POLITECNICO DI TORINO

Master's Degree in Physics of Complex Systems

Master's Degree Thesis

Many-body localization in open quantum system

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Abstract

We consider a spin chain coupled to a bath of independent harmonic oscillators. We will use the bosonization for the spins and treat the bath in the Caldeira-Leggett framework. Our goal is to compute transport properties and determine if the bath can induce localization in the spin chain.

Acknowledgements

This thesis is dedicated to the memory of Philip W. Anderson (1923-2020), Nobel prize theoretical physicist

I would like to thank my two supervisors for the support and the guidance, during these months of internship. I would like also to thank prof. Montorsi for her useful advice.

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Acronyms

MBL

Many-Body Localization

ETH

Eigenstate Thermalization Hypothesis

RG

Renormalization Group

Chapter 1

Introduction

The study of the localization in open dissipative quantum systems is indeed a promising and innovative one.
In general, the phenomenon of localization is related to the question of how a many-body system thermalizes and whether it maintains the memory of the initial state. For this sake, we distinguish between two different types of behaviour: the ergodic phase, well described by the eigenstate thermalization hypothesis, and the many-body localized phase (MBL)(FIG.1.1).

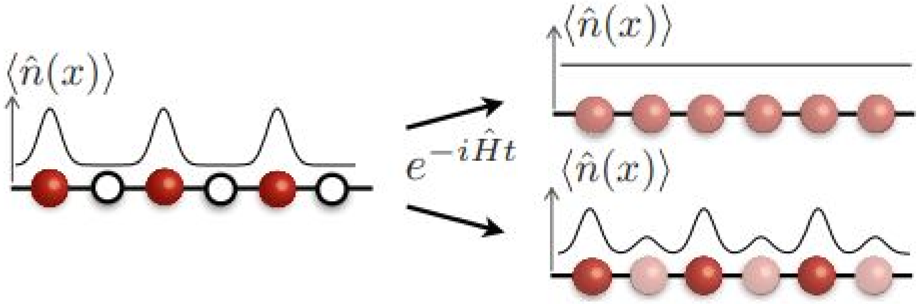

Figure 1.1: Ergodic vs. Localized behaviour with unitary quantum dynamics

Starting from a definite initial quantum state of preparation, the system evolves according to an unitary evolution governed by the Hamiltonian $\hat{H}$. At large times, it can lose all the memory of the initial state (ergodic phase, above) or retain its memory (MBL phase, below)(from [1]).

In the ergodic phase, the system cancels all the memory of the initial state. At large times, it reaches thermal equilibrium and can be characterized using few macroscopic quantities. This case is indeed well described by equilibrium quantum statistical mechanics.

Let us now consider an isolated quantum system with an Hamiltonian $\hat{H}$. The system is in the initial state $|\psi(0)\rangle$, which can be expanded in the eigenstates of the Hamiltonian as:

$$|\psi(0)\rangle = \sum_{\alpha} A_{\alpha}|\alpha\rangle \tag{1.1}$$

After the temporal evolution, the quantum state becomes:

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle = \sum_{\alpha} A_{\alpha} e^{-iE_{\alpha}t}|\alpha\rangle \tag{1.2}$$

However, the temporal evolution does not modify the probabilities associated to the eigenstates.
The concept of thermalization implies that the time average of the observables corresponds to the ensemble average :

$$\langle O\rangle_{\infty} = \lim_{T\to\infty}\frac{1}{T}\int_0^T \langle\psi(t)|\hat{O}|\psi(t)\rangle = \sum_{\alpha} p_{\alpha}\langle\alpha|\hat{O}|\alpha\rangle \tag{1.3}$$

The value of the observable at infinite time is determined by the probabilities $p_{\alpha} = |A_{\alpha}|^2$ and the expectation values of the operator over the eigenstates. The off-diagonal elements at infinite time are washed away, as they oscillate at different frequencies depending on the difference between the energy eigenstates. The ETH states that this diagonal elements agree with the microcanonical value of the observable O at fixed energy E_{α}, $\langle\alpha|\hat{O}|\alpha\rangle \approx O_{mc}(E_{\alpha})$.
The ETH has implications for the entanglement entropy. For any eigenstate $|\alpha\rangle$ that obeys ETH, this implies that the reduced density matrix is thermal in the associated small subsystem A: $\rho_A = \mathrm{Tr}_{\bar{A}}|\alpha\rangle\langle\alpha|$. Then, the von Neumann entanglement entropy

$$S_{ent}(A) = -\mathrm{Tr}\rho_A \log\rho_A \tag{1.4}$$

is equal to the thermal one and therefore scales as the volume

$$S_{ent}(A) \propto vol(A) \tag{1.5}$$

There are many ways in which a system fails to thermalize. The first example of localization has been studied in 1958 by Anderson [2]. He introduced the concept to describe a class of materials in which the insulating behaviour is driven by the presence of strong disorder. The Hamiltonian of the model he studied is:

$$(H,\psi)(i) = E_i\psi_i + \sum_i\sum_{i\neq k} V(|i-k|)\psi_k \tag{1.6}$$

where the energies E_i are stochastic variables distributed according to a probability distribution defined in the interval $[-W, W]$, and associated to the lattice sites i, W being the disorder strength and $V(|i-k|)$ a potential falling off as $1/r^2$ as $r \to \infty$. The spectrum of this model has been shown to be localized in 1 and 2 dimensions. In higher dimensions there may be a localized-delocalized phase transition as a function of the disorder strength, with the appearance of a mobility edge in the energy spectrum which separates the low-energy localized states from the extended ones at higher energies [3]. In the localized phase, the single-particle eigenstates are no more plane waves, but they are confined in some regions of the space. The ergodicity breaking determines a radical change of the transport properties of the system and the probability for a particle to make a transition between two spatially separated sites becomes vanishing. The system in the localized phase behaves as an insulator with zero DC conductivity, while in the delocalized one it has finite conductivity.

Subsequently, the localization has been demonstrated to be robust even to the presence of electron-electron interaction. The interactions tend to induce the hopping of the particles between the lattice sites and to counterbalance the effect of the disorder. However, it has been shown that the localized phase survives in presence of weak interactions. For example, it has been studied the subsequent model:

$$H_{XXZ} = J_{xy} \sum_i [\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y] + J_z \sum_i \sigma_i^z \sigma_{i+1}^z + \sum_i h_i \sigma_i^z \tag{1.7}$$

with h_i randomly distributed in the interval $[-W, W]$, where W is the disorder strength. It can be mapped using the Jordan-Wigner transformations into a fermionic model, the term with J_{xy} becomes a hopping term, J_z into a nearest-neighbour interaction and the field term into stochastic on-site energies. The recent studies has shown the presence of localization even in the context of non-zero temperature, as a dynamical phase of matter. Therefore, in this case the localization constitutes a robust phase of matter, called Many Body Localized phase, which is much different from the single-particle localization.

The entanglement entropy is an important signal of the presence of a MBL phase, as it is characterized by a logarithmic growth of entanglement entropy, in contrast to the linear growth associated to the ergodic phase.

In the MBL phase the system maintains the memory of the initial state (FIG. 1.1). In this phase, at T=0 the DC conductivity is zero while in the ergodic phase is finite and this is another important signal of localization. At finite temperature the behaviour is less clear.

Recently, the MBL phase has been shown to be integrable. It is, in fact, possible to construct a set of quasi-local integral of motion, which commute with the Hamiltonian. For example, in the above disordered Hamiltonian (1.7) with $J_{xy} = 0$ it has been demonstrated that the operators τ_i^z constitute a set of integral of motion. They are constructed as

$$\hat{\tau}_i^z = \hat{U}^\dagger \hat{\sigma}_z \hat{U} \tag{1.8}$$

with the unitary operator, which can be factorized over k-sites unitary operators:

$$\hat{U} = \prod_i ...\hat{U}^{(k)}_{i,i+1,...,i+k}...\hat{U}^{(2)}_{i,i+1} \tag{1.9}$$

with decreasing angles satisfying the property regarding the Frobenius norm:

$$||1 - \hat{U}^{(n)}_{i,i+1,...,i+n}||_F \leq e^{-n/\xi} \tag{1.10}$$

The integrability of the system implies an effective ergodicity breaking, as it constrains the dynamics of the systems. [1]
Our study is motivated by the results of the Leggett paper [4], which regard a two-state "spin-boson" interacting system. The model he studied was:

$$H = -\frac{1}{2}\hbar\Delta\sigma_x + \frac{1}{2}\epsilon\sigma_z + \frac{1}{2}q_0\sigma_z \sum_\alpha C_\alpha x_\alpha + \sum_\alpha \Big[\frac{1}{2}m_\alpha\omega_\alpha^2 x_\alpha^2 + \frac{p_\alpha^2}{2m_\alpha}\Big] \tag{1.11}$$

where x_α, p_α, m_α, ω_α are respectively the coordinates, momentum, mass and frequency of the α-th harmonic oscillator of the set. C_α is the strength of the coupling constant between the α-th harmonic oscillator and the spin variable. An important quantity considered in the paper is the spectral function $J(\omega)$ defined as:

$$J(\omega) = \frac{\pi}{2}\sum_\alpha \Big(C_\alpha^2/m_\alpha\omega_\alpha\Big)\delta(\omega - \omega_\alpha) \tag{1.12}$$

The author makes the assumption that $J(\omega)$ is of the form ω^s up to a frequency ω_c, which is much higher than Δ. So $J(\omega)$ must have the form $J(\omega) = A\omega^s e^{-\omega/\omega_c}$ with the conditions $\Delta << \omega_c$ and $k_B T << \hbar\omega_c$. At zero temperature, for example, the system is localized for $s < 1$ while for $s > 1$ is delocalized. As in the case of the disordered model, even in the spin-boson model, starting from the initial condition of spin up, the system may or may not maintain the memory of the initial state. The order parameter associated to this phase transition is $P(t \to \infty)$, where $P(t) = \langle\sigma_z(t)\rangle$, which is non-zero only in the localized phase.
The objective of our study is to investigate the possibility of localization induced on an interacting spin chain by the interaction with a bath model modelled in the Leggett framework as a collection of independent harmonic oscillators. For this end, we use the variational approximation to characterize the phase diagram of the system. This enables us to investigate the transport properties of the system, which are a major indicator of the presence of a localized phase. In particular, we distinguish between a "gapless" phase, where the DC conductivity is metallic, and a "gapped" one, in which the DC conductivity is zero and which corresponds to the MBL phase. The properties of the bath which are important are contained in the equal-position time correlation function of the harmonic oscillators $C(t - t')$. We examine only two extreme cases, in which the correlation function is a delta function and a constant.
The structure of the thesis is the following:

- In Chapter 2 we examine the bosonization procedure and we apply it to our model;
- In Chapter 3 we derive an effective theory from our model;
- In Chapter 4 we derive the Kubo formula and we use it to examine the contribution of the forward scattering term to the conductivity;
- In Chapter 5 we explain both the variational and the RG procedure and we apply them to study the conductivity in the case of a delta correlation function;
- in Chapter 6 we use the same variational procedure to study the constant correlation function case.

Chapter 2

The Bosonization of the model

We consider a 1/2 spin chain with a gapless spectrum, i.e. with no gap between the ground state and the excited ones, in contact with a bath of harmonic oscillators. The Hamiltonian writes:

$$\boxed{\mathrm{H} = \sum_n \left[J_{xy} \left(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y \right) + J_z S_n^z S_{n+1}^z \right] + g \sum_n S_n^z \sum_\alpha \lambda_\alpha (a_{\alpha n}^\dagger +}$$
$$\boxed{a_{\alpha n}) + \sum_\alpha \omega_\alpha \sum_n a_{\alpha n}^\dagger a_{\alpha n}} \tag{2.1}$$

with $-J_{xy} < J_z < J_{xy}$ also with $\hbar = 1$. Each spin is coupled to a collection of harmonic oscillators which are independent and with a position independent statistics. The spin operators have the usual commutation relations $[S_i, S_j] = i\epsilon_{ijk} S_k$ introducing the usual structure costants ϵ_{ijk}, which are anti-symmetric for any permutation of the indices starting from $\epsilon_{123} = 1$ and 0 when there are any equal indices.
The operators $a_{\alpha n}$ of the bath are a bosonic in character and they have the following commutation relations: $[a_{\alpha n}, a_{\alpha' n'}] = [a_{\alpha n}^\dagger, a_{\alpha' n'}^\dagger] = 0$ and $[a_{\alpha n}^\dagger, a_{\alpha' n'}] = i\delta_{nn'}\delta_{\alpha\alpha'}$.
The index n runs on the N spins in the spin chain, with periodic boundary conditions, while the index α runs on the different modes associated to the harmonic oscillators. The constant g is the coupling constant between the bath and the spin chain and regulates the strength of their interaction. λ_α represents the number of harmonic oscillators associated to every frequency. Finally, ω_α is the frequency of each mode of the harmonic oscillator.
The final result for the Hamiltonian, after applying the bosonization procedure, is:

$$\boxed{H = \frac{1}{2\pi} \int \mathrm{d}x \left[\frac{u}{K} \left(\partial_x \phi \right)^2 + K u \left(\partial_x \theta \right)^2 \right] - \frac{g}{\pi} \int \mathrm{d}x [(\partial_x \phi) -}$$

$$\boxed{a^{-1}(-1)^x \cos 2\phi] \sum_\alpha \lambda_\alpha (a^\dagger_{\alpha x} + a_{\alpha x}) + \sum_\alpha \omega_\alpha \int \mathrm{d}x a^\dagger_{\alpha x} a_{\alpha x}} \tag{2.2}$$

In this chapter firstly we discuss the phase diagram of the XXZ Heisenberg Spin Chain, then we introduce the transformations into a fermionic model and finally we apply to it the bosonization procedure.

2.1 Phase diagram of the XXZ Spin Chain

The XXZ spin chain is a generalization of the Heisenberg spin chain to account for an uni-axial anisotropy in the z direction. It is defined in general by the Hamiltonian:

$$\mathrm{H} = \sum_{n=1}^{N} \left[J_{xy} \left(S^x_n S^x_{n+1} + S^y_n S^y_{n+1} \right) + J_z S^z_n S^z_{n+1} \right] \tag{2.3}$$

in which if we set $J_z = J_{xy}$ we recover the usual isotropic Heisenberg model. In the plane xy, for $J_{xy} < 0$ we have a ferromagnetic ground state, while for $J_{xy} > 0$ an anti-ferromagnetic order is preferred. The effect of the J_z parameter is to regulate the strength of the z anisotropy.

Now we study the phase diagram for $J_{xy} > 0$, as the case $J_{xy} < 0$ can be simply obtained with a π rotation of every spin and by transforming J_z into $-J_z$. So, in this case we obtain three phases:

- For $J_z < -J_{xy}$ and zero magnetic field we have a ferromagnetic behaviour with a gap in the dispersion relation and in the limit $J_z \to \infty$ the ground state is $\prod_{j=1}^{N} |\uparrow_j\rangle$ and the low-energy excitations are spin waves;

- For $J_z > J_{xy}$ we have an anti-ferromagnetic gapped behaviour, as for $J_z \to -\infty$ the ground states are $|\uparrow\downarrow\uparrow\downarrow ...\rangle$ and $|\downarrow\uparrow\downarrow\uparrow ...\rangle$. The associated low-energy excitations are domain walls, in which the chain is split in two regions containing the two different ground states;

- In the interval $|J_z| < J_{xy}$, the phase is planar and paramagnetic. The ground state is paramagnetic and the low-energy excitations are the spinons, the excitations associated to this type of ground state;

- At the boundaries between the phases, at $J_z = -J_{xy}$ we have a gapless ferromagnet, while at $J_z = J_{xy}$ a gapless anti-ferromagnet.

The complete phase diagram is sketched in (FIG. 2.1). The solution of this model

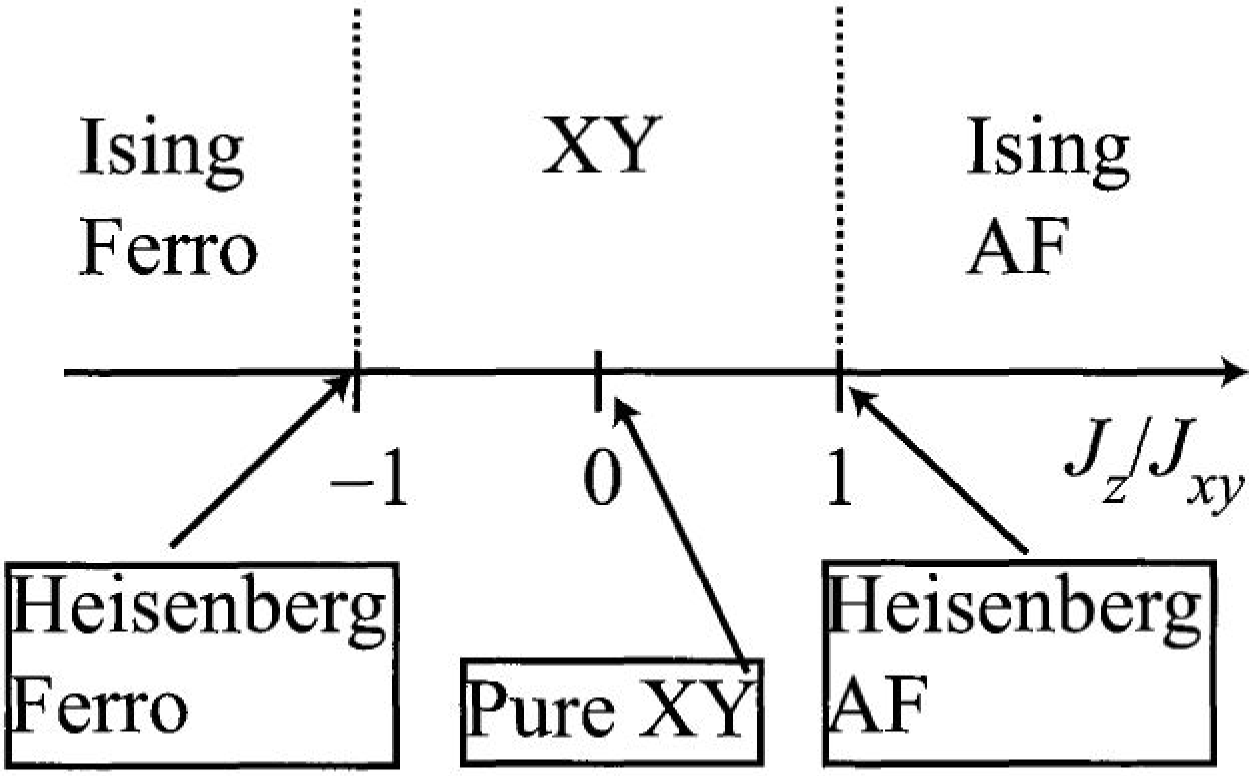

Figure 2.1: Phase Diagram of the XXZ spin chain as a function of J_z/J_{xy}

The XXZ Spin Chain has a rich phase diagram, with three different phases: anti-ferromagnetic, ferromagnetic and paramagnetic (from [5]).

can be obtained via a Bethe Ansatz methodology [6]. From now on, we fix into the zero magnetization phase of the spin chain and we consider no external magnetic field.

2.2 Fermionization

From the spin chain part of model, which is $\mathrm{H} = \sum_n \left[J_{xy} \left(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y \right) + J_z S_n^z S_{n+1}^z \right]$, we can map the spin degrees of freedom into fermionic ones. For this purpose, we use the celebrated Jordan-Wigner transformations:

$$S_n^+ = \psi_n^\dagger \exp\left(i\pi \sum_{m=1}^{n-1} \psi_m^\dagger \psi_m\right) \text{ and } S_n^z = \psi_n^\dagger \psi_n - \frac{1}{2} \tag{2.4}$$

with ψ_n and $\psi_n^\dagger$ are destruction/construction operators of a fermion on the site n, with the usual anti-commutation relations $\{\psi_n, \psi_m\} = \{\psi_n^\dagger, \psi_m^\dagger\} = 0$ and $\{\psi_n^\dagger, \psi_m\} = i\delta_{nm}$. The spin chain Hamiltonian becomes:

$$\boxed{H_{spin} = \sum_n \{-\frac{J_{xy}}{2}(\psi_n^\dagger \psi_{n+1} + \psi_n \psi_{n+1}^\dagger) + J_z(\psi_n^\dagger \psi_n - \frac{1}{2})(\psi_{n+1}^\dagger \psi_{n+1} - \frac{1}{2})\}} \tag{2.5}$$

Then, we pass from discrete fermionic operators to the left/right operators. In doing so, we linearize the spectrum because around the Fermi points, because we are dealing with low energy excitations, as shown in (FIG. 2.2). In order to derive

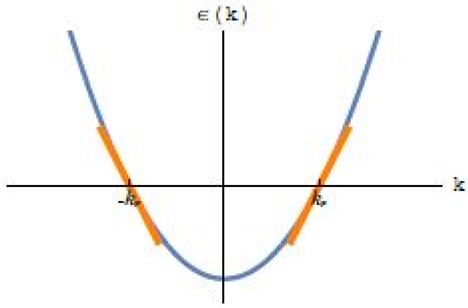

Figure 2.2: Linearization around the Fermi point (from [6])

This figure shows pictorially the transformation to the left/right fermionic operator through a linearization around the Fermi point.

the transformation, we expand the fermionic operator in Fourier space around them:

$$\psi_n \simeq e^{-ik_F x} \int_{-k_F-\Lambda}^{-k_F+\Lambda} e^{ikx} \psi_n(k) \frac{dk}{2\pi} + e^{+ik_F x} \int_{+k_F-\Lambda}^{+k_F+\Lambda} e^{ikx} \psi_n(k) \frac{dk}{2\pi}$$

$$\equiv e^{-ik_F x} \psi_+(x) + e^{+ik_F x} \psi_-(x) \tag{2.6}$$

The position index x is defined as $x = na$, where a is the lattice spacing. As we are in the zero magnetization phase, this means in the fermionic model that we are at half filling. Therefore, k_F being $\frac{\pi}{2a}$, the above relation reduces to:

$$\psi_n \Leftarrow \sqrt{a}\,[(-i)^x \psi_+(x) + i^x \psi_-(x)]$$

$$\psi_{n+1} \Leftarrow \sqrt{a}\left[(-i)^{x+1}\psi_+(x+a) + i^{x+1}\psi_-(x+a)\right] \tag{2.7}$$

The left/right fermionic operators have the fermionic anti-commutation relations: $\{\psi^\dagger_\pm(x), \psi_\pm(y)\} = i\delta(x-y)$, $\{\psi_\pm(x), \psi_\pm(y)\} = \{\psi^\dagger_\pm(x), \psi^\dagger_\pm(y)\} = 0$ and they anti-commute with each other $\{\psi^\dagger_\pm(x), \psi_\mp(y)\} = 0$.
The passage to the continuum limit for $\sum_n$ is $\sum_n \to \int dx a^{-1}$, where x becomes now a continuous variable.
The spin chain Hamiltonian can be rewritten in terms of the left/right operators as the sum of a kinetic part H_0 and an interacting part H_{int}, so $\boxed{H_{spin} = H_0 + H_{int}}$.
Dumping the terms which contain cross product of the left and the right fermions, which trivially anti-commute, we have:

$$H_0 = \frac{-iJ_{xy}}{2}\int dx[\psi^\dagger_+(x)\psi_+(x+a) - \psi^\dagger_-(x)\psi_-(x+a)$$

$$+\psi_+(x)\psi^\dagger_+(x+a) - \psi_-(x)\psi^\dagger_-(x+a)] \tag{2.8}$$

As the operators can be expanded as $\psi_\pm(x+a) \simeq \psi_\pm(x) + a\delta_x\psi_\pm(x)$ in the $a \to 0$ limit, we can rewrite it as:

$$H_0 = -\frac{iJ_{xy}}{2}\int dx[\psi^\dagger_+(x)(\psi_+(x) + a\frac{d}{dx}\psi_+(x)) - \psi^\dagger_-(x)(\psi_-(x) + a\frac{d}{dx}\psi_-(x)) +$$

$$\psi_+(x)(\psi^\dagger_+(x) + a\frac{d}{dx}\psi^\dagger_+(x)) - \psi_-(x)(\psi^\dagger_-(x) + a\frac{d}{dx}\psi^\dagger_-(x))] \tag{2.9}$$

Using the anti-commutation relations, the terms not involving a derivative are reduced to a constant. Furthermore, from:

$$H_0 = -\frac{iJ_{xy}a}{2}\int dx\left[\psi^\dagger_+\frac{d}{dx}\psi_+ + \psi_+\frac{d}{dx}\psi^\dagger_+ - \psi^\dagger_-\frac{d}{dx}\psi_- - \psi_-\frac{d}{dx}\psi^\dagger_-\right] \tag{2.10}$$

the last terms are integrated by part and the boundary term are neglected, finally giving:

$$\boxed{H_0 = -iJ_{xy}a\int dx\left[\psi^\dagger_+\frac{d}{dx}\psi_+ - \psi^\dagger_-\frac{d}{dx}\psi_-\right]} \tag{2.11}$$

The latter corresponds to the Hamiltonian of a massless Dirac fermion, where $\psi_\pm$ are the upper and lower component of the Dirac spinor. It is necessary to subtract the ground state expectation value of this operator as the vacuum of the Dirac Hamiltonian contains an infinite number of particles and therefore it is divergent. This procedure is called normal ordering and is indicated with the symbol :: applied to any operator A, with the following definition $: A := A - \langle 0|A|0\rangle$, $|0\rangle$ being the

Dirac vacuum. [7].
In one dimension, we can write the Dirac equation in Schrödinger form as:

$$i\frac{\partial\psi}{\partial t} = H\psi \tag{2.12}$$

where the Hamiltonian is

$$H = \alpha P + \beta m \tag{2.13}$$

with the momentum operator P and the mass m, $\alpha = \sigma_z$ and $\beta = \sigma_y$ defined in terms of Pauli matrices. If we focus on the massless case, the equation is then easily diagonalized in the two components $\psi_\pm$, obeying the commutation relations:

$$\psi^\dagger_\pm(x), \psi_\pm(y) = \delta(x-y) \tag{2.14}$$

The Hamiltonian in second quantization in terms of the two components is:

$$H = \int dx \psi^\dagger(x)(\alpha P)\psi(x)$$

$$= \int dx[\psi^\dagger_+(x)(i\partial_x)\psi_+(x) + \psi^\dagger_-(x)(-i\partial_x)\psi_-(x)] \tag{2.15}$$

which corresponds exactly to our Hamiltonian H_0
Replacing (2.6) in (2.3), we can write:

$$S^z(x) = a[:\psi^\dagger_+(x)\psi_+(x): + :\psi^\dagger_-(x)\psi_-(x): +$$

$$(-1)^n(:\psi^\dagger_+(x)\psi_-(x): + :\psi^\dagger_-(x)\psi_+(x):)] \tag{2.16}$$

We define the density operator and the staggered magnetization in terms of the two fields $\rho(x) =: \psi^\dagger_+(x)\psi_+(x) : + : \psi^\dagger_-(x)\psi_-(x) :$ and $M(x) =: \psi^\dagger_+(x)\psi_-(x) : + : \psi^\dagger_-(x)\psi_+(x) :$.
S^z becomes $S^z(x) = \rho(x) + (-1)^n M(x)$
In terms of this operators, neglecting the cross products between them, we get:

$$\boxed{H_{int} = -\frac{J_{xy}a}{2}\int dx[\rho(x)\rho(x+a) - M(x)M(x+a)]} \tag{2.17}$$

2.3 The Bosonization

Now that we have mapped the spin model into fermionic degrees of freedom, we finally apply the bosonization procedure, which maps the model into a bosonic scalar field theory.

We use the following mapping to the collective bosonic fields ϕ_+ and ϕ_- (see Appendix A):

$$\psi_\pm(x) = \frac{1}{\sqrt{2\pi a}} U_\pm e^{\pm i2\phi_\pm(x)} \tag{2.18}$$

The fermionic operators $U_\pm$, called Klein factors, are necessary to ensure that the fermionic anti-commutation relations between the fields of different species are maintained, once we have made sure that the ones between the Hamiltonian and the fields hold [8]. They are Majorana fermionic operator with a Clifford algebra, so they have the property that the creation and annihilation operators are equal to each other $U_\pm^\dagger = U_\pm$. Their effect is to change the total number of fermions by one, and therefore they do not commute with the number operator, while they commute with the bosonic fields ϕ_+ and ϕ_-. Furthermore they have also the following properties: $\{U_r^\dagger, U_{r'}^\dagger\} = \{U_r, U_{r'}\} = 0$, $\{U_r^\dagger, U_{r'}\} = 2\delta_{rr'}$ and $U_r^\dagger U_r = U_r U_r^\dagger = 1$.
In this case, the space of the Majorana operators has minimal dimension 2 (left and right) and can be represented in terms of Pauli matrices. For example, we may choose $U_+ = \sigma_x$ and $U_- = i\sigma_y$. The products of the two Klein factors are diagonal and, as $\Gamma^2 = 1$, if $\Gamma = i\sigma_x\sigma_y$, we can choose between the two eigenvalues ± 1 and we choose $+1$. This fixes the value of $U_+U_- = 1$, all the other products follow from the anti-commutation rules of the Clifford algebra. From now on, we don't take into account the action of the Klein factors when they don't contribute apart from an external constant factor which can be fixed to be equal to 1[9].
The fields $\phi_+(x)$ and $\phi_-(x)$, which we encounter above in the bosonization formula, have the following commutation relations : $[\phi_\pm(x), \phi_\pm(y)] = \frac{i\pi}{4}\epsilon(x-y)$, $\epsilon(x)$ being the step function, $[\phi_+(x), \phi_-(y)] = \frac{i\pi}{4}$ and the following expansion:

$$: \psi_\pm^\dagger(x)\psi_\pm(x+\epsilon) := \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} : \psi_\pm^\dagger(x)\partial_x^n\psi_\pm(x) := \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} j_n^\pm =$$

$$= \pm\frac{1}{2\pi i\epsilon}[: e^{\pm i2\sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}\partial_x^n\phi_\pm(x)} : -1] \tag{2.19}$$

whose r.h.s. is from:

$$: \psi_\pm^\dagger(x)\psi_\pm(x+\epsilon) := \frac{1}{2\pi a}[: e^{\pm i2(\phi_\pm(x+\epsilon)-\phi_\pm(x))} : -1] e^{4\langle\phi_\pm(x+\epsilon)\phi_\pm(x)-\phi_\pm^2(x)\rangle} =$$

$$\pm\frac{1}{2\pi i\epsilon}[: e^{\mp i2(\phi_\pm(x+\epsilon)-\phi_\pm(x))} : -1] \tag{2.20}$$

where we have applied : $e^A e^B :=: e^{A+B} : e^{\langle AB - \frac{A^2+B^2}{2}\rangle}$ and the correlation function:

$$G_+ = \langle \phi_\pm(x)\phi_\pm(0) - \phi_\pm^2(0)\rangle = \lim_{a\to 0} \frac{1}{4\pi} \ln \frac{a}{a \pm ix} \tag{2.21}$$

as $G_\pm(x) = \langle -\frac{(\phi(x)-\phi(0))^2}{2}\rangle = \langle \phi_\pm(x)\phi_\pm(0) - \phi_\pm^2(0)\rangle$, to which we apply the translational invariance property.
This follows from:

$$G_+ = \int_0^\infty \frac{dp}{\sqrt{4\pi|p|}} \int_0^\infty \frac{dq}{\sqrt{4\pi|q|}} \langle (\phi(p)\phi^\dagger(q))\rangle (e^{ipx} - 1)e^{-ap} = \frac{1}{4\pi} \ln \frac{a}{a - ix} \tag{2.22}$$

The boundary conditions used in this integral are $G_+(0) = 0$.
Then, using the relations concerning the fermionic current $j_n^\pm$, we can infer that:

$$\begin{aligned} \rho(x) = j_0^+ + j_0^- = -\frac{1}{\pi}\delta_x \phi(x) \\ H_0 \simeq \int dx[j_1^+ - j_1^-] = \frac{Ja}{\pi} \int dx[(\partial_x\phi_+)^2 + (\partial_x\phi_-)^2] \end{aligned} \tag{2.23}$$

which are the density operator and the Hamiltonian H_0. Introducing the dual fields $\phi = \phi_+ + \phi_-$ and $\theta = \phi_+ - \phi_-$, we can rewrite it as:

$$H_0 = \frac{J_{xy}a}{2\pi} \int dx[(\partial_x\theta(x))^2 + (\partial_x\phi(x))^2] \tag{2.24}$$

What we obtain for H_0 is a quadratic Hamiltonian associated to a bosonic scalar field theory.
The derivative of the dual field is the canonically conjugate variable of $\phi(x)$, as we can see from the commutation relations. In fact, we have that the conjugate variable of $\phi(x)$ is $\Pi(x) = \frac{1}{\pi}\partial_x\theta(x)$, for which $[\phi(x), \Pi(x')] = [\phi(x), \frac{1}{\pi}\partial_x\theta(x')] = i\delta(x - x')$, given $[\phi(x), \theta(y)] = i\pi\epsilon(x - y)$, where $\epsilon(x - y)$ is the step function (Appendix A).
For the $M(x)$, we use that $M(x) = \frac{1}{2\pi a}[U_+U_- : e^{-i2\phi_+(x)}e^{-i2\phi_-(x)} : -U_-U_+$ $h.c.] = U_+U_- \frac{1}{2\pi a}[: e^{-i2\phi(x)} : e^{\frac{1}{2}4\pi\frac{i}{4}} - h.c.] = \frac{i}{\pi a}\cos(2\phi(x))$
because of the commutation relations between ϕ_+ and ϕ_- and the anti-commutation relations between the fermionic $U_\pm$ operators. From this, applying Werner formulas and expanding in series, we can infer that:

$$\begin{aligned} \lim_{a\to 0} M(x)M(x+a) = \frac{-1}{(\pi a)^2}\cos(2\phi(x+a))\cos(2\phi(x)) \\ = \frac{-1}{2(\pi a)^2}[\cos(2(\phi(x+a) - \phi(x))) + \cos(2(\phi(x+a) + \phi(x)))] \\ \simeq \frac{1}{2\pi^2 a^2}[+2a^2(\partial_x\phi)^2 - \cos(4\phi(x))] \end{aligned} \tag{2.25}$$

The total spin Hamiltonian thus becomes:

$$\boxed{H_{spin} = \int dx \frac{J_{xy}a}{2\pi}[(\partial_x\theta(x))^2 + \left(1 + \frac{4J_z}{\pi J_{xy}}\right)(\partial_x\phi(x))^2] - \frac{J_z}{2\pi^2 a^2}\cos(4\phi(x))} \tag{2.26}$$

We can neglect cosine terms in the paramagnetic phase. For the interaction term with the bath, we can recall that $S^z(x) = \rho(x) + (-1)^n M(x)$ and substitute the above expressions. Then, (2.1) can be written in the bosonization language in the final form (2.2), in which the first term represents the spin Hamiltonian with

$$uK = J_{xy}a$$

$$u/K = J_{xy}a(1 + \frac{4J_z}{\pi J_{xy}}) \tag{2.27}$$

a being the lattice spacing [6].
The second term represents the interaction with the bath, while the third is the Hamiltonian of the harmonic oscillators.
Obviously, $(-1)^x$ in the continuum limit represents $\cos(2\phi - 2k_F x)$. For the last term, the continuum limit is trivial.
These expressions for uK and u/K are valid only in the XY limit for the spin chain, i.e. for J_z small. The exact solution can be obtained using the Bethe-Ansatz technique and is:

$$J_z/J_{xy} = -\cos(\pi\beta^2)$$

$$1/K = 2\beta^2$$

$$u = \frac{1}{1-\beta^2}\sin(\pi(1-\beta^2))\frac{J_{xy}}{2} \tag{2.28}$$

As a consequence, from the phase diagram of the XXZ spin chain (FIG. 2.1) we can infer that the Heisenberg point corresponds to $K = 1/2$, marking the phase transition between the gapped and the gapless phase for the spin chain XXZ. Furthermore, in the limit $J_z \to J_{xy}$, K diverges and u tends to 0, meaning that the ferromagnetic phase is indeed not Luttinger-like [5].

Chapter 3

Derivation of the effective action

Now that we have written the bosonized version of the model, our next goal is to write the partition function and to derive the effective action, which can adequately describe the physical phenomenon. The partition function of the model at inverse temperature β can be written using the path integral techniques in the imaginary time $0 < \tau < \beta$ (with $\hbar = 1$ from now on). From the Hamiltonian $H(\phi, \Pi, a)$, we can compute the partition function $Z = Tr[e^{-\beta H}]$.

The partition function of the spin part only can be expressed as the functional integral:
$Z_S = \int \mathrm{D}\phi(x,\tau)\mathrm{D}\Pi(x,\tau)\ e^{\int_0^\beta d\tau \int dx [i\Pi(x,\tau)\partial_x\phi(x,\tau) - H(\phi(x,\tau),\Pi(x,\tau))]}$.The path integral is on periodic paths in the configuration space, in which $\phi(x,\tau) = \phi(x,\tau+\beta)$.
We may write it in terms of $\phi(x)$ and $\theta(x)$.
It reads:

$$-S_S = \int_0^\beta dt \int dx[\frac{i}{\pi}\partial_x\theta(x,\tau)\partial_\tau\phi(x,\tau) - \frac{1}{2\pi}(uK(\partial_x\theta)^2 + \frac{u}{K}(\partial_x\phi)^2)] \tag{3.1}$$

.
It can be Fourier transformed, obtaining:

$$-S_S = \frac{1}{\beta\Omega}\sum_{\mathbf{q}}\Big[-\frac{ik\omega_n}{\pi}\phi(\mathbf{q})\theta(-\mathbf{q}) - \frac{uK}{2\pi}k^2\theta(\mathbf{q})\theta(-\mathbf{q}) - \frac{u}{2\pi K}k^2\phi(\mathbf{q})\phi(-\mathbf{q})\Big] \tag{3.2}$$

From now on, we use the notation $\mathbf{q} = (k, \omega_n/u)$ for the Matsubara discrete frequencies, which are $\omega_n = \frac{2\pi n}{\beta}$ for bosons and $\omega_n = \frac{(2n+1)\pi}{\beta}$ for the fermions. Then we complete the square in the integral:

$$-S_S = \frac{1}{\beta\Omega}\sum_{\mathbf{q}}\Big[-\frac{ik\omega_n}{\pi}\phi(\mathbf{q})\theta(-\mathbf{q}) - \frac{uK}{2\pi}k^2[\theta(\mathbf{q}) + \frac{i\omega_n\phi(\mathbf{q})}{uKk}][\theta(-\mathbf{q}) + \frac{i\omega_n\phi(-\mathbf{q})}{uKk}]$$

$$-\frac{u}{2\pi K}k^2\phi(\mathbf{q})\phi(-\mathbf{q})\Big] \tag{3.3}$$

After a gaussian integration on the variable $\tilde{\theta}(\mathbf{q}) = \theta(\mathbf{q}) + \frac{i\omega_n\phi(\mathbf{q})}{uKk}$ there is left only an average on ϕ on the action:

$$S_S = \frac{1}{2\pi K}\left[\frac{1}{u}\left(\partial_\tau\phi\right)^2 + u\left(\partial_x\phi\right)^2\right] \tag{3.4}$$

The action of the interaction between the bath and the system is obvious:

$$S_{\mathrm{SB}} = -\frac{g}{\pi}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}x \left[\partial_x\phi - a^{-1}\left(-1\right)^x \cos 2\phi\right] \sum_\alpha \lambda_\alpha (a_\alpha^*(\tau,x) + a_\alpha(\tau,x)) \tag{3.5}$$

The action of the bath S_B is the bosonic action of the Harmonic Oscillator. It can be derived from the action of a bosonic field $\hat{\psi}(x,t)$ with an Harmonic oscillator potential, which is

$$A(\hat{\psi},\hat{\psi}^\dagger) = \int dx \int_{t_a}^{t_b} dt \Big[i\psi^\dagger\partial_t\psi - \frac{\hbar^2}{2m}\psi^\dagger\partial_x^2\psi\Big] \tag{3.6}$$

The field $\hat{\psi}(x,t)$ describes an arbitrary number of particles, which is measured by the number operator, and satisfies the following commutation relations: $[\hat{\psi}(x,t),\hat{\psi}(x',t)] = [\hat{\psi}^\dagger(x,t),\hat{\psi}^\dagger(x',t)] = 0$ and $[\hat{\psi}(x,t),\hat{\psi}(x',t)] = \delta(x-x')$. By extremizing this action, we can obtain the Schrodinger equation. We can Fourier decompose the fields as:

$$\hat{\psi}(x,t) = \sum_p e^{ipx}\hat{a}_p(t) \tag{3.7}$$

The action becomes:

$$A(a,a^\dagger) = \sum_p \int_{t_a}^{t_b} dt \Big[a_p^\dagger i\partial_t a_p - \omega(p)a_p^\dagger a_p\Big] \tag{3.8}$$

The bosonic fields have the commutation relations: $[a_p(t),a_p'(t)] = [a_p^\dagger(t),a_p^{\dagger}{}'(t)] = 0$ and $[a_p^\dagger(t),a_p'(t)] = \delta_{p,p'}$

Then we can substitute the sum on the momenta with a sum on the modes α and introduce the position dependent fields $a_\alpha(t,x)$:

$$A(a,a^\dagger) = \int_{t_a}^{t_b} dt \sum_\alpha \Big[\int \mathrm{d}x a_\alpha^\dagger(t,x)(i\partial_t - \omega_\alpha)a_\alpha(t,x)\Big] \tag{3.9}$$

If we want to describe the quantum statistics of harmonic oscillators, we have to use the action with the Euclidean metric replacing $t \to -i\tau$ and integrating the time variable from 0 to β , as we identify $t_b - t_a = -i\beta$. We get [10]:

$$S_{\mathrm{B}} = \int_0^\beta \mathrm{d}\tau \sum_\alpha \left[\int \mathrm{d}x \; a_\alpha^*(\tau,x)(\partial_\tau + \omega_\alpha)a_\alpha(\tau,x)\right]$$

(3.10)

In total, for the partition function we obtain:

$$\boxed{Z = \int \mathrm{D}\phi(\tau,x)\mathrm{D}a(\tau,x)\ e^{-(S_{\mathrm{S}}+S_{\mathrm{S}B}+S_{\mathrm{B}})}} \tag{3.11}$$

with

$$\boxed{S_{\mathrm{S}} = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}x\ \frac{1}{2\pi K}\left[\frac{1}{u}\left(\partial_\tau\phi\right)^2 + u\left(\partial_x\phi\right)^2\right]}$$
$$\boxed{S_{\mathrm{S}B} = -\frac{g}{\pi}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}x \left[\partial_x\phi - a^{-1}\left(-1\right)^x \cos 2\phi\right] \sum_\alpha \lambda_\alpha (a^*_\alpha(\tau,x) + a_\alpha(\tau,x))}$$
$$\boxed{S_{\mathrm{B}} = \int_0^\beta \mathrm{d}\tau \sum_\alpha \left[\int \mathrm{d}x\ a^*_\alpha(\tau,x)(\partial_\tau + \omega_\alpha)a_\alpha(\tau,x)\right]} \tag{3.12}$$

If we are interested in observables of the the system only, we can integrate over the degrees of freedom of the bath.
It is useful to specify the unperturbed correlation function of the bath:

$$C(\tau-\tau',x-y) = \sum_\alpha \lambda_\alpha^2 \left\langle (a^*_\alpha(y,\tau) + a_\alpha(y,\tau))(a^*_\alpha(x,\tau') + a_\alpha(x,\tau'))\right\rangle_{\mathrm{B}} =$$

$$\delta(x-y)C(\tau-\tau') \tag{3.13}$$

where $C(\tau-\tau') = \sum_\alpha \lambda_\alpha^2 \left\langle (a^*_\alpha(x,\tau') + a_\alpha(x,\tau'))(a^*_\alpha(x,\tau) + a_\alpha(x,\tau))\right\rangle_{\mathrm{B}}$
The function $C(\tau)$ is periodic of period β and is at the equilibrium of the unperturbed bath. It is an input parameter we can fix to describe the characteristics of the bath (slow frequency bath or high frequency).
If we expand the partition function (normalized by the partition function of the bath) in power of the interaction term we obtain at the second order (the first averages to zero):

$$\frac{Z}{Z_B} = \frac{1}{Z_B}\int \mathrm{D}\phi(\tau,x)\mathrm{D}a(\tau,x)\ e^{-(S_{\mathrm{S}}+S_{\mathrm{S}B}+S_{\mathrm{B}})}$$

$$\simeq \frac{1}{Z_B}\int \mathrm{D}\phi(\tau,x)\mathrm{D}a(\tau,x)\ e^{-(S_{\mathrm{S}}+S_{\mathrm{B}})}\left(1 - S_{SB} + \frac{S_{SB}^2}{2}\right) \tag{3.14}$$

The second order term amounts to (up to constants):

$$\propto \frac{1}{Z_B}\int \mathrm{D}\phi(\tau,x)\mathrm{D}a(\tau,x)S_{SB}^2\; e^{-S_\mathrm{S}+S_\mathrm{B}} = \int \mathrm{D}\phi(\tau,x)e^{-S_\mathrm{S}}\frac{g^2}{\pi^2}\int dx\int dy$$

$$\int_0^\beta d\tau\int_0^\beta d\tau'[\partial_x\phi(x,\tau)-a^{-1}\,(-1)^x\cos 2\phi(x,\tau)][\partial_y\phi(y,\tau')-a^{-1}\,(-1)^y\cos 2\phi(y,\tau')]$$

$$C(\tau-\tau',x-y) = \int \mathrm{D}\phi(\tau,x)e^{-S_\mathrm{S}}\frac{g^2}{\pi^2}\int dx\int_0^\beta d\tau\int_0^\beta d\tau'[\partial_x\phi(x,\tau)-a^{-1}(-1)^x$$

$$\cos 2\phi(x,\tau)][\partial_x\phi(x,\tau')-a^{-1}\,(-1)^x\cos 2\phi(x,\tau')]C(\tau-\tau') = \int \mathrm{D}\phi(\tau,x)e^{-S_\mathrm{S}}$$

$$\frac{g^2}{\pi^2}\int \mathrm{d}x\int_0^\beta \mathrm{d}\tau\int_0^\beta \mathrm{d}\tau'[\partial_x\phi(x,\tau)\partial_x\phi(x,\tau')+a^{-2}\cos 2\phi(x,\tau)\cos 2\phi(x,\tau')]C(\tau-\tau') \tag{3.15}$$

We can re-exponentiate the second order term $\langle S_\mathrm{SB}^2\rangle_\mathrm{B}/2$ and obtain an exact effective action for the system degrees of freedom only:

$$\boxed{Z\propto\int \mathrm{D}\phi(\tau,x)e^{-S_\mathrm{eff}}} \tag{3.16}$$

with:

$$\boxed{S_\mathrm{eff} = \frac{1}{2\pi K}\int_0^\beta \mathrm{d}\tau\int \mathrm{d}x\;\left[\frac{1}{u}\,(\partial_\tau\phi)^2+u\,(\partial_x\phi)^2\right]-\frac{g^2}{2\pi^2}\int \mathrm{d}x\int_0^\beta \mathrm{d}\tau\int_0^\beta \mathrm{d}\tau'}$$
$$\boxed{\big[\partial_x\phi(x,\tau)\partial_x\phi(x,\tau')+a^{-2}\cos 2\phi(x,\tau)\cos 2\phi(x,\tau')]C(\tau-\tau')} \tag{3.17}$$

Where the oscillating terms has been neglected. The first term which multiplies g^2 represents the forward scattering in the language of fermionic left/right movers, as we may represent it as the bosonized version of the product of two of operators like$\rho(x) =: \psi_+^\dagger(x)\psi_+(x) : + : \psi_-^\dagger(x)\psi_-(x) :$, which do not change the type of the operators (left movers remain left movers and right movers the same). Instead, the latter represents the backward one, as the cosine terms are produced by $: \psi_+^\dagger(x)\psi_-(x) : + : \psi_-^\dagger(x)\psi_+(x) :$, which changes the left into right movers and vice versa.
To distinguish their contributions, we may write two different coupling costants for the forward and the backward scattering, g_1 and g_2.

Chapter 4

The Effect of the Forward Scattering

In this chapter we demonstrate that the effect of the forward scattering alone for the calculation of the conductivity is negligible, as it leaves unchanged the type of fermion (left or right). Therefore we set $g_2 = 0$ from now on.
In the first section we investigate how to calculate it using the Kubo formula. Then, we discuss the conductivity in 3 cases: the delta correlation function, the constant correlation function and the generic case.

4.1 Derivation of the Kubo formula

The Kubo formula (1959) is related to the calculation of DC electrical conductivity in the framework of the linear response theory. The meaning of linear response in general is that the signal measured is directly proportional to the external perturbation which is applied. Usually, this assumption is valid when the applied external perturbation is not too strong.
Applying an external field $E_\alpha^{(ext)}(\mathbf{r}, t) = \Xi_\alpha^{(ext)} e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}$, where the Greek letters α and β represent the spatial coordinates and i is the particle index, the linear response implies that the current in the solid is:

$$J_\alpha(\mathbf{r}, t) = \sum_\beta \sigma'_{\alpha\beta} \Xi_\beta^{(ext)} e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} \tag{4.1}$$

However, the $\sigma'_{\alpha\beta}$ is not the real conductivity, as the real conductivity is related to the electric field in the solid:

$$J_\alpha(\mathbf{r}, t) = \sum_\beta \sigma_{\alpha\beta} \Xi_\beta e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} \tag{4.2}$$

where we have defined the time-dependent electric field as:

$$E_\beta(\mathbf{r}, t) = \Xi_\beta e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} \tag{4.3}$$

The conductivity is complex, containing both a real and an imaginary part:

$$\sigma_{\alpha\beta} = \Re(\sigma_{\alpha\beta}) + i\Im(\sigma_{\alpha\beta}) \tag{4.4}$$

The Hamiltonian of the system is of the form:

$$H + H' \tag{4.5}$$

Where H' is the interaction between the system and the external perturbation. It can be written as:

$$H' = -\frac{1}{c}\int d^3r j_\alpha(\mathbf{r}) A_\alpha(\mathbf{r}, t) \tag{4.6}$$

Where we have used the vector potential:

$$A_\alpha(\mathbf{r}, t) = \frac{-ic}{\omega} E_\alpha(\mathbf{r}, t) \tag{4.7}$$

Here we are in the Coulomb gauge, i.e. the one in which $\nabla \cdot \mathbf{A} = 0$. The current operator is the summation over the velocities and the charges of all the particles:

$$j_\alpha = \frac{1}{2m}\sum_i e_i[\mathbf{p}_i\delta(\mathbf{r}-\mathbf{r}_i) + \delta(\mathbf{r}-\mathbf{r}_i)\mathbf{p}_i]_\alpha \tag{4.8}$$

The measured current, in terms of the particle velocities and the volume V is:

$$J_\alpha(\mathbf{r}, t) = \frac{e}{V}\sum_i \langle v_{i\alpha}\rangle \tag{4.9}$$

the velocity is:

$$\mathbf{v}_i = \frac{1}{m}\left[\mathbf{p}_i - \frac{e}{c}\mathbf{A}(\mathbf{r}_i)\right] \tag{4.10}$$

Then, the resulting current is:

$$J_\alpha(\mathbf{r}, t) = \frac{e}{mV}\sum_i \langle p_i\rangle - \frac{e^2}{mcV}\sum_i \mathbf{A}_\alpha(\mathbf{r}_i, t) \tag{4.11}$$

Knowing the proportionality between the momentum and the current operator, we may rewrite the current as:

$$J_\alpha(\mathbf{r}, t) = \langle j_\alpha(\mathbf{r}, t)\rangle + i\frac{n_0 e^2}{m\omega}\mathbf{E}_\alpha(\mathbf{r}_i, t) \tag{4.12}$$

replacing $\sum_i /V$ with the density n_0 and inserting the electric field. The current is thus separated in two terms: one proportional to the expectation value of the current operator, the other to the electric field. We call these two terms $J_\alpha^{(1)}$ and $J_\alpha^{(2)}$ and the current is the sum of the two:

$$J_\alpha = J_\alpha^{(1)} + J_\alpha^{(2)} \tag{4.13}$$

$$J_\alpha^{(1)} = i\frac{n_0 e^2}{m\omega}\mathbf{E}_\alpha(\mathbf{r}_i, t) \tag{4.14}$$

$$J_\alpha^{(2)} = \langle j_\alpha(\mathbf{r}, t)\rangle \tag{4.15}$$

Our job, then is to determine the expectation value of current operator. Now, it is necessary to obtain the expression for $J_\alpha^{(2)}$ in order to derive the Kubo formula. There are numerous possible derivations, but we will present a general one at finite temperature.

We start from the previously introduced time-dependent perturbation H'. Then we introduce the equilibrium density matrix in absence of H' $\rho_0 = e^{\beta(\Omega - H + \mu N)}$ in the gran-canonical ensemble, with Ω the gran-canonical potential and μ the chemical potential. We call the density matrix of the system with H' as $\rho(t)$. We assume that the system is described by ρ_0 at $t = -\infty$, then the perturbation is switched on. $\rho(t)$ obeys the following equation of motion :

$$\frac{d\rho(t)}{dt} = -i[H + H'(t), \rho(t)] \tag{4.16}$$

Solving this, we can compute the current $J_\alpha^{(2)}$ as:

$$J_\alpha^{(2)} = Tr\left[\rho(t) j_\alpha\right] \tag{4.17}$$

Then, we assume that $\rho(t) = \rho_0 + f(t)$ and we can write a differential equation for $f(t)$, as ρ_0 is time-independent:

$$i\frac{df}{dt} = [H, \rho_0] + [H, f] + [H', \rho_0] + [H', f] \tag{4.18}$$

The first commutator is zero, as ρ_0 is at equilibrium with respect to H. Since f assumed to be proportional to H', if we treat the perturbation H' to be small, we can neglect terms that are $O(H'^2)$. We are thus left with:

$$i\frac{df}{dt} = [H, f] + [H', \rho_0] + O(H'^2) \tag{4.19}$$

We can rewrite this equation as:

$$i\frac{df}{dt} - [H, f] = [H', \rho_0] \tag{4.20}$$

The left-hand side can be expressed as:

$$e^{-iHt}\, i\frac{d}{dt}(e^{iHt}f(t)e^{-iHt})\, e^{iHt} \tag{4.21}$$

In this way, we can integrate the equation:

$$i\frac{d}{dt}(e^{iHt}f(t)e^{-iHt}) = e^{iHt}[H', \rho_0]e^{-iHt} = [H'(t), \rho_0] \tag{4.22}$$

$$f(t) = f(-\infty) - ie^{-iHt}\left[\int_{-\infty}^{t} dt'[H'(t'), \rho_0]\right]e^{iHt} \tag{4.23}$$

$f(-\infty) = 0$ as the interaction H' is not present at $t = -\infty$. We can note that f is appropriately proportional to H'. Now, we insert this results in the expression of the current:

$$J_\alpha^{(2)} = Tr[\rho_0 j_\alpha(\mathbf{r})] + Tr[f(t)j_\alpha(\mathbf{r})] \tag{4.24}$$

The first term is 0 as there is no current without any applied external electric field. Substituting the previous result for f(t) :

$$J_\alpha^{(2)} = -iTr\left[e^{-iHt}\int_{-\infty}^{t} dt'[H'(t'), \rho_0]e^{iHt}j_\alpha(\mathbf{r})\right] \tag{4.25}$$

We can now use the property of the cyclicity of the trace:

$$J_\alpha^{(2)} = -iTr\left[\int_{-\infty}^{t} dt'[H'(t'), \rho_0]e^{iHt}j_\alpha(\mathbf{r})e^{-iHt}\right] \tag{4.26}$$

Recognizing the time evolution of the current operator:

$$J_\alpha^{(2)} = -iTr\left[\int_{-\infty}^{t} dt'[H'(t'), \rho_0]j_\alpha(\mathbf{r}, t)\right] \tag{4.27}$$

Using again its cyclical properties, we can rewrite the trace as:

$$Tr\left[\int_{-\infty}^{t} dt'\rho_0[j_\alpha(\mathbf{r}, t), H'(t')]\right] \tag{4.28}$$

This is the thermodynamical average of the commutator:

$$J_\alpha^{(2)} = -i\int_{-\infty}^{t} dt'\langle[j_\alpha(\mathbf{r}, t), H'(t')]\rangle \tag{4.29}$$

If we write the interaction H' in terms of the Fourier transform of the current operator as:

$$H' = \frac{i}{\omega}j_\alpha(\mathbf{q})\Xi_\alpha e^{-i\omega t} \tag{4.30}$$

the commutator results:

$$[j_\alpha(\mathbf{r},t),H'(t')] = \frac{i}{\omega}\Xi_\beta e^{-i\omega t'}[j_\alpha(\mathbf{r},t),j_\beta(\mathbf{q},t')]$$

$$= \frac{i}{\omega}E_\beta(\mathbf{r},t)e^{-i\mathbf{q}\cdot\mathbf{r}}e^{i\omega(t-t')}[j_\alpha(\mathbf{r},t),j_\beta(\mathbf{q},t')] \tag{4.31}$$

Inserting it in the current yields:

$$J^{(2)}_\alpha = \frac{1}{\omega}E_\beta(\mathbf{r},t)e^{-i\mathbf{q}\cdot\mathbf{r}}\int_{-\infty}^{t} dt' e^{i\omega(t-t')}\langle[j_\alpha(\mathbf{r},t),j_\beta(\mathbf{q},t')]\rangle \tag{4.32}$$

Now, we may write the total conductivity, which we recall to be the proportionality constant to the electric field, as:

$$\sigma_{\alpha\beta}(\mathbf{q},\omega) = \frac{1}{\omega}e^{-i\mathbf{q}\cdot\mathbf{r}}\int_{-\infty}^{t} dt' e^{i\omega(t-t')}\langle[j_\alpha(\mathbf{r},t),j_\beta(\mathbf{q},t')]\rangle + \frac{n_0e^2}{m\omega}i\delta_{\alpha\beta} \tag{4.33}$$

Averaging on the spatial coordinates, we get the real Kubo formula:

$$\sigma_{\alpha\beta}(\mathbf{q},\omega) = \frac{1}{\omega V}\int_{-\infty}^{t} dt' e^{i\omega(t-t')}\langle[j_\alpha(\mathbf{q},t),j_\beta(\mathbf{q},t')]\rangle + \frac{n_0e^2}{m\omega}i\delta_{\alpha\beta} \tag{4.34}$$

We may also express it shifting the time variable $t-t'\to t'$:

$$\sigma_{\alpha\beta}(\mathbf{q},\omega) = \frac{1}{\omega V}\int_{0}^{\infty} dt' e^{i\omega t'}\langle[j_\alpha(\mathbf{q},t'),j_\beta(\mathbf{q},0)]\rangle + \frac{n_0e^2}{m\omega}i\delta_{\alpha\beta} \tag{4.35}$$

The $\langle[j_\alpha(\mathbf{q},t'),j_\beta(\mathbf{q},0)]\rangle$ is a retarded Green function, called current-current correlation function [11].
This quite in general, but let us focus now on the case of our interest: the bosonized scalar free field theory, with Hamiltonian:

$$H = \frac{1}{2\pi}\int dx[uK(\pi\Pi(x))^2 + \frac{u}{K}(\nabla\phi(x))^2] \tag{4.36}$$

Let us assume that we have a 1d wire of length L, which is submitted to an electric field $E(t) = E_0e^{-i(\omega+i\delta)t}$, which is small and uniform, in the limit $\delta\to 0$. Using the Kubo formula in this case for the expectation value of the current operator we have:

$$\langle j(x,t)\rangle = \frac{E_0e^{-i(\omega+i\delta)t}}{i(\omega+i\delta)}\left[-D - \int_{-L/2}^{L/2} dx'dt' e^{i(\omega+i\delta)(t-t')}\langle j(x,t);j(x',t')\rangle\right] \tag{4.37}$$

where D corresponds to the diamagnetic term and $\langle;\rangle$ to the retarded correlation function. Once we have done the minimal substitution in the Hamiltonian

$\Pi(x) \to \Pi(x) - \frac{eA}{\pi}$ with the vector potential corresponding to the electric field, the diamagnetic term corresponds to:

$$D = -\frac{\partial^2 H}{\partial A \partial A} = \frac{e^2 uK}{\pi} \tag{4.38}$$

The diamagnetic term corresponds to $J^{(1)}$, while the other one to $J^{(2)}$. For the electrical current, we can use the conservation law:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j} \tag{4.39}$$

Then, using the bosonized form of the density:

$$j(x,t) = \frac{e}{\pi}\partial_t \phi(x,t) \tag{4.40}$$

Using the Heisenberg equation of motion for ϕ:

$$j(x,t) = euK\Pi(x,\tau) \tag{4.41}$$

As a consequence, the current-current retarded correlation function can be calculated as:

$$\chi(\tau - \tau') = -\langle j(x,\tau) j(x',\tau')\rangle = -\left(euK\right)^2 \langle \Pi(x,\tau)\Pi(x',\tau')\rangle \tag{4.42}$$

The functional integral over Π can be computed, using $\frac{1}{Z}(\frac{\partial Z}{\partial \beta})^2$, as:

$$\langle \Pi(x,\tau)\Pi(x',\tau')\rangle = -\frac{1}{(euK)^2}\langle \partial_\tau \phi(x,\tau)\partial'_\tau \phi(x',\tau')\rangle + \frac{1}{euK}\delta(x-x')\delta(\tau-\tau') \tag{4.43}$$

If we Fourier transform this, we obtain:

$$\langle \Pi(q,\omega_n)^* \Pi(q,\omega_n)\rangle = -\frac{1}{(euK)^2}\omega_n^2 \langle \phi(q,\omega_n)^* \phi(q,\omega_n)\rangle + \frac{1}{euK} \tag{4.44}$$

The last term perfectly cancels the diamagnetic term. Now, all we have to do is to insert this into the the (4.34) and we finally have the expression of the conductivity:

$$\boxed{\sigma(\omega) = -\frac{e^2}{\pi^2} i(\omega + i\delta)\langle \phi(q,\omega_n)^* \phi(q,\omega_n)\rangle_{i\omega_n \to \omega + i\delta}} \tag{4.45}$$

in the limit $L \to \infty$. [5]

4.2 Conductivity in the case $C(t-t') = C\delta(t-t')$

Following the results of the previous sections, we can see that in this case the action reduces to the free case:

$$S_{\text{eff}} = \frac{1}{2\pi K}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}x \, \left[\frac{1}{u}\left(\partial_\tau\phi\right)^2 + (u - \frac{g_1^2 KC}{\pi})\left(\partial_x\phi\right)^2\right] \tag{4.46}$$

Now, we can calculate the conductivity. But first we need to compute the $\phi\phi$ correlation function.
For a quadratic field theory, the most common technique is the diagonalization with the Fourier transform. In general we have:

$$\langle\phi(\mathbf{q}_1)^*\phi(\mathbf{q}_2)\rangle = \frac{\int \mathrm{D}\phi[q]\phi^*(q_1)\phi(q_2)e^{-\frac{1}{2}\sum_q A(q)\phi^*(q)\phi(q)}}{\int \mathrm{D}\phi[q]e^{-\frac{1}{2}\sum_q A(q)\phi^*(q)\phi(q)}} = \frac{1}{A(q_1)}\delta_{q_1q_2} \tag{4.47}$$

where $A(q)$ is a generic N*N matrix.
Applying this result we obtain for the correlation:

$$\langle\phi(\mathbf{q}_1)^*\phi(\mathbf{q}_2)\rangle = \frac{\pi K\delta_{\mathbf{q}_1\mathbf{q}_2}\Omega\beta}{\frac{\omega_n^2}{u} + (u - \frac{g_1^2KC}{\pi})k_1^2} \tag{4.48}$$

Then we can use the Kubo formula:

$$\sigma(\omega) = -\frac{e^2}{\pi^2}(\omega + i\delta)\langle\phi(k=0,\omega_n)^*\phi(k=0,\omega_n)\rangle_{i\omega_n\to\omega+i\delta}$$

$$= -\frac{e^2}{\pi^2}(\omega+i\delta)\frac{\pi uK}{\omega_n^2}|_{i\omega_n\to\omega+i\delta} \tag{4.49}$$

Now we can apply the theorem of Sokhotski-Plemelj to calculate the conductivity. Here we offer a simple proof of it $\forall$ f(x) (analyticity not required) :

$$\lim_{\epsilon\to 0^+}\int_a^b \frac{f(x)}{x\pm i\epsilon}dx = \mp i\pi \lim_{\epsilon\to 0^+}\int_a^b \frac{\epsilon}{\pi(x^2+\epsilon^2)}f(x)dx + \lim_{\epsilon\to 0^+}\int_a^b \frac{x^2}{x^2+\epsilon^2}\frac{f(x)}{x}dx \tag{4.50}$$

The first term is a delta function in the limit, the second into a Cauchy principal part:

$$\lim_{\epsilon\to 0^+}\int_a^b \frac{f(x)}{x\pm i\epsilon}dx = \mp i\pi f(0) + P\int_a^b \frac{f(x)}{x}dx \tag{4.51}$$

The resulting conductivity is:

$$\boxed{\sigma(\omega) = \frac{e^2}{\pi^2}\frac{iuK}{\omega+i\delta} = e^2uK\left[\delta(\omega) + iP\frac{1}{\pi\omega}\right]} \tag{4.52}$$

which is absolutely equal to the conductivity of the initial quadratic action, whose real part corresponds only to the Drude peak. [5][11].

4.3 Conductivity in the constant case: $C(t-t') = C$

In this case, the action is:

$$S_{\text{eff}} = \frac{1}{2\pi K}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}x \; \left[\frac{1}{u}\left(\partial_\tau\phi\right)^2 + u\left(\partial_x\phi\right)^2\right] - \frac{g^2C}{2\pi^2}\int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau'$$

$$[\partial_x\phi(x,\tau)\partial_x\phi(x,\tau')] \quad (4.53)$$

Now, we can Fourier transform the action, in order to diagonalize it. As before, we use the notation $\mathbf{q} = (k, \omega_n/u)$ and $\mathbf{r} = (x, u\tau)$, $\mathbf{r'} = (x, u\tau')$.
We already know how to diagonalize the gaussian term, so let's concentrate on $\int dx \int_0^\beta d\tau \int_0^\beta d\tau' [\partial_x\phi(x,\tau)\partial_x\phi(x,\tau')]$. We can Fourier transform it and obtain:

$$\int dx \int_0^\beta d\tau \int_0^\beta d\tau' [\partial_x\phi(x,\tau)\partial_x\phi(x,\tau')] =$$

$$\frac{1}{(2\pi)^2}\int dx \int_0^\beta d\tau \int_0^\beta d\tau' \left(\sum_{\mathbf{q_1}} ik_1 e^{i\mathbf{q_1 r}}\right)\left(\sum_{\mathbf{q_2}} ik_2 e^{i\mathbf{q_2 r'}}\right) =$$

$$\frac{1}{2\pi}\int_0^\beta d\tau \int_0^\beta d\tau' \sum_k \sum_{\omega_n} \sum_{\omega'_n} k^2 e^{i\omega_n\tau} e^{i\omega'_n\tau'} \phi(-k,\omega_n)\phi(k,\omega'_n) =$$

$$\sum_k \sum_{\omega_n} \sum_{\omega'_n} k^2 \delta_{\omega_n,0}\delta_{\omega'_n,0}\phi(-k,\omega_n)\phi(k,\omega'_n) = \sum_k k^2\phi(-k,0)\phi(k,0) \quad (4.54)$$

Then the correlation becomes:

$$\boxed{\langle\phi(\mathbf{q_1})\phi(\mathbf{q_2})\rangle = \frac{\pi K\delta_{\mathbf{q_1 q_2}}\Omega\beta}{\frac{\omega_n^2}{u} + uk_1^2 - \frac{g_1^2KC}{\pi}k_1^2\delta_{\omega_n,0}}} \quad (4.55)$$

As we can see the conductivity is

$$\boxed{\sigma(\omega) = \frac{e^2}{\pi^2}\frac{iuK}{\omega + i\delta} = e^2uK[\delta(\omega) + iP\frac{1}{\pi\omega}]} \quad (4.56)$$

and does not differ from the free case.

4.4 Conductivity in the general case $g_2 = 0$

The form of the action with the generic correlation function is:

$$S_{\text{eff}} = \frac{1}{2\pi K}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}x \; \left[\frac{1}{u}\left(\partial_\tau\phi\right)^2 + u\left(\partial_x\phi\right)^2\right] - \frac{g^2}{2\pi^2}\int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau'$$

$$[\partial_x\phi(x,\tau)\partial_x\phi(x,\tau')]C(\tau-\tau')(4.57)$$

Once again, we concentrate on the second term of the action:

$$\int dx \int_0^\beta d\tau \int_0^\beta d\tau' C(\tau-\tau')[\partial_x\phi(x,\tau)\partial_x\phi(x,\tau')] =$$

$$\frac{1}{(2\pi)^{5/2}}\int dx \int_0^\beta d\tau \int_0^\beta d\tau' [\sum_\omega C(\omega)e^{i\omega(\tau-\tau')}\sum_{\mathbf{q_1}} ik_1\phi(\mathbf{q_1})e^{i\mathbf{q_1}\mathbf{r}}\sum_{\mathbf{q_2}} ik_2\phi(\mathbf{q_2})e^{i\mathbf{q_2}\mathbf{r}'}] =$$

$$\frac{1}{(2\pi)^{3/2}}\int_0^\beta d\tau \int_0^\beta d\tau' [\sum_\omega C(\omega)e^{i\omega(\tau-\tau')}\sum_k\sum_{\omega_n}\sum_{\omega_n'} k^2 e^{i(\omega_n\tau+\omega_n'\tau')}\phi(-k,\omega_n)\phi(k,\omega_n')] =$$

$$\frac{1}{(2\pi)^{3/2}}\sum_\omega\sum_k\sum_{\omega_n}\sum_{\omega_n'} C(\omega)k^2\phi(-k,\omega_n)\phi(k,\omega_n')\int_0^\beta d\tau e^{i(\omega_n+\omega)\tau}\int_0^\beta d\tau' e^{-i(-\omega_n'+\omega)\tau'} =$$

$$\sum_\omega\sum_k C(\omega)k^2\phi(-k,-\omega)\phi(k,\omega) \tag{4.58}$$

All together, the correlation function is:

$$\boxed{\langle\phi(\mathbf{q}_1)^*\phi(\mathbf{q}_2)\rangle = \frac{\pi K\delta_{\mathbf{q_1}\mathbf{q_2}}\Omega\beta}{\frac{\omega_n^2}{u} + (u - \frac{g_1^2 KC(\omega_n)}{\pi})k_1^2}} \tag{4.59}$$

which means that in the general case the conductivity is unchanged.

Chapter 5

The case of delta correlation function

Now we analyze the phase diagram of the system in one of the extreme cases, in which the correlation function is a delta function, now setting $g_2 \neq 0$ to consider also the effect of the backward scattering. In the first section we use the variational approximation, which is less powerful then the Renormalization Group technique in investigating the critical properties of the model, but is very effective in exploring the phase diagram of it. In the second section we calculate the transport properties using the Kubo formula. In the third and last section we make a comparison between this technique and the R.G., in order to make sure that the two methods are in agreement between each other.

5.1 Variational Approximation

Setting $C(t-t') = C\delta(t-t')$ we can easily infer that the effective action becomes:

$$S_{\text{eff}} = \frac{1}{2\pi K}\int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \ \left[\frac{1}{u}\left(\partial_\tau\phi\right)^2 + \left(u - \frac{g_1^2 KC}{\pi}\right)\left(\partial_x\phi\right)^2\right]$$
$$-\frac{g_2^2 C}{2\pi^2 a^2}\int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \left[\cos 2\phi \cos 2\phi\right] \tag{5.1}$$

This action corresponds clearly to the action of the Sine-Gordon model:

$$S_{\text{SG}} = \frac{1}{2\pi K'}\int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \ \left[\frac{1}{u'}\left(\partial_\tau\phi\right)^2 + u'\left(\partial_x\phi\right)^2\right] - \frac{g_2^2 C}{4\pi^2 a^2}\int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \left[\cos 4\phi\right] \tag{5.2}$$

with the new costants $u'^2 = u\left[u - \frac{g_1^2 KC}{\pi}\right]$ and $K'^2 = \frac{uK^2}{u - \frac{g_1^2 KC}{\pi}}$.

In this model, we can expect the presence of a phase transition, driven by the

competition between the quadratic part, which tends to favour smooth functions, and the cosine, which tends to lock the system at $\phi = 0$.
Now we can introduce the variational principle that we are going to use to investigate the phase diagram of the model.
Choosing an appropriate variational ansatz S_0, in general we can write:

$$Z = \int \mathrm{D}\phi e^{-S} = \int \mathrm{D}\phi e^{-S_0} e^{-(S-S_0)} = Z_0 \langle e^{-(S-S_0)} \rangle_0 \tag{5.3}$$

where $\langle\rangle_0$ indicates the average with respect to S_0. The resulting free energy is:

$$F = F_0 - T \log[\langle e^{-(S-S_0)} \rangle_0] \tag{5.4}$$

The exponential, being a convex function, satisfies the Jensen's inequality:

$$\langle e^{-(S-S_0)} \rangle_0 \geq e^{-\langle (S-S_0) \rangle} \tag{5.5}$$

Thus, the variational free energy satisfies:

$$F \leq \boxed{F_{var} = F_0 + T\langle S - S_0 \rangle} \tag{5.6}$$

The best estimator is the action with the variational parameters that minimize the variational free energy. Given the effect of the cosine term, we choose as a variational action $S_0 = \frac{1}{2\beta\Omega} \sum_{\mathbf{q}} G^{-1}(\mathbf{q}) \phi^*(\mathbf{q}) \phi(\mathbf{q})$ and optimize on the Green function imposing $\frac{\partial F_{var}}{\partial G(\mathbf{q})} = 0$. The variational free energy reads:

$$F_{var} = -T \sum_{\mathbf{q}} \log(G(\mathbf{q})) + \frac{T}{2\pi K'} \sum_{\mathbf{q}} \left[\frac{\omega_n^2}{u'} + u'k^2 \right] G(\mathbf{q}) - T \frac{g_2^2 C \beta \Omega}{4\pi^2 a^2} e^{-\frac{8}{\beta\Omega} \sum_{\mathbf{q}} G(\mathbf{q})} \tag{5.7}$$

where we have used the identity: $\langle \cos(\phi) \rangle = e^{-\frac{1}{2}\langle \phi^2 \rangle}$.
Optimizing on the Green function leads to:

$$G^{-1}(\mathbf{q}) = \frac{1}{\pi K'} \left[\frac{\omega_n^2}{u'} + u'k^2 + \frac{\Delta^2}{u'} \right] \tag{5.8}$$

The gap Δ satisfies:

$$\frac{\Delta^2}{\pi K' u'} = \frac{4 g_2^2 C}{\pi^2 a^2} e^{-\frac{8}{\beta\Omega} \sum_{\mathbf{q}} \frac{\pi K' u'}{\omega_n^2 + u'^2 k^2 + \Delta^2}} \tag{5.9}$$

In the thermodynamic limit and zero temperature limit we have in the exponential:

$$\frac{8}{(2\pi)^2} \int d\mathbf{q} \frac{\pi K' u'}{\omega_n^2 + u'^2 k^2 + \Delta^2} = 4K' \int_0^{\Lambda} q dq \frac{1}{q^2 + (\frac{\Delta}{u'})^2}$$

$$\simeq 4K' \int_{\frac{\Delta}{u'}}^{\Lambda} \frac{dq}{q} = 4K' \log\left(\frac{u'\Lambda}{\Delta}\right) \tag{5.10}$$

With the assumption that the gap is much smaller than the cutoff frequency, which must be intended in the limit to ∞, $u'\Lambda \gg \Delta$, the self-consistent equation is :

$$\Delta^2 = \frac{4K'u'^3y^2\pi C}{a^2}\left(\frac{\Delta}{u'\Lambda}\right)^{4K'} \tag{5.11}$$

where we introduced the new variable $y = \frac{g_2}{\pi u'}$. Solving this for the gap we obtain:

$$\boxed{\Delta = u'\Lambda\left(\frac{4K'y^2u'^3C\pi}{a^2\Lambda^2}\right)^{\frac{1}{2-4K'}}} \tag{5.12}$$

This solution is acceptable only if $K' < \frac{1}{2}$, which implies $g_1^2 < \frac{\pi}{2C}[uK - u/K]$, putting back the original quantities. That's because otherwise it does not respect the aforementioned condition $u'\Lambda \gg \Delta$. We can see this clearly from this plot of how Δ varies with K' ($u'\Lambda = 6$ and $\frac{y^2u'^3C\pi}{2a^2\Lambda^2} = 0.5$) (FIG. 5.1).

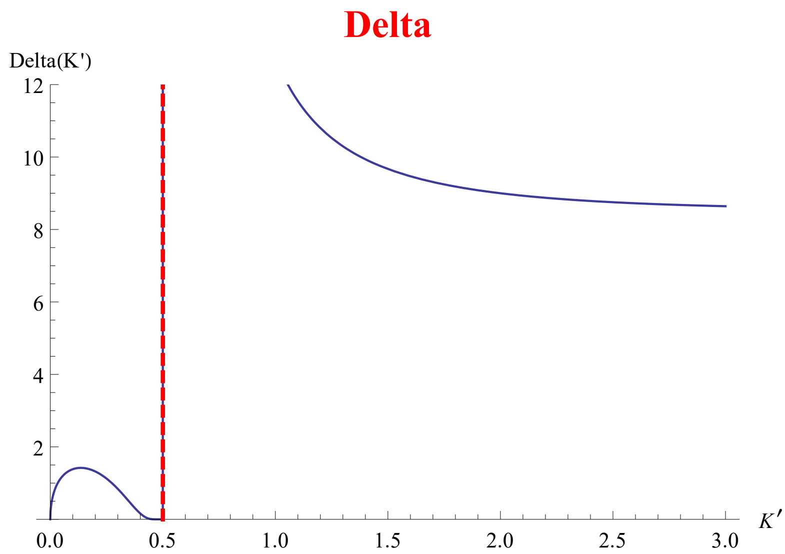

Figure 5.1: Plot of $\Delta(K')$
This plot shows the existence of two separate phases. Below $K' = \frac{1}{2}$ the solution is acceptable, while above that critical value it is not .

Solving for K we get the condition for the existence of the non-zero solution:

$$\boxed{K < \frac{-\frac{g_1^2C}{\pi} + \sqrt{\frac{g_1^4C^2}{\pi^2} + 16u^2}}{8u}} \tag{5.13}$$

The plot of the critical value of K, $K_C = \frac{-\frac{g_1^2 C}{\pi}+\sqrt{\frac{g_1^4 C^2}{\pi^2}+16u^2}}{8u}$, as a function of g_1 shows this graphically (FIG. 5.4).

5.2 The conductivity calculations

In order to investigate the effect of the interaction with the bath in this regime, we can use the variational approximation we have developed before with the following relation for the Green function:

$$G^{-1}(\mathbf{q}) = \frac{1}{\pi K'}[\frac{\omega_n^2}{u'} + u'k^2 + \frac{\Delta^2}{u'}] \tag{5.14}$$

Then the two-point correlation function reads:

$$\langle \phi(\mathbf{q_1})\phi(\mathbf{q_2}) \rangle = \frac{\pi K' \delta_{\mathbf{q_1 q_2}} \Omega \beta}{\frac{\omega_n^2}{u'} + u'k^2 + \frac{\Delta^2}{u'}} \tag{5.15}$$

For the conductivity, according to the Kubo formula, we have:

$$\sigma(\omega) = -\frac{e^2}{\pi^2}(\omega + i\delta)\langle \phi(k=0,\omega_n)^* \phi(k=0,\omega_n) \rangle_{i\omega_n \to \omega + i\delta} =$$

$$\left[-\frac{e^2}{\pi^2}(\omega + i\delta)\frac{\pi K' u'}{\omega_n^2 + \Delta^2} \right]_{i\omega_n \to \omega + i\delta} = -\frac{e^2}{\pi^2}(\omega + i\delta)\frac{\pi K' u'}{-(\omega + i\delta)^2 + \Delta^2} \tag{5.16}$$

Rationalizing this expression, we obtain:

$$-\frac{e^2}{\pi^2}(\omega + i\delta)\frac{\pi K' u'[\Delta^2 - \omega^2 + \delta^2 + 2i\omega\delta]}{(\Delta^2 - \omega^2 + \delta^2)^2 + (2\omega\delta)^2} \tag{5.17}$$

The real part of the conductivity is:

$$\boxed{Re(\sigma(\omega)) = -\frac{e^2 K' u'}{\pi}\frac{\omega(\Delta^2 - \omega^2 - \delta^2)}{(\Delta^2 - \omega^2 + \delta^2)^2 + (2\omega\delta)^2}} \tag{5.18}$$

From this we can see that the interaction opens a gap Δ, which shifts the real part of the conductivity. Therefore, as we can see from the graphic, in the gapped phase, where we have a non-zero solution for Δ, the conductivity is zero below Δ. Then the conductivity at $\omega = 0$ is zero in the gapped phase, while it is different from zero in the gapless one, where $\Delta = 0$. Therefore, in the first case we have an insulating behaviour, while in the second a metallic one. In both cases we can observe a power law decay at infinity. (FIG. 5.2 and 5.3)

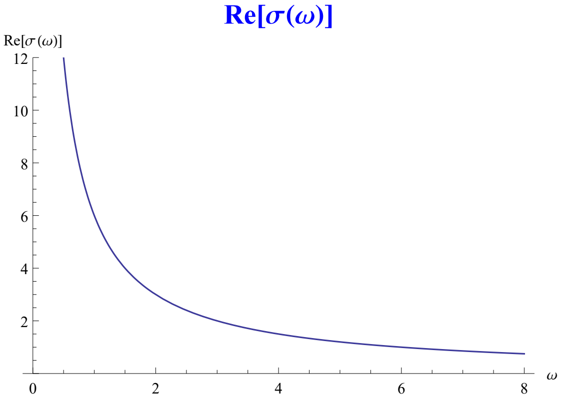

Figure 5.2: Plot of $Re[\sigma(\omega)]$ without any gap and with $\frac{e^2 K' u'}{\pi} = 6$
This figure shows the real part of the conductivity in the metallic phase. As we can see, it is finite in the origin.

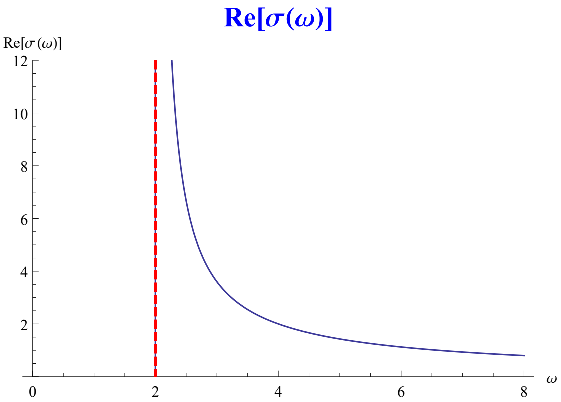

Figure 5.3: Plot of $Re[\sigma(\omega)]$ with a gap $\Delta = 2$ and with $\frac{e^2 K' u'}{\pi} = 6$
This figure shows the real part of the conductivity in the non-metallic phase. As we can see, it is zero in the origin and it is shifted of Δ with respect to the free case.

For the XXZ spin chain alone, when we have no interaction with the bath (FIG. 2.1), we have a phase transition between the gapless and the gapped phase at $J_z = J_{xy}$, which mapped using the Bethe-Ansatz relations (2.24) yields $K = 1/2$. Conversely, with the presence of the bath with a correlation function $C(t-t') = C\delta(t-t')$, we obtain that the critical point is at $K_C = \frac{-\frac{g_1^2 C}{\pi} + \sqrt{\frac{g_1^4 C^2}{\pi^2} + 16u^2}}{8u}$. Above that value we have a metallic gapless phase, while below we have an insulating gapped phase (FIG. 5.4). Setting the coupling constants g_1 and g_2 to 0 in (5.13), we recover the value $K = 1/2$.
It is easy to show that K_C is always below the value associated to the XXZ spin

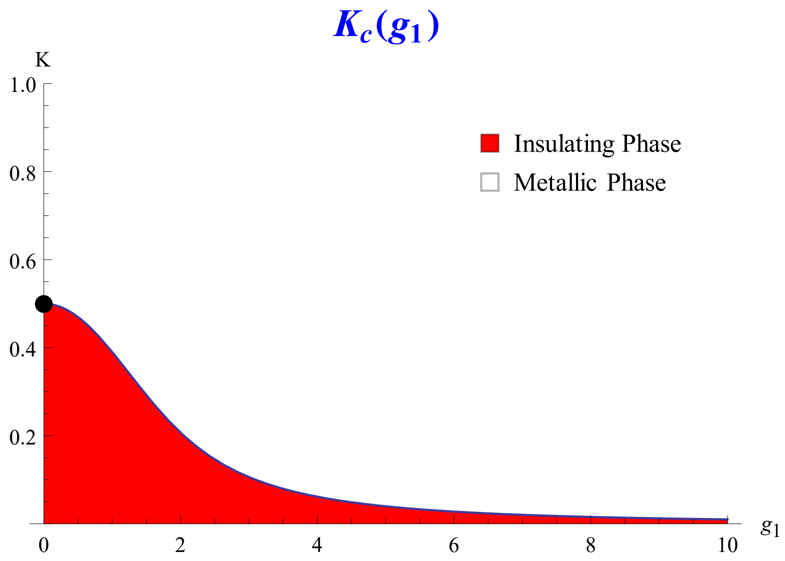

Figure 5.4: Plot of K_C as a function of g_1 with $u = 1$ and $C = \pi$
This plot shows the dependence of K_C from g_1. The red area corresponds to the non-metallic phase, while the white one to the metallic phase. The point represents the limit of zero coupling between the spin chain and the bath. We can see that the effect of the interaction with the bath is to lower the critical point.

chain alone. Therefore, we have demonstrated that the interaction with the bath favours the metallic behaviour and reduces the insulating phase.

5.3 The renormalization group technique

Now, we apply the renormalization group techniques, in order to investigate the critical properties of the system and to compare its results with the variational method.
The renormalization procedure is based on the idea of decomposing the fields into short-wavelength and large-wavelength components and integrating over the short ones, to get a new effective model with a new set of renormalized coupling constants. As this procedure is repeated many times, at each step we recover the form of the original action with different coupling constants. This enables to write the R.G. flow equations, which map the coupling constant to the ones corresponding to the low-energy phase. We will adopt the Wilson-Kadanoff perturbative procedure in the following.
We start from the partition function:

$$Z = \int \mathrm{D}\phi e^{-S} \tag{5.19}$$

For our Sine-Gordon action we have:

$$S_{\mathrm{SG}} = \frac{1}{2\pi K'}\int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \; \left[\frac{1}{u'}\left(\partial_\tau\phi\right)^2 + u'\left(\partial_x\phi\right)^2\right] - \frac{g_2^2 C}{4\pi^2 a^2}\int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \left[\cos 4\phi(x,\tau)\right] \tag{5.20}$$

We can write the field $\phi(x,\tau)$ as:

$$\phi(x,\tau) = \frac{1}{\beta\Omega}\sum_k \sum_{\omega_n} e^{i(kx-\omega_n\tau)}\phi(k,\omega_n) \tag{5.21}$$

As usual, the notation is $\mathbf{r} = (x, u'\tau)$ and $\mathbf{q} = (k, \omega_n/u')$. Then, we separate the low and high-frequency modes, defining a new frequency cutoff Λ', lower than the original one, Λ. The Brillouin zone is then split into two regions: $0 < |\mathbf{q}| < \Lambda'$ and $\Lambda' < |\mathbf{q}| < \Lambda$. The field ϕ is divided into the sum of the slow and the fast components:

$$\phi(\mathbf{r}) = \phi_<(\mathbf{r}) + \phi_>(\mathbf{r}) \tag{5.22}$$

where:

$$\phi_<(\mathbf{r}) = \frac{1}{\beta\Omega}\sum_{|\mathbf{q}|<\Lambda'} e^{i\mathbf{q}\cdot\mathbf{r}}\phi(\mathbf{q})$$

$$\phi_>(\mathbf{r}) = \frac{1}{\beta\Omega}\sum_{\Lambda'<|\mathbf{q}|<\Lambda} e^{i\mathbf{q}\cdot\mathbf{r}}\phi(\mathbf{q}) \tag{5.23}$$

The same holds for the quadratic action

$$S_0 = \frac{1}{2\pi K'}\frac{1}{\beta\Omega}\sum_{\mathbf{q}}\left[\frac{\omega_n^2}{u'} + u'k^2\right]\phi(\mathbf{q})^*\phi(\mathbf{q}) \tag{5.24}$$

which can be written as:

$$S_0 = S_0^< + S_0^> \tag{5.25}$$

separating the high frequency and low frequency components. However, this is obviously not possible for the cosine term and we need a perturbative expansion to treat it. The expansion yields:

$$\frac{Z}{Z_0} = \frac{1}{Z_0}\int \mathrm{D}\phi e^{-S_0^< - S_0^>}\left[1 - \frac{g_2^2 C}{4u'\pi^2 a^2}\int d^2r \cos(4(\phi_<(\mathbf{r}) + \phi_>(\mathbf{r})))\right.$$

$$\left. + \frac{g_2^4 C^2}{2u'^2(4\pi^2 a^2)^2}\int d^2r_1 \int d^2r_2 \cos(4(\phi_<(\mathbf{r}_1) + \phi_>(\mathbf{r}_1)))\cos(4(\phi_<(\mathbf{r}_2) + \phi_>(\mathbf{r}_2)))\right] \tag{5.26}$$

Now, we can average over the fast modes and remain only with the slow ones, using the identity demonstrated in the Appendix B $\langle\cos(\phi)\rangle = e^{-\frac{1}{2}\langle\phi^2\rangle}$

$$\frac{Z}{Z_0} = \frac{1}{Z_0^<}\int \mathrm{D}\phi e^{-S_0^<}\left[1 - \frac{g_2^2 C}{4u'\pi^2 a^2}\int d^2r \cos(4(\phi_<(\mathbf{r})))e^{-8\langle(\phi_>(\mathbf{r}))^2\rangle_>}\right.$$

$$+\frac{g_2^4C^2}{2u'^2(4\pi^2a^2)^2}\sum_{\epsilon=\pm}\int d^2r_1\int d^2r_2\cos(4(\phi_<(\mathbf{r}_1)+\epsilon\phi_<(\mathbf{r}_2)))e^{-8\langle(\phi_>(\mathbf{r}_1)+\epsilon\phi_>(\mathbf{r}_2))^2\rangle}\Big] \tag{5.27}$$

In order to obtain an effective action, we have to re-exponentiate the expression:

$$\frac{Z}{Z_0}=\frac{1}{Z_0^<}\int \mathrm{D}\phi e^{-S_0^<-\frac{g_2^2C}{4u'\pi^2a^2}\int d^2r\cos(4(\phi_<(\mathbf{r})))e^{-8\langle(\phi_>(\mathbf{r}))^2\rangle>}}$$

$$e^{\frac{g_2^4C^2}{2u'^2(4\pi^2a^2)^2}\int d^2r_1\int d^2r_2\sum_{\epsilon=\pm}[\cos(4(\phi_<(\mathbf{r}_1)+\epsilon\phi_<(\mathbf{r}_2)))e^{-8\langle(\phi_>(\mathbf{r}_1)+\epsilon\phi_>(\mathbf{r}_2))^2\rangle}]}$$

$$e^{-\frac{g_2^4C^2}{u'^2(4\pi^2a^2)^2}\int d^2r_1\int d^2r_2\cos(4(\phi_<(\mathbf{r}_1)))e^{-8\langle(\phi_>(\mathbf{r}_1))^2\rangle>}\cos(4(\phi_<(\mathbf{r}_2)))e^{-8\langle(\phi_>(\mathbf{r}_2))^2\rangle>}} \tag{5.28}$$

Now, we can re-scale the momentum with the new cutoff Λ', ensuring that the momenta k' maintain the cutoff Λ, as:

$$k'=\frac{\Lambda'}{\Lambda}k \tag{5.29}$$

and the same for ω.
The time variable and the distance transforms according to the inverse transformation, as:

$$x'=\frac{\Lambda}{\Lambda'}x \text{ and } \tau'=\frac{\Lambda}{\Lambda'}\tau \tag{5.30}$$

After this rescaling we recover a theory which is equivalent to the original one, but with new coupling constant:

$$g_2^2(\Lambda')=\Big(\frac{\Lambda}{\Lambda'}\Big)^2g_2^2(\Lambda)e^{-8\langle(\phi_>(\mathbf{r}))^2\rangle>}=\Big(\frac{\Lambda}{\Lambda'}\Big)^2g_2^2(\Lambda)e^{-\frac{8}{\beta\Omega}\sum_{\Lambda'<|\mathbf{q}|<\Lambda}\frac{\pi K'u'}{\omega_n^2+u'^2k^2}} \tag{5.31}$$

If we now perform the limit $L\to\infty$ and $\beta,\to\infty$ we can transform the sum into an integral :

$$g_2^2(\Lambda')=\Big(\frac{\Lambda}{\Lambda'}\Big)^2g_2^2(\Lambda)e^{-4\int_{\Lambda'<|\mathbf{q}|<\Lambda}\frac{K'}{q}}$$

$$=\Big(\frac{\Lambda}{\Lambda'}\Big)^2g_2^2(\Lambda)e^{-4K'\int_{\Lambda'}^{\Lambda}dq\frac{1}{q}}=\Big(\frac{\Lambda}{\Lambda'}\Big)^2g_2^2(\Lambda)e^{-4K'\log(\frac{\Lambda}{\Lambda'})} \tag{5.32}$$

To obtain the flow equation, we have to parametrize the cutoff according to the relation:

$$\Lambda(l)=\Lambda_0e^{-l} \tag{5.33}$$

where Λ_0 is the bare cutoff, and make an infinitesimal variation in order to obtain the expression for Λ':

$$\Lambda'(l) = \Lambda_0 e^{-l-dl} \tag{5.34}$$

The resulting equation for $g_2(l)$ is:

$$g_2^2(l + dl) = g_2^2(l)e^{(2-4K')dl} \tag{5.35}$$

Expanding the exponential we get the flow equation for $g_2^2(l)$:

$$\boxed{\frac{d(g_2^2(l))}{dl} = g_2^2(l)(2 - 4K')} \tag{5.36}$$

This flow equation confirms the presence of a phase transition at $K' = \frac{1}{2}$. In fact, g_2^2 is irrelevant for $K' > \frac{1}{2}$, as it decreases and flows to 0. This corresponds to a gapless phase. Instead, for $K' < \frac{1}{2}$ g_2^2 is relevant, as it grows and flows towards strong couplings, signalling the presence of a gapped phase.
[5]

Chapter 6

The case of constant correlation function

In this section, our aim is to study the other extreme and unphysical case, the one in which the correlation function is constant. Once again, we use the variational approximation, in order to study the phase diagram associated to this model. For this end, we calculate the conductivity in the same way as in the previous chapters to distinguish between a gapless phase and a gapped one.

6.1 Variational Approximation

We start from the following action:

$$S_{\text{eff}} = \frac{1}{2\pi K} \int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \, \left[\frac{1}{u} \left(\partial_\tau \phi\right)^2 + u \left(\partial_x \phi\right)^2 \right] - \frac{g_1^2 C}{2\pi^2} \int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau'$$

$$\partial_x \phi(x,\tau) \partial_x \phi(x,\tau') - \frac{g_2^2 C}{2\pi^2 a^2} \int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau' \cos 2\phi(x,\tau) \cos 2\phi(x,\tau') \tag{6.1}$$

We may separate the two backward scattering terms relative to the cosine of the sum and of the difference of the fields, which result from the Werner formulas, with two different coupling constants, g_2 and g_2':

$$S_{\text{eff}} = \frac{1}{2\pi K} \int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \, \left[\frac{1}{u} \left(\partial_\tau \phi\right)^2 + u \left(\partial_x \phi\right)^2 \right] - \frac{g_1^2 C}{4\pi^2} \int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau'$$

$$\partial_x \phi(x,\tau) \partial_x \phi(x,\tau') - \int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau' \frac{g_2^2 C}{4\pi^2 a^2} \cos 2(\phi(x,\tau) + \phi(x,\tau')) -$$

$$\int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau' \frac{g_2'^2 C}{4\pi^2 a^2} \cos 2(\phi(x,\tau) - \phi(x,\tau'))$$

(6.2)

Once again, we choose as variational action $S_0 = \frac{1}{2\beta\Omega}\sum_{\mathbf{q}} G^{-1}(\mathbf{q})\phi^*(\mathbf{q})\phi(\mathbf{q})$ and optimize on the Green function imposing $\frac{\partial F_{var}}{\partial G(q)} = 0$. The variational free energy reads:

$$F_{var} = -T\sum_{q>0} \log(G(\mathbf{q})) + \frac{T}{2\pi K}\sum_{\mathbf{q}} \left[\frac{\omega_n^2}{u} + uk^2\right] G(\mathbf{q}) - \frac{g_1^2 C}{2\pi^2} T \sum_{\mathbf{q}} k^2 G(\mathbf{q}) \delta_{\omega_n,0} -$$
$$-T\frac{g_2^2 C}{4\pi^2 a^2} \int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau' e^{-2\langle(\phi(x,\tau)+\phi(x,\tau'))^2\rangle}$$
$$-T\frac{g_2'^2 C}{4\pi^2 a^2} \int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau' e^{-2\langle(\phi(x,\tau)-\phi(x,\tau'))^2\rangle} \tag{6.3}$$

With a Fourier transform of the last term:

$$F_{var} = -T\sum_{\mathbf{q}} \log(G(\mathbf{q})) + \frac{T}{2\pi K}\sum_{\mathbf{q}} \Big[\frac{\omega_n^2}{u} + uk^2 - \frac{g_1^2 CK}{\pi} k^2 \delta_{\omega_n,0}\Big] G(\mathbf{q})$$
$$-T\frac{g_2^2 C}{4\pi^2 a^2} \int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau' e^{-\frac{2}{\beta\Omega}\sum_{\mathbf{q}}(2-2\cos\omega(\tau-\tau'))G(\mathbf{q})}$$
$$-T\frac{g_2'^2 C}{4\pi^2 a^2} \int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau' e^{-\frac{2}{\beta\Omega}\sum_{\mathbf{q}}(2+2\cos\omega(\tau-\tau'))G(\mathbf{q})} \tag{6.4}$$

Optimising on $G(\mathbf{q})$ one gets:

$$G^{-1}(\mathbf{q}) = \frac{1}{\pi K}\Big[uk^2 + \frac{1}{u}\omega_n^2 - \frac{g_1^2 CK}{\pi} k^2 \delta_{\omega_n,0}\Big] +$$
$$\frac{g_2^2 C}{2\pi^2 a^2 \beta\Omega} \int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau' (2 - 2\cos\omega(\tau-\tau')) e^{-\frac{2}{\beta\Omega}\sum_{\mathbf{q}}(2-2\cos\omega(\tau-\tau'))G(\mathbf{q})} +$$
$$\frac{g_2'^2 C}{2\pi^2 a^2 \beta\Omega} \int \mathrm{d}x \int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau' (2 + 2\cos\omega(\tau-\tau')) e^{-\frac{2}{\beta\Omega}\sum_{\mathbf{q}}(2+2\cos\omega(\tau-\tau'))G(\mathbf{q})} \tag{6.5}$$

Redefining a new more convenient variable $\tau'' = \tau - \tau'$:

$$G^{-1}(\mathbf{q}) = \frac{1}{\pi K}\Big[uk^2 + \frac{1}{u}\omega_n^2 - \frac{g_1^2 CK}{\pi} k^2 \delta_{\omega_n,0}\Big]$$

$$+\frac{g_2^2 C}{2\pi^2 a^2 \beta\Omega}\int_0^\beta \mathrm{d}\tau \int_\tau^{\tau+\beta} \mathrm{d}\tau''(2-2\cos(\omega\tau''))e^{-\frac{2}{\beta\Omega}\sum_{\boldsymbol{q}}(2-2\cos(\omega\tau''))\frac{\pi K u}{u^2 q^2+\Delta^2}}$$

$$+\frac{g_2'^2 C}{2\pi^2 a^2 \beta\Omega}\int_0^\beta \mathrm{d}\tau \int_\tau^{\tau+\beta} \mathrm{d}\tau''(2+2\cos(\omega\tau''))e^{-\frac{2}{\beta\Omega}\sum_{\boldsymbol{q}}(2+2\cos\omega(\tau''))\frac{\pi K u}{u^2 q^2+\Delta^2}} \tag{6.6}$$

Then, we can write the terms at the exponential in the continuum limit. We have:

$$G^{-1}(\mathbf{q}) = \frac{1}{\pi K}\Big[uk^2 + \frac{1}{u}\omega_n^2 - \frac{g_1^2 C K}{\pi}k^2\delta_{\omega_n,0}\Big]$$

$$+\frac{g_2^2 C}{2\pi^2 a^2 \beta\Omega}\int_0^\beta \mathrm{d}\tau \int_\tau^{\tau+\beta} \mathrm{d}\tau''(2-2\cos\omega(\tau''))e^{-\int dk\int d\omega 2(1-\cos(\omega\tau''))\frac{Ku^2}{u^2k^2+\omega^2+\Delta^2}}$$

$$+\frac{g_2'^2 C}{2\pi^2 a^2 \beta\Omega}\int_0^\beta \mathrm{d}\tau \int_\tau^{\tau+\beta} \mathrm{d}\tau''(2+2\cos\omega(\tau''))e^{-\int dk\int d\omega 2(1+\cos(\omega\tau''))\frac{u^2 K}{u^2k^2+\omega^2+\Delta^2}} \tag{6.7}$$

If we consider the limit $T \to 0$, the linear dependence of C with respect to the temperature (see Appendix C) implies that the last two terms are $O(1)$ with respect to T, as the time integral yields a factor β, while the term containing $\delta_{\omega_n,0}$ is $O(T)$, so we can neglect it.
To proceed, we can cast aside the oscillating terms, as their contribution to the time integral is small as $\beta \to \infty$. In order to justify this approximation, we can make a numerical plot of the result of the integral $I(\tau'') = \int dk \int d\omega \cos(\omega\tau'')e^{-\alpha\omega^2}\frac{u^2}{u^2k^2+\omega^2+\Delta^2}$. In the context of this integral related to the massive theory we have introduced a gaussian cutoff $e^{-\alpha\omega^2}$ to cure the divergence. This integral can be solved analytically in the k variable, then numerically in ω. We have made a plot of its estimation for integer values of τ'' (FIG 6.1)
We can observe from this plot that it decays exponentially fast to 0 as τ'' goes to infinity. The error made neglecting this term is discussed in Appendix D. In addition to that, we no longer distinguish between the two contributions from the cosine term ($g_2 = g_2'$), as the two terms play the same role. Thus, we are left with the subsequent self-consistent equation for the gap Δ:

$$\frac{\Delta^2}{\pi K u} = \frac{g_2^2 C}{\pi^2 a^2}2\beta e^{-2\int_0^\Lambda dq \frac{qKu^2}{u^2q^2+\Delta^2}} \tag{6.8}$$

As before, for the Green function, we have:

$$G^{-1}(\mathbf{q}) = \frac{1}{\pi K}[\frac{\omega_n^2}{u} + uk^2 + \frac{\Delta^2}{u}] \tag{6.9}$$

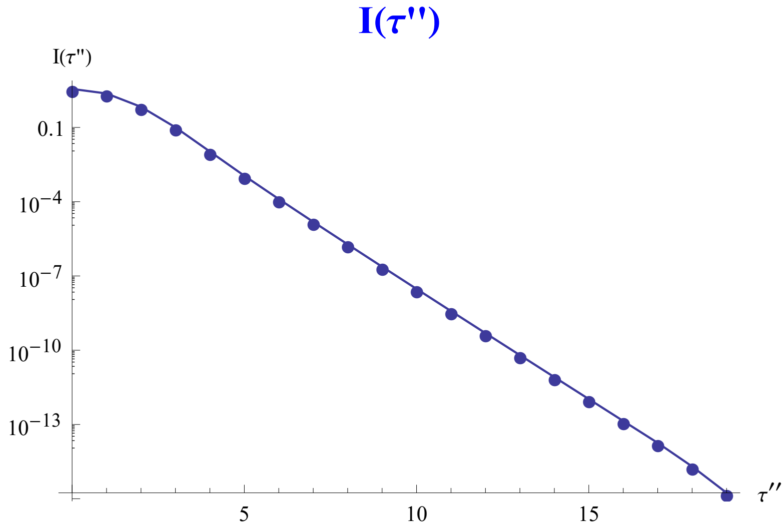

Figure 6.1: Plot of $I(\tau'')$, with $\Delta = 2$, $\alpha = 0.5$
This plot shows that the integral $I(\tau'')$ decays exponentially as $\tau'' \to \infty$.

For the gap we can apply the results from the previous subsection in the approximation $u\Lambda >> \Delta$ and we obtain (defining $C = \tilde{C} k_B T$):

$$\Delta^2 = \frac{2 g_2^2 \tilde{C} K u}{\pi a^2} \left(\frac{\Delta}{u\Lambda} \right)^{2K} \tag{6.10}$$

The result for the gap is:

$$\boxed{\Delta = u\Lambda \left(\frac{2\pi K y^2 u \tilde{C}}{a^2 \Lambda^2} \right)^{\frac{1}{2-2K}}} \tag{6.11}$$

and we can draw the same conclusions as in the previous chapter, but with a phase transition at $K = 1$ between the metallic and the non-metallic phase. Below we have a sketch of the phase diagram obtained in this case (FIG. 6.2). Unlike the previous one, in this case the coupling constant g_1 plays no role at all.
Using the results of the previous section concerning the variational approximation, we can start from the expression for the real part of the conductivity:

$$\boxed{Re(\sigma(\omega)) = -\frac{e^2 K' u'}{\pi} \frac{\omega(\Delta^2 - \omega^2 - \delta^2)}{(\Delta^2 - \omega^2 + \delta^2)^2 + (2\omega\delta)^2}} \tag{6.12}$$

As a consequence, we can observe the presence of the opening of the gap in the conductivity at $K = 1$. The situation is the same as (FIG. 5.2 and 5.3) with a gapped phase below $K = 1$ and a gapless one above. Therefore, the effect of the interaction with the bath is to induce an insulating behaviour in a region of parameters that otherwise would be conductive, as in the zero coupling case we have $K_C = \frac{1}{2}$.

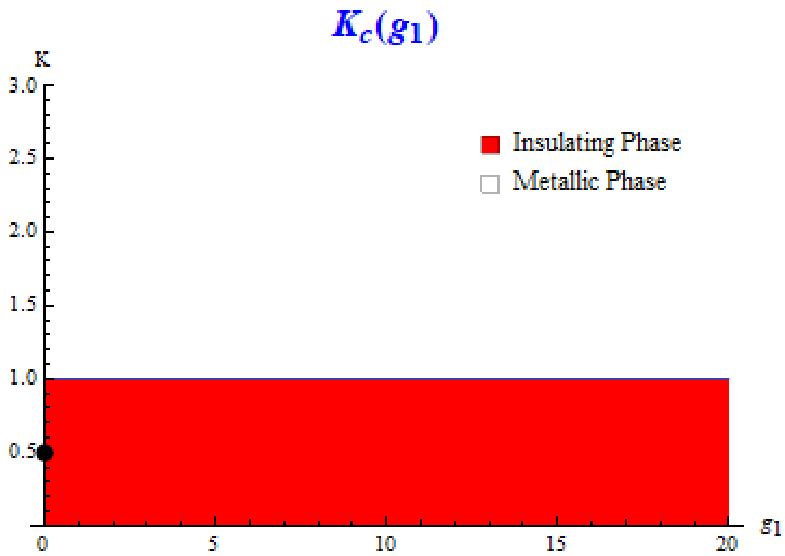

Figure 6.2: Plot of K_C as a function of g_1 with $u = 1$ and $C = \pi$
This plot shows the dependence of K_C from g_1, the coupling constant of the forward scattering term. The red area corresponds to the non-metallic phase, while the white one to the metallic phase. In this case, K_C is constant and equal to 1. The point at $g_1 = 0$ represents XXZ spin chain alone, in which the critical point is at $K_C = \frac{1}{2}$. So, we can see that the effect of the interaction with the bath is to raise the critical point.

Chapter 7

Conclusion

We have demonstrated that the effect of the bath of harmonic oscillators for the sake of the conductivity of the systems is relevant. In all the cases we have treated, the relevant parameter to discuss the transport properties is $K = \sqrt{\frac{1}{1+\frac{4J_z}{\pi J_{xy}}}}$ We have discussed these different situations:

- In absence of the bath of harmonic oscillators, we have an insulating phase for $K < \frac{1}{2}$ and a metallic phase for $K > \frac{1}{2}$.
- If we consider the case of fast harmonic oscillators, in which the correlation function is $C(t-t') = C\delta(t-t')$, the metallic phase extends under the value $K = \frac{1}{2}$ and the critical point is shifted below (FIG. 5.4).
- In the case of slow harmonic oscillators and correlation function $C(t-t') = C$, $K_C(g_2 \neq 0) = 1$ and the insulating phase extends over the value $K = \frac{1}{2}$, shifting above the critical point (FIG. 6.2).

This proves that, as in the Leggett paper, the bath can induce localization effects. This is, however, only the beginning of the study of our model. There is room to investigate what is the effect of the bath in the most general cases concerning the effect of the bath with a generic time-dependent correlation function. The study can be extended for non-equilibrium dynamical treatment and for non-zero temperature.
Another important direction of investigation can be the differences between the insulating phase and the localized phase typical of models with the presence of quenched disorder.

Appendix A

Derivation and Further Details about the Bosonization

The bosonization procedure is based on the bosonic nature of the low-energy particle-hole excitations in 1d quantum liquids (FIG. A.1). First of all, to derive it from the discrete fermionic operators we must start from the density fluctuation operator :

$$\rho_\eta(q) = \sum_k \psi^\dagger_{k+q,\eta}\psi_{k,\eta} \tag{A.1}$$

where $\eta = +, -$runs on the left/right type of operator. This operator creates a particle-hole excitation of momentum $q \neq 0$ around the Fermi momentum. It obeys the following algebra:

$$[\rho_\eta(q), \rho_{\eta'}(q')] = \mp\frac{qL}{2\pi}\delta_{\eta\eta'}\delta_{q,-q'} \tag{A.2}$$

So, we may define bosonic operators as

$$b_{q\eta} = \sqrt{\frac{2\pi}{Lq}}\rho_\eta(\mp q) \tag{A.3}$$

$$b^\dagger_{q\eta} = \sqrt{\frac{2\pi}{Lq}}\rho_\eta(\pm q) \tag{A.4}$$

From this, one can infer the bosonization mapping of the fermionic fields. We have the commutator:

$$[\rho_\eta(q), \psi_\eta(x)] = \frac{1}{\sqrt{\Omega}}\sum_{k,k_1} e^{ik_1x}[\psi^\dagger_{\eta,k+q}\psi_{\eta,k}, \psi_{\eta,k_1}] = -e^{-iqx}\psi_\eta(x) \tag{A.5}$$

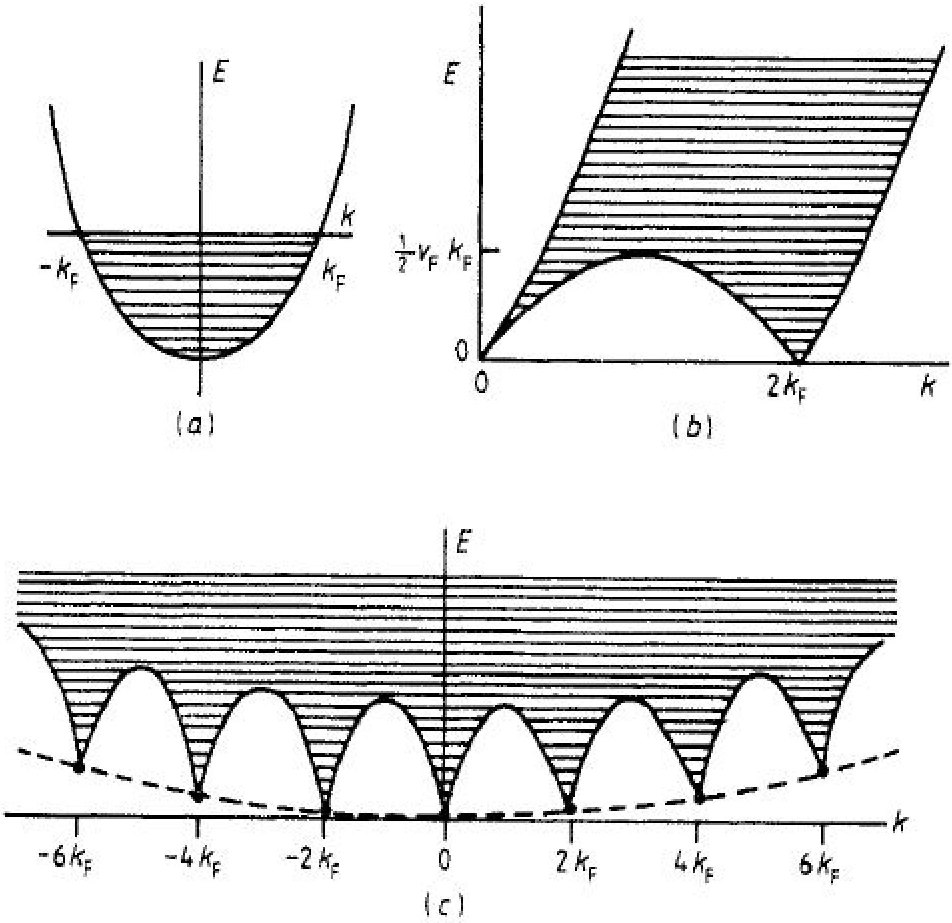

Figure A.1: Particle-Hole Spectrum of a 1D free Fermi gas theory

(a) Single particle spectrum (b) Particle-hole pair spectrum (c) Multiple particle-hole excitation spectrum. The last figure shows that the low energy excitations are located around the multiples of $2k_F$ and are indeed bosonic in character. In fact, they have a quadratic dispersion relation, which can be approximated as a linear sound-wave one as we are in a low-energy regime. (from [12])

Therefore, we could write an operator that yields the same commutator as:

$$\psi_\eta(x) = U_\eta e^{\sum_p e^{ipx}\rho^\dagger_\eta(-p)\frac{2\pi\eta}{\pi L}} \tag{A.6}$$

Including the Klein factor U_η this is a complete representation of the particle-hole excitations. From this relation we can derive the bosonization in terms of the bosonic operators, defining:

$$\phi(x) = \frac{i\pi}{L}\sum_{p\neq 0}\frac{1}{p}e^{-a|p|/2-ipx}(\rho_+(p)+\rho_-(p)) \tag{A.7}$$

$$\theta(x) = \frac{i\pi}{L}\sum_{p\neq 0}\frac{1}{p}e^{-a|p|/2-ipx}(\rho_+(p)-\rho_-(p)) \tag{A.8}$$

The factor $e^{-aq/2}$ is to be intended as a convergence factor as we must take the limit $a \to 0$ in the final results.
Inserting the bosonic operators we have:

$$\phi(x) = i\pi\sum_{q\neq 0}\frac{e^{-a|q|/2-iqx}}{q}\Big(\frac{|q|}{2L\pi}\Big)^{1/2}(b_q^\dagger + b_{-q}) \tag{A.9}$$

where we have used that $b_{q,-} = b_{-q}$.
The dual field $\theta(x)$ is:

$$\theta(x) = i\pi\sum_{q\neq 0}\frac{e^{-a|q|/2-iqx}}{|q|}\Big(\frac{|q|}{2L\pi}\Big)^{1/2}(b_q^\dagger - b_{-q}) \tag{A.10}$$

The field ϕ can be decomposed into the sum of two conjugate fields:

$$\phi(x) = \varphi^\dagger(x) + \varphi(x) \tag{A.11}$$

defined as:

$$\varphi(x) = i\pi\sum_{q\neq 0}\frac{e^{-a|q|/2-iqx}}{q}\Big(\frac{|q|}{2L\pi}\Big)^{1/2}b_{-q} \tag{A.12}$$

$$\varphi^\dagger(x) = i\pi\sum_{q\neq 0}\frac{e^{-a|q|/2-iqx}}{q}\Big(\frac{|q|}{2L\pi}\Big)^{1/2}b_q^\dagger \tag{A.13}$$

In all these expressions, the $q = 0$ contribution is negligible in the thermodynamic limit. As we have stated in the Chapter 2, the relation between the fermionic fields and the left/right fields is :

$$\psi_\eta(x) = \frac{1}{\sqrt{2\pi a}}U_\eta e^{\pm i2\phi_\eta(x)} \tag{A.14}$$

and the dual fields are defined as:

$$\phi(x) = \phi_+(x) + \phi_-(x) \tag{A.15}$$

$$\theta(x) = \phi_+(x) - \phi_-(x) \tag{A.16}$$

Consequently, we can construct the left/right fields from $\phi(x)$:

$$\phi(x) = i\sqrt{\pi}\sum_{q\neq 0}\frac{e^{-a|q|/2-iqx}}{q}\Big(\frac{|q|}{2L}\Big)^{1/2}(b_q^\dagger + b_{-q})$$

$$= i\sqrt{\pi}\Big[\sum_{q>0}\frac{e^{-a|q|/2-iqx}}{q}\Big(\frac{|q|}{2L}\Big)^{1/2}(b_q^\dagger + b_{-q}) + \sum_{q<0}\frac{e^{-a|q|/2-iqx}}{q}\Big(\frac{|q|}{2L}\Big)^{1/2}(b_q^\dagger + b_{-q})\Big]$$

$$= i\sqrt{\pi}\Big[\sum_{q>0}\frac{e^{-a|q|/2-iqx}}{\sqrt{2|q|L}}(b_q^\dagger + b_{-q}) - \sum_{q<0}\frac{e^{-a|q|/2-iqx}}{\sqrt{2|q|L}}(b_q^\dagger + b_{-q})\Big]$$

$$= i\sqrt{\pi}\sum_{q>0}\frac{e^{-aq/2}}{\sqrt{2qL}}\Big[e^{-iqx}(b_q^\dagger + b_{-q}) - e^{iqx}(b_{-q}^\dagger + b_q)\Big]$$

$$= \frac{i\sqrt{\pi}}{\sqrt{2L}}\sum_{q>0}\frac{e^{-aq/2}}{\sqrt{q}}(e^{-iqx}b_{q,-} - e^{iqx}b_{q,+} - e^{iqx}b_{q,-}^\dagger + e^{-iqx}b_{q,+}^\dagger)$$

As a consequence, for $\eta = +,-$ we can define the left/right fields as:

$$\phi_\eta(x) = \varphi_\eta(x) + \varphi_\eta^\dagger(x) = \frac{\mp i\sqrt{\pi}}{\sqrt{2L}}\sum_{q>0}\frac{e^{-aq/2}}{\sqrt{q}}(e^{\pm iqx}b_{q\eta} - e^{\mp iqx}b_{q\eta}^\dagger) \tag{A.17}$$

Where we have $b_{q,+} = b_q$ and $b_{q,-} = b_{-q}$ and the same for their hermitian conjugates. It follows the definition of the creation/annihilation components of the fields in terms of bosonic operators as:

$$\varphi_+(x) = \frac{-i\sqrt{\pi}}{\sqrt{2L}}\sum_{q>0}\frac{e^{iqx}}{\sqrt{q}}e^{-aq/2}b_{q,+} \tag{A.18}$$

$$\varphi_+^\dagger(x) = \frac{+i\sqrt{\pi}}{\sqrt{2L}}\sum_{q>0}\frac{e^{-iqx}}{\sqrt{q}}e^{-aq/2}b_{q,+}^\dagger \tag{A.19}$$

$$\varphi_-(x) = \frac{+i\sqrt{\pi}}{\sqrt{2L}}\sum_{q>0}\frac{e^{-iqx}}{\sqrt{q}}e^{-aq/2}b_{q,-} \tag{A.20}$$

$$\varphi_-^\dagger(x) = \frac{-i\sqrt{\pi}}{\sqrt{2L}}\sum_{q>0}\frac{e^{iqx}}{\sqrt{q}}e^{-aq/2}b_{q,-}^\dagger \tag{A.21}$$

Let us now compute the commutators between the bosonic fields:

$$[\phi(x),\phi(y)] = -\pi\sum_{q,q'\neq 0}\frac{e^{-a(|q|+|q'|)/2-i(qx+q'y)}}{q|q'|}\frac{\sqrt{|q||q'|}}{2L}[b_q^\dagger + b_{-q}, b_{q'}^\dagger + b_{-q'}] = 0 \tag{A.22}$$

$$[\theta(x),\theta(y)] = -\pi\sum_{q,q'\neq 0}\frac{e^{-a(|q|+|q'|)/2-i(qx+q'y)}}{qq'}\frac{\sqrt{|q||q'|}}{2L}[b_q^\dagger - b_{-q}, b_{q'}^\dagger - b_{-q'}] = 0 \quad \text{(A.23)}$$

$$[\phi(x),\theta(y)] = -\pi\sum_{q,q'\neq 0}\frac{e^{-a(|q|+|q'|)/2-i(qx+q'y)}}{q|q'|}\frac{\sqrt{|q||q'|}}{2L}[b_q^\dagger + b_{-q}, b_{q'}^\dagger - b_{-q'}]$$

$$= -\pi\sum_{q,q'\neq 0}\frac{e^{-a(|q|+|q'|)/2-i(qx+q'y)}}{q|q'|}\frac{\sqrt{|q||q'|}}{2L}2\delta_{q',-q}$$

$$= \frac{-\pi}{L}\sum_{q\neq 0}\frac{e^{-a(|q|)-iq(x-y)}}{q|q|}|q| = \frac{-\pi}{L}\sum_{q\neq 0}\frac{e^{-a(|q|)+iqR}}{q|q|}|q|$$

$$= \frac{-2i\pi}{L}\sum_{q>0}\frac{e^{-aq}\sin(qR)}{q} \rightarrow \boxed{L\rightarrow\infty} \rightarrow \frac{-2i\pi}{L\Delta q}\int_0^\infty dq\frac{e^{-aq}\sin(qR)}{q}$$

$$= -i\int_0^\infty dq\frac{e^{-aq}\sin(qR)}{q} \rightarrow$$

$$\boxed{a\rightarrow 0} \rightarrow -i\int_0^\infty dq\frac{\sin(qR)}{q} = \frac{-i\pi}{2}\epsilon(R) = \frac{i\pi}{2}\epsilon(x-y)$$

where we have made use of the auxiliary variable $R = y - x$, $\Delta q = \frac{2\pi}{L}$ and the representation of the step function:

$$\epsilon(x) = \int_0^\infty dq\frac{\sin(qx)}{2\pi q} \quad \text{(A.24)}$$

From these formulas, we can infer the commutation relations of the fields ϕ_η:

$$[\phi_\eta(x),\phi_\eta(y)] = \frac{1}{4}[\phi(x)\pm\theta(x),\phi(y)\pm\theta(y)] = \frac{\pm 1}{4}([\phi(x),\theta(y)] - [\phi(y),\theta(x)])$$

$$= \pm\frac{i\pi}{4}\epsilon(x-y) \quad \text{(A.25)}$$

and

$$[\phi_+(x),\phi_-(x)] = \frac{1}{4}([\phi(x),\theta(y)] + [\phi(y),\theta(x)]) = 0 \quad \text{(A.26)}$$

Now, let us work out the expression of the derivatives of the fields:

$$\nabla\phi(x) = \pi\sum_{q\neq 0}e^{-a|q|/2-iqx}\Big(\frac{|q|}{2L\pi}\Big)^{1/2}(b_q^\dagger + b_{-q}) \quad \text{(A.27)}$$

$$\nabla\theta(x) = \pi \sum_{q\neq 0} e^{-a|q|/2-iqx} \frac{q}{|q|}\Big(\frac{|q|}{2L\pi}\Big)^{1/2}(b_q^\dagger - b_{-q}) \tag{A.28}$$

We can derive the correspondent commutation relations:

$$[\nabla\phi(x), \nabla\phi(y)] = [\nabla\theta(x), \nabla\theta(y)] = 0 \tag{A.29}$$

$$[\nabla\phi(x), \phi(y)] = [\nabla\theta(x), \theta(y)] = 0 \tag{A.30}$$

$$[\nabla\phi(x), \theta(y)] = \nabla_x[\phi(x), \theta(y)] = \nabla_x \frac{i\pi}{2}\epsilon(x-y) = i\pi\delta(x-y) \tag{A.31}$$

$$[\phi(x), \nabla\theta(y)] = \nabla_y[\phi(x), \theta(y)] = \nabla_y \frac{i\pi}{2}\epsilon(x-y) = -i\pi\epsilon\delta(x-y) \tag{A.32}$$

$$[\nabla\phi(x), \nabla\theta(y)] = \nabla_x\nabla_y[\phi(x), \theta(y)] = \nabla_x\nabla_y \frac{i\pi}{2}\epsilon(x-y) = -\pi i \tag{A.33}$$

where we have used the distributional derivative of the delta function. These commutation relations can also be derived directly from the expression of the fields and its derivatives in terms of bosonic operators. For example,

$$[\phi(x), \nabla\theta(y)] = -i\pi \sum_{q,q'} \frac{e^{-a|q|-iq(x-y)}}{q} \frac{q}{|q|}\frac{|q|}{2L}2 = \frac{-i\pi}{L}\sum_q e^{-a|q|+iqR} \to \boxed{L\to\infty}$$

$$\to \frac{-i}{2}\int_{-\infty}^{\infty} dq e^{iqR-a|q|} \to \boxed{a\to 0} \to -\frac{i}{2}\int_{-\infty}^{\infty} dq e^{iqR} = -i\pi\delta(R) \tag{A.34}$$

where we have used the integral representation of the delta function. The same for the commutator between the derivatives:

$$[\nabla\phi(x), \nabla\theta(y)] = -\frac{\pi}{L}\sum_q e^{-a|q|+iqR} q \to \boxed{L\to\infty} \to -\frac{1}{2}\int_{-\infty}^{\infty} dq q e^{iqR-a|q|} \to$$

$$\boxed{a\to 0} \to -\frac{1}{2}\int_{-\infty}^{\infty} dq q e^{iqR} = -\frac{1}{2}\int_{-\infty}^{\infty} dq \frac{\partial}{\partial R}\frac{e^{iqR}}{i} = i\frac{\partial}{\partial R}\int_{-\infty}^{\infty} dq e^{iqR}$$

$$= i\pi\delta'(R) = -i \tag{A.35}$$

where we have used the derivative of the delta function $\delta'(R)$. [8] [5] [13]

Appendix B

Demonstration of the identity related to the average of the cosine

In this section we derive the identity $\langle\cos(\phi)\rangle = e^{-\frac{1}{2}\langle\phi^2\rangle}$.
We start from the general correlation function of the type:

$$I = \langle\prod_j e^{i(A_j\phi(\mathbf{r}_j)+B_j\theta(\mathbf{r}_j))}\rangle \tag{B.1}$$

where A and B are some coefficients and $\mathbf{r}_j = (x_j, u\tau_j)$. We can rewrite the term in the exponential in the Fourier space as:

$$\sum_j [A_j\phi(\mathbf{r}_j) + B_j\theta(\mathbf{r}_j)] = \frac{1}{\beta\Omega}\sum_{\mathbf{q}}[A(\mathbf{q})\phi(-\mathbf{q}) + B(\mathbf{q})\theta(-\mathbf{q})] \tag{B.2}$$

where

$$A(\mathbf{q}) = \sum_j A_j e^{-i(kx_j-\omega_n\tau_j)} \tag{B.3}$$

and similarly for B. The correlation function becomes:

$$\langle\prod_j e^{i(A_j\phi(\mathbf{r}_j)+B_j\theta(\mathbf{r}_j))}\rangle = \frac{1}{Z}\int \mathrm{D}\phi\mathrm{D}\theta$$

$$e^{-\frac{1}{2\beta\Omega}\sum_{\mathbf{q}}\left[\begin{pmatrix}\theta_{-\mathbf{q}} & \phi_{-\mathbf{q}}\end{pmatrix}M^{-1}\begin{pmatrix}\theta_{\mathbf{q}}\\ \phi_{\mathbf{q}}\end{pmatrix}-i[\begin{pmatrix}B(-\mathbf{q}) & A(-\mathbf{q})\end{pmatrix}\begin{pmatrix}\theta_{\mathbf{q}}\\ \phi_{\mathbf{q}}\end{pmatrix}+\begin{pmatrix}\theta_{-\mathbf{q}} & \phi_{-\mathbf{q}}\end{pmatrix}\begin{pmatrix}B(\mathbf{q})\\ A(\mathbf{q})\end{pmatrix}]\right]} \tag{B.4}$$

Where M is:

$$M = \frac{\pi}{k^2(u^2k^2+\omega_n^2)}\begin{pmatrix} k^2\frac{u}{K} & -ik\omega_n \\ -ik\omega_n & k^2uK \end{pmatrix} \tag{B.5}$$

Completing the square:

$$\langle\prod_j e^{i(A_j\phi(\mathbf{r}_j)+B_j\theta(\mathbf{r}_j))}\rangle = e^{-\frac{1}{2\beta\Omega}\sum_{\mathbf{q}}\begin{pmatrix} B(-\mathbf{q}) & A(-\mathbf{q})\end{pmatrix} M \begin{pmatrix} B(\mathbf{q}) \\ A(\mathbf{q})\end{pmatrix}} \tag{B.6}$$

Let us work out only the terms concerning A in the exponential only, which are useful for our purpose:

$$-\frac{1}{2\beta\Omega}\sum_{i,j}\sum_{\mathbf{q}} A_iA_j e^{i\mathbf{q}\cdot(\mathbf{r}_i-\mathbf{r}_j)}\frac{uK\pi}{\omega_n^2+u^2k^2}$$

$$= -\frac{1}{2\beta\Omega}\sum_{i,j}\sum_{\mathbf{q}} A_iA_j\cos(\mathbf{q}(\mathbf{r}_i-\mathbf{r}_j))\frac{uK\pi}{\omega_n^2+u^2k^2} \tag{B.7}$$

If we substitute the expression of $F_1(r) = \frac{1}{\beta\Omega}\sum_{\mathbf{q}}[1-\cos\mathbf{q}\cdot\mathbf{r}]\frac{2\pi u'}{\omega_n^2+u'^2k^2}$ in the above formula:

$$-\frac{1}{2\beta\Omega}\sum_{i,j}\sum_{\mathbf{q}} A_iA_j\cos(\mathbf{q}\cdot(\mathbf{r}_i-\mathbf{r}_j))\frac{uK\pi}{\omega_n^2+u^2k^2}$$

$$= -\frac{1}{2\beta\Omega}\sum_{i,j}\sum_{\mathbf{q}} A_iA_j[\cos(\mathbf{q}\cdot(\mathbf{r}_i-\mathbf{r}_j))-1]\frac{uK\pi}{\omega_n^2+u^2k^2} - \frac{1}{2\beta\Omega}\sum_{i,j}\sum_{\mathbf{q}} A_iA_j\frac{uK\pi}{\omega_n^2+u^2k^2}$$

$$= \frac{1}{4}\sum_{i,j} A_iA_jKF_1(\mathbf{r}_i-\mathbf{r}_j) - (\sum_i A_i)^2\frac{1}{2\beta\Omega}\sum_{\mathbf{q}}\frac{uK\pi}{\omega_n^2+u^2k^2} \tag{B.8}$$

Substituting $A_1 = 1$ and $A_i = 0$ with $i \neq 1$ and confronting with the correlation function

$$\langle\phi(\mathbf{q}_1)^*\phi(\mathbf{q}_2)\rangle = \frac{\pi K\delta_{\mathbf{q}_1\mathbf{q}_2}\Omega\beta}{\frac{\omega_n^2}{u}+uk_1^2} \tag{B.9}$$

we obtain the desired result.

Appendix C

Calculation of the correlation function of the quantum harmonic oscillator

Here we calculate the position correlation function of the quantum harmonic oscillator in order to justify the assumption of the linear dependence of C with respect to the temperature.
First of all, the density matrix of a system in the basis of the energy eigenfunctions with a temperature T described in the canonical ensemble is:

$$\rho = e^{-\beta E_n}|n\rangle\langle n| \tag{C.1}$$

The partition function is defined as:

$$Tr(e^{-\beta H}) \tag{C.2}$$

and can be computed as:

$$\sum_n e^{-\beta E_n} = \sum_n e^{-\beta\hbar\omega\left(n+\frac{1}{2}\right)} = \frac{e^{-\frac{\beta\hbar\omega}{2}}}{1-e^{-\beta\hbar\omega}} = \frac{1}{2senh(\frac{\beta\hbar\omega}{2})} \tag{C.3}$$

Now the value of C can be calculated using the average value of $\hat{x}^2$, which is:

$$\langle\hat{x}^2\rangle = Tr[\rho\hat{x}^2] = Tr[\hat{x}^2 e^{-\beta H}]/Z = \sum_n \langle n|\hat{x}^2|n\rangle e^{-\beta E_n}/Z = \frac{2\hbar}{m\omega}\sum_n (2n+1)e^{-\beta E_n}/Z$$

$$= \frac{2\hbar}{m\omega}\frac{2}{\beta\hbar}\frac{1}{Z}\frac{\partial}{\partial\omega}Z = \frac{\hbar}{2m\omega}cotgh\left(\frac{\hbar\omega}{K_B T}\right) \tag{C.4}$$

[14] Now we can compute the correlation function at different times, which should be equal in the constant limit. We can express it as:

$$C(t-t') = \sum_n p_n \langle n|U^\dagger(t)xU(t)U^\dagger(t')xU(t')|n\rangle = \sum_n p_n \langle n|xU(t-t)x|n\rangle e^{i\omega_n(t-t')} \tag{C.5}$$

Inserting an identity we can get

$$\sum_{mn} p_n \langle n|\hat{x}|m\rangle\langle m|\hat{x}|n\rangle e^{-i\omega_{nm}(t-t')} = \sum_{mn} p_n |\langle m|\hat{x}|n\rangle|^2 e^{-i\omega_{nm}(t-t')} \tag{C.6}$$

where we have used the unitary temporal evolution operator $U(t) = e^{-\frac{iHt}{\hbar}}$, the probabilities $p_n = exp(-\beta E_n)/Z$ and $\omega_{nm} = (E_n - E_m)/\hbar$. Since

$$A_{nm} = \langle m|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1}) \tag{C.7}$$

we have that:

$$C(\tau-\tau') = \frac{\hbar}{2m\omega}\sum_n p_n(ne^{-\omega(\tau-\tau')} + (n+1)e^{+\omega(\tau-\tau')}) \tag{C.8}$$

where we have passed to the euclidean time $\tau = it$. From this we can see that it reduces to the constant case in the limit $\omega \to 0$. As a consequence, we can perform a Taylor expansion of $\langle\hat{x}^2\rangle$ in this limit and we obtain:

$$\langle\hat{x}^2\rangle \simeq \frac{k_B T}{2m\omega^2} \tag{C.9}$$

which demonstrates that it depends linearly on the temperature in this regime.

Appendix D

Estimation of the error made neglecting the cosine term in the momentum space integral

We may estimate the relative error made in this approximation with the following expression:

$$\frac{\int_0^\beta \mathrm{d}\tau \int_\tau^{\tau+\beta} \mathrm{d}\tau'' 2e^{-\int d\omega \frac{e^{-\alpha\omega^2}\pi}{\sqrt{\Delta^2+\omega^2}}} \int d\omega \cos(\omega\tau'') \frac{e^{-\alpha\omega^2}\pi}{\sqrt{\Delta^2+\omega^2}}}{\int_0^\beta \mathrm{d}\tau \int_\tau^{\tau+\beta} \mathrm{d}\tau'' e^{-\int d\omega \frac{e^{-\alpha\omega^2}\pi}{\sqrt{\Delta^2+\omega^2}}}} \tag{D.1}$$

In the expression of the numerator, starting from:

$$\int_0^\beta \mathrm{d}\tau \int_\tau^{\tau+\beta} \mathrm{d}\tau''(2+2\cos\omega(\tau''))e^{-\int d\boldsymbol{q}(1+\cos(\omega\tau''))\frac{\pi K u}{u^2q^2+\Delta^2}} \tag{D.2}$$

and neglecting the oscillating term, one gets for the absolute error:

$$\int_0^\beta \mathrm{d}\tau \int_\tau^{\tau+\beta} \mathrm{d}\tau'' 2e^{-\int d\boldsymbol{q}(1+\cos(\omega\tau''))e^{-\alpha\omega^2}\frac{\pi K u}{u^2q^2+\Delta^2}} - \int_0^\beta \mathrm{d}\tau \int_\tau^{\tau+\beta} \mathrm{d}\tau'' 2e^{-\int d\boldsymbol{q} e^{-\alpha\omega^2}\frac{\pi K u}{u^2q^2+\Delta^2}} \tag{D.3}$$

Integrating analytically on the k variable one gets (u=K=1):

$$\int_0^\beta \mathrm{d}\tau \int_\tau^{\tau+\beta} \mathrm{d}\tau'' 2e^{-\int dk(1+\cos(\omega\tau''))\frac{\pi e^{-\alpha\omega^2}}{\sqrt{k^2+\Delta^2}}} - \int_0^\beta \mathrm{d}\tau \int_\tau^{\tau+\beta} \mathrm{d}\tau'' 2e^{-\int dk \frac{\pi e^{-\alpha\omega^2}}{\sqrt{k^2+\Delta^2}}} \tag{D.4}$$

As we assume the cosine integral at the exponential to be small, we can expand this term and we obtain for the absolute error:

$$\int_0^\beta \mathrm{d}\tau \int_\tau^{\tau+\beta} \mathrm{d}\tau'' 2e^{-\int d\omega \frac{e^{-\alpha\omega^2}\pi}{\sqrt{\Delta^2+\omega^2}}} \int d\omega \cos(\omega\tau'') \frac{e^{-\alpha\omega^2}\pi}{\sqrt{\Delta^2+\omega^2}} \tag{D.5}$$

The relative error tends to 0, because the denominator diverges as $\beta \to \infty$, while the numerator is finite

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