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Master's Degree in Physics of Complex Systems

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Decoding model for decision task under risk

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Abstract

In decision making under risk, probability of each outcome is known by all decision-makers. This is the case for instance, when one wants to choose between equestrian bets. There is a gain associated with each horses and an odds. This odds represents the horse's probability of winning. Historically, theoretical models assumed that decision makers were rational and therefore sought to maximize gain over the different bets. However, experiences have shown that human decision-maker deviates from rational choices. Different explanations have been proposed. First, a utility function was introduced which led to the so called "expected utility theory." A utility is assigned to each outcome depending on the subject. This explained risk aversion ie the preference of a sure gain over a risky gain even if statically, the risky gain is more advantageous. However, the theory fails to explain some behavior as the change in preferences when outcomes are reversed (changed sign). While, later prospect theory did. It is based on the fact that decision-makers tended to overestimate probabilities close to 0 and underestimate the one close to 1. Therefore, the brain does not use probabilities but rather distorted probabilities. Recently, based on that idea, Maloney gave a functional form for distorted probabilities. They used it in a new kind of model for decision making. Concretely, the brain constructs a noisy internal representation of outcomes. This phase is called coding. Then, from these internal representations, the brain makes a decision. Maloney examined coding for probabilities and not for outcomes. In this master thesis, we continue Maloney's work, focusing on decoding and building a model describing decoding of probabilities. We assumed that the subject makes his decision by maximizing his expected gain. We managed to derive an expression for the indifference point ie the probability for which the subject doesn't have any preference between the two prospects, under the assumptions of rational limit and optimal coding. This means that the noise that perturbs the internal representation of the probability tends to zero. And the optimal coding refers to the fact that the subject will code only a specific interval of probabilities which contains the rational indifference point (the rational point with a noise for the internal representation equals to zero). That expression depends on the prior parameters, therefore we were able to understand the effect of a change in mean and width prior on the indifference point. We also managed to derive an expression for the probability of choosing the sure prospect when the subject is faced to one sure prospect and one risky. From that expression, we were able to understand for which range of the prior parameters, the subject will be risk averse. This fact tells us also that perceptual bias can account for risk aversion and risk seeking. And we finally manage to get an expression for the distorted probability measured by Maloney. That expression is different from the one

they tried to construct and above all is derived from general considerations not by putting together ad-hoc ingredients.

Acknowledgements

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Acronyms

EUTR
Expected Utility Theory under Risk

VNM
von Neumann Morgenstern

FSD
First order Stochastic Dominance

LLO
Linear Log-Odds

BLO
Bounded Log-Odds

JRF
Judgement of Relative Frequencies

DMR
Decision Making under Risk

Chapter 1

Introduction

1.1 Contextual setting

Let's consider a prospect represented as follows: $(x_1, p_1; x_2, p_2; ...; x_n, p_n)$ where $n \in \mathbb{N}$, the $x_i \in \mathbb{R}$ are the different possible outcomes or payoffs, and the $p_i \in [0,1]$ their associated probabilities. The normalization of the p_i is set to 1, and usually, the possible null x_i are not noted. Another way of representing the prospect, easier to visualize, is in the form of a matrix: $\begin{pmatrix} x_1 & ... & x_n \\ p_1 & ... & p_n \end{pmatrix}$. Both notations are represented in the literature. I will keep the last one. A simple example of such a prospect would be a throw of a fair coin: there is a probability 0.5 to win 10 €, and a probability 0.5 to loose 5 € or equivalently to win -5 €. One has therefore the following representation: $\begin{pmatrix} 10 & -5 \\ 0.5 & 0.5 \end{pmatrix}$. Let's now consider the case where one has to choose between two prospects. This task is commonly called decision making under risk. Under risk means that the probability of each outcome is known by the decision maker. It is often encountered in the insurance context. For instance, a decision-maker considers the possibility of purchasing an insurance on his car. Let w be the wealth of the decision-maker, $-z$ the cost of repairing his car, p the probability of paying for the accident, $-y$ the cost of the insurance. Either the decision maker leaves his car uninsured. This may correspond to the following prospect: $\begin{pmatrix} w-z & w \\ p & 1-p \end{pmatrix}$. Or he purchases a regular insurance. In which case, he gives monthly money to the insurer. This is called a sure prospect: $\begin{pmatrix} w-y \\ 1 \end{pmatrix}$. It is now well known that decision-makers don't act rationally, have cognitive bias. Indeed, they have to make fast and frugal choices that are constrained by limited resources such as time, money, food, knowledge or computational effort [1]. In particular, it is pertinent to notice that several interpretations of a concrete situation (from the psychological point of view) can lead to different preferences. For instance, one may say that the decision-maker perceives the outcomes as gain or losses rather than final states. Therefore the decision-maker should choose between theses two prospects: $\begin{pmatrix} -z & 0 \\ p & 1-p \end{pmatrix}$ and $\begin{pmatrix} -y \\ 1 \end{pmatrix}$. These two ways of representing prospects can lead experimentally to different preferences. The interested reader will find some clues for a reflection on the representation of outcomes in prospects in this paper [2] written by two researchers in psychology.

A decision problem like the previous one is sometimes recast as the problem of choosing among two stochastic variables A, B. The latter takes the values of the outcomes of their associated prospects. Let F_A and F_B be respectively their cumulative distribution. Such distributions are commonly called lotteries or gambles. Following the last example, the decision-maker has to make a choice between the two gambles: $\forall x \in \mathbb{R},\ F_A(x) = \begin{cases} 0, & \text{if } x < -z \\ p, & \text{if } -z \leq x < 0 \\ 1, & \text{if } 0 \leq x \end{cases}$ and $F_B(x) = \begin{cases} 0, & \text{if } x < -y \\ 1, & \text{if } -y \leq x \end{cases}$. One can now identify the random variable to the prospect and speak about the expectation value of a prospect. Then $\mathbb{E}[A] = \mathbb{E}\left[\begin{pmatrix} -z & 0 \\ p & 1-p \end{pmatrix}\right] = -zp$. Be aware that, in the literature, people often interchange the terms "prospect," "probability distribution" and "random variable," for purposes of simplifying sentences.

1.2 History of the study of cognitive biases in decision-making under risk tasks

These last 400 years, a lot of work has been made in trying to build a theory of preferences and particularly in decision making under risk. As a first step, one may think that the brain just computes the expected value of each prospect as if he acts rationally. The choice of the decision-maker would be the prospect with the higher expected value. There is nevertheless a serious objection to that idea formulated under the name of St Petersburg game. Concretely, people were asked how much they would pay for the following game: if tails comes out of the first toss of a fair coin, they receive one dollar and stop the game, and if head comes out, they receive two dollars and stay in the game; if tails comes out of the second toss of the coin, they receive nothing and stop the game, and if head comes out, they receive four dollars and stay in the game; if tails comes out of the third toss of the coin, they receive nothing and stop the game, and if head comes out, they receive height dollars and stay in the game; and so on ad infinitum. One can translate this game in terms of prospect. Considering that decision-makers perceive outcomes as final states, the matrix representation of the prospect associated to this game is as follow: $\begin{pmatrix} 1 & 2 & 2^2 & 2^3 & \cdots \\ \frac{1}{2} & \frac{1}{2^2} & \frac{1}{2^3} & \frac{1}{2^4} & \cdots \end{pmatrix}$. His expected value reads $\sum_{n=1}^{\infty} 2^{n-1}\frac{1}{2^n} = +\infty$. Nevertheless, even if the expected gain is infinite, people always answer a finite number.

In 1738, Bernoulli resolved the paradox by assuming a logarithmic utility function of wealth [3]. Intuitively, the utility represents the satisfaction that consumers receive for having an outcome. It is therefore a function from the set of the outcomes of a prospect to $\mathbb{R}$. For clarity, let's write the expected utility of a prospect. Given a utility function U, a prospect $\begin{pmatrix} x & y \\ p & 1-p \end{pmatrix}$ associated with the random variable A, one calls expected utility of A and denotes $\mathbb{E}[U(A)]$ the following quantity: $U(x)p + U(y)(1-p)$. Back to the St Petersburg game, the expectation of the prospect is therefore equals to $\sum_{n=1}^{\infty} \log(2^{n-1})\frac{1}{2^n} = \log(2) < \infty$.

Moreover, it is interesting to notice that the logarithm is a concave function. Therefore, from the Jensen inequality, one can say that the utility of the expected value of a gamble is greater or equal than the expected value of the utility of the gamble. As the utility of the expected value can be seen as a sure prospect, a decision-maker characterized by a concave utility function, between a sure gain and a prospect, choose the sure gain. This decision-maker is therefore called risk averter. Similarly, a person with a convex utility function is called risk taker. There exist different and not necessarily equivalent ways of defining risk. See for more informations, the work of Stiglitz and Rotschild in 1970 [4]. All of these ideas are the beginnings of a expected utility theory under risk (EUTR).

In 1947, von Neumann and Morgenstern came up with an axiomatization of the EUTR [5] (see Appendix A.1.1). These axioms are widely believed to represent the essence of rational behaviour under risk. Briefly, a rigorous mathematical framework is set up for building a utility function U and ordering probability distributions (and so prospects). The fundamental theorem derived (also called von Neumann Morgenstern (VNM) theorem) states that there exists a function U from the set of outcomes to [0,1], such that for all P_1, P_2 in the set of distributions defined on a bounded interval, P_1 is preferred to P_2 if and only if $\mathbb{E}_{P_1}[U] > \mathbb{E}_{P_2}[U]$. This result can be generalised to distributions defined over unbounded intervals. This is a theorem, the equivalence is established from the axioms. Now, one can forget the axioms and see this equivalence as a definition for the statement " P_1 is preferred to P_2." Therefore, given two prospects, one can build a utility function, compute the expected utility values of the two prospects and compare them in order to find which one of the two prospects should be preferred by a decision-maker who's been asked to choose between the two. There was a lot of discussion afterwards about the axioms, see the works of Machina (axiom of independence ie the third axiom) [6], Aumann (axiom of completeness ie the first axiom) [7], Hausman (axioms of completeness and transitivity ie the second axiom) [8], Herstein and Milnor (all axioms) [9] among others.

Parallel to these discussions on the EUTR, another way of ordering prospects emerged with the stochastic dominance. Haddar and Russell are at the origin of the definition of first stochastic dominance [10]. One can define a partial order among the set of gambles. This partial order may be interpreted as preferences among gambles. Let F and G be two cumulative distributions. One can say that F first-order stochastically dominates G when : (1) $\forall x$ $F(x) \leq G(x)$ and (2) it exists an interval not empty where for all x inside that interval, $F(x) < G(x)$. Moreover, just recalling the other formulation of the expected value of a random variable in term of the cumulative[1], one can show an implication between expected utility and first order stochastic dominance (FSD) [11]. Let U be the utility function of a decision-maker. If U is strictly increasing, piecewise differentiable and cumulative F first-order stochastically dominates cumulative G, then $\mathbb{E}_F[U] > \mathbb{E}_G[U]$.

Although EUTR is a powerful tool for the analysis of decision under risk, it has long been known that decision-maker behavior, in both experimental and market settings, deviates from the predictions of EUTR. The most famous violations are the Allais paradox [12] (see Appendix A.1.2), the common ratio effect (see Appendix A.1.3)[2] . These violations of EUTR predictions were largely disregarded until the late 1970s, when a variety of alternatives to, and generalizations of, EUTR began to appear, most notably prospect theory of Kahneman and Tversky [2]. The central idea was that decision-makers tend to overweight low-probability events and underweight large-probability events. That idea was already present in some psychological papers [13]. They therefore replace probabilities p with weighting function $\pi(p)$ where it was assumed that π mapped the unit interval onto itself in such a way that $\pi(p) > p$ for small p, while $\pi(p) < p$ for p near 1. The graph of π is therefore an inverse S-shape (Fig. 1.1). And now, the expectation utility of a gamble involves the weight. Therefore as an example, the expected utility of this prospect $\begin{pmatrix} x & y \\ p & 1-p \end{pmatrix}$ is $U(x)\pi(p)+U(y)\pi(1-p)$. It is important to notice that $\pi(p)$ is not a probability measure. Nevertheless, the most important problem of prospect theory was that, since probability weights did not sum to one, the theory gave rise to violations of stochastic dominance, in the strong sense that one would prefer a prospect obviously not preferable (see Appendix A.1.4).

[1] Let X be a real, continuous random variable and $\mathbb{P}[X]$ be the probability distribution associated, $\mathbb{E}[X] = \int_{\mathbb{R}} \mathrm{d}x \; \mathbb{P}[X > x]$.

[2] The explanation of these violations of EUTR are quiet important to know and to keep in mind. They are explained in the appendix rather than the introduction because it's technical. So if you are not familiar with, please have a look to the appendix.

Dozens of generalized EU models appeared in the 1980s and early 1990s. The most important are: the Rank dependent model by Quiggin [14] which in turn gave birth to the Cumulative Prospect Theory [15] (awarded of the Nobel Memorial Prize in Economic Sciences in 2002), the betweenness models by Chew [16] and the regret-theoretic approaches by Loomes and Sugden [17]. Briefly, Quiggin proposed to make weighting function dependent on the rank-order of the outcomes which resolved the main problem of stochastic dominance. Chew chose a weighting function based on outcomes and probabilities. The motivation was mostly technical rather than intuitive. One can define the regret as the difference between the utility of a made decision and the one of the optimal decision. Rather than evaluating prospects in terms of a summary statistic like expected utility, Loomes and Sugden proposed that when facing a decision, decision-makers might anticipate regret and thus incorporate in their choice their desire to eliminate or reduce regret.

Three principal analytical expressions for the weighting functions have been found. They accommodate the common ratio effect and the common consequence effect (other name of the Allais paradox). The one from Tversky and Kahneman (1992) [15] was intuited rather than derived from a set of axioms, it reads: $\pi(p) = \frac{p^\beta}{\left(p^\beta+(1-p)^\beta\right)^{\frac{1}{\beta}}}$, where β is a free parameter. The one from Prelec (1998) [18] was derived from a set of axioms. It reads: $\pi(p) = e^{-\delta(-\ln(p))^\gamma}$, where δ and γ are free parameters. The main interest of the one of Tversky and Kahneman is that it has only one parameter, so it's easy to fit experimental data. The one from Lattimore, Baker and Witte is also an intuited function [19]. It reads: $\pi(p) = \frac{\delta p^\gamma}{\delta p^\gamma+(1-p)^\gamma}$, where γ and δ are free parameters.
A simple demonstration for the last expression is proposed by the two researchers in psychology Gonzales and Wu [20]. Let's consider two features of any weighting probability. One feature involves the degree of curvature of the weighting function (see Fig. 1.1 for a representation and see Appendix A.1.5 for more quantitative definitions). It can be interpreted as discriminability between probabilities or prospects. For instance, a child of 4 years old has a step function for weighting function [21]. Therefore, he can't discriminate between 0.4 and 0.7. Whereas, an adult can, so has more discriminability. Another feature involves the elevation of the weighting function, which can be interpreted as attractiveness to prospect (see Appendix A.1.5) (see Fig. 1.1). As an example, a decision-maker who knows all soccer players and nothing in politics will think that he's more likely to win a sport bet than a political bet. Therefore his weighting function for the prospect associated to the sport bet will be more elevated than the one for the prospect associated to the political bet. In psychology, this attitude is called "illusion of control".
Briefly, these two psychological properties of the weighting function are logically independent, then it should be possible to model π with two parameters such

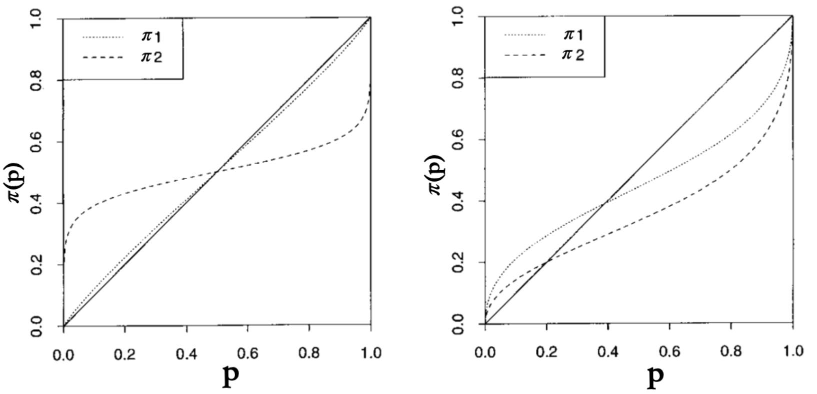

Figure 1.1: Left: Two weighting functions that differ primarily in curvature — $\pi 1$ is relatively linear and $\pi 2$ is almost a step function. **Right:** Two weighting functions that differ primarily in elevation — $\pi 1$ is over $\pi 2$ (adapted from [20]).

that one parameter represents curvature (discriminability) (γ) and the other parameter represents elevation (attractiveness) (θ). One way to do that is to change interval, to go for instance from [0,1] to $\mathbb{R}$ thanks to the log-odds function [22] or logit function [23] (two different names for the same function). The log-odds function is very common in the scientific literature. The same type of non linear function is found in the well-known Weber–Fechner law [24] and in neuroscience with the rethinal mapping which is the mapping of visual input from the retina to neurons [25]. Therefore, it makes sense to use it as follows:

$$\lambda(\pi(p)) = \gamma\lambda(p) + \theta \tag{1.1}$$

where $\lambda(p) = \ln\left(\frac{p}{1-p}\right)$ is the log-odds function or logit function.
Then setting $\delta = e^{\theta}$ leads to:

$$\pi(p) = \frac{\delta p^{\gamma}}{\delta p^{\gamma} + (1-p)^{\gamma}}. \tag{1.2}$$

One should remark that the change of scale does not preserve totally the independence of parameters. Indeed, in the probability scale, by continuity of the weighting function, one has $\pi(0) = 0$, $\pi(1) = 1$. This can be seen as boundary conditions. And therefore, given two different values of δ, varying γ will not have completely the same impact on the weighting function. The two parameters are no more completely independent.

Then Zhang and Maloney, with the linear log-odds model (LLO model), generalized the work of Lattimore, Baker and Witte to a variety of cognitive, perceptual, and motor tasks, not just decision-making [26]. They prefer to use the more general expression "distorted probability" rather than "weighting function." They used a two parameters model with a slope parameter (ω), and an intercept parameter (p_0) (see Fig.: 1.2). This family of distortion functions is defined by the implicit equation:

$$\lambda(\pi(p)) = \omega\lambda(p) + (1 - \omega)\lambda(p_0). \tag{1.3}$$

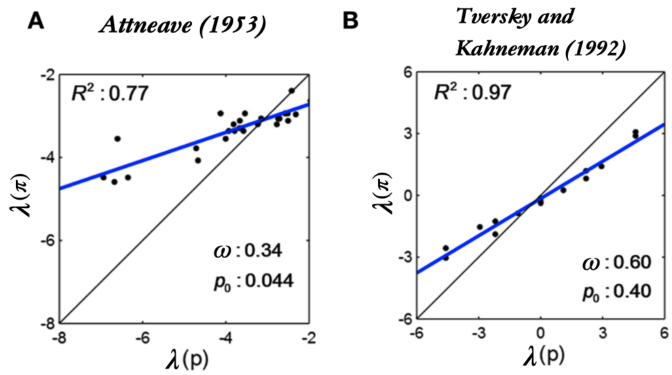

Figure 1.2: S-shaped distortions of frequency estimates: (A) Estimated relative frequencies of occurrence of letters in English text plotted versus actual relative frequency from Attneave (1953) [27]. **(B)** Subjective probability of winning a gamble (decision weight) plotted versus objective probability from Tversky and Kahneman (1992) [15]. R^2 denotes the proportion of variance accounted by the fit. (adapted from [26])

The parameter ω is the slope of the linear transformation and the remaining parameter p_0 is the fixed point of the distorted probability, the value of p which is mapped to itself ($\pi(p_0) = p_0$). Therefore, p_0 is also called crossover point. One can also show, directly from Eq. 1.3, that $\pi'(p_0) = \omega$ (see Fig.: 1.3). This family of distorted probabilities fits better experimental data[3] than all previous weighting functions. They also understood that ω and p_0 are not determined by the type of task and that ω is inversely proportional to the logarithm of the numerosity. For judgement of relative frequency, the numerosity is just the sample size. For prospect, the numerosity is the mean size of the outcomes. Therefore, ω decreases with increasing numerosity.

In 2017, Khaw, Li and Woodford, came up with the foundations of a quantitative model of the mental representation of a simple lottery choice problem which can explain risk aversion without the use of a utility function (or equivalently the utility function is considered as linear) [28]. It is based on the idea that the brain can only produce judgments based on the noisy information provided to it by sensory receptors. In the model, internal representation

[3]They reused experimental data from different groups of researchers. You can see an example in Fig.: 1.2. Experimental data are represented with black dots.

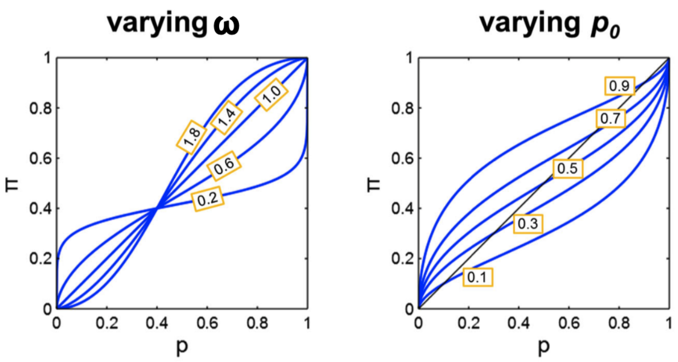

Figure 1.3: Demonstration of the effects of varying the parameters ω and p_0: Left: p_0 fixed at 0.4 and ω varied between 0.2 and 1.8. Note that the line at $\omega = 1$ overlaps with the diagonal line, i.e., no distortion of probability. **Right:** ω fixed at 0.6 and p_0 varied between 0.1 and 0.9. (adapted from [26])

of outcomes in prospects are considered noisy. The mental representation of numbers (used for non symbolic calculations like calculations with quantities) can indeed be represented by a quantity proportional to the logarithm of the numerical value encoded, plus a random error. As an example an outcome "30" will be encoded by a constant times the logarithm of "30" plus some noise (for more details see [29]). The logarithm is a simple concave function that just expresses the fact that people better distinguish two small quantities rather than two big quantities. And the noise stands for the degree of imprecision of the mental representation. However, the probabilities of the lotteries, are not taken noisy. They prefer to leave it for future work. And from these mental representations, the brain produces judgment. He does it optimally in the sens that the action is a solution of a maximisation problem (the subject has to maximize his proper wealth).
To be as clear as possible, let's consider the case where a decision-maker has to choose one of the two following prospects:

$$\begin{pmatrix} x & y \\ p & 1-p \end{pmatrix} \text{ or } \begin{pmatrix} z \\ 1 \end{pmatrix} \tag{1.4}$$

We repeat the task for many values of x, y, z and p. It is possible that the decision-maker is asked to choose between the same two prospects several times. Therefore, one can think about x, y, z and p as stimulus, and random variables. They are taken from prior distributions depending on the prospects shown to the subject. Let $\mathbb{P}[x]$[4] be the prior distribution for x, and let's use similar notation for the other random variables y and z. One argues that there are

[4]One uses the notation $\mathbb{P}$ for distribution instead of using $p()$ not to be confused with the p in prospects.

two phases during the decision process. The first is the **coding phase**, the second, the decoding phase. During coding, the brain, constructs noisy internal representations for x, y and z denoting respectively $r_x,\ r_y,\ r_z$. In mathematical terms, r_x is taken from the distribution $\mathbb{P}[r_x \mid x]$. One therefore can write: $r_x = \bar{x}(x) + \epsilon_x$ where $\bar{x}(x)$ is the expected value of r_x which should depend on x, and ϵ_x is the noise. One uses similar notation for the two other random variables. During **decoding**, given the internal representations, an optimal decision is taken according to the objective of maximizing the mathematical expectation of the subject's wealth. See the scheme 1.1 for an illustration.

Stimulus		**Internal representation**		**Decision**
x		r_x		
y	$\xrightarrow{\text{Coding}}$	r_y	$\xrightarrow{\text{Decoding}}$	0 1
z		r_z		

Table 1.1: Model synthesis scheme, the risky prospect is associated with the number 0, and the sure prospect with 1

Let's have a look to the decoding phase and state some interesting facts. Mathematically, maximizing his expected wealth can be stated as follows: (there are more general ways as explained in the article from Khaw, Li and Woodford but this one is intuitive)

- the subject chooses the risky prospect if and only if $\mathbb{E}[px + (1-p)y \mid r_x, r_y] > \mathbb{E}[z \mid r_z]$,
- the subject chooses the sure prospect if and only if $\mathbb{E}[px + (1-p)y \mid r_x, r_y] < \mathbb{E}[z \mid r_z]$.

Equivalently, assuming random variables are independent, the subject chooses the risky prospect if and only if $p\mathbb{E}[x \mid r_x] + (1-p)\mathbb{E}[y \mid r_y] > \mathbb{E}[z \mid r_z]$. And denoting by $\hat{x} = \mathbb{E}[x \mid r_x]$ (with analogous notation for y and z), one can rewrite the previous inequality as $p\hat{x} + (1-p)\hat{y} > \hat{z}$. Therefore, one can view $\hat{x}$, $\hat{y}$ and $\hat{z}$ as estimation of the stimulus, and based on these estimates, the subject make a decision. This is is synthesised in the scheme 1.2. What is also important to point out is that $\hat{x}$ is solution of the minimisation of the mean squared error problem (the same for y and z):

$$\hat{x}(r_x) = \min_{\hat{x}} \int_{\mathbb{R}} \mathrm{d}x \ \ (\hat{x}(r_x) - x)^2 \, \mathbb{P}[x \mid r_x].$$

Therefore the brain can be seen as a device that optimally perform the action of choosing one prospect, optimally in the sens that the action comes from both the maximisation of the perceived wealth computed from the estimates of the stimuli and the minimisation of the mean squared error between the estimates and the true stimuli.

Noisy internal representation of the stimulus	Optimally estimated stimulus	Optimal decision
$r_x \in \mathbb{R}$ $r_y \in \mathbb{R}$ $r_z \in \mathbb{R}$	$\hat{x}(r_x) \in \mathbb{R}$ $\hat{y}(r_y) \in \mathbb{R}$ $\hat{z}(r_z) \in \mathbb{R}$	• risky if $p\hat{x}(r_x)+(1-p)\hat{y}(r_y) > \hat{z}(r_z)$ • sure otherwise

Table 1.2: Decoding of $x,\ y,\ z$.

Nevertheless, even with an optimal estimation, the subject can make an error which is quantified by the bias (that can be computed explicitly in the model and measured experimentally):

$$\text{bias}(x) = \int_{\mathbb{R}} \mathrm{d}\hat{x} \ \ (\hat{x} - x)\, \mathbb{P}[\hat{x} \mid x]. \tag{1.5}$$

If $\text{bias}(x) = 0$, then the decision-maker is rational. If $\text{bias}(x) \neq 0$, then the decision-maker is not rational and will make a judgement from a perceptual bias. A fulfilled objective of this work is therefore to provide a possible explanation of how a perceptual bias can occur in the brain and what role this bias plays in decision making under risk.

1.3 Goal of the work

Based on the same ideas of noisy mental representation of quantities, Zhang and Maloney are currently working on an improved version of their LLO model: the bounded log-odds model (BLO model) [30]. It deals with the coding part in different tasks involving the estimation or use of probability. The general goal is therefore to build a general model of decision making under risk mainly from the work of Khaw, Li and Woodford and the one from Zhang and Maloney. This model should deal with coding and decoding, where the noisy variables are outcomes and probabilities. This model should finally offers an explanation for the paradoxes stated in prospect theory. But precisely, for this master thesis, the goal will be to do the decoding part for a decision task under risk between a risky prospect with two outcomes and a sure prospect, considering only the probability as a noisy random variable.

Chapter 2

Methods

2.1 BLO model

Zhang and Maloney are working on coding of probabilities (see Table 2.1). Their model is called bounded log-odds model (BLO model).

2.1.1 Two types of task

They considered two types of task: judgement of relative frequencies (JRF) task and decision making under risk (DMR) task. In **JRF task**, subjects were asked to estimate the proportion of black dots on a grey panel on which white and black dots were drawn. Here, two types of uncertainty are combined.

- There is an uncertainty in estimating the number of black dots. Let us call N the number of black dots estimated by the subject, and N_0 the total number of dots (black and white). In different trials that have the same number of dots in the visual stimulus, the value of N fluctuates.

- Once the value of N has been chosen (or equivalently the fraction,probability N/N_0), the brain records a noisy version of it.

In **DMR task**, subjects were asked to choose among two prospects the one that would maximize their gain. Either sure prospects or risky prospects with two outcomes were shown like the two following prospects:

$$\begin{pmatrix} z \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} x & y \\ p & 1-p \end{pmatrix}. \tag{2.1}$$

where $x, y, z \in \mathbb{R}$, and $p \in [0,1]$. Here, since the probability is already given, the only source of uncertainty comes from the internal representation of the probability.

Stimulus		**Internal representation**
$p \in [0,1]$	$\xrightarrow{\text{Coding}}$	L$\in \mathbb{R}$

Table 2.1: Model synthesis scheme

2.1.2 Coding steps

The steps are illustrated in Fig. 2.1 and explained below:

- Let p be either the proportion of black dots for JRF task or the probability of the highest outcome in a prospect for DMR task. This p is taken from a prior probability $\mathbb{P}[p]$. Some p are represented as short vertical lines in the first step in Fig. 2.1.

- Then, we pass from probability space to log-odds space threw the mapping: $\lambda(p) = \ln\left(\frac{p}{1-p}\right)$.

- For the third step is called bounding step. Zhang and Maloney argue that subjects are constrained by the resources it might take to find or compute the estimated probability or the optimal gain. They included this idea in their model saying the brain codes only a part of the log-odds scale from Δ^- to Δ^+, and that the smaller the interval, the more precisely the probabilities are coded. This last point is described in the model by the fifth and seventh steps. Let Δ be the half length of the interval: $\Delta = \frac{\Delta^+ - \Delta^-}{2}$. The points on that scale are now identified by $\Lambda(p)$. Λ is called the bounded log-odds function. One has:

$$\Lambda(p) = \begin{cases} \Delta^+ & \text{if } \lambda(p) > \Delta^+ \\ \lambda(p) & \text{otherwise} \\ \Delta^- & \text{if } \lambda(p) < \Delta^- \end{cases}. \tag{2.2}$$

- The fourth step is called "anchoring." The interval is shifted and compressed according to the transformation: $\omega_p \Lambda(p) + (1-\omega_p)\Lambda_0$, where $\forall p \in [0,1]$, $\omega_p \in [0,1]$, $\Lambda_0 \in \mathbb{R}$. $p \mapsto \omega_p$ is a non linear function of p. This function is determined thanks to what is called variance compensation. For JRF task, it integrates the uncertainty in the estimation of the frequency as explained in 2.1.1. See the next subsection 2.1.3 for more information. Let ω_+ be ω_p evaluated in p equals $\frac{1}{1+e^{-\Delta^+}}$. Let us denote $c^+ = \omega_+\Delta^+ + (1 - \omega_+)\Lambda_0$ and $c^- = \omega_-\Delta^- + (1-\omega_-)\Lambda_0$.

- The fifth step is the scaling or mapping to the Thurstone scale or interval. Each point on the precedent step is multiplied by $\tau \in \mathbb{R}_+^*$ to be mapped to the Thurstone scale. In that way, the smaller the coding interval, the more precisely the probabilities are coded. This can be seen as a microscope. The more you zoom in, the more precisely you can see but the smaller the size of the area being viewed. See the figure 2.2 for another pictorial representation of that idea. Let Ψ be the half length of the Thrustone scale. One has: $\Psi = \tau\omega\Delta$. See the next subsection 2.1.3 for more information. Let us denote $\Lambda_\omega(p)$ a point on this scale. One has: $\Lambda_\omega(p) = \tau\left[\omega_p\Lambda(p) + (1-\omega_p)\Lambda_0\right]$.

- The last step is the perturbation by some Gaussian noise ϵ_λ with mean $\Lambda_\omega(p)$ and variance σ_π^2. One denotes L a point on this scale. One has: $L = \Lambda_\omega(p) + \epsilon_\lambda$. L is the so called internal representation of p.

One can notice that fixing τ and Δ in the model is equivalent as fixing Ψ and Δ.

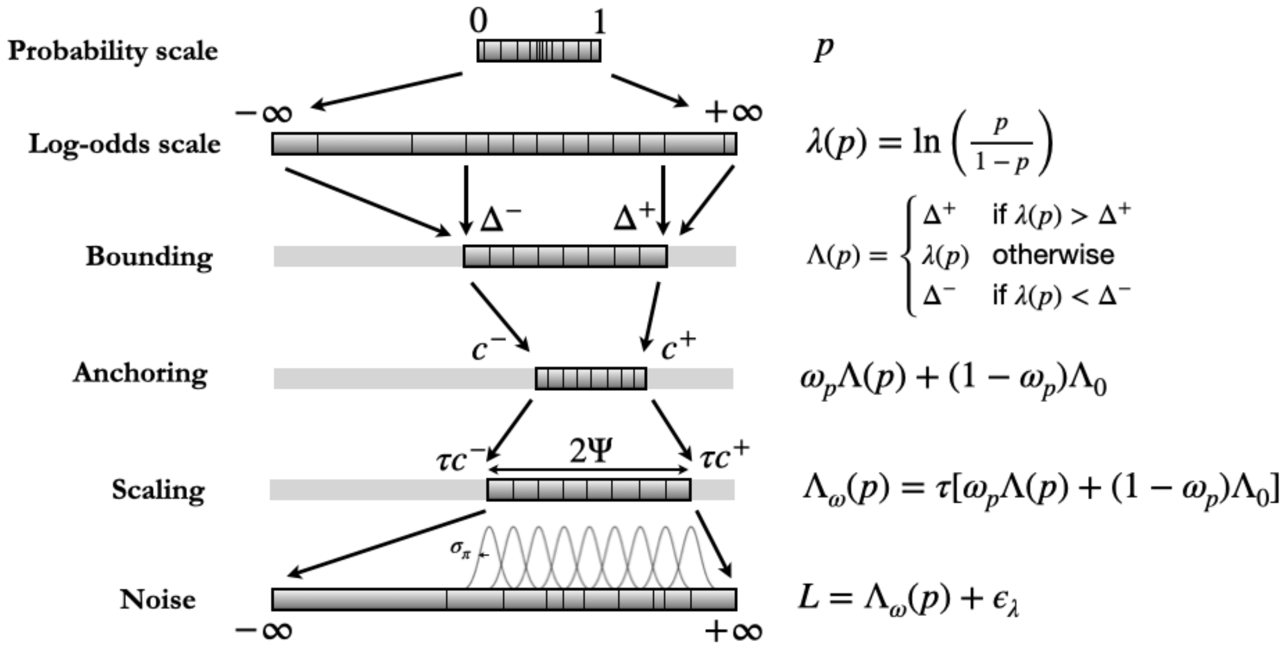

Figure 2.1: Scheme of the six coding steps. On the left-hand side is written the name of the step, in the middle are drawn illustrations of the steps, on the right-hand side, is written the literal expression of the small vertical lines drawn in the middle of the figure. (adapted from [31])

2.1.3 Three main assumptions

The model is based on three main assumptions:

- **log-odds representation** (second line in Fig. 2.1),
- **representation on a bounded Thurstone scale** (the fifth step in Fig. 2.1),

- **variance compensation** (the anchoring step in Fig. 2.1).

About **the second assumption**: the Thurstone scale [32] is a convenient mathematical structure in which information can be encoded and retrieved but now contaminated by Gaussian noise (with mean 0 and variance σ_π^2). It can model fallible memory [33]. In Fig. 2.1, one Gaussian curve drawn in the sixth step, symbolizes the encoding uncertainty induced by the Thurstone scale. As you can see in Fig. 2.2, there is a tension between the two limitations (density of encoded probability and size of the noise σ_π): the greater the log-odds range that needs to be encoded, the greater the density of the magnitudes along the Thurstone scale, and the greater the chances of confusion of nearby codes and vice versa. The challenge is to choose a transformation that is most beneficial to the organism or efficient for the organism. Intuitively, if the brain maximizes the transmitted information between the stimulus and the internal representation, over Δ^- and Δ^+, then it maximizes the mutual information between p and L. This is also known as efficient coding hypothesis [34] or maximization of the information channel capacity [35]. It is important to notice that σ_π is independent of any control by the subject, while Δ can be modified by the subject to optimize processing of probabilities.

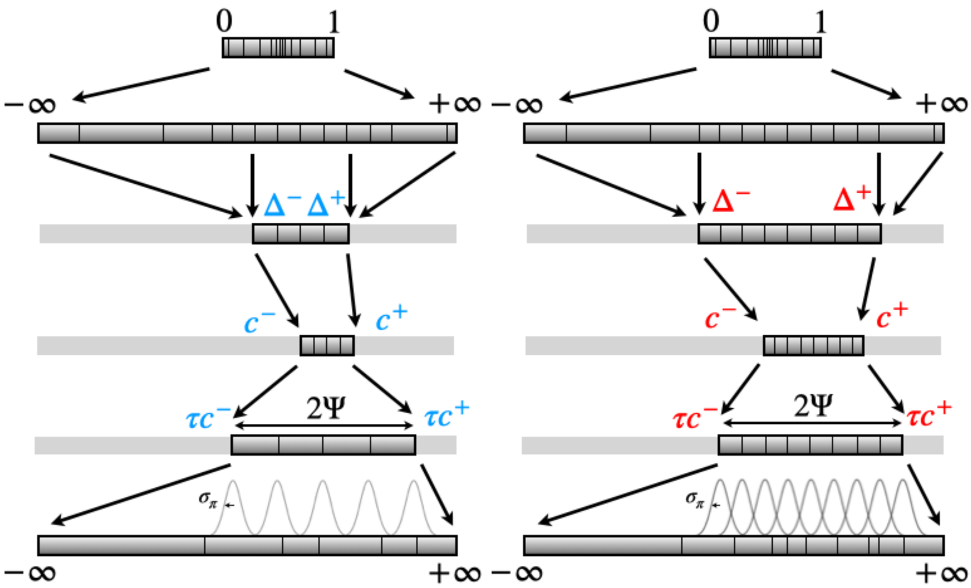

Figure 2.2: Left: Scheme of the coding steps in the case where Δ is small enough so that the Gaussian curves do not get tangled up in the sixth step. **Right:** Scheme of the coding steps in the case where Δ is big enough so that the Gaussian curves get tangled up in the sixth step.

The third assumption is technical. It concerns the choice of a functional form for ω_p, and the need to account for other sources of uncertainty (other than the Gaussian noise in the Thurstone scale). For JRF task (see subsection

2.1.1 for the comments on the different types of uncertainty), the other type of uncertainty is in the estimation of black dots N. Zhang and Maloney argue that the variance of N will locally change the internal representation of the estimation of the probability. The mathematical principle behind this transformation is called minimization of the variance of a cue combination (see [36], [37], [38] for more information). For DMR task, there is no other source of uncertainty. Therefore one can keep ω_p constant and equals to 1. It is also relevant to point out that one can fix Ψ to 1 because it doesn't contain any physics inside and it will then simplify the interpretation of the mathematical expressions. Finally, $\forall p,\ \Lambda_\omega(p) = \tau\Lambda(p) = \frac{\Lambda(p)}{\Delta}$. As you can see, it is also possible to think of all that model as simply first a distortion of the probability ie a mapping from [0,1] to $\mathbb{R}$, and then a perturbation by some noise that depend on the space. That noise should be dependant on some constrain so that it obeys the idea of the subject codes better some particular region of the space and less others. This perspective is left to a future work.

2.2 Setting up the decoding problem

We consider the following decision task with $x > z > y$:

$$\begin{pmatrix} x & y \\ p & 1-p \end{pmatrix} \text{ or } \begin{pmatrix} z \\ 1 \end{pmatrix}. \tag{2.3}$$

Internal representation		**Response**
$L \in \mathbb{R}$	$\xrightarrow{\text{Decoding}}$	0 1

Table 2.2: Decoding scheme, the risky prospect is associated with the number 0, and the sure prospect with 1.

The subject performs the decision task in order to maximize his or her monetary gain. Let $g(L)$ be the decision function which takes the value 0 if the risky prospect is chosen and 1 if the sure prospect is chosen. The subject wants to maximize over all the decision functions, his average gain. Let's write the average gain:

$$G = \int_0^1 \mathrm{d}p \int_{-\infty}^{+\infty} \mathrm{d}L \,[[xp + y(1-p)][1 - g(L)] + zg(L)]\, \mathbb{P}[L \mid p]\mathbb{P}[p]. \tag{2.4}$$

You can see that if $g(L) = 1$, then $G = z$ (this makes sense because the sure prospect has been chosen). If $g(L) = 0$, then $G = \int_0^1 \mathrm{d}p\ [xp + y(1-p)]\mathbb{P}[p]$ which is the average gain of the risky prospect. One is going to use a gaussian prior in log odds scale (that way, the computation is feasible). Therefore, $\forall p \in [0,1]$, the prior reads:

$$\mathbb{P}[p] = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\lambda(p)-\mu)^2}{2\sigma^2}} \frac{1}{p(1-p)}, \tag{2.5}$$

or equivalently:

$$\mathbb{P}[\lambda] = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}}. \tag{2.6}$$

The internal representation L of a probability p is corrupted by some Gaussian noise. The L are taken from the following distribution (called likelihood):

$$\mathbb{P}[L \mid p] = \frac{1}{\sqrt{2\pi\sigma_\pi^2}} e^{-\frac{\left(L-\frac{\Lambda(p)}{\Delta}\right)^2}{2\sigma_\pi^2}}, \tag{2.7}$$

where:

$$\Lambda(p) = \begin{cases} \Delta^+ & \text{if } \lambda(p) > \Delta^+ \\ \lambda(p) & \text{otherwise} \\ \Delta^- & \text{if } \lambda(p) < \Delta^- \end{cases}. \tag{2.8}$$

In order to define the problem correctly, the decision function must be parametrized correctly. To do that, let's notice:

$$\max_g G[g] = \int_{-\infty}^{+\infty} \mathrm{d}L \max\left\{ \int_0^1 \mathrm{d}p\ [xp + y(1-p)]\mathbb{P}[L \mid p]\mathbb{P}[p]\ , \int_0^1 \mathrm{d}p\ z\mathbb{P}[L \mid p]\mathbb{P}[p] \right\}. \tag{2.9}$$

Then, one can plot $\mathcal{I}_1(L) = \int_0^1 \mathrm{d}p\ [xp + y(1-p)]\mathbb{P}[L \mid p]\mathbb{P}[p]$ and $\mathcal{I}_2(L) = \int_0^1 \mathrm{d}p\ z\mathbb{P}[L \mid p]\mathbb{P}[p]$, for specific values of the parameters. One can see, that a change of maximum between $\mathcal{I}_1$ and $\mathcal{I}_2$ occurs only once. This is also the case also for other sets of parameters. We don't show the plots here because intuitively, it is obvious. Therefore, one can parametrize g as a threshold function:

$$g(L) = \begin{cases} 0, & \text{if } L > \frac{\theta}{\Delta} \\ 1, & \text{otherwise} \end{cases}. \tag{2.10}$$

Then, one can set the problem as follows: for which value of θ denoted θ^*, one has:

$$\frac{\mathrm{d}G}{\mathrm{d}\theta}(\theta^*) = 0. \tag{2.11}$$

To write $\frac{\mathrm{d}G}{\mathrm{d}\theta}(\theta^*)$, one starts to rewrite G using the definition of the decision function to make the integrals over L on $]-\infty, \frac{\theta}{\Delta}]$ (see Appendix B.1 for all the details). Then, one uses the Leibniz integral rule[1] to derive the gain with respect to θ. Let us denote θ^* the solution of the problem 2.10. One has: $\frac{\mathrm{d}G}{\mathrm{d}\theta}(\theta^*) = \frac{z-y}{\Delta}\int_0^1 \mathrm{d}p\ \mathbb{P}[L = \frac{\theta^*}{\Delta} \mid p]\mathbb{P}[p] + \frac{y-x}{\Delta}\int_0^1 \mathrm{d}p\ p\ \mathbb{P}[L = \frac{\theta^*}{\Delta} \mid p]\mathbb{P}[p]$. Then, one makes a change of variable to go from probability space to log-odds space and uses the definition of Λ. This splits each of the two integrals into three other integrals. An then putting the terms together, one finally gets:

$$0 = \mathfrak{I}_1(\theta^*) + \mathfrak{I}_2(\theta^*) + \mathfrak{I}_3(\theta^*), \tag{2.12}$$

where:

$$\mathfrak{I}_1(\theta^*) = \frac{e^{-\frac{\left(\theta^*-\Delta^+\right)^2}{2\sigma_\pi^2\sigma^2}}}{2\pi\sigma\sigma_\pi\Delta}\left[(z-y)\int_{\Delta^+}^{+\infty}\mathrm{d}\lambda\ e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}} + (y-x)\int_{\Delta^+}^{+\infty}\mathrm{d}\lambda\ \frac{e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}}}{1+e^{-\lambda}}\right], \tag{2.13}$$

$$\mathfrak{I}_2(\theta^*) = \frac{e^{-\frac{\left(\theta^*-\Delta^-\right)^2}{2\sigma_\pi^2\sigma^2}}}{2\pi\sigma\sigma_\pi\Delta}\left[(z-y)\int_{-\infty}^{\Delta^-}\mathrm{d}\lambda\ e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}} + (y-x)\int_{-\infty}^{\Delta^-}\mathrm{d}\lambda\ \frac{e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}}}{1+e^{-\lambda}}\right], \tag{2.14}$$

$$\mathfrak{I}_3(\theta^*) = \frac{z-y}{2\pi\sigma\sigma_\pi\Delta}\int_{\Delta^-}^{\Delta^+}\mathrm{d}\lambda\ e^{-\frac{(\theta^*-\lambda)^2}{2\sigma_\pi^2\Delta^2}}\ e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}} + \frac{y-x}{2\pi\sigma\sigma_\pi\Delta}\int_{\Delta^-}^{\Delta^+}\mathrm{d}\lambda\ \frac{e^{-\frac{(\theta^*-\lambda)^2}{2\sigma_\pi^2\Delta^2}}}{1+e^{-\lambda}}\ e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}}. \tag{2.15}$$

.

2.3 An interesting case

It is possible to solve Eq. 2.12 numerically. However, in some limiting case, an analytical expression for θ^* can be derived. Let us consider the case where the subject tends to be rational or equivalently the coding part is optimal and σ_π goes to 0.

Let us explain a bit. Let us denote p_0 the probability such that the two prospects are rationally equivalent. In log-odds space, that value is equal to $\ln\left(\frac{z-y}{x-z}\right)$. It is called the indifference point in log odds space. It corresponds to θ^* with $\sigma_\pi = 0$. Therefore σ_π goes to 0 is called the rational limit.
If we suppose that the coding part is optimal then θ^* is well coded. That is to say: Δ^- and Δ^+ are chosen (by the brain taking into account the noise) so that

[1]It is just derivative with integrals, nothing more.

they frame p_0 or θ^* with σ_π small enough. Quantitatively, one can assume:

$$\Delta^- + \alpha\Delta\sigma_\pi < \theta^* < \Delta^+ - \alpha\Delta\sigma_\pi, \tag{2.16}$$

where α is an integer which controls the distance of θ^* to the borders. One can rewrite these inequalities in the following form: $\left|\frac{\theta^*-\Delta^+}{\Delta\sigma_\pi}\right| > \alpha$ and $\left|\frac{\theta^*-\Delta^-}{\Delta\sigma_\pi}\right| > \alpha$ and $\theta^* \in [\Delta^-, \Delta^+]$. One can choose $\alpha = 3$, to be sure that θ^* is far from the borders. (Far from the borders, it's easier to get an analytical solution, and it's better for the understanding of the system).
Then using the optimal coding hypothesis, one can discard $\mathfrak{I}_2(\theta^*)$ and $\mathfrak{I}_3(\theta^*)$ in the equation 2.12. Therefore the equation that must be solve can be written:

$$0 = \frac{z-y}{2\pi\sigma\sigma_\pi\Delta}\int_{\Delta^-}^{\Delta^+} \mathrm{d}\lambda\; e^{-\frac{(\theta^*-\lambda)^2}{2\sigma_\pi^2\Delta^2}}\, e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}} + \frac{y-x}{2\pi\sigma\sigma_\pi\Delta}\int_{\Delta^-}^{\Delta^+} \mathrm{d}\lambda\; \frac{e^{-\frac{(\theta^*-\lambda)^2}{2\sigma_\pi^2\Delta^2}}}{1+e^{-\lambda}}\; e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}}. \tag{2.17}$$

Chapter 3

Results

3.1 Solution of Eq. 2.17

One wants to solve, in the limit σ_π goes to 0, the equation:

$$0 = \frac{z-y}{2\pi\sigma\sigma_\pi\Delta}\int_{\Delta^-}^{\Delta^+} \mathrm{d}\lambda\ e^{-\frac{(\theta^*-\lambda)^2}{2\sigma_\pi^2\Delta^2}} e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}} + \frac{y-x}{2\pi\sigma\sigma_\pi\Delta}\int_{\Delta^-}^{\Delta^+} \mathrm{d}\lambda\ \frac{e^{-\frac{(\theta^*-\lambda)^2}{2\sigma_\pi^2\Delta^2}}}{1+e^{-\lambda}}\ e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}}. \tag{3.1}$$

Let us rewrite this equation in a better form rewriting the product of the two Gaussians as the product of one Gaussian that does not depend on λ and another that depends on λ. Explicitly, it is possible to show that: for all a, b, c and d,

$$\frac{(\lambda-a)^2}{2b} + \frac{(\lambda-c)^2}{2d} = \frac{(a-c)^2}{2(b+d)} + \frac{\left(\lambda - \frac{bc+ad}{b+d}\right)^2}{2\frac{bd}{b+d}}. \tag{3.2}$$

Therefore Eq. 3.1 can be rewritten as follows:

$$\begin{aligned}0 = {} & \frac{z-y}{2\pi\sigma\sigma_\pi\Delta} e^{-\frac{(\mu-\theta^*)^2}{2(\sigma^2+\sigma_\pi^2\Delta^2)}} \int_{\Delta^-}^{\Delta^+} \mathrm{d}\lambda\ \exp\left[-\frac{\left(\lambda - \frac{\theta^*+\mu\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}\right)^2}{2\frac{\sigma_\pi^2\Delta^2}{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}}\right] \\ & + \frac{y-x}{2\pi\sigma\sigma_\pi\Delta} e^{-\frac{(\mu-\theta^*)^2}{2(\sigma^2+\sigma_\pi^2\Delta^2)}} \int_{\Delta^-}^{\Delta^+} \mathrm{d}\lambda\ \exp\left[-\frac{\left(\lambda - \frac{\theta^*+\mu\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}\right)^2}{2\frac{\sigma_\pi^2\Delta^2}{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}} - \ln\left(1+e^{-\lambda}\right)\right].\end{aligned} \tag{3.3}$$

Then by simplifying the exponential, one gets:

$$
\begin{aligned}
0 = {} & \frac{z-y}{2\pi\sigma\sigma_\pi\Delta}\int_{\Delta^-}^{\Delta^+} \mathrm{d}\lambda \ \exp\left[-\frac{\left(\lambda - \frac{\theta^* + \mu\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}\right)^2}{2\frac{\sigma_\pi^2\Delta^2}{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}}\right] \\
& + \frac{y-x}{2\pi\sigma\sigma_\pi\Delta}\int_{\Delta^-}^{\Delta^+} \mathrm{d}\lambda \ \exp\left[-\frac{\left(\lambda - \frac{\theta^* + \mu\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}\right)^2}{2\frac{\sigma_\pi^2\Delta^2}{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}} - \ln\left(1+e^{-\lambda}\right)\right].
\end{aligned}
\tag{3.4}
$$

3.1.1 General picture

We want to solve the previous equation in the rational limit: σ_π goes to 0. We are looking for θ^* in the form $a + b\sigma_\pi^2$ (where σ_π goes to 0, it is the rational limit). There is no therm of order one in σ_π, because only powers of degree two appear in the equations (then the taylor expansion consists only of even powers of σ_π).
Let's denote:

$$
\mathcal{I}_1 = \frac{1}{2\pi\sigma\sigma_\pi\Delta}\int_{\Delta^-}^{\Delta^+} \mathrm{d}\lambda \ \exp\left[-\frac{\left(\lambda - \frac{\theta^* + \mu\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}\right)^2}{2\frac{\sigma_\pi^2\Delta^2}{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}}\right], \tag{3.5}
$$

$$
\mathcal{I}_2 = \frac{1}{2\pi\sigma\sigma_\pi\Delta}\int_{\Delta^-}^{\Delta^+} \mathrm{d}\lambda \ \exp\left[-\frac{\left(\lambda - \frac{\theta^* + \mu\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}\right)^2}{2\frac{\sigma_\pi^2\Delta^2}{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}} - \ln\left(1+e^{-\lambda}\right)\right]. \tag{3.6}
$$

We are going to expand $\mathcal{I}_1$ and $\mathcal{I}_2$ up to second order in σ_π.

3.1.2 Expression for $\mathcal{I}_1$

We assume that:

- first the Gaussian is contained inside the interval $[\Delta^-, \Delta^+]$ ie quantitatively: $\frac{\theta^* + \mu\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2} \in [\Delta^-, \Delta^+]$,
- the width is sufficiently small ie quantitatively: $\frac{\sigma_\pi\Delta}{\sqrt{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}} \ll \Delta$.

We can then rewrite $\mathcal{I}_1$ by replacing Δ^- by $-\infty$ and Δ^+ by $+\infty$. We are then left with a simple Gaussian integral. We will have to check **the first assumption a posteriori**. And we can say in advance that it is closed to the efficient coding hypothesis. And by neglecting the first and second order in σ_π, it is exactly the efficient coding hypothesis. The **second assumption** is implied by the rational limit. Therefore, one gets:

$$\begin{aligned}\mathcal{I}_1 &\approx \frac{1}{2\pi\sigma\sigma_\pi\Delta}\sqrt{2\pi}\frac{\sigma\sigma_\pi\Delta}{\sqrt{\sigma^2+\sigma_\pi^2\Delta^2}} \\ &\approx \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\left[1-\frac{1}{2}\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2\right].\end{aligned} \tag{3.7}$$

The last step is obtained expanding to second order in σ_π.

3.1.3 Expression for $\mathcal{I}_2$

Saddle point approximation

We are going to use the saddle point method to compute the integral, being careful to use $a+b\sigma_\pi^2$ instead of θ^*. (Be aware that a saddle point method corresponds to a second order expansion in σ_π, so it's ok for us).
Let's denote $A=\frac{1}{\sigma_\pi^2}$, and

$$h(\lambda)=\frac{\left(\lambda-\frac{a+b\sigma_\pi^2+\mu\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}\right)^2}{2\frac{\Delta^2}{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}}+\sigma_\pi^2\ln\left(1+e^{-\lambda}\right). \tag{3.8}$$

We can rewrite the integral like this:

$$\mathcal{I}_2=\frac{1}{2\pi\sigma\sigma_\pi\Delta}\int_{\Delta^-}^{\Delta^+}\mathrm{d}\lambda\; e^{-Ah(\lambda)}. \tag{3.9}$$

Basically, we will expand $h(\lambda)$ around his minimum. Let's denote $\lambda^*=\operatorname{argmax}\{e^{-Ah(\lambda)}\}$. Then, one has up to second order in σ_π:

$$\mathcal{I}_2\approx\frac{e^{-Ah(\lambda^*)}}{\sqrt{2\pi}\sigma\Delta}\left[\frac{1}{\sqrt{h''(\lambda^*)}}+\frac{5}{24A}\frac{h'''(\lambda^*)^2}{\sqrt{h''(\lambda^*)^7}}-\frac{1}{8A}\frac{h''''(\lambda^*)}{\sqrt{h''(\lambda^*)^5}}\right]. \tag{3.10}$$

For all the detailed explanation of saddle point approximation, you can have a look to [39] and to the appendix C.1.1. We will then expand that expression up to second order in σ_π. The two last terms in the last equation are higher order in σ_π^2 so we can discard them and we are left with that expression:

$$\mathcal{I}_2\approx\frac{e^{-Ah(\lambda^*)}}{\sqrt{2\pi}\sigma\Delta\sqrt{h''(\lambda^*)}}. \tag{3.11}$$

Computation of λ^*

We have to solve:

$$h'(\lambda^*) = 0, \tag{3.12}$$

that is to say:

$$0 = \frac{\lambda^* - \frac{(a+b\sigma_\pi^2)\sigma^2+\mu\sigma_\pi^2\Delta^2}{\sigma^2+\sigma_\pi^2\Delta^2}}{\frac{\sigma^2\Delta^2}{\sigma^2+\sigma_\pi^2\Delta^2}} - \frac{\sigma_\pi^2}{1+e^{\lambda^*}}. \tag{3.13}$$

This is a transcendental equation. So, one way is to look for an approximate solution in the rational limit (σ_π goes to 0). A smarter way of proceeding is to look for a solution of the form:

$$\lambda^* = \frac{(a+b\sigma_\pi^2)\sigma^2+\mu\sigma_\pi^2\Delta^2}{\sigma^2+\sigma_\pi^2\Delta^2} + \frac{\sigma^2\Delta^2}{\sigma^2+\sigma_\pi^2\Delta^2}\lambda_1, \tag{3.14}$$

where λ_1 is to be determined. Now the unknown is λ_1. The equation is still transcendental but we can just add the hypothesis that λ_1 must be independent of σ_π not to have too high order.
Then, the equation reduces to:

$$\lambda_1 + \lambda_1 e^{\lambda^*} - \sigma_\pi^2 = 0. \tag{3.15}$$

λ_1 must be of order σ_π^2, we can therefore expand the exponential to order 0. Then, one has:

$$\lambda_1 + \lambda_1 e^{a} - \sigma_\pi^2 \approx 0. \tag{3.16}$$

Therefore, one gets:

$$\lambda_1 = \frac{1}{1+e^a}. \tag{3.17}$$

And finally, one has:

$$\lambda^* \approx \frac{(a+b\sigma_\pi^2)\sigma^2+\mu\sigma_\pi^2\Delta^2}{\sigma^2+\sigma_\pi^2\Delta^2} + \frac{\sigma^2\sigma_\pi^2\Delta^2}{\sigma^2+\sigma_\pi^2\Delta^2}\frac{1}{1+e^a}. \tag{3.18}$$

Expanding that expression up to second order in σ_π leads to the same expression the standard method would provide (with the standard method, you just suppose λ^* of the form $c + d\sigma_\pi^2$ and you expand everything in the rational limit).

Expression for $\mathcal{I}_2$

Now, one can do properly all the expansions:

$$e^{-h(\lambda^*)} \approx \frac{e^a}{1+e^a} + \frac{e^a\left[-2a\Delta^2 - 2a\Delta^2 e^a + 2\Delta^2\mu + 2\Delta^2\mu e^a + 2b\sigma^2 + \Delta^2\sigma^2 + 2be^a\sigma^2\right]}{2\left(1+e^a\right)^3}\left(\frac{\sigma_\pi}{\sigma}\right)^2, \tag{3.19}$$

$$\frac{1}{\sigma\sigma_\pi\Delta}\frac{1}{\sqrt{h''\left(\lambda^*\right)}} \approx \frac{1}{\sigma}\left[1 - \left(\frac{1}{2} + \frac{\sigma^2 e^a}{2\left(1+e^a\right)^2}\right)\left(\frac{\Delta\sigma_\pi}{\sigma}\right)^2\right]. \tag{3.20}$$

Finally, one gets:

$$\mathcal{I}_2 \approx \frac{1}{\sqrt{2\pi}\sigma}\frac{e^a}{1+e^a}\left[1 + \left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2\left[-\frac{1}{2} + \frac{\mu - a + b\left(\frac{\sigma}{\Delta}\right)^2}{1+e^a} + \frac{\sigma^2}{2}\frac{1-e^a}{\left(1+e^a\right)^2}\right]\right]. \tag{3.21}$$

3.1.4 Solution of Eq. 3.1 in closed form

Now, the equation reads:

$$0 = (z-y)\left[1 - \frac{1}{2}\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2\right] + \frac{y-x}{1+e^{-a}}\left[1 + \left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2\left[-\frac{1}{2} + \frac{\mu - a + b\left(\frac{\sigma}{\Delta}\right)^2}{1+e^a} + \frac{\sigma^2}{2}\frac{1-e^a}{\left(1+e^a\right)^2}\right]\right]. \tag{3.22}$$

We can now identify each term of the expansion, which leads to:

$$\begin{aligned} a &= \ln\left(\frac{z-y}{x-z}\right) \\ b &= \left(\frac{\Delta}{\sigma}\right)^2\left[\ln\left(\frac{z-y}{x-z}\right) - \mu + \sigma^2\frac{z - \frac{x+y}{2}}{x-y}\right]. \end{aligned} \tag{3.23}$$

Finally, one gets:

$$\theta^* = \ln\left(\frac{z-y}{x-z}\right) + (\sigma_\pi\Delta)^2\,\frac{z - \frac{x+y}{2}}{x-y} + \left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2\left[\ln\left(\frac{z-y}{x-z}\right) - \mu\right]. \tag{3.24}$$

3.2 Comments on Eq. 3.24

First, let us denote p_0 the probability such that the two prospects are rationally equivalent. It satisfies the following equation:

$$xp_0 + y(1-p_0) = z. \tag{3.25}$$

We have $p_0 = \frac{z-y}{x-y}$. In log-odds space, that value denoted λ_0 is equal to $\ln\left(\frac{z-y}{x-z}\right)$. It is called the indifference point in log-odds space. It corresponds to θ^* with $\sigma_\pi = 0$.

Then about the first order in σ_π^2 term, one can see that if Δ (the half length of the encoding interval), decreases, then θ^* becomes closer to θ_r rational (the one for $\sigma_\pi = 0$). This refers to the microscope picture. The smaller Δ is, the better the coding is.
Intuitively, assuming $z > \frac{x+y}{2}$ and μ close to 0.5 , if z increases and gets closer to x, people have impression that more winning sure prospect are shown, and so θ^* should increase.
One can also see that if σ goes to infinity (as if the prior was uniform), we retrieve the expression found in case of a uniform prior.
Intuitively, assuming $\mu \ll \lambda_0$, if μ decreases and moves away from λ_0, then more winning sure prospect are shown and so people will have the impression that all the winning prospects are the sure prospect, and so θ^* should increase. So, the μ-dependence in the expression of θ^* seems correct.

3.3 Risk Aversion

Let's compute the probability of choosing a sure prospect among all the pairs of prospects shown. One has:

$$\mathbb{P}[\text{sure}] = \int_0^1 \mathrm{d}p \; \mathbb{P}[\text{sure} \mid p] \; \mathbb{P}[p], \tag{3.26}$$

with:

$$\mathbb{P}[\text{sure} \mid p] = \mathbb{P}\left[L < \frac{\theta^*}{\Delta} \mid p\right] = \int_{-\infty}^{\frac{\theta^*}{\Delta}} \mathrm{d}L \; \frac{e^{-\frac{\left(L - \frac{\Lambda(p)}{\Delta}\right)^2}{2\sigma_\pi^2}}}{\sqrt{2\pi}\sigma_\pi}. \tag{3.27}$$

One can rewrite it as follows:

$$\mathbb{P}[\text{sure} \mid p] = \int_{-\infty}^{\theta^*} \mathrm{d}u \; \frac{e^{-\frac{(u - \Lambda(p))^2}{2\sigma_\pi^2 \Delta^2}}}{\sqrt{2\pi}\sigma_\pi \Delta}. \tag{3.28}$$

On fig. 3.1, that quantity is plotted for σ_π equals to 0 (the rational limit), and σ_π not equals to 0. The shift of threshold is called a **bias** for the indifference point (due to the **noisy** representation of the probability). The passage from a step function to a sigmoid accounts for the **variability** in choices for the same subject and so for a group of people (due to the **noise**).
And, we have:

$$\mathbb{P}[\text{sure}] = \int_{-\infty}^{\frac{\theta^*}{\Delta}} \mathrm{d}L \int_0^1 \frac{\mathrm{d}p}{p(1-p)} \; \frac{e^{-\frac{\left(L - \frac{\Lambda(p)}{\Delta}\right)^2}{2\sigma_\pi^2}}}{\sqrt{2\pi}\sigma_\pi} \frac{e^{-\frac{(\lambda(p) - \mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}. \tag{3.29}$$

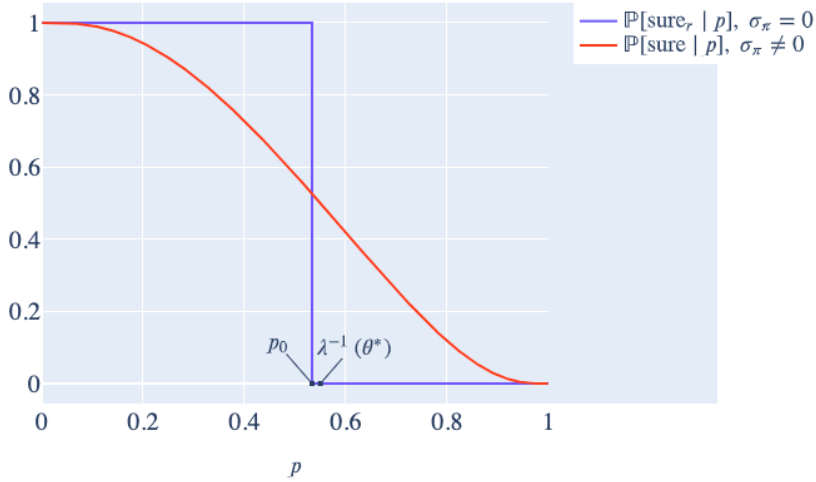

Figure 3.1: The curves are obtained with $\Delta^+ = 10$, $\Delta^- = -10$, $x = 40$, $y = 10$, $z = 26$, $\mu = 0.1$, $\sigma = 1$, $\sigma_\pi = 0.1$.

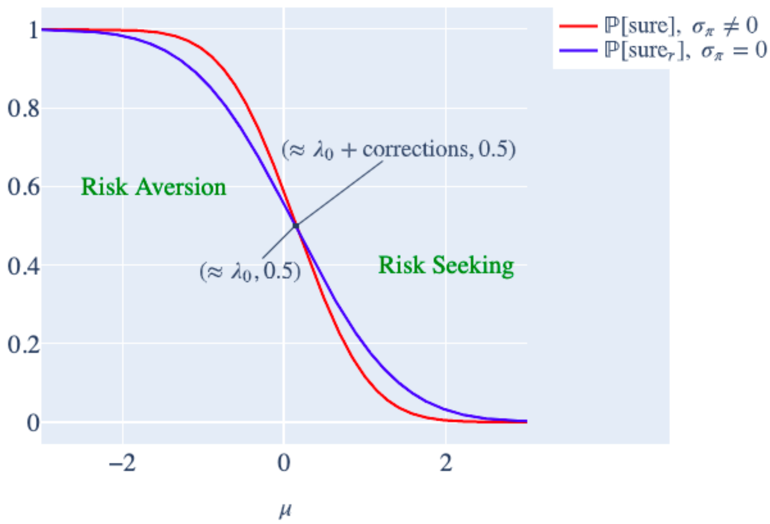

Figure 3.2: The curves are obtained with $\Delta^+ = 10$, $\Delta^- = -10$, $x = 40$, $y = 10$, $z = 26$, $\mu = 0.1$, $\sigma = 1$, $\sigma_\pi = 0.1$.

It is possible to write an expression in closed form for the probability of choosing the sure prospect. It is in the same spirit of the computation of θ^*. You can have a look to appendix C.2 for more details. One has:

$$\mathbb{P}[\text{sure}] \approx \frac{1}{2}\left[1 + \text{erf}\left(\frac{\frac{\sigma^2}{1+\frac{\sigma^2}{\sigma_\pi^2\Delta^2}}\frac{z-\frac{x+y}{2}}{x-y} + \ln\left(\frac{z-y}{x-z}\right) - \mu}{\sqrt{2}\sigma\frac{1}{\sqrt{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}}}\right)\right]. \tag{3.30}$$

Let us denote $\mathbb{P}[\text{sure}_r]$ the expression of $\mathbb{P}[\text{sure}]$ in the rational limit. We then see that:

$$\forall \mu < \lambda_0 + (\sigma_\pi \Delta)^2 \frac{z - \frac{x+y}{2}}{x-y}, \quad \mathbb{P}\left[\text{sure}\right] > \mathbb{P}\left[\text{sure}_r\right]. \tag{3.31}$$

On fig. 3.2, we tried to determine the dependence on μ of risk aversion. You see that for all μ below some threshold, the probability of choosing a sure prospect is above the probability of choosing a sure prospect being rational, and so there is risk aversion.

3.4 Distorted probability

We can go one step further finding an analytical expression for the distorted probability $\pi(p)$ mentioned in the introduction. Let us remind you what is the distorted probability and how is it measured. Let us consider a risky prospect where the subject can win x euros with a probability p and 0 with a probability $1-p$. (We place ourselves in the situation where $y = 0$ as it is often the case in the experiments conducted). Given a risky prospect and a sure one, there is rational equivalence between the two prospects when their average payoff is equal ie when $p = \frac{c}{x}$. Given p and x fixed, the value of c denoted c^* such that there is equivalence between the two prospects is called the certainty equivalence. In other terms, it is the amount of money for which the subject does not prefer one prospect over the other. Now if I vary p, this c^* also varies. Now, if we measure experimentally the certainty equivalence given p, and plot $\frac{c^*}{x}$ as a function of p, we don't get the identity at all as would be the case if subjects were rational but a curve which has an inverse s shape. This curve is called distorted probability.
Analytically, the certainty equivalent is the amount c^* such that the probability of choosing the sure prospect is equal to the probability of choosing the risky prospect, is equal to $\frac{1}{2}$:

$$\mathbb{P}[\text{sure} \mid p] = \mathbb{P}[\text{risky} \mid p] = \frac{1}{2}. \tag{3.32}$$

From the previous section, one has:

$$\mathbb{P}[\text{sure} \mid p] = \frac{1}{2}\left[1 + \text{erf}\left(\frac{r_p^* - \frac{\lambda}{\Delta}}{\sqrt{2}\sigma_\pi}\right)\right]. \tag{3.33}$$

Therefore, solving the equation, one gets $r_p^* = \frac{\lambda}{\Delta}$. r_p^* can here be seen as a function on c^*. As this function is strictly monotonous, there exists an inverse. So, by doing an expansion of the inverse function around σ_π equals to 0, one gets an anlytical expression of the certainty equivalent. Indeed, let us assume that:

$$c^* = c_0 + \sigma_\pi c_1 + \sigma_\pi^2 c_2. \tag{3.34}$$

Furthermore, $r_p^*(r_p^{*^{-1}}) = r_p^*(c_0 + \sigma_\pi c_1 + \sigma_\pi^2 c_2) = \frac{\lambda}{\Delta}$. By doing all the expansions, one gets:

$$\begin{aligned} c_0 &= \frac{x}{1+e^{-\lambda}}, \\ c_1 &= 0, \\ c_2 &= \Delta^2 x p(p-1)\left[p - \frac{1}{2} + \frac{\ln\left(\frac{p}{1-p}\right) - \mu}{\sigma^2}\right]. \end{aligned} \tag{3.35}$$

And finally, one obtains:

$$\pi(p) = p + (\sigma_\pi \Delta)^2 x p(p-1)\left[p - \frac{1}{2} + \frac{\ln\left(\frac{p}{1-p}\right) - \mu}{\sigma^2}\right]. \tag{3.36}$$

From that expression, we see that when σ_π goes to 0, then $\pi(p)$ goes to p, which is consistent with the rational limit taken. We also see that for $\mu = 0$, there is a fixed point at $p = \frac{1}{2}$ (see fig. 3.3).

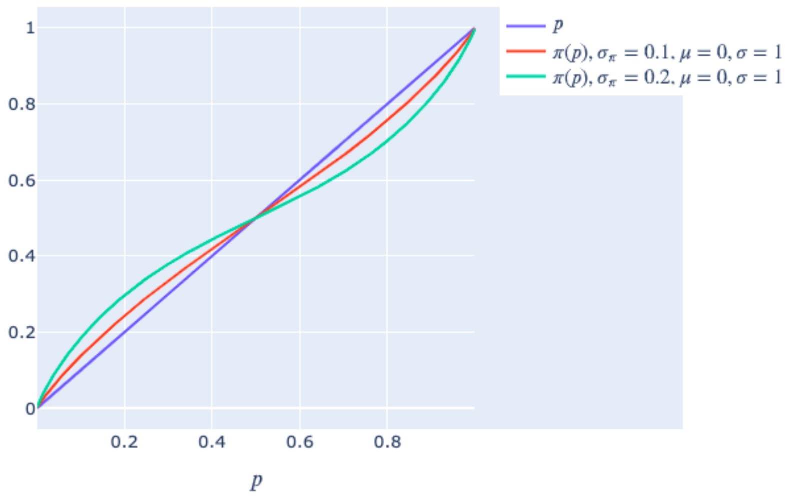

Figure 3.3: The curves are obtained with $\Delta^+ = 2$, $\Delta^- = -2$, $x = 10$, $y = 0$.

One can also try to find an expression for the crossover point p_0 ie the point p different from 0 and 1 such that $\pi(p) = p$. An analytical expression in the limit μ goes to 0 can be found:

$$p_0 \approx \frac{1}{1 + e^{-\frac{\mu}{1+\frac{\sigma^2}{4}}}}. \tag{3.37}$$

In the logodds space, p_0 is proportional to μ. The value of the crossover point increases with μ and decreases with σ. We can also get an expression for the slope of $\pi(p)$ at the crossover point in the limit μ goes to 0:

$$\pi'(p_0) \approx 1 - \frac{(\sigma_\pi \Delta)^2}{4} x \left[1 + \frac{4}{\sigma^2}\right]. \tag{3.38}$$

From that expression, we see that in the rational limit, the slope is equal to 1. And a slight perturbation in σ_π decreases the slope. Therefore, our model accounts for overweighting of small probabilities and underweighting of high probabilities. This analytical expressions are interesting and important because they can help us to design some experiments, and then easily compare the theory to the experiment.

Chapter 4

Conclusion/Discussion

We explored the decoding part of a decision task under risk between one risky prospect with two outcomes and one sure prospect. We used a Gaussian in log-odds space prior. We managed to find an expression for θ^* within the limit σ_π goes to 0, and the hypothesis of optimal coding that is to say θ^* is frame by Δ^+ and Δ^-. We noticed that the model was able to account for risk aversion for values of μ bigger than $\lambda_0 + (\sigma_\pi \Delta)^2 \frac{z - \frac{x+y}{2}}{x-y}$, and risk seeking for values of μ less than $\lambda_0 + (\sigma_\pi \Delta)^2 \frac{z - \frac{x+y}{2}}{x-y}$. We also managed to get an expression for the distorted probability. Then, one should try to see if this model fit better the data than all the others. Nevertheless, we can't say something about the Allai's paradox and the common ratio effect. Indeed, we should do the decoding work for two risky prospects with two and three outcomes. Moreover, we could study the effect of non symmetric prior on θ^* and so risk aversion, and the effect of the prior width on risk aversion. Moreover, we could lay the groundwork for a more general and pretty version of the model:

- First, one can discuss the use of efficient coding hypothesis for the model. Zhang and Maloney noticed that it wasn't entirely consistent with the experimental data. One might think that the bounded rational decision-maker would rather maximize his expected monetary gain minus a certain cost modelling the metabolic cost for the brain to compute the optimal decision. This maximization would be on Δ and on the form of the noise. We should indeed also use a noise depending on λ (the unit scale in the log-odds scale). This way of looking at the problem is already present in statistical physics where one usually have to minimize a free energy. Parallels between decision making theory and statistical physics have already been noted and formalized [40]. We could even think to a more general framework in which, instead of using Δ^+, Δ^- and the Thurstone scale, we use a non uniform noise over the whole log-odds space. And therefore, we should maximise the gain minus some constant which plays the role of the temperature time the mutual information between L the internal representation and the stimulus probability. Some efforts have already been made in that direction by mathematicians in case of a very simple gain function [41].

- Second, if p is given explicitly to the subject, then the only source of uncertainty comes from the noisy internal representation of the probability. But if instead the subject has to first estimate the probabilities (in an estimation of relative frequencies task for instance), the value of p may also be uncertain. Therefore, instead of choosing an ad-hoc functional form for ω as Zhang and Maloney did, one may introduce $\tilde{p} = p + \epsilon_{\tilde{p}}$ where $\epsilon_{\tilde{p}}$ is some noise.

- Third, one may use a sigmoid for the bounded log-odds function rather than a piecewise function. That sigmoid should be chosen making the computation of θ^* easier.

Appendix A

Introduction

A.1 History of the study of cognitive biases in decision-making under risk tasks

A.1.1 EUTR's axioms

Let $\mathcal{R}$ be the set of rewards (outcomes). Let $\mathcal{P}$ be the set of all probability distributions on $\mathcal{R}$.
We use the following notations (the same as the ones in the DeGroot book [42]): $\forall P_1, P_2 \in \mathcal{P}$,

$P_1 \prec^* P_2$: P_2 is preferred to P_1;

$P_1 \sim^* P_2$: P_1 is equivalent to P_2;

$P_1 \preceq^* P_2$: P_1 is not preferred to P_2.

Here are the four axioms:

Axiom 1: (Completeness)
$\forall P_1, P_2 \in \mathcal{P}$, exactly one of the following holds : $P_1 \prec^* P_2$, $P_1 \sim^* P_2$, or $P_2 \prec^* P_1$;

Axiom 2: (Transitivity)
$\forall P_1, P_2, P_3 \in \mathcal{P}$, if $P_1 \prec^* P_2$ and $P_2 \prec^* P_3$, then $P_1 \prec^* P_3$, and similarly for $\sim^*$;

Axiom 3: (Independence)
$\forall P_1, P_2 \in \mathcal{P}$, $P_1 \prec^* P_2$ implies that $\forall P_3 \in \mathcal{P}, \forall \alpha \in]0{,}1[, \quad \alpha P_1 + (1-\alpha)P_3 < \alpha P_2 + (1-\alpha)P_3$;

Axiom 4: (Continuity)
$\forall P_1, P_2, P_3 \in \mathcal{P}$, $P_1 \prec^* P_2 \prec^* P_3$ implies that $\exists \alpha \in]0{,}1[, \exists \beta \in]0{,}1[, \ \alpha P_1 + (1-\alpha)P_3 \prec^* P_2$ and $P_2 \prec^* \beta P_1 + (1-\beta)P_3$.

Now, let's recall some basic rules for manipulating probability distributions. Let P_1 be the probability distribution associated with the prospect $\begin{pmatrix} x & y \\ p_1 & 1-p_1 \end{pmatrix}$ and P_2 the probability distribution associated with that one $\begin{pmatrix} x & y \\ p_2 & 1-p_2 \end{pmatrix}$. One uses the following notation to represent P_1: $P_1 = \begin{bmatrix} x & y \\ p_1 & 1-p_1 \end{bmatrix}$. And similarly for P_2, one has: $P_2 = \begin{bmatrix} x & y \\ p_2 & 1-p_2 \end{bmatrix}$. Linear combination of probability distributions is defined as follows:

$$\forall \alpha \in [0,1],\ \alpha P_1 + (1-\alpha) P_2 = \begin{bmatrix} x & y \\ \alpha p_1 + (1-\alpha) p_2 & \alpha(1-p_1) + (1-\alpha)(1-p_2) \end{bmatrix}. \tag{A.1}$$

One can verify that the normalization is conserved.

A.1.2 Allais paradox

It is taken from [2]. Let's consider two decision problems. In each one, people have to choose between two prospects. Below each prospect is written the percentage of persons who chose the corresponding prospect, and above, the distribution associated with:

- first decision problem: $\overset{P_1}{\begin{pmatrix} 2500 & 2400 & 0 \\ 0.33 & 0.66 & 0.01 \end{pmatrix}}_{[18\%]}$ or $\overset{P_2}{\begin{pmatrix} 2400 \\ 1 \end{pmatrix}}_{[82\%]}$,

- second decision problem: $\overset{P_3}{\begin{pmatrix} 2500 & 0 \\ 0.33 & 0.67 \end{pmatrix}}_{[83\%]}$ or $\overset{P_4}{\begin{pmatrix} 2400 & 0 \\ 0.34 & 0.66 \end{pmatrix}}_{[17\%]}$.

According to EUTR (especially the VNM Theorem), $P_1 \preceq^* P_2 \Leftrightarrow \mathbb{E}_{P_1}[U] - \mathbb{E}_{P_2}[U] < 0 \Leftrightarrow \mathbb{E}_{P_3}[U] - \mathbb{E}_{P_4}[U] < 0 \Leftrightarrow P_3 \preceq^* P_4$
According to the percentages, $P_1 \preceq^* P_2$ and $P_4 \preceq^* P_3$. We have therefore a contradiction.
This paradox is sometimes also called "common consequence effect." As an explanation for that terminology, you can go from the first problem to the second erasing from prospect P_1 and P_2 the common consequence "2400," and readjusting the probability weights for the outcomes 0 and 2400.

A.1.3 Common ratio effect (with a sure gain)

It is taken from [2]. Let's consider two decision problems. In each one, people have to choose between two prospects:

- first decision problem: $\overset{P_1}{\begin{pmatrix} 4000 & 0 \\ 0.8 & 0.2 \end{pmatrix}}$ or $\overset{P_2}{\begin{pmatrix} 3000 \\ 1 \end{pmatrix}}$,
  [20%] [80%]

- second decision problem: $\overset{P_3}{\begin{pmatrix} 4000 & 0 \\ 0.2 & 0.8 \end{pmatrix}}$ or $\overset{P_4}{\begin{pmatrix} 3000 & 0 \\ 0.25 & 0.75 \end{pmatrix}}$.
  [65%] [35%]

Let P be the distribution associated to the prospect $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
According to EUTR (especially the axiom of independence with $\alpha = \frac{1}{4}$), $P_1 \preceq^* P_2 \Leftrightarrow \frac{1}{4}P_1 + \left(1 - \frac{1}{4}\right) P \preceq^* \frac{1}{4}P_2 + \left(1 - \frac{1}{4}\right) P \Leftrightarrow P_3 \preceq^* P_4$
According to the percentages, $P_1 \preceq^* P_2$ and $P_4 \preceq^* P_3$. We have therefore a contradiction.

A.1.4 Violation of stochastic dominance

Let's consider a decision task between the two prospects:

$$\overset{P_1}{\begin{pmatrix} 100 & 80 & 0 \\ 0.65 & 0.3 & 0.05 \end{pmatrix}} \text{ or } \overset{P_2}{\begin{pmatrix} 100 & 80 & 0 \\ 0.25 & 0.7 & 0.05 \end{pmatrix}}. \tag{A.2}$$

Obviously, any individual would choose the first prospect to maximize his monetary gain. As a remark: using the cumulative representation of the prospect, you can see geometrically directly that P_1 first order stochastically dominates P_2. One then see that FSD is intuitively the same concept as obviously preferred.

Let's now use prospect theory. One can recall that in prospect theory $U(0) = 0$. Let's choose a decision maker whose utility function satisfies $U(80) = 0.95 * U(100)$ (this is consistent with the fact that U is increasing), and whose weighting function satisfies $\pi(0.65) = 0.55$, $\pi(0.3) = 0.38$, $\pi(0.25) = 0.36$, $\pi(0.7) = 0.59$ (this is consistent with the property of subcertainty of π ie $\pi(p) + \pi(p') + \pi(1 - p - p') < 1$ and the S-shape of π). Then, $\mathbb{E}_{P_1}[U] = 0.911 * U(100)$, and $\mathbb{E}_{P_2}[U] = 0.9205 * U(100)$. Therefore $P_1 \preceq^* P_2$. FSD is therefore violated.

A.1.5 Discriminability and attractiveness

Given $q_1, q_2 \in [0,1]$ such that $q_1 < q_2$, and π_1, π_2 two weighting functions, one says that π_1 exhibits a higher discriminability than π_2 within the interval $[q_1, q_2]$ whenever : $\forall \epsilon > 0, \forall p \in [0,1]$ such that $p \in [q_1, q_2]$ and $p + \epsilon \in [q_1, q_2]$, one has $\pi_1(p + \epsilon) - \pi_1(p) > \pi_2(p + \epsilon) - \pi_2(p)$. In other words the variations within $[q_1, q_2]$ of π_1 are bigger than the ones of π_2.

Let π_1 and π_2 be two weighting functions associated with individual number one and number two respectively. One says that individual number one finds betting on the chance domain [0,1] more attractive than individual number two if $\forall p \in [0,1]$, $\pi_1(p) \geq \pi_2(p)$ and $\exists p \in [0,1]$, $\pi_1(p) > \pi_2(p)$.

Appendix B

Methods

B.1 Setting up the decoding problem

$$G = y + (x-y)\int_0^1 \mathrm{d}p\ p\ \mathbb{P}[p] + (z-y)\int_0^1 \mathrm{d}p \int_{-\infty}^{+\infty} \mathrm{d}L\ g(L)\mathbb{P}[L \mid p]\mathbb{P}[p] + \\ (y-x)\int_0^1 \mathrm{d}p \int_{-\infty}^{+\infty} \mathrm{d}L\ p\ g(L)\mathbb{P}[L \mid p]\mathbb{P}[p] \quad \text{(B.1)}$$

$$G = y + (x-y)\int_0^1 \mathrm{d}p\ p\ \mathbb{P}[p] + (z-y)\int_0^1 \mathrm{d}p \int_{-\infty}^{\frac{\theta}{\Delta}} \mathrm{d}L\ \mathbb{P}[L \mid p]\mathbb{P}[p] + \\ (y-x)\int_0^1 \mathrm{d}p \int_{-\infty}^{\frac{\theta}{\Delta}} \mathrm{d}L\ p\ \mathbb{P}[L \mid p]\mathbb{P}[p] \quad \text{(B.2)}$$

The first two terms are constant with respect to θ so we don't care of them.

$$\begin{aligned}
\frac{\mathrm{d}G}{\mathrm{d}\theta} &= \frac{z-y}{\Delta}\int_0^1 \mathrm{d}p\ \mathbb{P}[L = \frac{\theta}{\Delta} \mid p]\mathbb{P}[p] + \frac{y-x}{\Delta}\int_0^1 \mathrm{d}p\ p\ \mathbb{P}[L = \frac{\theta}{\Delta} \mid p]\mathbb{P}[p] \\
&= \frac{e^{-\frac{\left(\theta-\Delta^+\right)^2}{2\sigma_\pi^2\sigma^2}}}{2\pi\sigma\sigma_\pi\Delta}\left[(z-y)\int_{\Delta^+}^{+\infty} \mathrm{d}\lambda\ e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}} + (y-x)\int_{\Delta^+}^{+\infty} \mathrm{d}\lambda\ \frac{e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}}}{1+e^{-\lambda}}\right] \\
&+ \frac{e^{-\frac{\left(\theta-\Delta^-\right)^2}{2\sigma_\pi^2\sigma^2}}}{2\pi\sigma\sigma_\pi\Delta}\left[(z-y)\int_{-\infty}^{\Delta^-} \mathrm{d}\lambda\ e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}} + (y-x)\int_{-\infty}^{\Delta^-} \mathrm{d}\lambda\ \frac{e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}}}{1+e^{-\lambda}}\right] \\
&+ \frac{z-y}{2\pi\sigma\sigma_\pi\Delta}\int_{\Delta^-}^{\Delta^+} \mathrm{d}\lambda\ e^{-\frac{(\theta-\lambda)^2}{2\sigma_\pi^2\Delta^2}} e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}} + \frac{y-x}{2\pi\sigma\sigma_\pi\Delta}\int_{\Delta^-}^{\Delta^+} \mathrm{d}\lambda\ \frac{e^{-\frac{(\theta-\lambda)^2}{2\sigma_\pi^2\Delta^2}}}{1+e^{-\lambda}}\ e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}}
\end{aligned} \quad \text{(B.3)}$$

For the last step wee just used the definition of $\Lambda(p)$ to split the integrals, and made the change of variable $\lambda = \ln\left(\frac{p}{1-p}\right)$.

Appendix C

Results

C.1 Solution of eq. 2.17

C.1.1 Expression for $\mathcal{I}_2$

We would like to have an expansion of an integral of the form $\mathcal{I}_2 = \frac{1}{2\pi\sigma\sigma_\pi\Delta}\int_{\Delta^-}^{\Delta^+} \mathrm{d}\lambda\, e^{-Ah(\lambda)}$ when A goes to infinity. Let us first expand $h(\lambda)$ around λ^* (the maximum of the integrand which is assumed to belong to the interval $[\Delta^-, \Delta^+]$):

$$h(\lambda) = h(\lambda^*) + \frac{(\lambda-\lambda^*)^2}{2}h''(\lambda^*) + \frac{(\lambda-\lambda^*)^3}{6}h'''(\lambda^*) + \frac{(\lambda-\lambda^*)^4}{24}h''''(\lambda^*). \quad \text{(C.1)}$$

Let us now make a change of variable to deal with an integral between $-\infty$ and $+\infty$: $u \doteq (\lambda-\lambda^*)\sqrt{A}$. Therefore, one has:

$$Ah(\lambda) \simeq Ah(\lambda^*) + \frac{u^2}{2}h''(\lambda^*) + \frac{u^3}{6\sqrt{A}}h'''(\lambda^*) + \frac{u^4}{24A}h''''(\lambda^*). \quad \text{(C.2)}$$

It follows:

$$\begin{aligned} e^{-Ah(\lambda)} &\simeq e^{-Ah(\lambda^*)}e^{-\frac{u^2}{2}h''(\lambda^*)}e^{-\frac{u^3}{6\sqrt{A}}h'''(\lambda^*)}e^{-\frac{u^4}{24A}h''''(\lambda^*)} \\ &\simeq e^{-Ah(\lambda^*)}e^{-\frac{u^2}{2}h''(\lambda^*)}\left[1 - \frac{u^3}{6\sqrt{A}}h'''(\lambda^*) + \frac{u^6 h'''(\lambda^*)^2 - 3u^4 h''''(\lambda^*)}{72A}\right] \end{aligned} \quad \text{(C.3)}$$

For the last steps, we made an expansion for A goes to infinity. Finally, one has keeping in mind that the integration boundaries will be approximate to infinity:

$$
\begin{aligned}
\mathcal{I}_2 &\simeq \frac{e^{-Ah(\lambda^*)}}{2\pi\sigma\Delta} \int_{(\Delta^- - \lambda^*)\sqrt{A}}^{(\Delta^+ - \lambda^*)\sqrt{A}} \mathrm{d}u \; e^{-\frac{u^2}{2}h''(\lambda^*)} \\
&+ \frac{1}{72A}\frac{e^{-Ah(\lambda^*)}}{2\pi\sigma\Delta}\left[h'''(\lambda^*)^2 \int_{(\Delta^- - \lambda^*)\sqrt{A}}^{(\Delta^+ - \lambda^*)\sqrt{A}} \mathrm{d}u \; u^6 e^{-\frac{u^2}{2}h''(\lambda^*)} - 3h''''(\lambda^*) \int_{(\Delta^- - \lambda^*)\sqrt{A}}^{(\Delta^+ - \lambda^*)\sqrt{A}} \mathrm{d}u \; u^4 e^{-\frac{u^2}{2}h''(\lambda^*)}\right] \\
&\simeq \frac{e^{-Ah(\lambda^*)}}{2\pi\sigma\Delta} \int_{-\infty}^{+\infty} \mathrm{d}u \; e^{-\frac{u^2}{2}h''(\lambda^*)} \\
&+ \frac{1}{72A}\frac{e^{-Ah(\lambda^*)}}{2\pi\sigma\Delta}\left[h'''(\lambda^*)^2 \int_{-\infty}^{+\infty} \mathrm{d}u \; u^6 e^{-\frac{u^2}{2}h''(\lambda^*)} - 3h''''(\lambda^*) \int_{-\infty}^{+\infty} \mathrm{d}u \; u^4 e^{-\frac{u^2}{2}h''(\lambda^*)}\right] \\
&= \frac{e^{-Ah(\lambda^*)}}{\sqrt{2\pi}\sigma\Delta}\left[\frac{1}{\sqrt{h''(\lambda^*)}} + \frac{5}{24A}\frac{h'''(\lambda^*)^2}{\sqrt{h''(\lambda^*)^7}} - \frac{1}{8A}\frac{h''''(\lambda^*)}{\sqrt{h''(\lambda^*)^5}}\right].
\end{aligned}
\tag{C.4}
$$

C.2 Risk aversion

We want to compute:

$$
\mathbb{P}[\text{sure}] = \int_{-\infty}^{\frac{\theta^*}{\Delta}} \mathrm{d}L \int_0^1 \frac{\mathrm{d}p}{p(1-p)} \frac{e^{-\frac{\left(L - \frac{\Lambda(p)}{\Delta}\right)^2}{2\sigma_\pi^2}}}{\sqrt{2\pi}\sigma_\pi} \frac{e^{-\frac{(\lambda(p)-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}. \tag{C.5}
$$

Let us focus on the convolution term first. One has:

$$
\begin{aligned}
\mathbb{P}[L] &= \int_0^1 \mathrm{d}p \; \mathbb{P}[L \mid p] \; \mathbb{P}[p] \\
&= \int_{-\infty}^{+\infty} \mathrm{d}\lambda \; \frac{e^{-\frac{\left(L - \frac{\Delta}{\Delta}\right)^2}{2\sigma_\pi^2}}}{\sqrt{2\pi}\sigma_\pi} \frac{e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} \\
&= \frac{e^{-\frac{\left(L - \frac{\Delta^+}{\Delta}\right)^2}{2\sigma_\pi^2}}}{2\pi\sigma\sigma_\pi} \int_{\Delta^+}^{+\infty} \mathrm{d}\lambda \; e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}} \\
&+ \frac{e^{-\frac{\left(L - \frac{\Delta^-}{\Delta}\right)^2}{2\sigma_\pi^2}}}{2\pi\sigma\sigma_\pi} \int_{-\infty}^{\Delta^-} \mathrm{d}\lambda \; e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}} \\
&+ \frac{1}{2\pi\sigma\sigma_\pi} \int_{\Delta^-}^{\Delta^+} \mathrm{d}\lambda \; e^{-\frac{\left(L - \frac{\lambda}{\Delta}\right)^2}{2\sigma_\pi^2}} e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}}
\end{aligned}
\tag{C.6}
$$

The last integral is a convolution. Using the efficient coding hypothesis, one can remove the boundary terms. As previously, we can rewrite the product of two Gaussians in two Gaussians, one of which is independent of λ:

$$\mathbb{P}[L] \approx \frac{e^{-\frac{\left(L-\frac{\mu}{\Delta}\right)^2}{2\left(\frac{\sigma}{\Delta}\right)^2\left[1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2\right]}}}{2\pi\sigma\sigma_\pi} \int_{\Delta^-}^{\Delta^+} \mathrm{d}\lambda \, \exp\left[-\frac{\left(\lambda - \frac{\mu(\sigma_\pi\Delta)^2+\sigma^2 L\Delta}{\sigma^2+(\sigma_\pi\Delta)^2}\right)^2}{2\frac{(\sigma_\pi\Delta)^2}{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}}\right]. \tag{C.7}$$

Using the efficient coding hypothesis once again and the rational limit hypothesis, one can use the saddle point method to compute the integral. We get:

$$\mathbb{P}[L] \approx \frac{\Delta}{\sigma\sqrt{2\pi}} \frac{e^{-\frac{\left(L-\frac{\mu}{\Delta}\right)^2}{2\left(\frac{\sigma}{\Delta}\right)^2\left[1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2\right]}}}{\sqrt{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}}. \tag{C.8}$$

And finally, one has:

$$\begin{aligned}
\mathbb{P}[\text{sure}] &\approx \int_{-\infty}^{\frac{\theta^*}{\Delta}} \mathrm{d}L \, \frac{\Delta}{\sigma\sqrt{2\pi}} \frac{e^{-\frac{\left(L-\frac{\mu}{\Delta}\right)^2}{2\left(\frac{\sigma}{\Delta}\right)^2\left[1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2\right]}}}{\sqrt{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}} \\
&= \int_{-\infty}^{\theta^*} \mathrm{d}u \, \frac{1}{\sigma\sqrt{2\pi}} \frac{e^{-\frac{(u-\mu)^2}{2\sigma^2\left[1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2\right]}}}{\sqrt{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}} \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{\theta^*-\mu}{\sqrt{2}\sigma\sqrt{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}}} \mathrm{d}y \, e^{-y^2} \\
&= \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{\theta^*-\mu}{\sqrt{2}\sigma\sqrt{1+\left(\frac{\sigma_\pi\Delta}{\sigma}\right)^2}}\right)\right].
\end{aligned} \tag{C.9}$$

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