Politecnico di Torino

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MASTER THESIS

The role of disorder in the yielding transition of amorphous solids

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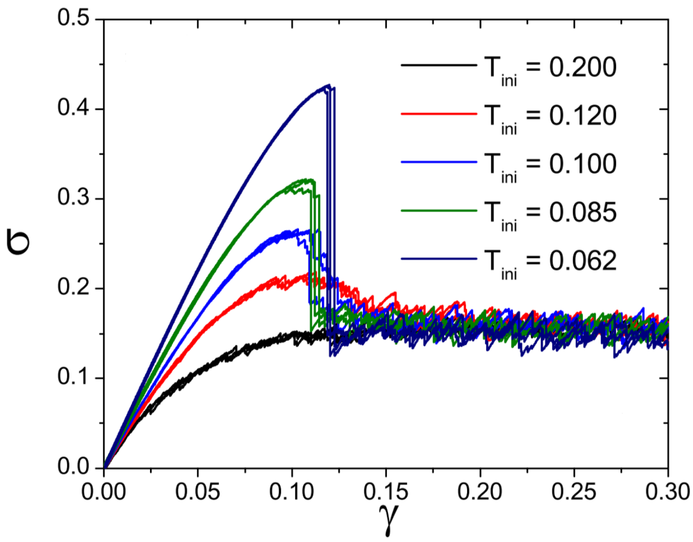
Abstract

Depending on their preparation, amorphous materials display different types of behavior under shear when passing from an elastic regime to a plastic one. This passage, known as the "yielding transition" can happen in a smooth way or in a sudden, catastrophic manner. This strongly depends on the preparation of the sample, and in particular on the annealing which it undergoes and which controls the initial disorder. The transition between a smooth (ductile) and a discontinuous (brittle) behavior is reminiscent of the one observed in quasi-statically driven Random Field Ising Model (RFIM) at zero temperature, where the magnetization can either undergo a macroscopic jump or smoothly vary when an applied magnetic field is slowly increased. Both transitions are dependent on the disorder strength.

The goal of this thesis has been to investigate what the effective strength of the disorder Δ_{eff} is at the transition of a driven RFIM and at the yielding transition of an EPM, and to make the connection between the two. The quantity we aim at computing is the so-called disconnected susceptibility, which is given by the variance of the sample-to-sample fluctuations of the magnetization or the stress and which characterizes the distribution of the effective disorder. It can indeed be shown (by means of field-theoretical arguments) that the disconnected susceptibility is proportional to the square of the so-called connected one, with a coefficient of proportionality precisely given by the effective disorder strength Δ_{eff}.

Figure 1: The different kinds of yielding transition obtained from simulations in [5].



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1 Introduction

Amorphous solids are not perfect solids, but neither simple liquids: some of these materials are indeed composed from a large percentage of liquid, but maintain a solid structure. Examples of these solids are foams (densely packed droplets dispersed in a liquid), colloidal glasses (suspensions of solid particles in a liquid) and metallic glasses (obtained by lowering the temperature of a liquid while avoiding crystallization). During this project we considered the application of simple shear to this class of materials. Other types of mechanical responses are also interesting, but simple shear is easier to study. The adjective "simple" points to the fact that in what follows we will make the approximation that the only response of the material to the applied force is a shift in the same direction of the latter. This will allow us to avoid the use of tensorial quantities.

When a shear force is applied to a solid it will undergo a displacement. The ratio between the displacement along the direction of the force Δl and the size l of the material in the perpendicular direction is called the shear strain γ

$$\gamma = \frac{\Delta l}{l}$$

The forced change in the shape will cause an internal stress σ to be created inside the material at the microscopic level through the mechanism of elastic deformations (mainly reversible) which involves the sliding of small blocks of material one with respect to the other. However when the stress is locally too high, the particles which constitute these blocks will undergo an irreversible rearrangement to find a more comfortable configuration. These rearrangements are called plastic events and will affect the particles even at a long distance. Due to this long range effect some other zones could undergo an irreversible rearrangement, creating what is called an avalanche. The avalanches can be of all range, thus also visible at a macroscopic level.

To understand better what happens at a microscopic level experiments with layers of bubbles have been performed in [1]. Under strain application one can understand which is the characteristic form of the majority of plastic events: the deformations of this form are called T1 events and involve the change of nearest neighbors, as shown in figure (2) where 1 and 4 are previously nearest neighbor and end up as next nearest neighbor, while the opposite happens for 2 and 3.

One may look at the behavior of the stress as a function of the strain. It is in general possible to identify two particular regions in the stress vs strain curve. In the first one, the stress is proportional to the strain through a coefficient which is called elastic modulus μ_2. This part, called elastic region, is characterized by the reversibility of the displacement and by the dominance of elastic deformation. However, also in the linear regime, the solid is only marginally elastic: the elastic events are indeed coupled with very few plastic ones. For this reason, even if the macroscopic stress and strain are reversible, the solid will not go back to the original configuration once the applied strain is removed (the energy difference between the initial and final configuration will however be infinitesimal). When the strain increases we observe the second region, the plastic one. Herein the stress takes a constant value and the plastic events are no longer present in a negligible number. This leads to the solid undergoing stationary flow and, possibly, failure.

The passage from the elastic to the plastic region takes the name of yielding transition, and is the region in the stress vs strain curve that we are interested in. This passage can happen in roughly three different ways: if the material is soft enough the stress will monotonically increase up to the constant value of the plastic region (called yield stress σ_Y). For more rigid materials one will still observe a smooth behavior in the transition, but the stress will reach a maximum and then descend to the plateau. These first two cases display what is called a ductile behavior. In the third one the stress will reach a maximum and then, after beginning the descent, will drop discontinuously to a lower value (through a macroscopic jump). This is instead called a brittle behavior.

For a long time the two possible ways of crossing the transient state to reach a steady one were studied separately: the plasticity to study the failure of hard solids and the rheology for the flow of soft material. What changed in the recent years is that we became able to reproduce both experimentally and computationally a whole set of shades in between these two behaviours within the same material and only by changing the initial conditions (in figure (1) one can observe the results of simulations where only the initial temperature is changed).

Our choice is to work in a strain-driven framework, which means that we are going to apply a strain and study the response of the stress. The other way round is also possible (applying stress and measuring

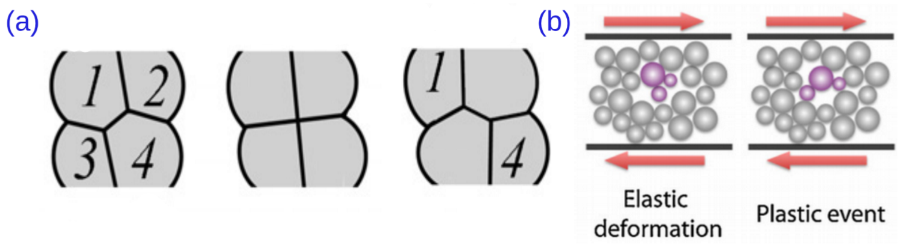

Figure 2: Deformations seen at a microscopic level: (a) shows a typical T1 event in a bubble layer (from [1]), while in (b) the differences between a plastic event and simple elastic deformation are displayed(from [4]).

strain) but more difficult to implement. We will also consider the athermal case. In general the plastic rearrangements could be triggered by the thermal fluctuations, but we will avoid considering this feature and stick to the stress increase as only cause of the plastic events.

The change in the behavior at the yielding transition is reminiscent of what we observe in the Random Field Ising Model. It is known from many years that the pure Ising model at $T = 0$ undergoes a sharp transition in the magnetization when the applied field changes sign. When disorder is added the situation is different: the disorder strength determine whether the transition in the sign of the magnetization occurs smoothly or through a macroscopic jump. This is the reason behind the goal of this work. We aim at finding an analogy between the Random Field Ising model and the behavior of amorphous solids under applied strain, based on the similarities that can be observed in the disorder-related behavior. The analogy will be carried on at a macroscopic level, by investigating how the bare disorder introduced in the system at the beginning of the dynamic evolves. This quantity is called effective disorder Δ_{eff}.

In what follows we will proceed in this way: in chapter 2 we will present the framework that we are going to use in the rest of the work and the formula to compute the effective disorder. In chapters 3 and 4 we will define the properties of the two models under study and the approximation chosen in each of them. We will then show the computations to derive the values of Δ_{eff} in both the cases. Finally in chapter 5 we will draw conclusions and present some ideas to further proceed with this work.

2 Models and effective disorder

In the following we are going to investigate the Random Field Ising Model (RFIM). However we also need a model for the amorphous solids. There is a whole class of models that have been devised for this purpose, each of them with an higher or lower degree of simplification. This class takes the name of "Elasto-Plastic models (EPM)". The specifications of our case are explained in chapter 4. In this chapter we will briefly introduce the framework in which we are going to study both the models (EPM and RFIM) and the quantity we are interested in. In doing so we use a terminology which is proper of the Ising case, but easily generalizable to EPM by substituting "external field" with "external strain" and "magnetization" with "1-stress" (we will introduce the order parameter for the EPM later). In both cases we will neglect the effect of the thermal fluctuations by considering the $T = 0$ case. This directly leads to the fact that once the configuration of the random field or the initial stress is known, the dynamics is fully deterministic. Moreover we will, for simplicity, consider the quasistatic limit in which the rate of change of the external field (or strain) tends to 0. This means that the field changes slowly enough that all the avalanches finish before it increases again. We will furthermore look at both models in a mean-field, fully connected approximation. Since the real interactions in the amorphous solids are long range, this choice will not be too abrupt.

The analogy we are trying to perform is not at all obvious and will be carried on at a macroscopic scale. This means that we are not interested in finding a direct mapping between the two at the level of the microscopic variables (spins and blocks, respectively) and interactions but that we want to see if the behavior the two models show for a macroscopic quantity can be compared and if some insight on the transient regime of the EPM can be gained by studying a model which is in principle very different. The reasons behind our interest in doing so is that, while the RFIM has been studied intensively in the last decades, we have less results for EPM. Finding a way to map the transient regime of EPM into RFIM would lead us to gain insight into the former.

The quantity we are going to analyze is the effective disorder Δ_{eff}. This quantity represents the renormalized disorder in a field theoretical framework. The formula to compute it is given by

$$\Delta_{eff}(H) = \frac{\chi_{disc}(H)}{\chi^2_{con}(H)} \tag{1}$$

Where χ_{dis} is the disconnected susceptibility and χ_{con} is the connected one given by

$$\chi_{dis}(H) = N\left[\overline{\langle m\rangle^2}(H) - \overline{\langle m\rangle}^2(H)\right] \qquad \chi_{con}(H) = \overline{\langle m^2\rangle(H) - \langle m\rangle^2(H)} \tag{2}$$

Here m is the magnetization, H is the external field, N the number of sites. The angle brackets represent the thermal average and the overbar represent the average over the disorder. To obtain equation (1) one can follow the procedure in [6] which holds in the RFIM at the equilibrium. In the next part we will give only an idea of how the actual derivation works. It is not in the aim of this work to study the problem at a field theoretical level.

One starts by defining the bare action of the RFIM in a field theoretical framework under the effect of a ϕ^4-potential. The effective Hamiltonian $S[\sigma; h]$ is of the kind

$$S[\sigma; h] = \int_{\boldsymbol{x}} \left\{ \frac{1}{2}\left(|\boldsymbol{\partial}\sigma(\boldsymbol{x})|^2 + \tau\sigma(\boldsymbol{x})^2\right) + \frac{u}{4!}\sigma(\boldsymbol{x})^4 - h(\boldsymbol{x})\sigma(\boldsymbol{x}) \right\}$$

Where $\sigma(\boldsymbol{x})$ is our (for simplicity, scalar) field in d-dimensional space and $h(\boldsymbol{x})$ is the random magnetic field. The random fields are uncorrelated and characterized by a Gaussian distribution of 0 mean and variance

$$\overline{h(\boldsymbol{x})h(\boldsymbol{y})} = \Delta\delta(\boldsymbol{x}-\boldsymbol{y})$$

As is usually done in statistical mechanics, to find the equilibrium properties we look at the partition function. In this case

$$Z[J; h] = \int \mathcal{D}\sigma \exp\left\{ -S[\sigma; h] - \int_{\boldsymbol{x}} \sigma(\boldsymbol{x})J(\boldsymbol{x}) \right\} \tag{3}$$

Where $J(\boldsymbol{x})$ is an external source linearly coupled to the fundamental field and plays the role of the external magnetic field. This source is added in such a way that allows us to find the cumulants of the field by differentiating with respect to it. The thermodynamic properties are given by the average over the sample to sample fluctuations of the thermodynamic potential associated to the partition function $\overline{W[J,h]} = \overline{\log(Z[J,h])}$. However, if one needs to compute quantities that are effects of the sample to sample fluctuations this approach is not enough, and the higher cumulants of $W[J,h]$ are required. To obtain this information one can restore to the replica trick. This trick is based on rewriting the logarithm as

$$\log(Z) = \lim_{n\longrightarrow 0} \frac{Z^n - 1}{n}$$

The averaging of a power of Z can be physically interpreted as the average over a number n of replicas of the same system (i.e. with the same realization of the disorder). Our original system has been replaced by one containing n replicated fields $\{\sigma_a\}$. In the usual replica trick while performing the limit one face the problem of the spontaneous breaking of the replica symmetry. In this case we will instead break explicitly the symmetry by applying a different source J_a to each replica for the same reasons we previously cited below equation (3). The new replicated action $S^{(n)}$ is defined by the equation

$$\int \prod_{a=1}^{n} \mathcal{D}\sigma_a \exp\left(S^{(n)}[\{\sigma_a\}]\right) = \int \prod_{a=1}^{n} \mathcal{D}\sigma_a \overline{\exp\left(\sum_{a=1}^{n} S[\sigma_a; h]\right)}$$

And in our case correspond to

$$S^{(n)}[\{\sigma_a\}] = \int_{\boldsymbol{x}} \left\{ \frac{1}{2}\sum_{a=1}^{n}\left[|\boldsymbol{\partial}\sigma_a(\boldsymbol{x})|^2 + \tau\sigma_a(\boldsymbol{x})^2 + \frac{u}{12}\sigma_a(\boldsymbol{x})^4\right] - \frac{1}{2}\sum_{a,b=1}^{n} \Delta\sigma_a(\boldsymbol{x})\sigma_b(\boldsymbol{x}) \right\}$$

With the partition function given by

$$Z^{(n)}[\{J_a\}] = \int \prod_{a=1}^{n} \mathcal{D}\sigma_a \exp\left\{-S^{(n)}[\sigma_a] - \int_{\boldsymbol{x}} \sum_{a=1}^{n} \sigma_a(\boldsymbol{x}) J_a(\boldsymbol{x})\right\}$$

Where $\{J_a\}$ are the external sources introduced before. The related thermodynamic potential is $W^{(n)}[\{J_a\}] = \log(Z^{(n)}[\{J_a\}])$ and by a Legendre transform it is possible to find the expression of the effective action $\Gamma^{(n)}$

$$\Gamma^{(n)}[\{\phi_a\}] = -W^{(n)}[\{J_a\}] + \sum_{a=1}^{n} \int_{\boldsymbol{x}} J_a(\boldsymbol{x})\phi_a(\boldsymbol{x})$$

Where the fields $\{\phi_a\}$ and the sources $\{J_a\}$ are related by:

$$\phi_a(\boldsymbol{x}) = \langle \sigma_a(\boldsymbol{x})\rangle = \frac{\delta W^{(n)}[\{J_a\}]}{\delta J_a(\boldsymbol{x})} \qquad J_a(\boldsymbol{x}) = \frac{\delta \Gamma^{(n)}[\{\phi_a\}]}{\delta \phi_a(\boldsymbol{x})}$$

Where $\langle X\rangle$ is the average with the weight given by the replicated partition function $Z^{(n)}$. It is possible to show that $W^{(n)}[\{J_a\}]$ can be expanded as a function of the cumulants of the random functional $W[J;h]$ as

$$W^{(n)}[\{J_a\}] = \sum_{a=1}^{n} W_1[J_a] + \frac{1}{2}\sum_{a,b=1}^{n} W_2[J_a, J_b] + \dots$$

With

$$\begin{aligned} W_1[J_a] &= \overline{W[J_a;h]} \\ W_2[J_a, J_b] &= \overline{W[J_a;h]W[J_b;h]} - \overline{W[J_a;h]} \cdot \overline{W[J_b;h]} \end{aligned}$$

Where the term of first order is a sum over all replicas, the one of second order over all couples of replicas etc. Since $W^{(n)}$ and $\Gamma^{(n)}$ are linked by a Legendre transform it is possible to expand also $\Gamma^{(n)}$ as a sum over replica indices.

$$\Gamma^{(n)}[\{\phi_a\}] = \sum_{a=1}^{n} \Gamma_1[\phi_a] - \frac{1}{2}\sum_{a,b=1}^{n} \Gamma_2[\phi_a, \phi_b]$$

By using the Legendre transform it is then possible to perform an analogy between all the terms of the same order in $W^{(n)}$ and $\Gamma^{(n)}$ and to obtain, for the second order term:

$$\Gamma_2[\phi_1, \phi_2] = W_2[J[\phi_1], J[\phi_2]]$$

By defining the renormalized random field as

$$\tilde{h}[\phi](\boldsymbol{x}) = -\frac{\delta}{\delta\phi(\boldsymbol{x})}\Big(W[J[\phi(\boldsymbol{x})];h] - \overline{W[J[\phi(\boldsymbol{x})];h]}\Big)$$

we can compute the local effective disorder, which is the second cumulant of the local renormalized random field just defined:

$$\begin{aligned} \overline{\tilde{h}[\phi_1](\boldsymbol{x})\tilde{h}[\phi_2](\boldsymbol{y})} &= \frac{\delta^2}{\delta\phi_1(\boldsymbol{x})\delta\phi_2(\boldsymbol{y})} \overline{\Big[W[J[\phi_1(\boldsymbol{x})];h] - \overline{W[J[\phi_1(\boldsymbol{x})];h]}\Big]\Big[W[J[\phi_2(\boldsymbol{y})];h] - \overline{W[J[\phi_2(\boldsymbol{y})];h]}\Big]} \\ &= \frac{\delta^2}{\delta\phi_1(\boldsymbol{x})\delta\phi_2(\boldsymbol{y})} \overline{\Big[W[J[\phi_1(\boldsymbol{x})];h]W[J[\phi_2(\boldsymbol{y})];h] - \overline{W[J[\phi_1(\boldsymbol{x})];h]W[J[\phi_2(\boldsymbol{y})];h]}\Big]} \\ &= \frac{\delta^2}{\delta\phi_1(\boldsymbol{x})\delta\phi_2(\boldsymbol{y})} W_2[J[\phi_1], J[\phi_2]] = \Gamma^{(1,1)}_{2,\boldsymbol{xy}}[\phi_1, \phi_2] \end{aligned}$$

In this way one finds the definition of the local effective disorder in the field theoretical framework. By using the equations coming from the functional renormalization flow it is possible to show that the term that gives the disconnected propagator $W_2^{(1,1)}[J_1, J_2]$ respect the following equation

$$W^{(1,1)}_{2,\boldsymbol{xy}}[J_1, J_2] = \int_{\boldsymbol{z},\boldsymbol{z'}} W^{(2)}_{1,\boldsymbol{xz}}[J_1]\Gamma^{(1,1)}_{2,\boldsymbol{zz'}}[\phi[J_1], \phi[J_2]]W^{(2)}_{1,\boldsymbol{z'y}}[J_2]$$

Where the term $W^{(2)}_{1,\boldsymbol{xz}}[J_1]$ is the connected propagator. By integrating over the variables $\boldsymbol{x}$ and $\boldsymbol{y}$ and choosing uniform sources $J_1 = J_2 = J$ one finally obtains the relation

$$\chi_{dis}(J) = \Delta_{eff}(J)\chi^2_{con}(J)$$

Note that the derivation we just carried on holds for the equilibrium case of RFIM. However we will use in what follows the same equation for Δ_{eff} when considering the systems out of equilibrium.

Once that we have given a justification for the formula giving the effective disorder, we can proceed by computing it in the case of the two models under study.

3 Quasistatically driven Random Field Ising Model at T=0

The first model we present is the Random Field Ising model. This chapter is organized in the following way: we start with a brief introduction on the model and the properties in and out of equilibrium both for hard and soft spins. We then choose to examine the second case out of equilibrium and we introduce the equation (8) that will allow us to compute not only the average quantities but also higher cumulants. After finding the first result (the average magnetization) we focus a bit on the discussion of it and on the connected susceptibility. We eventually pass to the computation of the disconnected susceptibility and of the effective disorder.

The Ising model has historically been used as the toy model for the magnetization. Herein we have a lattice with a spin s_i taking values ± 1 situated at site i of N total sites. The Hamiltonian in the pure case is

$$\mathcal{H}_{pure}(\{s_i\}) = -J\sum_{\langle i,j\rangle} s_i s_j - H\sum_{i}^{N} s_i$$

Where the notation $\langle i,j\rangle$ means here that the summation runs over the couples of i, j which are nearest neighbors. H is the external field and J represent the interaction between the spins and can be either positive or negative, favoring ferromagnetism or antiferromagnetism respectively (here we consider only the case $J > 0$). At equilibrium, when $d \geq 2$ and $T < T_c$ (where T_c is the Curie temperature), the pure Ising model shows a ferromagnetically ordered phase. If one consider the application of an external field H all the spins will point downward for $H < 0$ and upward for $H > 0$, with a discontinuity at $H = 0$.

In the following we instead consider the case in which the system is brought out of equilibrium by imposing a local dynamics. This dynamics is the same of [2], where each spin will flip only when the effective field at its site changes sign. Notice that in this way one takes into account the process of nucleation, but this is not the only possible choice. Other choices, which link the RFIM to the problem of interface depinning, are also possible.

The effective field is defined as

$$h_i^{eff} = -\sum_j J_{ij}s_j - H$$

Where the sum over j runs over the nearest neighbors of i. It is clear that now the magnetization is history dependent and the system will be in some metastable state instead of being in the ground one. However we did not take into consideration the random part yet. By allowing all types of disorder the most general Hamiltonian would be:

$$\mathcal{H}[\{s_i\}] = -\sum_{i,j} J_{ij}s_i s_j - \sum_{i=1}^{N} h_i s_i$$

The case we want to focus on is the one where the quantity J_{ij} is equal for every nearest neighbor couple and takes a positive value J. Moreover we consider the random fields h_i to be uncorrelated and drawn from a Gaussian distribution with 0 mean and variance Δ. It can be shown (for example in [3] and references therein) that choosing a different distribution does not change the critical behavior of the RFIM, as long as the distribution has a finite variance. Choosing a Gaussian leads to simpler computations and is somehow justified if one wants to think of the spins as coarse-grained quantities to which the central limit theorem apply. The Hamiltonian of this model is given by

$$\mathcal{H}[\{s_i\}] = -J\sum_{\langle i,j\rangle} s_i s_j - \sum_{i=1}^{N}(h_i + H)s_i$$

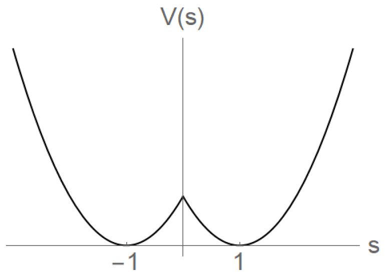

Figure 3: Potential acting on the soft spins. The minima at -1 and 1 favor the spins to take the hard case values. Plot for the case k=2.

When treating the equilibrium case it can be shown that this system has a critical curve in the $2D$ plane of temperature and disorder. At 0 temperature there could be an ordered phase if the strength of the disorder (its variance) is small enough. If the disorder has a large variance one will instead remain in the paramagnetic case. This can be interpreted in the following way: if at some time one of the spin flips, the effect of this flipping has a higher probability of flipping the others (and hence to create an avalanche) when the disorder is small.

Again we want to focus on the out of equilibrium case. We apply an external magnetic field H that increases quasistatically of an infinitesimal quantity dH. The disorder slightly modifies the equations for the dynamics into

$$h_i^{eff} = -\sum_j J_{ij} s_j - H - h_i$$

What mentioned above holds when spins are binary objects taking values ± 1. In what follows we instead take into consideration a different case, the one of soft spins. In this framework each spin s_i can take any real value between $-\infty$ and $+\infty$ and can be imagined as a coarse grained version of the hard spin. This choice is justified from the fact that in the other case, when the disorder is larger than the critical value, the magnetization curve follows the same path both in the ascending and in the descending field. This lack of memory is only an artifact of the hard spin model, and does not happen when instead one choose to study soft spins, where a history dependence is present for every value of the disorder.

To mimic the two spin states we assume that the spins move in a double well potential given by

$$V(s_i) = \begin{cases} \frac{k}{2}(s_i+1)^2 & s_i < 0 \\ \frac{k}{2}(s_i-1)^2 & s_i > 0 \end{cases} \tag{4}$$

The potential is shown in figure (3). To have a finite magnetization at each value of the field we have to impose the conditions $k > 0$ and $k > J$. $V(s_i)$ is an approximation of the usual quartic potential, but it gives close results and it is easier to treat.

Since we want to stand in the out of equilibrium regime, we have to choose the initial condition of the dynamics. Depending on whether we choose to start with $H \to +\infty$ or $H \to -\infty$ every spin will be pointing respectively up or down at the beginning. This choice divides the dynamic of the magnetization in two branches, the ascending and the descending one. For symmetry reasons one branch is equal to the other shifted by the difference of the minima in the potential (in our case 2), so we can restrict our study to only one of these cases. In particular we choose to focus on the ascending branch, so that $H_{in} \to -\infty$ at the beginning. As we anticipated, we study the problem at a mean field, fully connected level. In this case the Hamiltonian becomes

$$\mathcal{H}_{soft}[\{s_i\}] = -\sum_{i=1}^{N} \Big[\big(Jm + H + h_i\big) s_i - V(s_i) \Big]$$

Where m is the magnetization $m = \frac{1}{N}\sum_{i=1}^{N} s_i$.

Since we chose to study the increasing H case, our system will begin by being subject to the lower well Hamiltonian because $s_i < 0 \quad \forall i$. The Hamiltonian is then

$$\mathcal{H}^{(-)}_{soft}[\{s_i\}] = \sum_{i=1}^{N} \left[\frac{k}{2}(s_i+1)^2 - (H + Jm + h_i)s_i \right]$$

Which has a minimum for negative s_i. This minimum exists until the derivative of the Hamiltonian computed at $s_i = 0$ is greater or equal than 0. If this is the case, the Hamiltonian is increasing when reaching 0, but also tends to ∞ when $s_i \to -\infty$. This means that there is a minimum for $s_i < 0$. This condition, for a generic i, is given by

$$\frac{\delta}{\delta s_i}\left[\frac{k}{2}(s_i+1)^2 - \big(H + h_i + Jm\big)s_i\right]_{s_i=0} \geq 0$$

By finding the minima in the two cases and solving the equation above we can conclude that, for the ascending branch, the mean field equations for each spin s_i are given by:

$$s_i^\alpha = \begin{cases} \frac{H+h_i+Jm^\alpha}{k} + 1, & h_i > -Jm^\alpha - H + k, \\ \frac{H+h_i+Jm^\alpha}{k} - 1, & h_i < -Jm^\alpha - H + k. \end{cases} \tag{5}$$

Notice that we added a superscript α, until now neglected, to stress out the fact that s_i and m are both dependent on the particular realization of the disorder and hence on the particular sample α. From the expression of the single spin one can obtain a formula for the magnetization, that still depends on the sample under consideration

$$m^\alpha = \frac{1}{N}\sum_{i=1}^{N} s_i^\alpha = \frac{1}{N}\left\{\sum_{i=1}^{N}\frac{H+h_i+Jm^\alpha}{k} + \sum_{i=1}^{N}\left[-1 + 2\theta\big(h_i + H + Jm^\alpha - k\big)\right]\right\} \tag{6}$$

$$= -1 + \frac{H+Jm^\alpha}{k} + \frac{1}{N}\sum_{i=1}^{N}\left[\frac{h_i}{k} + 2\theta\big(h_i + H + Jm^\alpha - k\big)\right] \tag{7}$$

Now that we have the equation that regulate the dynamic of the system we can proceed by searching for a method to compute the cumulants of the magnetization. What we aim at is the effective disorder. We are then interested both in the computation of the disconnected and connected susceptibilities. We use the equation in (2) for the disconnected part. For χ_{con} the situation is a bit more complicate. Since we study both the systems at 0 temperature there are no thermal fluctuations, but we can still compute the susceptibility as

$$\chi_{con}(H) = \frac{d\overline{m^\alpha}(H)}{dH}$$

Then we need a way to compute the average of quantities (we need the average magnetization to obtain the connected part) but also higher order moments (for the disconnected). For this reason we devise a method that allows us to compute the average for a general function $f(m^\alpha)$. Any function of this kind can be rewritten as

$$f(m^\alpha) = \int_{-\infty}^{+\infty} dm f(m)\delta(m - m^\alpha)$$

The usefulness of the formula above is evident when one rewrites the delta function as the delta of a function which is 0 when $m = m^\alpha$ times the modulus of the Jacobian of that function: $\delta(m - m^\alpha) = \delta(\mathcal{F}(m)) \cdot |\mathcal{F}'(m)|$ if $\mathcal{F}(m) = 0$ when $m = m^\alpha$. In our case we have from equation (6) that

$$\mathcal{F}(m) = m + 1 - \frac{H+Jm}{k} - \frac{1}{N}\sum_{i=1}^{N}\left[\frac{h_i}{k} + 2\theta(h_i + H + Jm - k)\right]$$

Then we can proceed to rewrite the formula for $f(m^\alpha)$ as

$$f(m^\alpha) = \int_{-\infty}^{+\infty} dm f(m)\delta(\mathcal{F}(m)) \cdot |\mathcal{F}'(m)| \tag{8}$$

with

$$\mathcal{F}'(m) = 1 - \frac{J}{k} + \frac{2}{N}\sum_{i=1}^{N} J\delta(h_i + H + Jm - k)$$

Since we consider the case $k > J$ and the summation term scales as $1/N$ we omit the absolute value. Moreover, since this term is of order 1 we do not consider its contribution (it is however rather simple, but really long, to show that it is in fact negligible).

To reach our goal we now look at what happens when one tries to compute the average over all possible realizations of the disorder of the function $f(m^\alpha)$. This corresponds to compute:

$$\overline{f(m^\alpha)} = \int_{-\infty}^{+\infty} \prod_{i=1}^{N} dh_i \rho(\{h_i\}) f(m^\alpha)$$

Since the fields are taken to be independent identically distributed we have the relation $\rho(\{h_i\}) = \prod_{i=1}^{N} \rho(h_i)$. We can also express the delta function through its Fourier representation:

$$\delta(x) \propto \int_{-\infty}^{+\infty} dk e^{-ikx} \propto \int_{-i\infty}^{+i\infty} dk e^{kx}$$

Where we neglect the further i that would appear because it simplifies when one goes back to a real-valued k.

We choose $k = \lambda N$ and we neglect the constants that going forward appear in front of the integral. We will later ensure normalization by imposing $\overline{1} = 1$. Then we have

$$\overline{f(m^\alpha)} \propto \int_{-\infty}^{+\infty} \prod_{i=1}^{N} (dh_i \rho(h_i)) \int_{-\infty}^{+\infty} dm f(m) \int_{-i\infty}^{+i\infty} d\lambda e^{N\lambda \mathcal{F}(m)} \tag{9}$$

$$= \int_{-\infty}^{+\infty} dm f(m) \int_{-i\infty}^{+i\infty} d\lambda e^{N\lambda\left[m+1-\frac{H+Jm}{k}\right]} \int_{-\infty}^{+\infty} \prod_{i=1}^{N} (dh_i \rho(h_i)) e^{-\frac{\lambda}{k}\sum_{i=1}^{N} h_i - 2\lambda \sum_{i=1}^{N} \theta(h_i+H+Jm-k)} \tag{10}$$

$$= \int_{-\infty}^{+\infty} dm f(m) \int_{-i\infty}^{+i\infty} d\lambda e^{N\lambda\left[m+1-\frac{H+Jm}{k}\right]} \left[\int_{-\infty}^{+\infty} dh \rho(h) e^{-\frac{\lambda}{k}h - 2\lambda\theta(h+H+Jm-k)}\right]^N \tag{11}$$

Where the quantity between square brackets will be called $I(m, \lambda)$. After exploiting the θ function we have

$$I(m,\lambda) = \int_{-\infty}^{-H-Jm+k} dh e^{-\frac{\lambda}{k}h} \rho(h) + \int_{-H-Jm+k}^{+\infty} dh e^{-\frac{\lambda}{k}h - 2\lambda} \rho(h)$$

Then our average will be

$$\overline{f(m^\alpha)} \propto \int_{-\infty}^{+\infty} dm f(m) \int_{-i\infty}^{+i\infty} d\lambda e^{N\mathcal{G}(m,\lambda)}$$

with

$$\mathcal{G}(m,\lambda) = \lambda \left[m + 1 - \frac{H+Jm}{k}\right] + \log(I(m,\lambda))$$

Since we are interested in the thermodynamic limit, we will consider $N \to \infty$ and we can continue with a saddle node approximation. To do so we need to find the maximum value of the function $\mathcal{G}(m, \lambda)$. Since it is a function of two variables we maximize with respect to both of them. In the following the notation $\mathcal{G}^{(1,0)}(m, \lambda)$ will be used to denote the first derivative of $\mathcal{G}$ with respect to the first argument (m). One finds

$$\mathcal{G}^{(1,0)}(m,\lambda) = \lambda(1 - \frac{J}{k}) + \frac{1}{I(m,\lambda)}(e^{-2\lambda} - 1) J e^{\frac{\lambda}{k}(H+Jm-k)} \rho(H + Jm - k)$$

Imposing $\mathcal{G}^{(1,0)} = 0$ gives the trivial solution $\lambda^* = 0$. We now pass to the derivative with respect to λ

$$\mathcal{G}^{(0,1)}(m,\lambda) = m+1-\frac{H+Jm}{k} + \frac{\int_{-\infty}^{-H-Jm+k} dh \rho(h) e^{-\frac{\lambda}{k}h} \left(-\frac{h}{k}\right) + \int_{-H-Jm+k}^{+\infty} dh \rho(h) e^{-(\frac{\lambda}{k}h+2\lambda)} \left(-\frac{h}{k} - 2\right)}{I(m,\lambda)}$$

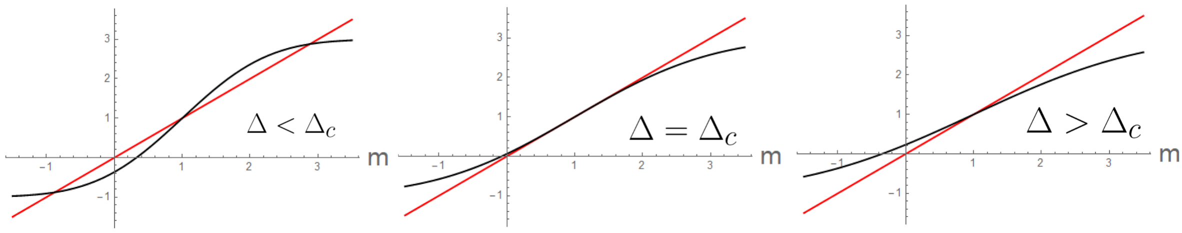

Figure 4: Plots of the left hand side (red) and right hand side (black) of equation (12) at fixed external field $H = 1$. The plots are performed taking $J = 1$ and $k = 2$.

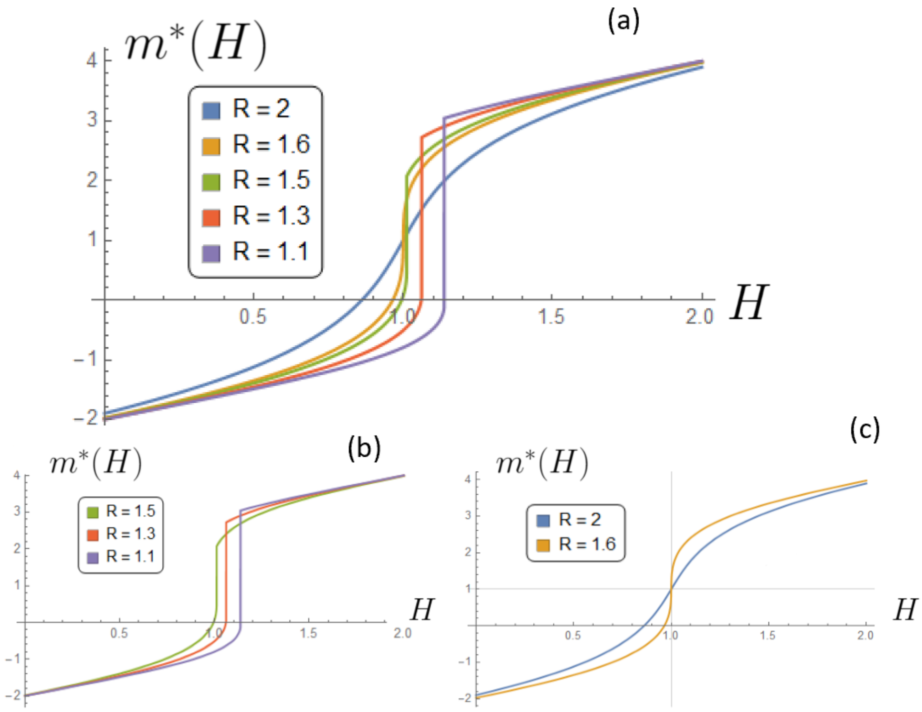

Figure 5: (a) Plots of the magnetization as a function of H for $R = \{2, 1.6, 1.5, 1.3, 1.1\}$. (b) magnetization when $R < R_c$ and (c) when $R > R_c$.

We evaluate this function at $\lambda = \lambda^* = 0$ to obtain the value of m that maximizes the function $\mathcal{G}$:

$$m^*(H) = \frac{k + H}{k - J} - \frac{2k}{k - J} \int_{-\infty}^{-H - Jm^*(H) + k} dh\rho(h) \tag{12}$$

The equation above provides a recursive relation for the value of m^* and is the same obtained in [2]. It is possbile to show, choosing $f(m) = m$, that the m^* obtained in this way represents the average value of the magnetization $\overline{m^\alpha}$ over the disorder since it is the one that maximizes the function $\mathcal{G}$ and hence the only one surviving when $N \to \infty$. One can further prove (we prove it later, equation (14)) that this relation gives an analytic result as long as the standard deviation of the noise $R = \sqrt{\Delta}$ is greater than $R_c = \sqrt{\Delta_c} = \sqrt{2/\pi} J(k/(k - J))$. When instead $\Delta < \Delta_c$ the equation has three possible results in a small range of the external field, two of which are stable and one is unstable (figure (4)). This leads to an unphysical behavior where the magnetization decreases with an increasing field. What happens in the physical system is that the average magnetization remains in the lower branch until it is stable and then it jumps to the other stable fixed point. The results for the case $J = 1$ and $k = 2$ are shown in figure (5). With our choice of J and k the resulting critical value for R is $R_c = 1.5957$, so we decide to use for our plots the following values for R:$\{2, 1.6, 1.5, 1.3, 1.1\}$.

One can notice the symmetry of the magnetization with respect to the point (1,1) in figure (5.(c)), where the strength of the disorder is large enough to prevent a macroscopic jump in the magnetization. In the same way one can observe the lack of symmetry in figure (5.(b)) where instead the disorder is weaker. The proof of this is given in appendix A). Another important property of the magnetization can be seen in this figure: the value of H at which the jump occurs, H_c, depends on the disorder. It increases with the weakness of the disorder until it reaches the value of $H_c = 2$ when the initial distribution is a δ centered in 0. The physical interpretation is that when the variance is smaller it takes an higher field to flip the spin with the smallest value of the local field.

Now that we have the value of the magnetization we can proceed to compute the first quantity we need, the connected susceptibility.

3.1 Connected susceptibility

The connected susceptibility is the average over the sample-to-sample fluctuations of the usual susceptibility we find in pure systems. To compute it we use the definition with the derivative of $m^*(H)$, which is the average magnetization, with respect to the external field:

$$\chi_{con} = \frac{dm^*(H)}{dH}$$

By using equation (12) for the value of m^* we have

$$\chi_{con} = \frac{dm^*}{dH} = \frac{1}{k-J} + \frac{2kJ}{k-J}(1 + J\frac{dm^*}{dH})\rho(H + Jm^* - k)$$

From which

$$\chi_{con} = \frac{1 + 2k\rho(H + Jm^* - k)}{k - J - 2kJ\rho(H + Jm^* - k)} \tag{13}$$

From this formula we can deduce a couple of properties of the magnetization. First of all we see that χ_{con} could diverge (the denominator can be 0). This divergence is directly correlated to the jump of the magnetization. Then we can infer that this jump is not always possible. As we said before, only for certain values of the disorder one can observe a discontinuity in the magnetization. If indeed the disorder is too strong, the peak of the distribution will not be large enough to cancel the denominator. To find the value of Δ_c one needs to impose that the peak of the Gaussian is at least large enough to let the denominator vanish. This gives the condition

$$\max\Big(\rho(H + Jm^*(H) - k)\Big) > \frac{k-J}{2kJ}$$

The max of a gaussian distribution is at the mean (0 in this case) and is given by $\frac{1}{\sqrt{2\pi\Delta}}$. We obtain the condition

$$\frac{1}{\sqrt{2\pi}R} > \frac{k-J}{2kJ} \quad \rightarrow \quad R_c = \sqrt{\frac{2}{\pi}}\frac{kJ}{k-J} \tag{14}$$

The same divergence is present also in the disconnected susceptibility and the effective disorder has indeed a finite value at every H. We can now proceed by computing the disconnected part of the susceptibility to eventually obtain the effective disorder.

3.2 Disconnected susceptibility and effective disorder

Up to now we computed the average magnetization and the connected susceptibility. In the next step towards the effective disorder we need to find the expression for the disconnected susceptibility. As we said before this quantity is defined as

$$\chi_{disc} = N\Big[\overline{m^2} - \overline{m}^2\Big]$$

We know that $\overline{m^2} = \overline{m}^2 + \mathcal{O}(\frac{1}{N})$ which means that χ_{dis} is of order 1. To compute the quantity inside the square brackets we need to find the expression for $\overline{m^2}$, but, in doing so, we need to consider its fluctuations which will be of order $1/N$. It follows that we should consider also the expansion around the fixed points of the function $\mathcal{G}$. We know that the fluctuations of m and λ around the fixed points will be of order $1/\sqrt{N}$, then we expand near (m^*, λ^*) as

$$m = m^* + \frac{\delta m}{\sqrt{N}}$$
$$\lambda = \lambda^* + \frac{\delta \lambda}{\sqrt{N}}$$

Then we have

$$\overline{f(m^\alpha)} \propto \int_{-\infty}^{+\infty} dm \int_{-i\infty}^{+i\infty} d\lambda f(m) e^{N\mathcal{G}(m,\lambda)} = \int_{K}^{K} d(\delta m) \int_{-iK'}^{iK'} d(\delta\lambda) f(m^* + \frac{\delta m}{\sqrt{N}}) e^{N\mathcal{G}(m^* + \frac{\delta m}{\sqrt{N}}, \frac{\delta\lambda}{\sqrt{N}})}$$

Where K and K' are two constants that will eventually be sent to ∞. To impose the correct normalization we only need to change our definition of $\overline{f(m^\alpha)}$ by dividing everything by

$$\mathcal{Z} = \int_{K}^{K} d(\delta m) \int_{-iK'}^{iK'} d(\delta\lambda) e^{N\mathcal{G}(m^* + \frac{\delta m}{\sqrt{N}}, \frac{\delta\lambda}{\sqrt{N}})}$$

In this way we have $\overline{1} = 1$. We now proceed to expand in power series of δm and of $\delta\lambda$

$$\begin{aligned} \overline{f(m^\alpha)} = \frac{1}{\mathcal{Z}} \int_{-K}^{K} d(\delta m) \int_{-iK'}^{iK'} d(\delta\lambda) \left[f(m^*) + \frac{f'(m^*)}{\sqrt{N}} \delta m + \frac{f''(m^*)}{2N} \delta m^2 \right] \cdot \\ \cdot \exp\left\{ N \left[\mathcal{G}_* + \frac{1}{2N} \left(\mathcal{G}_*^{(2,0)} \delta m^2 + 2\mathcal{G}_*^{(1,1)} \delta m \delta\lambda + \mathcal{G}_*^{(0,2)} \delta\lambda^2 \right) \right] \right\} \end{aligned} \tag{15}$$

Where we used the subscript $*$ to stress the fact that $\mathcal{G}$ and its derivatives are computed at the point (m^*, λ^*). For the same reason we avoid writing down the first derivatives, since they are 0 at the maximum. We now compute the second derivatives of $\mathcal{G}$.

- **Second derivative with respect to m:**

$$\mathcal{G}^{(2,0)}(m, \lambda) = \frac{1}{I(m, \lambda)^2} \left[I(m, \lambda) I^{(2,0)}(m, \lambda) - (I^{(1,0)}(m, \lambda))^2 \right]$$

  However one can easily show that both $I^{(2,0)}(m, \lambda)$ and $I^{(1,0)}(m, \lambda)$ are zero when computed at $\lambda^* = 0$, so that $\mathcal{G}_*^{(2,0)}$ is 0. This is good news since otherwise the integral in δm could diverge.

- **Mixed derivative:**

$$\mathcal{G}^{(1,1)} = \left(1 - \frac{J}{k}\right) + \frac{1}{I(m, \lambda)^2} \left[I(m, \lambda) I^{(1,1)}(m, \lambda) - I^{(1,0)}(m, \lambda) I^{(0,1)}(m, \lambda) \right]$$

  By computing it at the maximum we obtain

$$\mathcal{G}_*^{(1,1)} = \left(1 - \frac{J}{k}\right) - 2J\rho(H + Jm^* - k)$$

- **Second derivative with respect to λ:**

$$\mathcal{G}^{(0,2)}(m, \lambda) = \frac{1}{I(m, \lambda)^2} \left[I(m, \lambda) I^{(0,2)}(m, \lambda) - (I^{(0,1)}(m, \lambda))^2 \right]$$

  Which gives

$$\mathcal{G}_*^{(0,2)} = \frac{\Delta^2}{k^2} + 4 \int_{-H-Jm^*+k}^{+\infty} dh \rho(h) + 4 \int_{-H-Jm^*+k}^{+\infty} dh \rho(h) \frac{h}{k} - 4 \left[\int_{-H-Jm^*+k}^{+\infty} dh \rho(h) \right]^2$$

  Which is > 0

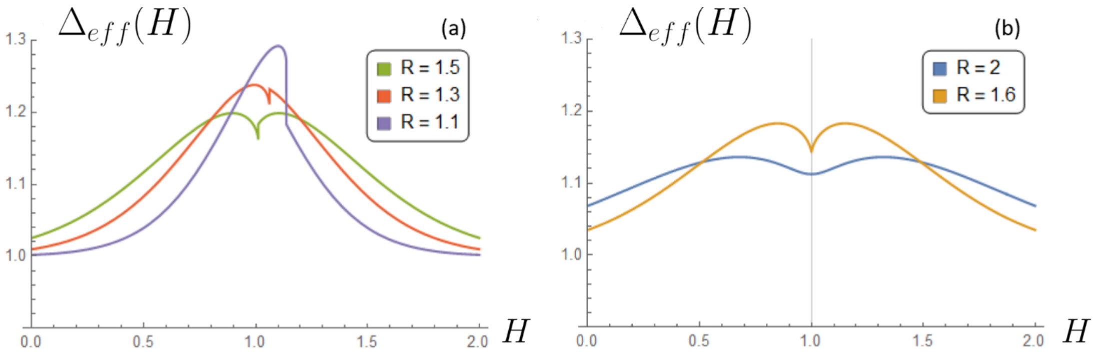

Figure 6: Plots of the effective disorder in the out of equilibrium RFIM. Divided in (a) supercritical strength of the disorder and (b) subcritical. In the figure (b) one can see that the axis $H = 1$ is an axis of symmetry for the effective disorder

We can now pass again to a real valued variable λ, by substituting $\lambda \to i\lambda$. The same is true for the part of exponential which is constant in δm and $\delta\lambda$. We then obtain:

$$\begin{aligned}\overline{f(m^\alpha)} = &\frac{1}{\mathcal{Z}'}\int_K^K d(\delta m)\int_{-K'}^{K'} d(\delta\lambda)\left[f(m^*) + \frac{f'(m^*)}{\sqrt{N}}\delta m + \frac{f''(m^*)}{2N}\delta m^2\right]\cdot \\ &\cdot \exp\left\{\frac{1}{2}\left[2i\mathcal{G}_*^{(1,1)}\delta m\delta\lambda - \mathcal{G}^{(0,2)}\delta\lambda^2\right]\right\}\end{aligned} \tag{16}$$

By sending K' to infinity we can compute the Gaussian integral. Since the extrema of the integral are symmetric the term in the first derivative of f gives 0 because it multiply an odd power of δm. We remain with:

$$\overline{f(m^\alpha)} = f(m^*) + \frac{1}{\mathcal{Z}'}\int_{-K}^{K} d(\delta m)\frac{f''(m^*)}{2N}\delta m^2 \exp\left\{-\frac{\left(\mathcal{G}_*^{(1,1)}\right)^2}{2\mathcal{G}_*^{(0,2)}}\delta m^2\right\}$$

We are now ready to substitute the f we want to compute, which is $f(m) = m^2$. This means that $f''(m^*) = 2$. Finally, sending also K to infinity we have:

$$\overline{(m^\alpha)^2} = \overline{m^\alpha}^2 + \frac{1}{N}\frac{\mathcal{G}^{(1,1)}(\overline{m^\alpha},0)^2}{2\mathcal{G}^{(0,2)}(\overline{m^\alpha},0)}$$

The disconnected susceptibility is then

$$\chi_{disc} = N\left(\overline{(m^\alpha)^2} - \overline{m^\alpha}^2\right) = \frac{\mathcal{G}^{(1,1)}(\overline{m^\alpha},0)^2}{2\mathcal{G}^{(0,2)}(\overline{m^\alpha},0)}$$

The final step of these long computations is the effective disorder. By using this results we obtained and the formula for the effective disorder we have, for the quasistatically driven, $T = 0$, Random Field Ising model:

$$\Delta_{eff}(H) = \frac{\chi_{disc}(H)}{\chi^2_{conn}(H)} = \frac{\left[\frac{\Delta}{k^2} + 4\int_{-H-Jm^*(H)+k}^{+\infty} dh\rho(h)\int_{-\infty}^{-H-Jm^*(H)+k} dh\rho(h) + 4\int_{-H-Jm^*(H)+k}^{+\infty} dh\frac{h}{k}\rho(h)\right]}{\left[\frac{1}{k} + 2\rho(H + Jm^*(H) - k)\right]^2}$$

One can see that in the extreme cases of $H \to \infty$ and $H \to -\infty$ the effective disorder is exactly equal to R^2 which is the bare one. To compare the results for the different values of R we plot the quantity $\tilde{\Delta}_{eff}(H) = \Delta_{eff}(H)/\Delta$. The plots are shown in figure (6). We see that, again, when the disorder is strong the effective disorder is symmetric with respect to $H = 1$. In the other case instead Δ_{eff} shows a jump at a value of H that increases with the weakness of the disorder. This is due to the jump in the magnetization.

Before passing to the main model, we can say something about the RFIM in the equilibrium case.

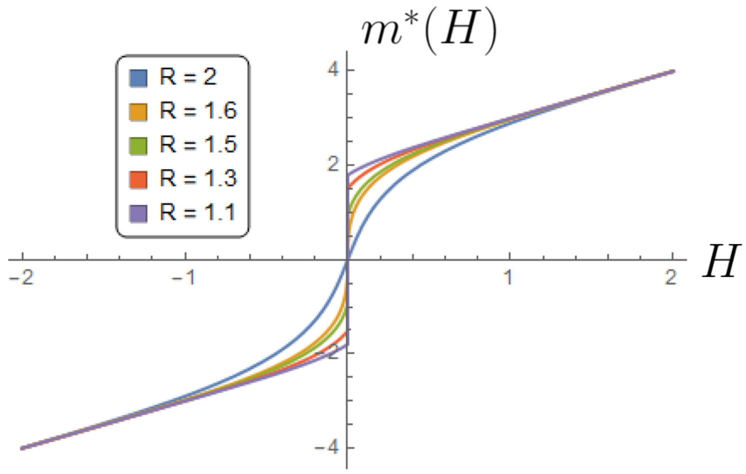

Figure 7: Plots of the magnetization at equilibrium as a function of H for $R = \{2, 1.6, 1.5, 1.3, 1.1\}$.

3.3 Equilibrium

After these computations one can wonder if it is possible to say something about the case of equilibrium. What changes is that in the previous part we had two different curves (hysteresis) depending on whether we decided to start with $H_{in} \to +\infty$ or $H_{in} \to -\infty$. In the case at equilibrium we instead have only one curve that represent the value of the spin as a function of the effective field, and this curve will, by symmetry, have a discontinuity at $h_{eff} = 0$. This changes equation (5) into

$$s_i^\alpha = \begin{cases} \frac{H+h_i+Jm^\alpha}{k} + 1, & h_i > -Jm^\alpha - H, \\ \frac{H+h_i+Jm^\alpha}{k} - 1, & h_i < -Jm^\alpha - H. \end{cases}$$

From here however, the same reasoning can be applied and the resulting equilibrium magnetization is then

$$m^*_{eq}(H) = \frac{k+H}{k-J} - \frac{2k}{k-J} \int_{-\infty}^{-H-Jm^*_{eq}(H)} dh\rho(h) \tag{17}$$

The resulting plots are shown in figure (7). Again also here the magnetization is symmetric ($m^*_{eq}(H) = -m^*_{eq}(-H)$), but this time the symmetry is not lost when $\Delta < \Delta_c$. The jump moreover is always at $H = 0$ for the same symmetry reasons. This slightly changes the equations we obtained before, but the methods we used are still applicable and lead to new connected and disconnected susceptibilities. The new effective disorder will be given by

$$\Delta_{eff}(H)\Big|_{eq} = \frac{\left[\frac{R^2}{k^2} + 4\int_{-H-Jm^*_{eq}(H)}^{\infty} dh\rho(h) \int_{-\infty}^{-H-Jm^*_{eq}(H)} dh\rho(h) + 4\int_{-H-Jm^*_{eq}(H)}^{\infty} dh\frac{h}{k}\rho(h)\right]}{\left[\frac{1}{k} + 2\rho(H + Jm^*_{eq}(H))\right]^2} \tag{18}$$

The relative plot is shown in figure (8). Also in this case it can be proven that the form of effective disorder grant symmetry between H and $-H$ (proof in appendix A). The difference with the previous case is that now this symmetry is maintained for all the values of the disorder. The jump in the magnetization is what cause the cusp to appear in the curve of the effective disorder. However now the effective disorder tends to the same value when $H \to 0$ from left and from right, while before this was not the case. The disorder is therefore continuous but its derivative it's not unless $\Delta > \Delta_c$.

Now that we have this result we can pass to treat the central topic of this work: the study of the amorphous materials through the Elasto-Plastic model.

4 Elasto-Plastic model

The study of amorphous materials differs consistently from the case of crystalline ones, where we have an underlying periodic lattice that simplifies the modeling. To face the problem of the mechanical response

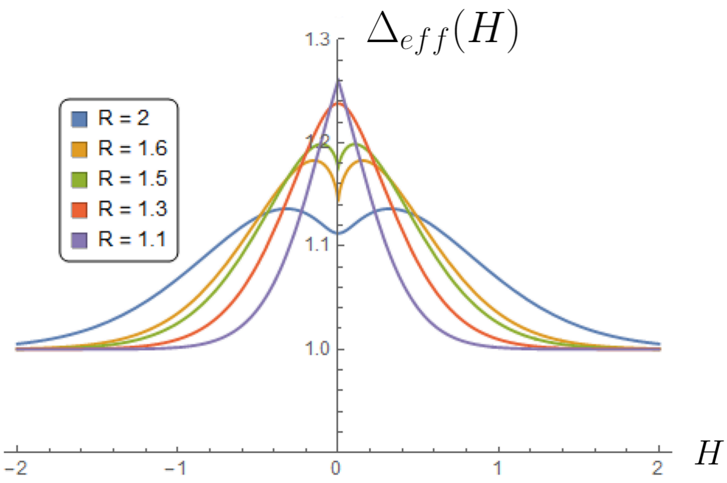

Figure 8: Plots of the effective disorder at equilibrium as a function of H for $R = \{2, 1.6, 1.5, 1.3, 1.1\}$.

of amorphous solids one can instead appeal to what are called Elasto-Plastic models. Lots of these models have been devised to obtain always more realistic properties, we will list some of the general key points before showing and explaining our choices and performing the calculations.

The idea behind these models is looking at the solid at a mesoscopic scale by dividing it into blocks which are small enough to be considered in the thermodynamical limit, but not enough to let quantum mechanics play a role. Each block (represented as a site in a d-dimensional lattice) has a value associated to it representing its local stress σ_i. Since we want to work in a strain controlled framework, the average stress $\sigma^\alpha(\gamma)$ of the material is our quantity of interest and is given by:

$$\sigma^\alpha(\gamma) = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^\alpha(\gamma)$$

Where, here and in what follows, the α means that the quantity is sample dependent.

The elastic deformation acts by increasing the local stress of each block linearly with the applied strain γ, with a coefficient which represents the elastic modulus of the material. The value of the local stress is upper-bounded by a local threshold σ_{th_i}. Once σ_i exceeds this value, the block is said to yield, and its stress will drop to a random value drawn from a distribution $g(\sigma)$. This feature is included to mimic the plastic rearrangements that occur locally and that decrease the stress of the yielding block while affecting also the value of other blocks. The procedure of yielding is carried on by the deformation of the block itself, i.e. by the change in the configuration of the particles composing it. The principal mechanism behind this deformation is the one called T1 event, which is shown in figure (2). Due to the particular anisotropic form of this kind of events, the blocks which are near the yielding one can see their stress increase or decrease depending on whether the particle at the boundary of the two blocks pushes or pulls the others. Then, in the EPM, one chooses the drop in the stress of the yielding block to be redistributed among the other blocks through what is called an elastic propagator. The choice for the propagator that better mimics what we said above is the Eshelby form:

$$G_{ij} = \frac{\cos(4\theta)}{|\boldsymbol{r_i} - \boldsymbol{r_j}|^d} \tag{19}$$

Where d is the dimensionality of the space we are in and θ is the angle between the two vectors $\boldsymbol{r_i}$ and $\boldsymbol{r_j}$. The anisotropy is clearly reproduced by the cosine at the numerator.

By considering all these features the model is pretty complete and realistic, but it surely is not simple. During this project we indeed chose to avoid adding too many details and we instead decided to rely on some approximations. First of all we decide to investigate the model by using the mean-field fully connected simplification. In this simplification one looses the concept of angle and the Eshelby form of interaction is no longer suitable. At this point we choose the rearrangements to always increase the stress of all the blocks uniformly. Another possibility would be the choice of randomly deciding whether to increase or decrease the value at each block after a plastic event, but in this case one would insert a further source of noise in the system, and it is not trivial to understand how the different sources interact between them (and in which order perform the average). This reduces the elastic propagator introduced before to the following form

$$G_{ij} = G = \frac{\mu_2}{N(\mu_1 + \mu_2)}$$

We will also adopt some additional hypotheses. First of all we study the system in a strain controlled framework, where the independent variable is γ and the variable under study is $\sigma(\gamma)$. Secondly we only choose a fixed threshold for each site $\sigma_{th_i} = \sigma_{th}$ and without loosing generality we choose $\sigma_{th} = 1$. Moreover we consider the case in which the stress of a yielding block drops to the same quantity $\overline{\sigma}$ every time it yields, which means $g(\sigma) = \delta(\sigma - \overline{\sigma})$ (a different choice is taken in [5] where instead $g(\sigma) = \exp(-\sigma/\overline{\sigma})$).

It is important to notice one thing at this point: all the simplifications we are listing have the effect of removing sources of disorder from the system. This allows us to only study the effect that the initial configuration has on the sample, but has the drawback of making our model much less realistic. Since the only source of disorder is the one present at the beginning (and then each block goes to the same value after undergoing a plastic event) the evolution is completely deterministic. In particular, after a while, we end up in a periodic dynamics which is clearly unphysical and meaningless. The choice of $g(\sigma) = \delta(\sigma - \overline{\sigma})$ hence prevents the solid from reaching a stationary state, but this is not a problem since we are only interested in the transient regime and not in the steady flow that appears at higher values of γ. In our case the randomness is indeed only given by the initial value of the stress at each site when $\gamma = 0$, $\sigma_i(\gamma = 0) \sim P_{ini}(\sigma)$.

Now that we have discussed the generalities about the Elasto-Plastic models, and we have defined our choices, we can proceed with the calculations.

The stress at each site responds linearly to an increase in the strain following the rule

$$\sigma_i \to \sigma_i + 2\mu_2\delta\gamma$$

The blocks interact among themselves through the elastic propagator G. When the block i undergoes yielding, it affects the other blocks as

$$\sigma_j \to \sigma_j + G(1 - \overline{\sigma})$$

Since we made the fixed-threshold choice, it is more simple to proceed on by working with the quantity $x_i = \sigma_{th} - \sigma_i = 1 - \sigma_i$ instead of the stress. This quantity is clearly bounded from below by $x_i = 0$ and the block i yields when this occur. After yielding the value of x_i goes to $\overline{x} = 1 - \overline{\sigma}$. The rules change as follows:

- $x_i \to x_i - 2\mu_2\delta\gamma$ when $x_i > 2\mu_2\delta\gamma$, otherwise i yields and $x_i \to \overline{x}$
- $x_j \to x_j - G\overline{x}$ if a block $i \neq j$ yields

We use as the order parameter the average value of this quantity which is given by:

$$m^\alpha(\gamma) = \frac{1}{N}\sum_{i=1}^{N} x_i^\alpha(\gamma) = 1 - \sigma^\alpha(\gamma)$$

Called m because it plays in the EPM model the same role of the magnetization in Ising.

The quantity we aim to compute is again the effective disorder, which can be obtained from the ratio between disconnected susceptibility and the connected one (squared). To reach this goal we introduce the function that express the fraction of sites i having a stress such that $0 < x_i < x$, called $f_\gamma^\alpha(x)$. Clearly this quantity will depend on the particular realization of the disorder we are looking at as well as at which strain we are considering it. Each variable x_i will indeed be a function of the applied strain $x_i = x_i(\gamma)$. We can express it by mean of theta functions as

$$f_\gamma^\alpha(x) = \frac{1}{N}\sum_{i=1}^{N} \theta(x - x_i^\alpha(\gamma)) \tag{20}$$

As we said before, at each step $\gamma \to \gamma + \delta\gamma$ the quantity x_i will change as $x_i \to x_i - 2\mu_2\delta\gamma$ for every i. But this infinitesimal change given by eternal applied strain, will cause the yielding of all the blocks that have a value of x between 0 and $2\mu_2\delta\gamma$. This phenomenon will affect the stress of the other blocks by increasing it (which means decreasing x) by $\mathcal{G}\overline{x}$. This means that

$$x_i \to x_i - \frac{\mu_2}{N(\mu_1+\mu_2)}\overline{x}\sum_{j=1}^{N}\theta(2\mu_2 d\gamma - x_j^\alpha(\gamma)) = x_i - x_c f_\gamma^\alpha(2\mu_2 d\gamma)$$

Where $x_c = \frac{\mu_2\overline{x}}{(\mu_1+\mu_2)}$. Since we have also the contribution of the other blocks, the stress of each block does not change always linearly with γ. For this reason is useful to introduce a new quantity which is called plastic strain y. This quantity represent how much x_i changes when going from γ to $\gamma + d\gamma$ taking into consideration also the yielded blocks. We now want to characterize and find a definition for the infinitesimal plastic strain dy. We know that all the blocks with $0 < x_i < 2\mu_2 d\gamma$ will yield after the infinitesimal strain $d\gamma$. The number of sites that respect this condition is exactly given by $f_\gamma^\alpha(2\mu_2 d\gamma)$. But, again, this stress increment will cause the yielding of the blocks inside the region $2\mu_2 d\gamma < x_i < 2\mu_2 d\gamma + x_c f_\gamma^\alpha(2\mu_2 d\gamma)$ which are in number equal to $f_\gamma^\alpha(2\mu_2 d\gamma + x_c f_\gamma^\alpha(2\mu_2 d\gamma)) - f_\gamma^\alpha(2\mu_2 d\gamma)$. Continuing with this reasoning we obtain, for the infinitesimal plastic strain:

$$dy = 2\mu_2 d\gamma + \cancel{x_c f_\gamma^\alpha(2\mu_2 d\gamma)} + x_c\left[f_\gamma^\alpha(2\mu_2 d\gamma + x_c f_\gamma^\alpha(2\mu_2 d\gamma)) - \cancel{f_\gamma^\alpha(2\mu_2 d\gamma)}\right] + \ldots$$

It is easy to see that this expression will, when iterated, lead to the implicit relation

$$dy = 2\mu_2 d\gamma + x_c f_\gamma^\alpha(dy) \tag{21}$$

This relation highlights the fact that one can write the stress as a function of the plastic strain as well as of the strain, since the two are linked. Using the plastic strain is surely easier computation wise but it is not something one can control externally: the only thing we can control from outside the system is indeed the strain γ. One could think about carrying on the calculations using y and then substitute the function $y(\gamma)$ at the end. This is however not so immediate because as we can see from (21) the relation between y and γ will be sample dependent. In what follows we will jump from considering y or γ as the independent variable. A big part of the interesting behavior of the system is indeed given by the relation from y and γ.

We will use interchangeably the notation f_γ^α and f_y^α, where with the latter we actually mean $f_{y(\gamma)}^\alpha(x)$.

We now proceed by considering y fixed and studying the function f when one perform the infinitesimal change $y \to y + dy$. Clearly all the sites that have $x_i < \overline{x}$ will remain in the interval $[0 - \overline{x}]$ for all the dynamics. This means that if we are counting the number of states that have stress up to x at $y + dy$ we have to distinguish two conditions: if $x > \overline{x}$ the blocks j with $x_j < \overline{x}$ will contribute, and also all the ones that have $x_j < x$. When y increase all the x_i will decrease, so that $f_{y+dy}^\alpha(x) = f_y^\alpha(x + dy)$. In the case of $x < \overline{x}$ the situation is slightly different, indeed not all the nodes j with $x_j < \overline{x}$ will contribute: the ones that undergo yielding will have $x_i = \overline{x} > x$. Then, considering both cases, we have

$$f_{y+dy}^\alpha(x) = \left[f_y^\alpha(x+dy) - f_y^\alpha(dy)\right]\theta(\overline{x} - x) + f_y^\alpha(x+dy)\theta(x - \overline{x})$$

Where we also know the (random) initial condition

$$f_0^\alpha(x) = \frac{1}{N}\sum_{i=1}^{N}\theta(x - x_i^\alpha(0))$$

Because we know how the $x_i^\alpha(0)$ are distributed. We can rewrite the equation we found before by subtracting $f_y^\alpha(x)$ from both sides as

$$f_{y+dy}^\alpha(x) - f_y^\alpha(x) = f_y^\alpha(x+dy) - f_y^\alpha(x) - f_y^\alpha(dy)\theta(\overline{x} - x) \tag{22}$$

It is better to avoid taking derivatives of f^α, because f is still a function that depend on the sample, and, since it is a summation of theta functions, it is defined by steps. We can then introduce a new quantity, Q_y, which is the average value of f computed over all the possible realizations of the samples. This new function will instead be smooth, and it will be easier to perform derivatives on it.

$$Q_y(x) = \overline{f_y^\alpha}(x)$$

It is clear that this quantity will represent the cumulative distribution of the stress at plastic strain y

$$Q_y(x) = \int_0^x P_y(x')dx'$$

With $P_y(x')$ the probability that a site has stress x' at plastic strain y. By averaging over the disorder the equation (22) and expanding in dy we obtain a new equation that describes the evolution of Q_y

$$\frac{\partial Q_y(x)}{\partial y} = \frac{\partial Q_y(x)}{\partial x} - \left.\frac{\partial Q_y(x)}{\partial x}\right|_{x=0} \theta(\overline{x} - x) \tag{23}$$

Where the term $Q_y(0)$ is 0 by the definition of Q. By deriving everything with respect to x we can obtain

$$\frac{\partial P_y(x)}{\partial y} = \frac{\partial P_y(x)}{\partial x} + P_y(0)\delta(x - \overline{x}) \tag{24}$$

These two equations give the evolution of P and Q. It would now be interesting to solve these equations if possible. They have been solved exactly in the case of exponentially distributed stress drop with average value $\overline{x}$ [5]. In our case instead the computations are more difficult, even if our situation is easier to imagine.

We now proceed in the calculations by searching for a solution for the probability $P_y(x)$. We again consider y as the independent variable, to pass later to the case of γ which is the realistic one. To start, we choose the initial distribution of x to have average value of 1, in such way the dynamics will start at 0 stress for $\gamma = 0$, which is the reasonable initial condition in the real, physical case. Then we considered equation (23) in the two cases of $x > \overline{x}$ and $x < \overline{x}$. Before starting it is important to remind the initial conditions for Q in the equations we are considering. In particular, since $Q_y(x)$ is a cumulative density function up to x, it should be 0 when $x = 0$ (since no site can have $\sigma > \sigma_{th}$) and 1 when x tends to infinity. Moreover we know the initial condition on the value of $Q_y(x)$ when $y = 0$, since it is given only by the initial cumulative distribution of the x which is:

$$Q_0(x) = \int_0^x P_0(x')dx' \tag{25}$$

Let's look at the two cases of x separately.

- $x > \overline{x}$

In this case we have the additional condition that $Q_y(x)$ tend to 1 when y tends to infinity. The reason behind this is that when the strain is huge, all the blocks have prior undergone yielding at least one time, and will have their value of x in the region between 0 and $\overline{x}$. Since our $Q_y(x)$ represent the cumulative density function, if $x > \overline{x}$ and $y \to \infty$ then $Q_y(x) \to 1$. The equation (23) reduces to

$$\frac{\partial Q_y(x)}{\partial y} = \frac{\partial Q_y(x)}{\partial x} \tag{26}$$

Which gives

$$\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right) Q_y(x) = 0 \to Q_y(x) \equiv \tilde{Q}(x + y)$$

This means that since the derivative with respect to y minus the derivative with respect to x gives 0, the function $Q_y(x)$ in instead function of the sum of $x + y$. By using the condition given by equation (25) we have

$$\tilde{Q}(x) = \int_0^x P_0(x')dx' \to Q_y(x) = \tilde{Q}(x + y) = \int_0^{y+x} P_0(x')dx'$$

Let's now consider the other case.

- $x < \overline{x}$

This time the solution is a bit more complicated, but we can use the initial condition at $x = 0$ to help us. Equation (23) becomes

$$\frac{\partial Q_y(x)}{\partial y} = \frac{\partial Q_y(x)}{\partial x} - \left.\frac{\partial Q_y(x)}{\partial x}\right|_{x=0} \tag{27}$$

By proceeding in a similar way we arrive to

$$\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right) Q_y(x) = -\left.\frac{\partial Q_y(x)}{\partial x}\right|_{x=0} = -f(y)$$

This means that $Q_y(x)$ will be a function of the variables y and $x + y$. In particular we have that

$$Q_y(x) = G(x + y) - F(y)$$

where $G(x + y)$ has to be determined and

$$F(y) = \int_0^y f(y')dy' = \int_0^y P_{y'}(0)dy'$$

By using the fact that $Q_y(0) = 0$ we have $G(y) - F(y) = 0$ that means $G(y) = F(y)$ and is true for every value of y. Then $Q_y(x) = F(y + x) - F(y)$, but we stil need to find what this function $F(y)$ is. One can check that the following solution satisfy the equation (27) and the initial conditions.

$$F(y) = Q_0(y) + \sum_{i=1}^{\infty} Q_0(y - n\overline{x})\theta(y - n\overline{x}) \tag{28}$$

And then

$$Q_y(x) = Q_0(y + x) - Q_0(y) + \sum_{n=1}^{\infty} (Q_0(y + x - n\overline{x})\theta(y + x - n\overline{x}) - Q_0(y - n\overline{x})\theta(y - n\overline{x}))$$

Now, merging together the two results we obtain (for a general value of x)

$$Q_y(x) = Q_0(y + x) + \theta(\overline{x} - x)\left[-Q_0(y) + \sum_{n=1}^{\infty} (Q_0(y + x - n\overline{x})\theta(y + x - n\overline{x}) - Q_0(y - n\overline{x})\theta(y - n\overline{x}))\right] \tag{29}$$

By deriving with respect to x we can find the exact expression of $P_y(x)$:

$$P_y(x) = P_0(y + x) + \theta(\overline{x} - x)\sum_{n=1}^{\infty} \theta(y + x - n\overline{x})P_0(y + x - n\overline{x}) \tag{30}$$

What is interesting to see is that this probability is exactly the beginning one at $y + x$ summed to the same thing shifted of $\overline{x}$, $2\overline{x}$, ... It is exactly here that we start to see the periodic behavior which is obviously non physical. This could be corrected with the introduction of a random stress drop distributed following $g(x)$ or with the choice of a random threshold for each block. The solution of this equation is plotted in figure (10.(b)). The initial distribution is a Gaussian and is chosen in figure (10.(a)).

It is also possible to solve equation (21) in the average case. By considering y fixed and $\gamma^\alpha(y)$ we obtain

$$dy = 2\mu_2 d\overline{\gamma^\alpha}(y) + x_c Q_y(dy) \tag{31}$$

Which gives

$$\frac{d\overline{\gamma^\alpha}(y)}{dy} = \frac{1}{2\mu_2}\left[1 - x_c \left.\frac{\partial Q_y(x)}{\partial x}\right|_{x=0}\right] = \frac{1}{2\mu_2}\left[1 - x_c P_y(0)\right] \tag{32}$$

And then

$$\overline{\gamma^\alpha}(y) = \frac{1}{2\mu_2}\left[y - x_c \int_0^y P_{y'}(0)dy'\right] \tag{33}$$

It would be interesting to find the inverse of this function to obtain also $\overline{y^\alpha}(\gamma)$. However, by looking at equation (32) one can notice that whether $\overline{\gamma^\alpha}$ is a monotonic function of y or not depends on the

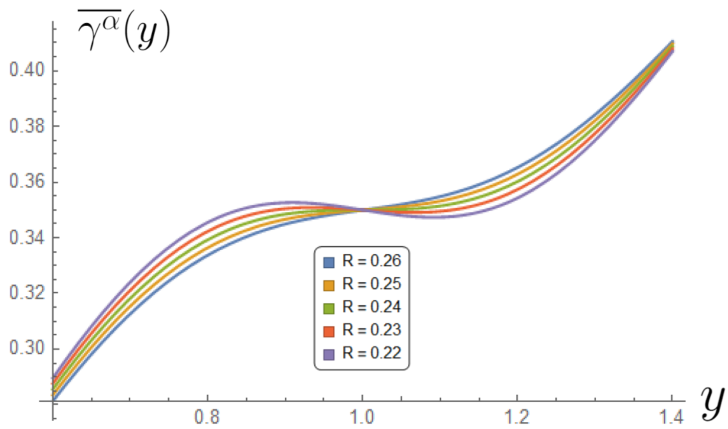

Figure 9: Plot of the solution of equation (33). When $R < R_2$ the function is no more monotonic.

probability $P_y(0)$. We can say that if at some point the function $P_y(0)$ will be equal or larger than $1/x_c$ the derivative will change sign and γ^α will not be monotonic anymore. We will later face this problem and try to find a solution. What we said about the function $\overline{\gamma^\alpha}(y)$ can be seen even better from figure (9).

Now that we have the equation for the evolution of the probability with y we can proceed by computing the two quantities that will lead us to the effective disorder: the connected and disconnected susceptibilities.

4.1 Connected susceptibility

The first quantity that we need to find the effective disorder is the connected susceptibility. This is the easier one, since it can be obtained by differentiating $m^\alpha(\gamma)$ with respect to γ. To do so we first need an expression of $m^\alpha(\gamma)$. We consider how much m^α change when γ is increased by $d\gamma$, keeping in mind the definition of f_γ^α given by equation (20). We have:

$$dm^\alpha(\gamma) = m^\alpha(\gamma + d\gamma) - m^\alpha(\gamma) = \frac{1}{N}\sum_{i=1}^{N}\left(x_i^\alpha(\gamma + d\gamma) - x_i^\alpha(\gamma)\right)$$

If x_i^α is not in the region between 0 and $dy(\gamma)$ then its value will only decrease by $dy(\gamma)$. If instead this is the case, it will be sent again to $\overline{x}$. Using this consideration in the previous equation we obtain:

$$dm^\alpha(\gamma) = \frac{1}{N}\sum_{i=1}^{N}\Big[\big(1-\theta(dy(\gamma)-x_i^\alpha(\gamma))\big)\big(\cancel{x_i^\alpha(\gamma)} - dy(\gamma) - \cancel{x_i^\alpha(\gamma)}\big) + \theta(dy(\gamma)-x_i^\alpha(\gamma))\big(\overline{x} - x_i^\alpha(\gamma)\big)\Big] =$$

$$= -dy(\gamma) + dy(\gamma) f_\gamma^\alpha(dy(\gamma)) + \overline{x} f_\gamma^\alpha(dy(\gamma)) - \frac{1}{N}\sum_{i=1}^{N} x_i^\alpha(\gamma)\theta(dy(\gamma) - x_i^\alpha(\gamma))$$

We can note that in the above expression the second and the fourth term are of second order (in the fourth survives only terms where $x_i^\alpha(\gamma) < dy(\gamma)$ and then multiply $dy(\gamma)$). We can neglect these terms and retain only the ones which are of the first order, but we still can not derive because of the particular form of $f_\gamma^\alpha(x)$. We remain with

$$dm^\alpha(\gamma) = -dy(\gamma) + \overline{x} f_\gamma^\alpha(dy(\gamma)) \tag{34}$$

We can however study the case of $\overline{m^\alpha}$ to find what the connected susceptibility looks like. We first consider y as the independent variable and then we will perform a change of variable.

$$\overline{m^\alpha}(y + dy) - \overline{m^\alpha}(y) = -dy + \overline{x} Q_y(dy) = -dy + \overline{x}\left.\frac{\partial Q_y(x)}{\partial x}\right|_{x=0} dy$$

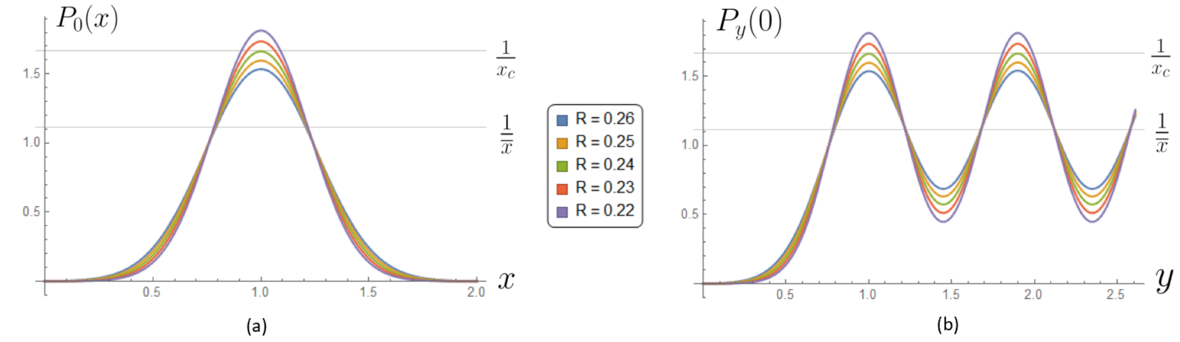

Figure 10: (a) Plot of the initial distribution of the value x at each site. (b) Plot of the probability of being in $x = 0$ at plastic strain y. One can notice that only for some values of R the distribution crosses the line $1/x_c$

which gives for the connected susceptibility with respect to γ:

$$\chi_{con}(\gamma) = \frac{d\overline{\sigma^\alpha}(\gamma)}{d\gamma} = -\frac{d\overline{m^\alpha}(y)}{dy}\frac{d\overline{y^\alpha}(\gamma)}{d\gamma} = 2\mu_2\frac{1-\overline{x}P_\gamma(0)}{1-x_cP_\gamma(0)} \tag{35}$$

Where the value of the derivative of $\overline{y^\alpha}(\gamma)$ with respect to γ is obtained by inverting equation (32) and hence the last equation holds only when y and γ are invertible. This is the first quantity we are interested in, the connected susceptibility. One can clearly understand that there exist two important values of γ in this function, which are the ones where either the numerator or the denominator is 0. The vanishing of the numerator happens at γ_1 such that $P_{\gamma_1}(0) = 1/\overline{x}$ and represents an extremum of the stress. Since $P_\gamma(0)$ is 0 when $\gamma = 0$ and then increases, this point γ_1 represents a maximum in the value of the stress (the overshoot). The denominator will instead reach 0 at the value γ_2, such that $P_{\gamma_2}(0) = 1/x_c$. At this point the slope of the stress diverges leading to a jump in its behavior. Since $x_c < \overline{x}$ we have $\gamma_1 < \gamma_2$ always. Whether such points exist or not depends on the function $P_\gamma(0)$ and hence also on the initial disorder (since we know that $P_\gamma(0)$ is a function of $P_0(x)$). If the distribution is wide enough, its height may not reach one of this value. When instead the distribution is narrower, i.e. the disorder is weaker, it is possible to observe either γ_1 alone or both the points. This explain why for strong disorder we notice a smooth transition from elastic to plastic region, while, when the strength is diminished we observe first an overshoot and then both an overshoot and a macroscopic jump.

The critical value for the disorder can be obtained in the same way we did in the RFIM. It is sufficient to impose that the peak of the Gaussian is larger than the value of $1/x_c$ or $1/\overline{x}$. One obtain respectively $R_1 = \overline{x}/\sqrt{2\pi}$ as the critical value of the standard deviation ($R = \sqrt{\Delta}$) for having the overshoot and $R_2 = x_c/\sqrt{2\pi}$ as the one for the macroscopic drop in the stress. In what follows we focus only in the cases when the disorder is weak enough to have at least overshoot. To obtain the plots we choose the following values for the constants: $\mu_1 = 0.5$, $\mu_2 = 1$, $\overline{x} = 0.9$. This gives $R_1 = 0.359$ and $R_2 = 0.239$, therefore we choose to evaluate for the values $R = \{0.26, 0.25, 0.24, 0.23, 0.22\}$. The plot of the

Now, before searching for the disconnected susceptibility, we deepen a bit the study of y and γ

4.2 Extended analysis of γ and y

We would like to focus our attention towards the computation of the disconnected susceptibility. Unfortunately, before trying to do so, it is necessary to find some more relations and to better understand the link between the plastic strain y and the strain γ. To show why this is important we will derive another possible expression for the value of m^α. By expressing f^α_γ in equation (21) as a function on $d\gamma$ and dy and inserting it in equation (34) we can obtain a new evolution equation for m^α which is:

$$dm^\alpha(y) = m^\alpha(y+dy) - m^\alpha(y) = -dy\left(1-\frac{\overline{x}}{x_c}\right) - 2\mu_2\frac{\overline{x}}{x_c}d\gamma^\alpha(y) \tag{36}$$

Which, after computing it for $y = Ndy$ and imposing $\gamma^\alpha(0) = 0$, gives:

$$m^\alpha(y) = m^\alpha(0) - y\left(1 - \frac{\overline{x}}{x_c}\right) - 2\mu_2\frac{\overline{x}}{x_c}\gamma^\alpha(y) \tag{37}$$

Where $m^\alpha(0)$ is the initial value of m^α. This means that the value of m^α is a function only of y and of $\gamma^\alpha(y)$, and we have hidden the dependence on the function f inside the two strains. This is only true if we consider y as the independent variable. To study instead the case of fixed γ we now take a look at the two functions $\gamma^\alpha(y)$ and $y^\alpha(\gamma)$. Since they are both depending on the sample we can make the following approximation, by considering only their average value and the fluctuations around it:

$$\gamma^\alpha = \overline{\gamma^\alpha}(y) + \frac{1}{\sqrt{N}}\Gamma^\alpha(y) \tag{38}$$

$$y^\alpha = \overline{y^\alpha}(\gamma) + \frac{1}{\sqrt{N}}Y^\alpha(\gamma) \tag{39}$$

We will now make the assumption that $\gamma^\alpha(y)$ (and obviously also $y^\alpha(\gamma)$) is invertible. We know from equation (32) that this is not always the case. However the region where $\gamma^\alpha(y)$ is not increasing is not physical, so the evolution of the system will jump to an higher value of y (loosing continuity). Under this assumption we have that

$$\gamma = \gamma^\alpha(y^\alpha(\gamma)) = \overline{\gamma^\alpha}\big(\overline{y^\alpha}(\gamma) + \frac{1}{\sqrt{N}}Y^\alpha(\gamma)\big) + \frac{1}{\sqrt{N}}\Gamma^\alpha\big(\overline{y^\alpha}(\gamma) + \frac{1}{\sqrt{N}}Y^\alpha(\gamma)\big)$$

In the first term we can expand in series, while at the argument of the second term we can neglect the second contribution since it would be of order $1/N$. Moreover, following our previous assumption, we can say that also $\overline{\gamma^\alpha}(y)$ is invertible and hence

$$\cancel{\gamma} = \cancel{\overline{\gamma^\alpha}(\overline{y^\alpha}(\gamma))} + \overline{\gamma^\alpha}'(\overline{y^\alpha}(\gamma))\frac{1}{\sqrt{N}}Y^\alpha(\gamma) + \frac{1}{\sqrt{N}}\Gamma^\alpha(\overline{y^\alpha}(\gamma))$$

So that we obtain a formula that links the fluctuations of y to the ones of γ. In the same way we can obtain the symmetric relation, we have:

$$\overline{\gamma^\alpha}'(\overline{y^\alpha}(\gamma))Y^\alpha(\gamma) = -\Gamma^\alpha(\overline{y^\alpha}(\gamma)) \tag{40}$$

$$\overline{y^\alpha}'(\overline{\gamma^\alpha}(y))\Gamma^\alpha(y) = -Y^\alpha(\overline{\gamma^\alpha}(y)) \tag{41}$$

Moreover, as long as $\gamma^\alpha(y)$ and $y^\alpha(\gamma)$ are invertible we can also insert $y = y^\alpha(\gamma)$ in equation (37) and rewrite m^α as

$$m^\alpha(\gamma) = m^\alpha(0) - y^\alpha(\gamma)\left(1 - \frac{\overline{x}}{x_c}\right) - 2\mu_2\frac{\overline{x}}{x_c}\gamma \tag{42}$$

Thank to this formula one can now plot the average value of the stress as a function of the strain. To perform this computation one choose the initial value of the stress to be 0 (notice that this is true up to fluctuations of order $1/N$ since the initial stress is a sum of random variables). The resulting plots are shown in figure (11) and one can notice that when $R < R_2$ a macroscopic jump appears. The value at which the jump is observed increases with the weakness of the disorder. Here only a region of the stress vs strain curve is shown. For higher strain the behavior is periodic.

Before moving on is better to have a formula for our fluctuations. Since we saw that the fluctuations of the two quantities γ and y are linked by equation (40) it is simpler to study only one of them. Our choice will fall on $\Gamma^\alpha(y)$ because from equation (33) we can find how it behaves on average. Then our fixed independent variable will be y and from equation (21) we have:

$$2\mu_2 d\gamma^\alpha(y) = dy - x_c f_y^\alpha(dy)$$

By using $d\gamma^\alpha(y) = \gamma^\alpha(y + dy) - \gamma^\alpha(y)$ and the fact that $\gamma^\alpha(0) = 0$ we then find for the M-th step of size dy:

$$\gamma^\alpha(M\,dy) = \frac{1}{2\mu_2}\left[M\,dy - \sum_{n=0}^{M} f^\alpha_{n\,dy}(dy)\right]$$

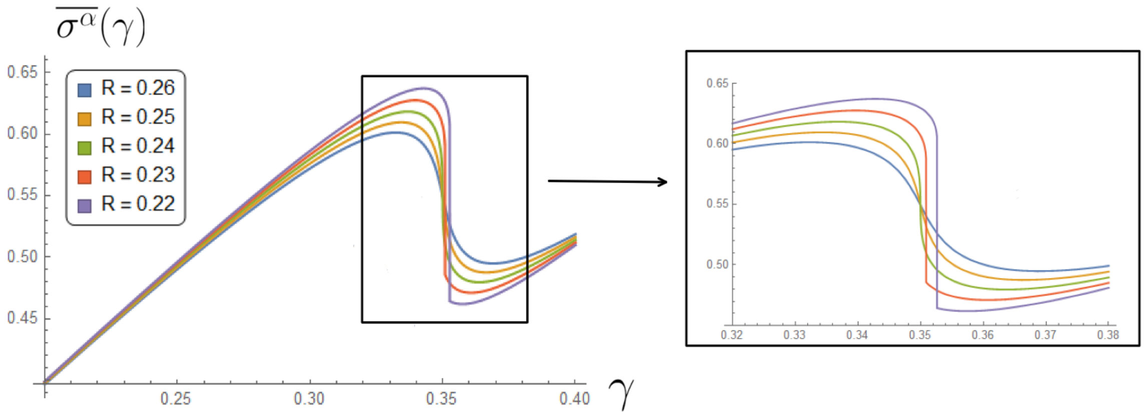

Figure 11: Plot of the average stress $\overline{\sigma^\alpha}(\gamma)$ as a function of the applied strain γ. A macroscopic jump appears for $R < R_2$. With values $\mu_1 = 0.5$, $\mu_2 = 1$, $\overline{x} = 0.9$ which give $R_2 \sim 0.2394$.

By averaging this equation we can find that for the average value of γ holds the same relation with Q instead of f. We then have:

$$\Gamma^\alpha(M\,dy) = \sqrt{N}\Big(\gamma^\alpha(M\,dy) - \overline{\gamma^\alpha}(M\,dy)\Big) = \frac{x_c\sqrt{N}}{2\mu_2}\sum_{n=1}^{M}\Big[Q_{n\,dy}(dy) - f^\alpha_{n\,dy}(dy)\Big] \tag{43}$$

It is important to underline the fact that we made the hypothesis of invertibility of y and γ. Then the formulas we will find in the following are true only when this is verified. From equation (32) one sees that this is true only when $\gamma > \gamma_1$. This is the reason why we restrict our considerations only to the region after the overshoot and before the minimum of the stress when computing the disconnected susceptibility.

We can now pass to computing χ_{dis}.

4.3 Disconnected susceptibility and effective disorder

We are finally ready to compute the disconnected susceptibility, which is the last step before obtaining the effective disorder. Since we want to look at the case of strain-driven experiment we will directly consider γ as the independent variable this time. Our formula of m^α for fixed increase in γ is given by equation (42). The disconnected susceptibility will be given by

$$\chi_{dis}(\gamma) = N\overline{\Big[m^\alpha(\gamma) - \overline{m^\alpha}(\gamma)\Big]^2}$$

Before moving on is key discuss on the role of $m^\alpha(0)$ in equation (42). Its value is given by the initial condition on the $x_i^\alpha(\gamma)$ and its mean is chosen equal to 0. Since it is the sum of random variables, it will itself be a random variable and, for large enough N, will have a Gaussian distribution thanks to the central limit theorem. The mean will be given by the mean of the single $x_i^\alpha(0)$ and the variance will be Δ/N. The disconnected susceptibility is then:

$$\begin{aligned}\chi_{dis} = N\overline{\left[m^\alpha(0) - 1 - \frac{1}{\sqrt{N}}\left(1 - \frac{\overline{x}}{x_c}\right)Y^\alpha(\gamma)\right]^2} &= \overline{\left[\tilde{m}^\alpha - \left(1 - \frac{\overline{x}}{x_c}\right)Y^\alpha(\gamma)\right]^2} = \\ = \Delta + \left(1 - \frac{\overline{x}}{x_c}\right)^2\overline{Y^\alpha(\gamma)^2} - 2\left(1 - \frac{\overline{x}}{x_c}\right)\overline{\tilde{m}^\alpha Y^\alpha(\gamma)}\end{aligned}$$

Where $\tilde{m^\alpha}$ is a random variable with a Gaussian distribution of 0 mean and variance Δ. We can choose Δ, but we still do not have the second and third terms.

We will first focus on $\overline{Y^\alpha(\gamma)^2}$. However, computing it directly from $y^\alpha(\gamma)$ and $\overline{y^\alpha}(\gamma)$ is difficult, then we can use the relation (40) and study $\Gamma^\alpha(y)$ instead. So we want to compute $\overline{\Gamma^\alpha(y)^2}$ and study it at

$y = y^\alpha(\gamma)$. We have from equation (43):

$$\overline{\Gamma^\alpha(y)^2} = \left(\frac{x_c}{2\mu_2}\right)^2 N \sum_{n=0}^{M}\sum_{n'=0}^{M}\left[Q_{n\,dy,n'\,dy}(dy,dy) - Q_{n\,dy}(dy)Q_{n'\,dy}(dy)\right] \tag{44}$$

With

$$Q_{y,y'}(x,x') = \overline{f_y^\alpha(x) f_{y'}^\alpha(x')} = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\overline{\theta\Big(x - x_i^\alpha(y)\Big)\theta\Big(x' - x_j^\alpha(y')\Big)}$$

Since both the functions are averages I can expand in series. The expansion of $Q_{y,y'}(x,x')$ involves two derivatives, but it is possible to notice that whenever $x = 0$ or $x' = 0$ also $Q_{y,y'}(x,x') = 0$. This holds also every time I derive with respect to a single value, so that also quantities like $Q_{y,y'}^{(k,0)}(x,x')$ will be zero when $x' = 0$ whatever the degree k of the derivative. The first term of order dy^2 that survives is then the derivative with respect to both x and x'. We obtain:

$$\begin{aligned}\overline{\Gamma^\alpha(y)^2} &= \left(\frac{x_c}{2\mu_2}\right)^2 N \int_0^y\int_0^y \left[\frac{\partial^2 Q_{y,y'}(x,x')}{\partial x \partial x'}\bigg|_{x=0,x'=0} - \frac{\partial Q_y(x)}{\partial x}\bigg|_{x=0}\frac{\partial Q_{y'}(x')}{\partial x'}\bigg|_{x'=0}\right] dy\, dy' = \\ &= \left(\frac{x_c}{2\mu_2}\right)^2 N \int_0^y\int_0^y \frac{\partial^2}{\partial x\partial x'}\Big[Q_{y,y'}(x,x') - Q_y(x)Q_{y'}(x')\Big]_{x=0,x'=0} dy\, dy'\end{aligned}$$

The only thing we miss to continue and finally obtain our result is the expression of $Q_{y,y'}(x,x')$. This can be found in a way similar to the one we used for the computation of $Q_y(x)$ that led to equation (29). It can be shown that $Q_{y,y'}(x,x')$ will behave as $Q_y(x)Q_{y'}(x')$ at order 1, but this will cancel with the other term inside the integral. The remaining terms will be of order $1/N$ and will simplify the N outside the integral. The computations are shown in the appendix B. The result is given by (in the case $x, x' < \overline{x}$):

$$Q_{y,y'}(x,x') - Q_y(x)Q_{y'}(x') = \frac{1}{N}\Big[h_{y,y'}(x,x') - Q_y(x)Q_{y'}(x')\Big]$$

With

$$\begin{aligned} h_{y,y'}(x,x') =& U(y+x, y'+x') - U(y+x, x') - U(x, y'+x') + U(x,x') \\ U(z,z') =& Q_0(\min(z,z')) + \sum_{n=1}^{\infty} Q_0(\min(z - n\overline{x}, z'))\theta\Big(z - n\overline{x}\Big) + \sum_{n=1}^{\infty} Q_0(z, \min(z' - n\overline{x}))\theta\Big(z' - n\overline{x}\Big) + \\ &+ \sum_{n=1}^{\infty}\sum_{n'=1}^{\infty} Q_0(\min(z - n\overline{x}, z' - n'\overline{x}))\theta\Big(z - n\overline{x}\Big)\theta\Big(z' - n'\overline{x}\Big)\end{aligned} \tag{45}$$

In the derivative of $h_{y,y'}(x,x')$ only the term $U(y+x, y'+x')$ will survive since it is the only one that depends on both x and x'. Deriving with respect to x is the same of deriving with respect to y in this function, so we obtain:

$$\frac{\partial^2}{\partial x\partial x'} h_{y,y'}(x,x')\Big|_{x=0,x'=0} = \frac{\partial^2}{\partial x\partial x'} U(y+x, y'+x')\Big|_{x=0,x'=0} = \frac{\partial^2}{\partial y\partial y'} U(y,y')$$

While for the product $Q_y(x)Q_{y'}(x')$ we have

$$\frac{\partial^2}{\partial x\partial x'} Q_y(x)Q_{y'}(x')\Big|_{x=0,x'=0} = \frac{\partial^2}{\partial x\partial x'} F(y+x)F(y'+x')\Big|_{x=0,x'=0} = \frac{\partial^2}{\partial y\partial y'} F(y)F(y')$$

Where $F(y+x)$ is the one defined in equation (28).

The result for our Γ^α is finally:

$$\overline{\Gamma^\alpha(y)^2} = \left(\frac{x_c}{2\mu_2}\right)^2 \Big(U(y,y) - F(y)^2\Big) \tag{46}$$

Then, using equation (40) to switch from Y to Γ, we obtain the form:

$$\chi_{dis}(\gamma) = \Delta + \left(1 - \frac{\overline{x}}{x_c}\right)^2 \left(\frac{x_c}{2\mu_2}\right)^2 \left(U(\overline{y^\alpha}(\gamma), \overline{y^\alpha}(\gamma)) - F(\overline{y^\alpha}(\gamma))^2\right) \left(\overline{y^\alpha}'(\gamma)\right)^2 - 2\left(1 - \frac{\overline{x}}{x_c}\right) \overline{\tilde{m}^\alpha Y^\alpha(\gamma)}$$

In this formula we still miss the computations for the last term which we should be able to find but we didn't because of time, we therefore avoid to consider it in the following. The effective disorder is given by

$$\Delta_{eff}(\gamma) = \Delta \left[\frac{1 - x_c P_\gamma(0)}{(1 - \overline{x}P_\gamma(0))2\mu_2}\right]^2 + \left(U(\overline{y^\alpha}(\gamma), \overline{y^\alpha}(\gamma)) - F(\overline{y^\alpha}(\gamma))^2\right) \left[\frac{\overline{x} - x_c}{(1 - \overline{x}P_\gamma(0))2\mu_2}\right]^2 \tag{47}$$

Where the two functions can be rewritten as

$$U(y, y) = Q_0(y) + 2\sum_{n=1}^{\infty} Q_0(y - n\overline{x})\theta(y - n\overline{x}) + \sum_{n,n'=1}^{\infty} Q_0(\min(y - n\overline{x}, y, n'\overline{x}))\theta(y - n\overline{x})\theta(y - n'\overline{x})$$

$$F(y)^2 = Q_0(y)^2 + 2Q_0(y)\sum_{n=1}^{\infty} Q_0(y - n\overline{x})\theta(y - n\overline{x}) +$$
$$+ \sum_{n,n'=1}^{\infty} Q_0(y - n\overline{x})Q_0(y - n'\overline{x})\theta(y - n\overline{x})\theta(y - n'\overline{x})$$

Which gives a simpler expression

$$\left(U(y, y) - F(y)^2\right) = \sum_{n=0}^{\infty} Q_0(y - n\overline{x})\left(1 - Q_0(y - n\overline{x})\right)\theta(y - n\overline{x}) + 2\sum_{n>n'=0}^{\infty} Q_0(y - n\overline{x})\left(1 - Q_0(y - n'\overline{x})\right)\theta(y - n\overline{x})$$

The solution to this equation is plotted in figure (12).

One can see that the disorder in the initial configuration of the system, which is described by $P_0(x)$, generates via the solution of the deterministic equation of evolution a renormalized disorder. This disorder takes the form of a random field of variance $\Delta_e ff(\gamma)$ and only makes physical sense when $1 - \overline{x}P_\gamma(0) > 0$, i.e., beyond the overshoot of the stress-strain curve when it is present. One can notice many properties of the effective disorder in this figure. First of all it has not been possible to display all the plots in the same graph because of the wide range spanned by Δ_{eff} when the disorder is changed. This is due to the fact that, as we said before, our treatment of the problem and in particular our formula for the effective disorder is only valid between the two zeros of the connected susceptibility. When the strength of the disorder decrease the values of γ at which χ_{con} vanishes get closer and closer, reducing our range of interest. Moreover, when χ_{con} vanishes, χ_{dis} does not. This means that the effective disorder will diverge at the two points where the connected susceptibility is 0.

Nonetheless is possible to comment on some features that appear from the figure. In particular, a strong resemblance with the case of RFIM is present when one looks at the minimum of Δ_{eff}. When the disorder is above the critical value (subcritical) the plot of Δ has a minimum at the value of γ such that half the blocks have undergone yielding. Since the initial distribution of x is a Gaussian with its maximum at 1, this is true when the plastic strain y is equal to 1. From equation (33) one has

$$\gamma_{min} = \frac{1}{2\mu_2}\left(1 - \frac{x_c}{2}\right)$$

Which, with our values of the constants, gives $\gamma_{min} = 0.3499$.

When instead $R < R_2$ a macroscopic jump appears, given by the jump of the stress. The value of γ at which this jump appears, as well as the amplitude of the jump, seems to increase with the weakness of the disorder.

We are now ready to draw our conclusions.

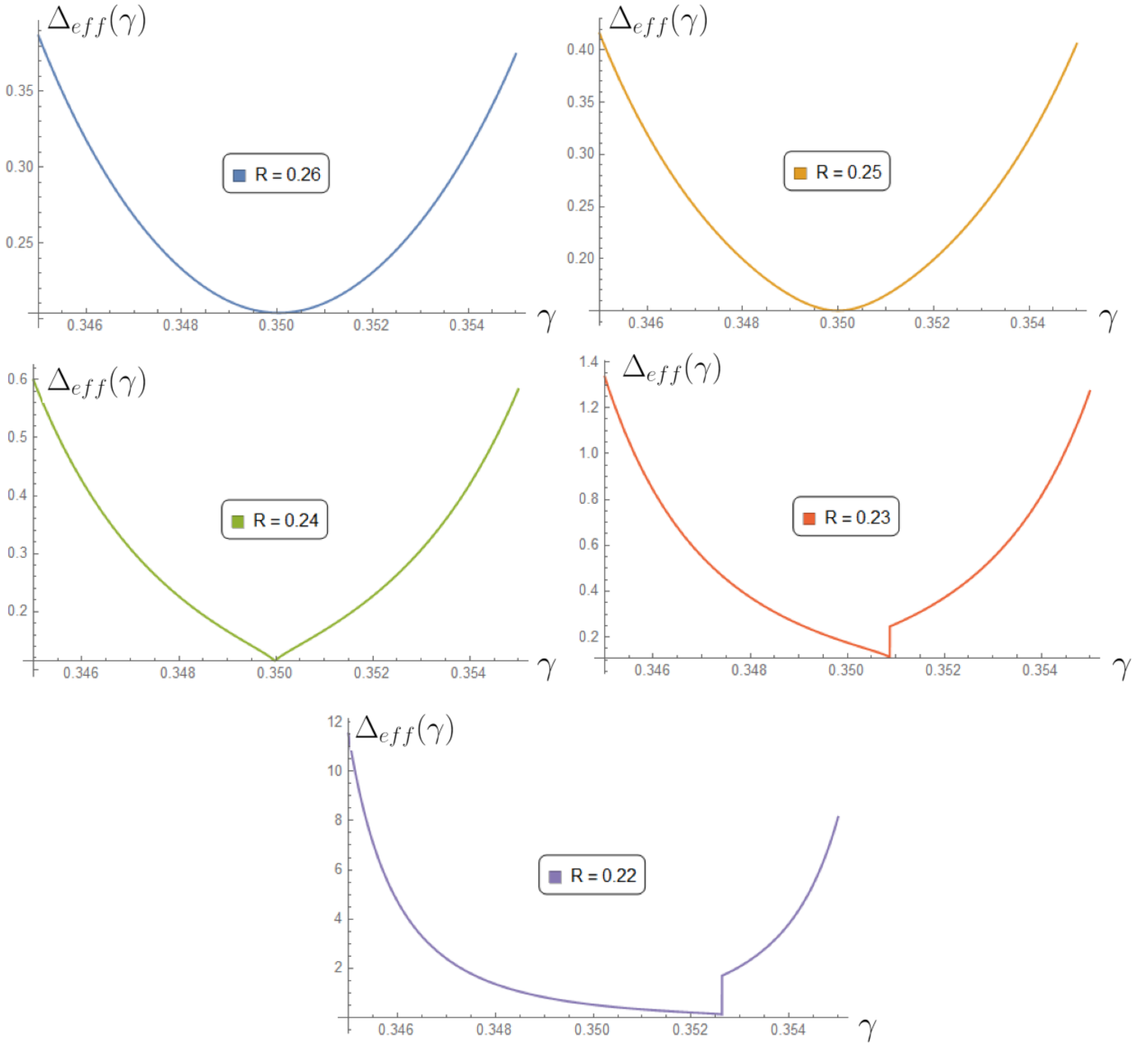

Figure 12: Plots of the effective disorder as a function of γ

5 Conclusion and future work

In this project we investigated the properties of the stress of amorphous solids in a strain controlled framework at 0 temperature. In particular we studied the behavior of the effective disorder, a quantity that represents the effect of the disorder at a coarse grained level, as the ratio between the disconnected susceptibility and the connected one squared. We also studied the same quantity in the archetypal model of statistical mechanics, the Ising model, with the addition of a random field located at each site. The study we carried on had the goal of finding an analogy between the two models at a macroscopic level, i.e. by studying the macroscopic properties. As we already said previously, this analogy can only be performed in the vicinity of the yielding transition of the solid (in particular between the maximum and the minimum of the stress), and not in the elastic region or in the steady state. Finding such a correspondence between the models would allow to an easier treatment of the solids, since the Ising model is a better known system. What we found is summarized in the figures (12) and (6), where we can observe the behavior of the disorder in both cases. We clearly see that some similarities exist, even if in the EPM case everything is biased by the divergence at the boundary of the region we investigated. However, if one zooms in, is possible to better explore a small region where a discontinuity is seen. From the plots it is evident that in both cases the disorder builds up during the dynamics increasing before and after the transition region.

As for the future works, there are a lot of improvement that can be done in both sides. As a first step, still at a mean field level, one could think about adding the possibility for the plastic events to both increase or decrease the stress on the neighbor blocks. At a mean field level this effect could be taken into consideration by choosing randomly at each rearrangement in which way to act. Another important step is to restore the parts of the dynamics that we neglected until now, for example by choosing a random stress drop or a random threshold, that are able to reach a stationary state which is close to the experimentally observed one. In general the improvements on the side of the EPM can be carried on by considering further sources of disorder and studying how they interact with the major source, the initial distribution of stress.

On the side of the RFIM it would be interesting to abandon the mean field hypothesis and insert an interaction between the spins given by the Eshelby kernel in equation (19). This is a long range interaction, but since it scales as d, which is exactly the dimension of the space we are in, it can be treated analitically. The study of this case should proceed by mean of non-perturbative functional renormalization group. By adding this kind of interaction one can hope to recover more aspects of the physics of amorphous solids at the yielding transition.

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A Symmetries in RFIM

In this appendix we deepen the discussion regarding symmetries in the RFIM. We start by showing that when the disorder satisfies $\Delta < \Delta_c$ the magnetization is symmetric with respect to the point $(m^*(H_s), H_s)$. From equation (12) we can study what happens to the function m^* in the two limits of $H \to \infty$ and $H \to -\infty$. We obtain that

$$m^*(H) \sim \frac{H}{k-J} + \frac{k}{k-J} \quad , \quad H \to \infty$$
$$m^*(H) \sim \frac{H}{k-J} - \frac{k}{k-J} \quad , \quad H \to -\infty$$

Since we are in the subcritical case, the curve of the magnetization will be smooth and there will be a point in the middle between this two regions. Let's call H_s the value of the field at which $m^*(H_s) = H_s/(k-J)$. Then to find H_s we must impose

$$\int_{-\infty}^{-H_s - Jm^*(H_s)+k} dh\rho(h) = \frac{1}{2} \to -H_s - Jm^*(H_s) + k = 0 \to k = H_s + J\left(\frac{H_s}{k-J}\right)$$

From which we obtain $H_s = k - J$. In what follows we prove that the magnetization is indeed symmetric with respect to this point, but to do so we substitute some values to k and J to simplify computations. This should not change the validity of the argument.

We choose the same values which we performed the simulations with: $k = 2$, $J = 1$. This means that the point we claim to be of symmetry is. We then rewrite equation (12) using two new variables: $\tilde{m} = m^*(H) - 1$ and $\tilde{H} = H - 1$. The equation becomes

$$\tilde{m} = 2 + \tilde{H} - 4\int_{-\infty}^{-\tilde{m}-\tilde{H}} dh\rho(h)$$

We now perform the following changes: $\tilde{m} \to -\tilde{m}$ and $\tilde{H} \to -\tilde{H}$ and obtain:

$$\tilde{m} = -2 + \tilde{H} + 4\left[1 - \int_{\tilde{H}+\tilde{m}}^{\infty} dh\rho(h)\right] = 2 + \tilde{H} - 4\int_{-\infty}^{-\tilde{H}-\tilde{m}} dh\rho(h)$$

Which is exactly the same relation with starte with, which means that $\tilde{m}(\tilde{H}) = -\tilde{m}(\tilde{H})$ and so that the function $m^*(H)$ is symmetric with respect to the point (1,1).

The same argument can be used to prove that also in the equilibrium case the magnetization is symmetric. At difference with what we just saw, at equilibrium the symmetry remains also when $\Delta < \Delta_c$.

We now prove another symmetry, the one of the effective disorder, which is even in H. We do this only for the equilibrium case which is simpler, but the same can be done for the $\Delta > \Delta_c$ case in the quasistatic regime. It can be done whenever the expression $x = H + Jm^*(H) - k$ is itself symmetric with respect to a change $\tilde{H} \to -\tilde{H}$ with $\tilde{H} = H - H_s$ for some H_s. From equation (17) we have the value of $m^*_{eq}(H)$ and we know that it is symmetric with respect to $H \to -H$. From equation (18) we know the form of the effective disorder. We can now study one term per time to prove its symmetry.

- **denominator**: In the denominator we only have a constant and the distribution $\rho(H + Jm^*)$. When going from H to $-H$ the argument of the function will become $-H - Jm^*(H)$, but the function $\rho(x)$ is even. The denominator is then even.
- **numerator, second term**: Let's define the new variable $x = H + Jm(H)$. When changing $H \to -H$ we will have $x \to -x$. This term will be

$$4\int_{x}^{\infty} dh\rho(h) \int_{-\infty}^{x} dh\rho(h)$$

Now, since $\rho(x)$ is even, it is clear that

$$\int_{x}^{\infty} dh\rho(h) = \int_{-\infty}^{-x} dh\rho(h)$$

The it is easy to see

$$4\int_x^\infty dh\rho(h)\int_{-\infty}^x dh\rho(h) = 4\int_{-\infty}^{-x} dh\rho(h)\int_{-x}^\infty dh\rho(h)$$

This shows that also this term in the numerator is even

- **numerator, last term**: In the last term in the numerator one can write again

$$4\int_x^\infty dh\frac{h}{k}\rho(h) = 4\left[\int_{-x}^\infty dh\frac{h}{k}\rho(h) - \int_{-x}^x dh\frac{h}{k}\rho(h)\right]$$

But, since the integrand is odd, the second term in the sum vanisches, showing that also this part of the numerator is even

In this way we have shown that the effective disorder is even with respect to the external field H. It is important to remember that this is true only when also the magnetization has a symmetry with respect to the origin or to some point $(m^*(H_s), H_s)$.

B Solution for $Q_{y,y'}(x,x')$

In this appendix we derive equation (45) by mean of the same method we used before to derive $Q_y(x)$. For a general couple y, y' one can see that

$$Q_{y,y'}(x,x') = \left(1-\frac{1}{N}\right)Q_y(x)Q_{y'}(x') + h_{y,y'}(x,x')$$

Where the function $h_{y,y'}(x,x')$ is what we want to find. For simplicity we will this time consider the case $x, x' > \overline{x}$ instead of examining all the cases (we will only use the expression for $Q_{y,y'}(x,x')$ at $x, x' = 0$). We start by finding the equations that $Q_{y,y'}(x,x')$ follows. From the definition of $Q_{y,y'}(x,x')$ we have that the change $y \to y + dy$ acts in the following way

$$\begin{aligned} Q_{y+dy,y'}(x,x') - Q_{y,y'}(x,x') &= \overline{f^\alpha_{y+dy}(x)f^\alpha_{y'}(x') - f^\alpha_y(x)f^\alpha_{y'}(x')} = \\ &= \overline{f^\alpha_{y'}(x')\Big[f^\alpha_{y+dy}(x) - f^\alpha_y(x)\Big]} \end{aligned}$$

Using equation (22) we rewrite it as

$$\begin{aligned} Q_{y+dy,y'}(x,x') - Q_{y,y'}(x,x') &= \overline{f^\alpha_{y'}(x')\Big[f^\alpha_y(x+dy) - f^\alpha_y(x) - f^\alpha_y(dy)\theta(\overline{x}-x)\Big]} = \\ &= Q_{y,y'}(x+dy,x') - Q_{y,y'}(x,x') - Q_{y,y'}(dy,x')\theta(\overline{x}-x) \end{aligned}$$

By noticing that the same can be done for y' and expanding in series we can obtain the differential equations that regulate the evolution of $Q_{y,y'}(x,x')$:

$$\begin{aligned} \frac{\partial Q_{y,y'}(x,x')}{\partial y} &= \frac{\partial Q_{y,y'}(x,x')}{\partial x} - \left.\frac{\partial Q_{y,y'}(x,x')}{\partial x}\right|_{x=0}\theta(\overline{x}-x) \\ \frac{\partial Q_{y,y'}(x,x')}{\partial y'} &= \frac{\partial Q_{y,y'}(x,x')}{\partial x'} - \left.\frac{\partial Q_{y,y'}(x,x')}{\partial x'}\right|_{x'=0}\theta(\overline{x}-x') \end{aligned}$$

Since we said that we consider the case where both x and x' are larger than 0, the theta-functions in the last terms will give 1. Since also $Q_y(x)$ and $Q_{y'}(x')$ respect the same equations, also $h_{y,y'}(x,x')$ will respect them. Let's now discuss the initial conditions of the function $h_{y,y'}(x,x')$. Since when either x or x' are 0 both the respective Q and $Q_{y,y'}(X,x')$ vanishes, then also $h_{y,y'}(x,0)$ or $h_{y,y'}(0,x')$ are 0. Moreover we can study what happens at $y, y' = 0$. In this case we have

$$\begin{aligned} Q_{y,y'}(x,x') &= \frac{1}{N^2}\sum_{i=1}^N\sum_{j=1}^N \overline{\theta\Big(x - x^\alpha_i(y)\Big)\theta\Big(x' - x^\alpha_j(y')\Big)} = \\ &= \frac{1}{N}\overline{\theta\Big(x - x^\alpha_1(y)\Big)\theta\Big(x' - x^\alpha_1(y')\Big)} + \left(1-\frac{1}{N}\right)\overline{\theta\Big(x - x^\alpha_1(y)\Big)\theta\Big(x' - x^\alpha_2(y')\Big)} \end{aligned}$$

Where the nodes 1 and 2 are two randomly chosen nodes (since the average is equal for every node I can perform the summation). Now, when $y, y' = 0$ the second term in the sum is exactly $Q_0(x)Q_0(x')$ so that the first term is exactly the $h_{0,0}(x, x')$ we were looking for. By looking better at this term we notice that if $x > x'$ this function will be different from 0 only when $x_1^\alpha < x'$ which means that $h_{0,0}(x, x') = Q_0(x')$ in this case. Clearly the inverse is true when $x < x'$ meaning that $h_{0,0}(x, x') = Q_0(\min(x, x'))$.

Now that we have the boundary conditions and the equations we can proceed to search for a solution. Starting from the equation with the derivative with respect to y:

$$\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right) h_{y,y'}(x, x') = -\frac{\partial}{\partial x} h_{y,y'}(x, x')\bigg|_{x=0} = -g(y; y', x')$$

Then we can say that

$$h_{y,y'}(x, x') = \tilde{h}(x + y, y; x', y') = H(x + y; x', y') - G(y; x' + y')$$

With $G(y; x' + y') = \int_0^y g(\tilde{y}; x', y')d\tilde{y}$. By using the initial condition at $x = 0$ we have:

$$0 = h_{y,y'}(0, x') = \tilde{h}(y, y; x', y') = H(y; x', y') - G(y; x' + y') \to H(y; x', y') = G(y; x', y') \quad \forall y$$

This means that H and G are the same function. Now since the function h is given by the sum of two functions, both the functions will satisfy the same differential equation that is satisfied by h. We can then write

$$\left(\frac{\partial}{\partial y'} - \frac{\partial}{\partial x'}\right) G(x + y; x', y') = -\frac{\partial}{\partial x'} G(x + y; x', y')\bigg|_{x'=0} = -u(x + y; y')$$

Then, as before

$$G(x + y; x', y') = T(x + y; x' + y') - U(x + y; y')$$

with $U(x + y; y') = \int_0^{y'} u(x + y; \tilde{y}')d\tilde{y}'$. Our initial function is now:

$$h_{y,y'}(x, x') = T(x + y; x' + y') - U(x + y; y') - T(y; x' + y') + U(y; y')$$

By imposing the initial condition at $x' = 0$ we end up with

$$0 = h_{y,y'}(x, 0) = T(x + y; y') - U(x + y; y') - T(y; y') + U(y; y')$$

Which is solved by the choice $T(x; y) = U(x; y)$. The function h is then rewritable as

$$h_{y,y'}(x, x') = U(x + y; x' + y') - U(x + y; y') - U(y; x' + y') + U(y; y')$$

It can eventually be proven that the correct form for $U(z, z')$ that solves the equation and respects all the boundary conditions is:

$$\begin{aligned} U(z, z') =& Q_0(\min(z, z')) + \sum_{n=1}^{\infty} Q_0(\min(z - n\overline{x}, z'))\theta(z - n\overline{x}) + \\ &+ \sum_{n=1}^{\infty} Q_0(\min(z, z' - n\overline{x}))\theta(z' - n\overline{x}) + \sum_{n,n'=1}^{\infty} Q_0(\min(z - n\overline{x}, z' - n'\overline{x}))\theta(z - n\overline{x})\theta(z' - n'\overline{x}) \end{aligned}$$