Master degree in Physics of Complex Systems

RENORMALIZATION GROUP APPROACH TO THE SOLUTION OF INTEGRALS AND SCHRÖDINGER EIGENVALUE EQUATIONS

Master thesis

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A mio padre e mia madre.

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Alberto Giuseppe Catalano

Contents

Introduction

The idea of renormalization was first introduced as an ad hoc method of eliminating unphysical divergences arising in the theoretical treatment of interacting many-body systems. Later, it found its fulfillment as a technique for coarse-graining the statistical description of systems [1, 2, 3, 4, 5, 6] with many length and time scales, and thus as a way to understand critical phenomena and the concept of universality. Mean field theory represents a first step towards a thorough understanding of critical phenomena. When fluctuations at all scales play a vital role one needs to go beyond the mean field approach. In this framework, the renormalization group and scaling theory provide a series of tool to systematically take into account the effects of fluctuations, providing a deep insight into critical phenomena. This is the reason why the renormalization group has had great influence on the development of statistical field theory, and has undoubtedly been one of the most successful developments in many-body theory in the last half-century.

The main topic of this thesis is the application of renormalization group techniques as a mathematical tool for the calculation of non trivial integrals and for the determination of the spectrum of the Schrödinger equation. The work is organized as follows.

Chapter 1 contains a brief introduction to the concepts of renormalization group, with some practical examples.

In *Chapter* 2 a modern non-perturbative formulation of the functional renormalization group (FRG) is presented in the framework of the effective average action approach.

Our original results are presented in *Chapters* 3 - 5. *Chapters* 3 and 4 are devoted to the application of the ideas of the RG to the calculation of non-trivial integrals, respectively of one variable and of many variables.

Chapter 5 contains applications of the RG concepts to quantum mechanics, and to the determination of the ground state of the Schrödinger equation for different potentials.

Chapter 1

Reminders on the Renormalization Group

1.1 Why renormalization?

The fundamental problem studied within statistical mechanics is the emergence of new collective properties in the macroscopic realm from the dynamics of the microscopics constituents of a system. This underlies the idea of reducing the amount of information necessary to describe a system. An exact microscopic description of a physical systems involves a very high number of degrees of freedom (e.g. position and momentum of each particle $\{\vec{q_i}, \vec{p_i}\}$). Instead, a macroscopic description of the system typically involves just few phenomenological parameters, such as pressure, temperature, density, magnetization, etc...

The bridge between these two descriptions of a physical system is *coarse-graining*. This idea, typical of RG approaches, consists in building an effective theory for the macroscopic degrees of freedom by progressively eliminating the microscopic ones. This procedure, together with rescaling, represent the main ingredient of a RG transformation, which generates a flow of effective theories. The properties of fixed points of flow equations allow for a complete description of the critical behaviour of a physical system, and to address the problem of universality. Indeed, only few parameters survive the flow, and determine the critical behaviour of the system, and the latter is a manifestation of the concept of *universality*.

In textbooks, one typically deals with examples from statistical mechanics, such as the Ising model. These examples will be covered in the introductory *Chapters* 1 and 2. However, FRG can be surprisingly useful also in other and simpler applications, such as the ones we are going to treat in *Chapters* 3 - 5.

1.2 Renormalization at work: the 2D Ising model

Consider a set of Ising spins $s_i = \pm 1$ aranged on a two dimensional square lattice. The dimensionless hamiltonian for such a system in an external magnetic field h reads

$$H = -K \sum_{\langle ij \rangle} s_i s_j - h \sum_i s_i, \tag{1.1}$$

where $\sum_{\langle ij \rangle}$ represents a sum over nearest neighbours pairs, and $K > 0$ is the dimensionless coupling of the ferromagnetic interaction among nearest neighbors spins. The first exact analytical solution of this model was proposed by Onsager in 1944 [7] for the case of zero field ($h = 0$). I will approach this problem from the renormalization group perspective, in order to introduce the mathematical apparatus of the RG while studying a concrete example [8, 9, 10].

In the previous section I have described a RG transformation as composed of two fundamental steps: coarse-graining and rescaling. Coarse-graining consists in integrating out some microscopic degrees of freedom in order obtain an effective model for the long range ones. The square lattice is a bipartite lattice, and thus it can be subdivided

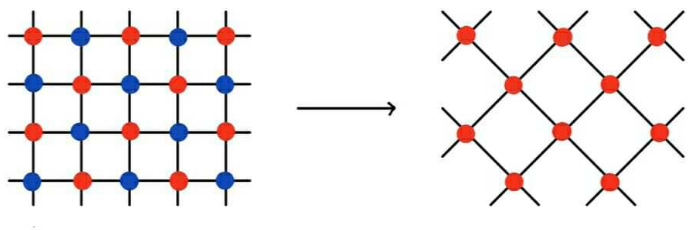


Figure 1.1: Graphical visualization of the coarse-graining on the 2D square lattice. The blue spins of sublattice B are integrated out.

into two sublattices A and B (respectively red and blue spins in *Figure 1.1*). Therefore, a coarse-graining strategy that can be fostered is to integrate out a whole sublattice, say sublattice B. This will leave us with a coarse grained lattice which is still a square lattice, but rotated by $\frac{\pi}{4}$ (right side of *Figure 1.1*). If we denote $S_L = \{s_i\}_{i\in L}$, $L = A, B$, then the coarse-graining procedure can be formally written as

$$Z_N(H) = \sum_{S_A}\sum_{S_B} e^{-H(S_A,S_B)} = \sum_{S_A} e^{-H'(S_A)} = Z_{N'}(H'), \qquad (1.2)$$

where $N' = N/2$ and $H' = -\log\left(\sum_{S_B} e^{-H(S_A,S_B)}\right)$ are respectively the new number of particles, and the coarse-grained effective Hamiltonian. In order to complete the RG transformation, one has to properly rescale the spatial scale in such a way to normalize the new lattice spacing to the original one. In this case, the scaling factor is clearly $b = \sqrt{2}$. Therefore, we can symbolically represent a full renormalization group transformation for the hamiltonian as

$$H' = \mathcal{R}_b(H). \qquad (1.3)$$

Despite of its name, such a transformation does not define mathematically a group, since information is burned out by coarse-graining, meaning that there is no possibility to define an inverse mapping. However, the set of RG transformations $\{\mathcal{R}_b\}$ do form a semi-group, since it possesses an identity transformation, corresponding to $b = 1$, and the composition of two transformations is still a RG transormations:

$$H'' = \mathcal{R}_{b_2}(H') = \mathcal{R}_{b_2}\mathcal{R}_{b_1}(H) = \mathcal{R}_{b_2 b_1}(H), \tag{1.4}$$

i.e. $\mathcal{R}_{b_2}\mathcal{R}_{b_1} = \mathcal{R}_{b_2 b_1}$.

At this point, the idea behind the RG approach is to understand the critical behaviour of the system by studying the asymptotic limit of infinitely many iterations of the renormalization process just from the analysis of a single renormalization step. Therefore, let's perform concretely the transformation (1.2), and compute H'. One interesting property of the bipartite lattice is that spins belonging to the same sublattice do not interact, since if $s_j \in B$, then all of its four neighbours s_i, $i = 1, \dots 4$ are in A. So, we can treat separately each plaquette of the kind shown in *Figure 1.2*. So, in zero field ($h = 0$) we have that

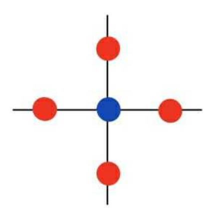


Figure 1.2: The central spin in sublattice B only interacts with four spins of sublattice A.

$$\sum_{s_j = \pm 1} \exp\left(K s_j \sum_{i=1}^{4} s_i\right) = 2\cosh\left(K \sum_{i=1}^{4} s_i\right). \tag{1.5}$$

Now, there is no reason to assume that H' is still of the form (1.1). Indeed, trying with an ansatz of the form $H' = -N' e'_0 - K' \sum_{\langle ik \rangle \in A} s_i s_k$, and imposing the condition

$$2\cosh\left(K \sum_{i=1}^{4} s_i\right) = \exp\left(e'_0 + \frac{K'}{2}(s_1 s_2 + s_2 s_3 + s_3 s_4 + s_4 s_1)\right), \tag{1.6}$$

and considering all possible combinations for $\{s_i\}_{i=1}^4$, we obtain

$$\begin{cases} 2\cosh(4K) = \exp(e_0' + 2K') \\ 2\cosh(2K) = \exp(e_0') \\ 2 = \exp(e_0') \\ 2 = \exp(e_0' - 2K') \end{cases} . \tag{1.7}$$

This system admits solution only for $K' = K = 0$, i.e. in absence of interaction. Moreover, since we have four independent equations, we need for parameters in H' in order to find a solution. Therefore, new kinds of interactions will appear in the hamiltonian after the RG transformation. This phenomenon is called the *proliferation* of interactions, and it reflectes the fact that even if a hamiltonian involves only a finite number of couplings, all correlation functions, involving an arbitrary number of spins, are non-trivial [11]. On a general footing, if $\{\hat{O}_n\}$ is the set of all the operators compatible with the constraints of our system (e.g. symmetries), then the hamiltonian can be written in its most general form as

$$H = \sum_n u_n \hat{O}_n, \tag{1.8}$$

where u_n are the coupling strengths corresponding to each operator $\hat{O}_n$. Therefore, since the set of compatible operators is fixed by the properties of the system, the renormalization group transformation will act on the couplings, and (3) can be rewritten as

$$\vec{u}' = \mathcal{R}_b(\vec{u}). \tag{1.9}$$

This is the *renormalization group flow equation*, and gives the law of change of the couplings in the hamiltonian after a single RG step. Fixed points of this transformation are points $\vec{u}^*$ in parameter space which are invariant, corresponding to scale invariant hamiltonians $H*$. Indeed, if we look at the trasformation of the correlation length under RG trasformations, we see that because of spatial rescaling it is given by

$$\xi(\vec{u}') = \frac{\xi(\vec{u})}{b}.$$

Therefore, at the fixed point we will have

$$\xi(\vec{u}^*) = \frac{\xi(\vec{u}^*)}{b}, \tag{1.10}$$

meaning that at the fixed point we either have $\xi = 0$ or $\xi \to \infty$, the latter case corresponding to a critical fixed point. This expresses the physical fact that at fixed point of the RG transformation there is no characteristic length, corresponding to the manifestation of scale invariance.

In general, the RG transformation $\mathcal{R}_b$ is nonlinear. Since at the end of the day we will be interested in studying the critical behaviour of physical quantities close to a critical point, we can just study the properties of the RG transformation sligthly away from the fixed point, and hence linearize the transformation:

$$\vec{u}' = \vec{u}^* + \delta\vec{u}' = \mathcal{R}_b(\vec{u}^* + \delta\vec{u}) = \vec{u}^* + T_b(\vec{u}^*)\delta\vec{u} + O((\delta\vec{u})^2).$$

Here, T_b is the jacobian matrix of the RG map evaluated at the fixed point. So, in the end we have to study the linearized equation

$$\delta\vec{u}' = T_b(\vec{u}^*)\delta\vec{u}. \tag{1.11}$$

In particular, critical phenomena will be characterized by the eigenvalues and eigenvectors of the linear operator T_b.
Regarding the eigenvalues, one can easily show that, because of the linearity of T_b and of the semigroup property of the RG transformation, they will have the form $\lambda_i = b^{y_i}$. Indeed, for the semigroup property we have $T_{b_1}T_{b_2} = T_{b_1 b_2}$. Then, because of the linearity of the operator T_b, the eigenvalues satisfy the functional equation

$$\lambda_i(b_1)\lambda_i(b_2) = \lambda_i(b_1 b_2),$$

whose solution will be indeed of the power law form $\lambda_i = b^{y_i}$. Since $b \geq 1$, the couplings corresponding to eigenvalues with positive y_i are amplified, while those corresponding to negative y_i are suppressed. Therefore, we distinguish respectively between *relevant* and *irrelevant* couplings. Those couplings with exponent $y_i = 0$ are instead called *marginal*, and are associated to logarithmic corrections to scaling.

Coming back to our study of the $2D$ Ising model, and considering that it is not possible in practice to really take into account the flow of all the possible couplings, we can propose the following ansatz for the form of H' which respects the symmetries of the original model:

$$H' = -Ne'_0 - K' \sum_{\langle ij \rangle} s_i s_j - K'_2 \sum_{\langle\langle ij \rangle\rangle} - K'_4 \sum_{[ijkl]} s_i s_j s_k s_l, \tag{1.12}$$

where the second sum is over second neighbours, while the third one is over square plaquettes. Repeating the procedure mentioned above, we arrive to the system

$$\begin{cases} 2\cosh(4K) = \exp(e'_0 + 2K' + 2K'_2 + K'_4) \\ 2\cosh(2K) = \exp(e'_0 - K'_4) \\ 2 = \exp(e'_0 - 2K'_2 + K'_4) \\ 2 = \exp(e'_0 - 2K' + 2K'_2 + K'_4) \end{cases},$$

whose solution gives the recursion relation for the couplings:

$$\begin{cases} K' = \frac{1}{4} \log\cosh(4K) \\ K'_2 = \frac{1}{8} \log\cosh(4K) \\ K'_4 = \frac{1}{8} \log\cosh(4K) - \frac{1}{2} \log\cosh(2K) \end{cases}. \tag{1.13}$$

At this point, in order to proceed, we need some approximations to deal with the proliferation of the interactions. In particular, when $K \to 0$, we can notice that K' and K'_2 vanishes as K^2, while K'_4 as K^4. Therefore, we could think of neglecting the square-plaquette interaction and set K'_4 to zero. Then, we replace K' and K'_2 with a new nearest-neighbours coupling $\bar{K} = K' + K'_2$, chosen in such a way to preserve the ground state of (1.12). Therefore, our flow equation simply becomes

$$\bar{K} = \frac{3}{8} \log \cosh(4K) = \mathcal{R}_b(K), \tag{1.14}$$

with $b = \sqrt{2}$. Equation (1.14) presents two attractive fixed points for

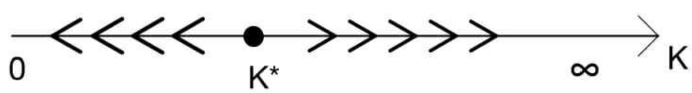


Figure 1.3: Renormalization group flow of nearest-neighbors interaction.

$K = 0$ and $K = \infty$, corresponding respectively to high temperature and low temperature, and a finite temperature repulsive fixed point at $K = K^* \approx 0.50698$, not too far from the exact result, whose numerical value is $K^* \approx 0.44069$.

Linearization of the flow equation about K^* gives that the exponent associated to the nearest-neighbours coupling is $y_K \approx 1.06996$, corresponding to a critical exponent $\nu = y_K^{-1} \approx 0.934614$ for the divergence of the correlation length (see next section for a demonstration of scaling relations). This is actually quite close to the exact value $\nu = 1$.

1.3 Scaling relations

In the previous section we have seen that it is possible to understand the critical behaviour of a system by studying the properties of the RG flow close to its fixed points. In particular, it is possible to relate the exponents of the eigenvalues of the linearized transformation to the critical exponents of the system.

Since physical information about the system can be extracted from the free energy, let's first see how the free energy behaves under the RG flow. Equation (1.2) implies that the partition function is unchanged under RG transformation, and hence also the free energy remains the same. However, since the number of particles changes as $N' = b^{-d}N$ in a d-dimensional lattice, we have that the free energy per particle will be modified. Upon introduction of the scaling parameter $t = \frac{T-T_c}{Tc}$, called the reduced temperature, which is proportional to bare parameter $k = K_c - K$, the scaling law for the free energy density is found to be

$$F(t,h) = F(t',h') = N'f(t',h'),$$

$$\Rightarrow f(t,h) = b^{-d}f(b^{y_t}t, b^{y_h}h).$$

Therefore, after n renormalization steps we will have

$$f(t,h) = b^{-nd}f(b^{ny_t}t, b^{ny_h}h). \tag{1.15}$$

For $t \neq 0$ we can choose n in such a way that $b^{ny_t}t = 1$. Then, the scaling law of the free energy reduces to

$$f(t,h) = t^{\frac{d}{y_t}}f(1, ht^{-\frac{y_h}{y_t}}) = t^{\frac{d}{y_t}}\phi(ht^{-\frac{y_h}{y_t}}), \tag{1.16}$$

where ϕ is a scaling function. Upon differentiating this equation we can establish the critical behaviour of the physical quantities of the system. For example, the specific heat in zero field is obtained by differentiating twice the free energy:

$$C(t,0) \propto \left.\frac{\partial^2 f}{\partial t^2}\right|_{(t,0)} \propto t^{\frac{d}{y_t}-2},$$

giving the critical exponent for the specific heat as

$$\alpha = 2 - \frac{d}{y_t}. \tag{1.17}$$

Similarly, the magnetization is found by differentiation with respect to h:

$$m(t,0) \propto \left.\frac{\partial f}{\partial h}\right|_{(t,0)} \propto t^{\frac{d-y_h}{y_t}},$$

giving

$$\beta = \frac{d - y_h}{y_t}. \tag{1.18}$$

One can play the same game for other quantities such as the magnetic susceptibility, correlation functions, etc..., and the results would be

$$\begin{cases} \alpha = 2 - \frac{d}{y_t} \\ \beta = \frac{d-y_h}{y_t} \\ \gamma = \frac{2y_h - d}{y_t} \\ \delta = \frac{y_h}{d-y_h} \\ \nu = \frac{1}{y_t} \\ \eta = d - 2y_h + 2 \end{cases} . \tag{1.19}$$

These formulas also allow to obtain other two important sets of relations between the critical exponents by elimination of y_t and y_h. The first are called the *thermodynamic scaling relations*:

$$\begin{cases} \alpha + 2\beta + \gamma = 2 \\ \gamma = \beta(\delta - 1) \end{cases} , \tag{1.20}$$

and relates the exponents that characterizes the singularities of the free energy at the critical point. The second are called the *hyperscaling relations*

$$\begin{cases} \alpha = 2 - d\nu \\ \beta = \nu\frac{d-2+\eta}{2} \\ \gamma = \nu(2 - \eta) \\ \delta = \frac{d+2-\eta}{d-2+\eta} \end{cases} , \tag{1.21}$$

connecting the exponents describing the critical behavior of the free energy to ν and η, which characterize the critical behaviour of the correlation function.

Chapter 2

The flow equation for the effective action

In the previous chapter we have shown a real space implementation of the renormalization group, i.e. the decimation of spin on a regular lattice. As we have seen, despite being transparent and effective, the decimation process generates new terms in the hamiltonian, forcing us to perform some uncontrolled approximations in order to deal with calculations. The alternative is to set up the RG transformation in momentum space, where the coarse-graining is performed by integrating out high momentum modes in order to obtain an effective theory for the slow modes. Since high momenta corresponds to short length scales, this amounts to a coarse-graining in real space.

In this chapter I will introduce the so called functional renormalization group (FRG), which is a nonperturbative implementation of momentum space RG which combines the functional methods of quantum field theory with the renormalization group ideas introduced by Wilson. Many formulations of the FRG have been developed, such as the Callan-Symanzik, Wegner-Houghton and Wilson-Polchinski formulations. Here I will present a formulation introduced based on the study of a formally exact flow equation for a scale dependent effective action $\Gamma_k[\varphi]$ [12], which has proven to be successful in devising non-perturbative approximation schemes [11, 13, 14].

2.1 The effective average action approach

Consider a statistical field ϕ whose dynamics is governed by an action $S[\phi]$. Physical information is contained in the n-points correlation functions, tha are generated by the generating functional

$$Z[j] = \int \mathcal{D}[\phi] e^{-S[\phi]+\int d^D x j(x)\phi(x)}, \tag{2.1}$$

where $\int \mathcal{D}[\phi]$ denotes integration over all field configurations. We assume this measure to be properly regularized by an UV-cutoff Λ. Therefore, the n-point correlation functions are obtained by taking the functional derivatives of $Z[j]$ with respect to the external source field $j(x)$:

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \int \mathcal{D}[\phi] \phi(x_1) \dots \phi(x_n) e^{-S[\phi]} = \left. \frac{\delta^n Z[j]}{\delta j(x_1) \dots \delta j(x_n)} \right|_{j=0}.$$

It is also customary to introduce the generating functional of the connected correlation functions, obtained by taking the logarithm of (2.1), called the *Schwinger functional*:

$$W[j] = \log Z[j],$$

and its Legendre transform, called the *effective action*

$$\Gamma[\varphi] + W[j] = \int d^D x j(x) \varphi(x),$$

where $\varphi = \frac{\delta W}{\delta j} = \langle \phi \rangle_j$.

The main idea of Wilson's RG is to compute the partition function by progressively integrating out fast momenta, corresponding to short-distance fluctuations. In the FRG approach, one builds a family of models indexed by a momentum scale k in such a way that fluctuations are taken into account as k is lowered from some reference UV scale to 0, corresponding to the thermodynamic limit. More concretely, the idea is to define a scale dependent effective action, called the *effective average action* Γ_k, which interpolates between the bare action of the model S at the UV scale Λ, and the true effective action Γ, the latter being approached when $k \to 0$.

$$\Gamma_\Lambda[\varphi] = S[\varphi], \quad \Gamma_{k\to 0}[\varphi] = \Gamma[\varphi] \tag{2.2}$$

To construct such interpolating action, we introduce a IR-cutoff term $\Delta S_k[\phi]$ to the bare action, obtaining scale dependent generating functionals:

$$Z_k[j] = \int \mathcal{D}[\phi] e^{-S[\phi]-\Delta S_k[\phi]+\int d^D x j(x)\phi(x)}, \quad W_k[j] = log Z_k[j].$$

The prototypical form of ΔS_k is that of a mass-like quadratic term, i.e.

$$\Delta S_k = \frac{1}{2}\int_x \int_y \phi(x) R_k(x-y)\phi(y) = \int_q \phi(-q) R_k(q) \phi(q), \tag{2.3}$$

where I have denoted $\int_x = \int d^D x$ and $\int_q = \int \frac{d^D q}{(2\pi)^D}$. R_k is called the *regulator function*, and it must be chosen in order to satisfy the requirements (2.2). Its properties are explained in detail in the next section, together with some examples of regulator functions.

The effective average action Γ_k is therefore obtained via modified Legendre transformation of the scale dependent Schwinger functional W_k:

$$\Gamma_k[\varphi] + W_k[j] = \int_x j(x)\varphi(x) - \Delta S_k[\varphi]. \tag{2.4}$$

The flow equation describing the evolution of the effective average action with the scale k is called the *Wetterich equation*, and reads

$$\partial_k \Gamma_k = \frac{1}{2} Tr\left(\frac{\partial_k R_k}{\Gamma_k^{(2)} + R_k}\right), \tag{2.5}$$

where the trace is performed over momenta and over the field index (e.g., for a $O(n)$ model we would have to sum over the n components of the field). A detailed derivation of (2.5) is presented in *Section 2.3.*

2.2 The regulator function

In the previous section we have introduced the IR-cutoff ΔS_k, with a momentum dependent mass R_k. Since it has to work as an IR regulator, which avoids the right hand side of (2.5) to blow up when $q \to 0$, the first property that we should require is that it must be finite and positive for small momenta:

$$\lim_{q \to 0} R_k(q) > 0. \tag{2.6}$$

Then, since when $k \to 0$, i.e. when all fluctuations have been integrated, we want $\Gamma_{k\to 0} = \Gamma$, we must also require that

$$\lim_{k \to 0} R_k(q) = 0 \ \ \forall q, \tag{2.7}$$

in such a way that $Z_k[j] = Z[j]$, and therefore also Γ_k flows towards the full effective action. The last requirement is that at the UV scale Λ we want to have $\Gamma_\Lambda[\varphi] = S[\varphi]$. It can be shown that this is achieved by requiring that R_k diverges at least as Λ^2:

$$\lim_{k \to \Lambda} R_k(q) = O(\Lambda^2) \ \ \forall q. \tag{2.8}$$

In order to understand why this is the case, consider the following. From (2.4) we have that[1]

$$j(x) = \frac{\delta \Gamma_k}{\delta \varphi(x)} + \int_y R_k(x-y)\varphi(y). \tag{2.9}$$

Exponentiating (2.4) we have that

$$e^{-\Gamma_k[\varphi]} = e^{W_k[j] - \int_x j(x)\varphi(x) + \frac{1}{2}\int_{x,y}\varphi(x)R_k(x-y)\varphi(y)}.$$

[1] As one can see by taking functional derivative of both sides of the equation with respect to $\phi(x)$.

So, substituting the definition of W_k in terms of Z_k and using (2.9) for $j(x)$, one obtains

$$e^{-\Gamma_k[\varphi]} = \int \mathcal{D}[\phi] e^{-S[\phi]+\int_x \frac{\delta\Gamma_k}{\delta\varphi(x)}(\phi(x)-\varphi(x))-\frac{1}{2}\int_{x,y}(\phi(x)-\varphi(x))R_k(x-y)(\phi(y)-\varphi(y))}.$$

Therefore, if R_k diverges for all q as $k \to \Lambda$, then the gaussian like exponential behaves has a functional Dirac delta, $\delta(\phi(x) - \varphi(x))$, and we have

$$\Gamma_k[\varphi] = S[\varphi] + const, \tag{2.10}$$

which also represents the initial condition to the flow equation.

Considering the constraints (2.6 - 2.7 - 2.8), a typical choice for the regulator function R_k is of the form

$$R_k(q^2) = q^2 r(q^2/k^2), \tag{2.11}$$

where $r(y)$ is a dimensionless function which determines the shape of the regulator. Some examples are [8]:

- the exponential regulator:

$$r_{exp}(y) = \frac{1}{e^{y^b} - 1};$$

- the power regulator:

$$r_{pow}(y) = y^{-b};$$

- the Litim regulator:

$$r_{opt}(y) = \left(\frac{1}{y} - 1\right)\Theta(1-y).$$

2.3 Derivation of the Wetterich equation

Let's now derive step-by-step the Wetterich equation (2.5). The starting point is (2.4), that we have to differentiate with respect to k. First of all, we should notice that (2.9) tells us that if we consider φ to be independent of k, then j is instead k-dependent. This means that the derivative with respect to k of the Schwinger functional has two contributions:

$$\partial_k W_k[j] = \int_x \frac{\delta W_k}{\delta j(x)} \partial_k j(x) + \partial_{k|j} W_k[j],$$

where in the second term the partial derivative is taken at fixed $j(x)$. Therefore, recalling that $\varphi(x) = \frac{\delta W_k}{\delta j(x)}$, we obtain that

$$\partial_k \Gamma_k = -\partial_{k|j} W_k[j] - \frac{1}{2} \int_{x,y} \varphi(x) \partial_k R_k(x-y) \varphi(y). \tag{2.12}$$

At this point we have to compute $\partial_{k|j} W_k[j]$:

$$\partial_{k|j} W_k[j] = \frac{\partial_{k|j} Z_k[j]}{Z_k[j]} = -\frac{1}{2} \int_{x,y} \partial_k R_k(x-y) \langle \phi(x) \phi(y) \rangle.$$

Therefore, (2.12) becomes

$$\partial_k \Gamma_k = \frac{1}{2} \int_{x,y} \partial_k R_k(x-y) \left[\langle \phi(x)\phi(y) \rangle - \langle \phi(x) \rangle \langle \phi(y) \rangle \right],$$

$$\Rightarrow \partial_k \Gamma_k = \frac{1}{2} \int_{x,y} \partial_k R_k(x-y) \frac{\delta^2 W_k}{\delta j(x) \delta j(y)}. \tag{2.13}$$

The last step we need to perform is to express the second functional derivative of W_k in terms of the effective average action Γ_k. One could notice that

$$\delta(x-z) = \frac{\delta \varphi(x)}{\delta \varphi(z)} = \frac{\delta^2 W_k}{\delta \varphi(z) \delta j(x)} = \int_y \frac{\delta^2 W_k}{\delta j(y) \delta j(x)} \frac{\delta j(y)}{\delta \varphi(z)}.$$

The second factor appearing in the integrand can then be evaluated differentianting (2.9), to obtain

$$\delta(x-z) = \int_y \frac{\delta^2 W_k}{\delta j(y)\delta j(x)} \left[\frac{\delta^2 \Gamma_k}{\delta\varphi(z)\varphi(y)} + R_(y-z) \right].$$

If we now denote the second derivatives as $W_k^{(2)}$ and $\Gamma_k^{(2)}$ respectively, the last expressions means that

$$W_k^{(2)} = \left[\Gamma_k^{(2)} + R_k\right]^{-1}, \tag{2.14}$$

i.e. the second functional derivative of the Schwinger functional is the inverse, in the operator sense, of the second derivative of the effective average action plus the regulator. This should not be surprising, since it resembles a quite common property of the Legendre transform. Inserting (2.14) into (2.13) finally gives the Wetterich equation (2.5).

2.4 The Local Potential Approximation

The Wetterich equation (2.5) is a non-linear functional integro-differential equation, which is rather complicated to solve in general. Therefore, some approximations are required.

Two main approximation procedures have been developed: the vertex expansion, and the derivative expansion. In both cases, the stategy consists in solving the equation in a restricted functional space rather than employing series expansions in small parameters. Therefore, the quality of the results strictly depends on the choice of the functional space on which we are projecting the flow equation [14]. Here I will describe the second strategy, i.e. the derivative expansion. A detailed description of the vertex expansion method can be found in [11]. For simplicity of notation I will consider a scalar field $\phi(x)$, but the concepts can be extended straightforwardly to vector fields.

The underlying idea of the derivative expansion is that since we are mostly interested in the long distance physics, than we can think to keep all the orders in a field expansion of the effective average action Γ_k, and retain only the lowest orders in the derivatives:

$$\Gamma_k[\phi] = \int_x \left[U_k(\varphi) + \frac{Z_k(\varphi)}{2}(\vec{\nabla}\varphi)^2\right] + O(\nabla^4). \tag{2.15}$$

The coefficient Z_k is called the *wavefunction renormalization*, and is needed in order to study the anamalous dimension η of the model. Higher order terms in the gradient are instead needed in order to get accurate results for the critical exponents. In the applications that I will present in this work, i.e. the computation of integrals and the determination of the ground state energy of the Schrodinger equation, the fundamental term of the derivative expansion is the so-called effective potential U_k. Therefore, I will neglect higher order gradients, and the wavefunction renormalization Z_k, which I will set equal to one. Moreover, the flow equations will also be projected onto uniform configurations for φ. This order of the derivative expansion is called the *local potential approximation* (LPA). At this level of approximation the effective average action is thus given by

$$\Gamma_k[\varphi] = \int_x \left(\frac{1}{2}(\vec{\nabla}\varphi)^2 + U_k(x)\right). \tag{2.16}$$

In order to project the Wetterich equation on the functional space that we have just selected, we first have to evaluate the second derivative of Γ_k. This is given by

$$\Gamma_k^{(2)} = (U_k'' - \nabla^2)\delta(x-x') = \int_p \left(U_k'' + p^2\right)e^{ip(x-x')},$$

$$\Rightarrow (\Gamma_k^{(2)} + R_k)(x-x') = \int_p \left\{U_k'' + p^2[1+r(y)]\right\}e^{ip(x-x')}.$$

Inserting this into the Wetterich equation, we find that

$$\partial_k\Gamma_k = \frac{1}{2}\int_x\int_{x'}\int_p\int_{p'} \frac{\partial_k R_k(p)}{U_k'' + p'^2[1+r(y)]}e^{i(p+p')(x-x')} \Rightarrow$$

$$\partial_k \Gamma_k = \frac{1}{2} \int_x \int_p \int_{p'} \frac{\partial_k R_k(p)}{U_k'' + p'^2[1 + r(y)]} \delta(p + p') \Rightarrow$$

$$\partial_k \Gamma_k = \frac{1}{2} \int_x \int_p \frac{\partial_k R_k(p)}{U_k'' + p^2[1 + r(y)]} = \int_x \partial_k U_k,$$

where the last identity comes from comparison with (2.16) evaluated for constant field configurations. Therefore, the flow equation for the effective potential is given by

$$\partial_k U_k = \frac{1}{2} \int_p \frac{\partial_k R_k(p)}{U_k'' + p^2[1 + r(y)]} \tag{2.17}$$

At this point, it is customary to change the integration variable from p to y:

$$\partial_k U_k = \frac{\Omega_d}{(2\pi)^d} \int_0^\infty \frac{\partial_k R_k(p)}{U_k'' + p^2[1 + r(y)]} p^{d-1} dp \Rightarrow$$

$$\Rightarrow \partial_k U_k = \frac{k^{d-1}\Omega_d}{(2\pi)^d} \int_0^\infty \frac{y^{\frac{d-1}{2}} \partial_k[yk^2 r(y)]}{U_k'' + yk^2[1 + r(y)]} \frac{k dy}{2\sqrt{y}}$$

$$\Rightarrow \partial_k U_k = \frac{k^d \Omega_d}{(2\pi)^d} \int_0^\infty \frac{\cancel{-2ykr(y) + 2ykr(y)} - 2ky^2 r'(y)}{U_k'' + yk^2[1 + r(y)]} y^{\frac{d-2}{2}} \frac{dy}{2}$$

$$\Rightarrow \partial_k U_k = -k^{d+1} \mu_d \int_0^\infty \frac{y^{\frac{d+2}{2}} r'(y)}{U_k'' + yk^2[1 + r(y)]} dy, \tag{2.18}$$

where $\mu_d = \Omega_d/(2\pi)^d$ and Ω_d is the d-dimensional solid angle.

2.5 Regulator dependence

To conclude this chapter, I would like to stress out that the Wetterich equation (2.5) is in principle an exact flow equations. This means that if we were able to solve it without resorting to any approximation, then we would obtain the exact result for the effective action Γ at the end of the flow, independently of the regulator we choose. In other words, although the specific flow followed by Γ_k depends intrisically on the choice of R_k, if we solve the Wetterich equation exactly then any dependence on the regulator disappears in the thermodynamic limit $k \to 0$, and all the different flows associated to different regulators converge to the full effective action Γ.

Unfortunately, as explained in *Section 2.4*, approximations are required in order to solve the Wetterich equation. This introduces an unwanted spurious dependence on the regulator function in the results of the flow. Therefore, a very important question arises: how can we choose the regulator function in order to produce the closest-to-reality results within a given approximation? In literature. several attempts to try to answer this question developing some optimization criteria are present.

A first one, called the principle of minimal sensitivity (*pms*) [15], states that we should seek for the values of a physical quantity which *depends less* on the regulator function. To be more explicit, imagine that we want to measure some observable O, and we compute it using a one parameter family of regulators $R_k(\alpha)$. Then, what we measure is a function $O(\alpha)$. Since the physical value O of the observable does not depend on the parameter α, the principle of minimal sensitivity suggests to take as the best estimate the value $O(\alpha^*)$ where the derivative with respect to α vanishes:

$$\alpha^* : \frac{dO}{d\alpha} = 0. \tag{2.19}$$

A different approach was proposed by Litim [16, 17]. In the denominator of (2.18) the quantity $y[1+r(y)]$ appears. This can be recognized to be the dimensionless inverse coarse-grained propagator at scale k and at vanishing field, which we can denote as $P^2 = 1/(k^2\Delta_k(q^2))$. So, equation (2.18) can be written as

$$\partial_k U_k = -k^{d+1}\mu_d \int_0^\infty \frac{F(y)}{U_k'' + k^2 P^2(y)} dy, \tag{2.20}$$

with $F(y) = y^{\frac{d+2}{2}} r'(y)/(1+r(y))$. Now, suppose that we consider one-parameter families of regulators, $r_\alpha(y)$. Then, we can ask ourselves how the convegence properties of the intregral in (2.20) depend on the regulator, and therefore for which value of the parameter α the best regulator is obtained. The idea is to expand the integral in powers of U_k'':

$$\int_0^\infty \frac{F(y)}{U_k''+k^2P^2(y)}dy = \int_0^\infty \frac{F(y)}{k^2P^2(y)}\frac{1}{1+\frac{U_k''}{k^2P^2}}dy =$$

$$= \int_0^\infty \frac{F(y)}{U_k''+k^2P^2(y)}dy = \int_0^\infty \frac{F(y)}{k^2P^2(y)}\sum_{n=0}^{\infty}\binom{-1}{n}\left(\frac{U_k''}{k^2}\right)^n P^{-2n} =$$

$$= \sum_{n=0}^{\infty}\frac{1}{k^2}\binom{-1}{n}\left(\frac{U_k''}{k^2}\right)^n \int_0^\infty F(y)P^{-2(n+1)} =$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{k^{2n+2}}a_n(U_k'')^n.$$

The radius of convergence of a series is defined as

$$C = \lim_{n\to\infty}\frac{a_n}{a_{n+1}}.$$

In our case, when $n \to \infty$, since $P^2(y) > 0\ \forall y \geq 0$ because of the IR regulation operated by $r(y)$, we have that the integral appearing in the coefficients a_n is dominated by the minimum of $P^2(y)$. Therefore, we find the radius of convergence to be

$$C_\alpha = k^2 \min_{y\geq 0}\{P^2(y)\}. \tag{2.21}$$

The subscript α signals the dependence on the parameter of the regulator. Therefore, the optimal value for α is the one that maximizes (2.21), and the optimal radius of convergence is

$$C_{optimal} = k^2 \max_{\alpha}\{\min_{y\geq 0}\{P^2(y)\}\}. \tag{2.22}$$

An important remark is that this procedure is model independent, since it depends only on the inverse propagator at zero field. As an example, let's apply the Litim's optimization criterium to the power law regulator

$$r_{pow}(y) = y^{-b}, \quad b > 1.$$

We have that the inverse propagator is given by

$$P^2(y) = y + y^{-b+1},$$

$$\Rightarrow \frac{dP^2}{dy} = 1 + (-b+1)y^{-b},$$

$$\Rightarrow \bar{C_b}^2 = \min_{y\geq 0}\{P^2(y)\} = b(b-1)^{\frac{1}{b}-1}.$$

This has to be maximized w.r.t. b.

$$\frac{d\bar{C}_b}{db} - (b-1)^{\frac{1}{b}-1}\log(b-1) \geq 0 \iff b \leq 2.$$

So, we find $\bar{C}_{opt} = 2$, and $b_{opt} = 2$.

[2] $\bar{C}_b = C_b/k^2$ is the adimensional radius of convergence.

Chapter 3

Integrals of one real variable

Despite being often presented in literature within the context of statistical mechanics (and as we did in *Chapter 1* and *2*), the renormalization group techniques can be employed to study other problems, such as turbolence [10] or deep learning [18]. The simplest, but still very interesting and non-trivial problem that can be treated using the RG tools is the calculation of integrals. Rather surprisingly, pedagogical presentations of this kind of application are quite rare in the literature, where one may refer to [19] for the study of the one variable gaussian integral.

In the following, the FRG tools introduced in *Chapter 2* will be specialized to the problem of the calculation of one variable integrals of the kind

$$Z = \int_{-\infty}^{+\infty} f(x)dx, \tag{3.1}$$

where $f : \mathbb{R} \to \mathbb{R}_0^+$ is a non-negative function defined over the real axis. As an original result, I will present the study of non-trivial quartic integrals, showing that the LPA is exact for this problem and explaining how one can obtain very good estimates, which agree with exact numerical results, by studying the convergence of the LPA truncation.

3.1 The flow equation

Since we have chosen $f(x) \geq 0$, then we can write the integral (3.1) as

$$Z = \int_{-\infty}^{+\infty} e^{-S(x)} dx, \tag{3.2}$$

where we have defined $S(x) = -\log f(x)$. With this notation, the analogy with statistical mechanics is straightforward: we can think of Z as the partition function for the distribution of a real random variable x[1] with Boltzmann wheight $e^{-S(x)}$. Therefore, if we add a source variable j, we can define the moments and cumulants generating functions, respectively

$$Z(j) = \int_{-\infty}^{+\infty} e^{-S(x)+jx} dx, \tag{3.3}$$

and

$$W(j) = \log \big[Z(j)\big]. \tag{3.4}$$

Again, the idea is to work with the Legendre transform of the cumulant generating function, called the effective action $\Gamma(\varphi)$:

$$\Gamma(\varphi) = j\varphi - W(j), \tag{3.5}$$

where $\varphi = \frac{dW}{dj} \equiv \langle x \rangle$.

In order to implement the Wilson idea of successive fluctuation modes elimination, we introduce a quadratic cut-off which depends on some scale parameter k:

$$\Delta S_k = \frac{1}{2} R_k x^2, \tag{3.6}$$

where $R_k \in \mathbb{R}^+$ is called the regulator function. Accordingly, we define a scale dependent moment generating function as

$$Z_k(j) = \int_{-\infty}^{+\infty} e^{-S(x)+jx-\frac{1}{2}R_k x^2} dx. \tag{3.7}$$

[1] To be on the same line with *Chapter 2*, a real random variable can also be considered as a $(0+0)$ - dimensional random field.

In particular, R_k is chosen in such a way that:

- when $k \gg 1$, then also $R_k \gg 1$, so that only a narrow window of the real axis contributes to $Z_k(j)$, since the contributions from the region outside the interval $[-R_k^{-1}, R_k^{-1}]$ are exponentially suppressed;

- when $k \to 0$ then also $R_k \to 0$, in such a way that we integrate all the contributions from the real axis, and we can recover the original integral.

Therefore, the idea is now to define a running effective action $\Gamma_k(\varphi)$, that again we call the effective average action, and to follow its evolution with the scale parameter k, until it eventually flows to the true effective action of the model Γ for $k \to 0$. This effective average action is defined as the modified Legendre transform of $W_k(j) = \log Z_k(j)$:

$$W_k(j) + \Gamma_k(\varphi) = j\varphi - \Delta S_k(\varphi). \tag{3.8}$$

In the same spirit of *Section 2.4* we are now in position to derive an evolution equation for Γ_k. Therefore, we need to differentiate (3.8) with respect to k. One point to note here is that if φ is assumed to be k-independent, then j is scale dependent. Indeed, differentiating (3.8) with respect to φ we see that

$$\frac{\partial}{\partial\varphi}\Gamma_k(\varphi) = j - R_k\varphi \tag{3.9}$$

and hence j depends on k. Therefore, the derivative of (3.8) with respect to k will be given by

$$\partial_k\Gamma_k = -\partial_j W_k \partial_k j + \varphi\partial_k j - \partial_{k|j} W_k - \frac{\partial_k R_k}{2}\varphi^2,$$

where $\partial_{k|j}$ denotes that the derivative is taken at fixed j. Thus, since

$$\partial_j W_k = \varphi = \langle x \rangle,$$

and

$$\partial_{k|j} W_k = \frac{\partial_{k|j} Z_k}{Z_k} = -\frac{\partial_k R_k}{2} \langle x^2 \rangle,$$

we conclude that

$$\partial_k \Gamma_k = \frac{\partial_k R_k}{2} \big(\langle x^2 \rangle - \langle x \rangle^2 \big) = \frac{\partial_k R_k}{2} \partial_j^2 W_k. \tag{3.10}$$

The last step is to express $\partial_j^2 W_k$ in terms of Γ_k. From (3.9) we have that

$$j = \partial_\varphi \Gamma_k + R_k \varphi.$$

Thus, differentiating both sides with respect to j we have that

$$1 = \partial_j \partial_\varphi \Gamma_k + R_k \partial_j \varphi = \big(\partial_\varphi^2 \Gamma_k + R_k \big) \partial_j \varphi = \big(\partial_\varphi^2 \Gamma_k + R_k \big) \partial_j^2 W_k,$$

and hence

$$\partial_j^2 W_k = \big(\partial_\varphi^2 \Gamma_k + R_k \big)^{-1}. \tag{3.11}$$

Inserting (3.11) in (3.10) we finally get

$$\partial_k \Gamma_k(\varphi) = \frac{1}{2} \partial_k R_k \bigg(\partial_\varphi^2 \Gamma_k + R_k \bigg)^{-1}, \tag{3.12}$$

that is the equivalent of the FRG equation (2.5) for the case of a one variable integral. We emphasize that we are then transforming the calculation of an integral intoto the solution of a partial differential equation, which later will be transformed into a set of ordinary differential equations.

The initial condition for equation (3.12) is given by examining the behaviour of Γ_k for large k:

$$e^{-\Gamma_k(\varphi)} = e^{W_k(j) - j\varphi + \Delta S_k(\varphi)} =$$

$$= \int_{-\infty}^{+\infty} e^{-S(x) + j(x-\varphi) + \Delta S_k(\varphi) - \Delta S_k(x)} dx$$

$$= \int_{-\infty}^{+\infty} e^{-S(x) + \Gamma_k'(\varphi)(x-\varphi) + R_k \varphi (x-\varphi) + \Delta S_k(\varphi) - \Delta S_k(x)} dx$$

$$= \int_{-\infty}^{+\infty} e^{-S(x)+j(x-\varphi)-\frac{1}{2}R_k(x-\varphi)^2} dx.$$

Therefore, since we have assumed that for $k \to \Lambda \gg 1 \Rightarrow R_k \to R_\Lambda \gg 1$, and recalling that

$$\delta(x-\varphi) = \lim_{\alpha\to\infty} \frac{\alpha}{\sqrt{2\pi}} e^{-\frac{1}{2}\alpha^2(x-\varphi)^2},$$

we get that

$$e^{-\Gamma_\Lambda(\varphi)} \approx \sqrt{\frac{2\pi}{R_\Lambda}} e^{-S(\varphi)}, \quad \Lambda \gg 1,$$

$$\Rightarrow \Gamma_\Lambda(\varphi) = S(\varphi) + \frac{1}{2}\log\left(\frac{R_\Lambda}{2\pi}\right). \tag{3.13}$$

Finally, once (3.12) is solved with the initial condition (3.13), the value of Z is obtained by taking $k = 0$ and $j = 0$, the latter corresponding to $\varphi = \varphi^* : \frac{d\Gamma}{d\varphi}(\varphi^*) = 0$ because of the Legendre transform definition. So, we will have

$$Z = e^{-\Gamma(\varphi^*)}. \tag{3.14}$$

At the end of the day, we have thus replaced the direct calculation of an integral (3.1) with the solution of a partial differential equation (3.12) for a properly defined function Γ_k.

In the following section the strategy for the solution of the Wetterich equation is explained with some pratical examples.

3.2 Gaussian integrals

Equation (3.12) is in general complicated to solve, and appropriate approximation schemes must be devised. A major case that can be treated analytically without any approximation is that of a gaussian integral, i.e. we choose

$$S(x) = \frac{1}{2\sigma^2}x^2. \tag{3.15}$$

In this case, it is indeed sufficient to consider a gaussian ansatz for the effective average action, with the same variance as in (3.15):

$$\Gamma_k(\varphi) = a_{0,k} + \frac{1}{2\sigma^2}\varphi^2. \tag{3.16}$$

Therefore, the Wetterich equation has to be projected onto the scale dependent parameter $a_{0,k}$, giving

$$\partial_k a_{0,k} = \frac{1}{2}\frac{\partial_k R_k}{\sigma^{-2} + R_k}, \tag{3.17}$$

with the initial condition $a_{0,\Lambda} = \frac{1}{2}\log\frac{R_\Lambda}{2\pi}$ fixed by (3.13). So, we only have to integrate (3.17):

$$\begin{aligned}
\partial_k a_{0,k} &= \frac{1}{2}\frac{\partial_k R_k}{\sigma^{-2} + R_k}, \\
\Rightarrow \frac{1}{2}\log\left(\frac{R_\Lambda}{2\pi}\right) - a_{0,0} &= \frac{1}{2}\int_0^\Lambda \frac{\partial_k R_k}{\sigma^{-2} + R_k}, \\
\Rightarrow a_{0,0} = \frac{1}{2}\log\left(\frac{R_\Lambda}{2\pi}\right) - \frac{1}{2}\log\left(\frac{\sigma^{-2} + R_\Lambda}{\sigma^{-2}}\right) &= \frac{1}{2}\log\left[\frac{\sigma^{-2}R_\Lambda}{2\pi(\sigma^{-2} + R_\Lambda)}\right].
\end{aligned}$$

Therefore, sending $\Lambda \to \infty$ we get $a_{0,0} = \frac{1}{2}\log\left(\frac{1}{2\pi\sigma^2}\right)$, and hence the integral is given by

$$Z = e^{-a_{0,0}} = \sqrt{2\pi\sigma^2}, \tag{3.18}$$

which corresponds to the very well known exact result!

3.3 Quartic integrals

Let's now consider the case of a quartic action

$$S(x) = \frac{b}{2}x^2 + \frac{c}{4!}x^4,$$

where c is a positive parameter, while b can be positive, negative or either zero. In this case there is no simple ansatz which allows for an exact solution of the flow equation (3.12). The main strategy is to find an approximated solution to the flow equation by expanding in Taylor series the effective average action ${\Gamma_k}^2$

$$\Gamma_k(\varphi) = \sum_{i=0}^{n} \frac{a_{2n,k}}{(2n)!}\varphi^{2n}, \tag{3.19}$$

and projecting (3.12) onto the scale-dependent coefficients $a_{2n,k}$. This procedure is equivalent to the LPA described in *Section 2.4*, but in the case of the calculation of integrals this would become exact if we were able to send $n \to \infty$ and follow the running of the infinitely many couplings generated along the flow, since there are no further gradient terms to include in a derivative expansion. For example, if we choose a 4^{th} - order truncation for the effective average action

$$\Gamma_k(\varphi) = a_{0,k} + \frac{a_{2,k}}{2}\varphi^2 + \frac{a_{4,k}}{4!}\varphi^4, \tag{3.20}$$

then the flow equations projected onto the expansion coefficients are

$$\partial_k a_{0,k} = \frac{1}{2}\frac{\partial_k R_k}{a_{2,k} + R_k}, \tag{3.21}$$

$$\partial_k a_{2,k} = -\frac{1}{2}\frac{\partial_k R_k}{(a_{2,k} + R_k)^2}a_{4,k}, \tag{3.22}$$

$$\partial_k a_{4,k} = 3\frac{\partial_k R_k}{(a_{2,k} + R_k)^3}a_{4,k}^2, \tag{3.23}$$

with the initial conditions $a_{0,\Lambda} = \frac{1}{2}\log\frac{R_\Lambda}{2\pi}$, $a_{2,\Lambda} = b$, $a_{4,\Lambda} = c$ fixed by (3.13).

3.3.1 Analytical results

Working within the LPA we have thus replaced the solution of (3.12) with the solution of a system of coupled non-linear differential equations for the expansion coefficients, which in general has to be solved

[2] Since the flow equation preserves the symmetries of our problem, we can keep only even powers of φ since we started with an even action S.

numerically. However, there is still some interesting property about the flow that we can study analytically.

For the moment, let's keep the regulator general, and consider the case of a finite but small 4^{th} - order coupling $c \ll 1$. In order to perform some analytical calculation, the simplest non trivial approximation that we can think of is a 4^{th} - order truncation of the Taylor expansion of the effective average action, in which we stop the running of the 4^{th} - order coupling by setting $a_{4,k} = c \ \forall k$. Then we can solve the resulting equations for $a_{0,k}$ and $a_{2,k}$

$$\begin{cases} \partial_k a_{0,k} = \frac{1}{2}\frac{\partial_k R_k}{a_{2,k}+R_k} \\ \partial_k a_{2,k} = -\frac{1}{2}\frac{\partial_k R_k}{(a_{2,k}+R_k)^2} c \end{cases} , \tag{3.24}$$

perturbatively in c. For starting, we substitute $a_{2,k} = b + a^{(1)}_{2,k} c + a^{(2)}_{2,k} c^2$ into the second equation and solve it perturbatively for the $a^{(i)}_{2,k}$. So, we find[3]

$$\partial_k a^{(1)}_{2,k} = -\frac{1}{2}\frac{\partial_k R_k}{(b+R_k)^2},$$

$$\Rightarrow a^{(1)}_{2,k} = \frac{1}{2}\frac{1}{b+R_k}, \tag{3.25}$$

while

$$\partial_k a^{(2)}_{2,k} = \frac{\partial_k R_k}{(b+R_k)^3} a^{(1)}_{2,k},$$

$$\Rightarrow a^{(2)}_{2,k} = -\frac{1}{6}\frac{1}{(b+R_k)^3}. \tag{3.26}$$

Now we shall expand perturbatively the first equation of (3.24), and using the results (3.25) and (3.26) we get[4]

$$\partial_k a^{(1)}_{0,k} = -\frac{1}{2}\frac{\partial_k R_k}{(b+R_k)^2} a^{(1)}_{2,k},$$

[3] The initial conditions are $a^{(i)}_{2,k} = 0$.

[4] To get the final result, the “UV" limit $\Lambda \to \infty$ has to be taken.

$$\Rightarrow a_{0,k}^{(1)} = \frac{1}{8b^2}, \tag{3.27}$$

while

$$\partial_k a_{0,k}^{(2)} = \frac{1}{2}\frac{\partial_k R_k}{b+R_k}\left[\frac{(a_{2,k}^{(1)})^2}{(b+R_k)^2} - \frac{a_{2,k}^{(2)}}{b+R_k}\right],$$

$$\Rightarrow a_{2,k}^{(2)} = -\frac{5}{96b^4}. \tag{3.28}$$

Since the zeroth order result for $a_{0,0}$ is nothing but the gaussian one, we find the following second order perturbative expansion in c:

$$a_{0,0} = \frac{1}{2}\log\left(\frac{b}{2\pi}\right) + \frac{c}{8b^2} - \frac{5c^2}{96b^4} + o(c^3). \tag{3.29}$$

Amazingly, we discover that the perturbative expansion does not depend on the choice of the regulator for one variable integrals.

At this point, the RG estimate of the integral is thus given to second order by

$$Z_{RG} = e^{-a_{0,0}} \approx \sqrt{\frac{2\pi}{b}}\cdot\left[1 - \frac{c}{8b^2} + \frac{23c^2}{384b^4} + o(c^3)\right]. \tag{3.30}$$

One can show (see *Appendix A*) that the quartic integral that we are trying to compute can also be expressed in closed form in terms of a modified Bessel function of second kind:

$$Z = \int_{-\infty}^{\infty} e^{-\frac{b}{2}x^2 - \frac{c}{4!}x^4}dx = \sqrt{\frac{3b}{c}}e^{\frac{3b^2}{4c}}K_{\frac{1}{4}}\left(\frac{3b^2}{4c}\right). \tag{3.31}$$

Its expansion for small c is given by

$$Z \approx \sqrt{\frac{2\pi}{b}}\cdot\left(1 - \frac{c}{8b^2} + \frac{35c^2}{384b^4} + o(c^3)\right) \tag{3.32}$$

Therefore, comparing (3.30) and (3.32) we see that the first order perturbative result obtained via the RG flow equation retrieves the exact one, while at second order the error on the expansion coefficient is of

34%. However, one can see that this error can be improved by including the running of $a_{4,k}$ and of the higher order couplings.

We are now ready to study the numerical solution of the flow equation.

3.3.2 Numerical analysis

The numerical solution of the LPA flow equations introduces two unwanted dependencies in the final result. One is on the cut-off scale Λ, which must take a finite value in order to perform the numerical integration, and one on the order of the truncation that we choose for Γ_k. However, these two dependencies can be easily removed following the procedure described below. All of the following numerical results are obtained using the regulator $R_k = k^2$, also called the Callan-Symanzik regulator [8].

In order to remove the Λ-dependence, we can fix the order of the polynomial truncation of the effective average action and study how the result of the numerical integration varies with Λ. In particular, we use a fitting procedure based on the relation

$$I = I_0 + \beta\left(\frac{1}{\Lambda}\right)^{\alpha}, \tag{3.33}$$

where I is the result of the numerical integration and I_0 is the value reached in the limit $\Lambda \to \infty$. The results obtained using a 14^{th} - order truncation, for $b = c = 1$ is shown in *Figure 3.1.* A fit with equation (3.33) gives $I_0 = 2.30338808(2)$.

This value I_0 is still a function of the truncation order n, where we again use the functional form

$$I_0 = Z + \beta\left(\frac{1}{n}\right)^{\alpha} \tag{3.34}$$

where now Z, i.e. the value obtained for $n \to \infty$, is the value of the integral that we want to compute, without the unwanted dependences.

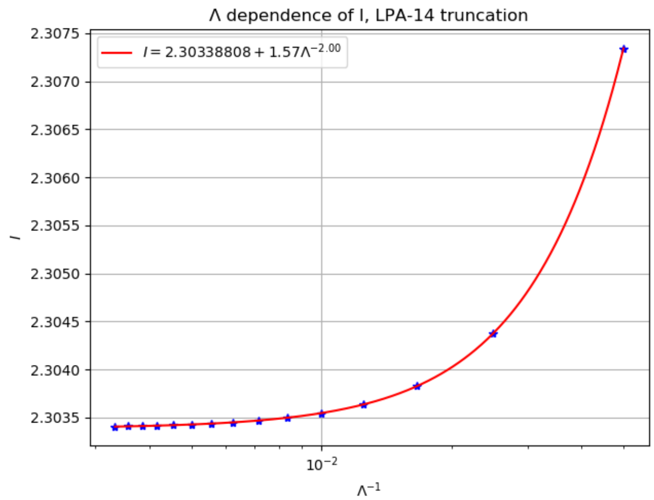


Figure 3.1: Λ dependence of the integral.

In order to eliminate also the dependence on the order of truncation, we should thus repeat the above procedure to find I_0 for different values of n. Then, a fit with (3.34) will give us the value for Z. A plot is shown in *Figure 3.2*, and the result $Z = 2.3033874(9)$ agrees at the fifth digit with the numerical results obtained using *Mathematica*, that is 2.3033881.

Once we have understood how to optimize our estimates removing all the unwanted dependences, it is still interesting to see how the quality of the estimate obtained via numerical integration of the flow equation varies with the strength of the 4^{th} - order coupling c for fixed $b = 1$. After elimination of the $\Lambda-$dependence, the results for a $4^{th}-$order truncation are shown in *Figure 3.3*. We can observe that already within a 4^{th} - order truncation the results are quite good in the small coupling regime, but they worsen with increasing c. Therefore, in order to get better estimates in the strong coupling regime we need to use higher order truncations. The results of a 20^{th} - order truncation

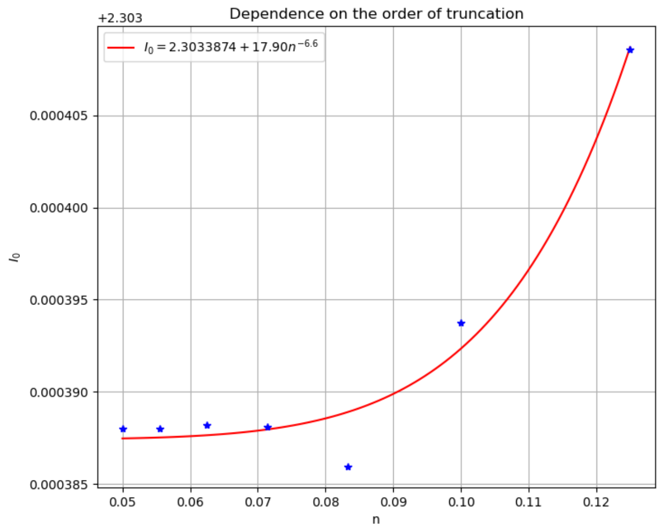


Figure 3.2: Truncation dependence of the integral

are shown in *Figure 3.4*, and confirm that the estimate significantly improves also in the strong coupling limit when the order of truncation is raised.

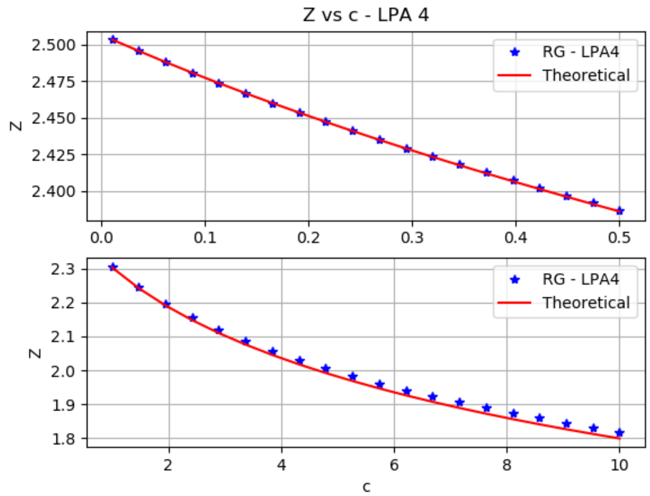


Figure 3.3: LPA-4 numerical results for different values of the coupling c.

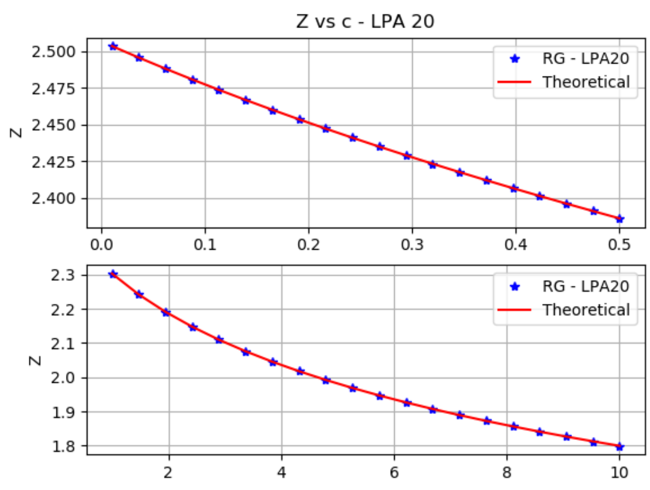


Figure 3.4: LPA-20 numerical results for different values of the coupling c.

Chapter 4

Multivariable integrals

In this chapter the concepts introduced for one variable integrals will be generalized to $N-$variable ones. Therefore, the integral that we want to compute takes now the general form

$$Z = \int d^N x \, e^{-S(\vec{x})}, \tag{4.1}$$

where $\vec{x} = (x_1, \ldots, x_N)^T \in \mathbb{R}^N$ is a $N-$dimensional vector, and the measure is given by $\int d^N x = \prod_{i=1}^{N} \int_{-\infty}^{\infty} dx_i$.

4.1 The flow equation

As done for the one variable case, we can think of Z as the partition function for the joint distribution of a set of real random variables $\vec{X} = \{X_i\}_{i=1}^{N}$ with Boltzmann weight $e^{-S(\vec{x})}$. Accordingly, we can define the moments and cumulants generating functions, respectively as

$$Z(\vec{j}) = \int d^N x \, e^{-S(\vec{x}) + \vec{j} \cdot \vec{x}},$$

and

$$W(\vec{j}) = \log \left[Z(\vec{j}) \right],$$

while the effective action $\Gamma(\vec{\varphi})$ is defined as usual via Legendre transform

$$\Gamma(\vec{\varphi}) = \vec{j} \cdot \vec{\varphi} - W(\vec{j}),$$

where $\varphi_i = \partial_{j_i} W \equiv \langle x_i \rangle$.

The renormalization group approach is again implemented upon introduction of a scale dependent cut-off function in the exponent of the integrand, with the prototypical quadratic form

$$\Delta S_k(\vec{x}) = \frac{1}{2}\vec{x} \cdot R_k \vec{x}.$$

Therefore, R_k is now a positive semi-definite matrix-valued function of the scale k, whose entries satisfies analogous constraints as for the one variable case in the limits $k \to 0$ and $k \to \infty$:

$$\lim_{k \to 0} R_k = 0,$$

and

$$\lim_{k \to \Lambda \gg 1} (R_k)_{ij} = (R_\Lambda)_{ij} \gg 1.$$

The scale dependent partition function is thus given by

$$Z_k(\vec{j}) = \int d^N x \, e^{-S(\vec{x}) + \vec{j} \cdot \vec{x} - \Delta S_k(\vec{x})}.$$

Therefore, if we suppose $R_k = k^2 \mathbb{I}$, where $\mathbb{I}$ denotes the $N \times N$ identity matrix, the role of the cut-off term becomes very clear: it suppresses exponentially the contribution to Z_k outside of the $N-$dimensional box of side $\frac{1}{k}$ centered at the origin.

The running effective action at scale k is given by the modified Legendre transform:

$$W_k(\vec{j}) + \Gamma_k(\vec{\varphi}) = \vec{j} \cdot \vec{\varphi} - \Delta S_k(\vec{\varphi}). \tag{4.2}$$

The derivation of the flow equation follows the same scheme used for the $N = 1$ case, with some more attention to pay to the fact that we are now dealing with vectors and matrices. We start by differentianting (4.2) with respect to k, which gives

$$\partial_k \Gamma_k = -\sum_{i=1}^{N} \left(\partial_k j_i \partial_{j_i} W_k - \varphi_i \partial_k j_i \right) - \partial_{k|\vec{j}} W_k - \frac{1}{2} \vec{\varphi} \cdot \partial_k R_k \vec{\varphi}, \quad (4.3)$$

where the chain rule has been used because to scale independent $\vec{\varphi}$ corresponds a scale dependent $\vec{j}$, with

$$j_i = \partial_{\varphi_i} \Gamma_k + \sum_{l=1}^{N} (R_k)_{il} \varphi_l \quad (4.4)$$

Since $\varphi_i = \partial_{j_i} W_k$ the first term in (4.3) vanishes, and the equation becomes

$$\partial_k \Gamma_k = \frac{1}{2} \sum_{i,l=1}^{N} (\partial_k R_k)_{il} \big(\langle x_i x_l \rangle - \langle x_i \rangle \langle x_l \rangle \big) = \frac{1}{2} \sum_{i,l=1}^{N} (\partial_k R_k)_{il} (W_k^{(2)})_{li},$$

$$\Rightarrow \partial_k \Gamma_k = \frac{1}{2} \operatorname{Tr} \left(\partial_k R_k W_k^{(2)} \right). \quad (4.5)$$

Here, $W_k^{(2)}$ is the hessian matrix of W_k, and must be written in terms of Γ_k in order to complete the derivation. Differentiating (4.4) with respect to j_m we get, upon using the chain rule on the right hand side

$$\delta_{im} = \sum_{l=1}^{N} \left[\partial_{\varphi_i} \partial_{\varphi_l} \Gamma_k - (R_k)_{il} \right] (W_k^{(2)})_{lm},$$

$$\Rightarrow W_k^{(2)} = \big(\Gamma_k^{(2)} + R_k \big)^{-1}, \quad (4.6)$$

where $\Gamma_k^{(2)}(\vec{\varphi})$ is the Hessian matrix of the running action. Therefore, the Wetterich equation for multivariable integrals finally reads

$$\partial_k \Gamma_k = \frac{1}{2} \operatorname{Tr} \frac{\partial_k R_k}{\Gamma_k^{(2)} + R_k}. \quad (4.7)$$

In order to establish the initial condition to (4.7) one can proceed as in the $N = 1$ case. After a few analogous steps, we reach the expression

$$e^{-\Gamma_k(\vec{\varphi})} = \int d^N x \, e^{-S(\vec{x}) + \vec{\nabla}\Gamma_k \cdot (\vec{x} - \vec{\varphi}) - \frac{1}{2}\left(\vec{x}\cdot R_k \vec{x} - 2\vec{x}\cdot R_k\vec{\varphi} + \vec{\varphi}\cdot R_k\vec{\varphi}\right)}$$

The latter term in the exponent can be rewritten in the following way:

$$\vec{x} \cdot R_k \vec{x} - 2\vec{x} \cdot R_k \vec{\varphi} + \vec{\varphi} \cdot R_k \vec{\varphi} = \sum_{i,j=1}^{N} (R_k)_{ij} \big(x_i x_j - 2x_i\varphi_j + \varphi_i\varphi_j\big)$$

$$= \sum_{i,j=1}^{N} (R_k)_{ij} \big[x_i(x_j - \varphi_j) - (x_i - \varphi_i)\varphi_j\big]$$

$$= \sum_{i,j=1}^{N} (x_i - \varphi_i)(R_k)_{ij}(x_j - \varphi_j) = (\vec{x} - \vec{\varphi}) \cdot R_k(\vec{x} - \vec{\varphi}),$$

where I have assumed R_k to be symmetric[1]. So, we get a multivariate gaussian term, in analogy with the $N = 1$ case. Therefore, for $k \to \Lambda \gg 1 \;\Rightarrow\; (R_k)_{ij} \to (R_\Lambda)_{ij} \gg 1$, then we get

$$e^{-\Gamma_\Lambda(\vec{\varphi})} \approx \left(\sqrt{(2\pi)^N \det R_\Lambda^{-1}}\right) e^{-S(\vec{\varphi})}, \quad \Lambda \gg 1,$$

and hence

$$\Rightarrow \Gamma_\Lambda(\vec{\varphi}) = S(\vec{\varphi}) + \frac{1}{2}\log\left(\frac{\det R_\Lambda}{(2\pi)^N}\right). \tag{4.8}$$

The value of Z is obtained by taking $k = 0$ and $\vec{j} = 0$, the latter corresponding to $\vec{\varphi} = \vec{\varphi}^* \,:\, \vec{\nabla}\Gamma(\varphi^*) = 0$. So, we will have

$$Z = e^{-\Gamma(\vec{\varphi}*)}. \tag{4.9}$$

[1] This is always possible since we have no constraints on the actual shape of the regulator matrix.

4.2 Multivariate gaussian integrals

As for the one variable case, also for the multivariate gaussian integral

$$Z = \int d^N x \, e^{-\frac{1}{2}\vec{x}\cdot\Sigma^{-1}\vec{x}}$$

the Wetterich equation (4.7) can be solved analytically. It is again sufficient to consider the following gaussian ansatz for the effective average action:

$$\Gamma_k(\varphi) = a_{0,k} + \frac{1}{2}\vec{\varphi}\cdot\Sigma^{-1}\vec{\varphi},$$

and the flow equation will be projected onto the coefficient $a_{0,k}$, with the initial condition fixed by (4.8). The flow equation thus becomes

$$\partial_k a_{0,k} = \frac{1}{2}\,\mathrm{Tr}\,\frac{\partial_k R_k}{\Sigma^{-1} + R_k}. \tag{4.10}$$

Integration of (4.10) yields

$$\frac{1}{2}\log\frac{\det R_\Lambda}{(2\pi)^N} - a_{0,0} = \frac{1}{2}\,\mathrm{Tr}\left[\log\left(R_k + \Sigma^{-1}\right) - \log\Sigma^{-1}\right].$$

Since $R_k + \Sigma^{-1}$ and Σ^{-1} are positive definite matrices, then we can exploit the property

$$\mathrm{Tr}\log A = \log\det(A).$$

Therefore, we get the following expression for $a_{0,0}$:

$$a_{0,0} = \frac{1}{2}\log\det\left(\frac{\Sigma^{-1}R_\Lambda}{2\pi(R_\Lambda + \Sigma^{-1})}\right),$$

which in the limit $\Lambda \to \infty$ becomes

$$a_{0,0} = -\frac{1}{2}\log\det(2\pi\Sigma). \tag{4.11}$$

Finally, the result for the integral is given by

$$Z = e^{-a_{0,0}} = \sqrt{\det(2\pi\Sigma)}, \tag{4.12}$$

which recovers again the well known exact result!

4.3 Multivariate quartic integrals

As a first non trivial example, let's focus on the extension of the quartic integral proposed in *Section 3.3*:

$$Z = \int d^N x \, e^{-\frac{1}{2}\vec{x}\cdot B\vec{x} - \frac{1}{4!}\sum_{i=1}^{N} c_i x_i^4}.$$

As for the one variable case, I will first consider analytically the case of small quartic couplings using a perturbative approach, and then I will evaluate the numerical results of the integration of the flow equation.

4.3.1 Perturbative approach

Let's consider the case where the couplings $\{c_i\}_{i=1}^N$ are small, in such a way that they can be treated as small perturbations to a gaussian model.

As for the $N = 1$ case, the simplest approximation that we could perform is to consider a polynomial truncation for the running effective action where the 4^{th} - order couplings do not flow, and the lower order ones are expanded perturbatively with respect to them. Hence, we consider a truncation of the form

$$\Gamma_k(\vec{\varphi}) = a_{0,k} + \frac{1}{2}\vec{\varphi}\cdot a_{2,k}\vec{\varphi} + \frac{1}{4!}\sum_{i=1}^{N} c_i\varphi_i^4,$$

with the initial condition

$$\Gamma_\Lambda(\varphi) = \frac{1}{2}\log\left(\frac{\det R_\Lambda}{(2\pi)^N}\right) + \frac{1}{2}\vec{\varphi}\cdot B\vec{\varphi} + \frac{1}{4!}\sum_{i=1}^{N} c_i\varphi_i^4.$$

As for the regulator, I will assume without loss of generality that it has a diagonal form, i.e. $R_k = r_k\mathbb{I}$. With this choice for the regulator, the flow equation for the effective action becomes

$$\partial_k\Gamma_k(\varphi) = \frac{\partial_k r_k}{2}\,\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + r_k\mathbb{I}\right)^{-1}\right]. \tag{4.13}$$

So, the first thing to do is to evaluate $\Gamma_k^{(2)}$ and then compute the trace. In particular, it is easy to understand that the Hessian will take the form

$$\Gamma_k^{(2)} = a_{2,k} + \Delta_{\vec{c}}(\vec{\varphi}), \tag{4.14}$$

where the latter term arises when $\vec{c} \neq \vec{0}$ from the derivatives of the quartic terms. It has a diagonal form, given by

$$\Delta_{\vec{c}}(\vec{\varphi}) = \frac{1}{2}\,\mathrm{diag}(c_1\varphi_1^2,\ \ldots\ , c_N\varphi_N^2).$$

The inverse of $\Gamma_k^{(2)} + r_k\mathbb{I}$ can be expressed perturbatively as

$$\frac{1}{\Gamma_k^{(2)} + r_k\mathbb{I}} = \frac{1}{a_{2,k} + \Delta_{\vec{c}}(\vec{\varphi}) + r_k\mathbb{I}} \approx \frac{1}{a_{2,k} + r_k\mathbb{I}}\left(\mathbb{I} - \frac{\Delta_{\vec{c}}(\vec{\varphi})}{a_{2,k} + r_k\mathbb{I}} + o(\vec{c}\cdot\vec{c})\right)$$

Inserting this expansion into (4.13) we can deduce the two flow equations projected onto $a_{0,k}$ and $a_{2,k}$:

$$\partial_k a_{0,k} = \frac{\partial_k r_k}{2}\,\mathrm{Tr}\left[\frac{1}{a_{2,k} + r_k\mathbb{I}}\right], \tag{4.15}$$

$$\frac{1}{2}\vec{\varphi}\cdot\partial_k a_{2,k}\vec{\varphi} = -\frac{\partial_k r_k}{2}\,\mathrm{Tr}\left[\frac{\Delta_{\vec{c}}(\vec{\varphi})}{(a_{2,k} + r_k\mathbb{I})^2}\right]. \tag{4.16}$$

Therefore, we have to solve first equation (4.16), and then use the result to solve equation (4.15). Because of the diagonal form of $\Delta_{\vec{c}}(\vec{\varphi})$, it is immediately understood that on the RHS of (4.16) there are no terms in $\varphi_i \cdot \varphi_j$ for $i \neq j$. This means that the off-diagonal terms remain constant along the flow within this truncation, keeping their initial value. Therefore, we can choose a first order perturbative expansion for $a_{2,k}$ in the form

$$a_{2,k} = B + a_{2,k}^{(1)}(\vec{c}) + o(c_i^2),$$

where $a_{2,k}^{(1)}(\vec{c}) = \mathrm{diag}(b_i^{(1)}c_i,\, i = 1, \ldots, N)$. Therefore, first order perturbation theory gives

$$\partial_k b_i^{(1)} = -\frac{\partial_k r_k}{2}\cdot\left[\frac{1}{(B + r_k\mathbb{I})^2}\right]_{ii}. \tag{4.17}$$

The above discussion holds for a generic number of variables N, but now on I will consider $N = 2$ in order to carry on the calculation explicitely. So, suppose that the initial condition is the 2×2 matrix

$$B = \begin{pmatrix} b_1 & d \\ d & b_2 \end{pmatrix}$$

corresponding to the 2-dimensional integral

$$Z = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 e^{-\frac{b_1}{2}x_1^2 - \frac{b_2}{2}x_2^2 - dx_1x_2 - \frac{c_1}{4!}x_1^4 - \frac{c_2}{4!}x_2^4}.$$

Then, standard matrices algebra leads us to the flow equations for b_1 and b_2:

$$\partial_k b_1^{(1)} = -\frac{\partial_k r_k}{2} \frac{(r_k + b_2^{(1)})^2 + d^2}{(r_k + b_1^{(1)})(r_k + b_2^{(1)}) - d^2},$$

while the other equation for $b_2^{(1)}$ is obtained by exchanging $1 \to 2$, $2 \to 1$ (this confirms the fact that the flow preserves the symmetry of the bare model). The integration of this equation yields

$$b_1^{(1)}(k) = \frac{1}{2} \frac{b_2 + r_k}{(b_1 + r_k)(b_2 + r_k) - d^2}. \tag{4.18}$$

We can immediately check the consistency of this result. Setting $d = 0$ the two integrals on x_1 and x_2 are now decoupled, and we retrieve the result found for the $N = 1$ perturbation theory (3.25), that is what one would indeed expect.

The last step is to insert this result into the equation for $a_{0,k}$ to compute the first order correction. Of course, at zeroth order, the Gaussian result (4.12) is recovered. The final result is

$$a_{0,0} = \log\left(\frac{\sqrt{detB}}{2\pi}\right) + \frac{b_2^2}{(b_1b_2 - d^2)^2}\frac{c_1}{8} + \frac{b_1^2}{(b_1b_2 - d^2)^2}\frac{c_2}{8}. \tag{4.19}$$

Once again, we recover the $N = 1$ first order correction (3.27) if we set $d = 0$. The perturbative expansion for the integral Z will then be:

$$Z_{RG} = \sqrt{\frac{(2\pi)^2}{\det B}}\left[1 - \frac{b_2^2}{(\det B)^2}\frac{c_1}{8} - \frac{b_1^2}{(\det B)^2}\frac{c_2}{8} + o(c_1^2) + o(c_2^2)\right], \tag{4.20}$$

where $\det B = b_1 b_2 - d^2$.

This can be compared with the perturbative expansion of Z for small c_1 and c_2. To first order we have

$$Z_{PT} = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 e^{-\frac{b_1}{2}x_1^2 - \frac{b_2}{2}x_2^2 - dx_1 \cdot x_2} \left(1 - \frac{c_1}{4!}x_1^4 - \frac{c_2}{4!}x_2^4\right).$$

These are now gaussian integrals, which can be easily evaluated. The result is

$$Z_{PT} = \sqrt{\frac{(2\pi)^2}{\det B}} \left[1 - \frac{b_2^2}{(\det B)^2}\frac{c_1}{8} - \frac{b_1^2}{(\det B)^2}\frac{c_2}{8} + o(c_1^2) + o(c_2^2)\right]. \quad (4.21)$$

So, as for the $N = 1$ case, the flow equation returns the correct result to first order in pertubation theory, independently of the choice of the regulator r_k.

4.3.2 Numerical analysis

Let's now analyze in the results obtained from numerical integration of the flow equation in the $N = 2$ case.

In particular, let's start by considering the case where $b_1 = b_2 = c_1 = c_2 = 1$, while the coupling d between the two variables x_1 and x_2 is small.
For small values of d the exponential can be expanded in power series as follows:

$$Z = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \sum_{n=0}^{\infty} \frac{(-d)^n}{n!}(x_1 x_2)^n e^{-\frac{1}{2}(x_1^2+x_2^2) - \frac{1}{4!}(x_1^4+x_2^4)},$$

$$\Rightarrow Z = \sum_{n=0}^{\infty} \frac{d^{2n}}{(2n)!} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 (x_1 x_2)^{2n} e^{-\frac{1}{2}(x_1^2+x_2^2) - \frac{1}{4!}(x_1^4+x_2^4)},$$

$$\Rightarrow Z = \sum_{n=0}^{\infty} \frac{d^{2n}}{(2n)!} \left(\int_{-\infty}^{\infty} dx x^{2n} e^{-\frac{1}{2}x^2 - \frac{1}{4!}x^4}\right)^2, \quad (4.22)$$

where the integrals involving odd powers of x disappeared because they are obviously equal to zero. So, for very small d values we expect a parabolic behaviour.

Numerical results coming from the integration of the RG flow equation have been compared with numerical integration and 2^{nd} - order perturbation theory, and the results are displayed in *Figure 4.1.* Very good agreement is obtained.

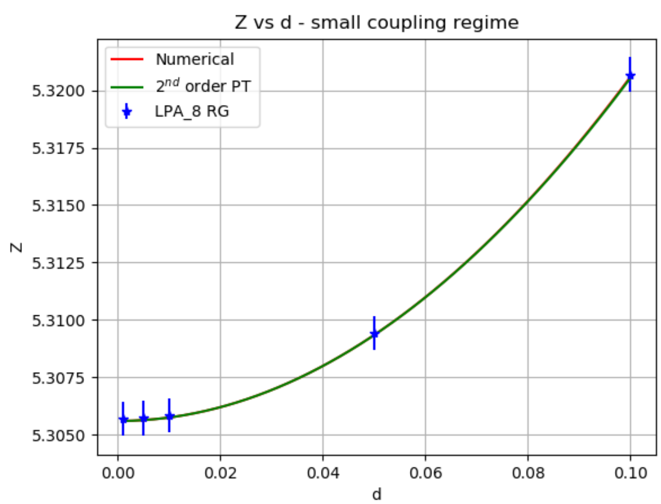


Figure 4.1: Numerical vs numerical RG vs PT results in the small coupling

For non perturbative values of the coupling d the effect of truncation. For a 4^{th} - order truncation, instead of increasing monotonically, the RG-computed integral reaches a maximum and then decreases. This is understood as an effect of the truncation. Indeed, *Figure 4.2* shows that increasing the order of truncation the agreement improves. So, to get better and better estimates of the integral we should go to higher order truncations for large d values.

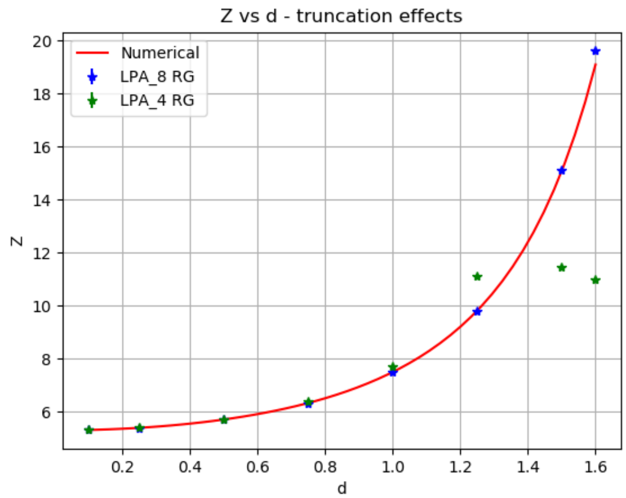


Figure 4.2: Truncation dependence

4.4 Integral of a homogeneous quartic form

In this section I will consider the integrals of homogeneous quartic form, i.e. with an action of the type

$$S(\vec{x}) = \sum_{i,j=1}^{N} x_i^2 A_{ij} x_j^2,$$

where A_{ij} is a symmetric $N \times N$ matrix. In *Appendix A* some strategies for the analytical solution of such integrals are reviewed and proposed, with calculations carried on explicity for the case $N = 2$.

The integral that I will study via the FRG approach is thus

$$Z = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 e^{-\frac{x_1^4}{4!} - \frac{x_2^4}{4!} - \frac{\epsilon}{4!} x_1^2 x_2^2}, \tag{4.23}$$

which can be expressed in closed form as

$$Z = \sqrt{24\pi} K_E \left(\frac{1}{2} - \frac{\epsilon}{4} \right), \tag{4.24}$$

where K_E is a special function, called the elliptic integral of second kind. This integral is a bit simpler to study than the quartic integral of the previous section because of symmetry reasons. Indeed, the bare action $S(x_1, x_2)$ possesses even symmetry, and hence the number of couplings generated along the flow is significantly reduced. To make this concept clear, consider the set of all 4^{th} - order operators in two variables. These are given by

$$x_1^4,\ x_1^3 x_2,\ x_1^2 x_2^2,\ x_1 x_2^3,\ x_2^4.$$

For an even action only three of them are generated along the flow, since the two presenting odd powers in x_1 and x_2 violates the even symmetry, and this is true at each order of truncation. Instead, when the symmetry of the action is just the exchange $x_1 \to x_2$, as for the previous quartic integral, all of the couplings are generated at each order since this time they all respect the exchange symmetry. Therefore, an integrand with even bare action is easier to be studied in the sense that we can reach higher order truncations with less effort, i.e. the number of flow equations to be studied is smaller.

The numerical results produced with a 10^{th} - order LPA truncation of the effective action are shown in *Figure 4.3*, and present very good agreement with (4.24) both in the perturbative (top figure) and non-perturbative (bottom figure) ϵ regimes. Such agreement suggests that in case of larger values of N, such as between 5 and 10, the FRG calculation of these integrals can be a convenient scheme with respect to other possible numerical methods.

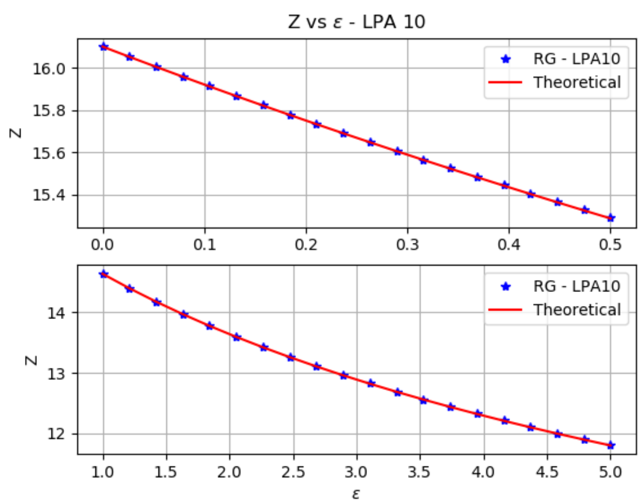


Figure 4.3: FRG results for the $r = 2$ non-gaussian form.

Chapter 5

Renormalization group approach to quantum mechanics

One of the fundamental problems studied in quantum mechanics is the solution of the time-independent Schrödinger equation

$$-\frac{1}{2}\frac{d^2\psi}{dq^2} + V(q)\psi = E\psi, \tag{5.1}$$

which allows to determine the eigenfunctions and the energy spectrum of a quantum system under the action of a potential $V(q)$. Of course, several analytical, semi-classical, approximate and numerical procedures have been developed in order to solve this problem.

This problem is intermediate between the solution of the integrals, presented in *Chapter 3* and *4*, and the calculation of the partition function of statistical mechanical models (e.g. the Ising model), which are described by $(d+1)$ - dimensional field theories. Indeed, quantum systems can be described through $(0+1)$ - dimensional field theories using the Euclidean path integral formalism, where the dynamics of the system is governed by the bare action

$$S_E[q(t)] = \int dt \left(\frac{\dot{q}^2}{2} + V(q)\right).$$

Therefore, the direct generalization of the integrals is obtained by promoting the variable q to a field $q(t)$, and correspondingly the action becomes a functional of this field.

In this chapter I will employ the functional renormalization group techniques described in *Chapter 2* for the determination of the ground state of the one dimensional Schrödinger equation (5.1) for different kinds of non-trivial potentials, such as the quartic anharmonic oscillator, the quartic double-well potential and a periodic cosine potential.

5.1 The LPA flow equation

In order to apply the FRG formalism of *Chapter 1*, we will thus consider the Euclidean path-integral formulation of quantum field theory (QFT) [8, 20], which describes quantum mechanics as a scalar field theory in $(0+1)$ - dimensions.

The physical information of the quantum system is contained in the n-points correlation functions, generated by the Schwinger functional

$$Z[j] = \int \mathcal{D}[q] e^{-S_E[q] + \int d\tau j(\tau) q(\tau)},$$

where $S_E[q]$, as introduced before, is the Euclidean action that describes the dynamics of the system, while $\int \mathcal{D}[q]$ denotes integration over all paths $q(\tau)$. We assume this measure to be properly regularized by an UV-cutoff Λ. Therefore, it shall now be clear that we can implement the Wetterich approach exactly as prescribed in *Chapter 2*, and study the flow of the effective average action Γ_k interpolating between the Euclidean bare action S_E in the UV and the full quantum action Γ when all the quantum fluctuations have been taken into account in the limit $k \to 0$ [20].

Since we are interested in the determination of the ground state, it will be sufficient to work within the framework of the LPA approximation, where we consider the following truncation for Γ_k[1]:

$$\Gamma_k = \int d\tau \left(\frac{\dot{x}^2}{2} + U_k(x) \right). \tag{5.2}$$

The flow equation for the effective potential U_k was given in (2.20) for a $d-$dimensional scalar field theory. Thus, for $d = 1$ we obtain

$$\partial_k U_k = -\frac{k^2}{2\pi} \int_0^\infty \frac{y^{3/2} r'(y)}{U_k'' + k^2 P^2(y)} dy, \tag{5.3}$$

where $P^2(y) = y[1 + r(y)]$.

An important physical correction has to be done here. If we start at the UV scale Λ with a free particle, then we would expect U_k not to flow, but to remain constant at every scale. However, this is not the content of equation (5.3), which, setting $U_k'' = 0$, tells us that

$$\partial_k U_k = -\frac{1}{2\pi} \int_0^\infty \frac{y^{3/2} r'(y)}{P^2(y)} dy \neq 0.$$

Hence, in order to obtain physically meaningful results we have to subtract this term from the flow equation, which becomes

$$\partial_k U_k = \frac{U_k''}{2\pi} \int_0^\infty \frac{y^{3/2} r'(y)}{[U_k'' + k^2 P^2(y)] P^2(y)} dy. \tag{5.4}$$

This is an instance of the renormalization of a physical theory, which is typically seen in the regularization of quantum field theories.

We are now in position to study some concrete application of the method.

5.2 The harmonic oscillator

The simplest problem that we can study is for sure the quantum harmonic oscillator, with hamiltonian

$$\hat{H} = -\frac{1}{2} \frac{d^2}{dq^2} + \frac{\omega^2}{2} q^2, \tag{5.5}$$

[1] Here $x \equiv \langle q \rangle$.

for which the exact determination of the energy spectrum and eigenfunctions is presented in every reference textbook for quantum mechanics.

With no surprise, we are able to solve analytically the Wetterich equation in this case. Indeed, the harmonic oscillator corresponds to a gaussian field theory and thus, as for the case of the integrals, the usual quadratic ansatz will be sufficient for the effective potential:

$$U_k(x) = E_{0,k} + \frac{\omega^2}{2}x^2, \tag{5.6}$$

with $E_{0,\Lambda} = 0$, and $E_{0,0}$ representing the quantum ground state energy. the flow equation (5.4) becomes

$$\partial_k E_{0,k} = \frac{\omega^2}{2\pi}\int_0^\infty \frac{y^{3/2}r'(y)}{[\omega^2 + k^2P^2(y)]P^2(y)}dy. \tag{5.7}$$

Keeping the regulator general, we shall first perform the integration over k:

$$0 - E_{0,0} = \frac{\omega^2}{2\pi}\int_0^\infty dy\frac{y^{3/2}r'(y)}{P^2(y)}\int_0^\infty dk\frac{1}{\omega^2 + k^2P^2(y)}, \tag{5.8}$$

$$\Rightarrow E_{0,0} = -\frac{\omega}{4}\int_0^\infty dy\frac{y^{3/2}r'(y)}{P^3(y)} = -\frac{\omega}{4}\int_0^\infty dyf(y). \tag{5.9}$$

Therefore, we are left with the calculation of the integral over y, which will be useful also in the next section:

$$\int_0^\infty dyf(y) = \int_0^\infty \frac{r'(y)dy}{[1+r(y)]^{3/2}},$$

$$\Rightarrow \int_0^\infty dyf(y) = -2\left[\frac{1}{\sqrt{1+r(y)}}\right]_0^\infty = -2, \tag{5.10}$$

where the last step comes from the fact that because of (28) and (29) we have $r(y) \to 0$ when $y \to \infty$, and $r(y) \to \infty$ when $y \to 0$. Finally, inserting (111) in (112) we get that

$$E_{0,0} = \frac{\omega}{2}, \tag{5.11}$$

that is the very well known exact result for the ground state energy of a particle in one dimensional quantum harmonic potential, in units where $\hbar = 1$.

5.3 The anharmonic oscillator

Let's now consider the problem of a quartic anharmonic oscillator, described by the Hamiltonian

$$\hat{H} = -\frac{1}{2}\frac{d^2}{dq^2} + \frac{\omega^2}{2}q^2 + \frac{\lambda}{4!}q^4. \tag{5.12}$$

In order to solve the flow equation (5.4) we need to resort to some further approximation, and the best-suited for this problem is a polynomial expansion for the effective potential

$$U_k(x) = \sum_{n=0}^{\infty} \frac{a_{2n,k}}{(2n)!}x^{2n}, \tag{5.13}$$

where only even terms are considered because of the even symmetry of the bare potential. Thus, the flow equation has to be projected onto the scale dependent coefficient $a_{2n,k}$. Therefore, using a 4^{th} - order truncation of the type

$$U_k(x) = E_{0,k} + \frac{\omega_k^2}{2}x^2 + \frac{\lambda_k}{4!}x^4, \tag{5.14}$$

we obtain three coupled differential equations for the scale dependent expansion coefficients:

$$\partial_k E_{0,k} = \frac{\omega_k^2}{2\pi}\int_0^{\infty} \frac{y^{3/2}r'(y)}{[\omega_k^2 + k^2P^2(y)]P^2(y)}dy, \tag{5.15}$$

$$\partial_k \omega_k^2 = \frac{k^2\lambda_k}{2\pi}\int_0^{\infty} \frac{y^{3/2}r'(y)}{[\omega_k^2 + k^2P^2(y)]^2}dy, \tag{5.16}$$

$$\partial_k \lambda_k = -\frac{3k^2\lambda_k}{\pi}\int_0^\infty \frac{y^{3/2}r'(y)}{[\omega_k^2 + k^2P^2(y)]^3}dy, \tag{5.17}$$

with the initial conditions $E_\Lambda = 0$, $\omega_\Lambda^2 = \omega^2$, $\lambda_\Lambda = \lambda$.

The solution of the system must be carried out numerically. As for the case of the quartic integrals, I will first propose some analytical insight about perturbative solution of the FRG equations, and then I will show the numerical results of the integration.

5.3.1 Perturbative approach

Consider the case where $\lambda_k = \lambda \ll 1\ \forall k$. Then, we can solve perturbatively (5.15) and (5.16) in λ. Starting from (5.16), let's consider the following perturbative expansion for the frequency

$$\omega_k^2 = \omega^2 + \Omega_k^{(1)}\lambda + O(\lambda^2). \tag{5.18}$$

Thus, the first order perturbation will evolve according to

$$\partial_k \Omega_k^{(1)} = \frac{k^2\lambda_k}{2\pi}\int_0^\infty \frac{y^{3/2}r'(y)}{(\omega^2 + k^2P_y^2)^2}dy. \tag{5.19}$$

Hence, we have

$$\Omega_k^{(1)} = -\frac{1}{2\pi}\int_0^\infty dy y^{3/2}r'(y)\int_k^\infty ds\frac{s^2}{(\omega^2+s^2P_y^2)^2},$$

$$\Rightarrow \Omega_k^{(1)} = -\frac{1}{2\pi\omega}\int_0^\infty dy f(y)\int_{k\frac{P_y}{\omega}}^\infty dz\frac{z^2}{(1+z^2)^2},$$

$$\Rightarrow \Omega_k^{(1)} = -\frac{1}{2\pi\omega}\int_0^\infty dy f(y) g\left(k\frac{P_y}{\omega}\right), \tag{5.20}$$

where I have denoted

$$g(s) = \int_s^\infty \frac{z^2}{(1+z^2)^2}. \tag{5.21}$$

This result has to be inserted into the equation (5.15) for the ground state energy, and we find that the first order correction to the energy is the solution of

$$\partial_k E_k^{(1)} = \frac{\Omega_k^{(1)} k^2}{2\pi} \int_0^\infty \frac{y^{3/2} r'(y)}{[\omega^2 + k^2 P^2(y)]^2} dy. \tag{5.22}$$

Thus, we have that

$$E_0^{(1)} = -\frac{1}{2\pi} \int_0^\infty dy y^{3/2} r'(y) \int_0^\infty dk \frac{\Omega_k^{(1)} k^2}{(\omega^2 + k^2 P_y^2)^2},$$

$$E_0^{(1)} = \frac{1}{4\pi^2\omega} \int_0^\infty dy y^{3/2} r'(y) \int_0^\infty dx f(x) \int_0^\infty dk \frac{g\left(k\frac{P_x}{\omega}\right) k^2}{(\omega^2 + k^2 P_y^2)^2},$$

$$E_0^{(1)} = \frac{1}{4\pi^2\omega^2} \int_0^\infty dy f(y) \int_0^\infty dx f(x) \int_0^\infty ds \frac{g\left(s\frac{P_x}{P_y}\right) s^2}{(1+s^2)^2},$$

$$E_0^{(1)} = -\frac{1}{4\pi^2\omega^2} \int_0^\infty dy f(y) \int_0^\infty dx f(x) \int_0^\infty ds g\left(s\frac{P_x}{P_y}\right) g'(s),$$

$$E_0^{(1)} = -\frac{1}{4\pi^2\omega^2} \int_0^\infty dy f(y) \int_0^\infty dx f(x) \int_0^\infty dz g(z) g'(z),$$

where in the last passage I have changed variable to $z = s\frac{P_x}{P_y}$. Thus, I finally obtain

$$E_0^{(1)} = \frac{1}{8\pi^2\omega^2} g^2(0) \left(\int_0^\infty dy f(y) \right)^2. \tag{5.23}$$

From (5.21) we can see that $g(0) = \frac{\pi}{4}$, and hence we finally get

$$E_0^{(1)} = \frac{1}{32\omega^2}. \tag{5.24}$$

So, the first order perturbative series for the ground state energy can be written as

$$E_0 = \frac{\omega}{2} + \frac{3\omega}{4} \left(\frac{\lambda}{4!\omega^3} \right) + O(\lambda^2), \tag{5.25}$$

which coincides with the result of standard perturbation theory in quantum mechanics [8, 20]. Once again, it is remarkable that this is achieved independently of the choice of the regulator function.

5.3.2 Numerical results

Let's now study the results obtained via numerical integration of the flow equation. For this purpose one can use the Litim regulator [8, 20]

$$r(y) = \left(\frac{1}{y} - 1\right)\theta(1-y).$$

With this choice of the regulator[2] the domain of integration of the integral in (5.4) is restricted to $y \in [0,1]$, and considering that $P^2(y) = 1$, while $r'(y) = -y^{-2}$, we have that

$$\partial_k U_k = -\frac{1}{2\pi}\frac{U_k''}{U_k'' + k^2}\int_0^1 y^{-1/2} dy,$$

and hence the flow equation for the effective potential is given by

$$\partial_k U_k = -\frac{1}{\pi}\frac{U_k''}{U_k'' + k^2}. \tag{5.26}$$

At 4^{th} - order truncation of the effective potential the flow equations are given by

$$\partial_k E_{0,k} = -\frac{1}{\pi}\frac{\omega_k^2}{\omega_k^2 + k^2}, \tag{5.27}$$

$$\partial_k \omega_k^2 = -\frac{1}{\pi}\frac{k^2}{(\omega_k^2 + k^2)^2}\lambda_k, \tag{5.28}$$

$$\partial_k \lambda_k = \frac{6}{\pi}\frac{k^2}{(\omega_k^2 + k^2)^3}\lambda_k^2. \tag{5.29}$$

The numerical solution of the RG flow equation for $\omega = 2$ and $\lambda \in (0, 100)$ can be compared with the ground state energy obtained via numerical solution of the Schŕodinger equation. *Figure 5.1* shows that already within 4^{th} - order LPA it is possible to obtain a good estimate for the ground state energy. More specifically, it can be noticed that the estimate improves for smaller values of the coupling λ.

[2]The flow equations for some different regulators are derived in *Appendix B*.

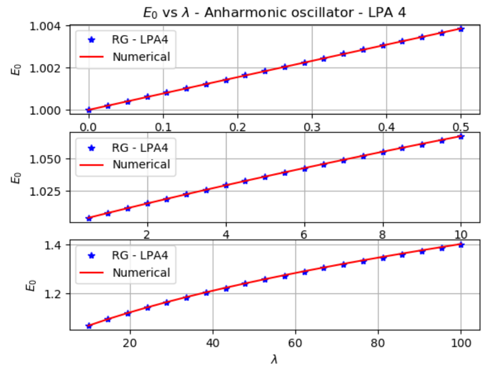


Figure 5.1: Comparison of RG and numerical results for different values of the quartic coupling Λ in LPA-4.

In order to improve our estimates also in the strong coupling regime we should consider higher order truncations of the effective potential. The new results obtained using a 20^{th} - order truncation are shown in the left side of *Figure 5.2*. In order to see explicitly how the estimate

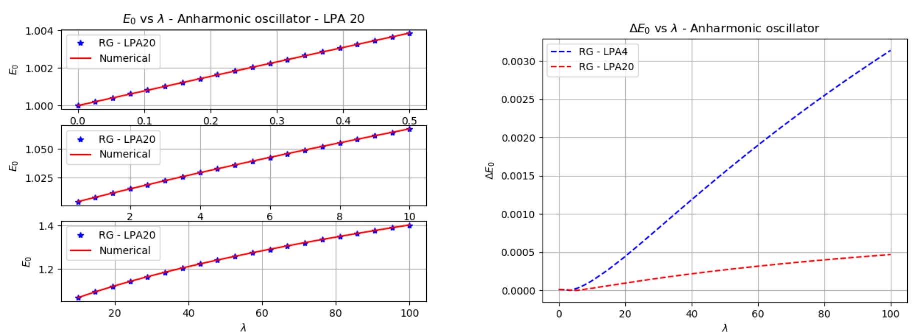


Figure 5.2: Left: Results from the LPA-20 trunctation. Right: Comparison of the quality of the two truncation orders.

has improved we can plot the difference between the RG and numerical ground state for both the truncation orders. This is done in the right side of *Figure 5.2*, and we can clearly observe that the estimate in the strong coupling regime indeed improves significantly, of about one order of magnitude.

5.4 The double well potential

We shall now consider the problem of a quantum particle inside a quartic double well potential, that is described through the Hamiltonian

$$\hat{H} = -\frac{1}{2}\frac{d^2}{dq^2} - \frac{\omega^2}{2}q^2 + \frac{\lambda}{4!}q^4, \tag{5.30}$$

which presents a change in the sign of the quadratic term with respect to the anharmonic potential in (5.12). This change in the sign corresponds to a completely new shape for the potential, which now presents a maximum at $q = 0$ instead of a minimum, and two symmetric minima at $q = \pm\frac{\sqrt{6}\omega}{\sqrt{\lambda}}$, as shown in *Figure 5.3.* Since the potential is still

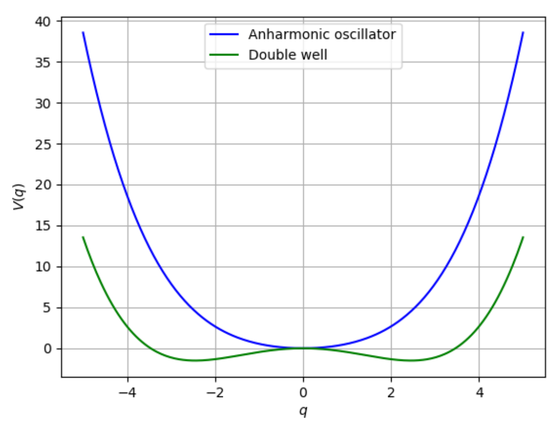


Figure 5.3: Profiles of the anharmonic and double well potentials for $\lambda = \omega = 1$.

polynomial, we shall resort to the same approximations used for the anharmonic oscillator in the previous section, and expand the effective potential in power series in order to solve the flow equation (5.4). Once again, since the problem possesses an even symmetry, only those couplings corresponding to even powers of the Taylor expansion will be generated along the flow. Therefore, it should be clear that the flow equations for the expansion coefficients will be exactly the same as for the case of the anharmonic oscillator.

The numerical solution of the RG flow equation is carried out for $\omega = 1$ and $\lambda \in (1, 100)$, and again the Litim regulator has been chosen for the integration. The results coming from 4^{th} and 20^{th} - order truncation are displayed in *Figure 5.4*. What we observe this time is

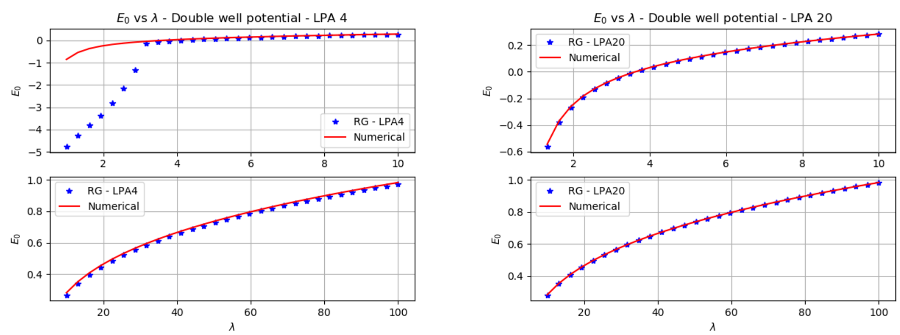


Figure 5.4: Comparison of RG and numerical results for different values of the quartic coupling Λ in LPA-4.

completely different from what we found for the case of the anharmonic oscillator. Indeed, this time we see that the estimate of the ground state energy obtained within a 4^{th} - order LPA is very bad for small values of the coupling λ, although the results improve when looking at the strong coupling regime. This global picture improves when we go to 20^{th} - order truncation. However, we still observe that the estimates are better in the strong coupling regime. This is better observed when plotting the difference between the RG and numerical ground state (*Figure 5.5*). The failure of the LPA truncation in the weak coupling regime can be explained in view of the shape of the potential. Indeed, we have said that the position of the minimum of the potential energy

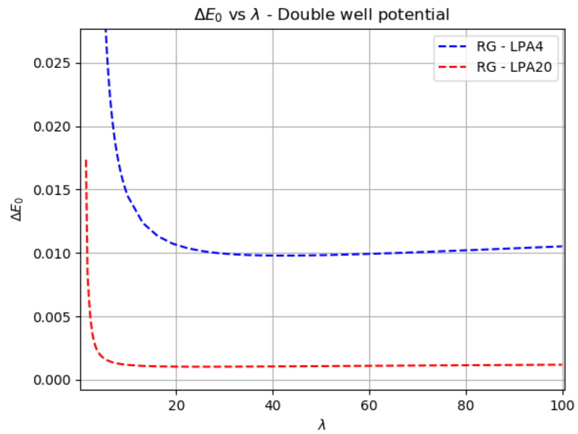


Figure 5.5: Comparison of the LPA-4 and LPA-20 truncations.

is proportional to $\lambda^{-1/2}$. This means that if we decrease λ, then we are moving the minima of the potential away from the origin. Therefore, a 4^{th} - order Taylor expansion about $x = 0$ is not enough to capture properly the behaviour of the potential close to its minima, and we are forced to consider higher order expansions in order to get accurate results.

As an alternative, one could think to solve the flow equation (5.4) numerically without expanding the potential in power series. This procedure is explained in [8], but it is not pursued here.

5.5 The cosine potential

The last problem that we will study is that of a quantum particle inside a periodic cosine potential, that is described by the Hamiltonian

$$\hat{H} = -\frac{1}{2}\frac{d^2}{dq^2} + u\cos(q). \tag{5.31}$$

In this case, the Schrödinger equation

$$-\frac{1}{2}\frac{d^2\psi}{dq^2} + u\cos(q)\psi(q) = E\psi(q) \tag{5.32}$$

corresponds to the so called *Mathieu equation.* This is a celebrated equation of mathematical physics which appears in the treatment of a wide class of problems [21]. Perturbative results for the eigenvalues of this equation can be found in literature [22].

A polynomial expansion of the effective potential is no more well suited for treating the problem. Indeed, the periodicity of the potential constitutes itself a symmetry of the problem, and this would be broken using a finite order polynomial expansion. Therefore, in order to allow for a convenient solution of the flow equation and preserve the periodicity the idea is to expand U_k in Fourier series [23]:

$$U_k(x) = \sum_{n=0}^{\infty} u_{n,k} \cos(nx). \tag{5.33}$$

In this framework, the simplest truncation for the effective potential is thus given by

$$U_k(x) = E_{0,k} + u_k \cos(x). \tag{5.34}$$

Thus, we need to project the flow equation (105) onto the coefficients $E_{0,k}$ and u_k. This requires to expand in Fourier series the term

$$\frac{U_k''}{U_k'' + k^2 P_y^2}.$$

Using the choice (5.34) for the truncation we get

$$\frac{U_k''}{U_k'' + k^2 P_y^2} = 1 - \frac{k^2 P_y^2}{[(k^2 P_y^2)^2 - u_k^2]^{1/2}} + \frac{2k^2 P_y^2}{u_k} \left(1 - \frac{k^2 P_y^2}{[(k^2 P_y^2)^2 - u_k^2]^{1/2}} \right) \cos(x).$$

Therefore, the flow equations for the two expansion coefficients become

$$\partial_k E_{0,k} = \frac{1}{2\pi} \int_0^{\infty} \frac{y^{3/2} r'(y)}{P^2(y)} \left(1 - \frac{k^2 P_y^2}{[(k^2 P_y^2)^2 - u_k^2]^{1/2}} \right) dy, \tag{5.35}$$

$$\partial_k u_k = \frac{k^2}{u_k \pi} \int_0^{\infty} y^{3/2} r'(y) \left(1 - \frac{k^2 P_y^2}{[(k^2 P_y^2)^2 - u_k^2]^{1/2}} \right) dy. \tag{5.36}$$

5.5.1 The small coupling regime

Truncation (5.34) allows to get some insights about the small coupling regime. Choosing again the Litim's regulator, the flow equations (5.35) and (5.36) becomes

$$\partial_k E_{0,k} = -\frac{1}{\pi}\left(1 - \frac{k^2}{[k^4 - u_k^2]^{1/2}}\right), \tag{5.37}$$

$$\partial_k u_k = -\frac{2k^2}{u_k \pi}\left(1 - \frac{k^2}{[k^4 - u_k^2]^{1/2}}\right). \tag{5.38}$$

When we consider the case $u \ll 1$, then these two coupled equations can be solved perturbatively considering $u_k \ll 1$. This is true because of the following physical consideration. For the one-dimensional quantum system that we are considering there is no symmetry-breaking, and therefore in the thermodynamic limit $k \to 0$ the true effective action of the model Γ should be convex. In turns, this means that we expect that the cosine-coupling flows towards zero under the RG flow, i.e. $u_{k=0} = 0$, in order to recover convexity in the thermodynamic limit. Therefore, if the initial condition $u_\Lambda = u \ll 1$, then it is reasonable to assume that $u_k \ll 1\ \forall k$. For $u_k \ll 1$ the flow equations reduce to

$$\begin{cases} \partial_k E_{0,k} = \frac{u_k^2}{2\pi k^4} \\ \partial_k u_k = \frac{u_k}{\pi k^2} \end{cases}. \tag{5.39}$$

The solution of the second equation gives

$$u_k = u e^{-\frac{1}{\pi k}}. \tag{5.40}$$

Inserting this result in the equation for the energy and integrating from 0 to ∞ we finally get

$$E_0 = -\frac{\pi^2}{8} u^2. \tag{5.41}$$

This shall be compared with the perturbative result for the ground state of the Mathieu equation, which is given by [22]

$$E_0(u) = -u^2 + \frac{7}{4}u^4 - \frac{58}{9}u^6 + O(u^8). \tag{5.42}$$

Therefore, we see that the first order in u is retrieved exactly also for this periodic potential (it is indeed zero), while the leading order parabolic behaviour is correctly recovered with a discrepancy of 23% for the expansion coefficient, which is not too bad considering that a similar error is found also when studying second order perturbation theory for the anharmonic oscillator using lowest order truncations [8, 20].

The results coming from full numerical integration of (5.37) and (5.38) are shown in *Figure 5.6*, in comparison with (5.42) and with the numerical solution of the Schrödinger equation (5.32). We see that in

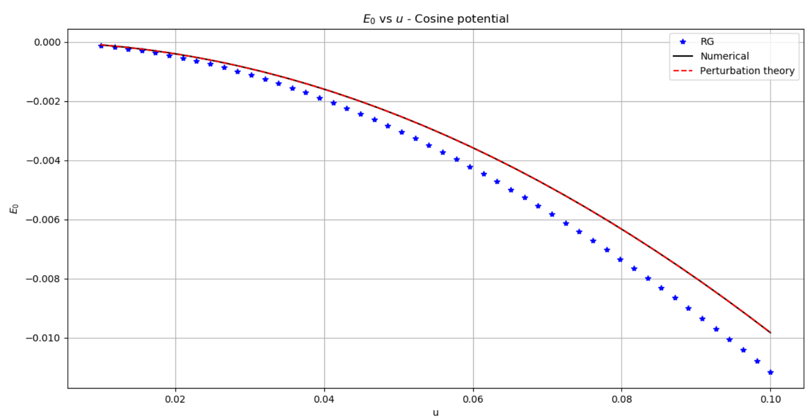


Figure 5.6: Ground state of the cosine potential in the small coupling regime.

this case the perturbative formula (5.42) produces more accurate results than the renormalization group procedure, at least to the truncation order that we have taken into account. Notice, indeed, that we are at the lowest order of the Fourier truncation scheme. Motivated by the results presented in *Chapters 3* and *4*, as well as those in the previous sections of this chapter, one can be confident that increasing the truncation scheme one can have a significant improvement of the obtained results.

Chapter 6

Conclusions

The aim of this thesis was to study the application of renormalization group techniques to two problems of interest in mathematics and physics: the calculation of integrals and the solution of the Schrödinger eigenvalue equation.

Using renormalization group (RG) tools and in particular the functional renormalization group (FRG) formalism, we have shown that both these problems can be reduced to the solution of a non-linear partial differential equation for a properly defined effective action, which is also known as the Ginzburg-Landau free energy in condensed matter contexts.

For the problems treated in this work we have studied how fast the local potential approximation (LPA) truncations hierarchy converges to the exact results (if these exist) or to numerically exact solutions.

Despite being rarely encountered in the RG literature, the problem of the calculation of integrals can be considered an excellent tool for a pedagogical introduction to the FRG. Studying this problem, we have shown that it is possible to estimate the convergence to exact results increasing the level of the LPA truncation. This is possible because the LPA is an exact approximation for this problem, since there are no gradients in the derivative expansion of this $(0+0)$-dimensional case.

Therefore, we have been able to obtain very good numerical estimates of the values of non-trivial quartic integrals both for one and two variables. The method can be easily generalized to more variables.

The solution of the Schrödinger equation can be seen as a generalization of the $(0+0)$-dimensional case of the integrals since quantum systems can be described through $(0+1)$ - dimensional field theories using the Euclidean path integral formalism. For this problem we have seen that it is not possible to converge to exact results using LPA. However, also within this approximation scheme one can obtain very good numerical estimates for the ground state energies, and study how rapidly convergence is approached for different potentials. Excellent results have been obtained in the case of the anharmonic oscillator, because of the rather simple shape of this potential. When the potential profiles become more complicated, such as for the quartic double well or the cosine potential, the results worsen and higher order truncations are needed in order to improve the estimates of the ground-state energy. In particular, a good direction for future work would be to include more Fourier modes in the expansion of the effective potential for the Mathieu equation, or to even include higher order gradients in the derivative expansions, i.e. to go beyond the LPA. Moreover, a possible future development could be the study of the regulator dependence of the flow, and to devise a general optimization criterium which allows to identify which regulator produces the best results within a given approximation scheme.

Appendix A

Analytical calculation of non-gaussian integrals

Here I will give a derivation of an analytical closed form for some non-gaussian integrals, that in the multivariable case will have an homogeneous action of the form

$$S(\vec{x}) = \sum_{i,j=1}^{N} x_i^r A_{ij} x_j^r,$$

where $A \in \mathbb{R}^{N,N}$ is a symmetric matrix.

In particular, I will pursue two different approaches for the non-trivial cases: integration by series and a differential approach introduced in [24].

A.1 Trivial examples

Let's start from some simple examples which do not require any special treatment.

A.1.1 $S(x) = \lambda x^{2r}$

Consider the one variable integral

$$Z = \int_{-\infty}^{+\infty} e^{-\lambda x^{2r}} dx.$$

Upon changing variable to $y = \lambda x^{2r}$ we get

$$Z = \frac{\lambda^{-\frac{1}{2r}}}{r} \int_0^{\infty} y^{\frac{1}{2r}-1} e^{-y} dy.$$

Here we can recognize the Euler gamma function, and we finally have

$$Z = \frac{\lambda^{-\frac{1}{2r}}}{r} \Gamma\left(\frac{1}{2r}\right). \tag{A.1}$$

Using the same idea we can also compute all the moments of the normalized distribution

$$p(x) = \frac{r\lambda^{\frac{1}{2r}}}{\Gamma\left(\frac{1}{2r}\right)} e^{-\lambda x^{2r}}.$$

The odd moments all vanish for symmetry reasons

$$\mathbb{E}[x^{2n+1}] = 0 \quad \forall\, n = 0, 1, \dots. \tag{A.2}$$

Instead the even moments are given by

$$\mathbb{E}[x^{2n}] = \frac{r\lambda^{\frac{1}{2r}}}{\Gamma\left(\frac{1}{2r}\right)} \int_{-\infty}^{+\infty} x^{2n} e^{-\lambda x^{2r}} dx,$$

$$\Rightarrow \mathbb{E}[x^{2n}] = \frac{\lambda^{-\frac{n}{r}}}{\Gamma\left(\frac{1}{2r}\right)} \int_0^{+\infty} x^{\frac{2n+1}{2r}-1} e^{-y} dy,$$

$$\Rightarrow \mathbb{E}[x^{2n}] = \lambda^{-\frac{n}{r}} \frac{\Gamma\left(\frac{2n+1}{2r}\right)}{\Gamma\left(\frac{1}{2r}\right)}. \tag{A.3}$$

A.1.2 Diagonal coupling matrix

The simplest case when we considere more than one variable is for a diagonal coupling matrix, $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. The integral we want to compute is thus

$$Z = \int d^N x e^{-\lambda_i x_i^{2r}},$$

where Einstein convetion for the sum over repeated indices is intended in the exponent. In such a case, all of the N integrals over x_i can be performed separately, since the variables are uncoupled. Each of them will give a contribution of the form $(A.1)$. So, the result will read

$$Z = \left[\frac{1}{r}\Gamma\left(\frac{1}{2r}\right)\right]^N \left(\prod_{i=1}^N \lambda_i\right)^{-\frac{1}{2r}} = \left[\frac{1}{r}\Gamma\left(\frac{1}{2r}\right)\right]^N \left(\det A\right)^{-\frac{1}{2r}}. \quad \text{(A.4)}$$

A.1.3 The quartic spherical case

Another case which is simple to treat is the case in which

$$A_{ij} = \sqrt{\lambda_i \lambda_j} \quad \forall\, i \neq j,$$

and $r = 2$. In such case, the determinant of A vanishes, and the quartic form in the exponent becomes a perfect square:

$$S(\vec{x}) = \sum_{i,j=1}^N \sqrt{\lambda_i \lambda_j} x_i^2 x_j^2 = \left(\sum_{i=1}^N \sqrt{\lambda_i} x_i^2\right)^2.$$

Therefore, upon changing variables to $\lambda_i^{1/4} x_i$, the integral becomes

$$Z = \left(\prod_{i=1}^N \lambda_i^{-1/4}\right) \int d^N x e^{-(\sum_{i=1}^N x_i^2)^2}.$$

This integral becomes trivial in spherical coordinates, where we have

$$Z = \Omega_N \left(\prod_{i=1}^N \lambda_i^{-1/4}\right) \int_0^{+\infty} d\rho e^{-\rho^4},$$

where Ω_N is the N-dimensional solid angle, and the integral over the radial coordinate is half of $(A.1)$ with $r=2$, resulting in

$$Z = \Omega_N \left(\prod_{i=1}^{N} \lambda_i^{-1/4} \right) \frac{1}{4} \Gamma\left(\frac{1}{4}\right). \tag{A.5}$$

A.2 Integration by series

We now switch to more complicated non-trivial integrals, and start by approaching them using the method of integration by series.

A.2.1 $S(x) = \epsilon x^r + \lambda x^{2r}$

Suppose that we want to compute the one variable integral

$$Z_{1|2r} = \int_{-\infty}^{\infty} e^{-\epsilon x^r - \lambda x^{2r}} dx, \tag{A.6}$$

with $r \in \mathbb{N}$.

A strategy to find a closed form for such an integral is to expand the x^r exponential, and then switch integration and series:

$$Z_{1|2r} = \sum_{i=0}^{\infty} \frac{(-\epsilon)^n}{n!} \int_{-\infty}^{\infty} x^{rn} e^{-\lambda x^{2r}} dx. \tag{A.7}$$

The remaining integral can now be easily expressed in terms of the Euler gamma function, exactly as we did *Appendix A.1.1*. We shall distinguish the even and odd r cases.

For $r = 2k$, $k \in \mathbb{N}$, then the integrals are non-vanishing $\forall n$, and we get

$$\int_{-\infty}^{\infty} x^{2kn} e^{-\lambda x^{4k}} dx = \frac{1}{2k} \lambda^{-\frac{2nk+1}{4k}} \Gamma\left(\frac{2nk+1}{4k}\right). \tag{A.8}$$

Therefore, we get

$$Z_{1|4k} = \frac{\lambda^{-1/4k}}{2k} \sum_{i=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\epsilon}{\sqrt{\lambda}}\right)^n \Gamma\left(\frac{2kn+1}{4k}\right). \tag{A.9}$$

Then, finally, the series can be summed, and we get

$$Z_{1|2r} = \frac{\lambda^{-1/2r}}{r}\Gamma\left(\frac{1}{2r}\right){}_1F_1\left(\frac{1}{2r},\frac{1}{2};\frac{\epsilon^2}{4\lambda}\right) + \dots$$

$$\dots - \frac{\epsilon}{\sqrt{\lambda}}\frac{\lambda^{-1/2r}}{r}\Gamma\left(\frac{1+r}{2r}\right){}_1F_1\left(\frac{1+r}{2r},\frac{1}{2};\frac{\epsilon^2}{4\lambda}\right). \tag{A.10}$$

with r even. In the $r = 2$ case this expression simplifies to

$$Z_{1|4} = \frac{1}{2}\sqrt{\frac{\epsilon}{\lambda}}e^{\frac{\epsilon^2}{8\lambda}}K_{\frac{1}{4}}\left(\frac{\epsilon^2}{8\lambda}\right), \tag{A.11}$$

corresponding to the formula given for the quartic integral in (3.31) with rescaled coefficients.

The $r = 2k-1$ case is treated identically, but this time only even n terms will survive the integration. Therefore, we have that

$$Z_{1|2(2k-1)} = \frac{\lambda^{-\frac{1}{2(2k-1)}}}{2k-1}\sum_{i=0}^{\infty}\frac{1}{(2n)!}\left(\frac{\epsilon}{\sqrt{\lambda}}\right)^{2n}\Gamma\left(n+\frac{1}{2(2k-1)}\right), \tag{A.12}$$

giving

$$Z_{1|2r} = \frac{\lambda^{-1/2r}}{r}\Gamma\left(\frac{1}{2r}\right){}_1F_1\left(\frac{1}{2r},\frac{1}{2};\frac{\epsilon^2}{4\lambda}\right), \tag{A.13}$$

for odd values of r.

A.2.2 $S(x,y) = 2\epsilon x^r y^r + \lambda_1 x^{2r} + \lambda_2 y^{2r}$

Consider now the two variables extension of $(A.6)$:

$$Z_{2|2r} = \int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy e^{-2\epsilon x^r y^r - \lambda_1 x^{2r} - \lambda_2 y^{2r}}, \tag{A.14}$$

with $r \in \mathbb{N}$. The procedure is analagous to the one variable case. Indeed, this time we will expand the interaction term, in such a way that the integrals decouples:

$$Z_r = \sum_{i=0}^{\infty}\frac{(-2\epsilon)^n}{n!}\left(\int_{-\infty}^{\infty}x^{rn}e^{-\lambda_1 x^{2r}}dx\right)\left(\int_{-\infty}^{\infty}y^{rn}e^{-\lambda_2 y^{2r}}dy\right), \tag{A.15}$$

where the integrals appearing in the summation have already been evaluated in *Appendix A.2.1*. Again, we shall distinguish the even and odd r cases.

For $r = 2k$ we get

$$Z_{2|4k} = \frac{(\lambda_1\lambda_2)^{-1/4}}{(2k)^2}\sum_{i=0}^{\infty}\frac{(-1)^n}{n!}z^n\Gamma^2\left(\frac{2nk+1}{4k}\right),$$

where $z = \frac{2\epsilon}{\sqrt{\lambda_1\lambda_2}}$. This series is expressed in closed form in terms of hypergeometric functions:

$$Z_{2|2r} = \frac{(\lambda_1\lambda_2)^{-1/2r}}{r^2}\Gamma^2\left(\frac{1}{2r}\right){}_2F_1\left(\frac{1}{2r},\frac{1}{2r};\frac{1}{2};\frac{z^2}{4}\right) + \dots$$

$$\dots - z\frac{(\lambda_1\lambda_2)^{-1/2r}}{r^2}\Gamma^2\left(\frac{1+r}{2r}\right){}_2F_1\left(\frac{1+r}{2r},\frac{1+r}{2r};\frac{3}{2};\frac{z^2}{4}\right), \quad \text{(A.16)}$$

for r even. In the case $r = 2$, corresponding to a quartic homogeneous integral, the expression simplifies to

$$Z_{2|4} = (\lambda_1\lambda_2)^{-1/4}\sqrt{\pi}K_E\left(\frac{1}{2} - \frac{\epsilon}{2\sqrt{\lambda_1\lambda_2}}\right), \quad \text{(A.17)}$$

where K_E is the complete elliptic integral of second kind.

For the odd r case, setting $r = 2k-1$, $k \in \mathbb{N}$ we instead get

$$Z_{2|2r} = \frac{(\lambda_1\lambda_2)^{-1/2r}}{r^2}\Gamma^2\left(\frac{1}{2r}\right){}_2F_1\left(\frac{1}{2r},\frac{1}{2r};\frac{1}{2};\frac{z^2}{4}\right). \quad \text{(A.18)}$$

Clearly, the case $r = 1$ corresponds to the gaussian integral, and indeed we get

$$Z_1 = \frac{\pi}{\sqrt{\lambda_1\lambda_2 - \epsilon^2}}. \quad \text{(A.19)}$$

A.3 Differential approach

Integral discriminants are a particular class of integrals of the form

$$\int d^N x e^{-S(\vec{x})},$$

where

$$S(\vec{x}) = \sum_{i_1 \dots i_r = 1}^{N} A_{i_1 \dots i_r} x_{i_1} x_{i_2} \dots x_{i_r}$$

i.e. it is a homogeneous form of degree r in the N variables $x_1 \dots x_N$. This class of integrals has been studied in [?] by Morozov and Shakirov, who proposed an approach based on the solution of some differential identies that the integral satisfies as a function of the parameters of A.

Here I will study a simpler subclass of integral discriminants of interest in physics, i.e. those where

$$S(\vec{x}) = \sum_{i,j=1}^{N} A_{ij} x_i^r x_j^r, \tag{A.20}$$

where $A \in \mathbb{R}^{N,N}$ is a symmetric matrix.

A.3.1 Ward identities

Let's consider the integral

$$Z_{N|2r} = \int d^N x e^{-S(\vec{x})},$$

where S is of the form (163).

The idea is to consider $Z_{N|2r}$ as a function of the parameters appearing in the matrix A, i.e. $Z_{N|2r} = Z_{N|2r}(\{A_{ij}\})$. Then, we can demostrate that Z satisfies some differential identities, called *Ward identities*, as a function of these parameters.

First of all, let's fix some notation here. I will denote

$$\langle f(\vec{x}) \rangle = \int d^N x f(\vec{x}) e^{-S(\vec{x})},$$

and hence it is clear that $Z_{N|2r} = \langle 1 \rangle$. We can observe that the following four relations holds true for any symmetric A:

$$\frac{\partial Z}{\partial A_{ii}} = -\langle x_i^{2r} \rangle,$$

$$\frac{\partial^2 Z}{\partial A_{ii}\partial A_{jj}} = \langle x_i^{2r} x_j^{2r}\rangle,$$

$$\frac{\partial Z}{\partial A_{ij}} = -2\langle x_i^r x_j^r\rangle,$$

$$\frac{\partial^2 Z}{\partial A_{ij}^2} = 4\langle x_i^{2r} x_j^{2r}\rangle.$$

Therefore, it follows that $Z_{N|2r}$ satisfies a first set of differential identities:

$$\left(\frac{\partial^2}{\partial A_{ij}^2} - 4\frac{\partial^2}{\partial A_{ii}\partial A_{jj}}\right) Z = 0. \tag{A.21}$$

Since this set of identities holds also for the integrand itself, they are too particular, and we can look for another set which instead holds only for Z.

An idea is to consider that

$$\int d^N x \frac{\partial}{\partial x_i}\left(f(\vec{x})e^{-S(\vec{x})}\right) = 0,$$

whenever $f(\vec{x})e^{-S(\vec{x})}$ vanishes at the integration boundaries. Then, performing the derivative inside the integral we get to

$$\left\langle \frac{\partial f}{\partial x_i}\right\rangle - \left\langle f\frac{\partial S}{\partial x_i}\right\rangle = 0.$$

Choosing $f(\vec{x}) = x_i$ we therefore obtain

$$Z_{N|2r} - \left\langle x_i \frac{\partial}{\partial x_i}\sum_{k,j=1}^{N} A_{kj} x_k^r x_j^r\right\rangle = 0,$$

$$\Rightarrow Z_{N|2r} - 2r\sum_{j=1}^{N} A_{ij}\langle x_i^r x_j^r\rangle = 0,$$

$$\Rightarrow Z_{N|2r} + r\sum_{j\neq i}^{N} A_{ij}\frac{\partial Z}{\partial A_{ij}} + 2rA_{ii}\frac{\partial Z}{\partial A_{ii}} = 0.$$

There are N such relations, one for each value of $i = 1 \dots N$. Therefore, summing over i we get

$$\sum_{i,j\leq i} A_{ij} \frac{\partial Z_{N|2r}}{\partial A_{ij}} = -\frac{N}{2r} Z_{N|2r}. \tag{A.22}$$

This is the other identity we where looking for.

From the latter expression we can immediately deduce an interesting property of Z. Indeed, $(A.22)$ is nothing but Euler's identity for homegenous functions, telling us that $Z_{N|2r}$ will be an homegenous function of the couplings, of degree $-\frac{N}{2r}$.

Therefore, instead of directly computing the integral, we can obtain $Z_{N|2r}$ by solving the two sets of differential equations $(A.21)$ and $(A.22)$.

A.3.2 $S(x) = \lambda x^{4r} + \epsilon x^{2r}$

Before proceeding to the multivariable case it is interesting to consider $S(x) = \lambda x^{4r} + \epsilon x^{2r}$. This is not a homogeneous form, but it is still interesting to study it because it is the simplest non-trivial case where we can apply the technique of the Ward identies. The two identities satisfied by Z in this case are

$$\begin{cases} \frac{\partial^2 Z}{\partial \epsilon^2} + \frac{\partial Z}{\partial \lambda} = 0 \\ \\ 4\lambda \frac{\partial Z}{\partial \lambda} + 2\epsilon \frac{\partial Z}{\partial \epsilon} = -Z \end{cases}.$$

We can easily understand the scaling form which satisfies the the second equation by considering that with the change of variable $x \to \lambda^{-1/4r} x$ the integral becomes

$$Z = \lambda^{-\frac{1}{4r}} \int_{-\infty}^{+\infty} e^{-x^{4r} - \frac{\epsilon}{\sqrt{\lambda}} x^{2r}} dx.$$

So, considering this emerging scaling form, and $(A.1)$, we can make the guess

$$Z = \frac{\Gamma\left(\frac{1}{4r}\right)}{2r} \lambda^{-\frac{1}{4r}} f\left(\frac{\epsilon^2}{\lambda}\right),$$

with $f(0) = 1$. Substituting this form into the first differential identity, then we obtain the following equation for f:

$$4tf''(t) + (2-t)f'(t) - \frac{f(t)}{4r} = 0.$$

Setting now $r = 1$, we can solve this equation and imposing the correct condition for recovering the Gaussian result in the limit $\lambda \to 0$, and the final result is

$$Z = \frac{1}{2}\sqrt{\frac{\epsilon}{\lambda}} e^{\frac{\epsilon^2}{8\lambda}} K_{\frac{1}{4}}\left(\frac{\epsilon^2}{8\lambda}\right).$$

A.3.3 The N=2 case

For the case of two coupled variables, the matrix A introduced in $(A.20)$ can be written as

$$A = \begin{pmatrix} \lambda_1 & \epsilon \\ \epsilon & \lambda_2 \end{pmatrix}.$$

The Ward identity $(A.22)$ now reads

$$\lambda_1 \frac{\partial Z}{\partial \lambda_1} + \lambda_2 \frac{\partial Z}{\partial \lambda_2} + \epsilon \frac{\partial Z}{\partial \epsilon} = -\frac{Z}{r},$$

telling us that $Z_{2|2r}$ is a homogeneous function of degree $-\frac{1}{r}$.

In order to understand the correct form of the scaling function, consider the following argument. The integral that we want to compute is

$$Z_{2|2r} = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy e^{-\lambda_1 x^{2r} - \lambda_2 y^{2r} - 2\epsilon x^r y^r}.$$

If we perform the change of variable

$$\begin{cases} x \to \lambda_1^{-\frac{1}{2r}} x \\ y \to \lambda_2^{-\frac{1}{2r}} y \end{cases},$$

then the integral becomes

$$Z_{2|2r} = (\lambda_1\lambda_2)^{-\frac{1}{2r}} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy e^{-x^{2r} - y^{2r} - 2\frac{\epsilon}{\sqrt{\lambda_1\lambda_2}} x^r y^r}$$

So, considering this result, together with $(A.1)$, a good starting ansatz for $Z_{2|2r}$ is

$$Z_{2|2r} = (\lambda_1\lambda_2)^{-\frac{1}{2r}}\left[\frac{1}{r}\Gamma\left(\frac{1}{2r}\right)\right]^2 f\left(\frac{\epsilon^2}{\lambda_1\lambda_2}\right), \tag{A.23}$$

where f is a homogeneous of order zero in the couplings, such that $f(0) = 1$.

In order to determine f we have to insert this ansatz into $(A.21)$ and derive a differential equation for it. After some tedious but straightforward algebra, a differential equation for $f(t)$ is obtained, and reads:

$$4t(1-t)f''(t) + 2\left(1 - 2\frac{1+r}{r}t\right)f'(t) - \frac{f(t)}{r^2} = 0 \tag{A.24}$$

where $t = \frac{\epsilon^2}{\lambda_1\lambda_2}$. Equation $(A.24)$ can be recognized as the Euler's hypergeometric differential equation, whose generic form is

$$t(1-t)f''(t) + \big[c - (a+b+1)t\big]f'(t) - abf(t) = 0.$$

In our case the parameters of the equation are $a = b = \frac{1}{2r}$ and $c = \frac{1}{2}$. So, the general solution satysfying the condition $f(0) = 1$ is given by

$$f(t) = {}_2F_1\left(\frac{1}{2r}, \frac{1}{2r}; \frac{1}{2}; t\right) + \alpha\sqrt{t}\,{}_2F_1\left(\frac{1}{2r} + \frac{1}{2}, \frac{1}{2r} + \frac{1}{2}; \frac{3}{2}; t\right), \tag{A.25}$$

where α is an integration constant to be determined.

The quartic case r=2

If we set $r = 2$, then we know also the condition for $t = 1$ from the result of the quartic spherical integral $(A.5)$. In particular, we find that taking the limit for $t \to 1$ this is finite only for

$$\alpha = -\frac{4\pi^2}{\Gamma\left(\frac{1}{4}\right)^4}.$$

This means that $(A.25)$ becomes

$$Z = (\lambda_1\lambda_2)^{-\frac{1}{4}}\left[\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\right]^2\left[{}_2F_1\left(\frac{1}{4},\frac{1}{4};\frac{1}{2};x\right)-\frac{4\pi^2}{\Gamma\left(\frac{1}{4}\right)^4}\sqrt{x}\,{}_2F_1\left(\frac{3}{4},\frac{3}{4};\frac{3}{2};x\right)\right].$$

The term in square brackets is proportional to a complete elliptic integral, and in particular we find

$$Z_{2|4} = \frac{\sqrt{\pi}}{(\lambda_1\lambda_2)^{1/4}}K_E\left(\frac{1}{2}(1-\sqrt{x})\right),$$

and hence

$$Z_{2|4} = \frac{\sqrt{\pi}}{(\lambda_1\lambda_2)^{1/4}}K_E\left(\frac{1}{2}-\frac{\epsilon}{2\sqrt{\lambda_1\lambda_2}}\right). \tag{A.26}$$

The odd r case

When r is odd we can exploit another condition to determine α. Indeed, we can argue that

$$\left.\frac{\partial Z}{\partial \epsilon}\right|_{\epsilon=0} = -2\langle x^r y^r\rangle_{\epsilon=0} = 0.$$

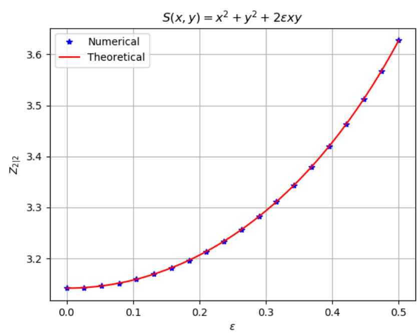
This condition is translated into the requirement that $f'(t)$ is finite when $t \to 0$. This is true only for $\alpha = 0$. Therefore, we have that

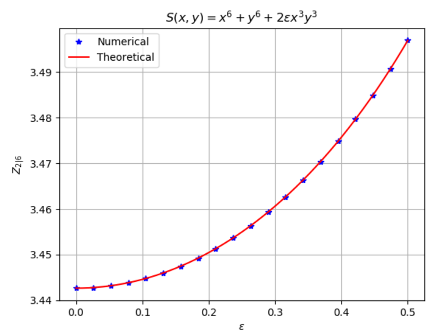
$$Z_{2|2r} = (\lambda_1\lambda_2)^{-\frac{1}{2r}}\left[\frac{1}{r}\Gamma\left(\frac{1}{2r}\right)\right]^2{}_2F_1\left(\frac{1}{2r},\frac{1}{2r};\frac{1}{2};\frac{\epsilon^2}{\lambda_1\lambda_2}\right), \tag{A.27}$$

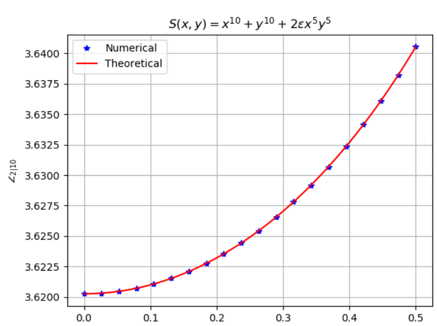
for $r = 2k-1$, $k \in \mathbb{N}$. An immediate consistency check is given by setting $r = 1$, in which case we find

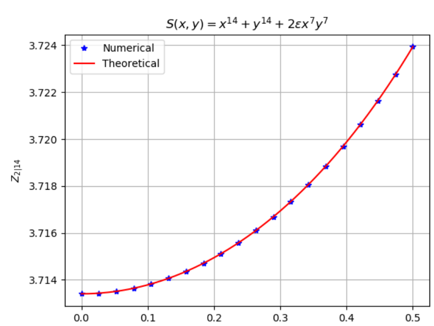
$$Z_{2|2} = \frac{\pi}{\sqrt{\lambda_1\lambda_2-\epsilon^2}},$$

which is the well known Gaussian result! The following plots illustrate that the expression $(A.27)$ agrees with the numerical estimates of the integrals.


$S(x, y) = x^2 + y^2 + 2\epsilon xy$
Numerical
Theoretical
$Z_{2|2}$
ϵ


$S(x, y) = x^6 + y^6 + 2\epsilon x^3 y^3$
Numerical
Theoretical
$Z_{2|6}$
ϵ


$S(x, y) = x^{10} + y^{10} + 2\epsilon x^5 y^5$
Numerical
Theoretical
$Z_{2|10}$
ϵ


$S(x, y) = x^{14} + y^{14} + 2\epsilon x^7 y^7$
Numerical
Theoretical
$Z_{2|14}$
ϵ

Appendix B

Flow equations for the anharmonic oscillator

In this appendix the flow equation for the quantum anharmonic oscillator problem is derived for some specific regulators, for which the integral appearing in (105) can be performed analytically.

B.1 The generalized Litim regulator with b=2

Another interesting regulator is the generalized form of the Litim one:

$$r(y) = \left(\frac{1}{y^b} - 1\right)\theta(1-y). \tag{B.1}$$

Introducing this regulator into equation (5.4) then we obtain the following flow equation:

$$\partial_k U_k = -\frac{bU_k''}{2\pi}\int_0^1 \frac{y^{b-\frac{3}{2}}}{k^2 + U_k'' y^{b-1}} dy. \tag{B.2}$$

Here we can make the substitution $t = \alpha_k y^{b-1}$, where $\alpha_k = U_k''/k^2$. Then we get

$$\partial_k U_k = -\frac{b}{2\pi(b-1)}\alpha_k^{-\frac{1}{2(b-1)}}\int_0^{\alpha_k} \frac{t^{\frac{1}{2(b-1)}}}{1+t} dt. \tag{B.3}$$

If we consider the case $b = 2$, then the flow equation will finally be given by

$$\partial_k U_k = -\frac{2}{\pi}\left[1 - \frac{k}{\sqrt{U_k''}} \arctan\left(\frac{\sqrt{U_k''}}{k}\right)\right]. \tag{B.4}$$

The latter can be projected on the coefficient of the 4^{th}-order local potential approximation, obtaining the following system of coupled differential equations:

$$\partial_k E_{0,k} = -\frac{2}{\pi}\left[1 - \frac{k}{\omega_k} \arctan\left(\frac{\omega_k}{k}\right)\right], \tag{B.5}$$

$$\partial_k \omega_k^2 = -\frac{1}{\pi}\left[- \frac{k^2}{\omega_k^2(k^2+\omega_k^2)} + \frac{k}{\omega_k^3} \arctan\left(\frac{\omega_k}{k}\right)\right]\lambda_k, \tag{B.6}$$

$$\partial_k \lambda_k = -\frac{3}{2\pi}\left[\frac{k^2(3k^2+5\omega_k^2)}{\omega_k^4(k^2+\omega_k^2)^2} - 3\frac{k}{\omega_k^5} \arctan\left(\frac{\omega_k}{k}\right)\right]\lambda_k^2. \tag{B.7}$$

B.2 The b=1 power regulator

Let's now change family of regulators, and consider the power law ones:

$$r(y) = y^{-b}. \tag{B.8}$$

Inserting the power regulator with $b = 1$ in (5.4) the flow equation becomes

$$\partial_k U_k = -\frac{U_k''}{2\pi}\int_0^\infty \frac{1}{U_k'' + k^2(1+y)} \frac{1}{1+y} \frac{1}{\sqrt{y}} dy,$$

and hence we get

$$\partial_k U_k = -\frac{1}{2} + \frac{1}{2}\frac{k}{\sqrt{U_k'' + k^2}}. \tag{B.9}$$

The LPA flow equations truncated to 4^{th} order now read:

$$\partial_k E_{0,k} = -\frac{1}{2} + \frac{1}{2}\frac{k}{(\omega_k^2 + k^2)^{1/2}}, \tag{B.10}$$

$$\partial_k \omega_k^2 = -\frac{\lambda}{4}\frac{k}{(\omega_k^2 + k^2)^{3/2}}, \tag{B.11}$$

$$\partial_k \lambda_k = -\frac{9\lambda^2}{8}\frac{k}{(\omega_k^2 + 2k^2)^{5/2}}. \tag{B.12}$$

B.3 The b=2 power regulator

As a final example, consider again the power regulator, but this time with $b = 2$. The flow equation reads

$$\partial_k U_k = -\frac{U_k''}{\pi}\int_0^\infty \frac{\sqrt{y}}{U_k'' y + k^2(1+y^2)}\frac{1}{1+y^2}dy,$$

which after integration yields

$$\partial_k U_k = -\frac{1}{\sqrt{2}}\left[1 - \frac{\sqrt{2}k}{\sqrt{U_k'' + 2k^2}}\right]. \tag{B.13}$$

The flow equations truncated to 4^{th} order now read:

$$\partial_k E_{0,k} = -\frac{1}{\sqrt{2}} + \frac{k}{(\omega_k^2 + 2k^2)^{1/2}}, \tag{B.14}$$

$$\partial_k \omega_k^2 = -\frac{\lambda}{2}\frac{k}{(\omega_k^2 + 2k^2)^{3/2}}, \tag{B.15}$$

$$\partial_k \lambda_k = -\frac{9\lambda^2}{4}\frac{k}{(\omega_k^2 + 2k^2)^{5/2}}. \tag{B.16}$$

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