint of the edge with endpoints $\boldsymbol{z}'$ and $\boldsymbol{z}''$.

From Beirão da Veiga et al. [2021], we have the following **Lemma**.

Lemma 2.5.1 (Local interpolation error vs hierarchical detail)**.** *For all E in $\mathcal{T}$ it holds the following relation.*

$$|v - \mathcal{I}_E v|^2_{1,E} \simeq \sum_{\boldsymbol{x}\in\mathcal{H}_E} d^2(v;\boldsymbol{x}), \qquad v \in \mathbb{V}_{E,2}, \tag{2.11}$$

where the hidden constants depend only on Λ.

Proof. From Beirão da Veiga et al. [2017], with the choice of stabilization (1.22), we have

$$|v - \mathcal{I}_E v|^2_{1,E} \simeq \sum_{\boldsymbol{x}\in\mathcal{H}_E} |v(\boldsymbol{x}) - \mathcal{I}_E(\boldsymbol{x})|, \qquad v \in \mathbb{V}_{E,2},$$

from the **Definition** 2.5.1, we have that $d(\mathcal{I}_E v;\boldsymbol{x}) = 0$. Then the relation (2.11) holds true if

$$\sum_{\boldsymbol{x}\in\mathcal{H}_E} d^2(v - \mathcal{I}_E v;\boldsymbol{x}) \simeq \sum_{\boldsymbol{x}\in\mathcal{H}_E} |v(\boldsymbol{x}) - \mathcal{I}_E(\boldsymbol{x})|, \qquad v \in \mathbb{V}_{E,2}.$$

Equivalently,

$$\sum_{\boldsymbol{x}\in\mathcal{H}_E} d^2(w;\boldsymbol{x}) \simeq \sum_{\boldsymbol{x}\in\mathcal{H}_E} |w(\boldsymbol{x})|\,, \qquad w \in \tilde{\mathbb{V}}_{E,2},$$

where $\tilde{\mathbb{V}}_{E,2} := \{v \in \mathbb{V}_{E,2} : v(\boldsymbol{x}) = 0, \forall \boldsymbol{x} \in \mathcal{V}_E\}$. These two quantities are equivalent norms in the finite dimension space $\tilde{\mathbb{V}}_{E,2}$ and depend only on the value of the function at the nodes of $\mathcal{H}_E$ and not at the shape of E. □

Corollary 2.5.2 (Global interpolation error vs hierarchical detail)**.** *It holds the following relation.*

$$|v - \mathcal{I}_{\mathcal{T}} v|^2_{1,\mathcal{T}} \simeq \sum_{\boldsymbol{x}\in\mathcal{H}} d^2(v;\boldsymbol{x}), \qquad v \in \mathbb{V}_{\mathcal{T},2},$$

where the hidden constants depend only on Λ.

Proof. Summing on each element $E \in \mathcal{T}$ the relation in **Lemma** 2.5.1 □

If $\boldsymbol{x}$ is a proper node, then $v(\boldsymbol{x}) = \left(\mathcal{I}^0_{\mathcal{T}} v\right)(\boldsymbol{x})$. For any $\boldsymbol{x} \in \mathcal{H}$, let us define

$$\delta(v,\boldsymbol{x}) := v(\boldsymbol{x}) - \left(\mathcal{I}^0_{\mathcal{T}} v\right)(\boldsymbol{x}),$$

that will be useful to estimate

$$|\mathcal{I}_{\mathcal{T}} v - \mathcal{I}^0_{\mathcal{T}} v|^2_{1,\mathcal{T}} = \sum_{E\in\mathcal{T}} |\mathcal{I}_{\mathcal{T}} v - \mathcal{I}^0_{\mathcal{T}} v|^2_{1,E} \simeq \sum_{E\in\mathcal{T}} \sum_{\boldsymbol{x}\in\mathcal{V}_E} (\mathcal{I}_{\mathcal{T}} v - \mathcal{I}^0_{\mathcal{T}} v)^2_{1,E}(\boldsymbol{x}).$$

Indeed, if $\boldsymbol{x} \in \mathcal{V}_E$, $(\mathcal{I}_E v)(\boldsymbol{x}) = (v)(\boldsymbol{x})$, then

$$|\mathcal{I}_{\mathcal{T}} v - \mathcal{I}^0_{\mathcal{T}} v|^2_{1,\mathcal{T}} \simeq \sum_{E\in\mathcal{T}} \sum_{\boldsymbol{x}\in\mathcal{V}_E} (v - \mathcal{I}^0_{\mathcal{T}} v)^2_{1,E}(\boldsymbol{x}) = \sum_{\boldsymbol{x}\in\mathcal{H}} \delta^2(v,\boldsymbol{x}).$$

From the **Corollary** 2.5.2, the (2.9) holds if the following it is true

$$\sum_{\boldsymbol{x}\in\mathcal{H}} \delta^2(v,\boldsymbol{x}) \lesssim \sum_{\boldsymbol{x}\in\mathcal{H}} d^2(v;\boldsymbol{x}), \qquad v \in \mathbb{V}_{\mathcal{T},2}. \tag{2.12}$$

As done in Beirão da Veiga et al. [2021], we fix v and we define $\delta(\boldsymbol{x}) := \delta(v,\boldsymbol{x})$, $d(\boldsymbol{x}) := d(v,\boldsymbol{x})$ and $v^* := \mathcal{I}^0_{\mathcal{T}} v$. Let

$$\boldsymbol{\delta} = (\delta(\boldsymbol{x}))_{\boldsymbol{x}\in\mathcal{H}}\,, \qquad \boldsymbol{d} = (d(\boldsymbol{x}))_{\boldsymbol{x}\in\mathcal{H}}\,,$$

the relation (2.12) reduces to

$$||\boldsymbol{\delta}||_{l^2(\mathcal{H})} \lesssim ||\boldsymbol{d}||_{l^2(\mathcal{H})}.$$

If v^* is on the segment $[\boldsymbol{x'}, \boldsymbol{x'''}]$, we have

$$\begin{aligned}\delta(\boldsymbol{x}) &= v(\boldsymbol{x}) - v^*(\boldsymbol{x}) = v(\boldsymbol{x}) + \frac{1}{8}v^*(\boldsymbol{x'''}) - \frac{3}{4}v^*(\boldsymbol{x''}) - \frac{3}{8}v^*(\boldsymbol{x'}) \\ &= v(\boldsymbol{x}) + \frac{1}{8}v(\boldsymbol{x'''}) - \frac{1}{8}\left(v(\boldsymbol{x'''}) - v^*(\boldsymbol{x'''})\right) - \frac{3}{4}v(\boldsymbol{x''}) \\ &\quad + \frac{3}{4}\left(v(\boldsymbol{x''}) - v^*(\boldsymbol{x''})\right) - \frac{3}{8}v(\boldsymbol{x'}) + \frac{3}{8}\left(v(\boldsymbol{x'}) - v^*(\boldsymbol{x'})\right) \\ &= d(\boldsymbol{x}) - \frac{1}{8}\delta(\boldsymbol{x'''}) + \frac{3}{4}\delta(\boldsymbol{x''}) + \frac{3}{8}\delta(\boldsymbol{x'}).\end{aligned}$$

Thus, we can build a matrix $\boldsymbol{W} : l^2(\mathcal{H}) \to l^2(\mathcal{H})$ such that $\boldsymbol{\delta} = \boldsymbol{W}\boldsymbol{d}$. We just need to prove that

$$||\boldsymbol{W}||_2 \lesssim 1.$$

We now organize the hanging nodes with respect to the global index $\lambda \in [1, \Lambda_{\mathcal{T}}]$. Calling $\mathcal{H}_\lambda = \{\boldsymbol{x} \in \mathcal{H} : \lambda(\boldsymbol{x}) = \lambda\}$, and $\mathcal{H} = \bigcup_{1 \le \lambda \le \Lambda_{\mathcal{T}}} \mathcal{H}_\lambda$. Matrix $\boldsymbol{W}$ can be factorized in lower triangular matrix $\boldsymbol{W}_\lambda$, that change the nodes of level λ, leaving the others unchanged. In particular,

$$\boldsymbol{W} = \boldsymbol{W}_{\Lambda_{\mathcal{T}}}\boldsymbol{W}_{\Lambda_{\mathcal{T}}-1} \dots \boldsymbol{W}_2\boldsymbol{W}_1,$$

where $\boldsymbol{W}_1 = \boldsymbol{I}$, the identity matrix, since if $\lambda = 1$, then $\delta(\boldsymbol{x'}) = \delta(\boldsymbol{x''}) = \delta(\boldsymbol{x'''}) = 0$. Each matrix $\boldsymbol{W}_\lambda$ differs from the identity only in the rows of block λ. Each of these rows contain all the elements equals to zero, but three entries with the coefficients previously found $(-\frac{1}{8}, \frac{3}{4}, \frac{3}{8})$ in the off-diagonal and one 1 in the diagonal. In order to estimate $\boldsymbol{W}_\lambda$, we use the Hölder inequality: $||\boldsymbol{W}_\lambda||_2^2 \le ||\boldsymbol{W}_\lambda||_1 ||\boldsymbol{W}_\lambda||_\infty$.

From the construction of $\boldsymbol{W}_\lambda$ have that

$$||\boldsymbol{W}_\lambda||_\infty \le \frac{1}{8} + \frac{3}{4} + \frac{3}{8} + 1 = \frac{9}{4} \qquad\qquad ||\boldsymbol{W}_\lambda||_1 \le 5\frac{3}{4} + 1 = \frac{19}{4},$$

where in the last inequality it has been used the fact that an hanging node of global index $< \lambda$, will appear at most 5 times in the relation between $\delta(\boldsymbol{x})$ and $d(\boldsymbol{x})$. These bring us to the following

$$||\boldsymbol{W}||_2 \le \prod_{2 \le \lambda \le \Lambda_{\mathcal{T}}} ||\boldsymbol{W}_\lambda||_2 \le \left(\frac{171}{16}\right)^{\frac{\Lambda - 1}{2}}.$$

Remark 5. We here want to recall that the proof is essentially the same to the one showed in Beirão da Veiga et al. [2021]. The main difference is in the values of the entries of matrix $\boldsymbol{W}$. In the case of polynomials of first order there were only two off-diagonal entries different from zero and they were both $\frac{1}{2}$. The values here found derive from the interpolating polynomial. Using higher polynomial degrees, as we will show in the next sections the proof still remains valid, but there will be more non null entries. This brings the estimates of the norms previously found to grow.

Because of this discussion, the following proposition showed in Beirão da Veiga et al. [2021] remains valid.

Proposition 2.5.3 (Comparison between interpolation operators)**.** *There exists a constant C_I depending only on Λ, such that*

$$|v - \mathcal{I}_{\mathcal{T}}^0 v|_1 \leq C_I |v - \mathcal{I}_{\mathcal{T}} v|_{1,\mathcal{T}}, \qquad \forall v \in \mathbb{V}_{\mathcal{T},2}.$$

The following proof, showed in Beirão da Veiga et al. [2021] and here reported only for the sake of completeness, where the only difference lies in the definition of the residual (2.5).

Proposition 2.5.4 (Bound of the stabilization term by the residual)**.** *There exists a constant C_B depending only on Λ, such that*

$$\gamma^2 S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}) \leq C_B \eta_{\mathcal{T}}^2(u_{\mathcal{T}}, \mathcal{D}).$$

Proof. From the definition (1.23), $\forall w \in \mathbb{V}_{\mathcal{T}}^0$ we have

$$S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}) = S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}} - w) = \mathcal{B}_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}} - w) - a_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}} - w) - m_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}} - w),$$

since in order to have the *k-Consistency* property (1.6) for $\mathcal{B}_{\mathcal{T}}$, we asked that if w is a polynomial, then $S_{\mathcal{T}}(u_{\mathcal{T}}, w) = 0$. We analyze now the single terms and using the definitions of the bilinear forms we have:

$$\mathcal{B}_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}} - w) = \mathcal{B}\left(\Pi_{\mathcal{T}}^{\nabla} u_{\mathcal{T}}, \Pi_{\mathcal{T}}^{\nabla}(u_{\mathcal{T}} - w)\right) = \left(f, \Pi_{\mathcal{T}}^{\nabla}(u_{\mathcal{T}} - w)\right)_{\Omega};$$

$$m_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}} - w) = m\left(\Pi_{\mathcal{T}}^{\nabla} u_{\mathcal{T}}, \Pi_{\mathcal{T}}^{\nabla}(u_{\mathcal{T}} - w)\right) = \left(c\Pi_{\mathcal{T}}^{\nabla} u_{\mathcal{T}}, \Pi_{\mathcal{T}}^{\nabla}(u_{\mathcal{T}} - w)\right)_{\Omega};$$

$$\begin{aligned} a_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}} - w) &= a\left(\Pi_{\mathcal{T}}^{\nabla} u_{\mathcal{T}}, \Pi_{\mathcal{T}}^{\nabla}(u_{\mathcal{T}} - w)\right) = \sum_{E \in \mathcal{T}} (A_E \nabla \Pi_E^{\nabla} u_{\mathcal{T}}, \nabla \Pi_E^{\nabla}(u_{\mathcal{T}} - w))_E \\ &= \sum_{E \in \mathcal{T}} (A_E \nabla \Pi_E^{\nabla} u_{\mathcal{T}} \cdot \boldsymbol{n}, u_{\mathcal{T}} - w)_{\partial E} - (\nabla \cdot A_E \nabla \Pi_E^{\nabla} u_{\mathcal{T}}, \Pi_E^{\nabla}(u_{\mathcal{T}} - w))_E. \end{aligned}$$

Given these and the definitions of $r_{\mathcal{T}}$ (2.5) and $j_{\mathcal{T}}$ (2.6), we have

$$\begin{aligned} S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}) &= (r_{\mathcal{T}}(E; u_{\mathcal{T}}, \mathcal{D}), \Pi_E^{\nabla}(u_{\mathcal{T}} - w))_{\Omega} + \sum_{e \in \mathcal{E}} (j_{\mathcal{T}}(e; u_{\mathcal{T}}, \mathcal{D}), u_{\mathcal{T}} - w)_e \\ &\leq \sum_{E \in \mathcal{T}} h_E ||r_{\mathcal{T}}(E; u_{\mathcal{T}}, \mathcal{D})||_{0,E} h_E^{-1} \left(||u_{\mathcal{T}} - w||_{0,E} + h_E |u_{\mathcal{T}} - w|_{1,E}\right) \\ &\quad + \frac{1}{2} \sum_{E \in \mathcal{T}} \sum_{e \in \mathcal{E}_{\mathcal{T}}} h_E^{1/2} ||j_{\mathcal{T}}(e; u_{\mathcal{T}}, \mathcal{D})||_{0,e} h_E^{-1/2} ||u_{\mathcal{T}} - w||_{0,e} \end{aligned}$$

From the definition of $\eta_{\mathcal{T}}$ (2.7), we obtain that, for any $\delta > 0$,

$$\gamma S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}) \leq \frac{1}{2\delta} \eta^2(u_{\mathcal{T}}, \mathcal{D}) + \frac{\delta}{2} \phi_{\mathcal{T}}(u_{\mathcal{T}} - w), \qquad \forall\, w \in \mathbb{V}_{\mathcal{T}}^0, \tag{2.13}$$

with

$$\begin{aligned}\phi_{\mathcal{T}}(u_{\mathcal{T}} - w) &= \sum_{E\in\mathcal{T}} \left(h_E^{-2} ||u_{\mathcal{T}} - w||_{0,E}^2 + |u_{\mathcal{T}} - w|_{1,E}^2 + \sum_{e\in\mathcal{E}_{\mathcal{T}}} h_E^{-1} ||u_{\mathcal{T}} - w||_{0,e}^2 \right) \\ &\lesssim \sum_{E\in\mathcal{T}} \left(h_E^{-2} ||u_{\mathcal{T}} - w||_{0,E}^2 + |u_{\mathcal{T}} - w|_{1,E}^2 \right).\end{aligned}$$

Choosing now $w = \mathcal{I}_{\mathcal{T}}^0 u_{\mathcal{T}}$, we can apply the Poincaré inequality (2.2.1), we have

$$\phi_{\mathcal{T}}(u_{\mathcal{T}} - \mathcal{I}_{\mathcal{T}}^0 u_{\mathcal{T}}) \lesssim |u_{\mathcal{T}} - \mathcal{I}_{\mathcal{T}}^0 u_{\mathcal{T}}|_1^2.$$

From the **Proposition** 2.5.3 and the property (1.19), we can end the proof by writing

$$\phi_{\mathcal{T}}(u_{\mathcal{T}} - \mathcal{I}_{\mathcal{T}}^0 u_{\mathcal{T}}) \leq C_\phi S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}).$$

Setting in (2.13) $\delta = \gamma / C_B$ and $C_\phi = C_B$, we obtain

$$\gamma^2 S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}) \leq C_B \eta_{\mathcal{T}}^2(u_{\mathcal{T}}, \mathcal{D}).$$

□

Using **Proposition** 2.5.4, **Proposition** 2.4.1 and **Corollary** 2.4.3, we have the last result.

Corollary 2.5.5 (Stabilization-free a posteriori error estimates)**.** *If γ is chosen as $\gamma^2 \geq \frac{C_B}{c_{apost}}$, it holds true*

$$(c_{apost} - C_B\gamma^2)\eta^2(u_{\mathcal{T}}, \mathcal{D}) \leq |u - u_{\mathcal{T}}|_1^2 \leq C_{apost}(1 + C_B\gamma^2)\eta^2(u_{\mathcal{T}}, \mathcal{D}).$$

2.6 Extension to higher polynomial degree

In order to extend the a posteriori error estimates to the case of the polynomials of degree three, we previously discuss what happens in terms of degrees of freedom when a refinement occurs. As already discussed in the case of $\mathbb{V}_{E,2}$, we consider only the degrees of freedom on the boundary of E. For clarity's sake, we suppose a triangulation made of two elements, as showed in Figure 2.7. When a refinement occurs two 'new' elements form and with them their 'new' degrees of freedom (showed with red squares in 2.8).

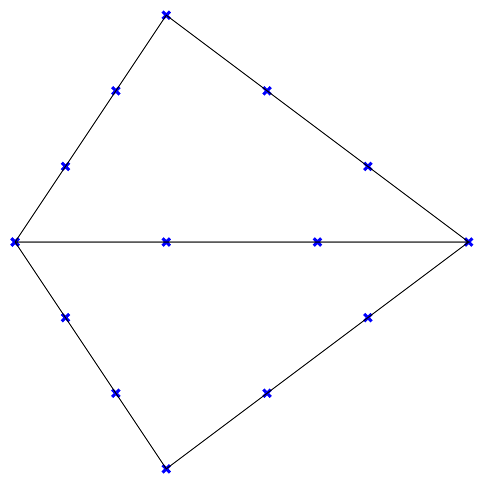


Figure 2.7: The figure shows the two elements and their degrees of freedom (represented with blue crosses).

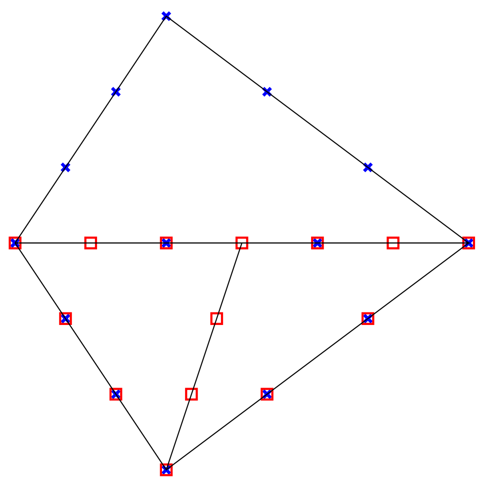


Figure 2.8: Triangulation after the first refinement. The degrees of freedom for the new elements are showed with red squares.

The first remark is that some nodes are degrees of freedom for both the 'old' triangles and the 'new' ones. And this is something that will be repeated with the following refinements. The situation is shown in Figure 2.9.

In the constructions of the details (used in Beirão da Veiga et al. [2021]), the global index of a node plays a fundamental role. In order to show this we previously recall how these details have been computed in the cases of lower polynomial degrees.

Let's consider the case of polynomials of degree 1. After the first refinement the detail is built as the difference of a linear function given by the two endpoints of the edge and the two linear functions each of them living in one half of the edge. As shown in Figure 2.10, this detail can be computed by the value of the polynomials at the midpoint (global

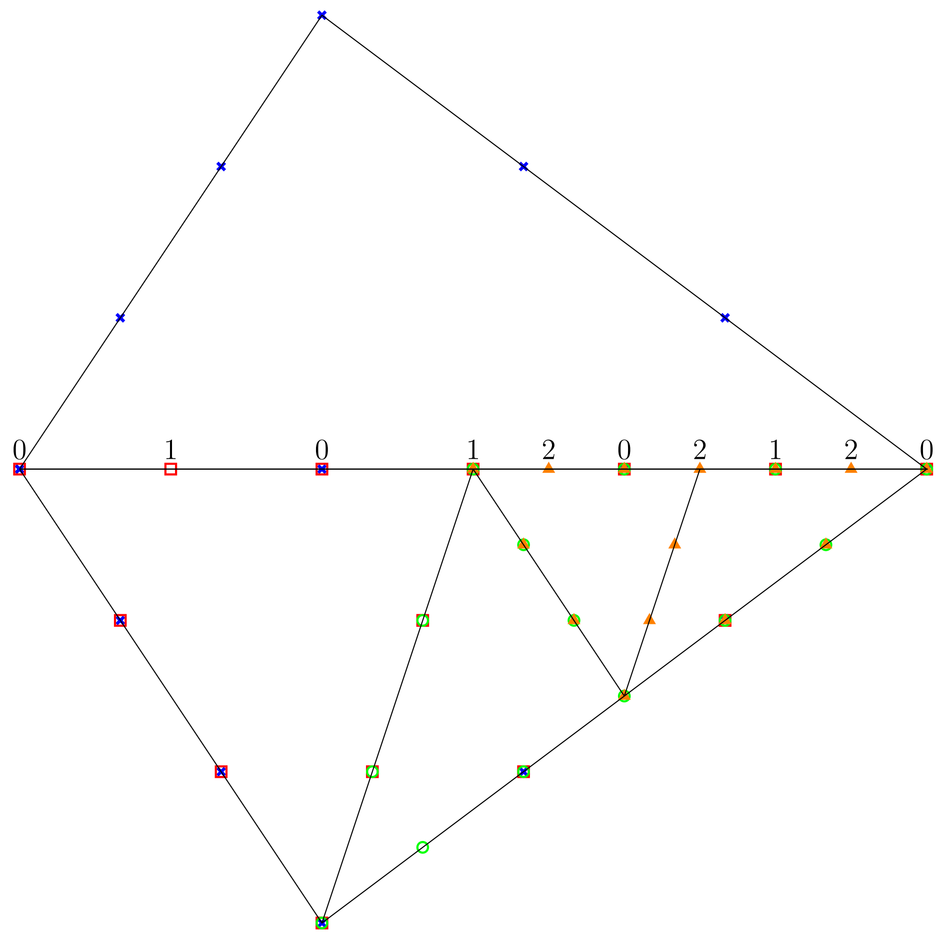


Figure 2.9: Triangulation after the three refinement. The original degrees of freedom are represented with blue crosses, the ones generated after the first refinement with red squares. Green circles and orange triangles are used for the degrees of freedom of the second and the third refinement respectively. The numbers on the horizontal line are the global indexes for the nodes.

index 1). The same happens with the second refinement (Figure 2.11), where now the difference is computed in the second half of the segment and the detail is the value of the function at the point with global index 2.

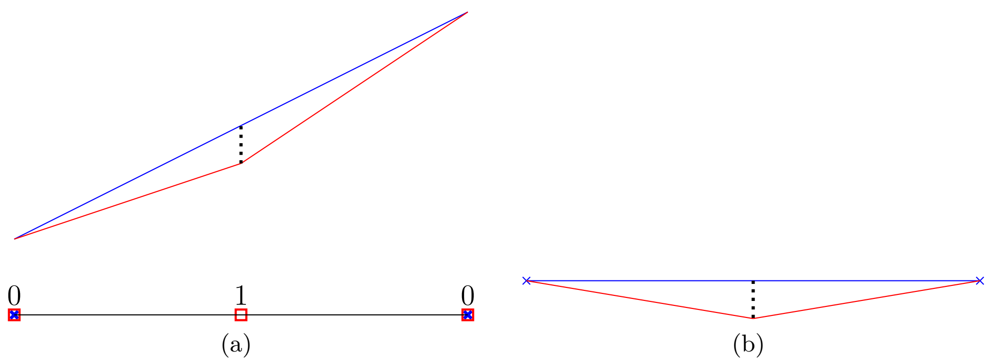


Figure 2.10: In (a) it is shown in blue the polynomial of degree 1 defined in the whole segment and in red the two polynomials of degree 1 living in half of the segment. (b) shows in black the detail of the first order.

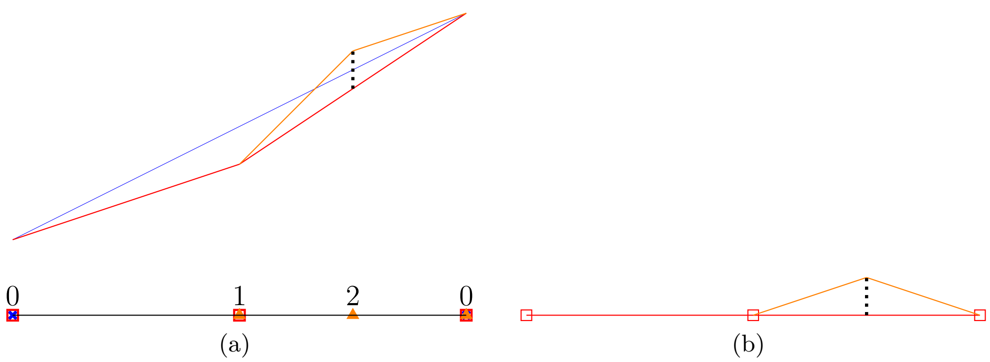


Figure 2.11: In (a) it is shown in blue the polynomial of degree 1 defined in the whole segment and in red the two polynomials of degree 1 living in half of the segment and in orange the polynomials that occur after the second refinement. (b) shows in black the detail of the second order.

Something analogous happens with the polynomials of degree 2. As shown in Figure 2.12 and 2.13, the number of details double with respect to the case of $\mathbb{P}_1$ and again for the first refinement it is computed in the values of the functions at the nodes with global index 1 and for the second refinement using the values of the polynomials at the nodes with global index 2.

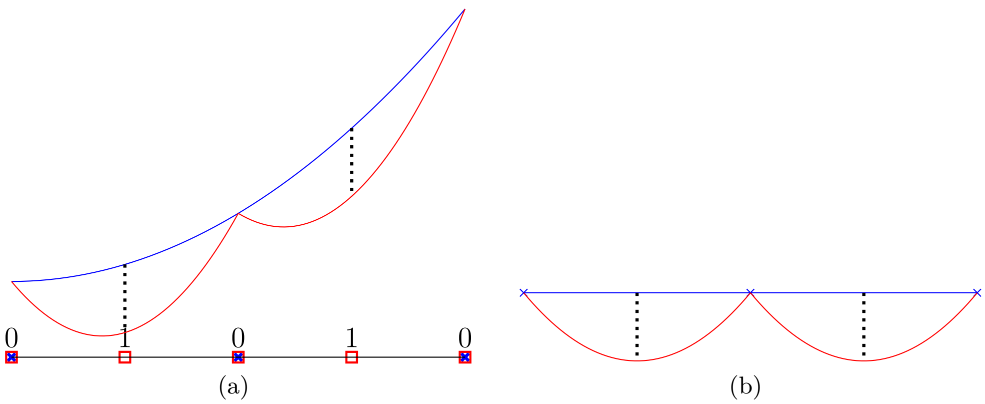


Figure 2.12: In (a) it is shown in blue the polynomial of degree 2 defined in the whole segment and in red the two polynomials of degree 2 living in half of the segment. (b) shows in black the detail of the first order.

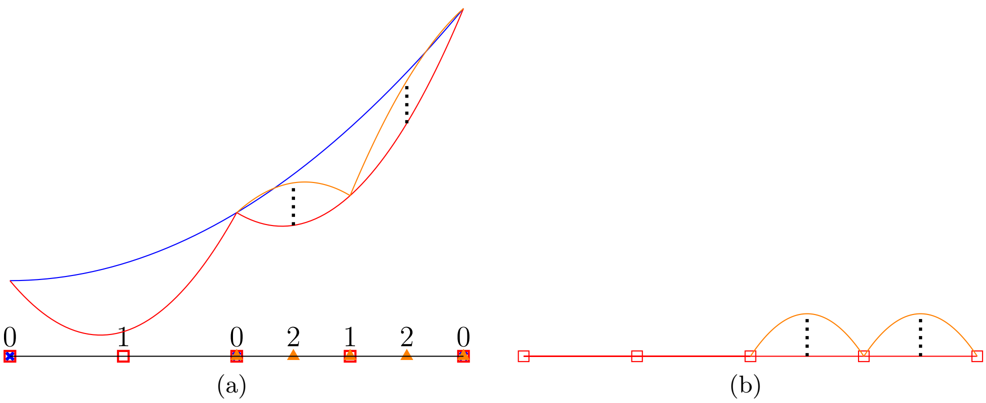


Figure 2.13: In (a) it is shown in blue the polynomial of degree 2 defined in the whole segment and in red the two polynomials of degree 2 living in half of the segment and in orange the polynomials that occur after the second refinement. (b) shows in black the detail of the second order.

The same is for the case with polynomials with degree 3, showed in Figure 2.14 and 2.15.

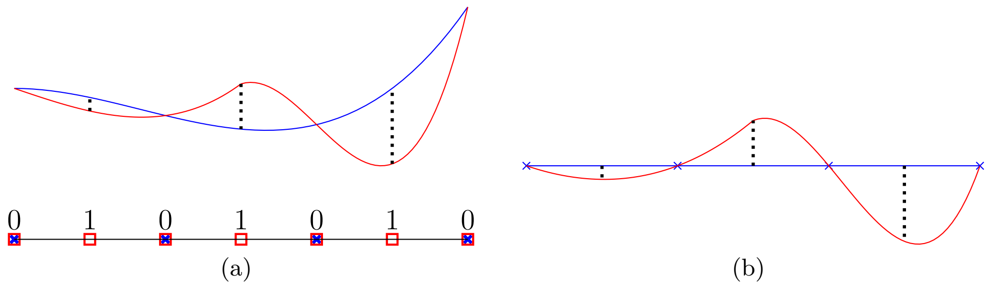


Figure 2.14: In (a) it is shown in blue the polynomial of degree 3 defined in the whole segment and in red the two polynomials of degree 3 living in half of the segment. (b) shows in black the detail of the first order.

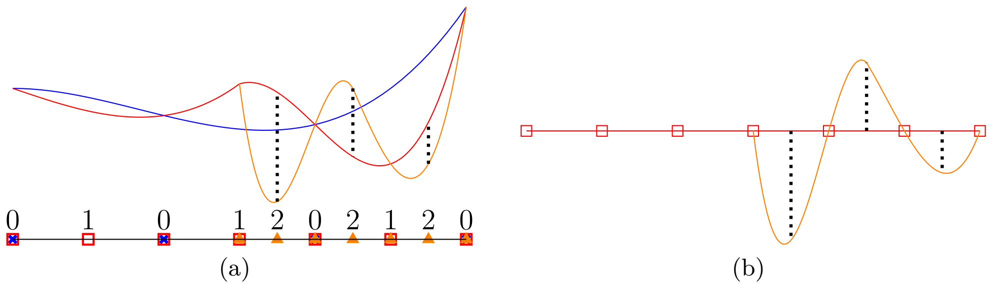


Figure 2.15: In (a) it is shown in blue the polynomial of degree 3 defined in the whole segment and in red the two polynomials of degree 3 living in half of the segment and in orange the polynomials that occur after the second refinement. (b) shows in black the detail of the second order.

2.6.1 Interpolating function in the case of polynomial of degree 3

In order to write the function of the *hierarchical detail of* v we need to determine the polynomial of degree 3, interpolating four nodes.

There are two possible situations. The first one is showed in Fig. 2.16.

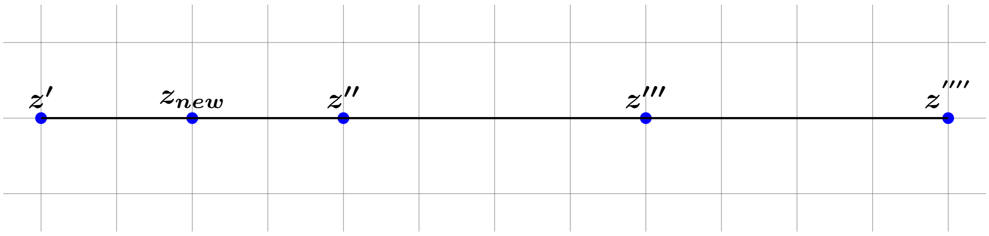


Figure 2.16: This figure shows the position of the nodes $\boldsymbol{z_{new}}$, $\boldsymbol{z'}$, $\boldsymbol{z''}$, $\boldsymbol{z'''}$ and $\boldsymbol{z''''}$.

In this case, we can summarize the relations among these nodes as:

$$\boldsymbol{z'} = \boldsymbol{z_{new}} - \Delta\boldsymbol{z}, \qquad \boldsymbol{z''} = \boldsymbol{z_{new}} + \Delta\boldsymbol{z},$$
$$\boldsymbol{z'''} = \boldsymbol{z_{new}} + 3\Delta\boldsymbol{z}, \qquad \boldsymbol{z''''} = \boldsymbol{z_{new}} + 5\Delta\boldsymbol{z}.$$

It is now possible to find a polynomial of degree 3 interpolating $v(\boldsymbol{z'})$, $v(\boldsymbol{z''})$, $v(\boldsymbol{z'''})$ and $v(\boldsymbol{z''''})$. For this purpose it is convenient to write the polynomial as:

$$\Pi(\boldsymbol{x}) = a\left(\frac{\boldsymbol{x}-\boldsymbol{z_{new}}}{\Delta\boldsymbol{z}}\right)^3 + b\left(\frac{\boldsymbol{x}-\boldsymbol{z_{new}}}{\Delta\boldsymbol{z}}\right)^2 + c\left(\frac{\boldsymbol{x}-\boldsymbol{z_{new}}}{\Delta\boldsymbol{z}}\right) + d. \tag{2.14}$$

We so have

$$v(\boldsymbol{z'}) = a\left(\frac{-\Delta\boldsymbol{z}}{\Delta\boldsymbol{z}}\right)^3 + b\left(\frac{-\Delta\boldsymbol{z}}{\Delta\boldsymbol{z}}\right)^2 + c\left(\frac{-\Delta\boldsymbol{z}}{\Delta\boldsymbol{z}}\right) + d = -a + b - c + d,$$
$$v(\boldsymbol{z''}) = a\left(\frac{\Delta\boldsymbol{z}}{\Delta\boldsymbol{z}}\right)^3 + b\left(\frac{\Delta\boldsymbol{z}}{\Delta\boldsymbol{z}}\right)^2 + c\left(\frac{\Delta\boldsymbol{z}}{\Delta\boldsymbol{z}}\right) + d = a + b + c + d,$$
$$v(\boldsymbol{z'''}) = a\left(\frac{3\Delta\boldsymbol{z}}{\Delta\boldsymbol{z}}\right)^3 + b\left(\frac{3\Delta\boldsymbol{z}}{\Delta\boldsymbol{z}}\right)^2 + c\left(\frac{3\Delta\boldsymbol{z}}{\Delta\boldsymbol{z}}\right) + d = 27a + 9b + 3c + d,$$
$$v(\boldsymbol{z''''}) = a\left(\frac{5\Delta\boldsymbol{z}}{\Delta\boldsymbol{z}}\right)^3 + b\left(\frac{5\Delta\boldsymbol{z}}{\Delta\boldsymbol{z}}\right)^2 + c\left(\frac{5\Delta\boldsymbol{z}}{\Delta\boldsymbol{z}}\right) + d = 125a + 25b + 5c + d.$$

The coefficients obtained are

$$a = \frac{v(\boldsymbol{z''''}) - 3v(\boldsymbol{z'''}) + 3v(\boldsymbol{z''}) - v(\boldsymbol{z'})}{48},$$
$$b = \frac{-v(\boldsymbol{z''''}) + 5v(\boldsymbol{z'''}) - 7v(\boldsymbol{z''}) + 3v(\boldsymbol{z'})}{16},$$
$$c = \frac{-v(\boldsymbol{z''''}) + 3v(\boldsymbol{z'''}) + 21v(\boldsymbol{z''}) - 23v(\boldsymbol{z'})}{48},$$
$$d = \frac{v(\boldsymbol{z''''}) - 5v(\boldsymbol{z'''}) + 15v(\boldsymbol{z''}) + 5v(\boldsymbol{z'})}{16}.$$

Posing the passage by $\boldsymbol{z_{new}}$, the polynomial obtained is

$$\Pi(\boldsymbol{z_{new}}) = \frac{v(\boldsymbol{z''''}) - 5v(\boldsymbol{z'''}) + 15v(\boldsymbol{z''}) + 5v(\boldsymbol{z'})}{16}. \tag{2.15}$$

The other possible situation is showed in Fig. 2.17, with the relations,

$$\boldsymbol{z'} = \boldsymbol{z_{new}} - 3\Delta \boldsymbol{z}, \qquad \boldsymbol{z''} = \boldsymbol{z_{new}} - \Delta \boldsymbol{z},$$
$$\boldsymbol{z'''} = \boldsymbol{z_{new}} + \Delta \boldsymbol{z}, \qquad \boldsymbol{z''''} = \boldsymbol{z_{new}} + 3\Delta \boldsymbol{z}.$$

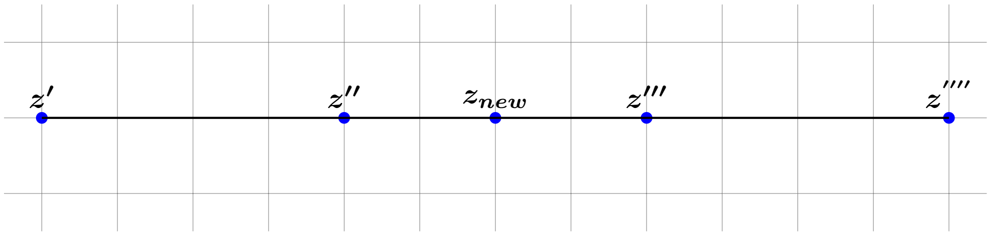


Figure 2.17: This figure shows the position of the nodes $\boldsymbol{z}_{new}$, $\boldsymbol{z'}$, $\boldsymbol{z''}$, $\boldsymbol{z'''}$ and $\boldsymbol{z''''}$.

Here we have

$$\begin{aligned}
v(\boldsymbol{z'}) &= a\left(\frac{-3\Delta \boldsymbol{z}}{\Delta \boldsymbol{z}}\right)^3 + b\left(\frac{-3\Delta \boldsymbol{z}}{\Delta \boldsymbol{z}}\right)^2 + c\left(\frac{-3\Delta \boldsymbol{z}}{\Delta \boldsymbol{z}}\right) + d = -27a + 9b - 3c + d,\\
v(\boldsymbol{z''}) &= a\left(\frac{-\Delta \boldsymbol{z}}{\Delta \boldsymbol{z}}\right)^3 + b\left(\frac{-\Delta \boldsymbol{z}}{\Delta \boldsymbol{z}}\right)^2 + c\left(\frac{-\Delta \boldsymbol{z}}{\Delta \boldsymbol{z}}\right) + d = -a + b - c + d,\\
v(\boldsymbol{z'''}) &= a\left(\frac{\Delta \boldsymbol{z}}{\Delta \boldsymbol{z}}\right)^3 + b\left(\frac{\Delta \boldsymbol{z}}{\Delta \boldsymbol{z}}\right)^2 + c\left(\frac{\Delta \boldsymbol{z}}{\Delta \boldsymbol{z}}\right) + d = a + b + c + d,\\
v(\boldsymbol{z''''}) &= a\left(\frac{3\Delta \boldsymbol{z}}{\Delta \boldsymbol{z}}\right)^3 + b\left(\frac{3\Delta \boldsymbol{z}}{\Delta \boldsymbol{z}}\right)^2 + c\left(\frac{3\Delta \boldsymbol{z}}{\Delta \boldsymbol{z}}\right) + d = 27a + 9b + 3c + d.
\end{aligned}$$

The coefficients now become

$$\begin{aligned}
a &= \frac{v(\boldsymbol{z''''}) - 3v(\boldsymbol{z'''}) + 3v(\boldsymbol{z''}) - v(\boldsymbol{z'})}{48},\\
b &= \frac{v(\boldsymbol{z''''}) - v(\boldsymbol{z'''}) - v(\boldsymbol{z''}) + v(\boldsymbol{z'})}{16},\\
c &= \frac{-v(\boldsymbol{z''''}) + 27v(\boldsymbol{z'''}) - 27v(\boldsymbol{z''}) + v(\boldsymbol{z'})}{48},\\
d &= \frac{-v(\boldsymbol{z''''}) + 9v(\boldsymbol{z'''}) + 9v(\boldsymbol{z''}) - v(\boldsymbol{z'})}{16}.
\end{aligned}$$

As before, posing the passage by $\boldsymbol{z_{new}}$, the polynomial obtained is

$$\Pi(\boldsymbol{z_{new}}) = \frac{-v(\boldsymbol{z''''}) + 9v(\boldsymbol{z'''}) + 9v(\boldsymbol{z''}) - v(\boldsymbol{z'})}{16}. \tag{2.16}$$

2.6.2 A posterirori error analysis in the case of polynomials of degree 3

We want to remark that all the results provided in the case of $\mathbb{V}_{E,2}$ do not depend on the space of polynomial used, but the **Lemma** 2.5.1. In this case we need to re-write the function of hierarchical details. In particular **Definition** 2.5.1 has to be reformulated, using the interpolating function built in the previous section, as follows.

Definition 2.6.1. To each function $v \in \mathbb{V}_{E,3}$ we associate a vector $d(v) = \{d(v;\boldsymbol{z})\}_{\boldsymbol{z}\in\mathcal{N}_E}$ that collects the following values, so called *hierarchical details of* v

$$d(v;\boldsymbol{z}) = \begin{cases} v(\boldsymbol{z}) & \text{if } \boldsymbol{z} \in \mathcal{P}_E, \\ v(\boldsymbol{z}) - \frac{1}{16}v(\boldsymbol{z}'''') + \frac{5}{16}v(\boldsymbol{z}''') - \frac{15}{16}v(\boldsymbol{z}'') - \frac{5}{16}v(\boldsymbol{z}') & \text{if } \boldsymbol{z} \in \mathcal{H}_E/\mathcal{M}_E, \\ v(\boldsymbol{z}) + \frac{1}{16}v(\boldsymbol{z}'''') - \frac{9}{16}v(\boldsymbol{z}''') - \frac{9}{16}v(\boldsymbol{z}'') + \frac{1}{16}v(\boldsymbol{z}') & \text{if } \boldsymbol{z} \in \mathcal{H}_E \cap \mathcal{M}_E, \end{cases} \tag{2.17}$$

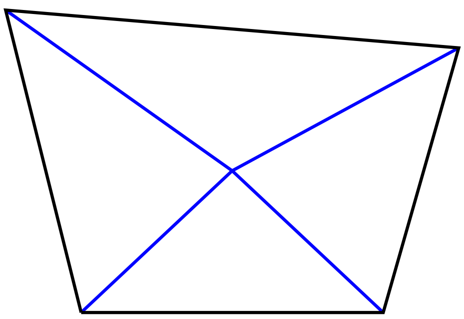
where $\mathcal{M}_E$ is the set of midpoints of edges of E and $\boldsymbol{z}'$, $\boldsymbol{z}''$ and $\boldsymbol{z}'''$, are defined as follows. $\boldsymbol{z}''$ and $\boldsymbol{z}'''$ split into three equal pats the edge with endpoints $\boldsymbol{z}'$ and $\boldsymbol{z}''''$.

By the **Remark** 5, the **Corollary** 2.5.2 holds also in the case of $\mathbb{V}_{E,3}$ and so the stabilization-free a posteriori error estimates (**Corollary** 2.5.5).

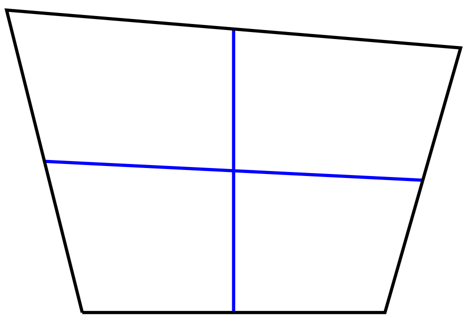
Chapter 3

Adaptive Virtual Element Methods on quadrangles

In the second chapter we have discussed the a posteriori error analysis considering a discretization made only of triangles. We have seen that a fundamental role has been played by the space $\mathbb{V}^0_{\mathcal{T}}$, made of functions in $\mathbb{V}_{\mathcal{T}}$ that are polynomials on each element E of the triangulation. In this chapter we want to extend the previous analysis to a different geometry: a discretization made of convex quadrangles. We here firstly remark that, in order to refine a quadrangle, a possible strategy is to trace the edges connecting the barycenter to each vertices, as it is showed in Figure 3.1 (a). This approach reduces the problem to the case of four triangles. Another strategy consists in tracing the lines connecting the midpoints of two opposite edges forming four new quadrangles, as showed in Figure 3.1 (b). Here we want to analyze this latter situation that brings us to redefine the VEM functional spaces. In particular, we want to define the space $\mathbb{V}^0_{\mathcal{T}}$ containing the functions that are 'polynomials' on each element of the discretization.



(a)



(b)

Figure 3.1: In (a) the refinement reduces the quadrangle to four triangles. Picture (b) shows the refinement reducing the quadrangle to four quadrangles.

The first trivial recall we need to make is that, given a quadrangle and supposing to consider as the space of the degrees of freedom the values of the function at the vertices, the space of the polynomials $\mathbb{P}_1$ is not determined by the values at the vertices of the quadrangle. Indeed, a general polynomial in two dimensions can be written as $p(x, y) = \alpha_1 x + \alpha_2 y + \alpha_3$, where α_i $i = 1,2,3$ are values in $\mathbb{R}$. The space of polynomials of degree 1 has then dimension three, while the set of the values at each vertex of the quadrangles has dimension four.

The idea is so to consider the space of the 'bilinear' functions on each element, where with the term 'bilinear' we mean a function $q(x, y) = \alpha_1 xy + \alpha_2 x + \alpha_3 y + \alpha_4$, linear in x and in y. But this function is not enough to describe the space $\mathbb{V}^0_{\mathcal{T}}$. Indeed, we want that

$$\mathbb{V}^0_{\mathcal{T}} \subseteq \mathbb{V}_{\mathcal{T}}.$$

If, as in the case of the triangles, $\mathbb{V}_{\mathcal{T}}$ contains functions that are polynomials on the boundaries of each element E of the discretization $\mathcal{T}$, then the description of the set of bilinear functions is not adequate. A bilinear function, when restricted to an edge, is not in general a polynomial of degree 1 in the arc-length (we are not requiring the quadrangles to have the edges parallel to the axis). The strategy used to go further this problem is to consider a unitary square as a **reference element**. Since here the edges are parallel to the axis, the bilinear functions restricted to an edge are polynomials, as it will be discussed in details in the following. The idea of the map between the reference element and the physical element is taken from Gordon and Hall [1973] and its properties will be analysed.

Moreover, when a refinement in the sense of Figure 3.1 (b) occurs, we want that the restriction of a bilinear function on a formed element E_i is a bilinear function on E_i. We will show that also this property holds using the reference element.

For clarity's sake, in the following we will discuss the classical Poisson problem, with Dirichlet condition at the boundary of the domain Ω,

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

with $f \in L^2(\Omega)$. The variational formulation reads as

$$\begin{cases} \text{find } u \in \mathbb{V} := H^1_0(\Omega) & \text{such that} \\ a(u, v) = (f, v), & \forall\, v \in \mathbb{V}, \end{cases} \tag{3.2}$$

with $a(u, v) = (\nabla u, \nabla v)$. This problem admits a unique solution, since $a(\cdot, \cdot)$ is continuous and coercive on Ω.

We finally remark that the definition of the Nabla Projector is not trivial and a discussion on it has been presented in the last section and needs further studies.

3.1 To the reference element via the Gordon-Hall function

As the first step, let us consider a single physical quadrangle E of the discretization. We call its vertices $\boldsymbol{P}_i$, $i = 0, \ldots, 3$, displaced as shown in Figure 3.2. We here want to make some assumptions on E. In order to do this, we define the following vectors:

$$\boldsymbol{v}_1 := \boldsymbol{P_1} - \boldsymbol{P_0}, \qquad \boldsymbol{v}_2 := \boldsymbol{P_2} - \boldsymbol{P_0},$$
$$\boldsymbol{v}_3 := \boldsymbol{P_3} - \boldsymbol{P_2}, \qquad \boldsymbol{v}_4 := \boldsymbol{P_3} - \boldsymbol{P_1}.$$

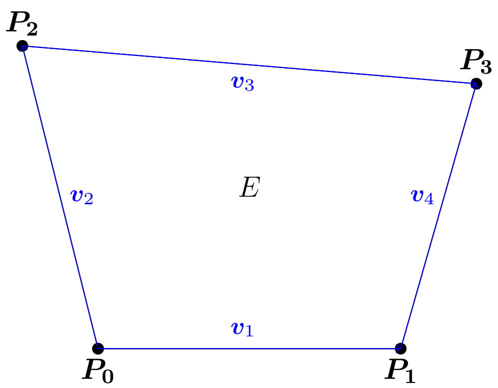


Figure 3.2: This figure shows a generic quadrangle of the partition.

Assumption 3.1.1. Given a partition $\mathcal{T}$, we assume that it is made of quadrangles E that do not degenerate. In particular, we ask the following: for each element $E \in \mathcal{T}$,

- E is convex. This implies that if we consider a point $\boldsymbol{P}$ in the interior of E, it can be written as a convex combination of the vertices of E;
- the angle forming between two adjacent vectors cannot vanish. In formula, there exists a $\rho > 0$ such that

$$\frac{||\boldsymbol{v}_4 \wedge \boldsymbol{v}_1||_2}{||\boldsymbol{v}_4||_2||\boldsymbol{v}_1||_2} \geq \rho \qquad \text{and} \qquad \frac{||\boldsymbol{v}_i \wedge \boldsymbol{v}_{i+1}||_2}{||\boldsymbol{v}_i||_2||\boldsymbol{v}_{i+1}||_2} \geq \rho, \ i = 1, \ 2, \ 3. \tag{3.3}$$

With the symbol '$\wedge$' we are referring to the classical cross product in $\mathbb{R}^3$.

We now want to define another object: the **reference element**. From a geometrical point of view the reference element is a unitary quadrangle $\hat{E} = [0,1]^2$, as showed in Figure 3.3. Inspired by Gordon and Hall [1973], we define a mapping that allows us to pass from $\hat{E}$ to E. We call this function used as the Gordon-Hall's one, defined as follows:

$$\boldsymbol{F} : \ \hat{E} \to E, \ \text{such that}$$

$$\boldsymbol{F}(s,t) = (1-s)(1-t)\boldsymbol{P_0} + s(1-t)\boldsymbol{P_1} + (1-s)t\boldsymbol{P_2} + st\boldsymbol{P_3}. \tag{3.4}$$

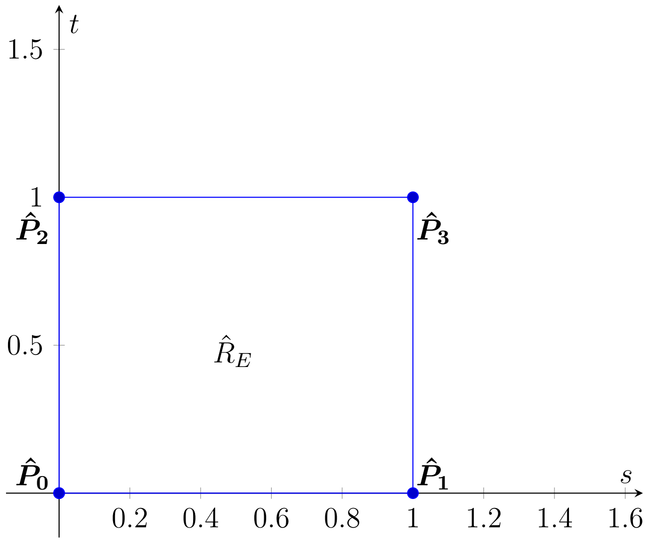


Figure 3.3: This figure shows the unitary quadrangle that has been used as reference element.

3.1.1 Properties of the Gordon-Hall function

In this section we want to discuss some properties of the the Gordon-Hall function as defined in (3.4). First of all, we see that the vertices of $\hat{E}$ correspond to those of E. Indeed

$$\begin{aligned} \boldsymbol{F}(\hat{\boldsymbol{P}}_{\boldsymbol{0}}) &:= \boldsymbol{F}(0,0) = \boldsymbol{P}_{\boldsymbol{0}}, & \boldsymbol{F}(\hat{\boldsymbol{P}}_{\boldsymbol{1}}) &:= \boldsymbol{F}(1,0) = \boldsymbol{P}_{\boldsymbol{1}}, \\ \boldsymbol{F}(\hat{\boldsymbol{P}}_{\boldsymbol{2}}) &:= \boldsymbol{F}(0,1) = \boldsymbol{P}_{\boldsymbol{2}}, & \boldsymbol{F}(\hat{\boldsymbol{P}}_{\boldsymbol{3}}) &:= \boldsymbol{F}(1,1) = \boldsymbol{P}_{\boldsymbol{3}}. \end{aligned} \tag{3.5}$$

Moreover, for instance, the edge with endpoints $\hat{\boldsymbol{P}}_{\boldsymbol{2}}$ and $\hat{\boldsymbol{P}}_{\boldsymbol{3}}$, as taken in Figure 3.3, can be written as

$$\hat{\boldsymbol{P}}_{\boldsymbol{3}} - \hat{\boldsymbol{P}}_{\boldsymbol{2}} = \{(s,1) : s \in [0,1]\}\,.$$

This edge corresponds to the edge with endpoints $\boldsymbol{P}_2$ and $\boldsymbol{P}_3$. Indeed, the restriction of function $\boldsymbol{F}$ to the edge with endpoints $\hat{\boldsymbol{P}}_{\boldsymbol{2}}$ and $\hat{\boldsymbol{P}}_{\boldsymbol{3}}$ results

$$\boldsymbol{F}(s,1) = (1-s)\boldsymbol{P}_{\boldsymbol{2}} + s\boldsymbol{P}_3, \qquad \text{with } s \in [0,1].$$

These remarks suggest us that $\boldsymbol{F}$ is bijective. In order to prove this we compute the jacobian matrix of $\boldsymbol{F}$.

$$\boldsymbol{J}_{\boldsymbol{F}(s,t)} = \begin{bmatrix} \frac{\partial}{\partial s}\boldsymbol{F}(s,t) \\ \\ \frac{\partial}{\partial t}\boldsymbol{F}(s,t) \end{bmatrix}, \text{ with}$$

$$\begin{aligned}
\frac{\partial}{\partial s}\boldsymbol{F}(s,t) &= -(1-t)\boldsymbol{P_0} + (1-t)\boldsymbol{P_1} - t\boldsymbol{P_2} + t\boldsymbol{P_3} \\
&= (1-t)(\boldsymbol{P_1} - \boldsymbol{P_0}) + t(\boldsymbol{P_3} - \boldsymbol{P_2}) \\
&= (1-t)\boldsymbol{v}_1 + t\boldsymbol{v}_3; \\
\frac{\partial}{\partial t}\boldsymbol{F}(s,t) &= -(1-s)\boldsymbol{P_0} - s\boldsymbol{P_1} + (1-s)\boldsymbol{P_2} + s\boldsymbol{P_3} \\
&= (1-s)(\boldsymbol{P_2} - \boldsymbol{P_0}) + s(\boldsymbol{P_3} - \boldsymbol{P_1}) \\
&= (1-s)\boldsymbol{v}_2 + s\boldsymbol{v}_4.
\end{aligned}$$

We can now compute the determinant of $\boldsymbol{J_F}$ showing that it is lower bounded by a quantity that does not vanish.

$$\begin{aligned}
|\det(\boldsymbol{J_F})| &= ||\,((1-t)\,\boldsymbol{v}_1 + t\boldsymbol{v}_3) \wedge ((1-s)\boldsymbol{v}_2 + s\boldsymbol{v}_4)\,||_2 \\
&= ||(1-t)(1-s)\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 + s(1-t)\boldsymbol{v}_1 \wedge \boldsymbol{v}_4 + t(1-s)\boldsymbol{v}_3 \wedge \boldsymbol{v}_2 + st\boldsymbol{v}_3 \wedge \boldsymbol{v}_4||_2.
\end{aligned}$$

From the choice of the vectors $\boldsymbol{v}_i$, $i = 1 \dots 4$ (see Figure 3.2) all the modules of the cross products are positive. Moreover, they are aligned in the direction that goes out of the plane; then, by using the **Assumption** 3.1.1 on the discretization,

$$\begin{aligned}
|\det(\boldsymbol{J}_F)| &= (1-t)(1-s)||\boldsymbol{v}_1 \wedge \boldsymbol{v}_2||_2 + s(1-t)||\boldsymbol{v}_1 \wedge \boldsymbol{v}_4||_2 + t(1-s)||\boldsymbol{v}_3 \wedge \boldsymbol{v}_2||_2 \\
&\quad + st||\boldsymbol{v}_3 \wedge \boldsymbol{v}_4||_2 \\
&\geq \rho((1-t)(1-s)||\boldsymbol{v}_1||_2||\boldsymbol{v}_2||_2 + s(1-t)||\boldsymbol{v}_1||_2||\boldsymbol{v}_4||_2 + t(1-s)||\boldsymbol{v}_3||_2||\boldsymbol{v}_2||_2 \\
&\quad + st||\boldsymbol{v}_3||_2||\boldsymbol{v}_4||_2) \\
&= \rho\,(((1-t)||\boldsymbol{v}_1||_2 + t||\boldsymbol{v}_3||_2)\,((1-s)||\boldsymbol{v}_2||_2 + s||\boldsymbol{v}_4||_2)) \\
&\geq \rho \min_{t\in[0,1]}\{(1-t)||\boldsymbol{v}_1||_2 + t||\boldsymbol{v}_3||_2\} \min_{s\in[0,1]}\{(1-s)||\boldsymbol{v}_2||_2 + s||\boldsymbol{v}_4||_2\} \\
&= \rho\min\{||\boldsymbol{v}_1||_2, ||\boldsymbol{v}_3||_2\}\min\{||\boldsymbol{v}_2||_2, ||\boldsymbol{v}_4||_2\} > 0.
\end{aligned}$$

We remark that we have bounded the absolute value of the determinant of E with a sort of minimal area. This is coherent to the case of the parallelogram whose edges are aligned only according to two different directions $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ and the area can be computed as

$$\text{Area}_{\square} = ||\boldsymbol{w}_1 \wedge \boldsymbol{w}_2||_2 = ||\boldsymbol{w}_1||_2||\boldsymbol{w}_2||_2|\sin\theta|,$$

where θ is the angle between the two vectors.

We see also that the $\det(\boldsymbol{J_F})$ can be bounded from above as follows:

$$\begin{aligned}
|\det(\boldsymbol{J}_F)| &= (1-t)(1-s)||\boldsymbol{v}_1 \wedge \boldsymbol{v}_2||_2 + s(1-t)||\boldsymbol{v}_1 \wedge \boldsymbol{v}_4||_2 + t(1-s)||\boldsymbol{v}_3 \wedge \boldsymbol{v}_2||_2 \\
&\quad + st||\boldsymbol{v}_3 \wedge \boldsymbol{v}_4||_2 \\
&\leq (1-t)(1-s)||\boldsymbol{v}_1||_2||\boldsymbol{v}_2||_2 + s(1-t)||\boldsymbol{v}_1||_2||\boldsymbol{v}_4||_2 + t(1-s)||\boldsymbol{v}_3||_2||\boldsymbol{v}_2||_2 \\
&\quad + st||\boldsymbol{v}_3||_2||\boldsymbol{v}_4||_2 \\
&\leq \max_{t\in[0,1]}\{(1-t)||\boldsymbol{v}_1||_2 + t||\boldsymbol{v}_3||_2\} \max_{s\in[0,1]}\{(1-s)||\boldsymbol{v}_2||_2 + s||\boldsymbol{v}_4||_2\} \\
&= \max\{||\boldsymbol{v}_1||_2, ||\boldsymbol{v}_3||_2\} \max\{||\boldsymbol{v}_2||_2, ||\boldsymbol{v}_4||_2\}.
\end{aligned} \tag{3.6}$$

We notice that the determinant of the jacobian matrix is fundamental to pass from an integral quantity computed on E to one computed on $\hat{R}_E$:

$$\int_E v = \int_{\hat{R}_E} v \circ \boldsymbol{F} \det(\boldsymbol{J_F}).$$

The considerations on the jacobian matrix allow us to state that $\boldsymbol{F}$ is injective. Indeed, if $\boldsymbol{F}(s_0, t_0) = \boldsymbol{F}(s_1, t_1)$, then using the *Mean value theorem* we know that there exists a point (s^*, t^*) such that

$$0 = \boldsymbol{F}(s_0, t_0) - \boldsymbol{F}(s_1, t_1) = \boldsymbol{J}_F(s^*, t^*) \begin{bmatrix} s_0 - s_1 \\ t_1 - t_0 \end{bmatrix}.$$

Since we have proved that $\det(\boldsymbol{J_F}) \neq 0$, necessarily $s_0 = s_1$ and $t_0 = t_1$ $\forall$ $(s_0, t_0), (s_1, t_1) \in [0,1]^2$.

Moreover, by the **Assumption** 3.1.1 on the discretization, E is convex. This implies that a point in E can be written as a convex combination of the vertices of the quadrangle. Then $\forall \boldsymbol{P} \in E$, there exist $(s^*, t^*) \in [0,1]^2$ such that

$$\boldsymbol{P} = (1-s^*)(1-t^*)\boldsymbol{P}_0 + s^*(1-t^*)\boldsymbol{P_1} + t^*(1-s^*)\boldsymbol{P_2} + s^*t^*\boldsymbol{P_4} = \boldsymbol{F}(s^*, t^*).$$

$\boldsymbol{F}$ is surjective, and so bijective.

3.2 Definition of the Functional Spaces

During the refinement procedure some hanging nodes can be generated. We define a reference element $\hat{R}_E$ that takes into account the presence of hanging nodes in E. The novelty here is that the hanging nodes are transported to the reference element via $\boldsymbol{F}^{-1}$, as showed in Figure 3.4.

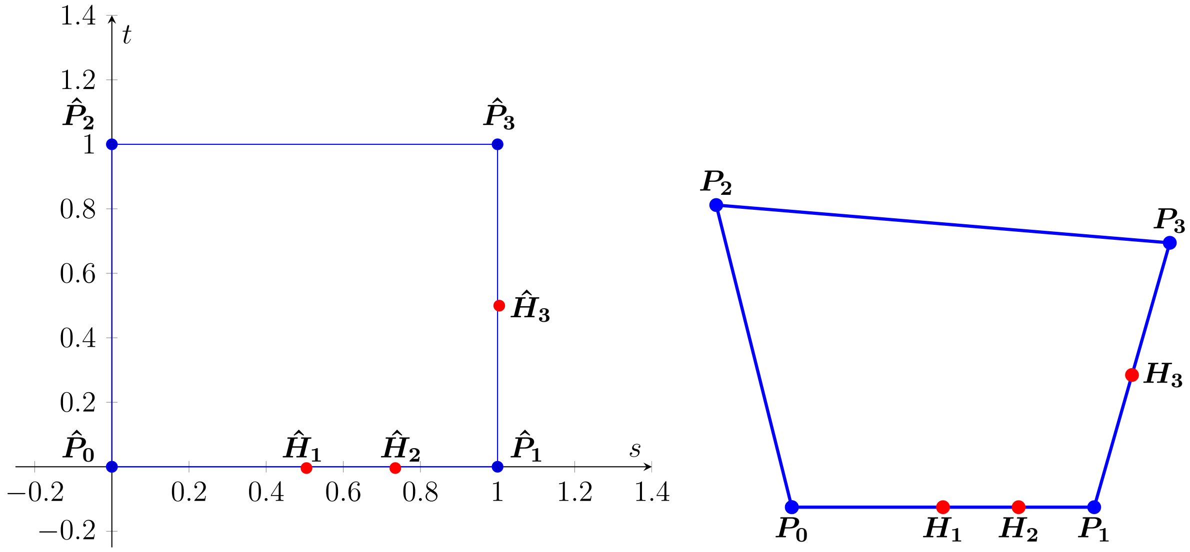


Figure 3.4: In the first picture it is shown the hanging nodes on the reference element, while in the second the corresponding hanging nodes are on the physical element.

We are now ready to define the functional spaces on $\hat{R}_E$. As we have already done in the introduction of VEM, we firstly define the space on the boundary of the reference element. We call

$$\mathbb{V}_{\partial \hat{R}_E} := \left\{ \hat{v} \in C^0 \left(\partial \hat{R}_E \right) : \ \hat{v}|_{\hat{e}} \in \mathbb{P}_1(\hat{e}), \ \forall \hat{e} \in \mathcal{E}(\hat{R}_E) \right\}.$$

This space contains the continuous functions living on the boundary of $\hat{R}_E$ that are polynomials of degree 1 on each edge. In the interior of $\hat{R}_E$, we define

$$\mathbb{V}_{\hat{R}_E} := \left\{ \hat{v} \in H^1 \left(\hat{R}_E \right) : \ \hat{v}|_{\partial \hat{R}_E} \in \mathbb{V}_{\partial \hat{R}_E} \text{ and } \hat{\Delta}_{(s,t)} \hat{v} = 0 \text{ in } \ \hat{R}_E \right\}.$$

These two definitions are the same of the classical VEM's ones given in Beirão da Veiga et al. [2013], but here the functions live on the reference elements.

The set of bilinear functions on $\hat{E}$,

$$\hat{\mathbb{Q}}_1(\hat{E}) = \{ \hat{q}(s,t) = \alpha_1 st + \alpha_2 s + \alpha_3 t + \alpha_4 : \ \alpha_i \in \mathbb{R}, \ i = 1, \ldots, 4 \},$$

is contained in $\mathbb{V}_{\hat{R}_E}$. Indeed, if $\hat{q} \in \hat{\mathbb{Q}}_1(\hat{E})$, then $\hat{\Delta}_{(s,t)} \hat{q} = 0$ and the restriction on each edge of the boundary is a linear function:

$$\hat{q}(0,t) = \alpha_3 t + \alpha_4 \qquad\qquad \hat{q}(1,t) = (\alpha_1 + \alpha_3) t + \alpha_2 + \alpha_4$$
$$\hat{q}(s,0) = \alpha_2 s + \alpha_4 \qquad\qquad \hat{q}(s,1) = (\alpha_1 + \alpha_2) s + \alpha_3 + \alpha_4,$$

defined by the four values α_i, $i = 1,\ldots,4$. A possible choice to determine the values α_i is to take the values of the functions at the four vertices. A general bilinear function on $\hat{E}$ can so be written as

$$\hat{q}(s,t) = (1-s)(1-t)\ \beta_0 + s(1-t)\beta_1 + (1-s)t\beta_2 + st\beta_3, \tag{3.7}$$

where we have defined

$$\beta_0 := \hat{q}(0,0), \qquad \beta_1 := \hat{q}(1,0), \qquad \beta_2 := \hat{q}(0,1), \qquad \beta_3 := \hat{q}(1,1).$$

We are now ready to define the functional spaces on E. We have

$$\mathbb{V}_{\partial E} := \left\{ v \in C^0(\partial E) : v|_e \in \mathbb{P}_1(e),\ \forall e \in \mathcal{E}(E) \right\} \tag{3.8}$$
$$= \left\{ v \in C^0(\partial E) : \hat{v} = v \circ \boldsymbol{F} \in \mathbb{V}_{\partial \hat{R}_E} \right\},$$
$$\mathbb{V}_E := \left\{ v \in H^1(E) :\ v|_{\partial E} \in \mathbb{V}_{\partial E} \text{ and } \hat{v} := v \circ \boldsymbol{F} \in \mathbb{V}_{\hat{R}_E} \right\}. \tag{3.9}$$

The difference with respect to the classical definitions of VEM is that the Laplacian is not required to vanish on the element, but on the reference element.

The dimension of $\mathbb{V}_E$ depends on the number of hanging nodes. It is equal to the number of hanging nodes plus four (the number of the geometrical vertices). As set of the degrees of freedom we take the set of the values of the function at these nodes.

As discussed in the first chapter, when a finite element is defined, the need of verifying if the degrees of freedom are unisolvent for $\mathbb{V}_E$ arises.

Proposition 3.2.1. *Let E be a quadrangle and let $\mathbb{V}_E$ be the space defined in (3.9). The degrees of freedom defined as the set of the values at the nodes of the element is unisolvent for $\mathbb{V}_E$.*

Proof. Let v be in $\mathbb{V}_E$, such that $v(\boldsymbol{P}_i) = 0$, $\forall i = 0,\ldots,3$ and possibly in some other hanging nodes formed on the geometric edges of $\hat{E}$. From (3.5), we have also that $v(\boldsymbol{P}_i) = v(\boldsymbol{F}(\hat{\boldsymbol{P}}_i)) = \hat{v}(\hat{\boldsymbol{P}}_i) = 0$. It implies that we have a function $\hat{v}$ which is equals zero in the nodes on the boundary of the reference element and whose Laplacian vanishes. The space defined on the reference element is the same of the classical VEM's one. This implies from **Proposition** 1.3.1 that $\hat{v} = 0$. Since $\boldsymbol{F}$ is a bijection, then also v is null. □

We define the space functions that are bilinear function on $\hat{E}$ as the function 'transported' on the reference element:

$$\mathcal{Q}_1(E) := \left\{ q \in H^1(E) :\ \hat{q} := q \circ \boldsymbol{F} \in \hat{\mathbb{Q}}_1(\hat{E}) \right\}. \tag{3.10}$$

We remind here that, if $\hat{q} \in \hat{\mathbb{Q}}_1(\hat{E})$, then it is linear on the boundary of $\hat{E}$. By the already discussed property of the Gordon-Hall function, the function $\boldsymbol{F}$ does not change the property of linearity along the boundaries. This means that $q = \hat{q} \circ \boldsymbol{F}^{-1}$ is linear on the boundary of E. We have so obtained the following inclusion:

$$\mathcal{Q}_1(E) \subseteq \mathbb{V}_E. \tag{3.11}$$

Proposition 3.2.2. *Let $\mathbb{P}_1(E)$ the space of polynomials of degree 1 defined on E and let $\mathcal{Q}_1(E)$ be defined in (3.10). The following inclusion holds:*

$$\mathbb{P}_1(E) \subset \mathcal{Q}_1(E). \tag{3.12}$$

Proof. Let's take a polynomial $p \in \mathbb{P}_1(E)$ that can be written as $p(x, y) = \alpha x + \beta y + \gamma$. On the reference element p becomes $\hat{p}(s, t) = p \circ \boldsymbol{F}(s, t) = \alpha F_1(s, t) + \beta F_2(s, t) + \gamma$, where $\boldsymbol{F}(s, t) = (F_1(s, t), F_2(s, t))$. Since $\boldsymbol{F}(s, t)$ is bilinear, then $F_i(s, t)$, $i = 1{,}2$ are scalar bilinear functions. This implies that $\hat{p} \in \mathbb{Q}_1(\hat{E})$ and so $p \in \mathcal{Q}_1(E)$. □

Remark 6. Let $a^E(\cdot, \cdot)$ be the restriction of the bilinear form $a(\cdot, \cdot)$, defined in (3.2), to the element E. The degrees of freedom defined are enough to compute $a^E(u, p)$ when $p \in \mathbb{P}_1(E)$ and $u \in \mathbb{V}_E$. Indeed,

$$a^E(u, p) = \int_E \nabla u \nabla p = -\int_E u \Delta p + \int_{\partial E} u \frac{\partial p}{\partial \boldsymbol{n}}, \tag{3.13}$$

requires only the value of u on the boundary of E, since $\Delta p = 0$ when p is a polynomial of degree 1. We want to underline here that, despite the definition $\mathbb{V}_E$ does not require that $\Delta v = 0$ on E as in the classical VEM, the previous integral can still be computed.

We now want to define a projector, Π_E^∇, that 'brings' a function from $\mathbb{V}_E$ to the space of polynomials. This definition derives directly from the work of Beirão da Veiga et al. [2013] for the classical VEM space. This choice is not optimal. It does not allow us to directly use the proofs contained in Beirão da Veiga et al. [2021]. We will highlight the points where the proofs fail and in the last section we propose other projectors, showing their strengths and weaknesses.

By now define $\Pi_E^\nabla : \mathbb{V}_E \to \mathbb{P}_1(E)$ that guarantees the following system to have a unique solution:

$$\begin{cases} \int_E \nabla \Pi_E^\nabla v \cdot \nabla q = \int_E \nabla v \cdot \nabla q & \forall q \ \in \mathbb{P}_1(E) \\ \\ \overline{\Pi_E^\nabla v} = \overline{v}, & \forall v \ \in \mathbb{V}_E, \end{cases} \tag{3.14}$$

where, if φ is a smooth function and $\{V_i\}_{i=1\ldots n}$ the set of vertices of E, we define

$$\overline{\varphi} := \frac{1}{n} \sum_{i=1}^{n} \varphi(V_i).$$

With this definition of the projector Π_E^∇, we have that if $p \in \mathbb{P}_1(E)$, then it holds:

$$\Pi_E^\nabla p = p, \tag{3.15}$$

as in the case of the classical VEM in (1.17). Moreover, the system (3.14) states that on each element of the discretization E we have

$$a^E(\Pi_E^\nabla v, q) = a^E(v, q), \tag{3.16}$$

As already done in the case of the classical VEM, we need a stabilization term in order to define the approximation bilinear form, $\mathcal{B}^E(\cdot,\cdot):\mathbb{V}_E\times\mathbb{V}_E\to\mathbb{R}$,

$$\mathcal{B}^E(v,w):=a^E(\Pi_E^\nabla v,\Pi_E^\nabla w)+s_E(v-\Pi_E^\nabla v,w-\Pi_E^\nabla w), \tag{3.17}$$

where $s_E:\mathbb{V}_E\times\mathbb{V}_E\to\mathbb{R}$ is a symmetric bilinear form such that

$$c_s|v|_{1,E}^2\le s_E(v,v)\le C_s|v|_{1,E}^2, \qquad \forall v\in\mathbb{V}_E/\mathbb{R}, \tag{3.18}$$

and c_s and C_s are positive constants independent of E and h_E

From the definition of $\mathbb{V}_E$ (3.9), we can now define the space $\mathbb{V}_\mathcal{T}$ on the whole discretization $\mathcal{T}$ as

$$\mathbb{V}_\mathcal{T}:=\left\{v\in H_0^1(\Omega):\ v|_E\in\mathbb{V}_E,\ \forall E\in\mathcal{T}\right\}, \tag{3.19}$$

then, summing on each element E of the discretization, we define the bilinear form $\mathcal{B}_\mathcal{T}(\cdot,\cdot):\mathbb{V}_\mathcal{T}\times\mathbb{V}_\mathcal{T}\to\mathbb{R}$ as

$$\mathcal{B}_\mathcal{T}(v,w):=a_\mathcal{T}(v,w)+\gamma S_\mathcal{T}(v,w), \tag{3.20}$$

where

$$a_\mathcal{T}(v,w):=\sum_{E\in\mathcal{T}}a^E(\Pi_E^\nabla v,\Pi_E^\nabla w),\qquad S_\mathcal{T}(v,w):=\sum_{E\in\mathcal{T}}s_E(v-\Pi_\mathcal{T}^\nabla v,w-\Pi_\mathcal{T}^\nabla w),$$

$\Pi_\mathcal{T}^\nabla$ is the operator that restricts to E is Π_E^∇ and $\gamma\ge\gamma_0$ fixed is a stabilization constant independent of $\mathcal{T}$.

Since (3.15) and (3.16) hold, following the same proof in the case of the classical VEM space (see Beirão da Veiga et al. [2013]), $\mathcal{B}_\mathcal{T}(\cdot,\cdot)$ satisfies the *k-Consistency* and the *Stability* properties and then the discrete problem admits a unique solution. Indeed the **Theorem** 1.2.1 holds and we have the following inequality

$$|u-u_h|\lesssim|u-u_I|_1+|u-u_\pi|_{1,\mathcal{T}}+||f-f_h||_{\mathbb{V}'}, \tag{3.21}$$

where u_h is the unique solution solving

$$\mathcal{B}_\mathcal{T}(u_h,v_h)=\langle f_h,v_h\rangle \qquad \forall v_h\in\mathbb{V}_\mathcal{T},$$

$u_I\in\mathbb{V}_\mathcal{T}$ is an approximation of u and u_π is an approximation of u that is piece-wise in $\mathbb{P}_1$ and

$$\langle f_h,v_h\rangle=\sum_{E\in\mathcal{T}_h}\int_E f\Pi_E^\nabla v_h, \qquad \forall v_h\in\mathbb{V}_\mathcal{T}.$$

For the construction of the right-hand side, we here have

$$\langle f_h,v_h\rangle-(f,v_h)=\sum_{E\in\mathcal{T}_h}\int_E f(\Pi_E^\nabla v_h-v_h) \tag{3.22}$$

$$\le\sum_{E\in\mathcal{T}_h}h_E^2|f|_{1,E}|v_h|_{1,E}\lesssim h\left(\sum_{E\in\mathcal{T}_h}|f|^2\right)^{1/2}|v_h|_1. \tag{3.23}$$

Then, we obtain that

$$||f_h - f||_{\mathbb{V}'_{\mathcal{T}}} \lesssim h \left(\sum_{E\in\mathcal{T}_h} |f|^2 \right)^{1/2} . \tag{3.24}$$

We remark that **Assumption** 1.4.1 holds by the definition of quadrangles. As done in the case of the classical VEM, we apply the Scott-Dupont theory (Brenner and Scott [2008]) and **Proposition** 1.4.2 and **Proposition** 1.4.3 hold. We then obtain that:

$$|u - u_h|_{1,\Omega} \lesssim h|u|_{2,\Omega},$$

with $h = \max_{E\in\mathcal{T}}\{h_E\}$.

3.3 The *enhanced* definition of $\mathbb{V}_E$

In the discussion of the bilinear form of a general symmetric elliptic problem, it has been necessary to introduce the *enhanced* definition of the VEM space. This space plays an important role in a lot of other proofs.

In this section we want to introduce an *enhanced* definition of $\mathbb{V}_E$ as done with the classical virtual elements in Ahmad et al. [2013]. In particular, we want to define a new space, here denoted as $\mathbb{W}_E$, such that the moments of $\Pi^\nabla_E v$ and v coincide for all $v \in \mathbb{W}_E$. The new definition passes through two steps: we firstly enlarge $\mathbb{V}_E$ and then we restrict it.

- **The enlargement.** We define the following spaces:

$$\tilde{\mathbb{V}}_{\hat{R}_E} := \left\{ \hat{v} \in H^1(\hat{R}_E) : \hat{v}|_{\partial\hat{R}_E} \in \mathbb{V}_{\partial\hat{R}_E} \text{ and } \hat{\Delta}_{(s,t)}\hat{v} \in \mathbb{P}_1(\hat{E}) \right\},$$

$$\tilde{\mathbb{V}}_E := \left\{ v : v|_{\partial E} \in \mathbb{V}_{\partial E} \text{ and } \hat{v} = v \circ \boldsymbol{F} \in \tilde{\mathbb{V}}_{\hat{R}_E} \right\}.$$

  In other terms we have introduced three new degrees of freedom on $\hat{\Delta}_{(s,t)}\hat{v}$.

- **The restriction.** We define $\mathbb{W}_E$ as the subspace of $\tilde{\mathbb{V}}_E$, such that for all the elements $v \in \mathbb{W}_E$ it holds:

$$\int_E (v - \Pi^\nabla_E v)p = 0, \qquad \forall\, p \in \mathbb{P}_1(E). \tag{3.25}$$

We now need to verify that the newly defined space $\mathbb{W}_E$ has the same dimension of $\mathbb{V}_E$. This means that the three new conditions on the moments of the restriction step are linearly independent. In order to show this via numerical experiments, we firstly define the followings:

- a basis $\{\hat{p}_i\}_{i=1,2,3}$ of $\mathbb{P}_1(\hat{E})$;

- the set of functions $\{\hat{v}_i\}_{i=1,2,3}$ as the solutions of the classical Poisson problem with Dirichlet boundary conditions and $\hat{p}_i$ as a forcing term.

$$\begin{cases} -\hat{\Delta}_{(s,t)}\hat{v}_i = \hat{p}_i & \text{in } \hat{E}, \\ \hat{v}_i = 0 & \text{on } \partial\hat{E}; \end{cases} \tag{3.26}$$

- the set of functions $\{v_i\}_{i=1,2,3}$, such that $\hat{v}_i = v_i \circ \boldsymbol{F}$;
- a basis $\{q_j\}_{j=1,2,3}$ of $\mathbb{P}_1(E)$;
- the 3×3 matrix $\boldsymbol{A}$ whose entries are computed as

$$\{\boldsymbol{A}\}_{i,j} = \int_E v_i q_j. \tag{3.27}$$

We need to show that $v = 0$ if and only if $\int_E vq = 0$, $\forall q \in \mathbb{P}_1(E)$. The first implication is trivial because clearly if $v = 0$, then the integral is 0. The other implication can be proved if we show that for a certain choice of $\{\hat{p}_i\}$ and $\{q_j\}$ the matrix $\boldsymbol{A}$ is not singular, meaning that if we want to find $\boldsymbol{\lambda} \in \mathbb{R}^3$ such that

$$\boldsymbol{A\lambda} = \boldsymbol{0},$$

then $\boldsymbol{\lambda} = \boldsymbol{0}$.

Indeed, any function $v \in \mathbb{W}_E$ whose Laplacian on the reference element is a polynomial can be written as $v = \sum_{i=1}^{3} \lambda_i v_i$, for some $\lambda_i \in \mathbb{R}$. If matrix $\boldsymbol{A}$ is not singular then $\forall j$

$$\int_E vq_j = \int_E \sum_i \lambda_i v_i q_j = \sum_i \lambda_i \int_E v_i q_j = 0,$$

implying that $\lambda_i = 0,\ i = 1,2,3$, and $v = 0$.

3.3.1 The numerical experiments

Here we present the following numerical experiments showing that matrix $\boldsymbol{A}$ is not singular. As the set of $\{\hat{p}_i\}$, we have chosen

$$\hat{p}_1 = 1 \qquad \hat{p}_2 = 2s - 1 \qquad \hat{p}_3 = 2t - 1.$$

The solutions of (3.26) have been computed with the software MATLAB, using a triangulation on the unitary quadrangle and the classical Courant element of order 1, $\mathbb{P}_1$. The solutions are showed in Figure 3.5.

In order to compute matrix $\boldsymbol{A}$ we previously need to define the set of polynomials q_i that forms a basis in $\mathbb{P}_1(E)$. The basis has been chosen as follows:

$$q_1(x,y) = c, \qquad q_2(x,y) = a_2x + b_2y + c, \qquad q_3(x,y) = a_3x + b_3y + c.$$

The first polynomial is constant on E. The same constant has been used for the other two polynomials; this guarantees us that the three polynomials are linearly independent.

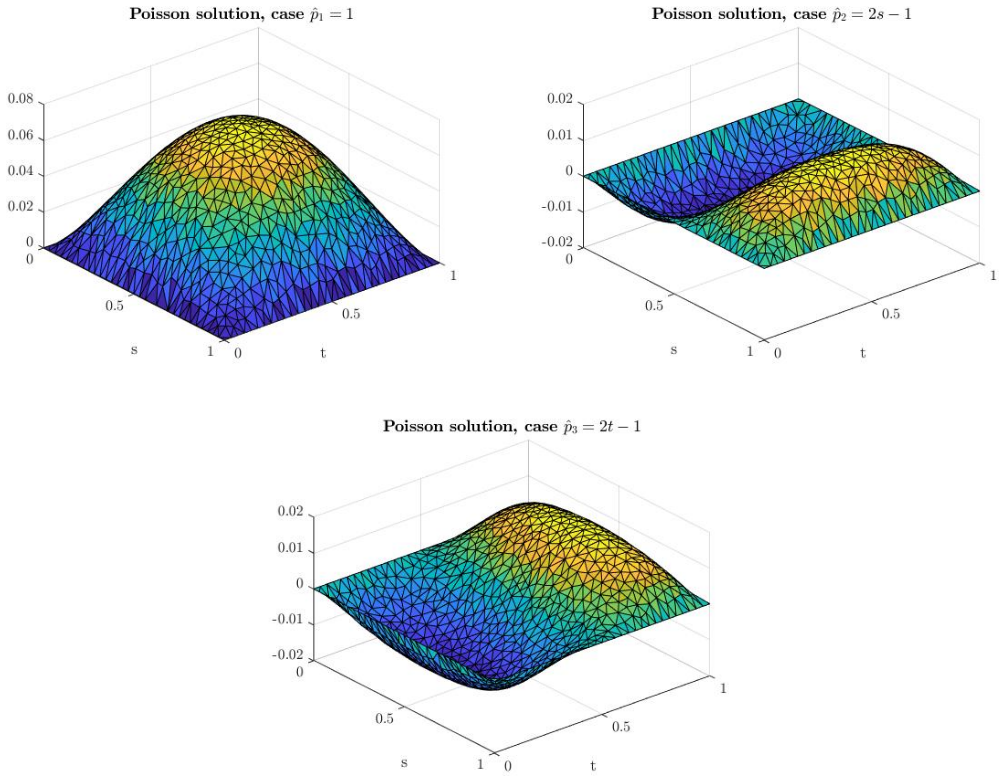


Figure 3.5: The figure shows the numerical solutions of problem (3.26) for different choices of the forcing term, $\hat{p}_i$. They are obtained using the Courant elements of order one.

The coefficients a_2 and b_2 are chosen such that q_2 vanishes when restricted to the segment with endpoints the midpoints $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_3$ of two opposite edges, as shown in Figure 3.6. Analogously, the coefficients a_3 and b_3 can be found imposing that q_3 has to vanish along the segment connecting the other two midpoints.

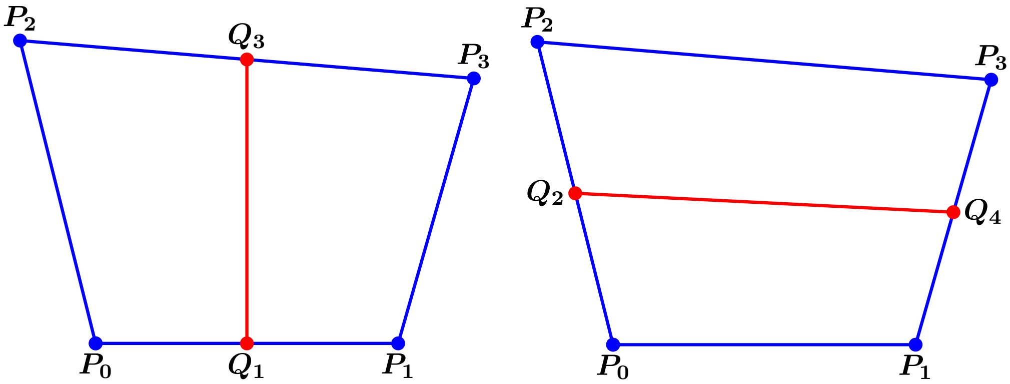


Figure 3.6: In the first picture it is shown the edge along which the polynomial q_2 vanishes. The second picture shows the edge where $q_3 = 0$.

Finally, in order to build the matrix $\boldsymbol{A}$, we compute the integrals (3.27), passing to the reference element.

$$\int_E v_i q_j = \int_E \hat{v}_i \hat{q}_j \det(\boldsymbol{J_F}) = \int_E \hat{v}_i q_j(\boldsymbol{F}(s,t)) \det(\boldsymbol{J_F}).$$

Since our purpose is to verify that the determinant of $\boldsymbol{A}$ is not zero, we have rescaled the polynomials q_j and the functions v_i with respect to their norms on E. In this way, the previous integrals are independent of the area of E.

As first experiment, we tested the position of points such that $\boldsymbol{F}$ becomes the identity function. With points $\boldsymbol{P}_i$,

$$\boldsymbol{P}_0 = (0,0) \qquad \boldsymbol{P}_1 = (1,0) \qquad \boldsymbol{P}_2 = (0,1) \qquad \boldsymbol{P}_3 = (1,1);$$

the matrix obtained is

$$\boldsymbol{A} = \begin{bmatrix} 0.8509 & 0.0003 & 0.0003 \\ 0.0001 & 0.7952 & 0.0001 \\ 0.0001 & 0.0001 & 0.7952 \end{bmatrix},$$

diagonally dominant by rows, the determinant is 0.5380. Let's now consider the quadrangles that present an acute angle. In particular, we fix the points $\boldsymbol{P_0}, \boldsymbol{P_1}$ and $\boldsymbol{P_2}$ and let the point $\boldsymbol{P}_3$ be $\boldsymbol{P_3} = (2^n, 2^n)$. Geometrically, we are considering the situation showed in Figure 3.7.

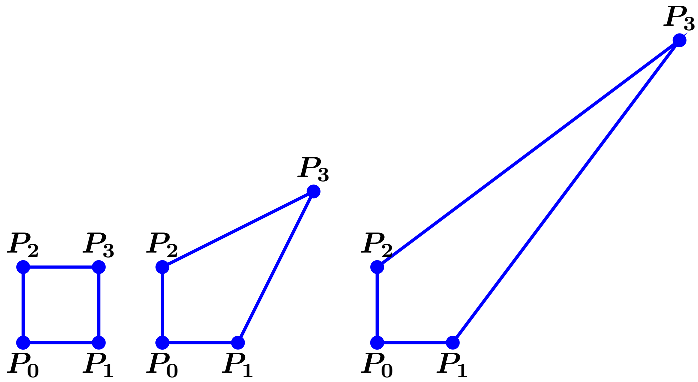


Figure 3.7: This picture shows the first three cases tested. Letting the two equal coordinates of $\boldsymbol{P_3}$ to grow, it arises an acute angle in vertex $\boldsymbol{P_3}$.

The results of our simulation are showed in Figure 3.8. Despite the presence of an angle which is smaller and smaller, when the coordinates of $\boldsymbol{P_3}$ grow, the determinant of $\boldsymbol{A}$ does not vanish, as wished.

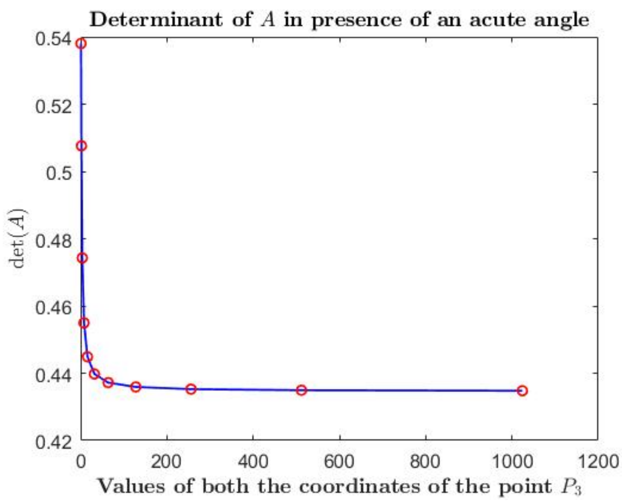


Figure 3.8: The figure shows how the determinant of $\boldsymbol{A}$ changes in presence of an acute angle obtained changing the coordinates of $\boldsymbol{P_3}$.

We now consider the case showed in Figure 3.9. Three vertices are fixed $\boldsymbol{P_0} = (0{,}0)$, $\boldsymbol{P_1} = (1{,}0)$ and $\boldsymbol{P_3} = (1.5{,}1.5)$. $\boldsymbol{P_2}$ is rotated, by posing $\boldsymbol{P_2} = (\cos(\theta), \sin(\theta))$ and letting θ varying. We remark that, since we are considering convex quadrangles, we tested $\theta \in [\frac{\pi}{2}, \pi)$. The results obtained are showed in Figure 3.10. Again the determinant of $\boldsymbol{A}$ does not go to zero.

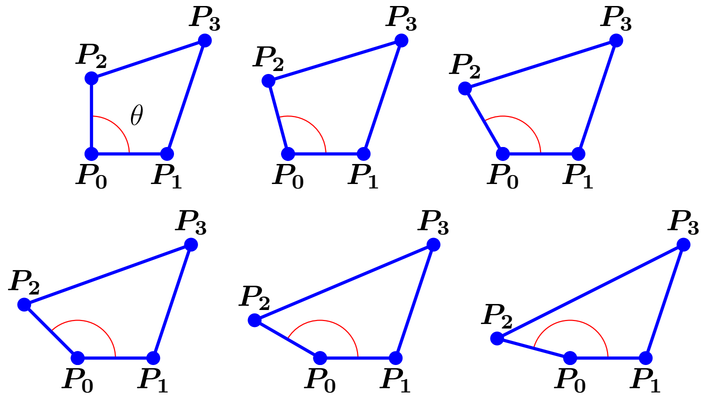


Figure 3.9: This picture shows the geometrical shapes of the second test. $\boldsymbol{P_0}, \boldsymbol{P_1}$ and $\boldsymbol{P_3}$ are fixed. $\boldsymbol{P_2}$ is 'rotated' bringing the quadrangle to be similar to a triangle.

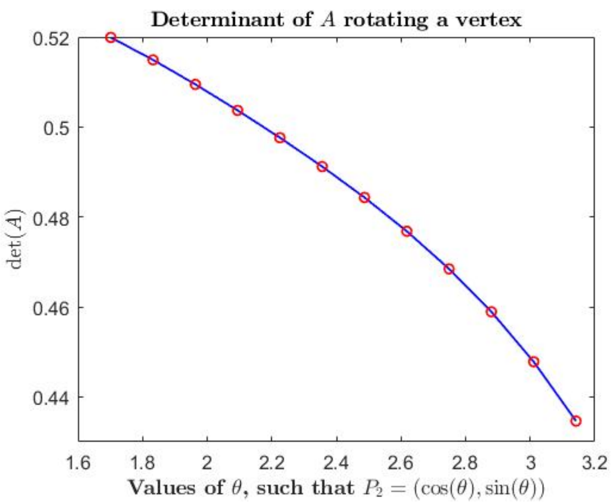
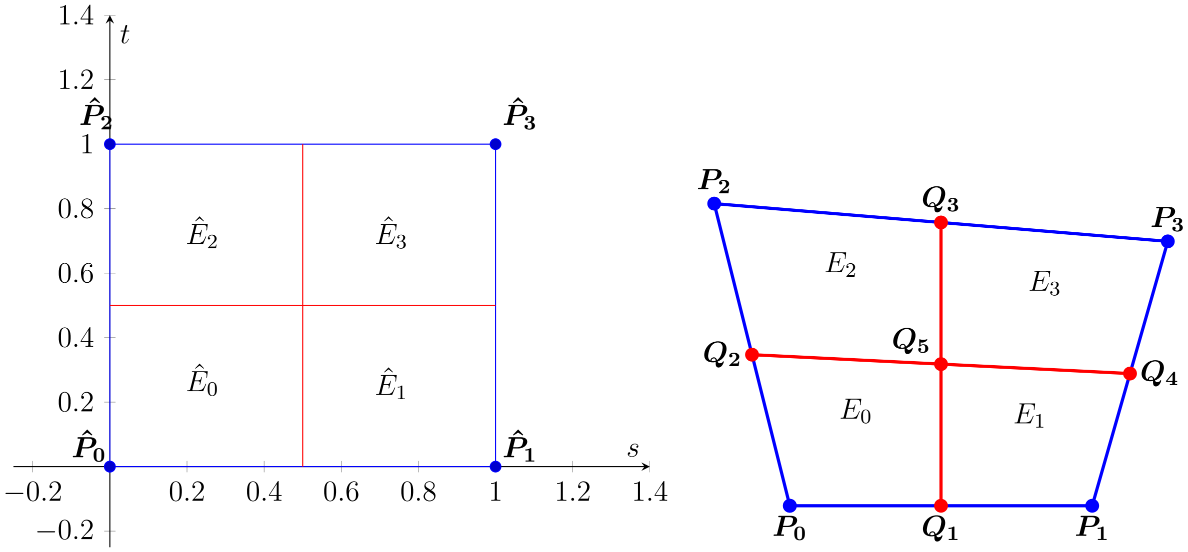


Figure 3.10: The figure shows how the determinant of $\boldsymbol{A}$ changes when rotating the vertex $\boldsymbol{P_2}$.

3.4 The refinement

As declared in the introduction of the chapter, our purpose is to split the quadrangles into four smaller quadrangles. As represented in Figure 3.11 (b), on the physical element E the partition introduces four quadrangles E_i, $i = 0, \ldots, 3$ and five nodes. In particular,



(a) (b)

Figure 3.11: In picture (a) it is shown the refinement on the reference element. (b) shows the refinement on the physical element E.

we have

$$\boldsymbol{Q}_1 := \frac{1}{2}\left(\boldsymbol{P}_0 + \boldsymbol{P}_1\right), \qquad \boldsymbol{Q}_2 := \frac{1}{2}\left(\boldsymbol{P}_0 + \boldsymbol{P}_2\right),$$

$$\boldsymbol{Q}_3 := \frac{1}{2}\left(\boldsymbol{P}_3 + \boldsymbol{P}_2\right), \qquad \boldsymbol{Q}_4 := \frac{1}{2}\left(\boldsymbol{P}_1 + \boldsymbol{P}_3\right), \qquad \boldsymbol{Q}_5 := \frac{1}{4}\sum_{i=0}^{3}\boldsymbol{P}_i.$$

We remark that if we fix the edge with the endpoints $\left(\frac{1}{2},\ 0\right)$ and $\left(\frac{1}{2},\ 1\right)$ on the reference element (that can be described as the set $\left\{\left(\frac{1}{2},\ t\right), t \in [0,1]\right\}$), using the definition of $\boldsymbol{F}$ we have:

$$\begin{aligned}\boldsymbol{F}\left(\frac{1}{2},\ t\right) &= \frac{(1-t)}{2}\left(\boldsymbol{P}_0 + \boldsymbol{P}_1\right) + t\frac{t}{2}\left(\boldsymbol{P}_2 + \boldsymbol{P}_3\right) \\ &= (1-t)\boldsymbol{Q}_1 + t\boldsymbol{Q}_3.\end{aligned}$$

Analogously for the other edge of the refinement, if $s \in [0,1]$,

$$\begin{aligned}\boldsymbol{F}\left(s,\ \frac{1}{2}\right) &= \frac{(1-s)}{2}\left(\boldsymbol{P}_0 + \boldsymbol{P}_2\right) + \frac{s}{2}\left(\boldsymbol{P}_1 + \boldsymbol{P}_3\right) \\ &= (1-s)\boldsymbol{Q}_2 + s\boldsymbol{Q}_4.\end{aligned}$$

This means that if $\hat{v}$ is linear on the new edge brought by the refinement on the reference element, then the function $\boldsymbol{F}$ does not modify the linearity on the edge on the physical element. Moreover, we obtain a fundamental property.

Proposition 3.4.1. *Let E be an element of the discretization $\mathcal{T}$ and E_i, $i = 0, \dots, 3$ obtained after a refinement, as shown in Figure 3.11, then*

$$\mathcal{Q}_1(E)|_{E_i} = \mathcal{Q}_1(E_i), \tag{3.28}$$

where $\mathcal{Q}_1(E)$ has been defined in (3.10).

Proof. A bilinear function on $\hat{E}$ can be defined by its values at vertices that we have called $\{\beta_i\}_{i=0,\dots,3}$, as discussed in (3.7). If we want to restrict the bilinear function to the smaller element, for instance, $\hat{E}_0 = [0, \frac{1}{2}]^2$, we have

$$\hat{q}(s,t) = (1-s)(1-t)\ \beta_0 + s(1-t)\beta_1 + (1-s)t\beta_2 + st\beta_3.$$

If we change the variables $\tilde{s} = 2s$ and $\tilde{t} = 2t$, then the previous function becomes, for $(\tilde{s}, \tilde{t}) \in [0,1]$

$$\hat{q}(\tilde{s},\tilde{t}) = \left(1 - \frac{1}{2}\tilde{s}\right)\left(1 - \frac{1}{2}\tilde{t}\right)\ \beta_0 + \frac{1}{2}\tilde{s}\left(1 - \frac{1}{2}\tilde{t}\right)\beta_1 + \left(1 - \frac{1}{2}\tilde{s}\right)\frac{1}{2}\tilde{t}\beta_2 + \frac{1}{4}\tilde{s}\tilde{t}\beta_3$$

$$= \left(1 - \frac{1}{2}\tilde{s} - \frac{1}{2}\tilde{t} + \frac{1}{4}\tilde{s}\tilde{t}\right)\beta_0 + \left(\frac{1}{2}\tilde{s} - \frac{1}{4}\tilde{s}\tilde{t}\right)\beta_1 + \left(\frac{1}{2}\tilde{t} - \frac{1}{4}\tilde{s}\tilde{t}\right)\beta_2 + \frac{1}{4}\tilde{s}\tilde{t}\beta_3$$

Adding and subtracting $\frac{1}{4}\tilde{s}\tilde{t}\ \beta_1$ and $\frac{1}{4}\tilde{s}\tilde{t}\ \beta_2$ and reorganizing the terms, we obtain

$$\hat{q}(\tilde{s},\tilde{t}) = (1-\tilde{s})\,(1-\tilde{t})\ \beta_0 + \tilde{s}\,(1-\tilde{t})\,\frac{(\beta_1+\beta_0)}{2} + (1-\tilde{s})\,\tilde{t}\frac{(\beta_0+\beta_2)}{2} + \tilde{s}\tilde{t}\frac{1}{4}\left(\sum_{i=0}^{3}\beta_i\right).$$

This last bilinear function is defined on $\hat{E}_0$ and, because of the linearity on each edge, the combinations of β_i that arise are nothing but the values of the bilinear functions on $\hat{E}$ at the midpoints formed with the refinement. Then we have that

$$\mathcal{Q}_1(E)|_{E_i} \subseteq \mathcal{Q}_1(E_i).$$

On the other hand, let q be a function in $\mathcal{Q}_1(E_0)$, the proof for the other elements $E_i, i = 1,2,3$ follows in the same way. By the definition (3.10), $\hat{q} = q \circ \boldsymbol{F}$ is a bilinear function in $\hat{E}_0$, then it can be written, following the names given in Figure 3.11, as

$$\hat{q}(s,t) = (1-s)\,(1-t)\ \beta_0 + (1-t)\,t\beta_4 + t\,(1-s)\,\beta_5 + st\beta_6,$$

where $s \in [0,1]$, $t \in\ [0,1]$ and

$$\beta_0 := \hat{q}\,(0,0)\,, \qquad \beta_4 := \hat{q}\left(\frac{1}{2},0\right), \qquad \beta_5 := \hat{q}\left(0,\frac{1}{2}\right), \qquad \beta_6 := \hat{q}\left(\frac{1}{2},\frac{1}{2}\right).$$

Let us define the variables $(\tilde{s},\tilde{t}) = (\frac{1}{2}s,\ \frac{1}{2}t)$, then

$$\hat{q}(\tilde{s},\tilde{t}) = (1-2\tilde{s})\,(1-2\tilde{t})\ \beta_0 + 2\tilde{s}\,(1-2\tilde{t})\,\beta_4 + 2\tilde{t}\,(1-2\tilde{s})\,\beta_5 + 4\tilde{s}\tilde{t}\beta_6,$$

for $\tilde{s} \in \left[0, \frac{1}{2}\right]$, $\tilde{t} \in \left[0, \frac{1}{2}\right]$.

Using now the definitions of $\{\beta_i\}_{i=0,\dots 3}$, given in (3.7) and the linearity on each edge, we have the following relations

$$\beta_4 = \frac{1}{2}(\beta_1 + \beta_0) \qquad \beta_5 = \frac{1}{2}(\beta_2 + \beta_0) \qquad \beta_6 = \frac{1}{4}\left(\sum_{i=0}^{3} \beta_i\right)$$

$$\begin{aligned}
\hat{q}(\tilde{s}, \tilde{t}) &= \left(1 - 2\tilde{t} - 2\tilde{t} + 4\tilde{s}\tilde{t}\right)\beta_0 + \left(\tilde{s} - 2\tilde{s}\tilde{t}\right)(\beta_1 + \beta_0) + \left(\tilde{t} - 2\tilde{s}\tilde{t}\right)(\beta_2 + \beta_0) + \tilde{s}\tilde{t}\left(\sum_{i=0}^{3} \beta_i\right) \\
&= \left(1 - \tilde{t} - \tilde{t} + \tilde{s}\tilde{t}\right)\beta_0 + \left(\tilde{s} - \tilde{s}\tilde{t}\right)\beta_1 + \left(\tilde{t} - \tilde{s}\tilde{t}\right)\beta_2 + \tilde{s}\tilde{t}\beta_3 \\
&= (1 - \tilde{s})(1 - \tilde{t})\,\beta_0 + \tilde{s}(1 - \tilde{t})\beta_1 + (1 - \tilde{s})\tilde{t}\beta_2 + \tilde{s}\tilde{t}\beta_3,
\end{aligned}$$

with $(\tilde{s}, \tilde{t}) \in \left[0, \frac{1}{2}\right]^2$. This implies that $q \in \mathcal{Q}_1(E)|_{E_0}$. Since this proof can be extended to the other elements E_i $i = 0, \dots, 3$, we have that

$$\mathcal{Q}_1(E_i) \subseteq \mathcal{Q}_1(E)|_{E_i},$$

that concludes the proof. □

We can now define the space $\mathbb{V}^0_{\mathcal{T}}$ as

$$\mathbb{V}^0_{\mathcal{T}} := \{v \in \mathbb{V}_{\mathcal{T}} : \; v|_E \in \mathcal{Q}_E, \; \forall E \in \mathcal{T}\}. \tag{3.29}$$

Proposition 3.4.2. *Let $\mathbb{V}^0_{\mathcal{T}}$ be defined as in (3.29) and $\mathbb{V}_{\mathcal{T}}$ as in (3.19). We have:*

$$\mathbb{V}^0_{\mathcal{T}} \subseteq \mathbb{V}^0_{\mathcal{T}^*},$$

where $\mathcal{T}^*$ is a discretization obtained as a refinement of $\mathcal{T}$.

Proof. From the previous **Proposition** 3.4.1 and the definition of $\mathbb{V}^0_{\mathcal{T}}$, it follows immediately the proof. □

3.5 Stabilization-free a posteriori error analysis

In this section we want to extend the work by Beirão da Veiga et al. [2021], in the case of the virtual element that we have defined. First of all, we recall the following set of nodes.

Definition 3.5.1. Given a discretization $\mathcal{T}$,

- the set of all the vertices of the quadrangles is denoted by $\mathcal{N}$. If $\boldsymbol{x} \in \mathcal{N}$, it is called as **node of the discretization**;
- the subset of $\mathcal{N}$ whose nodes are vertices of all the quadrangles containing them is called set of **proper nodes**, denoted as $\mathcal{P}$;
- the set of nodes $\boldsymbol{x}$ in $\mathcal{N}$ that are not in $\mathcal{P}$ is defined as set of the **hanging nodes**, $\mathcal{H}$.

Analogously, we can define the previous set, but related to a single element $E \in \mathcal{T}$. In particular,

$$\mathcal{N}_E\text{: subset of } \mathcal{N}\text{, with the nodes sitting on } \partial E,$$
$$\mathcal{P}_E\text{: set of the proper nodes of } E,$$
$$\mathcal{H}_E\text{: set of the hanging nodes of } E.$$

For clarity's sake, we here recall the **Definition** 2.1.3 of **Global index of a node**. The global index λ of a node $\boldsymbol{x} \in \mathcal{N}$ is recursively defined as follows:

- If $\boldsymbol{x}$ is a proper node, then set $\lambda(\boldsymbol{x}) := 0$;
- If $\boldsymbol{x}$ is a hanging node, with $\boldsymbol{x}', \boldsymbol{x}'' \in \mathcal{B}(\boldsymbol{x})$, then set $\lambda(\boldsymbol{x}) := \max\{\lambda(\boldsymbol{x}'), \lambda(\boldsymbol{x}'')\} + 1$, where $\mathcal{B}(\boldsymbol{x})$ indicates the set of the endpoints $\{\boldsymbol{x}',\ \boldsymbol{x}''\}$ of the edge which is bisected to create $\boldsymbol{x}$.

We will work under the **Assumption** 2.1.1, for which there exists a constant $\Lambda \geq 1$, such that

$$\Lambda_{\mathcal{T}} := \max_{\boldsymbol{x} \in \mathcal{N}} \lambda(\boldsymbol{x}) \leq \Lambda.$$

Remark 7. Figure 3.12 shows the global index of each node that arises with a refinement. As described in the previous section, the refinements 'creates' five new nodes. The node at the center of the quadrangles is by definition a proper node, since it is shared by all four quadrangles.

As done in Beirão da Veiga et al. [2021] and in the second chapter for the case of polynomials of higher degree, we firstly show the scaled Poincaré inequality.

Proposition 3.5.1 (Scaled Poincaré inequality in $\mathbb{V}_{\mathcal{T}}$)**.** *Given a discretization $\mathcal{T}$, there exists a constant $C_\Lambda > 0$ depending on Λ but independent on $\mathcal{T}$, such that*

$$\sum_{E \in \mathcal{T}} h_E^{-2} ||v||_{0,E}^2 \leq C_\Lambda |v|_1^2, \qquad \forall\, v \in \mathbb{V}_{\mathcal{T}} \text{ such that } v(\boldsymbol{x}) = 0,\ \forall\, \boldsymbol{x} \in \mathcal{P}.$$

Proof. Let $E \in \mathcal{T}$ be fixed. As seen in the **Remark** 7, at least one node of E is a proper node, then we immediately have:

$$h_E^{-2} ||v||_{0,E}^2 \lesssim |v|_{1,E}^2.$$

By summing on all the elements of the discretization, we conclude the proof. □

We now go back to the space $\mathbb{V}_{\mathcal{T}}^0$, that we have introduced in (3.29). A function $v \in \mathbb{V}_{\mathcal{T}}^0$ is a polynomial of degree 1 on each edge. Then, if $\boldsymbol{x}$ is hanging node and $\boldsymbol{x}'$ and $\boldsymbol{x}''$ are the endpoints of the edge containing $\boldsymbol{x}$ as a midpoint,

$$v(\boldsymbol{x}) = \frac{1}{2} \left(v(\boldsymbol{x}') + v(\boldsymbol{x}'') \right).$$

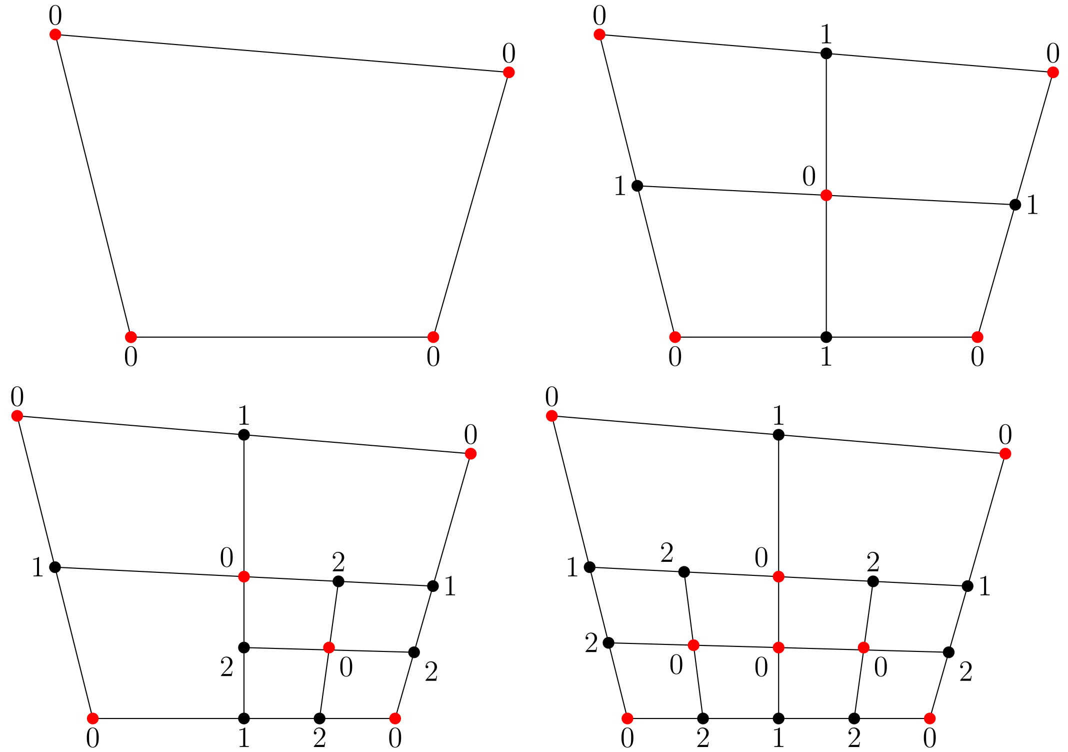


Figure 3.12: The picture shows the values of the global index at each node that arises when a refinement occurs. The red nodes are the proper nodes, while the black nodes are the hanging nodes.

We can now define a basis for $\mathbb{V}^0_{\mathcal{T}}$ as

$$\forall \boldsymbol{x} \in \mathcal{P}: \qquad \psi_{\boldsymbol{x}} \in \mathbb{V}^0_{\mathcal{T}} \text{ satisfies } \psi_{\boldsymbol{x}}(\boldsymbol{z}) = \begin{cases} 1 \text{ if } \boldsymbol{z} = \boldsymbol{x}, \\ 0 \text{ if } \boldsymbol{z} \in \mathcal{P} \setminus \{\boldsymbol{x}\}. \end{cases}$$

As done in Beirão da Veiga et al. [2021], we introduce the Lagrange interpolation operator

$$\mathcal{I}^0_{\mathcal{T}} : \mathbb{V}_{\mathcal{T}} \to \mathbb{V}^0_{\mathcal{T}} \text{ such that}$$
$$\mathcal{I}^0_{\mathcal{T}}(\xi) = \sum_{\boldsymbol{x} \in \mathcal{P}} v(\boldsymbol{x}) \psi_{\boldsymbol{x}}(\xi).$$

Moreover, we denote by $\overline{\mathcal{I}}^0_{\mathcal{T}}$ the classical Clément operator on $\mathbb{V}^0_{\mathcal{T}}$.

Lemma 2.3.1, as proved in Beirão da Veiga et al. [2021], holds and provide the following inequality:

$$\sum_{E \in \mathcal{T}} h_E^{-2} ||v - \overline{\mathcal{I}}^0_{\mathcal{T}} v||^2_{0,E} \lesssim |v|^2_1, \qquad \forall v \in \mathbb{V}.$$

We define now the **internal residual** over E as

$$r_{\mathcal{T}}(E; f) := f_E, \qquad \forall\, v \in \mathbb{V}. \tag{3.30}$$

Let E_1 and E_2 be two elements in $\mathcal{T}$ and let e be the edge shared by the two elements. We can define the jump over e as

$$j_{\mathcal{T}}(e;v) := [[\nabla\Pi^{\nabla}_{\mathcal{T}}v]]_e = (\nabla\Pi^{\nabla}_{E_1}v|_{E_1})\cdot \boldsymbol{n_1} + (\nabla\Pi^{\nabla}_{E_2}v|_{E_2})\cdot \boldsymbol{n_2}, \tag{3.31}$$

where $\boldsymbol{n_i}$ denotes the unit vector to e pointing outward with respect to E_i. If $e \in \partial\Omega$ we set $j_{\mathcal{T}}(e;v) = 0$. We then define the local residual estimator associated with E,

$$\eta^2_{\mathcal{T}}(E;v,f) := h_E^2||r_{\mathcal{T}}(E;f)||^2_{0,E} + \frac{1}{2}\sum_{e\in\mathcal{E}_E} h_E||j_{\mathcal{T}}(e;v)||^2_{0,e} \tag{3.32}$$

and the global residual as

$$\eta^2(v,f) := \sum_{E\in\mathcal{T}} \eta^2_{\mathcal{T}}(E;v,f).$$

The choice of the projector plays here a fundamental role. We would like the upper and lower bounds for the energy to follow from **Proposition** 2.4.1, given in Cangiani et al. [2017]. We here discuss where the 'proposition' fails.

'Proposition' 3.5.2 (Upper bound)**.** *There exists a constant C_{apost} depending only on Λ and f, such that*

$$|u-u_{\mathcal{T}}|_1 \leq C_{apost}\left(\eta^2_{\mathcal{T}}(u_{\mathcal{T}},f) + S_{\mathcal{T}}(u_{\mathcal{T}},u_{\mathcal{T}})\right).$$

Proof. Let $v \in H^1_0(\Omega)$ and $v_{\mathcal{T}} = \overline{\mathcal{I}}^0_{\mathcal{T}}v \in \mathbb{V}^0_{\mathcal{T}}$. We have

$$a(u-u_{\mathcal{T}},v) = ((f,v-v_{\mathcal{T}})_\Omega - a(u_{\mathcal{T}},v-v_{\mathcal{T}})) + a(u-u_{\mathcal{T}},v_{\mathcal{T}}) =: I + II.$$

The first term can be estimated as follows.

$$\begin{aligned} I = &\sum_{E\in\mathcal{T}}\{(f,v-v_{\mathcal{T}})_E - (\nabla\Pi^{\nabla}_E u_{\mathcal{T}},\nabla(v-v_{\mathcal{T}}))_E\} \\ &+ \sum_{E\in\mathcal{T}}\{(\nabla(\Pi^{\nabla}_E u_{\mathcal{T}} - u_{\mathcal{T}}),\nabla(v-v_{\mathcal{T}}))_E\} =: I_1 + I_2. \end{aligned}$$

Then, integrating by parts

$$\begin{aligned} |I_1| &\leq \sum_{E\in\mathcal{T}}\left|(f,v-v_{\mathcal{T}})_E - (\nabla\Pi^{\nabla}_E u_{\mathcal{T}},\nabla(v-v_{\mathcal{T}}))_E\right| \\ &= \sum_{E\in\mathcal{T}}\left|(f+\Delta\Pi^{\nabla}_E u_{\mathcal{T}},v-v_{\mathcal{T}})_E - (\nabla\Pi^{\nabla}_E u_{\mathcal{T}},v-v_{\mathcal{T}})_{\partial E}\right| \\ &= \sum_{E\in\mathcal{T}} h_E||r_{\mathcal{T}}(e;f)||_{0,E}h_E^{-1}||v-v_{\mathcal{T}}||_{0,E} + \frac{1}{2}\sum_{e\in\mathcal{E}_E} h_E||j_{\mathcal{T}}(e;v)||_{0,e}h_E^{-1}||v-v_{\mathcal{T}}||_{0,e} \\ &\lesssim \eta_{\mathcal{T}}(u_{\mathcal{T}},f)|v|_1, \end{aligned}$$

where in the last passage it has been used the Cauchy-Schwarz inequality and the **Lemma** 2.3.1. While for the second part

$$\begin{aligned}
|I_2| &\le \sum_{E\in\mathcal{T}} \left|(\nabla(\Pi_E^\nabla u_\mathcal{T} - u_\mathcal{T}), \nabla(v - v_\mathcal{T}))_E\right| \\
&\lesssim \sum_{E\in\mathcal{T}} \left(||\nabla(\Pi_E^\nabla u_\mathcal{T} - u_\mathcal{T})||_{0,E}\right) h_E^{-1} ||v - v_\mathcal{T}||_{0,E} \\
&\lesssim \left(\sum_{E\in\mathcal{T}} ||\nabla(\Pi_E^\nabla u_\mathcal{T} - u_\mathcal{T})||_{0,E}^2\right)^{1/2} \left(\sum_{E\in\mathcal{T}} h_E^{-2} ||v - v_\mathcal{T}||_{0,E}^2\right)^{1/2} \\
&\lesssim \left(\sum_{E\in\mathcal{T}} ||\nabla(\Pi_E^\nabla u_\mathcal{T} - u_\mathcal{T})||_{0,E}^2\right)^{1/2} |v_\mathcal{T}|_1,
\end{aligned}$$

where we used again **Lemma** 2.3.1, concluding

$$|I_2| \lesssim S_\mathcal{T}(u_\mathcal{T}, u_\mathcal{T})^{1/2}\, |v_\mathcal{T}|_1.$$

In order to conclude the proof we would like to estimate the last term II. But this is not possible. Indeed if $v_\mathcal{T}|_E \in \mathbb{P}_1(E)$, from (3.16),

$$a(u - u_\mathcal{T}, v_\mathcal{T}) = 0,$$

but $\mathbb{P}_1(E)$ is a proper subset of $\mathbb{V}_\mathcal{T}^0$. This property becomes so crucial, in the discussion of the projectors we will highlight it. □

From the **Proposition** showed in Cangiani et al. [2017] concerning the local lower bound, we would have the following inequality

$$\eta^2(E; u_\mathcal{T}, f) \lesssim \sum_{E'\in\omega_E} \left(|u - u_\mathcal{T}|_{1,E'}^2 + S_{E'}(u_\mathcal{T}, u_\mathcal{T})\right),$$

where $\omega_E := \{E' : |\partial E \cap \partial E'| \neq 0\}$. The hidden constant does not depend on γ, h, u and $u_\mathcal{T}$.

Then, as in the **Corollary** 2.4.3, there exists a constant $c_{apost} > 0$, depending on Λ, but independent of $u, \mathcal{T}, u_\mathcal{T}$ and γ such that

$$c_{apost}\eta^2(u_\mathcal{T}, f) \le |u - u_\mathcal{T}|_1^2 + S_\mathcal{T}(u_\mathcal{T}, u_\mathcal{T}).$$

3.5.1 Bound of the stabilization term by the residual

As discussed in the second chapter and in Beirão da Veiga et al. [2021], we firstly need to discuss the interpolation error in $\mathbb{V}_\mathcal{T}^0$, $\mathcal{I}_\mathcal{T}^0$, and in $\mathbb{V}_\mathcal{T}$, $\mathcal{I}_\mathcal{T}$. In particular, we need to show that the inequality 2.8, here reported:

$$|\mathcal{I}_\mathcal{T} v - \mathcal{I}_\mathcal{T}^0 v|_{1,\mathcal{T}} \lesssim |v - \mathcal{I}_\mathcal{T} v|_{1,\mathcal{T}}.$$

For this purpose, we introduce the *hierarchical detail of* v.

Definition 3.5.2 (Hierarchical detail of v)**.** To each function $v \in \mathbb{V}_E$ we associate a vector $d(v) = \{d(v; \boldsymbol{z})\}_{\boldsymbol{z}\in\mathcal{N}_E}$ that collects the following values, so called *hierarchical details of* v

$$d(v;\boldsymbol{z}) = \begin{cases} v(\boldsymbol{z}) & \text{if } \boldsymbol{z} \in \mathcal{P}_E, \\ v(\boldsymbol{z}) - \dfrac{1}{2}\left(v(\boldsymbol{z'}) + v(\boldsymbol{z''})\right) & \text{if } \boldsymbol{z} \in \mathcal{H}_E, \end{cases} \tag{3.33}$$

where $\boldsymbol{z'}$ and $\boldsymbol{z''}$, are the endpoints of the edge with $\boldsymbol{z}$ as a midpoint.

Lemma 2.5.1 holds and we have: the following:

$$|v - \mathcal{I}_E v|^2_{1,E} \simeq \sum_{\boldsymbol{x}\in\mathcal{H}_E} d^2(v;\boldsymbol{x}), \qquad v \in \mathbb{V}_{E,2}, \tag{3.34}$$

where the hidden constants depend only on Λ and from **Corollary** 2.5.2

$$|v - \mathcal{I}_\mathcal{T} v|^2_{1,\mathcal{T}} \simeq \sum_{\boldsymbol{x}\in\mathcal{H}} d^2(v;\boldsymbol{x}), \qquad v \in \mathbb{V}_\mathcal{T},$$

where again the hidden constants depend only on Λ. We now observe, as in Beirão da Veiga et al. [2021], that if $\boldsymbol{x}$ is a proper node, then $v(\boldsymbol{x}) = \left(\mathcal{I}^0_\mathcal{T} v\right)(\boldsymbol{x})$ and, for any $\boldsymbol{x} \in \mathcal{H}$, let us define

$$\delta(v,\boldsymbol{x}) := v(\boldsymbol{x}) - \left(\mathcal{I}^0_\mathcal{T} v\right)(\boldsymbol{x}),$$

that will be useful to estimate

$$|\mathcal{I}_\mathcal{T} v - \mathcal{I}^0_\mathcal{T} v|^2_{1,\mathcal{T}} = \sum_{E\in\mathcal{T}} |\mathcal{I}_\mathcal{T} v - \mathcal{I}^0_\mathcal{T} v|^2_{1,E} \simeq \sum_{E\in\mathcal{T}} \sum_{\boldsymbol{x}\in\mathcal{V}_E} (\mathcal{I}_\mathcal{T} v - \mathcal{I}^0_\mathcal{T} v)^2_{1,E}(\boldsymbol{x}).$$

Indeed, if $\boldsymbol{x} \in \mathcal{V}_E$, $(\mathcal{I}_E v)(\boldsymbol{x}) = (v)(\boldsymbol{x})$, then

$$|\mathcal{I}_\mathcal{T} v - \mathcal{I}^0_\mathcal{T} v|^2_{1,\mathcal{T}} \simeq \sum_{E\in\mathcal{T}} \sum_{\boldsymbol{x}\in\mathcal{V}_E} (v - \mathcal{I}^0_\mathcal{T} v)^2_{1,E}(\boldsymbol{x}) = \sum_{\boldsymbol{x}\in\mathcal{H}} \delta^2(v,\boldsymbol{x}).$$

From **Corollary** 2.5.2, (2.9) holds if the following it is true

$$\sum_{\boldsymbol{x}\in\mathcal{H}} \delta^2(v,\boldsymbol{x}) \lesssim \sum_{\boldsymbol{x}\in\mathcal{H}} d^2(v;\boldsymbol{x}), \qquad v \in \mathbb{V}_\mathcal{T}. \tag{3.35}$$

As done in Beirão da Veiga et al. [2021], we fix v and we define $\delta(\boldsymbol{x}) := \delta(v,\boldsymbol{x})$, $d(\boldsymbol{x}) := d(v,\boldsymbol{x})$ and $v^* := \mathcal{I}^0_\mathcal{T} v$. Let

$$\boldsymbol{\delta} = (\delta(\boldsymbol{x}))_{\boldsymbol{x}\in\mathcal{H}}, \qquad \boldsymbol{d} = (d(\boldsymbol{x}))_{\boldsymbol{x}\in\mathcal{H}},$$

the relation (3.35) reduces to

$$||\boldsymbol{\delta}||_{l^2(\mathcal{H})} \lesssim ||\boldsymbol{d}||_{l^2(\mathcal{H})}.$$

If v^* is on the segment $[\boldsymbol{x'}, \boldsymbol{x''}]$, we have

$$\begin{aligned}\delta(\boldsymbol{x}) = v(\boldsymbol{x}) - v^*(\boldsymbol{x}') &= v(\boldsymbol{x}'') - \frac{1}{2}\left(v^*(\boldsymbol{x'}) + v^*(\boldsymbol{x''})\right) \\ &= v(\boldsymbol{x}) - \frac{1}{2}v(\boldsymbol{x''}) + \frac{1}{2}\left(v(\boldsymbol{x''}) - v^*(\boldsymbol{x''})\right) \\ &\quad - \frac{1}{2}v(\boldsymbol{x'}) + \frac{1}{2}\left(v(\boldsymbol{x'}) - v^*(\boldsymbol{x'})\right) \\ &= d(\boldsymbol{x}) - \frac{1}{2}\left(\delta(\boldsymbol{x''}) + \delta(\boldsymbol{x'})\right).\end{aligned}$$

Thus, we can build a matrix $\boldsymbol{W} : l^2(\mathcal{H}) \to l^2(\mathcal{H})$ such that $\boldsymbol{\delta} = \boldsymbol{W}\boldsymbol{d}$. We just need to prove that

$$||\boldsymbol{W}||_2 \lesssim 1.$$

We now organize the hanging nodes with respect to the global index $\lambda \in [1, \Lambda_\mathcal{T}]$. Calling $\mathcal{H}_\lambda = \{\boldsymbol{x} \in \mathcal{H} : \lambda(\boldsymbol{x}) = \lambda\}$, and $\mathcal{H} = \bigcup_{1\leq\lambda\leq\Lambda_\mathcal{T}} \mathcal{H}_\lambda$. Matrix $\boldsymbol{W}$ can be factorized in lower triangular matrix $\boldsymbol{W}_\lambda$, that change the nodes of level λ, leaving the others unchanged. In particular,

$$\boldsymbol{W} = \boldsymbol{W}_{\Lambda_\mathcal{T}} \boldsymbol{W}_{\Lambda_\mathcal{T}-1} \dots \boldsymbol{W}_2 \boldsymbol{W}_1,$$

where $\boldsymbol{W}_1 = \boldsymbol{I}$, the identity matrix, since if $\lambda = 1$, then $\delta(\boldsymbol{x'}) = \delta(\boldsymbol{x''}) = 0$. Each matrix $\boldsymbol{W}_\lambda$ differs from the identity only in the rows of block λ. Each of these rows contain all the elements equals to zero, but two entries equal $\frac{1}{2}$ in the off-diagonal and one 1 in the diagonal. In order to estimate $\boldsymbol{W}_\lambda$, we use the Hölder inequality: $||\boldsymbol{W}_\lambda||_2^2 \leq ||\boldsymbol{W}_\lambda||_1 ||\boldsymbol{W}_\lambda||_\infty$.

From the construction of $\boldsymbol{W}_\lambda$ have that

$$||\boldsymbol{W}_\lambda||_\infty \leq \frac{1}{2} + \frac{1}{2} + 1 = 2 \qquad\qquad ||\boldsymbol{W}_\lambda||_1 \leq 4\frac{1}{2} + 1 = 3,$$

where in the last inequality it has been used the fact that hanging node of global index $< \lambda$, will appear at most 4 times in the relation between $\delta(\boldsymbol{x})$ and $d(\boldsymbol{x})$, as shown in Figure 3.13. These bring us to the following

$$||\boldsymbol{W}||_2 \leq \prod_{2\leq\lambda\leq\Lambda_\mathcal{T}} ||\boldsymbol{W}_\lambda||_2 \leq 6^{(\Lambda-1)/2}.$$

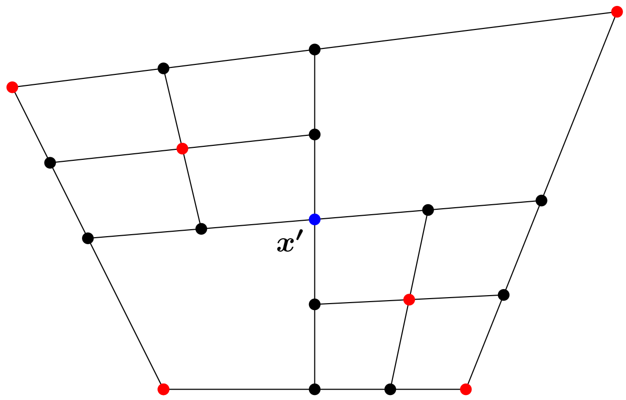


Figure 3.13: Given the blue node $\boldsymbol{x'}$ at the center of the quadrangle with $\lambda(\boldsymbol{x'}) < \lambda$, 4 hanging nodes $\boldsymbol{x}$ of global index λ are such that $\boldsymbol{x'} \in \mathcal{B}(\boldsymbol{x})$. In the figure the proper nodes are in red, while the hanging nodes in black.

Because of this discussion, the following proposition showed in Beirão da Veiga et al. [2021] remains valid.

Proposition 3.5.3 (Comparison between interpolation operators)**.** *There exists a constant C_I depending only on Λ, such that*

$$|v - \mathcal{I}^0_{\mathcal{T}} v|_1 \leq C_I |v - \mathcal{I}_{\mathcal{T}} v|_{1,\mathcal{T}}, \qquad \forall v \in \mathbb{V}_{\mathcal{T}}.$$

The discussion on the a posteriori error analysis can be ended following **Proposition** showed in Beirão da Veiga et al. [2021]. Here again the following **'Proposition'** does not hold, but suggests us another property we should ask for the projector.

'Proposition' 3.5.4 (Bound of the stabilization term by the residual)**.** *There exists a constant C_B depending only on Λ, such that*

$$\gamma^2 S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}) \leq C_B \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, f).$$

Proof. If we have that $\forall w \in \mathbb{V}^0_{\mathcal{T}}$ and $w|_E \in \mathbb{P}_1(E)$ we would have been able to solve the proof as it follows. We need in particular that

$$\Pi^{\nabla}_{\mathcal{T}} w = w, \qquad w \in \mathbb{V}^0_{\mathcal{T}}.$$

If the previous one does hold, then we have

$$\gamma S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}) = \gamma S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}} - w) = \mathcal{B}_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}} - w) - a_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}} - w),$$

since we have that if w is a polynomial, then $S_{\mathcal{T}}(u_{\mathcal{T}}, w) = 0$. We analyze now the single terms and using the definitions of the bilinear forms we have, from the definition of $\mathcal{B}_{\mathcal{T}}$ (3.20) and (3.22):

$$\mathcal{B}_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}} - w) = \left(f, \Pi^{\nabla}_{\mathcal{T}} (u_{\mathcal{T}} - w)\right)_{\Omega};$$

$$a_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}} - w) = a\left(\Pi_{\mathcal{T}}^{\nabla} u_{\mathcal{T}}, \Pi_{\mathcal{T}}^{\nabla}(u_{\mathcal{T}} - w)\right) = \sum_{E \in \mathcal{T}} (\nabla \Pi_E^{\nabla} u_{\mathcal{T}}, \nabla \Pi_E^{\nabla}(u_{\mathcal{T}} - w))_E$$
$$= \sum_{E \in \mathcal{T}} (\nabla \Pi_E^{\nabla} u_{\mathcal{T}} \cdot \boldsymbol{n}, u_{\mathcal{T}} - w)_{\partial E} - (\nabla \cdot \nabla \Pi_E^{\nabla} u_{\mathcal{T}}, \Pi_E^{\nabla}(u_{\mathcal{T}} - w))_E.$$

Given these and the definitions of $r_{\mathcal{T}}$ (3.30) and $j_{\mathcal{T}}$ (3.31), we have

$$S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}) = (r_{\mathcal{T}}(E; f), \Pi_E^{\nabla}(u_{\mathcal{T}} - w))_{\Omega} + \sum_{e \in \mathcal{E}} (j_{\mathcal{T}}(e; u_{\mathcal{T}}), u_{\mathcal{T}} - w)_e$$
$$\leq \sum_{E \in \mathcal{T}} h_E ||r_{\mathcal{T}}(E; f)||_{0,E} h_E^{-1} (||u_{\mathcal{T}} - w||_{0,E} + h_E |u_{\mathcal{T}} - w|_{1,E})$$
$$+ \frac{1}{2} \sum_{E \in \mathcal{T}} \sum_{e \in \mathcal{E}_{\mathcal{T}}} h_E^{1/2} ||j_{\mathcal{T}}(e; u_{\mathcal{T}})||_{0,e} h_E^{-1/2} ||u_{\mathcal{T}} - w||_{0,e}$$

From the definition of $\eta_{\mathcal{T}}$ (3.32), we obtain that, for any $\delta > 0$,

$$\gamma S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}) \leq \frac{1}{2\delta} \eta^2(u_{\mathcal{T}}, f) + \frac{\delta}{2} \phi_{\mathcal{T}}(u_{\mathcal{T}} - w), \qquad \forall\, w \in \mathbb{V}_{\mathcal{T}}^0, \tag{3.36}$$

with

$$\phi_{\mathcal{T}}(u_{\mathcal{T}} - w) = \sum_{E \in \mathcal{T}} \left(h_E^{-2} ||u_{\mathcal{T}} - w||_{0,E}^2 + |u_{\mathcal{T}} - w|_{1,E}^2 + \sum_{e \in \mathcal{E}_{\mathcal{T}}} h_E^{-1} ||u_{\mathcal{T}} - w||_{0,e}^2 \right)$$
$$\lesssim \sum_{E \in \mathcal{T}} \left(h_E^{-2} ||u_{\mathcal{T}} - w||_{0,E}^2 + |u_{\mathcal{T}} - w|_{1,E}^2 \right).$$

Choosing now $w = \mathcal{I}_{\mathcal{T}}^0 u_{\mathcal{T}}$, we can apply the Poincaré inequality (3.5.1), we have

$$\phi_{\mathcal{T}}(u_{\mathcal{T}} - \mathcal{I}_{\mathcal{T}}^0 u_{\mathcal{T}}) \lesssim |u_{\mathcal{T}} - \mathcal{I}_{\mathcal{T}}^0 u_{\mathcal{T}}|_1^2.$$

From the **Proposition** 3.5.3 and the property (3.18), we can end the proof by writing

$$\phi_{\mathcal{T}}(u_{\mathcal{T}} - \mathcal{I}_{\mathcal{T}}^0 u_{\mathcal{T}}) \leq C_{\phi} S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}).$$

Setting in (3.36) $\delta = \gamma / C_B$ and $C_{\phi} = C_B$, we obtain

$$\gamma^2 S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}) \leq C_B \eta_{\mathcal{T}}^2(u_{\mathcal{T}}, \mathcal{D}).$$

□

Using **'Proposition'** 3.5.4, **Proposition** 3.5.2 and **Corollary** 2.4.3, we have the last result.

Corollary 3.5.5 (Stabilization-free a posteriori error estimates)**.** *If γ is chosen as $\gamma^2 \geq \frac{C_B}{c_{apost}}$, it holds true*

$$(c_{apost} - C_B \gamma^2) \eta^2(u_{\mathcal{T}}, \mathcal{D}) \leq |u - u_{\mathcal{T}}|_1^2 \leq C_{apost}(1 + C_B \gamma^2) \eta^2(u_{\mathcal{T}}, \mathcal{D}).$$

3.6 Analysis of the projectors

The previous section has shown the limits of the given definition of Π_E^{∇}, even though it is the most natural way to define it since it comes directly from the Beirão da Veiga et al. [2013]. In particular, we have seen that all the propositions of Beirão da Veiga et al. [2021] are satisfied if

1. $\forall E \in \mathcal{T}$,

$$\Pi_E^{\nabla} q = q, \qquad \forall q \in \mathcal{Q}_1(E);$$

2. $\forall v \in \mathbb{V}_E$

$$a^E(\Pi_E^{\nabla} v, q) = a^E(v, q), \qquad \forall q \in \mathcal{Q}_1(E).$$

3. we can compute

$$a^E(v, \Pi_E^{\nabla} w), \qquad \forall v,\ w\ \in \mathbb{V}_E.$$

Property 3. is the only one which is satisfied by the Π_E^{∇} operator defined in (3.14), as showed with (3.13), while the first and the second are valid only for $\mathbb{P}_1(E)$ which is a proper subset of $\mathcal{Q}_1(E)$.

The first property suggests that Π_E^{∇} should be defined projecting $\mathbb{V}_E$ on $\mathcal{Q}_1(E)$. Let $\Pi_E^{\nabla} : \mathbb{V}_E \to \mathcal{Q}_1(E)$ be now the operator that guarantees the following system to have a unique solution:

$$\begin{cases} \int_E \nabla \Pi_E^{\nabla} v \cdot \nabla q = \int_E \nabla v \cdot \nabla q & \forall q\ \in \mathcal{Q}_1(E) \\ \overline{\Pi_E^{\nabla} v} = \overline{v}, & \forall v\ \in \mathbb{V}_E. \end{cases}$$

The first two properties are satisfied by the definition of Π_E^{∇}, but the third is not. Indeed, given $q = \Pi_E^{\nabla} w \in \mathcal{Q}_1(E)$, we have

$$a^E(v, q) = \int_E \nabla v \nabla q = -\int_E v \Delta q + \int_{\partial E} v \frac{\partial q}{n},$$

but here in general $\Delta q \neq 0$, by definition of $\mathcal{Q}_1(E)$ we know only that $\hat{\Delta} \hat{q} = 0$ on the reference element $\hat{E}$.

An alternative definition of Π_E^{∇} wants to use the condition on the Laplacian on the reference element. We define $\Pi_E^{\nabla} : \mathbb{V}_E \to \mathcal{Q}_1$ and $\hat{\Pi}_{\hat{E}}^{\hat{\nabla}} : \mathbb{V}_{\hat{R}_E} \to \hat{\mathbb{Q}}_1(\hat{E})$ such that, given $\hat{v} \in \mathbb{V}_{\hat{R}_E}$, $\hat{v} = v \circ \boldsymbol{F}$,

$$\Pi_E^{\nabla} v = (\hat{\Pi}_{\hat{E}}^{\hat{\nabla}} \hat{v}) \circ \boldsymbol{F}^{-1}. \tag{3.37}$$

$\hat{\Pi}_{\hat{E}}^{\hat{\nabla}}$ is defined such that the following system admits a unique solution

$$\begin{cases} \int_{\hat{E}} \hat{\nabla}\hat{\Pi}_{\hat{E}}^{\hat{\nabla}}\hat{v} \cdot \hat{\nabla}\hat{q} = \int_{\hat{E}} \hat{\nabla}\hat{v} \cdot \hat{\nabla}\hat{q} & \forall \hat{q} \ \in \mathbb{Q}_1(\hat{E}) \\ \overline{\hat{\Pi}_{\hat{E}}^{\hat{\nabla}}\hat{v}} = \overline{\hat{v}}, & \forall \hat{v} \ \in \mathbb{V}_{\hat{R}_E}. \end{cases}$$

Firstly we remark that if $\hat{q} \in \hat{\mathbb{Q}}_1(\hat{E})$, then $\hat{\Pi}_{\hat{E}}^{\hat{\nabla}}\hat{q} = \hat{q}$. By the definition of the space, we have that given $q \in \mathcal{Q}_1(E)$, it exists $\hat{q} \in \hat{\mathbb{Q}}_1(\hat{E})$, such that $q = \hat{q} \circ \boldsymbol{F}^{-1}$. From (3.37), the first property for this definition of Π_E^{∇} is satisfied, indeed,

$$\Pi_E^{\nabla} q = (\hat{\Pi}_{\hat{E}}^{\hat{\nabla}}\hat{q}) \circ \boldsymbol{F}^{-1} = \hat{q} \circ \boldsymbol{F}^{-1} = q.$$

On the reference element the second property holds by the definition of the operator. Also the third property is valid on the $\hat{E}$: we have

$$\int_{\hat{E}} \hat{\nabla}\hat{v}\hat{\nabla}\hat{q} = -\int_{\hat{E}} \hat{v}\hat{\Delta}\hat{q} + \int_{\partial\hat{E}} \hat{v}\frac{\partial \hat{q}}{\partial \boldsymbol{n}} = \int_{\partial\hat{E}} \hat{v}\frac{\partial \hat{q}}{\partial \boldsymbol{n}},$$

since $\hat{q}$ is a bilinear function, the previous integral can be computed knowing only the value of $\hat{v}$ on the border of $\hat{E}$. These properties cannot be transferred on E. Indeed, let $\boldsymbol{x}$ be a point in E and $\hat{\boldsymbol{x}} = \boldsymbol{F}^{-1}(\boldsymbol{x})$ the corresponding point in $\hat{E}$, $\hat{\varphi} := \hat{\Pi}_{\hat{E}}^{\hat{\nabla}}\hat{v}$ and

$$\varphi(\boldsymbol{x}) := \hat{\varphi}(\boldsymbol{F}^{-1}(\boldsymbol{x})),$$

deriving

$$\frac{\partial \varphi}{\partial x_i}(\boldsymbol{x}) = \sum_{j=1}^{2} \frac{\partial \hat{\varphi}}{\partial \hat{x}_j}\frac{\partial \boldsymbol{F}^{-1}}{\partial x_i}$$
$$\nabla\varphi(\boldsymbol{x}) = (\boldsymbol{J}_{\boldsymbol{F}^{-1}})^T \hat{\nabla}\hat{\varphi}(\hat{\boldsymbol{x}}),$$

where $\boldsymbol{J}_{\boldsymbol{F}^{-1}}$ is the jacobian matrix of $\boldsymbol{F}^{-1}$. The integral required for the property 3,

$$\int_E \nabla v \nabla \Pi_E^{\nabla} w = \int_{\hat{E}} (\boldsymbol{J}_{\boldsymbol{F}^{-1}})^T \nabla \hat{v} \cdot (\boldsymbol{J}_{\boldsymbol{F}^{-1}})^T \hat{\nabla}\hat{\Pi}_{\hat{E}}^{\hat{\nabla}}\hat{w} \det(\boldsymbol{J}_{\boldsymbol{F}^{-1}}),$$

cannot be computed.

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