POLITECNICO DI TORINO

Master Course
in Mathematical Engineering

Master Thesis

Social dynamics with quantized states: simulations and analytical results

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To mom and dad, who
have believed in me from
my first day on earth.

Summary

Over the years, interest in studying the evolution of social dynamics over graphs has greatly increased: different models and types of dynamics have been proposed to analyze how the nodes of a graph interact. These nodes can be conceivable as human individuals interacting, animals, cells or any other element whose behavior can be understood if placed inside a network and analysed through some sort of dynamics on this network. This thesis aims at reviewing many of the models proposed to describe opinion formation under some social influence, hence analysing a recent model of social influence in which the real opinions are concealed to the other nodes of the network and only seen through a filter, a quantizer. Indeed, in many situations, individuals may not be able to fully show their opinion which can only be assessed through their displayed behaviors (social networks posts, people followed, tweets, etc.). The difficulty in this last dynamics arises as the dynamics involves a discontinuous vector field coming from the quantization of the individual opinions. First of all, the problem of how to generalize solutions for these discontinuous dynamics will be addressed, then some properties of the solutions will be analyzed with the aim of analytically extending current results to different graphs. Simulating these dynamics has got a great part in the analysis: numerical simulations may lead to some results and may for instance suggest convergence properties but they might also fail in capturing all the possible solutions. In our case numerical methods have shown solutions compatible with the theory which had not been found analytically. These solutions show periodic patterns and may not converge to consensus and in general may not converge at all. Generalised solutions which have been considered are Carathéodory and Krasovskii, the latter requiring some theory about differential inclusions, has been briefly retrieved in the dissertation. The study about the dynamics follows a microscopic approach, whether it must be stated that another approach to social dynamics is often considered: the macroscopic one, in which the study doesn't focus on the single individual's opinion evolution. In the macroscopic setting, the evolution of some statistic upon the entire network (the total mean of vote preference for instance) is studied and not the single individual votes.

Acknowledgements

It seems standard to thank our supervisor for the work done together but in my case I want to thank the professor who supervised me, Francesca Ceragioli, not only for the great support she has given me, the help through some technicalities and the patience shown in repeating and underlying some aspects of the topics we have covered but I want to thank her also for the support in this period of my life characterized by so many choices to make: attending a PhD, starting working in the school sector, trying to grow up in a company. She has helped me keeping all these possibilities open, reassuring me that any choice would be great and, if you want, you can always try a different path, even if you left it behind before. I want to thank also my co-supervisor Paolo Frasca who has guided me through the literature regarding the topics I covered in my dissertation and for all his suggestions about my work which I, like a bad student, followed only in small part. Thank you both for your help in carrying out the end for this big chapter of my life: my master's degree.

The essence of mathematics lies in its freedom.

[G. Cantor]

Introduction

This master thesis develops over four chapters: the first chapter of preliminaries, the second chapter of review of classical dynamics over networks, the third chapter, on which this work focuses, with a complete description of a particular quantized dynamics and the analysis of some properties of the dynamics, and the last chapter, with the aim of giving the reader some convergence properties and present some directions towards which pointing the future research. The first chapter aims at establishing the notation that will be used over this work and at reviewing some fundamentals related to graph theories and some important theorems that shall be used during the dissertation. The aim of the first chapter is therefore that of giving the tools for a good comprehension of the dissertation. The second chapter has been written to gain a complete overview of which models have been proposed to describe social interactions and the phenomenon of people's opinions evolution. Despite some models being not quite recent, the second chapter is fundamental to gain an insight to what are the differences between the classical models and the quantized model which will be presented later. This second chapter gives some classical model for people interactions and some extensions in order to generalize those models. Properties will be discussed and some results will be presented, with the help of numerical simulations to observe the trajectories for these models applied through simple examples. The third chapter provides a complete description of this more recent dynamics often referred to as "quantized dynamics", as people's opinions are observed through a map q from real to integer numbers. In this chapter, the dynamics will be introduced, discussed and the main differences with respect to the classical ones presented in chapter two will be discussed. Then a numerical part will be briefly discussed to understand how MATLAB has been used in the drafting of the thesis. A brief discussion over generalised solutions will be carried out as in this new dynamics classical solutions cannot exists in the majority of the cases. Some results on particular graphs will be eventually discussed and we will present some solutions which had not been found analytically. In the last chapter some convergence results, mainly found in [3], will be discussed, with a quick overview of what directions could be explored in some successive work and some questions which are still unanswered in the literature of the field.

Chapter 1

Preliminaries

This chapter reviews some preliminaries about graph theory to pave the way towards a full understanding of the definitions and properties used in the following parts of the paper. This chapter has also the aim of making the reader familiar with the notation that will be used later in the dissertation.

1.1 Graphs

We start from the definition of a graph, to later recall the definition of adjacency matrix and laplacian of a graph. The definitions and the main preliminaries to be covered are essentially taken from [2]. A graph is essentially a set of nodes called vertices which are connected with edges. These may be oriented (directed graph) or not (undirected graph). More formally:

Definition 1.1.1 (**Graph**). A *graph* is a pair $\mathcal{G} = (V, E)$, with $V = (v_1, v_2, \dots, v_n)$ and $E \subseteq V \times V$.

Observation 1. *To reduce the notation, in the following, the n vertices of a graph will always be denoted by only the indexes* 1,2, $\dots$, n.

This abstract definition allows this instrument to be a powerful tool in different models. The set of nodes can indeed represent individuals, factories, warehouses, cities, etc. while edges can model interactions, traffic routes, influence, road fees, etc. Our interpretation is that of nodes as individual social agents interacting with other agents. The edges represent this interaction. It should be observed that in the following we will be considering finite graphs, therefore the sets V and E are finite. To define a graph just an *adjacency matrix* is necessary. This contains all the information to define the graph. In concrete:

Definition 1.1.2 (**Adjacency Matrix**). Given the graph $\mathcal{G} = (V, E)$, a matrix $A = (a_{ij})_{i,j \in V}$ with *nonnegative* entries ($a_{ij} \geq 0$) is called *weighted adjacency matrix* for $\mathcal{G}$ if

$$a_{ij} \begin{cases} > 0 & \textit{if } (i,j) \in E \\ = 0 & \textit{if } (i,j) \notin E \end{cases}$$

In this way, to any graph $\mathcal{G}$ can be associated a representation given by the adjacency matrix A. The following examples show some graphs which will be considered in the analysis of the discontinuous dynamics and will also be used in the numerical simulations. Given the adjacency matrix, it is natural to draw a diagram of the nodes and the connecting edges. An edge is drawn starting at node j and ending at node i if $a_{ij} > 0$, i.e. $(i, j) \in E$.

We do need graphs because they are a powerful instruments to keep track of the interactions between social individuals, animals, robotic sensors and so on. In our dissertation we shall think of nodes as individuals whose interactions are marked by the connecting edges.

When the edge starting at node j and ending at node i is drawn, an interaction between two individuals is modelled and the strength of the interaction is measured by the a_{ij} coefficient. It means that node j influences node i, or, alternatively, that i receives information about the opinion of node j. For a better visualization we'll use a *binary* adjacency matrix with:

$$a_{ij} = \begin{cases} = 1 & \textit{if}\,(i,j) \in E \\ = 0 & \textit{if}\,(i,j) \notin E. \end{cases}$$

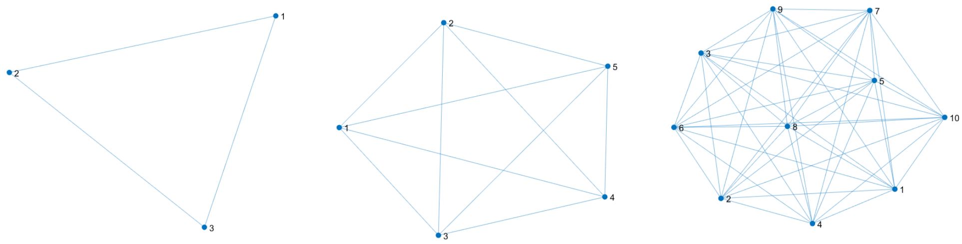
Observation 2. *In literature sometimes the matrix A with entries 1 or 0 is referred to adjacency matrix while the more general one with $a_{ij} \geq 0$ is known as weighted adjacency matrix*

Observation 3. *We highlight that if $a_{ij} > 0$ we'll draw an edge from j to i, meaning that j has influence over node i.*

Example 1.1.1 (Complete Graph). If the matrix is

$$A = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ \vdots & 1 & \ddots & & \\ \vdots & \vdots & & \ddots & \\ 1 & 1 & \dots & & 0 \end{bmatrix} \tag{1.1}$$

then we have $a_{ij} > 0$ for $i \neq j$. In this graph all nodes are therefore connected to all others without self-connections (selfloops). A graph associated with this adjacency matrix is called **complete**. In this network every individual observes the behaviors of any other individual.



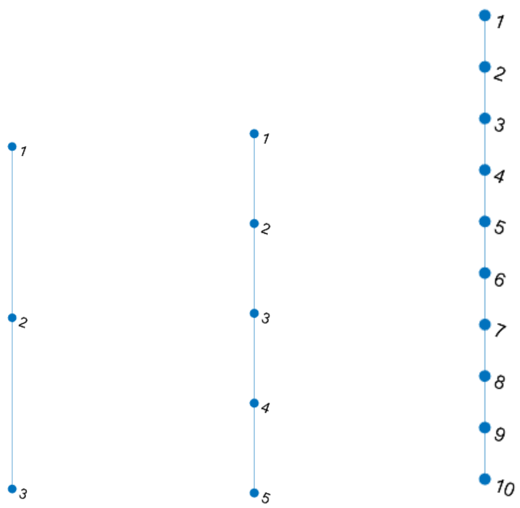
(a) (b) (c)

Figure 1.1: Complete graphs with respectively 3,5,10 nodes

Example 1.1.2 (Path Graph). If the matrix is in the form

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ 0 & 1 & \ddots & 1 & 0 \\ \vdots & \dots & 1 & 0 & 1 \\ 0 & \dots & \dots & 1 & 0 \end{bmatrix} \tag{1.2}$$

then the graph is called **path**. In this kind of interaction any individual observes the behaviors only of two people (i.e. his two best friends, two people taken as model), except for the first and the last, which are only influenced by one person.



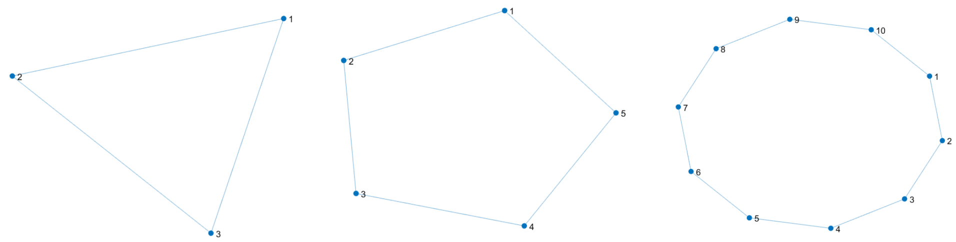
(a) (b) (c)

Figure 1.2: Path graphs with respectively 3,5,10 nodes.

Example 1.1.3 (Cycle Graph). If the matrix is in the form

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 1 \\ 1 & 0 & 1 & \dots & 0 \\ 0 & 1 & \ddots & 1 & 0 \\ \vdots & \dots & 1 & 0 & 1 \\ 1 & \dots & \dots & 1 & 0 \end{bmatrix} \tag{1.3}$$

then the graph is called **cycle**. This interaction is similar to that considered in (1.1.2) with a direct interaction between the last node and the first node. For this graph, each node "sees" the behaviors of two other individuals.



(a) (b) (c)

Figure 1.3: Cycle graphs with respectively 3,5,10 nodes

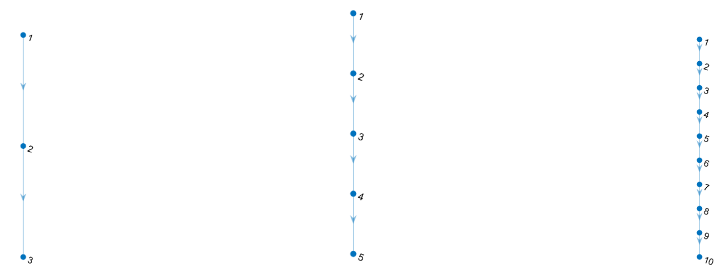
Without explicitly stating it, in the previous examples we have only seen *undirected graphs*, i.e. a graphs where all edges where bidirectional. For an undirected graph:

$$(i, j) \in E \iff (j, i) \in E.$$

In the following also directed graphs will be considered, i.e. graphs with also monodirectional edges, for example. To gain intuition with directed graphs we present the path graph of 1.1.2 in its directed counterpart.

Example 1.1.4 (Directed Path). If the matrix is in the form

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & 0 & \ddots & 1 & 0 \\ \vdots & \dots & 0 & 0 & 1 \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix} \tag{1.4}$$



(a) (b) (c)

Figure 1.4: Directed paths with respectively 3,5,10 nodes

then the graph is called a **directed path**. It is easily noticeable than one can build an incredible number of examples of graphs while here we have presented only three of the most well-known in the sector literature.

1.1.1 Degree Matrix and Laplacian of a graph

We are now recalling two matrices which are connected to a great part of dynamics over graphs: the degree matrix and the laplacian matrix. The first one contains in its diagonal elements d_{ii} the degree of a certain node, that is the number of edges having i as extremity.

Definition 1.1.3 (**Degree Matrix**)**.** A diagonal matrix $D = d_{ij}$ is called *degree matrix* of a graph $\mathcal{G}$ if:

$$d_{ij} = \begin{cases} \sum_{j \neq i} a_{ij} & \text{\textit{if} } i = j \\ 0 & \text{\textit{if} } i \neq j. \end{cases}$$

For a directed graph either the *in-degree* or the *out-degree* matrices can be considered.

OUT-DEGREE $d_{ij} = \begin{cases} \sum_{j \neq i} a_{ij} & \text{\textit{if} } i = j \\ 0 & \text{\textit{if} } i \neq j. \end{cases}$ IN-DEGREE $d_{ij} = \begin{cases} \sum_{i \neq j} a_{ij} & \text{\textit{if} } i = j \\ 0 & \text{\textit{if} } i \neq j. \end{cases}$

Definition 1.1.4 (**Laplacian Matrix**)**.** For a weighted graph $\mathcal{G} = (V, E, A)$, its Laplacian matrix is defined by:

$$L[A] = D - A$$

where D is the degree matrix and A is the adjacency matrix. For brevity we'll denote this matrix with L when it will be evident from the context that the laplacian is referred to matrix A.

In particular, the components of L_{ij} are:

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j \\ \sum_{j \neq i} a_{ij}, & i = j \end{cases}$$

After recalling the definition of path,cycle and walk in a graph, we also give the definition of *strong* graph, which will be useful in the next chapter.

Definition 1.1.5 (**Walk**)**.** A *walk* of length k connecting node i to node i' is a sequence of nodes $i_0, \ldots, i_k \in V$, where $i_0 = i$ and $i_k = i'$ and adjacent nodes are connected by arcs: $(i_{m-1}, i_m) \in E$ for any $m = 1, \ldots, k$.

Definition 1.1.6 (**Cycle**)**.** A walk from a node $i = i_0$ to itself ($i' = i_0$) is called a *cycle.*

Definition 1.1.7 (**Path**)**.** A walk without self-intersections ($i_m \neq i_l, \text{for}\, m \neq l$) is a path.

Definition 1.1.8 (**Periodic Graph**)**.** A graph is *periodic* if it has at least one cycle and the length of any cycle is divided by some integer $h > 1$. The maximal h which divides the length of any cycle is called the *period* of the graph.

Definition 1.1.9 (**Strong Graph**)**.** A graph is called *strongly connected* or *strong* if a walk between any two nodes exists.

Without explicitly stating it, in the previous examples we have only considered *undirected graphs*, i.e. graphs where all edges were bidirectional. For an undirected graph:

$$(i, j) \in E \iff (j, i) \in E.$$

In the dissertation also directed graphs will be considered, i.e. graphs with monodirectional edges, for instance.
To introduce these kind of graphs let us start with an example.

Example 1.1.5 (**Directed Path**)**.** If the matrix is in the form

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & 0 & \ddots & 1 & 0 \\ \vdots & \ldots & 0 & 0 & 1 \\ 0 & \ldots & \ldots & \ldots & 0 \end{bmatrix} \tag{1.5}$$

then the graph is called a **directed path**. It is easily noticeable than one can build an incredible number of examples of graphs while here we have presented only some of the most well-known in the sector literature.

1.1.2 Degree Matrix and Laplacian of a graph

We are now recalling two matrices which are connected to a great part of dynamics over graphs: the degree matrix and the laplacian matrix. The first one contains in its diagonal elements d_{ii} the degree of a certain node, that is the number of edges having i as extremity.

Definition 1.1.10 (Degree Matrix). A diagonal matrix $D = d_{ij}$ is called *degree matrix* of a graph $\mathcal{G}$ if:

$$d_{ij} = \begin{cases} \sum_{j \neq i} a_{ij} & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

For a directed graph either the *in-degree* or the *out-degree* matrices can be considered.

OUT-DEGREE $d_{ij} = \begin{cases} \sum_{j \neq i} a_{ij} & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$ IN-DEGREE $d_{ij} = \begin{cases} \sum_{i \neq j} a_{ij} & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

Definition 1.1.11 (Laplacian Matrix). For a weighted graph $\mathcal{G} = (V, E, A)$, its Laplacian matrix is defined by:

$$L[A] = D - A$$

where D is the degree matrix and A is the adjacency matrix.
In particular, the components of L_{ij} are:

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j \\ \sum_{j \neq i} a_{ij}, & i = j \end{cases}$$

1.2 Properties and Theorems

Here we analyze some useful definitions and theorems which will help us through the analysis of convergence of some simple continuous time dynamics, such as Abelson's. The exposition follows the approach of [2].

Definition 1.2.1 (M-Matrix). A square matrix Z is an M-matrix if it admits a decomposition Z = sI A, with $s \geq \rho(A)$ and where the matrix A is nonnegative.

For example, if A is a stochastic matrix (matrix with non negative entries and either rows or columns summing up to one), then $\rho(A) = 1$ and any matrix $Z = sI - A$, with $s \geq 1$ is an $M - matrix$.
As it is often done, we give a characterization of $Mmatrices$ on the base of the following lemma.

Lemma 1.2.1 (Characterization of an M-matrix). *If a matrix $Z = (z_{ij})$ satisfies the following two conditions:*

1. $z_{ij} \leq 0$ *when* $i \neq j$;
2. $z_{ii} \geq \sum_{j \neq i} |z_{ij}|$,

then, Z is an M-matrix; more precisely, $A = sI - Z$ is nonnegative and $\rho(A) \leq s$ whenever $s \geq max_i z_{ii}$.

Proof. If $s \geq max_i z_{ii}$ then, by the first requirement $A = sI - Z$ is nonnegative and

$$\rho(A) \leq \max_i \left(s - z_{ii} + \sum_{j \neq i} |z_{ij}| \right) \leq s$$

thanks to Gershgorin Disc Theorem. $\square$

By noticing that the eigenvalues of Z and A are in the following one-to-one correspondence $\lambda \mapsto s - \lambda$ it is possible to state the following:

Corollary 1.2.1 (Eigenvalues of an M-matrix). Any M-matrix $Z = sI - A$ has a real eigenvalue $\lambda = s - \rho(A) \geq 0$, whose algebraic and geometric multiplicities coincide. For this eigenvalue there exist nonnegative right and left eigenvectors v and p:

$$Zv = \lambda_0 v$$

and

$$p^\top Z = \lambda_0 p^\top.$$

These vectors are positive if the graph $\mathcal{G}[-Z]$ is *strongly connected*. For any other eigenvalue λ one has $\mathrm{Re}\lambda > \lambda_0$, hence Z is non-singular if and only if $s > \rho(A)$.

We have seen the definition of *Laplacian* matrix of a graph (1.1.11). This is an M-matrix due to (1.2.1): indeed its non-diagonal entries are all negative and $l_{ii} = \sum_{j \neq i} a_{ij} = \sum_{j \neq i} |l_{ij}| = \sum_{j \neq i} |a_{ij}|$.

We observe that $\lambda_0 = 0$ is always an eigenvalue of L, in fact

$$L\mathbb{1}_n = (D - A)\mathbb{1}_n = 0.$$

Furthermore we give three equivalent conditions which will be useful in the study of some continuous-time dynamics.

Lemma 1.2.2 (Equivalent Conditions for the Laplacian Matrix).

1. $\lambda_0 = 0$ *is an algebraically simple eigenvalue of the matrix* L*,*
2. *if* $Lv = 0$*,* $v \in \mathbb{R}^n$ *then* $v = c\mathbb{1}_n$ *for some* $c \in \mathbb{R}$*, i.e.* 0 *has geometric multiplicity* 1.
3. *The graph* $\mathcal{G}[A]$ *is quasi-strongly connected.*

This previous lemma, which we state without proof, will be used in the following chapter.

Chapter 2

Opinion Dynamics over a graph: review of the classical models

In this chapter we analyse some classic dynamics which have been already studied and are well characterized. This will be useful to understand the limitations of these but also the difficulties in passing to the quantized dynamics that will be presented in the next chapter. Here, in the first section, we'll briefly mention the French-De Groot discrete dynamical model for opinions to get an intuition of how the model is created and how it changes from the discrete to the continuous case. In the second section the Abelson model, a continuous counterpart of the French-De Groot model, will be considered and examined in detail.

2.1 Discrete Dynamic over a graph: the French-DeGroot opinion pooling dynamics

In this section we briefly introduce a discrete-time dynamics that will help us for the intuition and understanding of the continuous-time version.

French-De Groot Opinion Pooling [1]

Everyone agrees that the French-DeGroot model has been a pioneeristic representation of social interactions in a mathematical language. It was proposed by the social psychologist French in [7], in which he tried to bind social network analysis and system theory. The model, generalised by DeGroot [6] allows for reaching consensus under quite mild hypotheses. The goal of French was actually to model the social power rather than finding

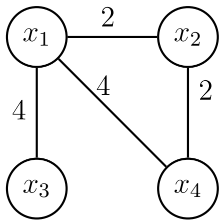
[1] The analysis follows that of [2]

an hyper-realistic model of how opinions evolve over a group of individuals. The social power of an individual or a node (if we think to the World Wide Web for instance) is a measure of the influence of the node over the net. This model has connected the two side of the coin: opinion formation and influence measures.

The Model

We consider a group of n agents, whose opinions are scalars denoted by $x_1, \ldots, x_n$. The parameters of the model are the *influence weights*, collected in a matrix $(W)_{ij}$. The weight w_{ij} measures how the agent i is influenced by j and the w_{ij} sum up to one along j, i.e. $\sum_{j=1}^{n} = w_{ij} = 1$.

Example 2.1.1. For example, we may imagine a group of 4 individuals communicating according to the following graph:



Building a normalized matrix of influence

In the graph above, the numbers on each edge represent the reciprocal influence that nodes have. $a_{13} = a_{31} = 4$ is a way of stating that the reciprocal influence of 1 and 3 and 1 and 4 are the strongest within all nodes. To normalize all the influence measures a_{ij}, the matrix W representing the dynamics is often built as $W = D^{-1}A$ where A is the adjacency matrix and $D = diag(w)$, where $w_i = \sum_j = w_{ij}$ is the (out) degree of the i-th node.

In the example above (2.1.1) the weighted matrix W would be:

$$W = \begin{bmatrix} 0 & \frac{2}{10} & \frac{4}{10} & \frac{4}{10} \\ \frac{2}{4} & 0 & \frac{2}{4} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{4}{6} & \frac{2}{6} & 0 & 0 \end{bmatrix}. \tag{2.1}$$

2.1.1 The DeGroot Dynamics

The mechanism is quite simple: the opinion at time $k+1$ is given by a weighted sum of the opinions of the "neighbours" of a node with weights w_{ij}. In matrix form the dynamics obeys the law:

$$x(k+1) = Wx(k), \qquad k = 0,1, \ldots \tag{2.2}$$

where we have denoted by $x(k) = (x_1(k), \ldots, x_n(k))^\top$ the vector of opinions at time k. Therefore, the DeGroot dynamics for the i-th individual is:

$$x_i(k+1) = \sum_{j=1}^{n} w_{ij} x_j(k), \qquad \forall i, \qquad k = 0,1, \ldots, n. \tag{2.3}$$

The meaning of self-influence w_{ii}

The weight w_{ii} can be seen as a measure of openness towards other opinions/behaviors. When $w_{ii} = 1$ the node is referred to as *stubborn node* and never changes opinion during the dynamics; from (2.3) one would get for the stubborn node i:

$$x_i(k+1) = w_{ii}x_i(k) = x_i(k)$$

therefore the opinion of the stubborn agent x_i is constant. A node with $w_{ii} = 0$ is conceivable as a node who completely relies on external opinions, a completely open-minded person.

One might look for a definition of convergence of a dynamics, in the sense that one might ask if the opinion of the individuals "stabilize", "converge" to a limit opinion. In this context we'll use the following:

Definition 2.1.1 (Convergence). The model (2.3) is said to be convergent if *for any* initial condition $x(0)$ it exists the limit:

$$x(\infty) \doteq \lim_{k\to\infty} x(k) = \lim_{k\to\infty} W^k x(0).$$

Definition 2.1.2 (Consensus). A convergent model is said to *reach consensus* if

$$x_1(\infty) = x_2(\infty) = \cdots = x_n(\infty).$$

It is possible to prove that the discrete-time dynamics defined in (2.1.1) reaches convergence and the convergence is a consensus configuration with $x_\infty \approx \left(\frac{12}{10}, \frac{12}{10}, \frac{12}{10}, \frac{12}{10}\right)$.

For this dynamics, convergence and consensus criteria are well known and we only state the main results of convergence without proving them. We'll see more details for the continuous-time counterpart of this model, as it will be useful to compare it to the quantized model.

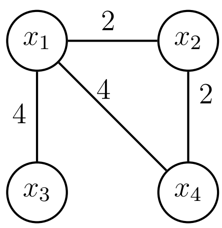
Lemma 2.1.1 (Convergence and Consensus). *The DeGroot pooling model* (2.2) *reaches convergence if and only if* $\lambda = 1$ *is the only eigenvalue of* W *on the unit circle:* $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$. *The model* (2.2) *reaches consensus if and only if the eigenvalue* $\lambda = 1$ *is simple.*

The consensus criterion can also be stated as:

Lemma 2.1.2. *If the graph* $\mathcal{G} = \mathcal{G}[\mathcal{W}]$ *is strong, then the model* (2.2) *reaches consensus if and only if the graph* $\mathcal{G}$ *is aperiodic. Otherwise the model is not convergent and opinions oscillate for almost all* $x(0)$.

As already affirmed, we'll give the details for the continuous time case. Here we show two dynamics: the first over the strong graph of example (2.1.1) which reaches convergence, and the second, built on a periodic graph, which doesn't converge.

Example 2.1.2. For the graph of (2.1.1)



the dynamics $x(k+1) = Wx(k)$, with initial opinion vector $[0{,}0{,}5{,}0]^\top$ has the following trajectories:

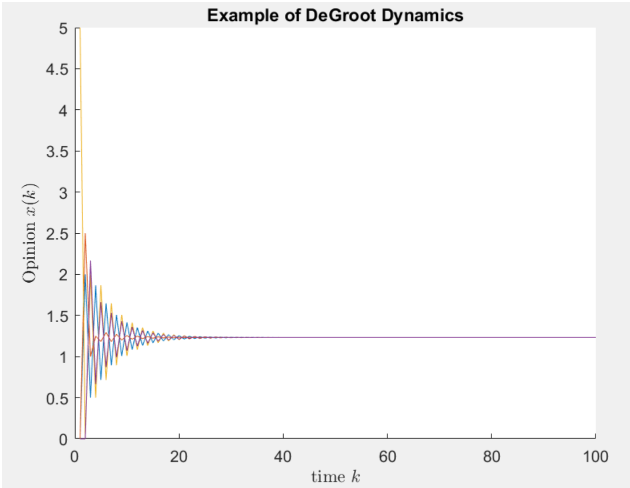
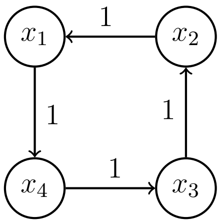


Figure 2.1: Trajectories for the DeGroot dynamics in example (2.1.1)

Example 2.1.3. For the following dynamics on the periodic graph of period 3



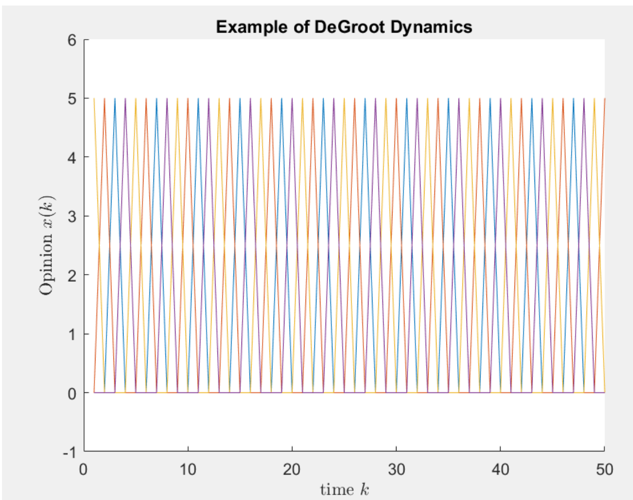


Figure 2.2: Trajectories for a DeGroot dynamics over a periodic graph

The trajectories shown in figure 2.1 and 2.2 show continuous trajectories just for visualization reasons, opinions actually change only at discrete steps.

It's indeed easy to see that a dynamics with a W of the form of example (2.1.3) cannot converge as W is a periodic matrix with period 4, i.e.

$$W^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} W^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} W^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} W^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} W^5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Indeed W^4 is the identity matrix and the dynamics keeps oscillating. At each time instant each individual i observes that of $i + 1 \mathrm{mod}$ n-th individual and *"copies"* its opinion.

Given the idea of how each individual updates its opinion, it is now necessary to see how a continuous model can be built.

2.2 Continuous Models

2.2.1 Abelson Model

Abelson proposed in [1] a continuous counterpart of the DeGroot model. Let's build this continuous time model from DeGroot's with the following reasoning:

Just recalling that $1 - w_{ii} = \sum_{j \neq i} w_{ij}$ we arrive to:

$$x_i(k+1) - x_i(k) = \sum_{j \neq i} w_{ij} x_j(k) + w_{ii} x_i(k) - x_i(k) = \sum_{j \neq i} w_{ij} x_j(k) - (1 - w_{ii}) x_i(k) = \sum_{j \neq i} w_{ij} (x_j(k) - x_i(k)).$$

We have arrived to:

$$\underbrace{x_i(k+1) - x_i(k)}_{\Delta x_i(k)} = \sum_{j \neq i} \underbrace{w_{ij} \left[x_j(k) - x_i(k) \right]}_{\Delta_{(j)} x_i(k)}, \qquad \forall i.$$

Supposing that the time of this variation is small, we can justify the continuous model:

$$\dot{x}_i(t) = \sum_{j \neq i} a_{ij} \left(x_j(t) - x_i(t) \right), i = 1, \ldots, n \tag{2.4}$$

Here $A = (A_{i,j})$ describes the influence weight related to the link which connects j to i and again is interpretable as the measure of change $dx_i(t) = \dot{x}_i(t) dt$ corresponding to an instantaneous distance in opinions $a_{ij}(x_j(t) - x_i(t))$, hence how the node j's opinion influences node i's. In Abelson's model the a_{ij} are used, therefore the stochasticity requirement is not required.

We could also define dynamics with time-dependant weights of the form

$$\dot{x}_i(t) = \sum_{j \neq i} a_{ij}(t) \left(x_j(t) - x_i(t) \right), \quad \forall i.$$

Which would describe systems in which the measure of influence between nodes varies over time. This leads to non autonomous systems, which will not be analysed in this dissertation. In general many changes are possible for dynamics of this type: we shall see some other changes in some of the following paragraphs.

We are now interested in the analysis of (2.4) which could easily be rewritten in the form:

$$\dot{x}(t) = -Lx(t)$$

in fact from (2.4)

$$\begin{aligned}
\dot{x}_i(t) &= \sum_{j \neq i} a_{ij} \left(x_j(t) - x_i(t) \right) \\
\dot{x}_i(t) &= \sum_{j \neq i} a_{ij} (x_j(t)) - \sum_{j \neq i} a_{ij} (x_i(t)) \\
\dot{x}_i(t) &= a_{i,\cdot} x - d_i x_i(t) = [(A - D)x]_i
\end{aligned}$$

As stated by [2], this dynamics has been rediscovered in multi-agent control theory as a continuous-time consensus algorithm.

Properties of the model

We now analyse some properties of Abelson model, such as stability, convergence and consensus, depending on the characteristics of the graph $\mathcal{G}$.

We recall the definition of Lyapunov stability and show that this property holds for the Abelson Dynamics.

Definition 2.2.1 (**Lyapunov Stability**)**.** An equilibrium x_0 for a vector field $\mathbf{f} : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^n$, is said to be *Lyapunov stable* if for every neighbourhood U of x_0 there exists a neighbourhood $V \subset U$ such that orbits starting from points inside V, stay inside U for all $t > 0$.

More formally: An equilibrium x_0 is said to be Lyapunov stable if

$$\forall \varepsilon > 0, \exists \delta > 0 s.t. \|x(0) - x_0\| < \delta \Rightarrow \|x(t) - x_0\| < \varepsilon, \forall t > 0.$$

We also recall the definition of asymptotic stability below.

Definition 2.2.2 (**Asymptotic Stability**)**.** An equilibrium x_0 is said to be asymptotically stable if it is Lyapunov stable and

$$\exists \delta > 0 \text{ such that if } \|x(0) - x_0\| < \delta, \text{ and } \lim_{t \to \infty} \|x(t) - x_0\| = 0.$$

In easier terms, an asymptotic stable equilibrium is an equilibrium and, if the initial condition $x(0)$ is in a neighborhood of the equilibrium, the the system reaches asymptotically the equilibrium x_0

We talk of *global* Lyapunov stability and *global* asymptotic stability whenever δ can be chosen arbitrarily in $\mathbb{R}$. If the δ is a real number then we talk of *local* asymptotic stability.

Without giving a detailed dissertation of stability for dynamical systems, we recall that for linear systems, if the matrix describing the linear dynamics is Hurwitz, then Lyapunov stability and asymptotic stability holds.

Theorem 2.2.1 **(Stability).** *The model* (2.4) *on a quasi-strongly connected graph is Lyapunov stable.*

Proof. It sufficient to observe from (1.2.1) that since $\lambda_0 = 0$ has algebraic multiplicity equal to the geometric multiplicity, all Jordan blocks associated to λ_0 are trivial and every other eigenvalue is such that $\mathrm{Re}\lambda > 0$. □

Observation 4. *The model* (2.4) *is not asymptotically stable (as there are eigenvalues λ such that $|\lambda| = 0$)*

Corollary 2.2.1 (Convergence of Abelson Model)**.** For any nonnegative matrix A, the limit $P^\infty = \lim_{t\to\infty} e^{-Lt}$ exists and the vector of opinions $x(t)$ converges to $x(t) \xrightarrow[t\to\infty]{} x^\infty = P^\infty x(0)$.

Proof. It is sufficient to observe that all eigenvalues of $-L$ are nonpositive, therefore $\lim_{t\to+\infty} e^{-Lt}$ exists and therefore it also exists $x^\infty = P^\infty x(0)$. □

Let us consider again the two examples (2.1.1) and (2.1.3), to complete the intuition of the differences in the convergence properties: For the dynamics of the (2.1.1), the Abelson's dynamics

$$\dot{x}(t) = -Lx(t),$$

with same initial condition as before $x(0) = [0,0,5,0]^\top$ produces the following trajectories

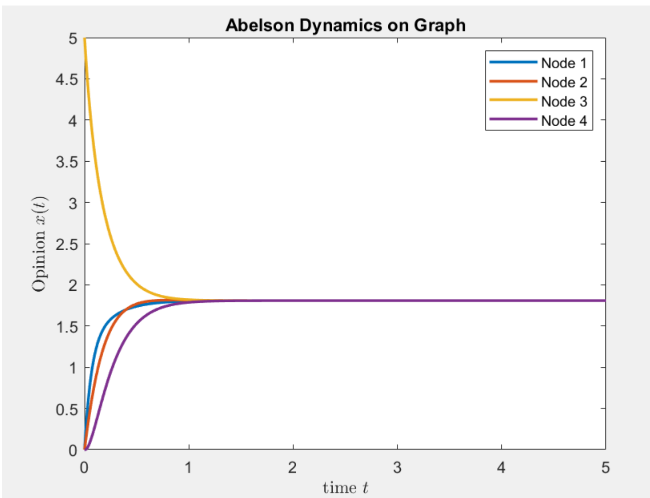


Figure 2.3: Abelson Dynamics over the graph of example (2.1.1)

We now see that also in the case of the graph in (2.1.3), convergence is reached: as the trajectories for the continuous-time dynamics in the figure 2.4 below show.

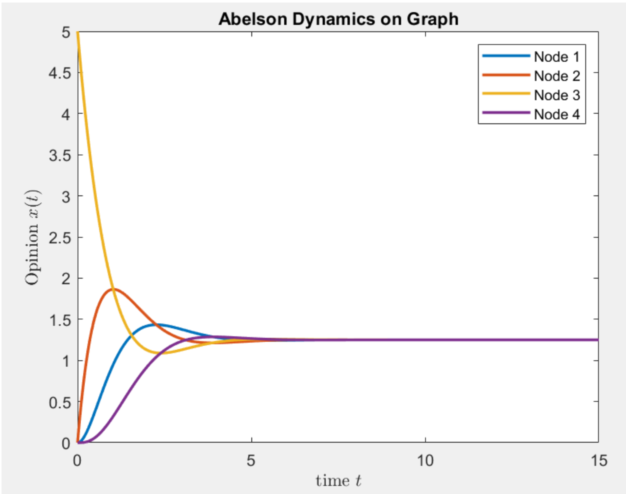


Figure 2.4: Trajectories for an Abelson Dynamics over the graph of example (2.1.3)

Consensus for a continuous time dynamics

The definition of *consensus* is analogous to that of the discrete case

Definition 2.2.3 (Consensus for a continuous dynamics). A continuous dynamics reaches consensus if the final *opinions* coincide:

$$x_1^\infty = \ldots = x_n^\infty \text{ for any initial condition } x(0).$$

We again give a characterization of consensus in the case of an Abelson dynamics. For the Abelson dynamics, the condition under which convergence is reached is quite simple: the dynamics always reaches consensus if the graph has a directed spanning tree. We shall see that this behaviour of the dynamics fails to capture the persistence of dissensus between individuals, while the dynamics we'll see in the following chapter shows a persistence of disagreement even on strong graph (those with a directed spanning tree). We state this in the following theorem.

Theorem 2.2.2. *The Abelson Model reaches consensus if and only if $\mathcal{G}[A]$ is quasi-strongly connected (i.e. has a directed spanning tree). The opinions, in this case, converge to the limit:*

$$\lim_{t\to\infty} x_1(t) = \ldots = \lim_{t\to\infty} x_n(t) = p_\infty^\top x(0)$$

Proof. Reaching consensus means that the following condition must hold:

$$P^\infty x_0 = x^\infty = c\mathbb{1}_n. \tag{2.5}$$

Therefore all the columns of P^∞ are a multiple of the unitary column $\mathbb{1}_n$. In other words, the following condition is a consequence of (2.5):

$$P^\infty = \mathbb{1}_n p_\infty^\top. \tag{2.6}$$

Now, by using the fact that $\mathbb{1}_n$ is an equilibrium we arrive to:

$$1\mathbb{1}_n = P^\infty \mathbb{1}_n = \mathbb{1}_n p_\infty^\top \mathbb{1}_n \implies \mathbb{1}_n p_\infty^\top = 1. \tag{2.7}$$

Therefore:

$$P^\infty x_0 = \mathbb{1}_n p_\infty^\top x_0. \tag{2.8}$$

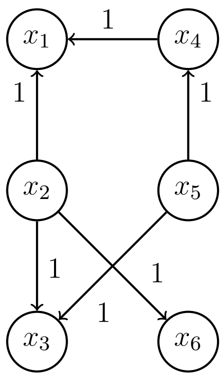
The (2.8) component-wise reads:

$$\lim_{t\to\infty} x_i(t) = p_\infty^\top x(0), \qquad \forall i \in 1, \ldots, n.$$

□

We conclude by showing an example of a graph without the property of quasi-strongly connection and we show that trajectories do not converge to a consensus.

Example 2.2.1. We now consider a dynamics over a graph of the form represented below.



We see that in this case there exists an initial condition s.t. consensus is not reached. To explain why consensus may not be reached for all initial conditions x_0 we observe that the nodes x_2 and x_5 are not influenced by any other node's opinion, therefore never change their state x_i (in fact $\dot{x_2}(t) = 0$ and $\dot{x_5}(t) = 0$. So for every initial condition with $x_2(0) \neq x_5(0)$, the dynamics doesn't reach consensus. Indeed taken the two nodes x_2 and x_5 it doesn't exist one node from which they are both reachable.

What is surprising is that in such dynamics it is sufficient to have a graph spanned by a directed tree (i.e. one node which can influence, albeit indirectly, any other node) to gain consensus. The Abelson model therefore doesn't explain the persistence of dissensus.

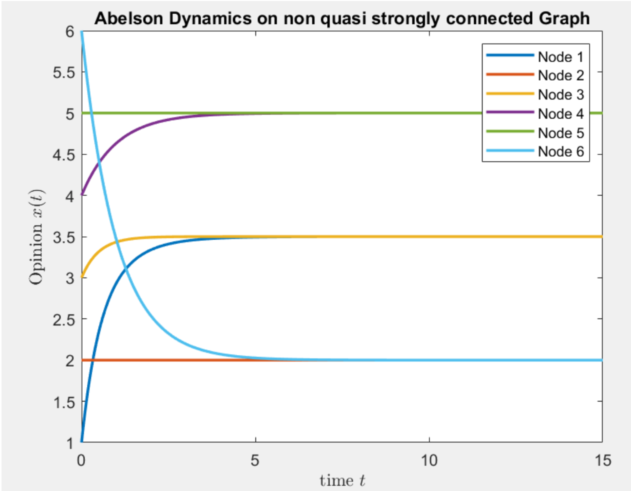


Figure 2.5: Trajectories for Abelson Dynamics on graph of example (2.2.1)

2.3 Unreality of consensus: Generalisations of Abelson and Taylor's model

Reaching a total agreement in a group of n individuals is quite unreal and in everyday life the persistence of disagreement between individuals is often observed. Looking for *social cleavage* has lead to generalisations of Abelson's model and the introduction of new models.

2.3.1 Taylor's model: social cleavage and prejudices

In [8] , Taylor generalised Abelson's model proposing the introduction of nodes with *static* opinion s_i. They would be interpretable as *communication sources*, for example the mass media.

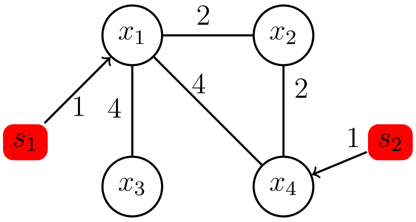
The model (2.4) is easily generalised to

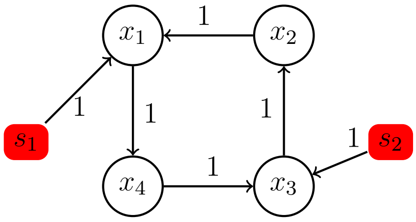
$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij} \left(x_j(t) - x_i(t)\right) + \sum_{k=1}^{m} b_{ik} \left(s_k - x_i(t)\right) \tag{2.9}$$

where m is the number off *communication sources* and b_{ik} measures how the i-th node is influenced by the k-th source.

In this model we have the nonnegative matrix $A = (a_{ij})$ containing the influence weights of arcs between individuals and the nonsquare $n \times m$ matrix $B = (b_{ik})$ of *external static influence*, also known in literature ([8]) as matrix of *persuasibility constants*. An agent is free of external influence if $b_{i1} = \cdots = b_{im} = 0$, while agents with $\sum_{k=1}^{m} b_{ik} > 0$ are influenced by communication sources.

To see how the evolution of the dynamics is changed let us consider again our examples (2.1.1) and (2.1.3), with the introduction of two communication sources, i.e. the new graphs become:





$$A = \begin{bmatrix} 0 & 2 & 4 & 4 \\ 2 & 0 & 2 & 0 \\ 4 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 \end{bmatrix} B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

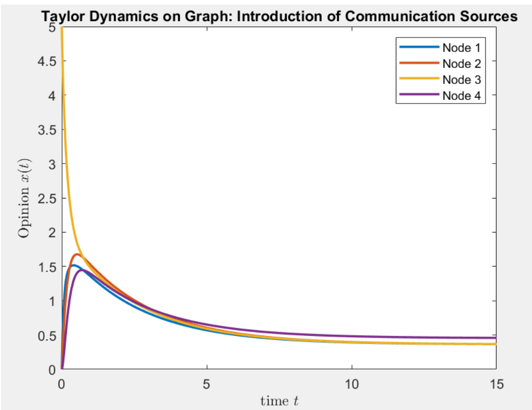


Figure 2.6: Taylor Dynamics over graph of example (2.1.1)

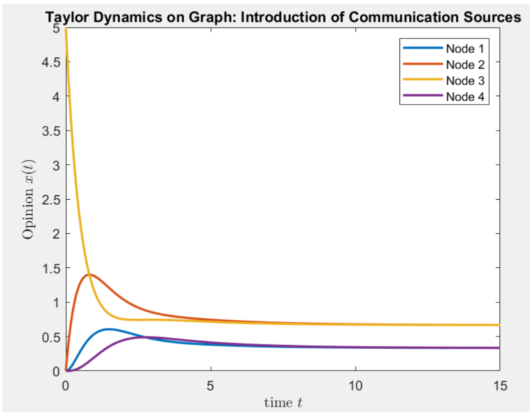


Figure 2.7: Taylor Dynamics over graph of example (2.1.3)

In [8], Taylor showed the *asymptotically stability* of model (2.9) and characterised the only equilibrium in terms of $s_1, \dots, s_k$.

Here we only prove the result for the existence of the equilibrium in Taylor's model.

Theorem 2.3.1. *Taylor dynamics is always convergent:*

$$\textit{For any } x(0) \textit{ and } s_1, \dots, s_k \textit{ there exists } x(\infty) = \lim_{t \to \infty} x(t).$$

Proof. Taylor's model can be considered as an Abelson model with $n + k$ agents, where the last k never update their opinion, therefore $x_{n+i} = s_i$, for $i = 1, \dots, k$. Here, corollary (2.2.1) can be applied and model (2.9) is always convergent. □

2.3.2 Prejudiced Agents

To conclude the chapter we introduce one last generalization of Abelson dynamics. It is similar to the introduction of *communication sources*, but in this kind of dynamics each agent has got its own "prejudice", i.e. some self belief.

The introduction of prejudice is carried out through this model:

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}\left(x_j(t) - x_i(t)\right) + \gamma_i\left(u_i - x_i(t)\right) \qquad \forall i \in \{1, \dots, N\}. \tag{2.10}$$

This model can be obtained by (2.9) just defining:

$$\gamma_i \triangleq \sum_{m=1}^{k} b_{im} \geq 0 \text{ and } u_i \triangleq \gamma_i^{-1} \sum_{m=1}^{k} b_{im} s_m,$$

with $u_i = 0$ if $\gamma_i = 0$.

In literature an agent is called *prejudiced* if $\gamma_i > 0$. The eternal inputs u_i are referred to in literature as *prejudices*. Of course an individual can be totally close to other people influence, therefore having $a_{ij} = 0$ with dynamics:

$$\dot{x}_i = \gamma_i(u_i - x_i(t)).$$

It is immediate that in this case its opinion converges to its prejudice:

$$x_i(t) \xrightarrow[t\to\infty]{} u_i$$

If its prejudice coincides with its initial opinion $u_i = x_i(0)$ the agent is stubborn and never changes its opinion (it works as a communication source).

We prove stability for Taylor model following the same argument used in [2]. This will also help to see how proving stability results for the dynamics described in the following chapter will require some more effort.

2.3.3 Stability in Taylor's Model

To analyse the stability properties for system (2.9) we just consider (2.10) obtained by the former one.

We define $\Gamma = \operatorname{diag}(\gamma_1, \dots, \gamma_n)$ and rewrite model (2.10) as

$$\dot{x}(t) = -(L + \Gamma)x(t) + \Gamma u. \tag{2.11}$$

To determine the stability properties of the system we split the agents in two classes: those with a prejudice or influenced by a prejudiced agent, called *P-dependent* and all the others *P-independent*. By just renumbering the agents we obtain agents $1, \dots, r$ P-dependent and agents $r+1, \dots, n$ P-independent (and it may happen $r = n$. Since P-independence constitutes a partition in the opinion vector $x_{(t)}$, denoting with $x_1(t)$ and $x_2(t)$ the P-dependent and P-independent nodes opinions we can decompose (2.10) as:

$$\dot{x}^1(t) = -\left(L^{11} + \Gamma^{11}\right)x^1(t) - L^{12}x^2(t) + \Gamma^{11}u^1 \tag{2.12}$$

$$\dot{x}^2(t) = -L^{22}x^2(t). \tag{2.13}$$

where the dynamics is evidently described by $-L^{22}$ for the P-independent nodes while for a P-dependent node the dynamics is

$$\dot{x}_i = \sum_{\substack{j\neq i\\ j\in\text{P-indep}}} a_{ij}(x_j^1(t)-x_i(t))+\sum_{\substack{j\neq i\\ j\in\text{P-dep}}} a_{ij}(x_j^2(t)-x_i(t))+\gamma_i(u_i-x_i(t)) = -[L^{11}+\Gamma^{11}x^1(t)]_i-[L^{12}x^2(t)]_i.$$

Observation 5. *The matrix L^{22} is a laplacian matrix of size $(n-r)\times(n-r)$, while matrix L^{11} is not in general a laplacian matrix, having $L^{11}\mathbb{1}_r \geq 0$.*

Having decomposed the dynamics within this partition of nodes we may state and prove the theorem below.

Theorem 2.3.2 (Convergence for a Taylor Dynamics). *Let the set of individuals be partitioned into $r\geq 1$ P-dependent nodes and $n-r\geq 0$ P-independent nodes. Then the dynamics (2.12) is asymptotically stable, i.e. the matrix $-(L^{11}+\Gamma^{11})$ is Hurwitz. The limit opinion to which the P-dependent vectors coincide is*

$$x^1(\infty) = M\begin{bmatrix} u^1 \\ x^2(\infty)\end{bmatrix}, \text{ with } M \triangleq \left(L^{11}+\Gamma^{11}\right)^{-1}\begin{bmatrix}\Gamma^{11} & L^{12}\end{bmatrix} \tag{2.14}$$

Proof. The proof follows from the properties of M-matrices. Indeed, using (1.2.1), we may state that $L^{11}+\Gamma^{11}$ is an M-matrix (we recall that Γ^{11} is diagonal). We now have to prove that the eigenvalue λ_0 of (1.2.1) is >0. Let us suppose $\lambda_0=0$ and let p be the nonnegative left eigenvalue $p^\top(L^{11}+\Gamma^{11})=0$. By multiplying by $\mathbb{1}_r$ and observing that $L_{11}\mathbb{1}_r\geq 0$ we arrive to

$$p^\top\Gamma^{11}\mathbb{1}_r = 0 \Rightarrow p^\top\Gamma^{11}=0 \Rightarrow p_i = 0 \text{ when } \gamma_i>0,$$

hence $p_i=0$ for all prejudiced agents.
Since $p^\top L^{11}=0$ for any j such that $p_j=0$ we get

$$\sum_{i\neq j} p_i a_{ij} = p_j\sum_{j\neq i} a_{ji}.$$

which is $p_i=0$ whenever $a_{ij}>0$. If node j is connected to node i and $p_j=0$ then $p_i=0$. This clearly implies $p=0$ which contradicts the choice of p. We conclude that choosing $\lambda_0=0$ leads to a contradiction, therefore it must be $\lambda_0>0$. ☐

We shall now discuss how the final opinion in (2.10) is composed by stating without proof (which can be found in [2]) the following theorem.

Theorem 2.3.3. *The Matrix M is stochastic, therefore the final opinion in model (2.10) is a convex combination of u^1 and $x^2(\infty)$, i.e. of the prejudice and on the final opinions of the P-independent agents.*

Chapter 3

Discontinous Dynamics: Quantized Behaviour

3.1 Introduction

In the previous chapter we dealt with dynamics of the form

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}(i)} a_{ij} \left[x_j(t) - x_i(t) \right], \qquad i \in \mathcal{I} = \{1, \dots, N\},$$

where $a_{ij} \in \mathbb{R}^+$ were somehow measures of the strength of the influence of the node j over i and we denote with $\mathcal{N}(i)$ the set of neighbors of node i. We considered the main known generalisations of this model with the introduction of prejudiced agents and communication sources, reaching Taylor's model. It has also been observed that in the basic version of Abelson model (2.4), when an individual can influence all the others (although indirectly) then, *consensus* is asymptotically achieved, where we have defined consensus as the situation in which

$$\exists \alpha \in \mathbb{R} : x_i(t) \xrightarrow{t \to +\infty} \alpha, \forall i \in \mathcal{I},$$

namely the situation in which the state vector tends to a constant vector of the form $\alpha \mathbb{1}^T = \{\alpha, \dots, \alpha\}^T$. We are now introducing a new model first presented in [3], where opinions (or behaviors) can be thought as displayed only through a filter which is modelled through a function $\mathbf{q} : \mathbb{R} \to \mathbb{Z}$ called quantizer, as it values only *integer* behaviors. Therefore this model describes how people are influenced by the discrete behaviors of other individuals.

Let us introduce more formally the definition of a quantizer and present the model:

Definition 3.1.1 (Quantizer). A *quantizer* $\mathbf{q} : \mathbb{R} \to \mathbb{Z}$ is a map taking real valued values and transforming them into integers.

Ideally, any function $f : \mathbb{R} \to \mathbb{Z}$ can be considered a quantizer. In our dissertation we choose

$$q(s) = \left\lfloor s + \frac{1}{2} \right\rfloor. \tag{3.1}$$

In this dynamics j can only influence i (with strength proportional to the weight a_{ij} as before) but the node i observes only the filtered $q(x_j(t))$ behavior at time t. We therefore define:

Definition 3.1.2 (Quantized Dynamics). A dynamics of the form

$$\dot{x}_i\left(t\right) = \sum_{j \in \mathcal{N}(i)} a_{ij}\left[q(x_j\left(t\right)) - x_i\left(t\right)\right], \qquad i \in \mathcal{I} = \{1, \ldots, N\}, \tag{3.2}$$

is called *quantized dynamics*.

Alternative forms of the Quantized Dynamics

It might be useful rephrase the dynamics (3.2) in matrix form: this is easily accomplished. From (3.2):

$$\begin{aligned}
\dot{x}_i(t) &= \sum_j a_{ij}\left[q\left(x_j(t)\right) - x_i(t)\right] \\
&= \sum_{j \neq i} a_{ij}\left(q\left(x_j(t)\right) - x_j(t) + x_j(t) - x_i(t)\right) \\
&= \sum_{j \neq i} a_{ij}\left(x_j(t) - x_i(t)\right) + \sum_{j \neq i} a_{ij}\left(q\left(x_j(t)\right) - x_j(t)\right) \\
&= -Lx + A\left(\boldsymbol{q}(x) - x\right)
\end{aligned} \tag{3.3}$$

The dynamics can easily also be seen as

$$\dot{x} = -Dx + A\boldsymbol{q}(x) \tag{3.4}$$

by just recalling that $L = D - A$.

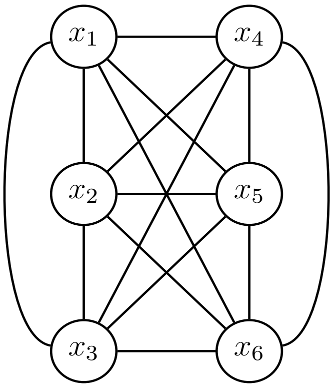
We give one last interpretation of (3.3) by writing:

$$\begin{aligned}
\dot{x} &= -Lx + A(\mathbf{q}(x) - x) \\
&= -L\left(x - x_a\mathbf{1}\right) + A(\mathbf{q}(x) - x)
\end{aligned} \tag{3.5}$$

where $x_a(t) = \frac{1}{N}\mathbf{1}^\top x(t)$ is the average of the vector of opinions $x(t)$. This just follows from the fact that $L\mathbb{1} = 0$ and the dynamics (3.2) can be seen as a classical consensus system which is perturbed by the other states' quantization errors.

To see how the quantized dynamics (3.2) differs from the classical ones presented in the previous chapter, we could start by investigating what happens to the state of opinions on two particular undirected graphs: the complete and the path graphs (1.1.1 and 1.1.2) by looking at two numerical simulations.

Complete Graph



Starting with random initial conditions in [0,10] we obtain this evolution of behaviors/opinions.

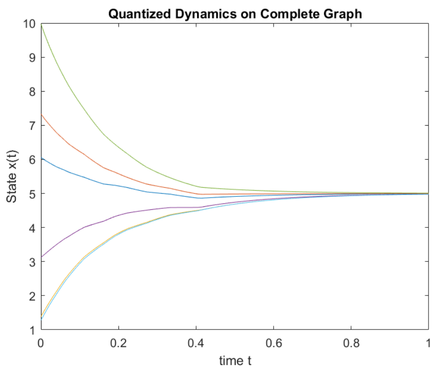
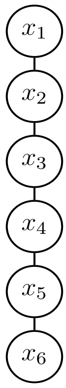


Figure 3.1: Evolution of Quantized Dynamics on Complete Graph with 6 nodes

Path Graph



Starting with random initial conditions in [0,10] we obtain this evolution of behaviors/opinions.

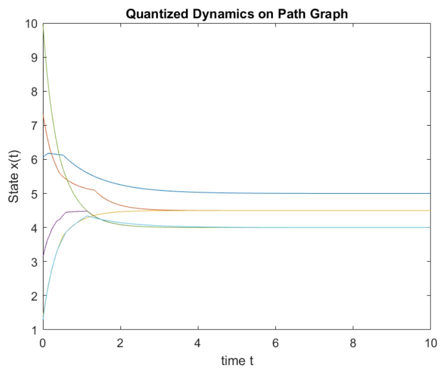


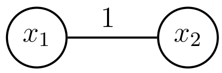
Figure 3.2: Evolution of Quantized Dynamics on Path Graph with 6 nodes

Despite both graphs being strongly connected, consensus is reached only in the case of the complete graph, while in the case of the dynamics on the path graph, consensus is not reached. The interaction chosen for this model then leads to the persistence of disagreement without the need for introduction of either stubborn agents with static opinions or prejudiced agents.

3.2 Generalised solutions for discontinuous ODEs

It is easily noticeable that the vector field beneath the dynamics in (3.2) is discontinuous. Let's consider the simplest example of graph and define the quantized dynamics over it.

Example 3.2.1 (Bidirectional Interaction)**.** Let's consider the following dyadic interaction:



The vector field for this simple dynamics is discontinuous along the set: $\{(x_1, x_2) \in \mathbb{R}^2$ s.t. $x_1 = \frac{k_1}{2}$ or $x_2 = \frac{k_2}{2}$, being k_i integers$\}$.

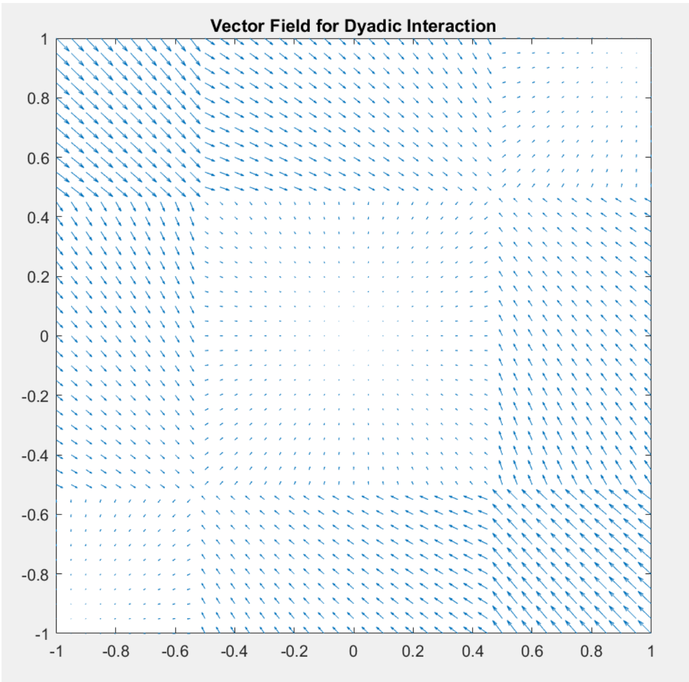


Figure 3.3: Vector field for the quantized dynamics of (3.2.1)

The nice property of this vector field is its affinity in $\mathbb{R}^n$ hypercubes; more precisely, if we define

$$S_{\mathbf{k}} = \left\{ x \in \mathbb{R}^N : k_i - \frac{1}{2} \leq x_i < k_i + \frac{1}{2}, i = 1, \ldots, N \right\}. \tag{3.6}$$

we observe that for every $\boldsymbol{k} \in \mathbb{Z}^n$, $q(x_i)$ is constant on $S_{\mathbf{k}}$.

From now on we'll denote with $\mathbf{q}(x)$ the vector whose components are $q(x_i)$, being $q(s) = \lfloor s + \frac{1}{2} \rfloor$ as defined before. Let's state these considerations in:

Observation 6. *The system* (3.2) *is affine if restricted to each set* $S_{\mathbf{k}}$ *and its right-hand side is discontinuous on the set* $\Delta = \bigcup_{\mathbf{k} \in \mathbb{Z}^N} \partial S_{\mathbf{k}}$*, having denoted by* $\partial S_{\mathbf{k}}$ *the boundary of* $S_{\mathbf{k}}$.

Here we have to face a countable infinite union of hyperplanes over which the vector field is not continuous. Indeed whenever $x_i = k_i + \frac{1}{2}$ we have an hyperplane over which the vector field f is discontinuous. For those value of x_i we have a discontinuity in the ith component of the vector field.

Here the dynamics, in the form $\dot{x} = f(x)$ is described by first order ODEs which, in a *classical* context with a continuous vector field would require $C^1([t_0, +\infty)$ solutions, but this kind of solutions cannot solve the ODEs in the classical sense, as we'll see in the following paragraph. Then we have to introduce *generalized* solutions. We introduce two notions and then try to whether such generalized solutions exist.

3.3 Generalised Solutions

We now follow the approach of [3] in describing the definitions of generalised solutions for a discontinuous dynamics. Some of the examples are again inspired by [3] and [5]. We shall start by explaining why it is not enough to look for solutions in a distributional sense and therefore why we need to find a way to somehow "*extend*" our vector field in correspondence of discontinuities points, moving towards the definition of Krasovskii solutions.

Before going through the paragraph we recall the definition of classical solution of an ODE which will be mentioned in various considerations. We limit to ODEs of the first order.

Definition 3.3.1 (**Classical solution of an ODE**)**.** Given a differential equation of the first order

$$F(t, x, \dot{x}) = 0,$$

a function $x_{sol}(t) : I \subseteq \mathbb{R} \to \mathbb{R}$ is a classical solution if it is of class $C^1(I)$ and

$$F(t, x, \dot{x}) = 0 \ \forall x \in I.$$

Solutions for a discontinuous ODE

Observing the dynamics (3.2) we might wonder if classical solutions satisfy the equations in (3.2) in a classical sense. By Darboux theorem, if $x(t)$ is a classical solution the $\dot{x}(t)$ cannot have jump discontinuities.

Let's consider the following simple example :

Example 3.3.1 (Non-existence of classical solutions to a discontinuous ODE)**.**

$$\dot{x} = \begin{cases} 1 & x \in [1,2) \\ 2 & x \in [2,3]. \end{cases} \tag{3.7}$$

If a $C^1([1,3])$ solution $x(t)$ existed, then this should satisfy Darboux theorem, i.e. $\forall z \in [f'(1), f'(3)]$, there should exist $t \in [1,3]$ s.t. $\dot{x}(t) = z$. This contradicts the differential equation (3.7). When introducing discontinuous RHS we must therefore extend the search for solutions to generalised solutions. The class of function which is natural to consider is that of the *absolutely continuous functions*, whose definition we briefly recall from [9].

Definition 3.3.2 (**Absolutely Continuous Function**). A function f defined on an interval $I = [a, b]$ is said to be *absolutely continuous* on I, if for every $\epsilon > 0$ there exists $\delta > 0$ such that:

$$\sum_{i=1}^{k} |f(b_i) - f(a_i)| < \varepsilon \tag{3.8}$$

for *any* finite collection of nonoverlapping intervals $[a_1, b_1], [a_2, b_2], \ldots, [a_k, b_k]$ in I satisfying $\sum_{i=1}^{k} |b_i - a_i| < \delta$.

The definition may seem a bit abstract but it can be shown that these are the most general function which satisfy the fundamental theorem of calculus which we recall below, as in [9].

Theorem 3.3.1 (Fundamental Theorem of Calculus)**.** *A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous if and only f' exists almost everywhere on (a, b), $f'$is integrable on (a, b), and*

$$f(x) - f(a) = \int_a^x f'(t)d\lambda \quad \mathit{for} \quad x \in [a, b]. \tag{3.9}$$

we use $d\lambda$ to indicate the Lebesgue integral of f with respect to a measure λ.

The importance of this theorem relies in the fact that the widest class of functions for which the fundamental theorem of calculus holds is that of the absolutely continuous ones. For this reason the first type of solutions we introduce for the dynamics in (3.2) is that of Carathéodory solutions.

Definition 3.3.3 (**Carathéodory Solution**). Let $I \subset \mathbb{R}$ be an interval of the form $(0, T)$. An *absolutely continuous* function $x : I \to \mathbb{R}^N$ is a *Carathéodory Solution* of $\dot{x} = f(x)$ if satisfies the equation *almost everywhere*, or equivalently, if it is a solution of the following integral equation:

$$x(t) = x_0 + \int_0^t f(x(s))\mathrm{d}s. \tag{3.10}$$

where we have substituted the generic measure λ with the Lebesgue measure on $\mathbb{R}$.

Let us consider again (3.7): we have observed that if a classical solution existed, it would obey at the some moment the ODE and Darboux theorem, but this leads to an absurd as $\dot{x}(t)$ can only attain the values 1 or 2 and no value between the two.

However we get that the function

$$x(t) = \begin{cases} t & t \in [1{,}2) \\ 2t & t \in [2{,}3] \end{cases} \tag{3.11}$$

is a Carathéodory solution as it solves the differential equation in the integral form of definition 3.3.3.

Example 3.3.2 (Carathéodory Solutions for a discontinuous dynamics)**.** Another example that can be consdiered is the following one:

$$\dot{x} = \begin{cases} 1 - x, & x > \frac{1}{2} \\ \frac{1}{2}, & x = \frac{1}{2} \\ -x, & x < \frac{1}{2} \end{cases} \tag{3.12}$$

We observe that no classical solution satisfying $x(0) = \frac{1}{2}$ exists in a neighborhood of $t = 0$ as left and right derivatives would be different, but $x(t) = -\frac{1}{2}e^{-t} + 1$ is a Carathéodory solution of (3.12) starting from $x_0 = \frac{1}{2}$ and $x(t) = \frac{1}{2}e^{-t}$ is a Carathéodory solution of (3.12) starting from $x_0 = \frac{1}{2}$. We also notice that both these solutions are defined over the interval $I = [0, +\infty]$, i.e. are *complete*.

We have to further notice that solutions of this kind, albeit being quite intuitive and rather simple, may lead to strange phenomenons, for instance they might converge to point which are not equilibria of the vector field as we shall see in a more elaborate example (3.4.1).

Another question which comes natural is why not introducing distributional solutions, well known in the field of discontinuous ODEs thanks to the solid theory about distributional derivatives.

3.3.1 Distributional Solutions

We briefly recall the definition of a distribution to understand in what sense it could be a solution to the ODE in (3.2).

Definition 3.3.4 (**Space of test functions**)**.** Let $\Omega \subseteq \mathbb{R}^n$ an open set, then

$$\mathcal{D}(\Omega) = \{\phi \in C^\infty(\Omega) : supp(\phi) = K \subset \Omega \text{ is compact}\}$$

is called space of test functions.

In the above definition we have formally stated that the space of test functions $\mathcal{D}(\Omega)$ consist of all the infinitely times derivable (*smooth*) functions having a compact support in Ω.

Given this definition we are ready to introduce the space of distributions $\mathcal{D}'(\Omega)$ as:

Definition 3.3.5 (**Space of distributions**)**.** The space of distributions on Ω, denoted by $\mathcal{D}'(\Omega)$ is the set of all the linear and continuous functionals $T : \mathcal{D}(\Omega) \to \mathbb{R}$.

Recalling all the properties of the distributions would not be inherent to the dissertation and we refer to [9] for a more complete overview on distributions. What it is important here is that when facing discontinuous ODEs we could think the equation in $\mathcal{D}'$. Let's make an example:

Example 3.3.3 (Discontinuous ODE)**.** Let's imagine we want to solve the differential equation

$$x'(t) = H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geqslant 0 \end{cases}. \tag{3.13}$$

We can affirm that the solution of this ODE is $x(t) = t \cdot H(t)$ thought as distributions, more specifically we mean that:

$$\int_{\mathbb{R}} [s \cdot H(s)]' \phi(s) ds = -\int_{\mathbb{R}} s \cdot H(s)[\phi(s)]' ds = -\int_0^{+\infty} s\phi'(s) ds \tag{3.14}$$

$$= \int_{\mathbb{R}} H(s)\phi(s) ds, \qquad \forall \phi \in \mathcal{D}(\mathbb{R}). \tag{3.15}$$

Therefore the derivative of $x(s) = s \cdot H(s)$ in $\mathcal{D}'$ is $H(s)$ and the ODE is solved.

The need to look for new solutions

The problem in looking for distributions as solutions to the ODE in (3.2) is that we would still end up with function distributions by solving the ODE. Let's indeed recall the following theorem as stated in [9] (345.1).

Theorem 3.3.2. *Suppose $x \in L^1_{loc}(a, b)$. Then x is equal almost everywhere to an absolutely continuous function if and only if the derivative of the distribution corresponding to x is a function.*

The consequence of this theorem is that solutions in the distributional sense of (3.2) are almost everywhere equal to functions solving the ODE. Therefore the functions solving the dynamics in the Carathéodory sense are the same distributions which solve the dynamics in the distributional sense.

We now give a sketch of the proof of theorem (3.3.2).

Proof. It is easy to see that if x is (equal almost everywhere to) an absolutely continuous function then, integrating by parts as done in (3.3.3):

$$T'_x(\varphi) = -\int_a^b \varphi' x d\lambda = \int_a^b \varphi x' d\lambda \tag{3.16}$$

where the steps follow by the compactness of the support of the φ functions and the usual integration by parts. Therefore the distribution T' is a function. This first part of the proof shows that if x is (equal almost everywhere to) an absolutely continuous function then its derivative is again a distribution corresponding to a function.

The converse follows by supposing that the derivative of the distribution is a function, hence $T' = \int_a^b \varphi x'(t) dt$, $\forall \phi \in \mathcal{D}$. We now define:

$$h(y) = \int_a^y x(t) dt \tag{3.17}$$

and observe that $h(y)$ is an absolutely continuous function (it follows by the definition of Lebesgue integral and the definition in 3.3.2. By calling S the distribution corresponding to h:

$$S_h = \int h(y)\phi(y), \forall \phi \in \mathcal{D},$$

then

$$S'_h = -\int h(y)\phi'(y) = \int h'(y)\phi(y) = \int x(y)\phi(y).$$

therefore $S' = x$ in the distributional sense. From the theory of distributions it holds true that since $S' = x$ and $T' = x$ then $T = S + k$ in the distributional sense and this implies:

$$\begin{aligned}\int x(t)\varphi(t)dt = T_x = S(\varphi) + \int k(t)\varphi(t)dt \\ = \int h(t)\varphi(t) + k(t)\varphi(t)dt \\ = \int (h(t) + k(t))\varphi(t).\end{aligned} \tag{3.18}$$

We have then proved that $x(t)$ is equal almost everywhere to the absolutely continuous function $h(t) + k$ □

3.3.2 Carathéodory Solutions

Carathéodory solutions are a natural extension of classical solutions: we just require a function solving a dynamics $\dot{x} = f(x)$ almost everywhere. Let us be more formal and state this as a definition.

3.3.3 Krasovskii Solutions

The idea behind these solutions is that, whenever a discontinuity is met, the behaviour of the function at the discontinuity point is determined by the derivatives of the function in the nearby points. We therefore extend the vector field at a discontinuity point, considering the directions in a neighbourhood of the discontinuity point.

Stating it in a more formal way:

Definition 3.3.6 (**Krasovskii Solution**). An absolutely continuous function $x : I \to \mathbb{R}^N$ is a *Krasovskii* solution of $\dot{x} = f(x)$ if, for almost every $t \in I$, it satisfies

$$\dot{x}(t) \in \mathcal{K}f(x(t)), \tag{3.19}$$

where:

$$\mathcal{K}f(x) = \bigcap_{\delta>0} \overline{\text{co}}(\{f(y) : y \text{ such that } \|x - y\| < \delta\}). \tag{3.20}$$

Let us consider a simple example to see how *Krasovskii* solutions work. The example is taken from [4].

Example 3.3.4. If we consider the vector field for the dynamics $\dot{x} = f(x)$:

$$f(x_1, x_2) = \begin{cases} (0{,}0) & \text{if } (x_1, x_2) = (0{,}0) \\ (1{,}0) & \text{if } (x_1, x_2) \neq (0{,}0) \end{cases}$$

With $t_0 = 0$ and $x_0 = (0{,}0)$. Looking for a Krasovskii solution we notice that the admissible vector field at (0,0) is the set of all the convex combinations of (0,0) and (1,0), i.e. (1,0), hence $x(t) = (t{,}0)$ is a Krasovskii solution.

Sometimes in mathematical literature *Philippov Solutions* are considered. In our settings, for the specific dynamics (3.2), the two solutions coincide. We also observe that in general any Carathéodory solution is also a Krasovskii solution. Before going through examples and building results for the dynamics (3.2), we investigate whether existence and uniqueness of solutions exist.

3.3.4 Properties of solutions

We state the following theorem to be able to guarantee existence, boundedness and completeness for the system described in (3.2).

Theorem 3.3.3 (Properties of Solutions). *The following properties of solutions hold:*

(i) (Existence) For any initial condition x_0 there exists a Carathéodory solution and a Krasovskii solution of (3.2)

(ii) (Boundedness) Any Carathéodory solution of (3.2) *is bounded on its domain.*

(iii) (Completeness)Any Carathéodory or Krasovskii solution starting at $t_0 \in \mathbb{R}$ is defined on the set $[t_0, +\infty)$.

The theorem 3.3.3 confirms the well posedness of the problem as Carathéodory and Krasovskii solutions for a dynamics of the form (3.2) do always exist. Furthermore these solutions cannot explode (see also proposition 3.3.1). Any (Carathéodory or Krasovskii) solution of (3.2) is also defined on set of the form $[t_0, +\infty]$, a quite good property.

Observation 7 (Uniqueness of solutions). *We have already seen in* (3.12) *that uniqueness of solutions is lost. In general, Carathéodory or Krasovskii solutions are not unique.*

If we extend example (3.12) to the following dyadic dynamics:

$$\dot{x}_1 = q\left(x_2\right) - x_1,$$
$$\dot{x}_2 = q\left(x_1\right) - x_2,$$

Considering as initial condition $\left(x_1(0), x_2(0)\right)^\top = (\frac{1}{2}, \frac{1}{2})^\top$, the solutions

$$x(t) = \left(1 - \frac{1}{2}e^{-t}, 1 - \frac{1}{2}e^{-t}\right)$$
$$x(t) = \left(\frac{1}{2}e^{-t}, \frac{1}{2}e^{-t}\right)$$

both satisfy the equation in the Carathéodory sense, start at $x_0 = (\frac{1}{2}, \frac{1}{2})^\top$ and are complete. The former asymptotically tends to $x^1_{lim} = (1,1)$ while the latter reaches $x^2_{lim} = (0,0)$. This example shows that uniqueness is not guaranteed and in general, taking initial conditions on the set $\Delta = \bigcup_{\mathbf{k} \in \mathbb{Z}^N} \partial S_{\mathbf{k}}$, the dynamics leads to multiple solutions.

When considering a dynamics, results of boundedness and monotonicity for the smallest and largest components of $x(t)$ are stated and proved. For example it often happens that the smallest component is nondecreasing while the largest component is nonincreasing. Here we give a property of monotonicity and a limit for the minimum and maximum quantization level for Carathéodory solutions of (3.2).

Proposition 3.3.1 (Monotonicity and limit for minimum and maximum quantization lever for Carathéodory solutions)**.** Let $x(t)$ be a Carathéodory solution and define

$$x_m(t) = \min \{x_i(t), i \in \mathcal{I}\} \tag{3.21}$$
$$x_M(t) = \max \{x_i(t), i \in \mathcal{I}\} \tag{3.22}$$

the minimum and the maximum component in the opinion vector $x(t)$ and let's define

$$q_m(t) = q\left(x_m(t)\right) \tag{3.23}$$
$$q_M(t) = q\left(x_M(t)\right) \tag{3.24}$$

the quantized levels corresponding to $x_m(t)$ and $x_M(t)$. Then:

- q_m is nondecreasing
- q_M is nonincreasing
- both q_m and q_M are definitively constant

The proof follows that of [3].

Proof. Let's consider a Carathéodory solution $x(t)$ of the dynamics in (3.2) and let m be any index such that

$$x_m(t) = \min \{x_i(t), i \in \mathcal{I}\},$$

then for any $i \in \mathcal{I}$, $x_i(t) \geq x_m(t)$.

It is now just necessary to consider the lowest quantization level related to $x_m(t)$, i.e. $q(x_m(t))$. Now:

- if $x_m(t) \in \left[q_m(t) - \frac{1}{2}, q_m(t)\right]$ then: $q(x_i(t)) \geq x_m(t)$, $\forall\, i \in \mathcal{I}$, hence $\dot{x}_m(t) = \sum_j a_{mj}\left[q\left(x_j(t)\right) - x_m(t)\right] \geq 0$. It follows that the solution is non decreasing and q_m is non decreasing.
- if $x_m(t) \in \left[q_m(t), q_m(t) + \frac{1}{2}\right]$ then $x_m(t)$ may be decreasing but only up to a certain point as when it decreases down to $x_m(t) = q_m(t)$ we fall in the case on the left and $\dot{x}_m(t) \geq 0$.

We have then found out that $x_m(t)$ is bounded below by $\min\{x_m(0), q_m(x_0)\}$. The same happens with a likewise proof for the index M such that $x_M(t) \geq x_i(t)$, $\forall i \in \mathcal{I}$.

This also means that $x_m(t)$ cannot reach the quantized level below and can only reach the above one and $x_M(t)$ cannot reach the quantized level above and can only reach the below one.

We have proved that the smallest quantization level $q_m(t)$ can only be nondecreasing and the biggest quantization level $q_M(t)$ can only be nonincreasing. Furthermore $q_m(t)$ is bounded and takes values in $\mathbb{Z}$, therefore it must be definitively constant, i.e. there must exist a $T^* \in \mathbb{R}$ and a $q_m^* \in \mathbb{Z}$ such that for any $t \geq T^*$ we have

$$\min\{q(x_i(t),\, i = 1, \ldots, N\} = q_m^*.$$

The same must similarly hold for $x_M(t)$. □

The importance of the proposition stays in the important fact that the number of quantization levels assumed during the dynamics is finite, fixed an initial condition. In particular the number of all the possible quantization levels is

$q_M(0) - q_m(0) + 1$ for each component i and the number of possible vector fields followed by the dynamics with n individuals is $[q_M(0) - q_m(0) + 1]^n$.

Considering a dynamics of the form (3.2), with three nodes having initial opinions $x(0) = (0,1,2)^\top$ and corresponding quantization $q(x(0)) = (0,1,2)^\top$, we know that at most the dynamics will be described by $3^3 = 27$ different affine vector fields.

3.4 Equilibria

In the classical setting an equilibrium is defined as:

Definition 3.4.1 (Equilibrium of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$). A point x^* is said to be a Carathédory (Krasovskii) equilibrium for $\dot{x} = f(x)$ if $x(t) \equiv x^*$ is a Carathéodory (Krasovskii) solution of $\dot{x} = f(x)$.

In the following we'll denote by E_C the set of Carathéodory equilibria and by E_K the set of Krasovskii equilibria.

We look for Carathéodory equilibria by finding constant function satisfying $f(x^*) = 0$ while Krasovskii equilibria are points such that $0 \in \mathcal{K}f\,(x^*)$: indeed starting from x^* if the possible null direction is admissible, then $x(t) \equiv x^*$ is a Krasovskii solution for the dynamics.

We observe that consensus points are always Carathéodory and Krasovskii equilibria of (3.2). It is indeed easy to see that starting from a point $\bar{x} = k\mathbb{1}$ the vector field at $\bar{x}$ reduces to

$$\dot{x} = -D\,k\mathbb{1} + A\,k\mathbb{1} \tag{3.25}$$

which leads to the null vector by the definitions of D and A. We briefly

Furthermore all equilibria are locally asymptotically stable as all the points of the form $x^* = k\mathbb{1}$ belong to the interior of $S_{\boldsymbol{k}\mathbb{1}}$. It happens indeed that, in correspondence of these points (which belong to the interior of $S_{k\mathbb{1}}$), the system (3.2) is in the form:

$$\dot{x} = -Dx + Ak\mathbb{1} \tag{3.26}$$

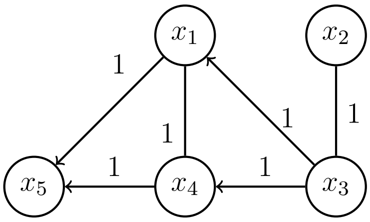
and since $-D$ is Hurwitz, locally asymptotic stability follows as recalled in chapter 2.

Observation 8. *Since, in general, Carathéodory solutions are also Krasovskii solution, it holds that any Carathéodory equilibrium is also a Krasovskii equilibrium, i.e.* $E_C \subset E_K$.

The inverse doesn't necessarily hold, i.e. there are Krasovskii equilibria which are not Carathéodory equilibria. We just need to consider the dynamics (3.2) with initial condition $x^* = \left(k_0 + \frac{1}{2}, k_0 + \frac{1}{2}, \dots, k_0 + \frac{1}{2}\right)^\top$ and observe that $x(t) \equiv x^*$ cannot be a Carathéodory solution of (3.2) while $0 \in \mathcal{K}f(x^*)$.

Let's gather intuition of the properties and observations stated by considering an example on a 6-node graph.

Example 3.4.1. Let us suppose that 6 agents interact according to the graph below:



and let's explicitly write the quantized dynamics for these group of individuals:

$$
\begin{aligned}
\dot{x_1} &= q(x_3) + q(x_4) - 2x_1 \\
\dot{x_2} &= q(x_3) - x_2 \\
\dot{x_3} &= q(x_2) - x_3 \\
\dot{x_4} &= q(x_1) + q(x_3) - 2x_4 \\
\dot{x_5} &= q(x_1) + q(x_4) - 2x_5
\end{aligned}
$$

We immediately see that $x^* = (k, \dots, k)$ are Carathéodory and Krasovskii equilibria. The point $x^* = \left(\frac{1}{2}, 0, 0, \frac{1}{2}, 1\right)^\top$ is a Carathéodory equilibrium which lies on the boundary of $S_{(1,0,0,1,1)}$ since $x_1 = \frac{1}{2}$, $x_4 = \frac{1}{2}$. We might investigate whether the Carathéodory equilibrium is Lyapunov stable and we immediately see that for a small perturbation in the initial condition $\tilde{x}_1(0) = \frac{1}{2} - \alpha$ and $\tilde{x}_1(0) = \frac{1}{2} + \alpha$ with $\alpha > 0$ the solution follows the vector field $f(x) = (1 - 2x_1, -x_2, -x_3, -2x_4, 1 - 2x_5)$ towards the point $\left(\frac{1}{2}, 0, 0, 0, \frac{1}{2}\right)^\top$ which is not a Carathéodory equilibrium.

Within this example we have shown that:

- We find Carathéodory solutions which converge to points that are not Carathéodory equilibria.

This has lead the authors of [3] to introduce the notion of extended equilibrium, presented in the definition below.

Definition 3.4.2. [Extended Equilibrium] Let $\boldsymbol{k} \in \mathbb{Z}$. and $f_{\boldsymbol{k}}$ be the vector field whose components are

$$(f_{\mathbf{k}})_i\,(x) = \sum_{j \neq i} a_{ij}\,(k_j - x_i)$$

which coincides with f on the set $S_{\boldsymbol{k}}$. We define the point $x^* \in \mathbb{R}^n$ an *extended equilibrium* of the dynamics (3.2) if there exists $\boldsymbol{k} \in \mathbb{Z}^n$ such that $f_{\boldsymbol{k}}(x^*) = 0$ and $x^* \in \overline{S_{\mathbf{k}}}$.

It is evident that $E_C \subset E_e \subset E_K$, having denoted

- E_C the set of *Carathéodory equilibria*
- E_e the set of *extended equilibria*
- E_K the set of *Krasovskii equilibria*

If we consider again example (3.4.1) we have that:

$$x^*_{ext} = \left(\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2}\right)$$

is an extended equilibrium since

$$x^*_{ext} \in \overline{S_{(1,0,0,0,1)}}$$

and

$$f_{(1,0,0,0,1)}\,(x^*_{ext}) = 0,$$

but it cannot be a Carathéodory equilibrium as

$$f_5(x^*_{ext}) = 2 - 2x_5 = 1$$

which is not 0. It also holds that

$$x_K = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)^\top$$

is a Krasovskii solution, as it can be shown that $0 \in \mathcal{K}f\,(x_K)$ but this point is neither a Carathéodory equilibrium (as $f(x_K) \neq 0$) nor an extended equilibrium as it should happen that both x_{2_K} and x_{3_K} should be integers, while they are both $\frac{1}{2}$.

The example we have considered shows that there exist extended equilibria like x^*_{ext} which are not consensus points. It also happens that there exist Carathéodory solutions which converge to extended equilibria. In our example it is sufficient to consider any initial condition $x_0 \in S_{(1,0,0,1,1)}$ and, since $f(x_0) = (1-2x_1, -x_2, -x_3, -2x_4, 1-2x_5)$ and there are Carathéodory solutions issuing from an initial condition in $S_{(1,0,0,1,1)}$ converging to x^*_{ext}, for instance the Carathéodory solution:

$$x(t) = \left(\frac{1}{2} - \frac{1}{2}e^{-2t},0,0,\frac{1}{2} - \frac{1}{2}e^{-2t},\frac{1}{2} + \frac{1}{2}e^{-2t}\right)$$

converges to the extended equilibrium $x^*_{ext} = \left(\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2}\right)$.

To gather confidence with the Krasovskii solution we also consider the differential inclusion related to Krasovskii solution in correspondence of the point $x^*_{ext} = \left(\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2})\right)$:

$$\mathcal{K}f\left(x^*_{ext}\right) = \overline{\text{co}}\left\{\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}-1\\0\\0\\-1\\1\end{pmatrix},\begin{pmatrix}-1\\0\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\0\\-1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\0\\-1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix},\begin{pmatrix}-1\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\-1\\1\end{pmatrix}\right\} \tag{3.27}$$

and observe that $\mathbf{0} \in \mathcal{K}f(x^*_{ext})$, therefore the extended equilibrium is also Krasovskii's.

3.5 Numerical Simulations

In this section we briefly discuss the approach used to simulate the quantized dynamics (3.2), implemented in MATLAB from scratch.

3.5.1 The solve_dynamics and solve_dynamics_adams_bashforth functions

In the MATLAB function *solve_dynamics* we have implemented the explicit Euler method to build the solution of the dynamics. Using the article *the* is improper as we have seen that in general the solution for this kind of dynamics is not unique. We then briefly recall the Euler Method to understand how the discontinuity is handled in the numeric approach.

Definition 3.5.1 (**Euler's Method**)**.** This method is built for the construction of the solution to a Cauchy problem of the form:

$$y'(t) = f(t, y(t)) \quad y\left(t_0\right) = y_0.$$

By a discretization of the t variable we may define $t_n = t_0 + nh$ and approximate $y_{n+1} = y_n + hf\left(t_n, y_n\right)$ which requires an evaluation of the vector field at a point computed in a previous step.

The Adams-Bashforth method is similar to the one described above but uses a further approximation: to be more precise we refer to the following definition:

Definition 3.5.2 (**Adams-Bashforth's method**)**.** This method is built for the construction of the solution to a Cauchy problem of the form:

$$y'(t) = f(t, y(t)) \quad y\left(t_0\right) = y_0.$$

The recursive formula for the construction of the points giving the numerical solution is:

$$y_{n+2} = y_{n+1} + \frac{3}{2}hf\left(t_{n+1}, y_{n+1}\right) - \frac{1}{2}hf\left(t_n, y_n\right).$$

From the recursive formula in (3.5.2) it is easy to see that the two previously computed approximations are needed to compute the $n+2$-th approximation.

3.5.2 Matlab Implementation

The two methods have been implemented in the two MATLAB functions already mentioned, solve_dynamics and solve_dynamics_adams_bashforth. This last function has been used as a second order test to confirm the solutions obtained with the first order method in (3.5.1).

These two methods have been fed with the vectorial field f describing the quantized dynamics as presented in (3.1). All the commented scripts can be found in https://github.com/lucacat97/quantized_behaviors_thesis or by scanning the following QR code:

Figure 3.4: QR code to access all the code developed for the numerical simulations in this thesis.

3.5.3 The quantized dynamics app

To ease the process related to the numerical simulations, a MATLAB app has been created. In this paragraph we briefly present the user-friendly interface of the app.

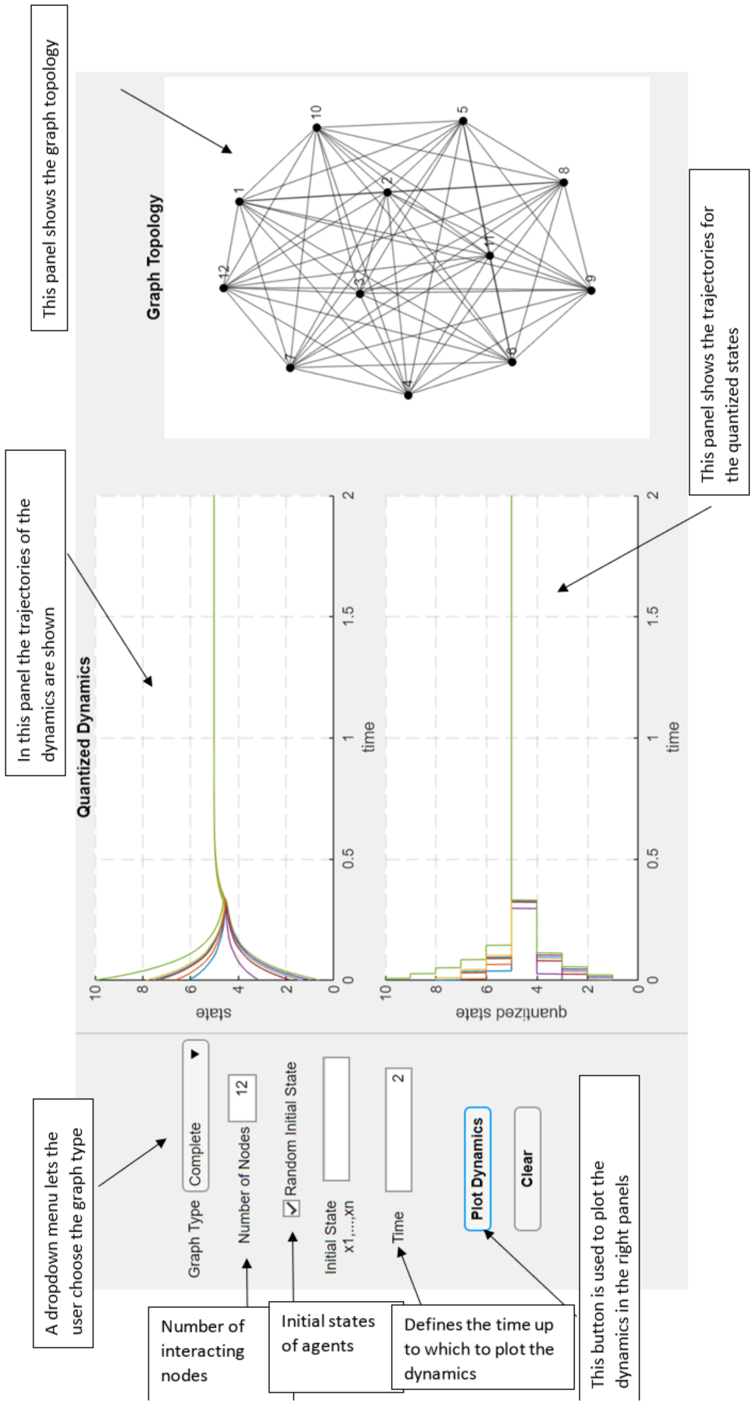


Figure 3.5: Interface for the quantized behaviors app in MATLAB

The pictures above show the interface and the options which can be selected by the user. In particular, the interface contains:

- Graph dropdown menu: allows for the selection of the graph type.
- Number of Nodes box: here the user can type how many agents are interacting
- Random Initial State tick box: when ticked the initial conditions are chosen randomly in the range $0 - 10$
- Initial State box: when the "random initial state tick box" is not ticked here the user can specify the agents initial opinions in the form $x_0 1, x_0 2, \ldots$
- Time box: the user can type the time up to which simulate the dynamics
- Plot Dynamics button: when clicking on this button the dynamics is simulated
- Clear button: it clears the options specified in a previous simulation and allows for a new one to be performed.

and the plot interface is characterized by three panels:

- Upper Central: here the trajectories for individual opinions of the nodes are shown
- Lower Central: here the quantized trajectories for the individual opinions (i.e. $q(x_i(t))$) are shown
- Right: the graph topology is displayed through nodes ad oriented edges.

3.5.4 Some final considerations of the numeric approach

As already suggested to the reader, numerical solutions for this kind of dynamics must be considered with "analytical wisdom". When the dynamics is described by a continuous vector field f, the Euler method has got a local truncation error which decreases with the decrease of the step h. When simulating dynamics described by a discontinuous field, uniqueness is in general lost. The numerical solution obtained is one and is often one of the many obtained for a particular vector field (as we shall see in the elaborate case of the graph cycle). Let's consider the simple of two nodes interacting as in 3.2.1. The system reduces to

$$\dot{x_1}(t) = q(x_2) - x_1(t) \tag{3.28}$$
$$\dot{x_2}(t) = q(x_1) - x_2(t) \tag{3.29}$$

by starting at $x_0 = (\frac{1}{2}, 1)$ we get that

$$x_1(t) = -\frac{1}{2}e^{-t} + 1 \tag{3.30}$$
$$x_2(t) = e^{-t} \tag{3.31}$$

is a Carathéodory solution which is not found analytically. Indeed the numerical simulation only gives an approximation for the following solution:

$$x_1(t) = -\frac{1}{2}e^{-t} + 1 \tag{3.32}$$

$$x_2(t) = 1 \tag{3.33}$$

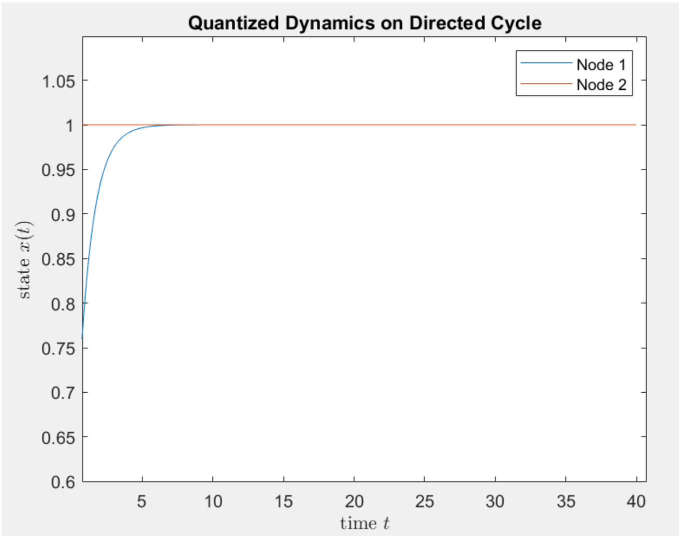


Figure 3.6: The only numerical solution found for the dyadic interaction example. The solution shown is both a Carathéodory and Krasovskii solution, but it is not the only one

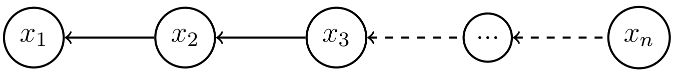
3.6 Equilibria on specific Graphs

In this section we look for equilibria on specific graphs, with a double aim: it will be useful to see how numerical simulations differ in finding such equilibria and it shows how the quantized dynamics may somehow be tricky when using such microscopic approach consisting of following each node's opinion $x_i(t)$.

3.6.1 Directed Line

3.6.2 Carathéodory Equilibria

Let's start by considering a directed line:



For this graph topology the dynamics is of the form:

$$\begin{cases} \dot{x_1} = q(x_2) - x_1 \\ \quad \vdots \\ \dot{x}_{n-1} = q(x_n) - x_{n-1} \\ \dot{x_n} = 0 \end{cases}$$

Carathéodory equilibria are points which have to satisfy $f(x^*) = 0$, hence, for a generic point $x^* \in S_{\boldsymbol{k}}$ it must hold that:

$$\begin{cases} x_1^* &= k_2 \\ x_2^* &= k_3 \\ \vdots \\ x_{n-1}^* &= k_n \\ x_n^* &= x_n(0) \end{cases}$$

By imposing the condition $x^* \in S_{\boldsymbol{k}}$, it must hold that:

$$\begin{cases} k_1 - \frac{1}{2} \leq x_1^* < k_1 + \frac{1}{2} \\ k_2 - \frac{1}{2} \leq x_2^* < k_2 + \frac{1}{2} \\ \quad \vdots \\ k_{n-1} - \frac{1}{2} \leq x_{n-1}^* < k_{n-1} + \frac{1}{2} \end{cases} \tag{3.34}$$

Combining (3.6.2) into (3.34) we arrive to:

$$\begin{cases} k_1 - \frac{1}{2} \leq k_2 < k_1 + \frac{1}{2} \\ k_2 - \frac{1}{2} \leq k_3 < k_2 + \frac{1}{2} \\ \quad \vdots \\ k_{n-1} - \frac{1}{2} \leq k_n < k_{n-1} + \frac{1}{2} \end{cases} \tag{3.35}$$

which leads to

$$\begin{cases} k_1 = k_2 \\ k_2 = k_3 \\ \quad \vdots \\ k_{n-1} = k_n \end{cases}$$

being all the k_i integers. Observing that $x_n(t) \equiv x_n^0$ we get that

$$\begin{cases} x_1^* = q(x_n^0) \\ x_2^* = q(x_n^0) \\ \quad \vdots \\ x_n = x_n^0 \end{cases}$$

Therefore the only extended equilibria are of the form

$$(q(x_n^0), q(x_n^0), \ldots, q(x_n^0), x_n^0)$$

which are consensus if and only if $x_n^0 \in \mathbb{Z}$.

3.6.3 Extended Equilibria

In the setting of extended equilibria, we replace the condition $x^* \in S_{\boldsymbol{k}}$ with the condition $x^* \in \bar{S_{\boldsymbol{k}}}$, therefore, condition (3.34) is the same with a $\leq$ for the second inequality which leads to $k_1 = k_2 = \cdots = k_n$ but here a difference arises as when $x_n = h + \frac{1}{2}$, for some $h \in \mathbb{Z}$ we have that both $\boldsymbol{k} = (h, \ldots, h)$ and $\boldsymbol{k} = (h+1, \ldots, h+1)$ are such that:

$$x^* = \left(h, \ldots, h, h + \frac{1}{2}\right)$$

belongs to $\overline{S_{(h,\ldots,h)}}$ and $f(h, \ldots, h, h + \frac{1}{2}) = 0$ but it also holds true that:

$$x^* = \left(h+1, \ldots, h+1, \ldots, h + \frac{1}{2}\right)$$

belongs to $\overline{S_{(h+1,\ldots,h+1,h)}}$ and $f\left(h+1, \ldots, h+1, h + \frac{1}{2}\right) = 0$. We lastly observe that the points of the form:

$$x^* = \left(h+1, \ldots, h+1, \ldots, h + \frac{1}{2}\right)$$

are not Carathéodory equilibria as $\dot{x}_{n-1} = h - (h+1) = -1$.

3.6.4 Krasovskii Equilibria

In the case of Krasovskii equilibria we observe that for points which are not of the form

$$x = (x_1, \ldots, x_n), \text{ s.t.} x_i = h + \frac{1}{2}, \text{ for some } h \in \mathbb{Z},$$

we have $\mathcal{K}f(x) = f(x)$ therefore and in this case we only have Krasovskii equilibria of the form:

$$(q(x_n^0), q(x_n^0), \ldots, q(x_n^0), x_n^0), \qquad x_n(0) \neq h + \frac{1}{2}, \text{ for some } h \in \mathbb{Z}.$$

For all the other case we consider the case with 3 nodes to gather intuition. If we consider a point of the form

$$x = (h + \frac{1}{2}, h, h)$$

then $\mathcal{K}f(x) = f(x)$ and x cannot be an equilibrium, while if we consider points of the form

$$x = (h, h + \frac{1}{2}, h)$$

, then

$$\mathcal{K}f\left(x\right) = \overline{\text{co}}\left\{\begin{pmatrix} (h+1)-h \\ h-(h+\frac{1}{2}) \\ 0 \end{pmatrix}, \begin{pmatrix} h-h \\ h-(h+\frac{1}{2}) \\ 0 \end{pmatrix}\right\} = \overline{\text{co}}\left\{\begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 0 \end{pmatrix}\right\} \tag{3.36}$$

and $0 \notin \mathcal{K}f(x)$ which happens as the third component is not $h + \frac{1}{2}$.

But also by considering points of the form: $(h, h + \frac{1}{2}, h + \frac{1}{2})$ we have that:

$$\mathcal{K}f\left(x\right) = \overline{\text{co}}\left\{\begin{pmatrix} 1 \\ \frac{3}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{3}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}\right\} \tag{3.37}$$

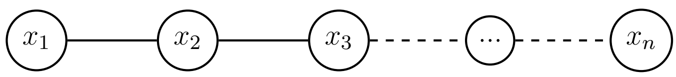
and it is easily noticeable that $0 \notin \mathcal{K}f(x)$.

To obtain the null vector $\mathbf{0}$ as a vector of the convex combination we need $x_1, x_2 \in \mathbb{Z}$ and also $x_1 = x_2 = h$, with only the last component equal to either h or $h - \frac{1}{2}$, or $h + \frac{1}{2}$.

3.7 Undirected Line

We now analyse the directed version of the precedent graph and observe how introducing bidirectional observations can lead to different equilibria and increase the difficulty of the dynamics.

Let us consider an undirected line:



Supposing again unitary weights, the dynamics for this graph topology is the following:

$$\begin{cases} \dot{x_1} = q(x_2) - x_1 \\ \dot{x_2} = q(x_1) + q(x_3) - 2x_2 \\ \quad \vdots \\ \dot{x_{n-1}} = q(x_{n-2}) + q(x_n) - 2x_{n-1} \\ \dot{x_n} = q(x_{n-1}) - x_{n-1}. \end{cases}$$

3.7.1 Carathéodory Equilibria

Carathéodory equilibria are points satisfying the two conditions:

$$\begin{cases} x^* \in S_k \\ f(x^*) = 0. \end{cases}$$

Hence it must hold that:

$$\begin{cases} x_1^* = k_2 \\ x_2^* = \frac{k_1+k_3}{2} \\ \quad\vdots \\ x_{n-1}^* = \frac{k_{n-2}+k_n}{2} \\ x_n^* = k_{n-1} \end{cases} \tag{3.38}$$

and also (from $x^* \in S_{\boldsymbol{k}}$):

$$\begin{cases} k_1 - \frac{1}{2} \leq x_1^* < k_1 + \frac{1}{2} \\ k_2 - \frac{1}{2} \leq x_2^* < k_2 + \frac{1}{2} \\ \quad\vdots \\ k_{n-1} - \frac{1}{2} \leq x_{n-1}^* < k_{n-1} + \frac{1}{2} \end{cases} \tag{3.39}$$

leading to something similar to (**??**):

$$\begin{cases} k_1 - \frac{1}{2} \leq k_2 < k_1 + \frac{1}{2} \\ k_2 - \frac{1}{2} \leq \frac{k_1+k_3}{2} < k_2 + \frac{1}{2} \\ k_3 - \frac{1}{2} \leq \frac{k_2+k_4}{2} < k_3 + \frac{1}{2} \\ \quad\vdots \\ k_{n-1} - \frac{1}{2} \leq \frac{k_{n-2}+k_n}{2} < k_{n-1} + \frac{1}{2} \\ k_n - \frac{1}{2} \leq k_{n-1} < k_n + \frac{1}{2} \end{cases} \tag{3.40}$$

From the first two inequalities, being k_1 and k_2 integers we obtain $k_1 = k_2$, therefore the second inequality becomes:

$$\frac{1}{2}k_1 - \frac{1}{2} \leq \frac{k_3}{2} < \frac{1}{2}k_1 + \frac{1}{2}$$

leading to either

$$k_3 = k_1 \text{ or } k_3 = k_1 - 1.$$

By recursively applying this reasoning we arrive to this following characterization of Carathéodory equilibria for a directed line:

$$q_1 = h \tag{3.41}$$

$$q_i = q_{i-1} - 1 \text{ or } q_i = q_{i-1} \tag{3.42}$$

$$\tag{3.43}$$

For example, for the case of 4 nodes we have the following quantization states and corresponding equilibria:

$$\begin{cases} q_{eq} = (h, h, h, h) \\ x_{eq} = (h, h, h, h) \end{cases}$$

and

$$\begin{cases} q_{eq} = (h, h, h-1, h-1) \\ x_{eq} = (h, h-\frac{1}{2}, h-\frac{1}{2}, h-1) \end{cases}$$

3.8 Complete Graph

Complete graphs are given a strong importance in interaction models as they capture the phenomenon in which every individual somehow can observe everybody's opinion. For this reason we look for the equilibria of this particular graph. We use the same notation of [3] to describe the equilibria of the dynamics on this graph.

The dynamic has the form:

$$\dot{x}_i = \sum_{j \neq i} \left(q\left(x_j\right) - x_i \right) \quad \forall i \tag{3.44}$$

we now show that for this dynamics it holds that all the equilibria are of the form:

$$E_C = E_e = \left\{ x \in \mathbb{Z}^N : \exists h \in \mathbb{Z} \text{ such that } x_i = h \forall i = 1, \ldots, N \right\}. \tag{3.45}$$

It is evident that all points of the form in (3.45) are Carathéodory equilibria. On the other hand extended equilibria are points such that there exists $\mathbf{k} \in \mathbb{Z}^N$ such that $f_{\mathbf{k}}\left(x^*\right) = 0$ and $x^* \in \overline{S_{\mathbf{k}}}$.

From the first condition mentioned it follows that:

$$\sum_{j \neq i} \left(q\left(x_j\right) \right) - (N-1) x_i = 0$$

therefore x_i^* must satisfy:

$$x_i^* = \frac{\sum_{j=1}^{N} k_j - k_i}{N-1}. \tag{3.46}$$

By defining $K = \sum_{j=1}^{N} k_j$ and from the second condition:

$$k_i - \frac{1}{2} \leq x_i^* \leq k_i + \frac{1}{2} \tag{3.47}$$

$$(N-1)\left(k_i - \frac{1}{2}\right) \leq K - k_i \leq (N-1)\left(k_i + \frac{1}{2}\right) \tag{3.48}$$

$$\frac{K}{N} - \frac{N-1}{2N} \leq k_i \leq \frac{K}{N} + \frac{N-1}{2N}. \tag{3.49}$$

these bounds show that each k_i is below some number $\frac{K}{N} + \alpha$ and above $\frac{K}{N} - \alpha$ with $\alpha < 1$, which imply that $k_1 = k_2 = \cdots = k_n$ and $x^* \in S_{(h,\ldots,h)}$ for some $h \in \mathbb{Z}$. From this it follows that the ith component of the vector field defining the dynamics is $f\left(x^*\right)_i = (N-1)h - (N-1)x_i^*$ and easily $x^* = h \; \forall i = 1, \ldots, N$.

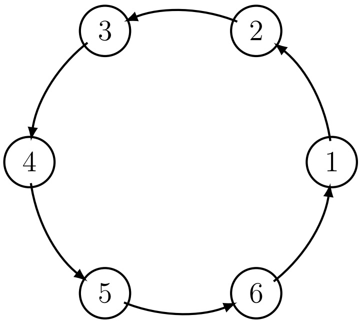
Observation 9. *We observe that the equilibria found for the dynamics on the complete graph are both Lyapunov stable and locally asymptotically stable as the vector field in a neighborhood of an equilibrium point reduces to $f(x) = -(N-1)\left(x - x^*\right)$. Local asymptotic stability follows from the negativity of $-(N-1)$.*

3.9 Cycle Graph: solutions with periodic behavior

3.9.1 The Dynamics on the Cycle Graph

Simulating the dynamics with some specific initial conditions leads to *numerical* solutions that seem periodic. We then now investigate what happens on such graph, trying to approach some analytical result.

The directed cycle graph has the form below:



therefore the system (3.2) reduces to:

$$\begin{cases} \dot{x_1}(t) & = q(x_2) - x_1(t) \\ \dot{x_2}(t) & = q(x_3) - x_2(t) \\ \dot{x_3}(t) & = q(x_4) - x_3(t) \\ \dot{x_4}(t) & = q(x_5) - x_4(t) \\ \dot{x_5}(t) & = q(x_6) - x_5(t) \\ \dot{x_6}(t) & = q(x_1) - x_6(t) \end{cases} \tag{3.50}$$

or, in vector form:

$$\dot{x_i}(t) = q(x_{i+1}(t)) - x_i(t). \tag{3.51}$$

Such dynamics has a quite nice closed form solution whose i-th component reduces to:

$$x_i(t) = (q(x_{i+1}) - x_{0i})\, e^{-t} + q(x_{i+1}). \tag{3.52}$$

This solution is a classical one of the ODE, up to the first point in which $q(x_{i+1}(t))$ changes its value. For this reason, since (3.52) solves (3.51) almost everywhere, we may state that (3.52) is a Carathédory solution and a Krasovskii solution of the dynamics in (3.51).

If we run some numerical simulations, we observe the following trajectories:

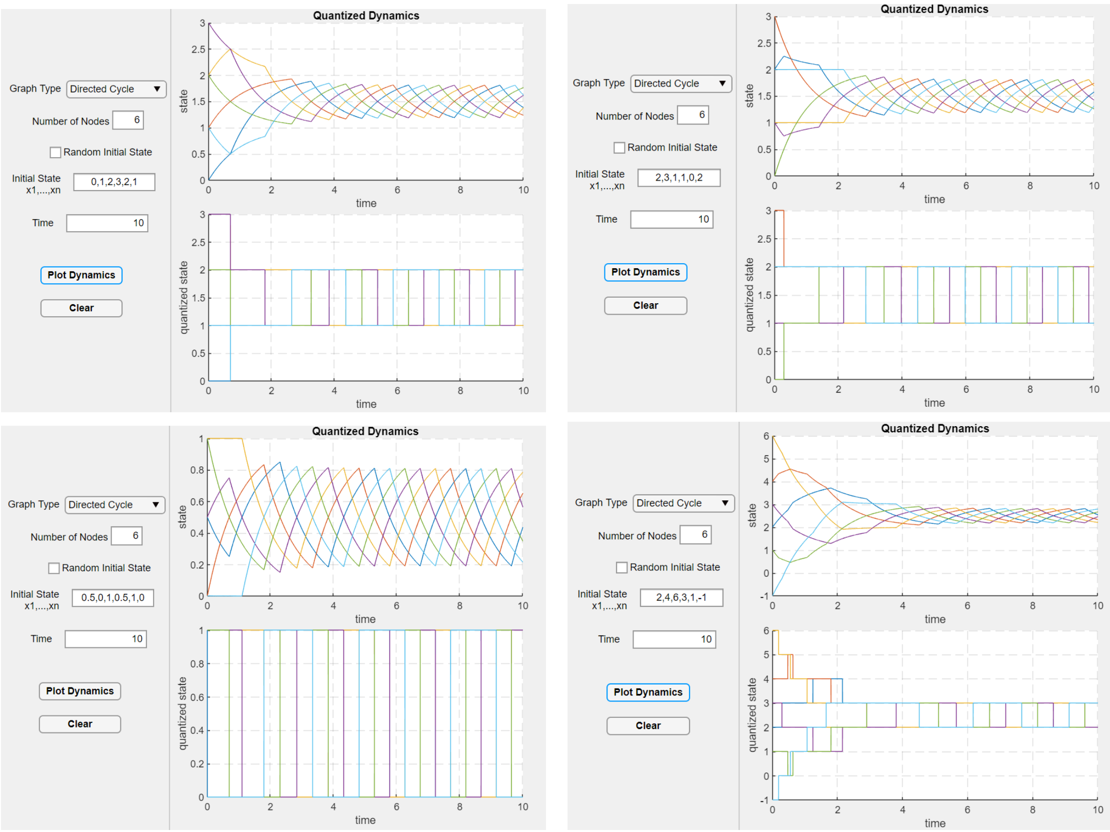


Figure 3.7: Trajectories of the dynamics simulated using the quantized behaviors app in MATLAB over the cycle graph. The four different trajectories correspond to different initial conditions for which the same periodic behavior occurs.

which have given the idea to look for periodic solutions.

The numeric trajectories show this behavior with specific initial conditions when the number of agents n is even, and are all of the form:

$$x_0 = \left\{ (x_0)_i \ s.t. \ (x_0)_i + (x_0)_{i+n/2} = 2k+1, \ \text{for some } k \in \mathbb{Z} \right\} \tag{3.53}$$

i.e. all those initial conditions for which $(x_0)_i$ and $(x_0)_{i+n/2}$ sum to an odd number.

We observe that, unless specified otherwise, all indexes are to be intended modulo n.

We might wonder if such solutions are the results of some numerical issues or have some analytical correspondence.

Proving the existence of periodic solutions is the aim of the following paragraph.

3.9.2 Proof that periodic Carathéodory solutions exist

Let's consider the system (3.50) with initial conditions of the form specified in (3.53). In particular if we consider:

$$x_0 = \left(\frac{3}{2}+\delta, \frac{3}{2}, \frac{3}{2}-\beta, \frac{3}{2}-\delta, \frac{3}{2}, \frac{3}{2}+\beta\right) \tag{3.54}$$

with:

$$\delta = \frac{1}{4}(\sqrt{5}-1) \qquad \beta = \frac{1}{4}(3-\sqrt{5}) \tag{3.55}$$

and

$$q(x_0) = (2,1,1,1,2,2)\,. \tag{3.56}$$

If we define

$$T = \log\left(\frac{\frac{1}{2}+\delta}{\frac{1}{2}-\delta}\right) = \log\left(2+\sqrt{5}\right) \tag{3.57}$$

we get, from (3.52) that:

$$\begin{aligned}
x_1(t) &= \left(\frac{3}{2}+\delta-1\right)e^{-t}+1 = \left(\frac{1}{2}+\delta\right)e^{-t}+1\\
x_2(t) &= \left(\frac{3}{2}-1\right)e^{-t}+1 = \frac{1}{2}e^{-t}+1\\
x_3(t) &= \left(\frac{3}{2}-\beta-1\right)e^{-t}+1 = \left(\frac{1}{2}-\beta\right)e^{-t}+1\\
x_4(t) &= \left(\frac{3}{2}-\delta-2\right)e^{-t}+2 = \left(-\frac{1}{2}-\delta\right)e^{-t}+2\\
x_5(t) &= \left(\frac{3}{2}-2\right)e^{-t}+2 = -\frac{1}{2}e^{-t}+2\\
x_6(t) &= \left(\frac{3}{2}+\beta-2\right)e^{-t}+2 = \left(-\frac{1}{2}+\beta\right)e^{-t}+2
\end{aligned} \tag{3.58}$$

is a (classical, hence Carathéodory and Krasovskii) solution for every t in the open interval $0 < t < \frac{T}{3}$.

We observe that $q(x_0)$ does not strictly follow the definition given in (3.1), but we might consider either the upper or the lower quantization at discontinuities points. The $x_i(t)$ have been chosen by substituting the value $q(x_{02})$ with the value 1, a limit value for $u \to \frac{3}{2}^-$.

We observe that, when evaluating the solutions at $\frac{T}{3}$, we obtain a *shift* of the nodes opinions, in the sense that, combining (3.58), (3.55), (3.53), we get to:

$$
\begin{aligned}
x_1\left(\frac{T}{3}\right) &= \left(\frac{1}{2}+\delta\right)e^{-\frac{T}{3}}+1=\frac{3}{2}=x_2(0)\\
x_2\left(\frac{T}{3}\right) &= \frac{1}{2}e^{-\frac{T}{3}}+1=\frac{1}{4}(3+\sqrt{5})=\frac{3}{2}-\frac{3-\sqrt{5}}{4}=\frac{3}{2}-\beta=x_3(0)\\
x_3\left(\frac{T}{3}\right) &= \left(\frac{1}{2}-\beta\right)e^{-\frac{T}{3}}+1=\frac{1}{4}(7-\sqrt{5})=\frac{3}{2}-\frac{\sqrt{5}-1}{4}=\frac{3}{2}-\delta=x_4(0)\\
x_4\left(\frac{T}{3}\right) &= \left(-\frac{1}{2}-\delta\right)e^{-\frac{T}{3}}+2=\frac{3}{2}=x_5(0)\\
x_5\left(\frac{T}{3}\right) &= -\frac{1}{2}e^{-\frac{T}{3}}+2=x_6(0)\\
x_6\left(\frac{T}{3}\right) &= \left(-\frac{1}{2}+\beta\right)e^{-\frac{T}{3}}+2=\frac{1}{4}(5+\sqrt{5})=\frac{3}{2}+\frac{\sqrt{5}-1}{4}=\frac{3}{2}+\delta=x_1(0)
\end{aligned}
\tag{3.59}
$$

This last computation in (3.59) shows that, with the T defined in (3.57), at each $T_k = k\frac{T}{3}$, $k \in \mathbb{N}$ there is a switch in the initial condition and the system obeys to the same differential equations with initial conditions:

$$
x_i\left(k\frac{T}{3}\right) = x_{i+k}(0). \tag{3.60}
$$

It must be observed that, up to a switch time, the vector field does not change. This follows by the monotonicity of the solutions in (3.58) which we have in explicit form and lead to a change in the quantization level only at T_k.

We also have got a numerical confirm of what stated above. Let's consider the trajectories below:

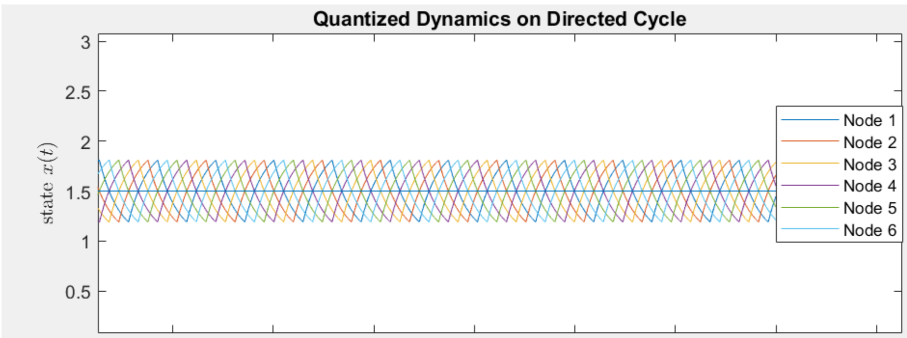


Figure 3.8: Numerical Solutions Corresponding to the Dynamics on the Cycle Graph with x_0 as in (3.54)

If we let the system start with initial conditions analogue to the one of the proof, the numerical simulations confirm what proved before as at each of the $k\frac{T}{3}$, a switch in the nodes opinions occur.

We also observe, when zooming in, that after each interval of time of $\frac{T}{3}$ a switch occurs.

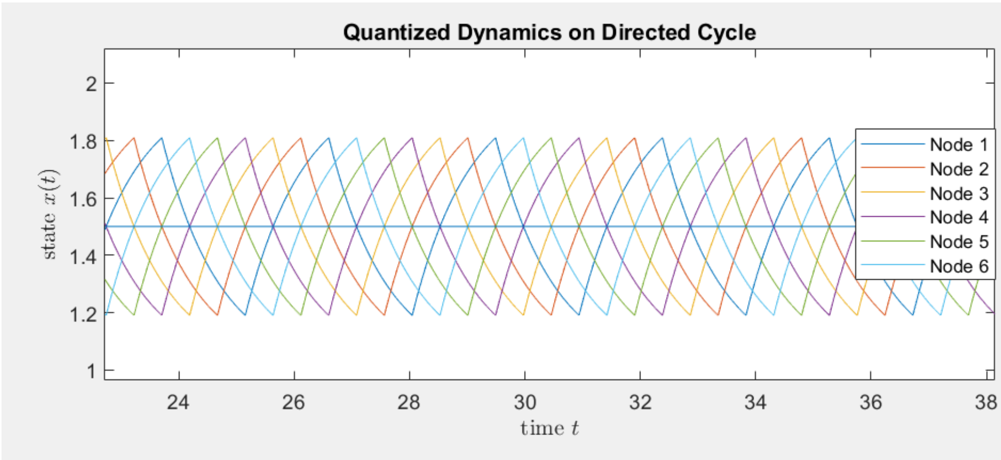


Figure 3.9: Switch in individual opinions - zoom

This last construction paves the way towards an important question an its relative answer: does a quantized dynamics (even if a continuous-time one) converge in any case? As this last paragraph has shown, the answer is no. We leave to the next paragraph the discussion about convergence for the quantized dynamics (3.2) and some results about convergence.

Chapter 4

Convergence Results for quantized dynamics

We have witnessed weird behaviors for the quantized dynamics of the form (3.2). We have produced examples showing solutions converging to consensus, which are (Carathéodory and Krasovskii) equilibria of the vector field, but also (see example (3.4.1) Carathéodory solutions converging to points which are not Carathéodory equilibria. In addition we have built an example of dynamics not converging at all, as in the cycle dynamics (3.50), we have constructed a solution with a limit cycle. We might then wonder whether under some conditions or on some particular graphs, convergence can be guaranteed analytically or not. The aim of this chapter is that of presenting some convergence results which have been found mainly in [3]. We retrieve here the results they presented to complete our discussion about a dynamics of quantized type, with the awareness that these results cannot fully characterize the graphs over which convergence can be obtained and only refer to some graphs in particular or are valid under specific hypotheses.

4.1 Main Convergence Results

4.1.1 Convergence of the dynamics to a set

As we already stated, convergence is a strong requirement when discussing about opinions evolution and in this dynamics it is not reached in general. This reflect human behaviors: it is quite difficult that at the of some discussion both individuals share a common intermediate opinion. What might happen also in the real world is that somehow individuals opinion "*get closer*". We have seen that, as one expects, the minimum and maximum level of quantization are definitively constant and individual opinions cannot overcome these two barriers definitively but also a more specific result holds. It is the following, as stated in [3]:

Theorem 4.1.1 (**Convergence to a set**). *Assume that the graph with adjacency matrix A is weight balanced and weakly connected. If $x(t)$ is any Carathéodory or Krasovskii* *solution of* (3.2) *and we define* λ_* *as the smallest nonzero eigenvalue of* $\mathrm{Sym}(L) = \frac{L+L^T}{2}$ *s.t.*

$$x^\top Lx \geq \lambda_* \left\| x - x_a \mathbf{1} \right\|^2, \ \forall x \in \mathbb{R}^N.$$

M as:

$$M = \left\{ x \in \mathbb{R}^N : \inf_{\alpha \in \mathbb{R}} \| x - \alpha \mathbf{1} \| \leq \frac{\|A\|}{\lambda_*} \frac{\sqrt{N}}{2} \right\} \tag{4.1}$$

then $dist(x(t), M) \to 0$, *as* $t \to +\infty$.

The importance of this theorem relies in the fact that it is possible to obtain a sort of "*distance to consensus*", in fact the set M represents a sort of "neighborhood" of a consensus and the theorem states that, as t approaches higher values, the trajectories approach this set M.

On some special graphs this result gives interesting bounds. Let's consider a complete graph.

Example 4.1.1 (Convergence to M **on a complete graph).** On complete graphs, the quantized dynamics is:

$$\dot{x}_i = \sum_{j \neq i} \left(q\left(x_j\right) - x_i \right) \quad \forall i \tag{4.2}$$

and, the matrix of the graph is as in 1.1.1, therefore $\|A\| = N - 1$ and $\lambda^* = N$, so, for $t \to +\infty$:

$$\frac{1}{\sqrt{N}} \left\| x - \frac{1}{N} \sum_{i=1}^{N} x_i \mathbf{1} \right\| \leq \frac{1}{2} \tag{4.3}$$

therefore, the dynamics asymptotically moves towards points which are close to consensus. In the next section we shall actually state a stronger result: it holds that these limit points are actually consensus points.

This bound can be tighter or looser. Let's consider for instance a path graph. We carry out the same reasoning performed in [3] to show the minor efficacy of this bound in some case.

It holds that $\lambda_* = 1 - \cos\left(\frac{\pi}{N}\right)$ and $\|A\| \leq 2$ by Gershgorin's disk lemma.

Therefore, from 4.1.1,

$$\begin{aligned} \frac{1}{\sqrt{N}} \left\| x - \frac{1}{N} \sum_{i=1}^{N} x_i \mathbf{1} \right\| &\leq \frac{2}{2\left(1 - \cos\frac{\pi}{N}\right)} \\ &= \frac{1}{\frac{\pi^2}{2N^2} - \frac{\pi^4}{4N^4} + o\left(\frac{1}{N^4}\right)} \\ &= \frac{2}{\pi^2} \frac{N^2}{1 - \frac{\pi^2}{2N^2} + o\left(\frac{1}{N^2}\right)} = \frac{2}{\pi^2} N^2 + O(1) \qquad \text{as } N \to \infty. \end{aligned} \tag{4.4}$$

This is just an approximation for the bound of 4.1.1.

As anticipated after theorem 4.1.1, a convergence result has been found in [3] in the form we state below:

Theorem 4.1.2 (Convergence on Complete Graph)**.** *Any Carathéodory or Krasovskii solution of the dynamics* (4.2)*, converges to a consensus point. Furthermore, if $x(t)$ is Carathéodory, then the limit point is necessarily of the form $(h, \dots, h)^\top$ with $h \in \mathbb{Z}$. If $x(t)$ is instead a Krasovskii solution, then the limit may be of the form $\left(h + \frac{1}{2}, \dots, h + \frac{1}{2}\right)^\top$.*

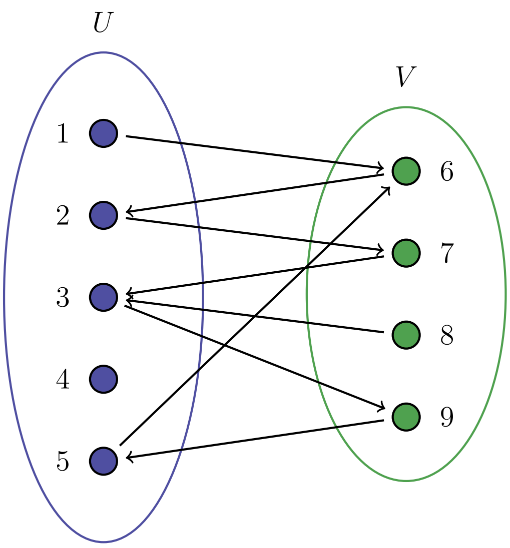
Finding these results of convergence is not easy as the proofs rely on the special structure of these graphs. However convergence properties complete somehow the theory about the quantized dynamics (3.2) as finding the (Carathéodory or Krasovskii) equilibria does not tell us anything about the trajectories. It depends on the graph structure whether the $x_i(t)$ converge to equilibria or not. As already mentioned, obtaining convergence results can be quite challenging. For this reason we limit to state another result of convergence again found by [3].

The results is about convergence for the quantized dynamics over the complete bipartite graph.

We recall the definition of a complete bipartite graph, write down how the dynamics reads on this graph and finally state the convergence result.

Definition 4.1.1. A graph whose vertices can be partitioned into two subsets V_1 and V_2 where no edge has both endpoints in the same subset, and all the edges connecting vertices in the two subsets do exist, is called complete bipartite graph.

A graph like the one in the example below is a complete bipartite graph



The dynamics on graphs of the form above reduces to:

$$\dot{x}_i = \sum_{h \in \mathcal{V}_\infty} \left[q\left(x_h\right) - x_i \right] \quad \dot{x}_h = \sum_{i \in \mathcal{V}_\in} \left[q\left(x_i\right) - x_h \right]. \tag{4.5}$$

Theorem 4.1.3 (Convergence on Complete Bipartite)**.** *Any Carathéodory or Krasovskii solution of* (4.5) *converges to a consensus point.*

We have stated two important convergence results about some well known graphs but the reader may have understood that it is a more difficult task producing results that adapt to more general graphs, for example based on the degrees of their nodes or some connectivity measure.

4.2 Open Questions,Generalisations and Future Research

The careful reader might have noticed that this work has brought to the attention some quite interesting results on some graphs but lacks of general results valid for very general graphs. For the writing of this thesis, a quite powerful numerical tool, the quantized_behavior_app, has been developed, but we must highlight that this tool only simulates the dynamics on special predefined graphs and can only value some solutions, not every solution, in general. Without any doubt the numerical tool has helped to build results about the existence of periodic solutions and has confirmed some converge properties presented in this chapter without leading to very general results though. This is one direction of possible future research, looking for properties of the graph (number of nodes, connectivity, completeness properties of sub-graphs) under which this field can be extended. Throughout all the paper, the approach adopted has been a microscopic one, as we have been following the individual nodes' opinions trajectories. Another strategy might be pursued: that of finding macro quantities (the mean opinion, the mean of two opposite agents opinion in a cycle) which are preserved for instance. The thesis also has not investigated the limits of the feasibility of the quantized model proposed: the reader might argue under with circumstance the quantized model for social interactions can be applied and, in the case, how to statistically predict the measures of interactions (the $a_{ij}s$) between two nodes. There is not a lot in the literature of the field about this more predictive aspect and it could be interesting investigating in that sense. Last but not least, all these discontinuous dynamics, for example in the form (3.4), find many applications in the branch of neural network and artificial intelligence which might exploit nonlinear functions and discontinuous ones to train the networks in image recognition and data pattern extraction, for example.

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