POLITECNICO DI TORINO

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Master's Degree in Mechatronic Engineering

Master Thesis

Output regulation and adaptive control of uncertain linear systems: an overview and an adaptive passification approach

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to myself

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Contents

List of Figures

Chapter 1

Introduction

Model Reference Adaptive Control (MRAC) is one of the main approaches to adaptive control, where the goal is that of making a system, with uncertain or even unknown parameters, to follow a time-varying but bounded reference trajectory. A reference model is designed to generate a desired trajectory which is in general a filtered version of the reference signal. The MRAC closed-loop is made of an ordinary feedback control law that contains the plant, a parameterized controller, and an adjustment mechanism that generates the controller parameter estimates online. This is called a direct adaptive approach as we want to directly estimate the controller parameters, without estimating the unknown plant parameters. In this scenario, the reference model basically describes the input-output properties we want for the closed-loop system. It is guaranteed that, for some classes of systems, the plant output is able to asymptotically follow the output of the reference model (which in turn may be able to follow the reference signal). If we perfectly knew the parameters of the plant, adaptation would not be needed and this problem is known in the literature as the reference model following problem or Model Reference Control (MRC): MRAC is indeed the adaptive version of MRC. Specifically, following the so-called certainty equivalence principle, we design ideal gains (from MRC, as if we knew the parameters of the plant) that could lead the plant to the desired input-output behaviour and thus reference model tracking. However, since these gains are unknown in view of the unknown dynamics, we design adaptive laws for these gains that lead the plant to follow the reference model output.

The problem of Output Regulation (OR) refers to the design of a control system able to achieve asymptotic tracking of prescribed trajectories and/or asymptotic rejection of disturbances, in a robust way. The main feature that distinguishes the output regulation problem from conventional tracking and rejection problems is the presence of an autonomous system, known as exosystem, that generates the exogenous inputs, which includes disturbances (to be rejected) and/or references (to be tracked). The well-known internal model principle states that, in order to solve the problem in a robust way, a copy of the exosystem must be incorporated in the feedback controller. Obviously, this in general requires perfect knowledge of the exosystem dynamics. In this scenario, the problem is solved by first augmenting the plant with an internal model unit, designed to properly incorporate the model of the exosystem and, then, the cascade of the plant and the internal model unit is stabilized by properly designing a stabilizer (depending on the location of the internal model unit we distinguish between pre-processing and post-processing schemes). A challenging problem arises when the dynamics of the exosystem are not assumed anymore to be perfectly known: in this case, adaptive techniques are required and the problem is referred to as the adaptive output regulation problem.

Both in MRAC and OR problem, as well in various control systems, the passivity property plays an important role in the design of stabilizers. In particular, the incremental passivity is being studied and applied to more and more cases, rising interesting points in the design of a stabilizer.

The aim of this work is to study the two different adaptive approaches related to linear time-invariant systems. In light of the recent advances in the field of adaptive output regulation, the primary contribution of this thesis is that of comparing the two aforementioned approaches in terms of control goal, assumptions and robustness properties, and to show to what extent they are similar. Moreover, a benchmark problem is carefully crafted so that the two approaches can both be used to solve the same problem, in order to have a comparison also in terms of performance. Finally, the possibility to merge the two techniques is evaluated, discussing what would be the advantages of doing so. Throughout this work, the stability and convergence analysis is performed via Lyapunov-like techniques, and numerical examples demonstrate the effectiveness of the protocols.

The comparison is focused on two particular cases. For the OR problem, it is considered the general framework with references and disturbances generated by a non-autonomous exosystem, where the stability of the system and the convergence are achieved by decoupling the external input from the regulation error. For the MRAC problem a variant of the problem shown in [Serrani, 2013] is presented: the reference model and the plant to be controlled are both driven by a signal generated by an autonomous exosystem. The stability analysis and the convergence of the error to zero, are demonstrated by means of the LaSalle-Yoshizawa theorem.

Then the the merging of the OR problem and the adaptive design is done studying the incremental passivity property. It is shown how to design an output feedback controller that makes the plant to be controlled incrementally passive. In the case the plant is perfectly known, it is possible to obtain a nominal value for the static feedback by means of a regular storage function and satisfying the condition of incremental passivity. In presence of uncertainties, an adaptive law is implemented to estimate the values of the feedback.

Considering the same problem, the estimated value will be different from the nominal one, relaxing the constrain on the value.

1.1 Thesis outline

The thesis is organized as follows.

Since one of the main topic in control system is stability, in Chapter 2 a review of the general notions useful for the thesis is presented, as well as an introduction on the passivity.

In Chapter 3 the Output regulation problem is studied, starting from the analysis at steady-state, to the special case in which we consider a nonautonomous exosystem.

Chapter 4 is dedicated to the other problem, Model Reference Adaptive Control; after an introduction on the simpler cases, the attention is focused on the approach shown in [Serrani, 2013] and on the special case of the input generated by an autonomous exosystem.

In Chapter 5 it is presented the incremental passivity property in the OR problem, with the adaptive design of the controller. It is also provide a numerical example in order to show the theoretical results.

In the end, in Chapter 6, a comparison between the two approaches of OR and MRAC is presented, as well as few considerations on the results of the numerical example on the incremental passivity.

Chapter 2

Basic Tools

2.1 Stability

Stability analysis plays a crucial role in system theory. All the results and definition in the following sections can be found in all books of control theory.

2.1.1 Norms and $\mathcal{L}_p$ Spaces

In order to measure the size of a signal, is necessary to introduce the norm function $\|x\|$, which satisfy the following properties:

- The norm of a signal is strictly positive, except the case where the signal is identically zero: $\|x\| \geq 0$ with $\|x\| = 0$ if and only if $x = 0$.
- For any positive constant λ and signal x the following relation is valid: $\|\lambda x\| = \lambda\|x\|$.
- The triangle inequality $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$ is valid for any signals x_1 and x_2.

The term $\|\cdot\|$ is used for the norm of a signal, while $|\cdot|$ for the norm of vectors and matrices.

It is possible to consider different space of signals, such the space of piecewise continuous, bounded function, and piecewise continuous, square-integrable functions.

Considering the two spaces of functions, it is possible to show two definitions of norm. In the space of piecewise continuous, bounded function, the space is denoted by $\mathcal{L}_\infty$ and the norm is defined as

$$\|x\|_\infty \triangleq \sup_{t\geq 0} |x\,(t)\,| < \infty \tag{2.1}$$

If the norm $\|x\|_\infty$ exists, then it is possible to say that $x \in \mathcal{L}_\infty$.

The space of piecewise continuous, square-integrable functions is denoted by $\mathcal{L}_p^m$, where the subscript p stand for the type of norm that is used and the superscript m for the dimension of the signal. The norm is defined as

$$\|x\|_{\mathcal{L}_p} \triangleq \left(\int_0^\infty \|x\left(\tau\right)\|^p d\tau\right)^{1/p} < \infty \tag{2.2}$$

for $p \in [1, \infty)$, and, as done before, if the norm $\|x\|_p$ exists, then $x \in \mathcal{L}_p$.

A particular case of the $\mathcal{L}_p^m$ norm is the $\mathcal{L}_2$ norm, defined as follows

$$\|x\|_2 = \sqrt{\left(\int_0^\infty x\left(\tau\right)^T x\left(\tau\right) d\tau\right)} \tag{2.3}$$

2.1.2 Properties of Function

Definition 2.1.1 (Continuity). *A function $f : [0, \infty) \to \mathbb{R}$ is* ***continuous*** *on $[0, \infty)$ if for any given $\epsilon_0 > 0$ there exists a $\delta\left(\epsilon_0, t_0\right)$ such that $\forall t_0, t \in [0, \infty)$ for which $|t - t_0| < \delta\left(\epsilon_0, t_0\right)$ we have $|f\left(t\right) - f\left(t_0\right)| < \epsilon_0$.*

Definition 2.1.2 (Uniform Continuity). *A function $f : [0, \infty) \to \mathbb{R}$ is* ***uniformly continuous*** *on $[0, \infty)$ if for any given $\epsilon_0 > 0$ thereexists a $\delta\left(\epsilon_0\right)$ such that $\forall t_0$,$t \in [0, \infty)$ for which $|t - t_0| < \delta\left(\epsilon_0\right)$ we have $|f\left(t\right) - f\left(t_0\right) < \epsilon_0$.*

Definition 2.1.3 (Piecewise Continuity). *A function $f : [0, \infty) \to \mathbb{R}$ is* ***piecewise continuous*** *on $[0, \infty)$ if, for any finite interval $[t_0, t_1] \subset [0, \infty)$, f is continuous on $[t_0, t_1]$ except for a finite number of points.*

Definition 2.1.4 (Lipschitz). *A function $f : [a, b] \to \mathbb{R}$ is* ***Lipschitz*** *on $[a, b]$ if $|f\left(x_1\right) - f\left(x_2\right)| \leq k|x_1 - x_2|, \forall x_1, x_2 \in [a, b]$, where $k \geq 0$ is a constant referred to as the* ***Lipschitz constant****.*

Definition 2.1.5 (Locally Lipschitz). *A function $f\left(x\right)$ is said to be* ***locally Lipschitz*** *on a domain $D \subset \mathbb{R}^n$ if each point of D has a neighbourhood D_0 such that f satisfies the Lipschitz condition (Definition 2.1.4) for all points in D_0 with some Lipschitz constant k.*

Definition 2.1.6 (Globally Lipschitz). *A function $f\left(x\right)$ is said to be* ***globally Lipschitz*** *if it is Lipschitz on $\mathbb{R}^n$.*

Lemma 2.1.1 (Barbalat's Lemma). *Consider a function $f\left(t\right)$ being a uniformly continuous function in the interval $[0, \infty)$, if the limit $\lim_{t\to\infty} \int_0^t f\left(\tau\right) d\tau$ exists, then*

$$f\left(t\right) \to 0 \quad as \quad t \to \infty$$

Definition 2.1.7 (Persistency of excitation). *A piecewise continuous signal $u(t)$ is* ***persistently exciting*** *(PE), if $\exists \alpha_1, \alpha_2, T > 0$ such that*

$$\alpha_1 I \leq \int_t^{t+T} u(\tau)\, u(\tau)^\top d\tau \geq \alpha_2 I, \quad \forall t \geq 0 \tag{2.4}$$

The property of persistent excitation is fundamental in order to guarantee parameter convergence in many adaptive schemes.

2.2 Lyapunov's Stability

Consider an autonomous system described by the following differential equation

$$\dot{x} = f(x) \tag{2.5}$$

where $f : \mathbb{D} \to \mathbb{R}^n$ is a locally Lipschitz map from the domain $\mathbb{D} \subset \mathbb{R}^n$ into $\mathbb{R}^n$.

Definition 2.2.1. *$\bar{x} \in \mathbb{D}$ is said to be an* ***equilibrium point*** *of (2.5) if*

$$f(\bar{x}) = 0 \tag{2.6}$$

Definition 2.2.2. *The set of points in the state space generated by the solution is called the trajectory (or orbit) of the system corresponding to the input signal $u(t)$ and the initial condition $x(0)$.*

Let $x(t)$ be the nominal solution of a system corresponding to given input signal and initial condition $x(0)$. Let $x_p(t)$ be the perturbed solution of the system with same input signal as before, but different initial condition $x_p(0) \neq x(0)$.

Definition 2.2.3. *The solution $x(t)$ is* ***stable*** *if $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$ such that*

$$\|x(0) - x_p(0)\| < \delta \;\Rightarrow\; \|x(t) - x_p(t)\| < \epsilon,\ \forall t \geq 0$$

Definition 2.2.4. *The solution $x(t)$ is* ***asymptotically stable*** *if it is stable and*

$$\lim_{t\to\infty} \|x(t) - x_p(t)\| = 0$$

Definition 2.2.5. *The solution $x(t)$ is* ***unstable*** *if it is not stable.*

2.2.1 Lyapunov's Direct Method

The Lyapunov's Direct Method is the method analyse stability properties based on the form of $f(x)$ rather then the solutions.

Let us define a continuously differentiable function $V : \mathbb{D} \to \mathbb{R}$ in the domain $\mathbb{D} \subset \mathbb{R}^n$, that contains the origin. The time derivative of V along the solutions of system (2.5) is given by

$$\dot{V}(t) = \frac{\partial V}{\partial x} f(x) \tag{2.7}$$

The *Lyapunov's stability theorem* is stated as follows

Theorem 2.2.1 (Lyapunov's Stability Theorem). *Let $x = 0$ be an equilibrium point of (2.5) and $\mathbb{D} \in \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : \mathbb{D} \to \mathbb{R}$ be a continuously differentiable function such that*

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } \mathbb{D} - \{0\}$$

Then, $x = 0$ is stable if

$$\dot{V}(x) \leq 0 \text{ in } \mathbb{D}$$

Moreover, $x = 0$ is asymptotically stable if

$$\dot{V}(x) < 0 \text{ in } \mathbb{D} - \{0\}$$

Definition 2.2.6 (Lyapunov function). *A function $V : \mathbb{D} \to \mathbb{R}$ is a* ***Lyapunov function*** *for the system (2.5) if, in some ball $B_R = x \in \mathbb{R}^n : \|x\| \leq R \subset \mathbb{D}$,*

(i) V is positive definite and has continuous partial derivatives.

(ii) $\dot{V}$ is negative semi-definite.

In the case of linear systems, it is possible to write the Lyapunov function in its quadratic form as follows

$$V = x^T P x \tag{2.8}$$

where P is a real symmetric positive definite matrix $P = P^\top > 0$.

Definition 2.2.7. *A function $V(x)$ is said to be* ***positive definite*** *if $x^T P x > 0, \ \forall x \in \mathbb{R}^n - \{0\}$.*

Due to the condition $\dot{V}(x) \leq 0$, when the trajectory crosses the surface $V(x) = 0$, it moves inside the set $\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq x\}$ and can never come out.

Definition 2.2.8. *A set M is said to be an* ***invariant set*** *with respect to (2.5) if*

$$x(0) \in \mathbb{M} \Rightarrow x(t) \in \mathbb{M}, \ \forall t \in \mathbb{R}$$

This means that if a solution belongs to M at some time instant, then it belongs to M for all future and past time. The set M is said to be ***positively invariant*** *if*

$$x(0) \in \mathbb{M} \Rightarrow x(t) \in \mathbb{M}, \ \forall t \geq 0$$

It is possible to state two important theorem for the study of the stability. Considering time-invariant system, the LaSalle Invariance Principle is adopted.

Theorem 2.2.2 (LaSalle). *Let $\Omega \subset \mathbb{D}$ be a compact set that is positively invariant with respect to (2.5). Let $V : \mathbb{D} \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.*

For the study of non-autonomous system, is possible to use the LaSalle-Yoshizawa theorem, in which differently from the LaSalle Invariance Principle, the set can be variant.

Theorem 2.2.3 (LaSalle-Yoshizawa). *Let $\bar{x} = 0$ be the equilibrium point of the nonautonomous system*

$$\dot{x} = f(t, x), \quad x(0) = x_0 \tag{2.9}$$

and assume that $f(t, x)$ is locally Lipschitz in x. Let $V : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function $V(t, x)$ satisfying

(i) $V(t, x) > 0$ and $V(0) = 0$ (positive definite).

(ii) $V(t, x) \to \infty$ as $\|x\| \to \infty$ (radially unbounded).

(iii) $\dot{V}(t, x) = \dfrac{\partial V(t, x)}{\partial t} + \dfrac{\partial V(t, x)}{\partial x} f(t, x) \leq -W(x) \leq 0$

where $W(x)$ is a positive continuous function. Then all solutions $x(t)$ of (2.9) are uniformly globally bounded and

$$\lim_{t \to \infty} W(x(t)) = 0 \tag{2.10}$$

and the system (2.9) is uniformly globally stable.

2.3 Passivity

The energy dissipation is a known concept in the study of dynamical system. The passivity property characterizes the energy consumption of a system. Consider a dynamical system in the following affine form

$$\begin{aligned} \dot{x} &= f(t, x) + g(t, u) \\ y &= h(t, x) \end{aligned} \tag{2.11}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$.

Definition 2.3.1 (Passivity). *The system (2.11) is said to be passive if there exists a smooth nonnegative function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (usually called a storage function) satisfying*

$$\begin{aligned}\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f\left(t, x\right) \leq 0 \\ \frac{\partial V}{\partial x} g\left(t, x\right) = h^T\left(t, x\right)\end{aligned}$$

for all $t \in \mathbb{R}_{\leq 0}$ and all $x \in \mathbb{R}^n$.

It can also be said that the system (2.11) is passive if V satisfies

$$\dot{V} \leq y^\top u \tag{2.12}$$

Definition 2.3.2 (Strict passivity). *The system (2.11) is said to be strictly passive if there exists a smooth positive definite storage function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and a positive definite function $\alpha\left(\cdot\right)$ (called dissipation rate) satisfying*

$$\begin{aligned}\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f\left(t, x\right) \leq -\alpha\left(x\right) \\ \frac{\partial V}{\partial x} g\left(t, x\right) = h^T\left(t, x\right)\end{aligned}$$

for all $t \in \mathbb{R}_{\leq 0}$ and all $x \in \mathbb{R}^n$.

First let us consider the definition of a proper rational transfer function.

Definition 2.3.3. *A proper rational transfer function $G\left(s\right)$, where $s = \sigma + j\omega$, is called positive real if*

(i) $G\left(s\right)$ is positive for real values of s.

(ii) $Re\left[G\left(s\right)\right] \geq 0$ for all $Re\left[s\right] > 0$.

Assume that $G\left(s\right)$ is not identically zero for all s. Then $G\left(s\right)$ is called strictly positive real if $G\left(s - \epsilon\right)$ is positive real for some $\epsilon > 0$.

Definition 2.3.4. *Consider the system described by*

$$y = k\left(u\right) \tag{2.13}$$

with input $u \in \mathbb{R}^p$ and output $y \in \mathbb{R}^p$. The system (2.13) is passive if the function $k\left(\cdot\right)$satisfies the following condition

$$k\left(u\right)^\top u \geq 0 \tag{2.14}$$

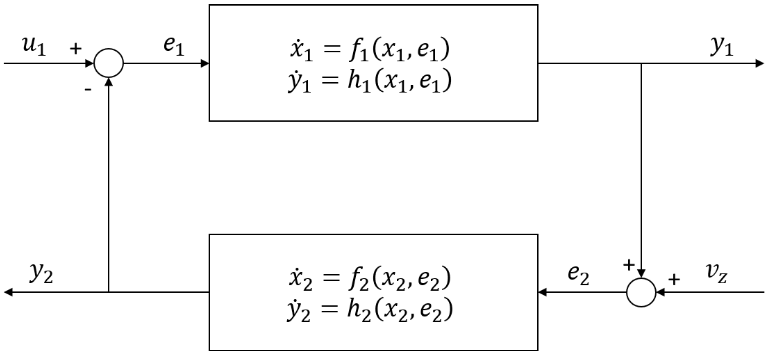


Figure 2.1: Feedback Interconnections between two passive systems

Lemma 2.3.1. *Consider the two system*

$$\begin{aligned} \dot{x}_1 &= f_1\left(x_1, e_1\right), \qquad & y_1 &= h_1\left(x_1, e_1\right) \\ \dot{x}_2 &= f_2\left(x_2, e_2\right), \qquad & y_2 &= h_2\left(x_2, e_2\right) \end{aligned} \tag{2.15}$$

Both system are passive, and interconnected through $e_1 = u_1 - y_2$ *and* $e_2 = u_2 + y_1$*. The interconnected system, shown in Figure 2.1 is passive.*

Proof. Each subsystem has its own storage function $V_1\left(x_1\right)$ and $V_2\left(x_2\right)$. From the definition of passivity and consider both the subsystems, the inqeuality (2.12) becomes

$$\dot{V}_1 + \dot{V}_2 \leq y_1^\top e_1 + y_2^\top e_2 = y_1^{|} top\left(u_1 - y_2\right) + y_2^\top\left(u_2 + y_1\right) \tag{2.16}$$

Hence,

$$\dot{V}_1 + \dot{V}_2 \leq y_1^\top u_1 + y_2^\top u_2 \tag{2.17}$$

Then the interconnected system is passive. □

Consider now the case of linear systems. In order to study the passivity properties and to show the relation between positive realness and passivity, it is necessary to also highlight the *Kalman-Yakubovich-Popov* lemma.

Lemma 2.3.2 (Positive Real). *Let* $G\left(s\right) = C\left(s\,I - A\right)^{-1} B$ *be a transfer matrix where* (A, B) *is controllable and* (C, A) *is observable[Appendix A]. Then* $G\left(s\right)$ *is positive real if and only if there exist a symmetric positive definite matrix* P *and a vector* q *such that*

$$\begin{aligned} P\,A + A^T\,P &= -\,q\,q^T \\ P\,B &= C^T \end{aligned} \tag{2.18}$$

Lemma 2.3.3 (Kalman-Yakubovich-Popov). *Let* $G\left(s\right) = C\left(s\,I - A\right)^{-1} B$ *be a transfer matrix where* (A, B) *is controllable and* (C, A) *is observable. Then* $G\left(s\right)$

is positive real if and only if and only if for any positive definite matrix L, there exist a symmetric positive definite matrix P, a scalar $\nu > 0$ and a vector q such that

$$\begin{aligned} P\,A + A^T\,P &= -\,q\,q^T - \nu\,L \\ P\,B &= C^T \end{aligned} \tag{2.19}$$

Consider the system

$$\begin{aligned} \dot{x} &= A\,x + B\,u \\ y &= C\,x \end{aligned} \tag{2.20}$$

If there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a symmetric positive semi-definite matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$\begin{aligned} &A^T\,P + P\,A \leq -Q \\ &P\,B = C^T \end{aligned}$$

then the above system is *passive*, and the pair (C, A) is detectable if and only if the pair (A, B) is stabilizable. In addition, if $Q > 0$, then the system is *strictly passive.*

Lemma 2.3.4. *The linear-time invariant minimal realization [Appendix A]*

$$\begin{aligned} \dot{x} &= A\,x + B\,u \\ y &= C\,x \end{aligned}$$

with $G\,(s) = C\,(s\,I - A)^{-1}\,B$ is

- *passive if $G\,(s)$ is positive real.*
- *strictly passive if $G\,(s)$ is strictly positive real.*

2.3.1 Incremental Passivity

Let us give a definition of incremental passivity and show some basic results.

Definition 2.3.5. *Consider the system*

$$\begin{aligned} \dot{x} &= L\,(x, u, t) \\ y &= G\,(x, t) \end{aligned} \tag{2.21}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^p$ and output $y \in \mathbb{R}^q$. It is possible to say that the system (2.21) is incrementally passive if there exists a storage function $V\,(t, x_1, x_2) : \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^+$ such that for any two inputs $u_1\,(t)$ and $u_2\,(t)$ and any two solutions of system (2.21) $x_1\,(t)$, $x_2\,(t)$ corresponding to these inputs, the *respective outputs* $y_1(t) = G(x_1(t), t)$ *and* $y_2(t) = G(x_2(t), t)$ *satisfy the inequality*

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1} L(x_1, u_1, t) + \frac{\partial V}{\partial x_2} L(x_2, u_2, t) \leq (y_1 - y_2)^T (u_1 - u_2) \tag{2.22}$$

Definition 2.3.6. *Consider the system described by*

$$y = k(u, t) \tag{2.23}$$

with input $u \in \mathbb{R}^p$ *and output* $y \in \mathbb{R}^p$. *The system (2.23) is incrementally stable if the function* $k(\cdot, \cdot)$ *satisfies the following condition*

$$(k(u_1, t) - k(u_2, t))^T (u_1 - u_2) \geq 0 \tag{2.24}$$

Lemma 2.3.5. *Let us consider the two systems*

$$\begin{aligned} \dot{x} &= G_x(x, u_x, t), & y_x &= L_x(x, t) \\ \dot{z} &= G_z(z, u_z, t), & y_z &= L_z(z, t) \end{aligned} \tag{2.25}$$

Both systems are incrementally passive. They are interconnected through $u_{xi} = \alpha\, y_{xi} + v_{xi}$ *and* $u_{zi} = -\alpha^T y_{zi} + v_{zi}$, *where* $\alpha \in \mathbb{R}^{p \times p}$ *is a square matrix gain. Then the interconnected system, shown in Figure 2.2, is incrementally passive with input* $v = \left(v_x^T, v_z^T\right)^T$ *and output* $y = \left(y_x^T, y_z^T\right)^T$.

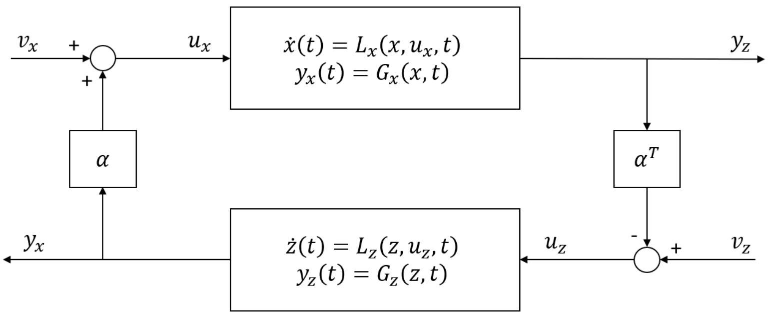


Figure 2.2: Feedback Interconnections between two incrementally passive systems

Proof. Each subsystem has its own storage function $V_x(t, x_1, x_2)$ and $V_z(t, z_1, z_2)$: it is possible to define the common storage function as the sum of the two: $W = V_x + V_z$.

From the definition of incremental passivity and considering both the subsystem, the inequality (2.22) becomes

$$\dot{W} \leq (y_{x1} - y_{x2})^T (u_{x1} - u_{x2}) + (y_{z1} - y_{z2})^T (u_{z1} - u_{z2}) \tag{2.26}$$

Replacing the interconnections, we obtain

$$\dot{W} \leq (y_1 - y_2)^T (v_1 - v_2) \tag{2.27}$$

where $v_i = \left(v_{xi}^T, v_{zi}^T\right)^T$ and $y_i = \left(y_{xi}^T, y_{zi}^T\right)^T$. Then the interconnected system is incrementally passive. □

Definition 2.3.7. *A storage function $V(t, x_1, x_2)$ is called regular if for any sequence (t_k, x_{1k}, x_{2k}), $k = 1,2,\ldots$, such that x_{2k} is bounded, t_k tends to infinity and $|x_{1k}| \to +\infty$, it holds that $V(t, x_1, x_2) \to +\infty$, as $k \to +\infty$.*

Lemma 2.3.6. *Consider the system described by*

$$\begin{aligned} \dot{x} &= G(x, u, w(t)) \\ y &= L(x, w(t)) \end{aligned} \tag{2.28}$$

Let $w(t)$ and $\dot{w}(t)$ be bounded on $\mathbb{R}^+$.

Suppose that the system (2.28) is incrementally passive with input u and output y. Suppose that for $u = 0$ the system has a bounded solution $\bar{x}(t)$ with zero output $\bar{y}(t) = 0$ for $t \geq t_0$. Considering a feedback of the type $u = -K\,y$, where K is positive semidefinite, all the solution of the system (2.28) are defined and bounded, and satisfy $y(t)^T K\, y(t) \to 0$ as $t \to +\infty$.

Proof. Consider the closed loop system (2.28) with $u = -K\,y$. From the previous lemma, there exist a solution $\bar{x}(t)$ that is bounded with the output $\bar{y}(t) = 0, t \geq t_0$. The input corresponding to this solution is $\bar{u}(t) - K\,\bar{y}(t) = 0$.

Considering the inequality expressed in (2.22), and the two solutions, $\bar{x}(t)$, $\bar{y}(t) = 0$ and $\bar{u}(t) = 0$ for x_2 , y_2 and u_2, and $x(t)$, $y(t)$ and $u(t) = -K\,y$ for x_1, y_1 and u_1, the derivative of the storage function becomes

$$\dot{V}(t, x(t), \bar{x}(t)) \leq -y(t)^T K\, y(t) \tag{2.29}$$

By integrating it, it is possible to obtain

$$V(t, x(t), \bar{x}(t)) - V(t_0, x(t_0), \bar{x}(t_0)) \leq -\int_{t_0}^{t} y(s)^T K\, y(s)\, ds \leq 0 \tag{2.30}$$

Therefore, $V(t, x(t), \bar{x}(t)) \leq V(t_0, x(t_0), \bar{x}(t_0))$ for all t from the maximal interval of existence of $x(t)$. From the previous lemma is known that $\bar{x}(t)$ is bounded, and since V is regular, these two conditions imply boundedness of $x(t)$ on its maximal interval of existence. The boundedness on $x(t)$ implies the boundedness on $\dot{x}(t)$ on $\mathbb{R}^+$. Together with the conditions on $w(t)$ and $\dot{w}(t)$, this implies that $\dot{y}(t)$ is bounded on $\mathbb{R}^+$. Hence, $y(t)^T K\, y(t)$ is uniformly continuous on $\mathbb{R}^+$. From (2.30)

$$\int_{t_0}^{+\infty} y(s)^T K\, y(s)\, ds \leq V(t_0, x(t_0), \bar{x}(t_0)) \leq +\infty \tag{2.31}$$

By Barbalat's lemma (Lemma 2.1.1), $y(t)^T K\, y(t) \to 0$ as $t \to 0$. □

The properties that have been defined for the non-linear systems, are also valid for the case of linear systems. Furthermore a passive linear system is also incrementally passive.

Lemma 2.3.7. *System of the form*

$$\begin{aligned} \dot{\tau} &= \Phi\,\tau + \alpha\,\Gamma\,e \\ v &= \Gamma^{\top}\,\tau, \end{aligned} \tag{2.32}$$

where Φ is a skew-symmetric matrix and $\alpha > 0$, is incrementally passive.

Proof. Consider the storage function

$$V\left(\tau_1, \tau_2\right) = \frac{1}{2\,\alpha}|\tau_1 - \tau_2|^2 \tag{2.33}$$

A system is incrementally passive if the following inequalities holds

$$\dot{V}\left(x_1, x_2\right) \leq \left(y_1 - y_2\right)^{\top}\left(u_1 - u_2\right)$$

In the considered case, the inequality becomes

$$\dot{V}\left(\tau_1, \tau_2\right) \leq \left(v_1 - v_2\right)^{\top}\left(e_1 - e_2\right) \tag{2.34}$$

The time derivative of the storage function can be written as

$$\dot{V}\left(\tau_1, \tau_2\right) = \frac{1}{\alpha}\left(\tau_1 - \tau_2\right)^{\top}\Phi\left(\tau_1 - \tau_2\right) + \left(\tau_1 - \tau_2\right)^{\top}\Gamma\left(e_1 - e_2\right) \tag{2.35}$$

The first element of the time derivative of the storage function can be deleted because Φ is a skew-symmetric matrix. The inequality (2.34) can be rewritten as

$$\dot{V}\left(\tau_1, \tau_2\right) \leq \left(\tau_1 - \tau_2\right)^{\top}\Gamma\left(e_1 - e_2\right) = \left(v_1 - v_2\right)^{\top}\left(e_1 - e_2\right) \tag{2.36}$$

Then the system (2.32) is incrementally passive. □

Chapter 3

Output Regulation

The Output Regulation problem regards a kind of problems where we want to achieve simultaneous reference tracking and disturbance rejection. The difference between the conventional tracking and rejection problem is that the signals to be tracked and/or rejected are generated by a known autonomous system, called *exosystem*. This kind of framework is referred as *problem of asymptotic disturbance rejection and/or tracking*, or just as *problem of output regulation*. There are three

different approaches, based on the knowledge of the exosystem: tracking by dynamic inversion, tracking via internal models, and adaptive tracking. Tracking by *dynamic inversion* consists computing precise initial state and a precise control input; in this case it is necessary the "perfect knowledge" of the entire trajectory to be tracked and of the model of the plant. This type of approach is not suitable in presence of uncertainties on the plant parameters as well on the reference signal. *Internal model-based* tracking is able to secure asymptotic decay to zero of the tracking error, in presence of disturbances on both the plant and the reference. *Adaptive* tracking consists in tuning the parameters of the control input computed via dynamic inversion in such a way to guarantee asymptotic convergence to zero of the tracking error. This method can handle parameter uncertainties, but still presupposes the knowledge of the trajectory to be tracked. The purpose of this chapter is to present the design of the internal model-based method.

3.1 Problem formulation

Consider a linear plant written in the following form

$$\begin{aligned} \dot{x} &= A\,x + B\,u + P\,w \\ e &= C_e\,x + Q_e\,w \\ \bar{y} &= C_{\bar{y}}\,x + Q_{\bar{y}}\,w \end{aligned} \tag{3.1}$$

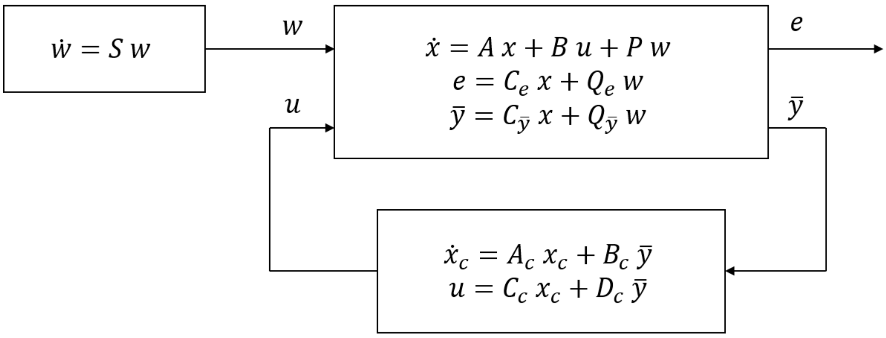


Figure 3.1: Closed-Loop System [Isidori, 2017].

The first equation represents the state $x \in \mathbb{R}^n$ of the plant, subject to the control input $u \in \mathbb{R}^p$ and the exogenous input $w \in \mathbb{R}^d$ including both the disturbances to reject and the reference to track. The second equation represents a set of regulated variables (or regulation errors) $e \in \mathbb{R}^m$. The third equation defines a set of measured variables $\bar{y} \in \mathbb{R}^q$.

The exogenous input is part of the family of signals that are solutions of the so-called *exosystem*: in general the exogenous signals are generated by an autonomous linear system described by the following differential equation:

$$\dot{w} = S\,w \tag{3.2}$$

for every initial conditions $w\,(0)$. In the analysis of the problem it is assumed that the matrix S is a marginally stable square matrix.

Control Objective The control problem is to design a controller able to make the close-loop signals bounded and make the regulation error e decay asymptotically to zero as $t \to \infty$, for any initial condition of the plant and the exosystem, and for every exogenous input in a prescribed family of functions.

It is considered the following model of the controller, in order to provide a feedback control action

$$\begin{aligned} \dot{x}_c &= A_c\,x_c + B_c\,\bar{y} \\ u &= C_c\,x_c + D_c\,\bar{y} \end{aligned} \tag{3.3}$$

where $x_c \in \mathbb{R}^{n_c}$ is the state and $\bar{y}$ is the measured information.

The resulting closed-loop system is shown in Figure 3.1 and described by the following equations

$$\begin{aligned} \dot{x} &= (A + B\,D_c\,C_{\bar{y}})\,x + B\,C_c\,x_c + (P + B\,D_c\,Q_{\bar{y}})\,w \\ \dot{x}_c &= B_c\,C_{\bar{y}}\,x + A_c\,x_c + B_c\,Q_{\bar{y}}\,w \\ e &= C_e\,x + Q_e\,w \end{aligned} \tag{3.4}$$

in which the input is w and the output e.

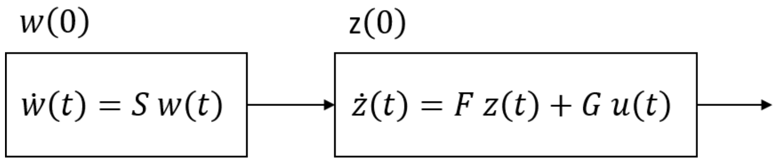


Figure 3.2: Steady-state analysis for an autonomous exosystem.

3.2 Steady-state Analysis

An important tool in the study of the output regulation problem is the steady-state analysis. Asymptotically stable linear system - driven by an harmonic signal as input - generates as output an harmonic signal having the same frequency but different amplitude and phase.

Consider the system shown in Figure 3.2, with the plant described by

$$\dot{z}\,(t) = F\,z\,(t) + G\,u\,(t) \tag{3.5}$$

where $S \in \mathbb{R}^{d \times d}$ is critically stable and $F \in \mathbb{R}^{n \times n}$ is Hurwitz.

The trajectories of the system can be written as

$$\begin{aligned} w\,(t) &= e^{S\,t} w\,(0) \\ z\,(t) &= e^{F\,t} z\,(0) + \int_0^t e^{F(t-\tau)} G\,w\,(t)\,d\tau \end{aligned} \tag{3.6}$$

When $t \to \infty$, the exogenous signal does not decay to zero, due to the matrix S that is marginally stable. The first component of the second equation instead, due to the fact that F is Hurwitz, goes to zero asymptotically leaving only the second component.

It is possible to define the steady-state behaviour of the the state z as a linear combination of ω:

$$z_{ss}\,(t) = \Pi\,w\,(t) \tag{3.7}$$

where $\Pi \in \mathbb{R}^{n \times d}$.

Replacing the steady-state (3.7) in (3.5) is possible to rewrite the equation only in function of w

$$\Pi\,\dot{w} = F\,\Pi\,w + G\,w \tag{3.8}$$

The resulting relation is the so called *Sylvester equation* or *regulator equation*

$$\Pi\,S = F\,\Pi + G \tag{3.9}$$

Proposition 3.2.1. *The state z converges to the steady-state (3.7) if there exists a matrix Π solution of the Sylvester equation (3.9).*

Proof. Define the variable error $\tilde{z}$ as

$$\tilde{z} = z - \Pi\, w$$

Taking the time derivative, the previous equation becomes

$$\dot{\tilde{z}} = \dot{z} - \Pi\, \dot{w} = F\,(\tilde{z} + \Pi\, w) + G\, w - \Pi\, S\, w$$

The last three terms are essentially the regulator equation, so they can be eliminated, and the previous equation becomes

$$\dot{\tilde{z}} = F\, \tilde{z} \;\rightarrow\; \tilde{z}\,(t) = e^{F\,t}\, \tilde{z}\,(0)$$

Since F is Hurwitz, the value of $\tilde{z}$ goes asymptotically to 0. It is possible to conclude that at steady-state z converges to

$$z\,(t) = \Pi\, w\,(t) \tag{3.10}$$

□

The existence of Π is guaranteed if and only if

$$\sigma\,(F) \cap \sigma\,(S) = 0$$

that means that the two spectra are disjoint. In this case the condition holds because S is marginally stable and F is Hurwitz.

3.3 Full Information Problem

Consider the system described by

$$\begin{aligned} \dot{x} &= A\, x + B\, u + P\, w \\ e &= C\, x + Q\, w \end{aligned} \tag{3.11}$$

with state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^p$, the exogenous signal $w \in \mathbb{R}^d$ generated by the exosystem (3.2) and the regulation errors $e \in \mathbb{R}^m$.

Assume that both the state x and the exogenous signal w are available for measurement and we have complete knowledge on the exosystem and in particular on the matrix S: this is the so called *full-information problem.*

In order to simplify the solution, instead of (3.3), it is considered the following form for the control law

$$u = K\, x + L\, w \tag{3.12}$$

The complete control scheme is shown in Figure 3.3.

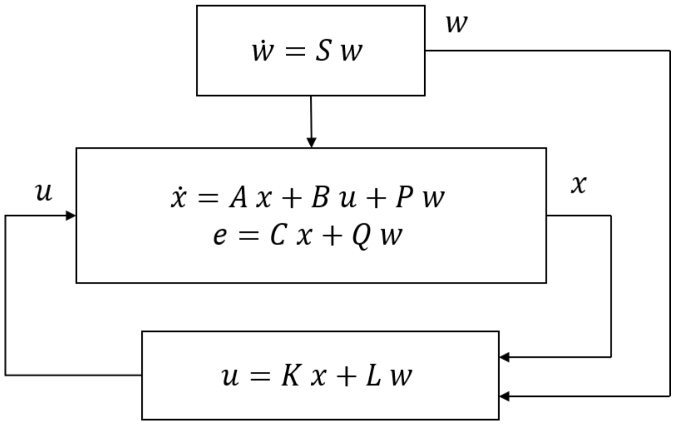


Figure 3.3: Full information control scheme.

Proposition 3.3.1. *The problem of output regulation in the case of full-information has a solution if and only if:*

(i) (A, B) *is stabilizable, due to the fact that the closed-loop trajectory must be bounded, so a stabilizing control action should exist.*

(ii) exists a couple of (Π, Ψ) *that is a solution for the Francis's equations*

$$\begin{aligned} \Pi\,S &= A\,\Pi + B\,\Psi + P \\ 0 &= C\,\Pi + Q \end{aligned} \tag{3.13}$$

Proof. Similarly to before, the steady-state of x can be expressed as follows

$$x_{ss}\,(t) = \Pi\,w\,(t) \tag{3.14}$$

In the case of full-information problem it is possible to derive also the steady-state of the signal u, replacing (3.14) in the control law (3.12) obtaining

$$u = K\,\Pi\,w + L\,w \tag{3.15}$$

Defining

$$\Psi = K\,\Pi + L$$

the (3.15) can be written as

$$u_{ss}\,(t) = \Psi\,w\,(t) \tag{3.16}$$

The ideal steady-state is defined as the target trajectories where the state x and the input u have to converge in order to have the problem fulfilled.

The Francis's equations are obtained considering the steady-state of x and u and replacing them in (3.1) as follows

$$\begin{aligned} \Pi\dot{w} &= A\,\Pi\,w + B\,\Psi\,w + P\,w \\ e &= C\,\Pi\,w + D\,w \end{aligned} \tag{3.17}$$

Considering the second equation, the objective of the control problem is to have the regulation error go to zero asymptotically. This is only possible if exists a Π such that $C\,\Pi + D = 0$.

[Sufficiency] Suppose that the pair of matrices (A, B) is stabilizable and exist a matrix K that is able to make the close-loop system stable. The control law can be written replacing $L = \Psi - \Pi\,K$ obtaining

$$u = \Psi\,w + K\,(x - \Pi\,w)$$

As done before, at steady-state the value of x should be equal to $\Pi\,w$, so it is possible to make a change of coordinates as follows

$$\tilde{x} = x - \Pi\,w$$

Deriving and replacing x

$$\dot{\tilde{x}} = (A + B\,K)\,\tilde{x} + \underbrace{(A\,\Pi + B\,\Psi + P - \Pi\,S)}_{=0}\,w$$

$$\dot{\tilde{x}} = (A + B\,K)\,\tilde{x}$$

Choosing K such that the matrix $(A + B\,K)$ is Hurwitz, the variable error goes to 0 asymptotically

$$\lim_{t\to\infty} \tilde{x}\,(t) = 0$$

Consider now the regulator equation error and the same change of coordinates done before:

$$e = C\,\tilde{x} + C\,\Pi\,w + Q\,w$$

The last two elements represent the second equation of (3.13) and the sum is equal to zero. The remaining term is function of $\tilde{x}$ that goes to zero asymptotically. Hence, at steady-state, the regulation error will be equal to zero.

This way we can say that

$$\lim_{t\to\infty} \tilde{x}\,(t) = 0 \qquad \lim_{t\to\infty} e\,(t) = 0 \tag{3.18}$$

□

Remark 3.3.1. *If the parameter matrices of the plant are unknown, it is impossible to find a solution pair* (Π, Ψ) *to the Francis' equations (3.13).*

Lemma 3.3.1 (Nonresonance condition). *The Francis' equation (3.13) have a solution for any* $(P\,Q)$ *if and only if*

$$rank \begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix} = \#rows \quad \forall \lambda \in \sigma(S) \tag{3.19}$$

If this is the case and the matrix on the left-hand size of (3.19) is square, then the solution pair (Γ, Ψ) *is unique.*

The *nonresonance condition* is considered as a necessary and sufficient condition if there exists a solution for any pair of matrices (P, Q).

Robust Output Regulation

Consider the case in which the matrices of the plant are dependent on a vector μ of uncertain parameters:

$$\begin{aligned} \dot{w} &= S\,w \\ \dot{x} &= A(\mu)\,x + B(\mu)\,u + P(\mu)\,w \\ e &= C_e(\mu)\,x + Q_e(\mu)\,w \\ \bar{y} &= C_{\bar{y}}(\mu)\,x + Q_{\bar{y}}(\mu)\,w \end{aligned} \tag{3.20}$$

Considering the case in which the control law is obtained with values of the plant matrices different from the real ones. In this case the regulation error will never go to zero. Then the control law defined in (3.12) is not robust against this kind of uncertainties.

The regulation error can also grow unbounded due to the fact that, if the matrices used to design the control law have huge differences with the real ones, the eigenvalues of the close-loop system can also be positive, making the closed-loop system unstable.

3.4 Measurement Feedback Problem

Consider now the case in which the states are unavailable for feedback, and are accessible only on measured variables like the *regulated output* and a supplementary set of independent measured variables $\bar{y} \in \mathbb{R}^q$.

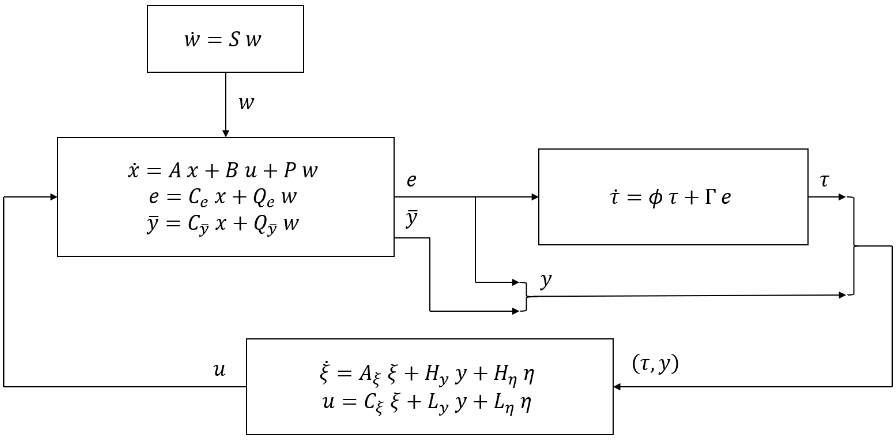


Figure 3.4: Measurement feedback control scheme.

The system is modelled as follows:

$$\begin{aligned}
\dot{w} &= S\,w \\
\dot{x} &= A\,x + B\,u + P\,w \\
e &= C_e\,x + Q_e\,w \\
\bar{y} &= C_{\bar{y}}\,x + Q_{\bar{y}}\,w
\end{aligned} \tag{3.21}$$

where we know that $x \in \mathbb{R}^n$, $e \in \mathbb{R}^m$ and $w \in \mathbb{R}^d$. In order to solve the problem of measurement feedback, it is necessary to consider a different control scheme from the ones used before.

Due to the fact that the states are not available, a post-processor is added after the plant to be controlled: this element is usually called an *internal model unit* (IMU of the exosystem).

It is usually characterized by the following equation

$$\dot{\tau} = \Phi\,\tau + \Gamma\,e \tag{3.22}$$

where $\tau \in \mathbb{R}^{m \cdot d}$.

In order to compute the two matrices Φ and Γ of the IMU, let us consider the minimal polynomial of S:

$$\Psi\,(s) = \alpha_0 + \alpha_1\,s + \cdots + \alpha_{d-1}s^{d-1} + s^d = 0$$

and build the following square matrix

$$\Phi = \begin{pmatrix}
0 & I & 0 & \cdots & 0 \\
0 & 0 & I & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & I \\
-\alpha_0\,I & -\alpha_1\,I & -\alpha_2\,I & \cdots & -\alpha_{d-1}\,I
\end{pmatrix} \tag{3.23}$$

and the matrix

$$\Gamma = \begin{pmatrix} 0 & 0 & \cdots & 0 & I \end{pmatrix}^\top \tag{3.24}$$

All the elements are blocks $m \times m$.

Proposition 3.4.1. *Since the matrix Φ embeds the minimal polynomial of the matrix S that characterizes the exosystem, the post-processor (3.22) is usually called an **internal model** (of the exosystem). It has the following properties:*

(i) the pair of matrices (Π, Γ) is controllable.

(ii) exists a matrix T that is nonsingular, such that

$$T\,\Phi\,T^{-1} = \begin{pmatrix} S & 0 & \cdots & 0 \\ 0 & S & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

We consider now the augmented system composed by the plant and the internal model, given by

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\tau} \end{pmatrix} &= \begin{pmatrix} A & 0 \\ \Gamma\, C_e & \Phi \end{pmatrix} \begin{pmatrix} x \\ \tau \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u \\ y_a &= \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ \tau \end{pmatrix} \end{aligned} \tag{3.25}$$

where $C = col\,(C_e, C_{\bar{y}})$.

Proposition 3.4.2. *The augmented system is stabilizable and detectable if the following properties are valid:*

(i) if (C, A) is detectable, then (x, τ) (the augmented system) is detectable.

(ii) if (A, B) is stabilizable and the non resonance condition

$$rank \begin{pmatrix} A - \lambda\, I & B \\ C_e & 0 \end{pmatrix} = \#rows(n + m)$$

then (x, τ) is stabilizable.

Checked the conditions for stabilizability and detectability, [Isidori, 2017], it is possible to stabilize the system. The design of the controller is based on the following model:

$$\begin{aligned} \dot{\xi} &= A_\xi\, \xi + H_y\, y + H_\tau\, \tau \\ u &= C_\xi\, \xi + L_y\, y + L_\tau\, \tau \end{aligned} \tag{3.26}$$

In order to define the closed-loop system, we rewrite the equations of the states x and τ:

$$\dot{x} = A\,x + B\,(C_\xi\,\xi + L_y\,(C\,x + Q\,w)) + B\,L_\tau\,\tau + P\,w$$
$$\dot{\tau} = \Phi\,\tau + \Gamma\,(C_e\,x + Q_e\,w)$$

and write the closed loop system in matrix form:

$$\begin{pmatrix} \dot{w} \\ \dot{x} \\ \dot{\tau} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} S & 0 & 0 & 0 \\ P + B\,L_y\,Q & A + B\,L_y\,C & B\,L_\tau & B\,C_\xi \\ \Gamma\,Q_e & \Gamma\,C_e & \Phi & 0 \\ H_y\,Q & H_y\,C & H_\tau & A_\xi \end{pmatrix} \begin{pmatrix} w \\ x \\ \tau \\ \xi \end{pmatrix} \tag{3.27}$$

The closed-loop system can be rewritten, for simplicity, in the following form

$$\begin{pmatrix} \dot{w} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} S & 0 \\ \underline{G} & \underline{F} \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \tag{3.28}$$

with $z = \mathrm{col}\,(x, \tau, \xi)$.

We can consider the closed loop system as we have done in the steady-state analysis, Section (3.2): considering that the augmented system is stabilizable and detectable, we can choose the elements of $\underline{F}$, then:

- $\sigma\,(S) \cap \sigma\,(\underline{F}) = 0$;
- exist Π such that is a solution of the Sylvester equation $\Pi\,S = \underline{F}\,\Pi + \underline{G}$;
- the closed loop system has a steady-state that is asymptotically stable.

Consider now the solution of the Sylvester equation: we obtained $\Pi = \begin{pmatrix} \Pi_x \\ \Pi_\tau \\ \Pi_\xi \end{pmatrix}$

It is possible to define the relation between the three elements of Π: we define

$$\Pi_e = C_e\,\Pi_x + Q_e$$

but due to the fact that we know that at steady-state $e \to \Pi_e\,w$, so in order to solve the output regulation problem we need $\Pi_e = 0$. In order to prove it, consider the following relation, taking into account that Π_x and Π_τ are the solutions of

$$\Pi_\tau\,S = \Phi\,\Pi_\tau + \Gamma\,\Pi_e$$

Keeping in mind the structure of Φ and G it derives that

$$\Pi_{\tau_1}\left[S^d + \alpha_{d-1}\,S^{d-1} + \cdots + \alpha_q\,S + \alpha_0\,I_p\right] = \Pi_e \tag{3.29}$$

From the Cayley-Hamilton theorem, the polynomial inside the square brakes is equal to 0, and then $\Pi_e = 0$.

Remarks

Differently from the full-information case, in these kind of problems, the solution obtained is robust with respect to the parameter uncertainties vector μ, due to the fact that the stabilizing system does not rely on the knowledge of the system matrices.

3.5 Generalized Output Regulation

Until now it was considered an autonomous exosystem of the following form

$$\dot{w}(t) = S\,w(t), \qquad w(t_0) = w_0 \tag{3.30}$$

and the system described by

$$\dot{z}(t) = F\,z(t) + G\,w(t), \qquad z(t_0) = z_0 \tag{3.31}$$

where F is Hurwitz. The solutions of system (3.30)-(3.31) can be written as

$$\begin{aligned} w(t) &= e^{S(t-t_0)}w_0 \\ z(t) &= e^{F(t-t_0)}z_0 + \int_{t_0}^{t} e^{F(t-\tau)}G\,w(\tau)\,d\tau \\ &= e^{F(t-t_0)}z_0 + \left[\int_{t_0}^{t} e^{F(t-\tau)}G\,e^{S(\tau-t_0)}d\tau\right] w_0 \end{aligned} \tag{3.32}$$

At steady-state, the state $z(t)$ converge to $z_{ss}(t)$, defined as

$$z_{ss}(t) = \Pi\,w(t) \tag{3.33}$$

where Π is the unique solution of the Sylvester equation

$$\Pi\,S = F\,\Pi + G$$

3.5.1 Steady-state Analysis

Consider now a non-autonomous exosystem described by

$$\dot{w}(t) = S\,w(t) + R\,r(t), \qquad w(t_0) = w_0 \tag{3.34}$$

where r is an external signal, piecewise continuous in time, and the system described in Figure 3.5, with the plant

$$\dot{z} = F\,z + G\,w, \qquad z(t_0) = z_0 \tag{3.35}$$

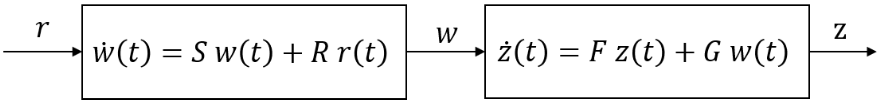


Figure 3.5: Steady-State analysis for a non-autonomous exosystem.

As done before it is possible to write the trajectory of the exosystem as

$$w\,(t) = e^{S\,(t-t_0)} w_0 + \int_{t_0}^{t} e^{S(t-\tau)} R\,r\,(\tau)\,d\tau \tag{3.36}$$

and write the steady-state of z_{ss} as follows

$$\begin{aligned} z_{ss}\,(t) &= \int_{t_0}^{t} e^{F(t-\tau)} G\,w\,(\tau)\,d\tau \\ &= \left[\int_{t_0}^{t} e^{F(t-\tau)} G\,e^{S(\tau-t_0)} d\tau\right] w_0 + \int_{t_0}^{t} e^{F(t-\tau)} G \int_{t_0}^{\tau} e^{S(\tau-\xi)} R\,r\,(\xi)\,d\xi d\tau \end{aligned} \tag{3.37}$$

Differently from before, it is more complicated to write a steady-state trajectory of the form (3.33).

Suppose that (3.37) can be rewritten as

$$z_{ss}\,(t) = \Pi\,w\,(t) + \Gamma\,r\,(t) \tag{3.38}$$

Theorem 3.5.1. *The problem of Output Regulation has a solution if there exists a pair* (Π, Γ) *that is a solution of the Sylvester equation*

$$\Pi\,S\,w + \Pi\,R\,e + \Gamma\,\dot{r} = F\,\Pi\,w + F\,\Gamma\,r + G\,w \tag{3.39}$$

Proof. Consider the change of coordinates $\tilde{z} = z - \Pi\,w - \Gamma\,r$. Deriving in time, it is possible to obtain

$$\dot{\tilde{z}} = F\,\tilde{z} + \underbrace{F\,\Pi\,w + F\,\Gamma\,r - \Pi\,S\,w - \Pi\,R\,r - \Gamma\,\dot{r} + G\,w}_{=0} \tag{3.40}$$

Due to the fact that F is Hurwitz, the variable error goes to zero asymptotically. This way it is possible to say that

$$\lim_{t\to\infty} \tilde{z} = 0$$

□

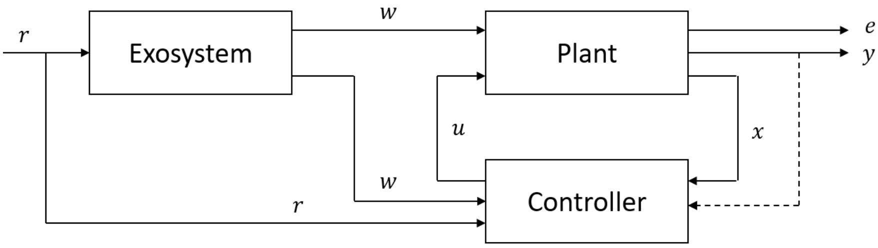


Figure 3.6: Output regulation scheme proposed in [Saberi et al., 2001].

3.5.2 Full-Information Problem

Consider now the Full-Information Problem described by the system

$$\begin{aligned} \dot{x} &= A\,x + B\,u + P\,w \\ e &= C\,x + Q\,w \end{aligned} \tag{3.41}$$

and the exosystem in (3.34). Consider the case in which every information about the input r are unknown.

In the solution of the problem it is highlighted an obstacle: suppose it is possible to describe the steady-state of x as $x_{ss} = \Pi\,w + \Gamma\,r$ and that there is a control law of the form $u = K\,x + L\,w$. The regulator equations can be written as follows

$$\begin{aligned} \Pi\,S\,w + \Pi\,Rr + \Gamma\,\dot{r} &= A\,\Pi\,w + A\,\Gamma\,r + B\,K\,\Gamma\,r + B\,\Psi\,w + P\,w \\ 0 &= C\,\Pi\,w + C\,\Gamma\,r + Q\,w \end{aligned} \tag{3.42}$$

where $\Psi = K\,\Pi + L$. It is evident, how the knowledge of the input r and its derivative $\dot{r}$ are necessary to solve the regulator equations.

A solution to the problem is given in [Saberi et al., 2001], with the control scheme shown in Figure 3.6.

Theorem 3.5.2. *Consider the system*

$$\begin{aligned} \dot{w} &= S\,w + R\,r \\ \dot{x} &= A\,x + B\,u + P\,w \\ e &= C\,x + D\,u + Q\,w \end{aligned} \tag{3.43}$$

The problem of output regulation is solvable if:

(i) There exists a pair (Π, Ψ) *that solves the regulator equations*

$$\begin{aligned} \Pi\,S &= A\,\Pi + B\,\Psi + P \\ 0 &= C\,\Pi + D\,\Psi + Q \end{aligned} \tag{3.44}$$

(ii) The following property holds

$$Im\Pi R \subseteq \mathcal{V}^{-}(A,B,C,D) \tag{3.45}$$

with the following control law

$$u = K\,x + L\,w \tag{3.46}$$

Proof. Consider the second assumption. The idea behind is to decouple the input from the observable part. Consider the system described in (3.35) and the exosystem in (3.34).

Applying a change of coordinate $\tau = H\,z$ and applying a Kalman decomposition, the plant can be described as

$$\begin{pmatrix} \dot{\tau} \\ \dot{\xi} \end{pmatrix} = \underbrace{\begin{pmatrix} \psi_{11} & 0 \\ \psi_{21} & \psi_{22} \end{pmatrix}}_{\Psi} \begin{pmatrix} \tau \\ \xi \end{pmatrix} + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} w \tag{3.47}$$

Suppose $r = 0$, the steady-state can be written as $\begin{pmatrix} \tau \\ \xi \end{pmatrix} = \Pi\,w$, and there exist a solution Π to the Sylvester equation

$$\Pi\,S = \Psi\,\Pi + G$$

Applying the change of coordinate $\begin{pmatrix} \tilde{\tau} \\ \tilde{\xi} \end{pmatrix} = \begin{pmatrix} \tau \\ \xi \end{pmatrix} - \begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix} w$ the (3.47) becomes

$$\begin{pmatrix} \dot{\tilde{\tau}} \\ \dot{\tilde{\xi}} \end{pmatrix} = \Psi \begin{pmatrix} \tilde{\tau} \\ \tilde{\xi} \end{pmatrix} - \Pi\,R\,r + \underbrace{\Psi\,\Pi\,w - \Pi\,S\,w + G\,w}_{=0}$$

$$\begin{pmatrix} \dot{\tilde{\tau}} \\ \dot{\tilde{\xi}} \end{pmatrix} = \begin{pmatrix} \psi_{11} & 0 \\ \psi_{21} & \psi_{22} \end{pmatrix} \begin{pmatrix} \tilde{\tau} \\ \tilde{\xi} \end{pmatrix} - \begin{pmatrix} \Pi_1\,R \\ \Pi_2\,R \end{pmatrix} r \tag{3.48}$$

In order to have convergence to zero of $\tilde{\tau}$, it is necessary to have $\Pi_1\,R = 0$. Writing the steady-state of the two dynamics,

$$\begin{aligned} \dot{\tilde{\tau}} &= \psi_{11}\,\tilde{\tau} & &\rightarrow \quad \tilde{\tau}(t) = e^{\psi_{11}(t-t_0)}\tilde{\tau}(t_0) \\ \dot{\tilde{\xi}} &= \psi_{22}\,\tilde{\xi} + \psi_{21}\,\tilde{\tau} - \Pi_2\,R\,r & &\rightarrow \quad \tilde{\xi}(t) = e^{\psi_{22}(t-t_0)}\tilde{\xi}(t_0) + \int \cdots \end{aligned} \tag{3.49}$$

This allows to have an autonomous dynamics for τ, such that it guarantees the convergence to zero.

Consider now the system (3.41). Suppose $r = 0$, in such case, the steady-state is given by $x_{ss}(t) = \Pi\, w(t)$ and $u_{ss}(t) = \Psi\, w(t)$, and the regulator equations are the same as in the previous section

$$\begin{aligned} \Pi\, S &= A\,\Pi + B\,\Psi + P \\ 0 &= C\,\Pi + Q \end{aligned} \tag{3.50}$$

with solution pair given by (Π, Ψ).

Let $\tilde{x} = x - \Pi\, w$ and $\tilde{u} = u - \Psi\, w$, then the system (3.41) becomes

$$\begin{aligned} \dot{\tilde{x}} &= A\,\tilde{x} + B\,\tilde{u} - \Pi\, R\, r \\ e &= C\,\tilde{x} \end{aligned} \tag{3.51}$$

while the control law (3.46) can be rewritten as follows

$$\tilde{u} + \Psi\, w = K\,(\tilde{x} + \Pi\, w) + L\, w$$

$$\tilde{u} = K\,\tilde{x} \tag{3.52}$$

Clearly the tracking error e does not go to zero due to the presence of the term $\Pi\, R\, r$. In order to show this problem, it is possible to replace (3.52) in (3.51) to obtain

$$\dot{\tilde{x}} = (A + B\,K)\,\tilde{x} - \Pi\, R\, r$$

Choosing K such that matrix $A + B\,K$ is Hurwitz, is not sufficient to have

$$\lim_{t\to\infty} \tilde{x}(t) = 0$$

It is then possible to modify system (3.41), adding the term $D\,u$ to the second equation

$$\begin{aligned} \dot{x} &= A\,x + B\,u + P\,w \\ e &= C\,x + D\,u + Q\,w \end{aligned} \tag{3.53}$$

Considering again $r = 0$, the regulator equations become

$$\begin{aligned} \Pi\, S &= A\,\Pi + B\,\Psi + P \\ 0 &= C\,\Pi + D\,\Psi + Q \end{aligned} \tag{3.54}$$

Letting $\tilde{x} = x - \Pi\, w$, $\tilde{u} = u - \Psi\, w$, and replacing $\tilde{u} = K\,\tilde{x}$, it is possible to obtain

$$\begin{aligned} \dot{\tilde{x}} &= A\,\tilde{x} + B\,\tilde{u} - \Pi\, R\, r \\ e &= C\,\tilde{x} + D\,K\,\tilde{x} \end{aligned} \tag{3.55}$$

Due to the second condition of Theorem 3.5.2, K must be chosen in order to not have any influence of the input on the regulation error. This way even if $\dot{\tilde{x}}$ is not equal to zero at steady-state, K allows to have

$$\lim_{t\to\infty} e(t) = 0 \tag{3.56}$$

□

Chapter 4

Model Reference Adaptive Control

Adaptive control refers to a dynamic control method used by a controller that must be able to obtain a robust behaviour in presence of uncertainties on the model.

The Model Reference Adaptive Control (MRAC) is a direct adaptive strategy with some adjustable controller parameters and an adjusting mechanism able to update continuously these parameters.

The MRAC is an evolution of a simpler problem, the Model Following Control or Model Reference Control (MRC). In the MRC problem, the main objective is to find a feedback control law such that the I/O properties of the closed loop function of the system match with the ones of a given Reference Model. The structure of a MRC scheme for a LTI, SISO plant is shown in Figure 4.1.

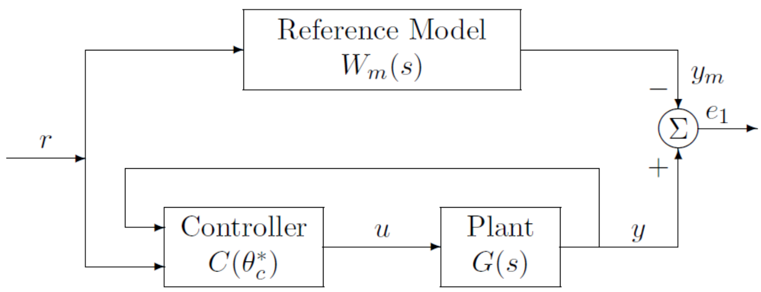


Figure 4.1: Model Reference Control [Ioannou and Sun, 2012].

The output of the Reference Model $W_m(s)$ is $y_m(t)$, which is the desired output that we want and that $y(t)$ must follow in order to obtain the desired I/O properties. The problem is to design a controller $C(\theta_c^*)$, such that the closed loop

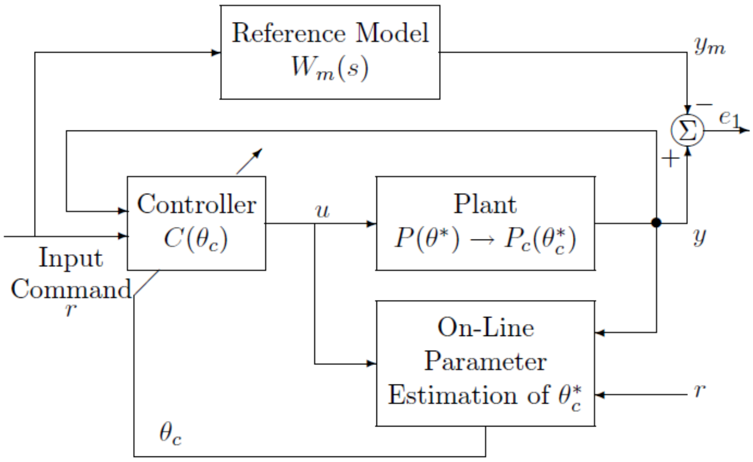


Figure 4.2: Direct MRAC [Ioannou and Sun, 2012].

transfer function from r to y is equal to the one of the Reference Model W_m.

This match of the transfer function means that for any given reference input $r(t)$, the tracking error $e_1 \triangleq y - y_m$, that represents the difference between the desired value of the output or the state and the real one, goes to zero asymptotically.

In order to achieve this property, we need to have a minimum phase transfer function for the plant: the matching condition of the two transfer functions is obtained due to the cancellation of the zeros of the plant transfer function, replacing them with the zeros of the Reference Model transfer function. The cancellation is possible only if the zeros are stable, in order to avoid unbounded signals due to the cancellation of unstable zeros, so the transfer function must be minimum phase.

This means that a good understanding of the plant and the knowledge of the specific θ^* are mandatory in order to solve this problem.

When we do not have any information on the plant, regarding its parameter vector θ^*, we cannot use the MRC scheme due to the fact that we cannot calculate θ_c^*. Due to the impossibility to obtain θ_c^*, it is possible to use the estimate $\theta(t)_c$. The resulting control scheme is the so called Model Reference Adaptive Control.

It is possible to distinguish between two types of MRAC schemes: the direct and indirect MRAC. In the indirect adaptive control the plant parameters are estimated on-line and then are used to obtain the controller parameters. In the direct adaptive control, the controller parameters are estimated without using the plant parameter estimates.

This design approach, using $\theta_c(t)$ (for the direct case) or $\theta(t)$ (for the indirect case) as if they are the true parameters, is called *certainty equivalence*. The idea

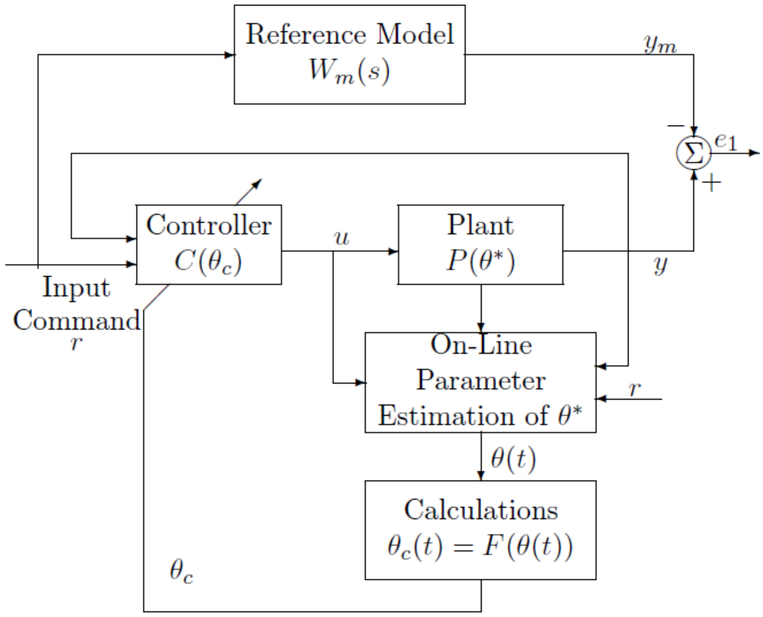


Figure 4.3: Indirect MRAC [Ioannou and Sun, 2012].

behind the certain equivalence approach is that as the estimates converge to the true value of the parameters, the performances of the adaptive controller $C\left(\theta_c\right)$ tends to be equal to the ones of the controller $C\left(\theta\right)$ in the case of known parameters. In reality the values of the estimates will never be equal to the real one, and the convergence to the real values is no more a requirements for the convergence of the tracking error.

We have talked about on-line estimates in the adaptive control: the use of *on-line estimators* is necessary in the cases where there are parameters that change with time or that are unknown. In the simple case of stable linear system with parameters fixed, is possible to use *off-line estimators* in order to compute the parameters. The on-line estimator can be used in either stable and unstable plants.

4.1 Model Reference Control

In order to understand better the framework, we start the study of the Model Reference Adaptive Control considering the simpler case of the Model Reference Control for a scalar case and for a vector case.

4.1.1 MRC: Scalar example

We consider the following scalar plant:

$$\dot{x} = a\,x + b\,u, \quad x\,(0) = x_0 \tag{4.1}$$

where $a, b \in \mathbb{R}$ are constant and known, and the reference model given by:

$$\dot{x}_m = a_m\,x_m + b_m\,r \tag{4.2}$$

where $a_m < 0$, b_m are known and x_m,r are measured at each time t. The signal r is a bounded piecewise continuous signal and u is the control input.

Control Objective The control objective is to find a control law u such that all signals in the closed-loop plant are bounded, x tracks the desired state x_m and the tracking error $e = x - x_m$ goes to zero. To achieve this we consider the following control law:

$$u = -k^*\,x + l^*\,r \tag{4.3}$$

We can rewrite the (4.1) and (4.2) as follows:

$$\dot{x} = a\,x + b\,(-k^*\,x + l^*\,r) = (a - b\,k^*)\,x + b\,l^*\,r$$

$$\dot{x}_m = a_m\,x_m + b_m\,r$$

and in order to have the same behaviour, we calculate k^* and l^* as

$$k^* = \frac{a - a_m}{b} \qquad l^* = \frac{b_m}{b} \tag{4.4}$$

We consider $b \neq 0$ in order to have a plant that is controllable. These two definition of k^* and l^* guarantees that

$$\begin{cases} x = x_m, \forall t \geq 0 & \text{when } x\,(0) = x_m\,(0) \\ |x - x_m| \to 0 \text{ exponentially fast} & \text{when } x\,(0) \neq x_m\,(0) \end{cases} \tag{4.5}$$

Numerical example

We consider now an example considering the plant model (4.1) and the reference model (4.2), where we have $a = 1$, $b = 3$, $a_m = -2$ and $b_m = 4$. The Simulink model is shown in Figure (4.4).

The plant is controllable due to the fact that $b \neq 0$ and using (4.4) we obtain $k^* = 1$ and $l^* = 1.33$. We simulate the system considering the two cases: one where we have the same initial condition for the plant and the reference model and one where are different.

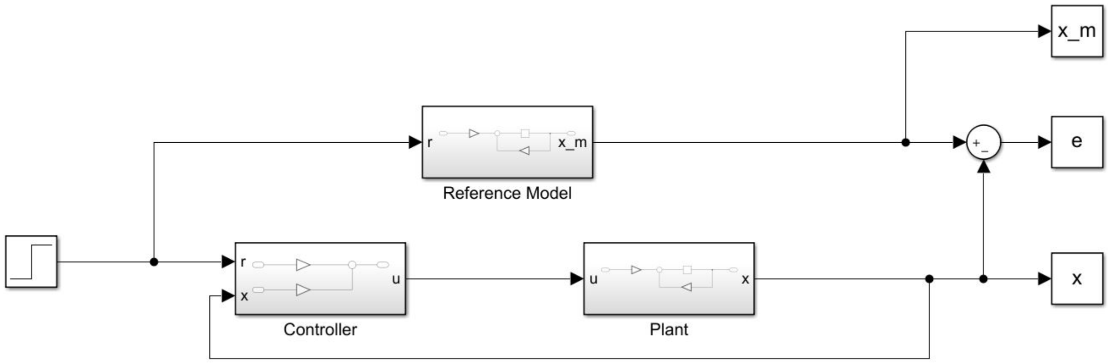


Figure 4.4: Simulink model for MRC example (4.1), (4.2), (4.4).

Looking at the simulations in Figure 4.5, the relations in (4.5) are guaranteed and the error tends to zero exponentially fast.

4.1.2 MRC: Full-state measurement example

Let now consider the case where we have a n-th order plant, given by:

$$\dot{x} = A\,x + B\,u \tag{4.6}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ and the state $x \in \mathbb{R}^n$. The reference model is defined as follows:

$$\dot{x}_m = A_m\,x_m + B_m\,r \tag{4.7}$$

where $A_m \in \mathbb{R}^{n\times n}$ is a stable matrix, $B_m \in \mathbb{R}^{n\times p}$ and, as before, $r \in \mathbb{R}^p$ is a bounded vector reference signal.

Control Objective As before, the control objective is to find a control law u such that the all the signals are bounded, x tracks x_m, and the tracking error e goes to zero. Considering the case where the matrices A and B are known, we can use the following control law:

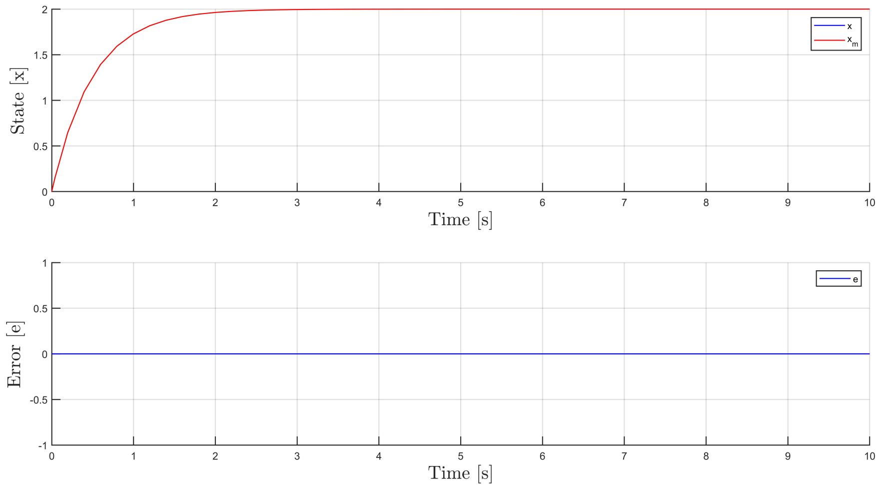
$$u = -{K^*}^\top x + L^*\,r \tag{4.8}$$

where $K^* \in \mathbb{R}^{n\times p}$ and $L^* \in \mathbb{R}^{p\times p}$ and rewrite the equation (4.6) obtaining the close-loop equation:

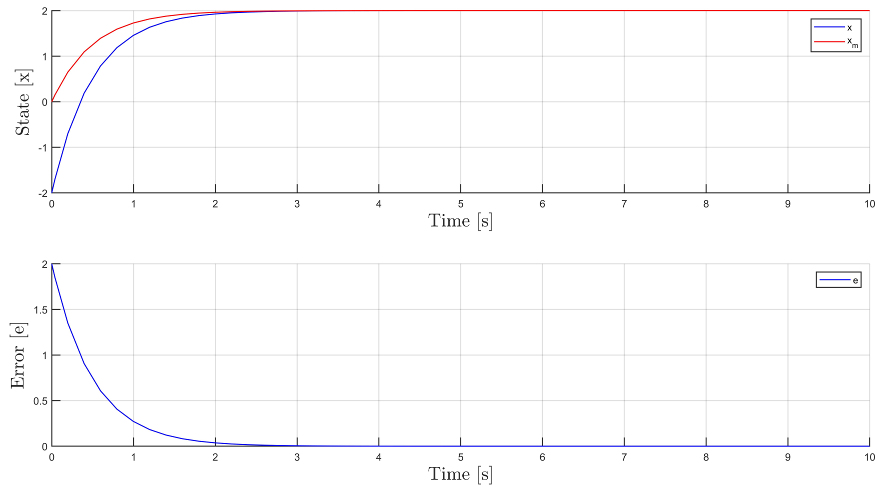
$$\dot{x} = \left(A - B\,{K^*}^\top\right) x + B\,L^*\,r$$

Hence, due to the fact that the close-loop transfer function must be equal to the transfer function of the reference model (4.7), and being the plant model controllable, we can write

$$A - B\,{K^*}^\top = A_m \qquad B\,L^* = B_m \tag{4.9}$$



(a)



(b)

Figure 4.5: Simulations for different initial conditions in the scalar case: (a) $x\,(0) = x_m\,(0) = 0$; (b) $x\,(0) = -2$ and $x_m\,(0) = 0$.

This way we can choose $K^{*\top}$ and L^* in order to satisfy (4.9). The chosen matrices allow to obtain an exponential convergence of x to x_m for any bounded reference signal and the convergence to zero of the error $e = x - x_m$.

Numerical example

Consider the following transfer function representation for the plant

$$y_p = \frac{s+1.5}{s^2+0.75s+2.5}\, u_p \tag{4.10}$$

and the following transfer function representation for the reference model

$$y_m = \frac{s+50}{s^2+15s+50}\, r \tag{4.11}$$

It is possible to transform the two transfer function representations in state-space representation in the so called controllable canonical form; due to the fact that both the transfer functions are strictly proper it is possible to write them in order to reveal all the coefficients in both the numerator and denominator

$$G(s) = \frac{n_1\, s^3 + n_2\, s^2 + n_3\, s + n_4}{s^4 + d_1\, s^3 + d_2\, s^2 + d_3\, s + d_4} \tag{4.12}$$

and then insert the coefficients directly into the state-space model as follows

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -d_4 & -d_3 & -d_2 & -d_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} n_4 & n_3 & n_2 & n_1 \end{bmatrix} x(t) \end{aligned} \tag{4.13}$$

This way it is possible to obtain the state-space representation for the plant with

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -2.5 & -0.75 \end{bmatrix} \quad & B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} 1.5 & 1 \end{bmatrix} & D &= 0 \end{aligned} \tag{4.14}$$

and for the reference model with

$$\begin{aligned} A_m &= \begin{bmatrix} 0 & 1 \\ -50 & -15 \end{bmatrix} \quad & B_m &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C_m &= \begin{bmatrix} 50 & 1 \end{bmatrix} & D_m &= 0 \end{aligned} \tag{4.15}$$

In order to respect the (4.9) we choose

$$K^{*\top} = \begin{bmatrix} 47.5 & 14.25 \end{bmatrix} \quad L^* = 1 \tag{4.16}$$

Considering as initial condition

$$x(0) = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} \qquad x_m(0) = \begin{bmatrix} -0.2 \\ 1 \end{bmatrix}$$

the resulting behaviour is shown in Figure 4.6.

Using the same initial conditions instead, allows to have a perfect tracking and the error e remains zero during the simulation.

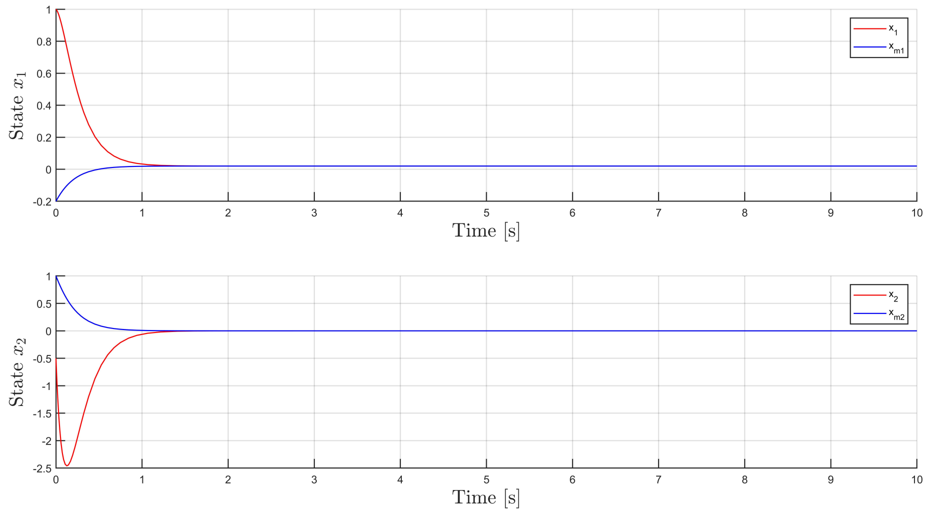


Figure 4.6: Full-state measurement case.

4.2 Model Reference Adaptive Control

The implementation of the MRAC is mandatory when we do not have informations about the plant to be controlled. In this case, it is necessary to design a dynamical update mechanism in order to obtain the parameters of the control law: we talk about the Adaptive Law. Again, as done before in the case of the MRC, we consider a scalar case and a vector case.

4.2.1 MRAC: Scalar example

We consider the following plant model

$$\dot{x} = a\,x + b\,u, \quad x\,(0) = x_0 \tag{4.17}$$

In this case the parameters a, $b \in \mathbb{R}$ are unknown and we assume the the $sgn\,(b)$ to be known. The reference model is described by

$$\dot{x}_m = a_m\,x_m + b_m\,r \tag{4.18}$$

where $a_m < 0$,b_m are known and we assume that the state x_m and the bounded piecewise continuous signal r are available for measurements at each time t. In order to track the desired state x_m for any reference signal r, we need to have the same transfer function for the reference model and the plant to control.

Control Objective Again, we want the state x to track the state x_m and the exponential convergence of the error e to zero. We choose as a control law the following expression

$$u = -k^* x + l^* r$$

The two parameters k^* and l^* can not be computed as done in the MRC case (4.4) due to unknown parameters a and b. It is necessary to use another control law

$$u = -k(t)\,x + l(t)\,r \tag{4.19}$$

where $k(t)$ and (t) are respectively the estimate of k^* and l^* at time t. In order to generate $k(t)$ and $l(t)$ on-line it is necessary to implement an adaptive law. The knowledge of the sign of b means that also the sign of l^* is known.

Adaptive Law The on-line estimation of the parameters is given by the two following differential equation

$$\begin{cases} \dot{k}(t) & = f_1(e, x, r, u) \\ \dot{l}(t) & = f_2(e, x, r, u) \end{cases} \tag{4.20}$$

For a better understanding, the tracking error and the parameters errors are defined as follows

$$\begin{aligned} e &= x - x_m \\ \tilde{k} &= k(t) - k^* \\ \tilde{l} &= l(t) - l^* \end{aligned} \tag{4.21}$$

The use of the SPR-Lyapunov approach allows us to design the two adaptive laws. We consider the following candidate Lyapunov function

$$V\left(e, \tilde{k}, \tilde{l}\right) = \underbrace{\frac{e^2}{2}}_{V_1} + \underbrace{\frac{\tilde{k}^2}{2\,\gamma_1\,|l^*|}}_{V_2} + \underbrace{\frac{\tilde{l}^2}{2\,\gamma_2\,|l^*|}}_{V_3} \tag{4.22}$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$ are the adaptive gains to be tuned. The time derivative of the Lyapunov function is given by

$$\dot{V}\left(e, \tilde{k}, \tilde{l}\right) = \underbrace{e\,\dot{e}}_{\dot{V}_1} + \underbrace{\frac{\tilde{k}}{\gamma_1\,|l^*|}\,\dot{\tilde{k}}}_{\dot{V}_2} + \underbrace{\frac{\tilde{l}}{\gamma_2\,|l^*|}\,\dot{\tilde{l}}}_{\dot{V}_3} \tag{4.23}$$

From the (4.21) we compute the time derivative of the tracking error

$$\dot{e} = \dot{x} - \dot{x}_m \tag{4.24}$$

where, taking into account the definition of the parameter errors, we obtain

$$\begin{aligned}\dot{x} &= a\,x + b\,u = a\,x + (-k\,x + l\,r)\\ &= a\,x - b\,k^*\,x - b\,\tilde{k}\,x + b\,l^*\,r + b\,\tilde{l}\,r\\ &= \underbrace{(a - b\,k^*)}_{a_m}\,x + \underbrace{b\,l^*}_{b_m}\,r - b\,\tilde{k}\,x + b\,\tilde{l}\,r\end{aligned}$$

so we have

$$\begin{aligned}\dot{x} &= a_m\,x + b_m\,r - b\,\tilde{k}\,x + b\,\tilde{l}\,r\\ \dot{x}_m &= a_m\,x_m + b_m\,r.\end{aligned} \tag{4.25}$$

(4.24) can be rewritten as

$$\begin{aligned}\dot{e} &= a_m\,x + b_m\,r - b\,\tilde{k}\,x + b\,\tilde{l}\,r - a_m\,x_m - b_m\,r\\ &= a_m\,e - b\,\tilde{k}\,x + b\,\tilde{l}\,r.\end{aligned} \tag{4.26}$$

Defined the time derivative of the tracking error e, it is possible to rewrite the first component of the time derivative of the Lyapunov function

$$\dot{V}_1 = e\,\dot{e} = a_m\,e^2 - e\,b\,\tilde{k}\,x + e\,b\,\tilde{l}\,r. \tag{4.27}$$

In order to have V (4.22) decreasing, we have to follow the Lyapunov's stability theorem exploited in (2.2.1): so it must be positive definite (in order to have asymptotic stability) or positive semi-definite (stability). In order to satisfy the Lyapunov's stability theorem, for simplicity we can say that the desired time derivative is

$$\dot{V}\left(e, \tilde{k}, \tilde{l}\right) = a_m\,e^2. \tag{4.28}$$

The stability is proved also by the Barbalat's Lemma. This way we can write the other two component of $\dot{V}$ as follows

$$\begin{aligned}\dot{V}_2 &= \frac{\tilde{k}}{\gamma_1\,|l^*|}\,\dot{\tilde{k}} = e\,b\,\tilde{k}\,x\\ \dot{V}_3 &= \frac{\tilde{l}}{\gamma_2\,|l^*|}\,\dot{\tilde{l}} = -e\,b\,\tilde{l}\,x.\end{aligned} \tag{4.29}$$

Exploting the following relation $|l^*| = sgn\,(l^*)\,l^*$ the two previous equations allow to obtain the adaptive laws

$$\begin{aligned}\dot{k} &= +sgn\,(l^*)\,\gamma_1\,e\,b_m\,x\\ \dot{l} &= -sgn\,(l^*)\,\gamma_2\,e\,b_m\,r.\end{aligned} \tag{4.30}$$

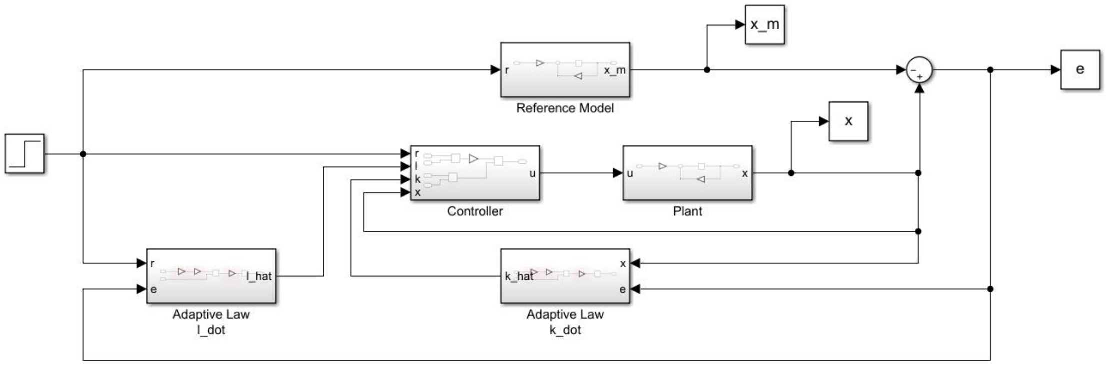


Figure 4.7: Simulink model for the MRAC example, (4.17), (4.18), (4.19), (4.30).

It is used the previous notation with just k and not $\tilde{k}$ due to the fact that the time derivative of the parameter error is

$$\dot{\tilde{k}} = \dot{k}$$

since k^* is a scalar. This is also valid for l. This way is possible to obtain boundedness for all signals in the closed-loop system. Moreover the state $x\,(t)$ tracks asymptotically the reference model state x_m. It is important to notice that the estimated values of $k\,(t)$ and $l\,(t)$ do not converge to the values k^* and l^* as $t \to \infty$.

Numerical example

Let us consider the same example studied in Section 4.1.1 with the same values of the parameters, but considering only the case where both the plant and the reference model have the same initial condition $x\,(0) = x_m\,(0) = 0$.

The Simulink model is shown in Figure (4.7).

Simulating the system is possible to see the convergence of the state x to the state x_m in Figure 4.8. It is also clear how the values of the estimated parameters are different from the ones obtained in the MRC example.

4.2.2 MRAC: Full-state measurement example

Consider the n-th order plant

$$\dot{x} = A\,x + B\,u \tag{4.31}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ and the state $x \in \mathbb{R}^n$. The two matrices A and B are unknown, but the pair (A, B) is controllable. The reference model is defined as follows:

$$\dot{x}_m = A_m\,x_m + B_m\,r \tag{4.32}$$

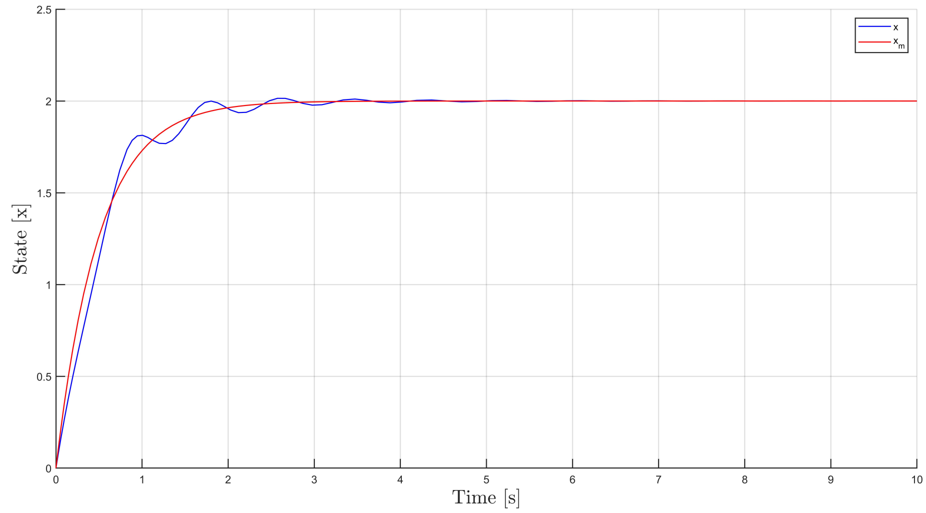


Figure 4.8: State behaviours in the scalar case of the MRAC.

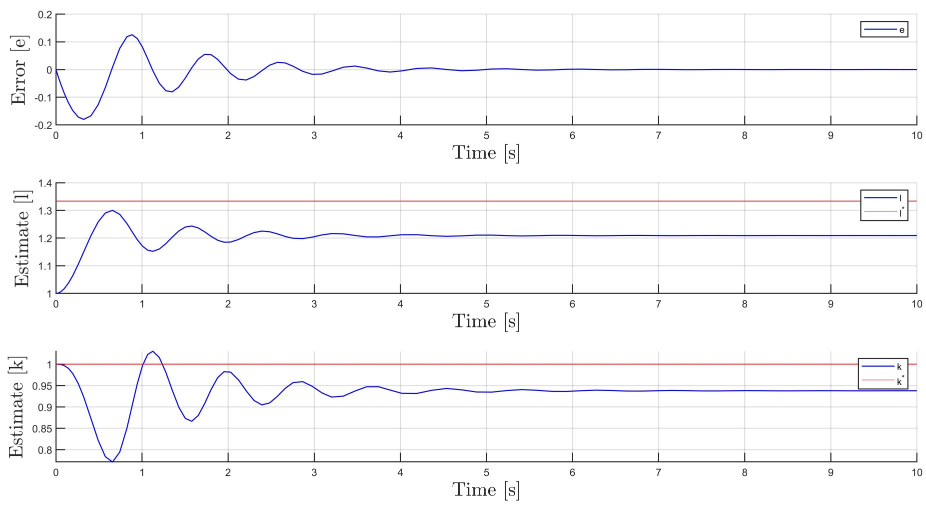


Figure 4.9: Error behaviours in the scalar case of the MRAC.

where $A_m \in \mathbb{R}^{n \times n}$ is a stable matrix, $B_m \in \mathbb{R}^{n \times p}$ and, as before, $r \in \mathbb{R}^p$ is a bounded vector reference signal.

Control Objective As in the scalar case, the control objective is to find a control law such as

$$u = -{K^*}^\top x + L^* r \tag{4.33}$$

in order to have tracking by x of the state x_m and asymptotic convergens of e to zero, where $K^* \in \mathbb{R}^{n \times p}$ and $L^* \in \mathbb{R}^{p \times p}$. If the matrices A and B were known it would be possible to obtain K^* and L^* as in (4.9). The parameters of the control law are unknown, then it is necessary to use a similar approach to the one used in the scalar case. The control law is modified with the estimate of the parameters as follows

$$u = -K(t)\, x + L(t)\, r \tag{4.34}$$

The two estimates are computed used an appropriate adaptive law.

Adaptive Law We choose the following Lyapunov's candidate function

$$V\left(e, \tilde{K}, \tilde{L}\right) = \underbrace{e^\top P\, e}_{V_1} + \underbrace{\frac{\tilde{K}^\top \tilde{K}}{\gamma_1 |L^*|}}_{V_2} + \underbrace{\frac{\tilde{L}^2}{\gamma_2 |L^*|}}_{V_3} \tag{4.35}$$

with

$$\begin{cases} \tilde{K}(t) & = K(t) - K^* \\ \tilde{L}(t) & = L(t) - L^* \end{cases} \tag{4.36}$$

and where $P \in \mathbb{R}^{n_p \times n_p}$ is a symmetric positive definite matrix that satisfies the Lyapunov equation

$$P\, A_m + {A_m}^\top P = -Q \tag{4.37}$$

for some $Q = Q^\top > 0 \in \mathbb{R}^{n_p \times n_p}$. The choice of the matrix Q is arbitrary and it will not affect the asymptotic behaviour but just the transient response. As done before, the time derivative of the Lyapunov's candidate function is

$$\dot{V}\left(e, \tilde{K}, \tilde{L}\right) = \underbrace{2\, e^\top P\, \dot{e}}_{\dot{V}_1} + \underbrace{\frac{2\, \tilde{K}^\top}{\gamma_1 |L^*|}\, \dot{\tilde{K}}}_{\dot{V}_2} + \underbrace{\frac{2\, \tilde{L}^\top}{\gamma_2 |L^*|}\, \dot{\tilde{L}}}_{\dot{V}_3} \tag{4.38}$$

As in (4.24) we obtain the time derivative of the tracking error e

$$\dot{e} = A_m\, e + B\, \tilde{K}\, x + B\, \tilde{L}\, r \tag{4.39}$$

Consider now only the first component of (4.38), it can be rewritten as follows

$$\begin{aligned} \dot{V}_1 = 2\, e^\top P\, \dot{e} &= 2\, e^\top P\, A_m\, e - 2\, e^\top P\, B\, \tilde{K}\, x + 2\, e^\top P\, B\, \tilde{L}\, r \\ &= e^\top P\, A_m\, e + e^\top {A_m}^\top P\, e - 2\, e^\top P\, B\, \tilde{K}\, x + 2\, e^\top P\, B\, \tilde{L}\, r \\ &= -e^\top Q\, e - 2\, e^\top P\, B\, \tilde{K}\, x + 2\, e^\top P\, B\, \tilde{L}\, r \end{aligned} \tag{4.40}$$

We again can rewrite the other two components of (4.38) as

$$\begin{aligned} \dot{V}_2 &= \frac{2\,\tilde{K}^\top}{\gamma_1|L^*|}\,\dot{\tilde{K}} = -2\,e^\top\,P\,B\,\tilde{K}\,x \\ \dot{V}_3 &= \frac{2\,\tilde{L}^\top}{\gamma_2|L^*|}\,\dot{\tilde{L}} = 2\,e^\top\,P\,B\,\tilde{L}\,r \end{aligned} \tag{4.41}$$

and remembering that $|L^*| = sgn\,(L^*)\,L^*$ it is possible to obtain the following update law

$$\begin{aligned} \dot{K}^\top &= +sgn\,(L^*)\,\gamma_1\,e^\top\,P\,B_m\,x^\top \\ \dot{L} &= -sgn\,(L^*)\,\gamma_2\,e^\top\,P\,B_m\,r \end{aligned} \tag{4.42}$$

4.3 Output feedback

Let us consider the SISO LTI system

$$\begin{aligned} \dot{x} &= A\,(\mu)\,x + B\,(\mu)\,u \\ y &= C\,(\mu)\,x \end{aligned} \tag{4.43}$$

with the state $x \in \mathbb{R}^n$, the control input $u \in \mathbb{R}$ and the output $y \in \mathbb{R}$. The matrices $A\,(\cdot)$, $B\,(\cdot)$ and $C\,(\cdot)$, are considered dependent on a vector μ of uncertainty parameters, that ranges in a compact set $\mathcal{K}_\mu \subset \mathbb{R}^p$. As shown in [Serrani, 2013] and in [Serrani, 2018] is it is possible to describe the system (4.43) in a state-space setting, making certain assumptions:

Assumption 4.3.1. *The pair $(A\,(\mu)\,, B\,(\mu))$ is controllable, and the pair $(C\,(\mu)\,, A\,(\mu))$ is observable for any $\mu \in \mathcal{K}_\mu$.*

Assumption 4.3.2. *The system in (4.43) has relative degree equal to one for any $\mu \in \mathcal{K}_\mu$.*

Assumption 4.3.3. *The sign of the high-frequency gain $b\,(\mu) = C\,(\mu)\,B\,(\mu)$ is known and constant for any $\mu \in \mathcal{K}_\mu$.*

These three assumptions are much stronger than needed, due to the fact that in order to solve this problem, the stabilizability and detectability are sufficient. As shown in (A.2) it is better to change the coordinates of the system in order to easily exploit the similarities with the output regulation problem.

Due to the fact that the system has a relative degree equal to one, from Assumption 4.3.2, after the change of coordinates, the system can be described in its normal form:

$$\begin{aligned} \dot{z} &= A_{11}\,(\mu)\,z + A_{12}\,(\mu)\,y \\ \dot{y} &= A_{21}\,(\mu)\,z + a_{22}\,(\mu)\,y + b\,(\mu)\,u \end{aligned} \tag{4.44}$$

where $z \in \mathbb{R}^{n-1}$; it is assumed that $b(\mu) \geq b_0$, for a given constant $b_0 > 0$, for all $\mu \in \mathcal{K}$. $b(\mu)$ is the so-called *high-frequency gain.*

As in all the MRAC problem, we have a reference model given by a minimum-phase and internally stable system, that describes the desired input/output behaviour of the system (4.43)

$$\begin{aligned} \dot{x}_m &= A_m \, x_m + B_m \, r \\ y_m &= C_m \, x_m \end{aligned} \tag{4.45}$$

with the state $x_m \in \mathbb{R}^m$ and r as a bounded piecewise differentiable function.

Due to the dependency of the system on the uncertainty parameter μ, it is necessary to look for a certainty-equivalence adaptive controller of the form

$$\begin{aligned} \dot{\xi} &= F_c(\hat{\theta})\,\xi + G_c(\hat{\theta})\,\mathbf{y} \\ u &= H_c(\hat{\theta})\,\xi + K_c(\hat{\theta})\,\mathbf{y} \end{aligned} \tag{4.46}$$

with state $\xi \in \mathbb{R}^{\nu}$ and input $\mathbf{y} = col\,(x_m, r, e)$. For the parameter $\hat{\theta}$ a suitable update law is defined by

$$\dot{\hat{\theta}} = \phi\,(\hat{\theta}, \xi, \mathbf{y}) \tag{4.47}$$

with $\hat{\theta} \in \mathbb{R}^{\bar{p}}$.

It is known that in order to solve the classic problem of MRC, the plant (4.43) needs to be minimum phase, for all $\mu \in \mathcal{K}_\mu$. This means that we can consider the following assumption.

Assumption 4.3.4. *There exist a continuous, symmetric and positive definite matrix-valued function $P : \mathbb{R}^p \rightarrow \mathbb{R}^{(n-1)\times(n-1)}$, and positive constants a_1, a_2 satisfying*

$$\begin{gathered} a_1\, I \leq P(\mu) \leq a_2\, I \\ P(\mu)\, A_{11}(\mu) + A_{11}{}^{\top}(\mu)\, P(\mu) \leq -I \end{gathered}$$

for all $\mu \in \mathcal{K}_\mu$.

4.3.1 Certainty-Equivalence controller

The design of the controller is done under the certainty-equivalence paradigm. Under certain assumptions the controller is derived: the knowledge of the uncertainty parameter μ and the availability of the state z for feedback.

In order to design the certainty-equivalence controller, the problem must be shifted from a tracking problem to a regulation one. This is done introducing the tracking error $e = y - y_m$. Deriving it in time and replacing the derivatives of the outputs y and y_m the equation of the tracking error becomes

$$\dot{e} = \dot{y} - \dot{y}_m = A_{21}(\mu)\, z + a_{22}(\mu)\, y + b(\mu)\, u - C_m\,(A_m\, x_m + B_m\, r)$$

Adding and subtracting $a_m\, e$, where a_m is a desired gain parameter, the previous equation becomes

$$\dot{e} = -a_m\, e + A_{21}\,(\mu)\, z + (a_{22}\,(\mu) + a_m)\, y - \bar{y}_r + b\,(\mu)\, u$$

where $\bar{y}_r = (C_m\, A_m + a_m\, C_m)\, x_m + C_m\, B_m\, r$, with $a_m > 0$. From the equation above it is possible to define the control law that allows to have convergence of the tracking error

$$u = -\frac{1}{b\,(\mu)}\left[A_{21}\,(\mu)\, z + (a_{22}\,(\mu) + a_m)\, y - \bar{y}_r\right] \tag{4.48}$$

The resulting closed-loop system, replacing y, can be described as follows

$$\begin{aligned} \dot{x}_m &= A_m\, x_m + B_m\, r \\ \dot{z} &= A_{11}\,(\mu)\, z + A_{12}\,(\mu)\, C_m\, x_m + A_{12}\,(\mu)\, e \\ \dot{e} &= -a_m\, e \end{aligned} \tag{4.49}$$

Considering this closed-loop system, it is possible to highlight that the trajectories of the last subsystem converge asymptotically if all the other trajectories are bounded, and this is guaranteed if $r \in \mathcal{L}_\infty$.

Remark 4.3.1. *The (4.48) can be shown as the sum of two contributions $u = u_{zd} + u_{st}$, defined as*

$$u_{zd} = -\frac{1}{b\,(\mu)}\left[A_{21}\,(\mu)\, z + A_{22}\,(\mu)\, x_m - C_m\, B_m\, r\right], \quad u_{st} = -\frac{1}{b\,(\mu)}\,(a_{22}\,(\mu) + a_m)\, e \tag{4.50}$$

where $A_{22}\,(\mu) = a_{22}\,(\mu)\, C_m - C_m\, A_m$.

The first component of the control action u_{zd} makes invariant the zero dynamics of the augmented system (4.43)-(4.45). The forced trajectories are given by

$$\begin{aligned} \dot{x}_m &= A_m\, x_m + B_m\, r \\ \dot{z} &= A_{11}\,(\mu)\, z + A_{12}\,(\mu)\, C_m\, x_m \end{aligned} \tag{4.51}$$

The second component of the control action u_{st} allows to make the subspace of the zero dynamics, globally attractive. The fact that the trajectories are forced on the zero dynamics explains why it is necessary for the system (4.45) to be minimum-phase.

Remark 4.3.2. *The second component of the control action can be defined in a different way, with just a simple high-gain feedback of the form $u = -k\, e$, with $k > 0$. There exists a $k^\star > 0$ such that $a_{22}\,(\mu) - b\,(\mu)\, k < 0$ for any $k \geq k^\star$.*

Until now we considered the availability of $z(\cdot)$ for feedback. Consider now the case in which the state $z(\cdot)$ is not available for feedback. In order to design the controller, it is necessary to modify the control law (4.48), previously obtained, introducing a reduced-order observer for $z(\cdot)$. The chosen observer can be describe by the following relation

$$\dot{\xi} = F_o(\mu)\,\xi + G_{o1}(\mu)\,y + G_{o2}(\mu)\,u \tag{4.52}$$

with state $\xi \in \mathbb{R}^{n-1}$, and the matrices F_o and $G_o = \begin{pmatrix} G_{o,1} & G_{o,2} \end{pmatrix}$ directly dependent on μ. The introduction of the reduced-order observer implies the definition of the observation error as follows

$$\chi = z - \xi - L(\mu)\,y \tag{4.53}$$

where $L(\mu) \in \mathbb{R}^{(n-1)\times 1}$ is an output-injection gain, arbitrarily chosen.

Replacing z with the available signals $\xi + L(\mu)\,y$, the control law (4.48) becomes

$$u = -\frac{1}{b(\mu)}\left[A_{21}(\mu)\,\xi + \left(a_{22}(\mu) + a_m + A_{21}(\mu)\,L(\mu)\right)y - \bar{y}_r\right] \tag{4.54}$$

Since the pair $(C(\mu), A(\mu))$ is assumed to be observable, for any $\mu \in \mathcal{K}_\mu$, applying a simple PHB test (A.3.3), is possible to say that also the pair $(A_{21}(\mu), A_{11}(\mu))$ is observable, for any $\mu \in \mathcal{K}_\mu$. Now it is possible to define the matrix $L(\mu)$: given any Hurwitz polynomial

$$p_d(\lambda) = \lambda^{n-1} + d_{n-2}\,\lambda^{n-2} + \cdots + d_1\,\lambda + d_0$$

exist a matrix $L(\mu)$, such that the following relation is valid

$$\det\left(A_{11}(\mu) - L(\mu)\,A_{21}(\mu) - \lambda I\right) = p_d(\lambda)$$

Deriving the (4.53) and replacing z it is possible to obtain (omitting the parameter μ to simplify the notation)

$$\begin{aligned}
\dot{\chi} = \dot{z} - \dot{\xi} - L\,\dot{y} &= A_{11}\,z + A_{12}\,y - F_o\,\xi \\
&\quad - G_{o1}\,y - G_{o2}\,u - L\left[A_{21}\,z + a_{22}\,y + b\,u\right] \\
&= \left[A_{11} - L\,A_{21}\right]\chi + \left[A_{11} - F_o - L\,A_{21}\right]\xi \\
&\quad + \left[A_{11}\,L + A_{12} - G_{o1} - L\,A_{21}\,L - L\,a_{22}\right]y - \left[G_{o2} + L\,b\right]u
\end{aligned}$$

In order to have an internally stable system for describing the observation error, the matrices of the reduced-order observer are selected as follows

$$\begin{aligned}
F_o(\mu) &= A_{11}(\mu) - L(\mu)\,A_{21}(\mu) \\
G_{o1}(\mu) &= A_{12}(\mu) + F_o(\mu)\,L(\mu) - a_{22}(\mu)\,L(\mu) \\
G_{o2}(\mu) &= -b(\mu)\,L(\mu)
\end{aligned} \tag{4.55}$$

The closed-loop system (4.49) can be rewritten, using the coordinates χ for the observer instead of ξ and adding the parameter $A_{21}(\mu)\,\chi$ to the tracking error, as follows

$$\begin{aligned}\dot{x}_m &= A_m\, x_m + B_m\, r\\ \dot{z} &= A_{11}(\mu)\, z + A_{12}(\mu)\, C_m\, x_m + A_{12}(\mu)\, e\\ \dot{\chi} &= F_o(\mu)\,\chi\\ \dot{e} &= A_{21}(\mu)\,\chi - a_m\, e\end{aligned}\tag{4.56}$$

The first term of the tracking error goes to zero when the observer reaches the real values of $z(\cdot)$, leaving only $-a_m\, e$.

Replacing (4.54) in (4.52) we obtain the controller in the desired form (4.46), with the matrices

$$\begin{aligned}F_c(\hat{\theta}) &= A_{11}(\hat{\theta}), & G_c(\hat{\theta}) &= \begin{pmatrix} G_{c,1}(\hat{\theta}) & G_{c,2}(\hat{\theta}) & G_{c,3}(\hat{\theta})\end{pmatrix}\\ H_c(\hat{\theta}) &= -\frac{A_{21}(\hat{\theta})}{b(\hat{\theta})}, & K_c(\hat{\theta}) &= \begin{pmatrix} K_{c,1}(\hat{\theta}) & K_{c,2}(\hat{\theta}) & K_{c,3}(\hat{\theta})\end{pmatrix}\end{aligned}\tag{4.57}$$

where the submatrices $G_{ci}(\cdot)$ and $K_{ci}(\cdot)$ are given by

$$\begin{aligned}G_{c,1}(\hat{\theta}) &= A_{11}(\hat{\theta})\, L(\hat{\theta})\, C_m + A_{12}(\hat{\theta})\, C_m - L(\hat{\theta})\, C_m\, A_m\\ G_{c,2}(\hat{\theta}) &= -L(\hat{\theta})\, C_m\, B_m\\ G_{c,3}(\hat{\theta}) &= A_{11}(\hat{\theta})\, L(\hat{\theta}) + A_{12}(\hat{\theta}) + a_m\, L(\hat{\theta})\\ K_{c,1}(\hat{\theta}) &= -\frac{1}{b(\hat{\theta})}\left(A_{21}(\hat{\theta})\, L(\hat{\theta})\, C_m + a_{22}(\hat{\theta})\, C_m - C_m\, A_m\right)\\ K_{c,2}(\hat{\theta}) &= \frac{1}{b(\hat{\theta})}\left(C_m\, B_m\right)\\ K_{c,3}(\hat{\theta}) &= -\frac{1}{b(\hat{\theta})}\left(A_{21}(\hat{\theta})\, L(\hat{\theta}) + a_{22}(\hat{\theta}) + a_m\right)\end{aligned}\tag{4.58}$$

where $\hat{\theta}$ is the tunable vector. The 'true value' of the parameter is given by the identity $\theta(\mu) = \mu$.

Remark 4.3.3. *Due to the fact that $A_{11}(\mu)$ is Hurwitz for all $\mu \in \mathcal{K}_\mu$, the controller described in (4.46) is internally stable.*

4.3.2 Controller Parametrization and System Immersion

The choice of the controller in (4.46) is not suitable for the design of the update law (4.47), due to the dependency on the tunable parameter $\hat{\theta}$. The design of the control law is much more complicated due to the dependency of the dynamics on the tunable parameter. In order to solve this problem, a solution is to confine the dependency on the tunable parameter to the output map of the controller.

In the classic MRAC problem, this is solved by using the nonminimal realization of the certainty-equivalence controller.

Internal model property

Let us consider $\bar{x}_m(t) = (L_1\, r)(t)$ and $\bar{z}(t) = (L_2\, \bar{x}_m)(t)$, where L_1 and L_2 are linear mappings, $L_1 : \mathbb{R} \to \mathbb{R}$ and $L_2 : \mathbb{R}^m \to \mathbb{R}^{n-1}$, defined as follows

$$
\begin{aligned}
(L_1\, \eta_1)(t) &= \int_{-\infty}^{t} e^{A_m(t-\tau)} B_m\, \eta_1(\tau)\, d\tau \\
(L_2\, \eta_2)(t) &= \int_{-\infty}^{t} e^{A_{11}(\mu)(t-\tau)} A_{12}(\mu)\, C_m\, \eta_2(\tau)\, d\tau
\end{aligned}
$$

Consider the *extended exosystem*

$$
\dot{w} = S(\mu)\, w + P\, r \tag{4.59}
$$

where $w(t) = col\,(\bar{x}_m(t), \bar{z}(t)) \in \mathbb{R}^q$ and the matrices are defined as follows

$$
S(\mu) = \begin{pmatrix} A_m & 0 \\ A_{12}(\mu)\, C_m & A_{11}(\mu) \end{pmatrix}, \qquad P = \begin{pmatrix} B_m \\ 0 \end{pmatrix} \tag{4.60}
$$

Letting $\tilde{x}_m = x_m - \bar{x}_m$ and $\tilde{z} = z - \bar{z}$ it is possible to rewrite the (4.49) obtaining the *augmented error system*

$$
\begin{aligned}
\dot{w} &= S(\mu)\, w + P(\mu)\, r \\
\dot{\tilde{x}}_m &= A_m\, \tilde{x}_m \\
\dot{\tilde{z}} &= A_{11}(\mu)\, \tilde{z} + A_{12}(\mu)\, C_m\, \tilde{x}_m + A_{12}(\mu)\, e \\
\dot{e} &= A_{21}(\mu)\, \tilde{z} + A_{22}(\mu)\, \tilde{x}_m + a_{22}(\mu)\, e + b(\mu)\, [u - Q(\mu)\, w - R(\mu)\, r]
\end{aligned} \tag{4.61}
$$

where

$$
\begin{aligned}
Q(\mu) = \begin{pmatrix} Q_1(\mu) & Q_2(\mu) \end{pmatrix} &= \begin{pmatrix} -\dfrac{1}{b(\mu)} A_{22}(\mu) & -\dfrac{1}{b(\mu)} A_{21}(\mu) \end{pmatrix} \\
R(\mu) &= \frac{1}{b(\mu)} C_m\, B_m
\end{aligned}
$$

It is possible to define a control law that solves the MRC problem as $u = u_{ff} + u_{st}$, where we have stabilization thanks to u_{st} and the error that became invariant thanks to the feedforward control $u_{ff} = Q(\mu)\, w + R(\mu)\, r$. Due to the fact that it is not possible to measure the signal $w(\cdot)$, it is necessary to embed a suitable *internal model* of the system

$$\begin{aligned} \dot{w} &= S(\mu) w + P(\mu) r \\ u_{ff} &= Q(\mu) w + R(\mu) r \end{aligned} \tag{4.62}$$

in the controller that provides regulation. It is interesting to note that the controller, designed in the previous section, is characterized by the *internal model property* when $\hat{\theta} = \mu$.

Consider the system and the control law defined in (4.45) and (4.46). It is possible to rewrite the system, considering $e = 0$, as follows

$$\begin{aligned} \begin{pmatrix} \dot{x}_m \\ \dot{\xi} \end{pmatrix} &= \begin{pmatrix} A_m & 0_{m\times(n-1)} \\ G_{c,1}(\mu) & F_c(\mu) \end{pmatrix} \begin{pmatrix} x_m \\ \xi \end{pmatrix} + \begin{pmatrix} B_m \\ G_{c,2}(\mu) \end{pmatrix} r \\ u &= \begin{pmatrix} K_{c,1}(\mu) & H_c(\mu) \end{pmatrix} \begin{pmatrix} x_m \\ \xi \end{pmatrix} + K_{c,2}(\mu) r \end{aligned} \tag{4.63}$$

it is evident the similarity with (4.62). It is possible to define a parametrized of mappings $\Sigma(\mu) : \mathbb{R}^q \to \mathbb{R}^q$, where

$$\Sigma(\mu) = \begin{pmatrix} I_m & 0_{m\times(n-1)} \\ -L(\mu) C_m & I_{n-1} \end{pmatrix}$$

Throw easy calculation the following relations are shown

$$\Phi(\mu)\Sigma(\mu) = \Sigma(\mu) S(\mu), \quad \Gamma(\mu) = \Sigma(\mu) P$$

$$\Psi(\mu)\Sigma(\mu) = Q(\mu), \quad K_{c,2}(\mu) = R(\mu)$$

where

$$\Phi(\mu) = \begin{pmatrix} A_m & 0_{m\times(n-1)} \\ G_{c,1}(\mu) & F_c(\mu) \end{pmatrix}, \quad \Gamma(\mu) = \begin{pmatrix} B_m \\ G_{c,2}(\mu) \end{pmatrix}$$

$$\Psi(\mu) = \begin{pmatrix} K_{c,1}(\mu) & H_c(\mu) \end{pmatrix}$$

Result 4.3.1. *Considering the certain-equivalence controller defined in (4.46), setting $\hat{\theta} = \mu$, when $e = 0$, the trajectories of (4.45) and (4.46), are diffeomorphic to those of the exosystem (4.62). The controller is a diffeomorphic copy of the exosystem.*

Remark 4.3.4. *This is valid also for the control law described by (4.52) and (4.54).*

It is evident how the internal model design has an explicit dependence on the uncertain parameter μ. The ideal situation is when it is possible to have a system, independent from μ, able to reproduce the output trajectories of (4.62).

The solution to this problem is the so called *system immersion*.

Definition 4.3.1. *A system, defined by the matrices* (S, P, Q, R)*, with* $(\mathcal{X}_1, \mathcal{U}, \mathcal{Y})$ *as state space, input space and output space, is said to be* immersed *in the system defined by the matrices* $(\Phi, \Gamma, \Psi, \Upsilon)$*, with* $(\mathcal{X}_2, \mathcal{U}, \mathcal{Y})$ *as state space, input space and output space, if there exist a mapping* $\Pi : \mathcal{X}_1 \to \mathcal{X}_2$ *such that*

$$\Pi\, S = \Phi\, \Pi, \quad \Pi\, P = \Gamma, \quad Q = \Psi\, \Pi, \quad R = \Upsilon$$

The immersion is regular *if the pair* (Ψ, Φ) *is observable and* weak *if it is detectable but not observable.*

Definition 4.3.2. *Given a parametrized family of systems* $(S(\mu), P(\mu), Q(\mu), R(\mu))$ *with parameter space* $\mathcal{P}_1$*, the parametrized family of systems* $(\Phi(\theta), \Gamma, \Psi(\theta), \Upsilon(\theta))$ *with parameter space* $\mathcal{P}_2$ *is said to be an internal model of* $(S(\mu), P(\mu), Q(\mu), R(\mu))$ *with canonical parametrization in feedback form if*

(i) There exists a Hurwitz matrix F *such that* $\Phi(\theta) = F + \Gamma\, \Psi(\theta)$

(ii) $\Psi(\theta)$ *and* $\Upsilon(\theta)$ *are affine functions of the parameter vector* θ

(iii) There exists a continuous mapping $\mathcal{P}_1 \to \mathcal{P}_2$*,* $\mu \mapsto \theta(\mu)$*, and a parametrized family of immersion mappings* $\Pi(\theta)$ *such that the identities*

$$\begin{aligned} \Pi(\theta(\mu))\, S(\mu) = \Phi(\theta(\mu))\, \Pi(\theta(\mu)), \quad & \Pi(\theta(\mu))\, P(\mu) = \Gamma \\ Q(\mu) = \Psi(\theta(\mu))\, \Pi(\theta(\mu)), \quad & R(\mu) = \Upsilon(\theta(\mu)) \end{aligned} \tag{4.64}$$

hold for every $\mu \in \mathcal{P}_1$.

Weak immersion

Let us consider the certainty-equivalence controller based on the observer, considering (4.52) and (4.54), as follows

$$\begin{aligned} \dot{x}_m &= A_m\, x_m + B_m\, r \\ \dot{\xi} &= F_o(\mu)\, \xi + G_{o,1}(\mu)\, y + G_{o,2}(\mu)\, u \\ u &= H_o(\mu)\, \xi + K_{o,1}(\mu)\, y + K_{o,2}(\mu)\, \bar{y}_r \end{aligned} \tag{4.65}$$

where F_o, $G_{o,1}$ and $G_{o,2}$ are defined in (4.55), and H_o, $K_{o,1}$ and $K_{o,2}$ are

$$\begin{aligned} H_o(\mu) &= -\frac{1}{b(\mu)} A_{21}(\mu) \\ K_{o,1}(\mu) &= -\frac{1}{b(\mu)} \left[a_{22}(\mu) + a_m + A_{21}(\mu) L(\mu) \right] \\ K_{o,2}(\mu) &= \frac{1}{b(\mu)} \end{aligned} \tag{4.66}$$

Let $W(s,\mu) = (W_1(s,\mu), W_2(s,\mu))$ be the transfer function of the triplet (H_o, F_o, G_o), where

$$\begin{aligned} W_1(s,\mu) &= H_o(\mu)(s\,I - F_o(\mu))^{-1} G_{o,1}(\mu) \\ W_2(s,\mu) &= H_o(\mu)(s\,I - F_o(\mu))^{-1} G_{o,2}(\mu) \end{aligned}$$

Lemma 4.3.1. *Assume that $p_d(\lambda)$ and the characteristic polynomial of $A_{11}(\mu)$ are coprime for all $\mu \in \mathcal{K}_\mu$, then the triplet $(H_o(\mu), F_o(\mu), G_{o,2}(\mu))$ is a minimal realization of $W_2(s,\mu)$*

The pair $(F_o, G_{o,2})$ is controllable, and consequently, also the pair (F_o, G_o) it is controllable. Together with Lemma 4.3.1, it is possible to say that (H_o, F_o, G_o) is a minimal realization of $W(s,\mu)$. The minimal realization of a transfer function can be rewritten using the controller canonical form, as follows

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -d_0 & -d_1 & -d_2 & \cdots & -d_{n-2} \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

The transfer function are modified considering the parametrized families of vectors $\theta_1(\mu) \in \mathbb{R}^{n-1}$ and $\theta_2(\mu) \in \mathbb{R}^{n-1}$ becoming

$$W_1(s,\mu) = \theta_1(\mu)^\top (s\,I - \Phi)^{-1} \Gamma, \qquad W_2(s,\mu) = \theta_2(\mu)^\top (s\,I - \Phi)^{-1} \Gamma$$

such that the resulting realization of $W(s,\mu)$ can be written as a $2(n-1)$ dimensional system

$$\begin{aligned} \dot{\zeta}_1 &= \Phi\,\zeta_1 + \Gamma\,y \\ \dot{\zeta}_2 &= \Phi\,\zeta_2 + \Gamma\,u \\ u_1 &= \theta_1(\mu)^\top \zeta_1 + \theta_2(\mu)^\top \zeta_2 \end{aligned} \tag{4.67}$$

Defining

$$\theta_3(\mu) = K_{o,1}(\mu), \qquad \theta_4(\mu) = K_{o,2}(\mu) \tag{4.68}$$

it is possible to derive the certainty-equivalence controller where the dynamic is free from the uncertainty parameter μ

$$
\begin{aligned}
\dot{x}_m &= A_m\, x_m + B_m\, r \\
\dot{\zeta}_1 &= \Phi\, \zeta_1 + \Gamma\, y \\
\dot{\zeta}_2 &= \Phi\, \zeta_2 + \Gamma\, u \\
u &= \hat{\theta}_1^\top \zeta_1 + \hat{\theta}_2^\top \zeta_2 + \hat{\theta}_3\, y + \hat{\theta}_4\, \bar{y}_r
\end{aligned}
\tag{4.69}
$$

The true value of the estimate of the parameter vector $\hat{\theta} = col\left(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4\right)$ is $\theta(\mu) = col\left(\theta_1(\mu), \theta_2(\mu), \theta_3(\mu), \theta_4(\mu)\right)$.

The knowledge of the two matrices Φ and Γ allows to make some assumptions. Due to the fact that the pair

$$
\begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix}, \quad \begin{pmatrix} \Gamma & 0 \\ 0 & \Gamma \end{pmatrix}
$$

is *completely controllable*, the pair

$$
\begin{pmatrix} \theta_1^\top & \theta_2^\top \end{pmatrix}, \quad \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix}
$$

is necessarily completely unobservable. It is possible to define the unobservable subspace $\mathcal{V}_{no}$, that has the same dimension of the minimal realization $W(s, \mu)$, that is $n-1$. The unobservable subspace can be written as

$$
\mathcal{V}_{no} = \operatorname{im}\begin{pmatrix} I \\ X(\mu) \end{pmatrix}
$$

where $X(\mu) \in \mathbb{R}^{(n-1)\times(n-1)}$ is a matrix that satisfy the condition in which $\mathcal{V}_{no}$ is the largest invariant subset of (4.67)

$$
\begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix}\begin{pmatrix} I \\ X(\mu) \end{pmatrix} = \begin{pmatrix} I \\ X(\mu) \end{pmatrix} V, \quad \begin{pmatrix} \theta_1^\top & \theta_2^\top \end{pmatrix}\begin{pmatrix} I\, X(\mu) \end{pmatrix} = 0
$$

where the matrix V is a generic matrix such that $V \in \mathbb{R}^{(n-1)\times(n-1)}$. In order to satisfy the previous relations, it is necessary that $V = \Phi$, and that $X(\mu)$ is the solution of

$$
\begin{aligned}
\Phi\, X(\mu) &= X(\mu)\, \Phi \\
\theta_2^\top\, X(\mu) &= -\theta_1^\top
\end{aligned}
$$

By means of a change of coordinates $\bar{\zeta}_2 = \zeta_2 - X(\mu)\,\zeta_1$, it is possible put the (4.67) in its Kalman canonical form

$$
\begin{aligned}
\begin{pmatrix} \dot{\zeta}_1 \\ \dot{\bar{\zeta}}_2 \end{pmatrix} &= \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \bar{\zeta}_2 \end{pmatrix} + \begin{pmatrix} \Gamma & 0 \\ \bar{\Gamma}(\mu) & \Gamma \end{pmatrix}\begin{pmatrix} y \\ u \end{pmatrix} \\
u_1 &= \begin{pmatrix} 0 & \theta_2^\top \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \bar{\zeta}_2 \end{pmatrix}
\end{aligned}
\tag{4.70}
$$

where $\bar{\Gamma}(\mu) = -X(\mu)\,\Gamma$.

Due to the fact that the previous subsystem is a minimal realization of $W(s,\mu)$, means that it is related to the matrices (H_o, F_o, G_o) through an invertible matrix $Y(\mu) \in \mathbb{R}^{(n-1)\times(n-1)}$, that satisfies the following relations

$$Y(\mu)^{-1}\,\Phi\,Y(\mu) = F_o(\mu)\,, \qquad Y(\mu)^{-1}\,\bar{\Gamma} = G_{o,1}(\mu)$$
$$Y(\mu)^{-1}\,\Gamma = G_{o,2}(\mu)\,, \qquad \theta_2^\top\,Y(\mu) = H_o(\mu)$$

and applying the following change coordinates
$\eta = \zeta_1, \eta_2 = Y(\mu)^{-1}\,\zeta_2 - Y(\mu)^{-1}\,X(\mu)\,\zeta_1$, the system (4.69) becomes

$$\begin{aligned}\dot{x}_m &= A_m\,x_m + B_m\,u_r\\ \dot{\eta}_1 &= \Phi\,\eta_1 + \Gamma\,y\\ \dot{\eta}_2 &= F_o(\mu)\,\eta_2 + G_{o,1}(\mu)\,y + G_{o,2}(\mu)\,u\\ u &= H_o(\mu)\,\eta_2 + K_{o,1}(\mu)\,y + K_{o,2}(\mu)\,\bar{y}_r\end{aligned} \tag{4.71}$$

Theorem 4.3.1. *In the case in which $\hat{\theta} = \theta(\mu)$, the new controller embeds a diffeomorphic copy of the original controller.*

Let us consider the mapping $L_3 : \mathcal{C}^0(\mathbb{R}^m) \to \mathcal{C}^1(\mathbb{R}^{n-1})$ described by

$$(L_3\,\eta_3)(t) = \int_{-\infty}^{t} e^{\Phi(t-\tau)}\Gamma\,C_m\,\eta_3(\tau)\,d\tau$$

Considering $\bar{\zeta}_1(t) = (L_3\,\eta_3)(t)$, and writing the dynamics

$$\dot{\bar{\zeta}}_1 = \Phi\,\bar{\zeta}_1 + \Gamma\,C_m\,\bar{x}_m$$

it is possible to do the following change of coordinates

$$\begin{aligned}&\zeta_1 \mapsto \chi_1 := \zeta_1 - \bar{\zeta}_1\\ &\zeta_2 \mapsto \chi_2 := \bar{z} - L(\mu)\,C_m\,\bar{x}_m - Y(\mu)^{-1}\,\zeta_2 - Y(\mu)^{-1}\,X(\mu)\,\zeta_1 + \tilde{z} - L(\mu)\,e\end{aligned}$$

The resulting *augmented error system* is described by

$$\begin{aligned}\dot{w} &= S(\mu)\,w + P\,r\\ \dot{\tilde{x}}_m &= A_m\,\tilde{x}_m\\ \dot{\tilde{z}} &= A_{11}(\mu)\,\tilde{z} + A_{12}(\mu)\,C_m\,\tilde{x}_m + A_{12}(\mu)\,e\\ \dot{\chi}_1 &= \Phi\,\chi_1 + \Gamma\,C_m\,\tilde{x}_m + \Gamma\,e\\ \dot{\chi}_2 &= F_o(\mu)\,\chi_2\\ \dot{e} &= A_{21}(\mu)\,\chi_2 - a_m\,e\end{aligned} \tag{4.72}$$

It is evident how the extended exosystem and the augmented error system are decoupled; it is possible then, let $\mathbf{e} = col\,(\tilde{x}_m, \tilde{z}, \chi_1, \chi_2, e)$, the autonomous subsystem

$$\dot{\mathbf{e}} = \mathcal{A}\,(\mu)\,\mathbf{e} \tag{4.73}$$

where $\mathcal{A}\,(\mu)$ is a Hurwitz matrix for all $\mu \in \mathcal{K}_\mu$. $\mathcal{A}\,(\mu)$ is defined as follows

$$\mathcal{A}\,(\mu) = \begin{pmatrix} A_m & 0 & 0 & 0 & 0 \\ A_{12}\,(\mu)\,C_m & A_{11}\,(\mu) & 0 & 0 & A_{12}\,(\mu) \\ \Gamma\,C_m & 0 & \Phi & 0 & \Gamma \\ 0 & 0 & 0 & F_o\,(\mu) & 0 \\ 0 & 0 & 0 & A_{21}\,(\mu) & -a_m \end{pmatrix} \tag{4.74}$$

4.3.3 Adaptive design

In order to simplify the problem, it is possible to denote with $\phi\,(\zeta, y, x_m)$ the *regressor*

$$\phi\,(\zeta, y, x_m) = \begin{pmatrix} \zeta_1^\top & \zeta_2^\top & y & \bar{y} \end{pmatrix}^\top$$

and rewrite the control law in the following form

$$u = \phi^\top\,(\zeta, y, x_m)\,\hat{\theta}$$

Introducing the "parameter estimation error" $\tilde{\theta} = \hat{\theta} - \theta$, and the two matrices

$$\mathcal{B}\,(\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 & b\,(\mu) \end{pmatrix}^\top, \quad \mathcal{C} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

it is possible to rewrite the error dynamics as follows

$$\begin{aligned} &\dot{\mathbf{e}} = \mathcal{A}\,(\mu)\,\mathbf{e} + \mathcal{B}\,(\mu)\,\phi^\top\,(\zeta, y, x_m)\,\tilde{\theta} \\ &e = \mathcal{C}\,\mathbf{e} \end{aligned} \tag{4.75}$$

The pair $(\mathcal{A}\,(\mu))$ is stabilizable and, consequently, the pair $(\mathcal{C}, \mathcal{A}\,(\mu))$ is detectable. The triplet is strictly passive and there exists a matrix function $\mathcal{Q}\,(\mu)$ and positive constant parameters c_1 and c_2 such that the following relations hold

$$c_1\,I \le \mathcal{Q}\,(\mu) \le c_2\,I$$

$$\begin{aligned} &\mathcal{A}^\top\,(\mu)\,\mathcal{Q}\,(\mu) + \mathcal{Q}\,(\mu)\,\mathcal{A}\,(\mu) \le -I \\ &\mathcal{B}^\top\,(\mu)\,\mathcal{Q}\,(\mu) = \mathcal{C} \end{aligned}$$

Then the triplet $(\mathcal{A}\,(\mu), \mathcal{B}\,(\mu), \mathcal{C})$ is strictly passive for all $\mu \in \mathcal{K}_\mu$.

The candidate Lyapunov function is the following

$$V\,(\tilde{\theta}, \mathbf{e}) = \mathbf{e}^\top\,\mathcal{Q}\,(\mu)\,\mathbf{e} + \frac{\tilde{\theta}^\top\,\tilde{\theta}}{\gamma} \tag{4.76}$$

The time derivative of the candidate Lyapunov function is the following

$$\begin{aligned} \dot{V}\left(\tilde{\theta}, \mathbf{e}\right) &= 2\,\mathbf{e}^\top \mathcal{Q}\left(\mu\right) \dot{\mathbf{e}} + \frac{2\,\tilde{\theta}^\top}{\gamma} \dot{\tilde{\theta}} = \\ &= -\mathbf{e}^\top I\, \mathbf{e} + 2\,\tilde{\theta}^\top \phi\left(\xi, y, x_m\right) \mathcal{B}^\top\left(\mu\right) \mathcal{Q}\left(\mu\right) e + 2\frac{\tilde{\theta}^\top}{\gamma} \dot{\tilde{\theta}} \end{aligned} \tag{4.77}$$

For the update law is possible to use a gradient-based adaptive law

$$\dot{\tilde{\theta}} = -\gamma\, \phi\left(\zeta, y, x_m\right) e$$

Then the system can be described by

$$\begin{aligned} \dot{w} &= S\left(\mu\right) w + P\, r \\ \dot{\mathbf{e}} &= \mathcal{A}\left(\mu\right) \mathbf{e} + \mathcal{B}\left(\mu\right) \phi^\top\left(\zeta, y, x_m\right) \tilde{\theta} \\ \dot{\tilde{\theta}} &= -\gamma\, \phi\left(\zeta, y, x_m\right) \mathcal{C}\, \mathbf{e} \end{aligned} \tag{4.78}$$

The origin is an uniformly stable equilibrium of (4.78) due to the *LaSalle-Yoshizawa theorem* (2.2.3) and the error decays to zero.

4.4 MRAC with Autonomous Exosystem

Consider the SISO LTI system

$$\begin{aligned} \dot{x} &= A\left(\mu\right) x + B\left(\mu\right) u \\ y &= C\left(\mu\right) x \end{aligned} \tag{4.79}$$

where the state is $x \in \mathbb{R}^n$, the control input is $u \in \mathbb{R}$ and the related output is $y \in \mathbb{R}$. The matrices of the system are dependent on a vector μ of uncertainty parameters, that ranges in a compact set $\mathcal{K}_\mu \subset \mathbb{R}^p$. The analysis of the problem is done similarly to the theory shown in the previous section.

In order to solve the problem it is sufficient that the system is stabilizable and detectable, but a number of assumptions are made:

Assumption 4.4.1. *The pair* $\left(A\left(\mu\right), B\left(\mu\right)\right)$ *is controllable, and the pair* $\left(C\left(\mu\right), A\left(\mu\right)\right)$ *is observable, for any* $\mu \in \mathcal{K}_\mu$.

Assumption 4.4.2. *The system (4.79) has relative degree equal to one for any* $\mu \in \mathcal{K}_\mu$.

Assumption 4.4.3. *The sign of the high-frequency gain* $b\left(\mu\right) = C\left(\mu\right) B\left(\mu\right)$ *is known and constant for any* $\mu \in \mathcal{K}_\mu$.

The relative degree equal to one allows to rewrite the system in its normal form

$$\begin{aligned}\dot{z} &= A_{11}(\mu)\, z + A_{12}(\mu)\, y\\ \dot{y} &= A_{21}(\mu)\, z + a_{22}(\mu)\, y + b(\mu)\, u\end{aligned} \tag{4.80}$$

where $z \in \mathbb{R}^{n-1}$; it is assumed that $b(\mu) > 0$, for all $\mu \in \mathcal{K}_\mu$.

As in all MRAC problem, it is given a reference model that is minimum phase and internally stable, that describe the desired input/output behaviour

$$\begin{aligned}\dot{x}_m &= A_m\, x_m + B_m\, r\\ y_m &= C_m\, x_m\end{aligned} \tag{4.81}$$

where the state is $x_m \in \mathbb{R}^m$.

Differently from the previous case, the reference signal is generated by an exosystem described as follows

$$\begin{aligned}\dot{w}(t) &= S\, w(t)\\ r(t) &= R\, w(t)\end{aligned} \tag{4.82}$$

where $w \in \mathbb{R}^p$.

Control Objective The control objective is to design an adaptive controller of the form

$$\begin{aligned}\dot{\xi} &= F_c(\hat{\theta})\, \xi + G_c(\hat{\theta})\, \mathbf{y}\\ u &= H_c(\hat{\theta})\, \xi + K_c(\hat{\theta})\, \mathbf{y}\end{aligned} \tag{4.83}$$

where $\xi \in \mathbb{R}^v$ and $\mathbf{y} = col\,(x_m, w, e)$. The suitable adaptive law is defined as

$$\dot{\hat{\theta}} = \phi(\hat{\theta}, \xi, \mathbf{y}) \tag{4.84}$$

4.4.1 Certainty-Equivalence controller

In order to design the certainty-equivalence controller, it is necessary first to define the tracking error $e = y - y_r$ and its derivative in time

$$\dot{e} = \dot{y} - \dot{y}_m$$

In the same way as it was done in the previous section, adding and subtracting $a_m\, e$, the tracking error can be written as

$$\dot{e} = -a_m\, e + A_{21}(\mu)\, z + (a_{22}(\mu) + a_m)\, y - \bar{y}_r + b(\mu)\, u$$

where $\bar{y}_r = (C_m\, A_m + a_m\, C_m)\, x_m + C_m\, B_m\, r$, with $a_m > 0$. In order to achieve convergence of the tracking error, the following input is chosen

$$u = -\frac{1}{b(\mu)}\left[A_{21}(\mu)\, z + (a_{22}(\mu) + a_m)\, y - \bar{y}_r\right] \tag{4.85}$$

The resulting closed-loop system is described by

$$
\begin{aligned}
\dot{w} &= S\,w \\
r &= R\,w \\
\dot{x}_m &= A_m\,x_m + B_m\,r \\
\dot{z} &= A_{11}\left(\mu\right)z + A_{12}\left(\mu\right)C_m\,x_m + A_{12}\left(\mu\right)e \\
\dot{e} &= -a_m\,e
\end{aligned}
\tag{4.86}
$$

As before, considering the state z not available for feedback, it is necessary to modify the control law (4.85) introducing the reduced-order observer

$$
\dot{\xi} = F_o\left(\mu\right)\xi + G_{o,1}\left(\mu\right)y + G_{o,2}\left(\mu\right)u \tag{4.87}
$$

where $\xi \in \mathbb{R}^{n-1}$. We introduce, then, the observation error $\chi = z - \xi - L\left(\mu\right)y$. Replacing z with $\xi + L\left(\mu\right)y$, the control law (4.85) becomes

$$
u = -\frac{1}{b\left(\mu\right)}\left[A_{21}\left(\mu\right)\xi + \left(a_{22}\left(\mu\right) + a_m + A_{21}\left(\mu\right)L\left(\mu\right)\right)y - \bar{y}_r\right] \tag{4.88}
$$

Deriving the observation error, it is possible to define the matrices of the observer as follows

$$
\begin{aligned}
F_o\left(\mu\right) &= A_{11}\left(\mu\right) - L\left(\mu\right)A_{21}\left(\mu\right) \\
G_{o,1}\left(\mu\right) &= A_{12}\left(\mu\right) + F_o\left(\mu\right)L\left(\mu\right) - a_{22}\left(\mu\right)L\left(\mu\right) \\
G_{o,2}\left(\mu\right) &= -b\left(\mu\right)L\left(\mu\right)
\end{aligned}
\tag{4.89}
$$

The observation error is described by the following autonomous asymptotically stable system

$$
\dot{\chi} = \left(A_{11}\left(\mu\right) - L\left(\mu\right)A_{21}\left(\mu\right)\right)\chi \tag{4.90}
$$

The closed-loop system can be rewritten again as follows

$$
\begin{aligned}
\dot{w} &= S\,w \\
r &= R\,w \\
\dot{x}_m &= A_m\,x_m + B_m\,r \\
\dot{z} &= A_{11}\left(\mu\right)z + A_{12}\left(\mu\right)C_m\,x_m + A_{12}\left(\mu\right)e \\
\dot{\chi} &= F_o\left(\mu\right)\chi \\
\dot{e} &= A_{21}\left(\mu\right)\chi - a_m\,e
\end{aligned}
\tag{4.91}
$$

Naturally, the controller expressed in the desired form (4.83) is the same described in (4.57) and (4.58).

4.4.2 Controller parametrization

Consider the controller described in (4.85), and write it in the following form

$$\begin{aligned} \dot{\xi} &= F_o(\mu)\,\xi + G_{o,1}(\mu)\,y + G_{o,2}(\mu)\,u \\ u &= H_o(\mu)\,\xi + K_{o,1}(\mu)\,y + K_{o,2}(\mu)\,\bar{y}_r \end{aligned} \tag{4.92}$$

where F_o, $G_{o,1}$ and $G_{o,2}$ are defined in (4.89), and

$$\begin{aligned} H_o(\mu) &= -\frac{1}{b(\mu)}A_{21}(\mu) \\ K_{o,1}(\mu) &= -\frac{1}{b(\mu)}\left[a_{22}(\mu) + a_m + A_{21}(\mu)\,L(\mu)\right] \\ K_{o,2}(\mu) &= \frac{1}{b(\mu)} \end{aligned} \tag{4.93}$$

As done in the Section (4.3.2), it is possible to find a realization of $W(s) = (W_1(s)\,W_2(s))$ such that it is possible to rewrite the certainty equivalence controller as

$$\begin{aligned} \dot{\zeta}_1 &= \Phi\,\zeta_1 + \Gamma\,y \\ \dot{\zeta}_2 &= \Phi\,\zeta_2 + \Gamma\,u \\ u &= \theta_1^\top\,\zeta_1 + \theta_2^\top\,\zeta_2 + \theta_3\,y + \theta_4\,\bar{y}_r \end{aligned} \tag{4.94}$$

Considering the change of coordinates

$$\eta_1 = \zeta_1, \qquad \eta_2 = Y^{-1}\,\zeta_2 - Y^{-1}\,X\,\zeta_1 \tag{4.95}$$

with the considerations done in the previous section, the closed-loop system can be rewritten as

$$\begin{aligned} \dot{w} &= S\,w \\ r &= R\,w \\ \dot{x}_m &= A_m\,x_m + B_m\,r \\ \dot{\eta}_1 &= \Phi\,\eta_1 + \Gamma\,y \\ \dot{\eta}_2 &= F_o(\mu)\,\eta_2 + G_{o,1}(\mu)\,y + G_{o,2}(\mu)\left[H_o(\mu)\,\eta_2 + K_{o,1}(\mu)\,y + K_{o,2}(\mu)\,\bar{y}_r\right] \\ \dot{z} &= A_{11}(\mu)\,z + A_{12}(\mu)\,C_m\,x_m + A_{12}(\mu)\,e \\ \dot{y} &= A_{21}(\mu)\,z + a_{22}(\mu)\,y + b(\mu)\left[H_o(\mu)\,\eta_2 + K_{o,1}(\mu)\,y + K_{o,2}(\mu)\,\bar{y}_r\right] \end{aligned} \tag{4.96}$$

Applying the change of coordinates

$$\chi_1 = \eta_1, \qquad \chi_2 = z - \eta_2 - L\,y, \qquad e = y - y_m$$

the previous system can be written as

$$
\begin{aligned}
\dot{w} &= S\,w \\
r &= R\,w \\
\dot{x}_m &= A_m\,x_m + B_m\,r \\
\dot{z} &= A_{11}\,(\mu)\,z + A_{12}\,(\mu)\,C_m\,x_m + A_{12}\,(\mu)\,e \\
\dot{\chi}_1 &= \Phi\,\chi_1 + \Gamma\,C_m\,x_m + \Gamma\,e \\
\dot{\chi}_2 &= F_o\,(\mu)\,\chi_2 \\
\dot{e} &= A_{21}\,(\mu)\,\chi_2 - a_m\,e
\end{aligned}
\tag{4.97}
$$

Introducing the deviations from the steady-state as new state variables

$$
\tilde{x}_m = x_m - \bar{x}_m\,(t)\,, \quad \tilde{z} = z - \bar{z}\,(t)\,, \qquad \tilde{\chi}_1 = \chi_1 - \bar{\chi}_1\,(t)
$$

where $\bar{x}_m\,(t)$, $\bar{z}\,(t)$ and $\bar{\chi}_1\,(t)$ are the steady-state of the signals defined as follows:

$$
\bar{x}_m = \Pi\,w \tag{4.98}
$$

where Π is a matrix that satisfies

$$
\Pi\,S = A_m\,\Pi + B_m\,R \tag{4.99}
$$

For $\bar{z}$ and $\bar{\chi}_1$ let us consider, at steady state, $e = 0$ and the vector $v = \mathrm{col}\,(\bar{z}, \bar{\chi}_1)$. It is possible to rewrite

$$
\begin{aligned}
\dot{z} &= A_{11}\,(\mu)\,z + A_{12}\,(\mu)\,C_m\,\Pi\,w \\
\dot{\chi}_1 &= \Phi\,\chi_1 + \Gamma\,C_m\,\Pi\,w
\end{aligned}
\tag{4.100}
$$

as follows

$$
\dot{v} = \underbrace{\begin{pmatrix} A_{11}\,(\mu) & 0 \\ 0 & \Phi \end{pmatrix}}_{A_v} v + \underbrace{\begin{pmatrix} A_{12}\,(\mu)\,C_m\,\Pi \\ \Gamma\,C_m\,\Pi \end{pmatrix}}_{B_v} w \tag{4.101}
$$

The steady state of v can be expressed as

$$
v = Q\,w \tag{4.102}
$$

with Q such that is valid the following relation

$$
Q\,S = A_v\,Q + B_v \tag{4.103}
$$

It is possible to rewrite the previous system as

$$
\begin{aligned}
\dot{w} &= S\, w \\
r &= R\, w \\
\dot{\tilde{x}}_m &= A_m\, \tilde{x}_m \\
\dot{\tilde{z}} &= A_{11}\,(\mu)\, \tilde{z} + A_{12}\,(\mu)\, C_m\, \tilde{x}_m + A_{12}\,(\mu)\, e \\
\dot{\tilde{\chi}}_1 &= \Phi\, \tilde{\chi}_1 + \Gamma\, C_m\, \tilde{x}_m + \Gamma\, e \\
\dot{\chi}_2 &= F_o\,(\mu)\, \chi_2 \\
\dot{e} &= A_{21}\,(\mu)\, \chi_2 - a_m\, e
\end{aligned}
\tag{4.104}
$$

Considering $\mathbf{e} = col(\tilde{x}_m, \tilde{z}, \tilde{\chi}_1, \chi_2, e)$, it is possible to see that the exosystem is decoupled from the rest, and the matrix

$$
\mathcal{A}\,(\mu) = \begin{pmatrix}
A_m & 0 & 0 & 0 & 0 \\
A_{12}\,(\mu)\, C_m & A_{11}\,(\mu) & 0 & 0 & A_{12}\,(\mu) \\
\Gamma\, C_m & 0 & \Phi & 0 & \Gamma \\
0 & 0 & 0 & F_o\,(\mu) & 0 \\
0 & 0 & 0 & A_{21}\,(\mu) & -a_m
\end{pmatrix}
\tag{4.105}
$$

is Hurwitz for all $\mu \in \mathcal{K}_\mu$

4.4.3 Adaptive design

The design of the adaptive law follows the same rules as the previous section. In the end the system is described by the following equations

$$
\begin{aligned}
\dot{w} &= S\, w \\
r &= R\, w \\
\dot{x}_m &= A_m\, x_m + B_m\, r \\
\dot{\mathbf{e}} &= \mathcal{A}\,(\mu)\, \mathbf{e} + \mathcal{B}\,(\mu)\, \phi^\top\,(\zeta, y, x_m)\, \tilde{\theta} \\
\dot{\tilde{\theta}} &= -\gamma\, \phi\,(\zeta, y, x_m)\, \mathcal{C}\, \mathbf{e}
\end{aligned}
\tag{4.106}
$$

Consider the following candidate Lyapunov equation

$$
V\,(\,\tilde{\theta}, \mathbf{e}\,) = \mathbf{e}^\top\, \mathcal{Q}\,(\mu)\, \mathbf{e} + \frac{\tilde{\theta}^\top\, \tilde{\theta}}{\gamma} \tag{4.107}
$$

the time derivative is given by

$$
\dot{V}\,(\,\tilde{\theta}, \mathbf{e}\,) \;= -|e|^2 \tag{4.108}
$$

The trajectories of the error dynamics converge to the invariant set in which the time derivative of the Lyapunov function assume value equal to 0. Taking into account the previous formulation, the only case in which the time derivative is equal to zero, is when the vector $\mathbf{e}$ is zero ($\mathbf{e} = 0$).

Replacing the value of $\mathbf{e}$ inside (4.106), the resulting error dynamics become

$$\begin{aligned} \dot{w} &= S\, w \\ x_m &= \Pi\, w \\ \dot{\mathbf{e}} &= 0 \\ \dot{\tilde{\theta}} &= 0 \end{aligned} \tag{4.109}$$

The fact that $\dot{\tilde{\theta}} = 0$ and $\tilde{\theta}$ is uniformly continuous means that $\tilde{\theta}$ is a constant.

At each time instant t the constant $\tilde{\theta}$ belongs to $\ker\left(\phi^\top(t)\right)$. Due to the fact that $\phi(t)$ is not constant in time, we are interested in studying the conditions such that

$$\mathcal{B}(\mu)\, \phi^\top(\zeta, y, x_m)\, \tilde{\theta} = 0, \qquad \forall t \geq 0 \tag{4.110}$$

yields $\tilde{\theta} = 0$.

Studying the behaviour of $\phi(t)$ in a time interval of length T, for all $t \geq 0$

$$\int_t^{t+T} |\phi^\top(s)\, \tilde{\theta}|^2 ds = 0 \tag{4.111}$$

it is possible to see that $\tilde{\theta}$ can be taken out of the integral due to the fact that it is constant. Consequently, in order to obtain $\tilde{\theta} = 0$, equation (4.111) yields a condition of persistent excitation of the form

$$\int_t^{t+T} \phi(s)\, \phi^\top(s)\, ds \geq c \tag{4.112}$$

for a positive scalar c. In the previous relation there is no upper bound due to the fact that w is bounded.

In order to verify the PE property on the exogenous signal it is possible to define the relation

$$\phi = M\, w \tag{4.113}$$

in order to map a PE condition for ϕ into a PE condition for w.

Replacing (4.113) in (4.112), the integral becomes

$$\int_t^{t+T} M\, w(s)\, w^\top(s)\, M^\top ds \geq c > 0 \tag{4.114}$$

At steady state, considering $\mathbf{e} = 0$, is possible to write

$$\begin{aligned} x_m &= \Pi\, w \\ y = y_m &= C_m\, \Pi\, w \\ \zeta_1 &= \chi_1 \\ \zeta_2 &= Y\,(z - L\, y) + X\, \chi_1 \end{aligned} \tag{4.115}$$

The last two equation can be rewritten as follows

$$\begin{aligned} \zeta_1 &= \bar{\chi}_1 \\ \zeta_2 &= Y\,\bar{z} - Y\,L\,C_m\,\Pi\,w + X\,\bar{\chi}_1 \end{aligned} \tag{4.116}$$

Considering the regressor in the following form

$$\phi\,(\zeta, y, x_m) = \begin{pmatrix} \zeta_1^\top & \zeta_2^\top & y & \bar{y}_r \end{pmatrix}^\top \tag{4.117}$$

is possible to write it the product between the matrix M and the exogenous signal w

$$\phi\,(\zeta, y, x_m) = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ y \\ \bar{y}_r \end{pmatrix} = \begin{pmatrix} \bar{\xi}_1 \\ Y\,(\bar{z} - L\,C_m\,\Pi\,w) + X\,\bar{\chi}_1 \\ C_m\,\Pi\,w \\ (C_m\,A_m + a_m\,C_m)\,\Pi\,w + C_m\,B_m\,R\,w \end{pmatrix} \tag{4.118}$$

Recalling (4.102), and considering $Q = col\,(Q_1, Q_2)$, the matrix M can be written as

$$M = \begin{pmatrix} Q_2 \\ Y\,(Q_1 - L\,C_m\,\Pi) + X\,Q_2 \\ C_m\,\Pi \\ (C_m\,A_m + a_m\,C_m)\,\Pi + C_m\,B_m\,R \end{pmatrix} \tag{4.119}$$

Consequently the conditions in order to have the (4.114) positive definite and to guarantee the PE on the exogenous signal are:

(i) It holds
$$\int_t^{t+T} w\,(s)\,w^\top\,(s)\,ds \geq c > 0 \tag{4.120}$$

(ii) $\mathrm{rank}\,(M) = \dim(\tilde{\theta})$: this means that the input w must be sufficiently rich in order, greater than or equal to the dimension of $\tilde{\theta}$.

Chapter 5

Passivity

Passivity is a property of dynamical system, well diffused in thermal and electric systems. Is possible to define the passivity as the energy consumption of the system.

The study of incrementally passive system is strictly related to the analysis of passivity.

The tool of incremental passivity is useful in the study of the OR problem, in order to design an internal model-based regulator. The regulator has the task to make the plant to be controlled incrementally passive. Together with an internal model that is incrementally passive, is possible to show how the interconnected system is incrementally passive and the problem is solvable.

The aim of the chapter is to define the main steps in the design of the stabilizer aforementioned, in a general case and when the parameters of the plant are affected by uncertainties. In this case, the implementation of a suitable adaptive law, allows to estimate the parameters of the stabilizer and make the plant incrementally passive.

In the end a numerical example is carried out in order to prove the results obtained.

5.1 The output regulation problem

Consider the system described by

$$\begin{aligned} \dot{x} &= A\,x + B\,u + P\,w \\ e &= C\,x + Q\,w \end{aligned} \tag{5.1}$$

where the state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}$ and the regulation error $e \in \mathbb{R}$. The exogenous input is generated by the exosystem described by

$$\dot{w} = S\,w, \qquad w\,(0) \in \mathcal{W} \tag{5.2}$$

where $w \in \mathbb{R}^m$ and $\mathcal{W}$ is a positively invariant set of initial conditions. In order to solve the classic problem of output regulation, it is necessary to find a stabilizer of the following form

$$\begin{aligned} \dot{\zeta}_s &= A_c\,\zeta_s + B_c\,e \\ u_s &= C_c\,\zeta_s + D_c\,e \end{aligned} \tag{5.3}$$

with $\zeta \in \mathbb{R}^q$, such that all trajectories starting in $(x(0), \zeta(0)) \in \mathbb{R}^{n+q}$ and $w(0) \in \mathcal{W}$ are bounded and $e(t) \to 0$ as $t \to 0$. In order to solve the problem are necessary the following assumptions.

Assumption 5.1.1. *There exist a solution pair* (Π, Ψ) *to the regulator equations*

$$\begin{aligned} \Pi\,S &= A\,\Pi + B\,\Psi + P \\ 0 &= C\,\Pi + Q \end{aligned} \tag{5.4}$$

and such that the steady state of the state x *and input* u *can be described respectively as*

$$\begin{aligned} x_{ss} &= \Pi\,w \\ u_{ss} &= \Psi\,w \end{aligned} \tag{5.5}$$

In order to prove the incremental passivity of the system, it is necessary to have the plant (5.1) incrementally passive, or to design a stabilizer (5.3) able to make the interconnected system incrementally stable.

Due to the fact that we are studying the OR problem, we have an internal model-based stabilizer. Then consider the following assumption.

Assumption 5.1.2. *There exists an internal model unit that is a copy of the exosystem, in the following form*

$$\begin{aligned} \dot{\tau} &= \Phi\,\tau + \alpha\,\Gamma\,\tilde{e} \\ \tilde{v} &= \Gamma^\top\,\tau \end{aligned} \tag{5.6}$$

that is incrementally passive.

The incremental passivity property of such IMU is proven in Section 2.3.1. It is also shown how different incrementally passive systems, through proper interconnections, make the overall interconnected system incrementally passive.

So the main problem in the study of the OR framework is to find the stabilizer.

Consider the system described by (5.1), (5.3) and (5.6), and suppose that the stabilizer (5.3) makes the plant (5.1) incrementally passive.

The plant and the stabilizer are interconnected by

$$u = u_s + v \tag{5.7}$$

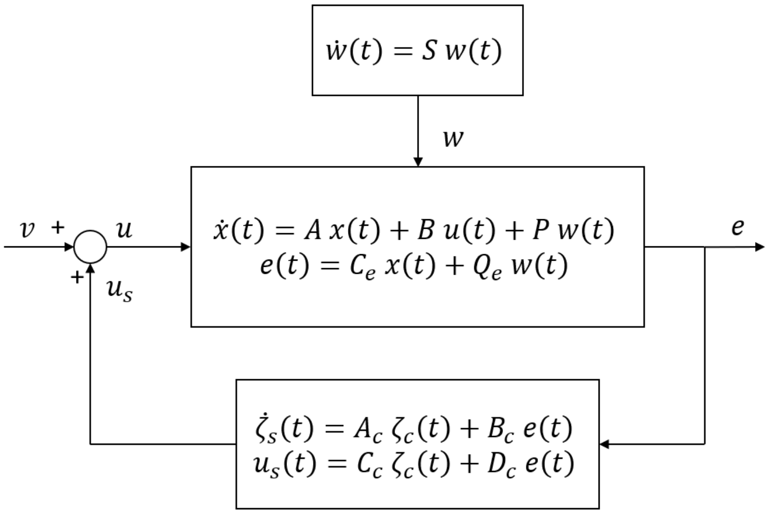


Figure 5.1: Augmented System with control law.

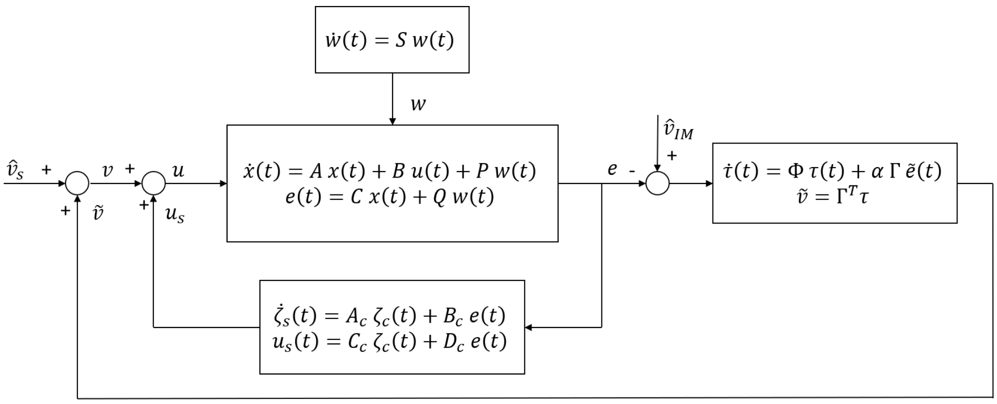


Figure 5.2: Augmented System with IMU.

and the overall system is shown in Figure 5.1, with input v and output e. This system is incrementally passive with a regular storage function.

Next is necessary to interconnect the previous system with the internal model (5.6) through

$$\begin{aligned} v &= \tilde{v} + \hat{v}_s \\ \tilde{e} &= -e + \hat{v}_{IM} \end{aligned} \tag{5.8}$$

The resulting system, with input $\hat{v} := \left(\hat{v}_s^\top, \hat{v}_{IM}^\top\right)^\top$ and output $\hat{e} := \left(e^\top, \tilde{v}^\top\right)^\top$, is the one shown in Figure 5.2.

The overall system shown in Figure 5.2, with these interconnections, is incrementally passive with a regular storage function.

In the end is possible to complete the scheme and closed the external loop, by

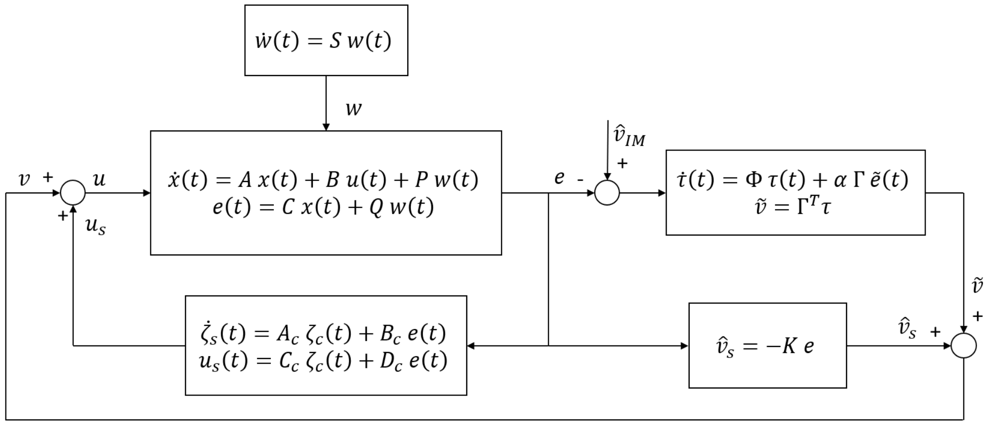


Figure 5.3: Augmented System with all the interconnections.

means of a simple feedback of the form $\hat{v}_s = -K\,e$. The interconnected system is shown in Figure 5.3. The static feedback can be rewritten, considering the input $\hat{v}$, as $\hat{v} = -\hat{K}\,\hat{e}$, where $\hat{K} = \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} \geq 0$.

The definition of the matrix $\hat{K} \geq 0$, means that the system described in Figure 5.2 can be already incrementally passive and the feedback to close the loop is not necessary.

Thanks to Assumption 5.1.1 and Assumption 5.1.2, for $\hat{v} = 0$, the system has a solution which is bounded and in which $e\,(t) = 0$. With Lemma 2.3.6 it is also possible to say that thanks to the static feedback, all the solutions are bounded and $\hat{e}^\top\,(t)\,K\,\hat{e}\,(t) \to 0$ as $t \to +\infty$. Consequently $e^\top\,(t)\,K\,e\,(t) \to 0$ as $t \to +\infty$, and since K is positive definite, $e \to 0$ as $t \to +\infty$.

Then, in order to have an overall system incrementally passive, there are two conditions that have to be satisfied: the first one is that the internal model has to be incrementally passive, and it was demonstrate in Section 2.3.1; the second one is to design a stabilizer that makes (5.1) incrementally passive.

5.2 How to make a system incrementally passive

Consider the system described by

$$\begin{aligned} \dot{x} &= A\,x + B\,u + R\,w \\ e &= C\,x + Q\,w \end{aligned} \tag{5.9}$$

where the state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}$, the regulation error $e \in \mathbb{R}$ and $w\,(t) \in \mathcal{W} \subset \mathbb{R}^m$.

Lemma 5.2.1. *System (5.9) is incrementally passive with a regular storage function if there exist $P = P^\top > 0$ such that*

$$\begin{aligned} &P\,A + A^\top P \leq -\alpha I \\ &P\,B = C^\top \end{aligned} \tag{5.10}$$

with $\alpha > 0$.

Proof. Considering the regular storage function

$$V\,(x_1, x_2) := 1/2\,(x_1 - x_2)^\top P\,(x_1 - x_2)$$

The derivative along any two solution $x_1\,(t)$ and $x_2\,(t)$, corresponding to the inputs $u_1\,(t)$ and $u_2\,(t)$, is

$$\dot{V} = (x_1 - x_2)^\top P\,(A\,x_1 - A\,x_2) + (x_1 - x_2)^\top P\,(B\,u_1 - B\,u_2) \tag{5.11}$$

The second term becomes

$$(x_1 - x_2)^\top P\,B\,(u_1 - u_2) = (C\,x_1 - C\,x_2)^\top (u_1 - u_2) = (e_1 - e_2)^\top (u_1 - u_2) \tag{5.12}$$

The first term can be rewritten as follows

$$(x_1 - x_2)^\top P\,A\,(x_1 - x_2) = \frac{1}{2}\,(x_1 - x_2)^\top J\,(x_1 - x_2) \tag{5.13}$$

where $J := P\,A + A^\top P$.

Considering Lemma 5.2.1, $(x_1 - x_2)^\top P\,(A\,x_1 - A\,x_2) \leq -\alpha I$. Consequently it is possible to write that $\dot{V} \leq (e_1 - e_2)^\top (u_1 - u_2)$, that means that the system (5.9) is incrementally passive.

□

Suppose that the following assumption holds for the system (5.9).

Assumption 5.2.1. *The system (5.9) is minimum phase and has relative degree equal to one.*

It is possible to apply a change of coordinates and write (5.9) in its normal form

$$\begin{aligned} \dot{z} &= A_{11}\,z + A_{12}\,e + R_1\,w \\ \dot{e} &= A_{21}\,z + A_{22}\,e + b\,u + R_2\,w \end{aligned} \tag{5.14}$$

where e,$u \in \mathbb{R}$, $z \in \mathbb{R}^{n-1}$ and $w \in \mathcal{W} \subset \mathbb{R}^m$.

Theorem 5.2.1. *Consider the system (5.14). Suppose that there exists a constant matrix $Q = Q^\top > 0$ and the constant values α_1, α_2, α_3 and α_4, where $\alpha_1 > 0$, and the following inequalities hold*

$$Q\, A_{11} + A_{11}^\top\, Q \leq -\alpha_1\, I_{n-1} \tag{5.15}$$

$$\|A_{12}\| \leq \alpha_2, \qquad \|A_{21}\| \leq \alpha_3, \qquad \|A_{22}\| \leq \alpha_4 \tag{5.16}$$

Then there exists a static output feedback of the form $u = b^{-1}\,(-L\,e)$, such that the system (5.14) is incrementally passive.

Proof. Consider the following control law

$$u = b^{-1}\,(-L\,e + \tilde{v}) \tag{5.17}$$

Together with (5.14), it is possible to rewrite the closed loop system as (5.9), where $x = \left(z^\top, e^\top\right)^\top$, $C = (0, 1)$, $B = (0, 1)^\top$, $Q = 0$ and $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. Replacing u, the matrix A becomes $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} - L \end{bmatrix}$.

Considering the matrix P described by $P = \begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix}$. Notice that the choice of P in this form, allows to satisfy the second condition of (5.10). The resulting Jacobian is

$$J = \begin{bmatrix} Q\, A_{11} + A_{11}^\top\, Q & Q\, A_{12} + A_{21}^\top \\ A_{21} + A_{12}^\top\, Q & A_{22} + A_{22}^\top - L - L^\top \end{bmatrix} = - \begin{bmatrix} F & M \\ M^\top & N \end{bmatrix} \tag{5.18}$$

The Jacobian must be negative definite for all $(z, e) \in \mathbb{R}^n$, and all $w \in \mathcal{W}$. In order to guarantee this result, it is studied the Schur complement of the second block of (5.18).

The Jacobian is negative definite if $F > 0$ and $N - M^\top\, F^{-1}\, M > 0$.

The first condition is satisfied by (5.15). In order to satisfy the second condition, it is necessary to replace the elements with the Jacobian, obtaining

$$\begin{aligned} L + L^\top > & \left(A_{22} + A_{22}^\top\right) \\ & - \left(\left(A_{21} + A_{12}^\top\, Q\right)\left(Q\, A_{11} + A_{11}^\top\, Q\right)^{-1}\left(Q\, A_{12} + A_{21}^\top\right)\right) \end{aligned} \tag{5.19}$$

The inequality can be written in the "worst case", considering the norms of each element, as follows

$$L + L^\top > \left(\|A_{22} + A_{22}^\top\| + \frac{1}{\alpha_1}\|A_{21} + A_{12}^\top\, Q\|\, \|Q\, A_{12} + A_{21}^\top\|\right) \tag{5.20}$$

If the previous relations are satisfied, then the chosen static output feedback ensure the incremental passivity of the plant.

□

5.3 Passivity and Adaptive control

In case the parameters of the plant are affected by uncertainties, is not possible to compute a nominal value of L, and it is possible to implement an adaptive law. The study of this case is analogue to the one done in the previous section.

Consider the system

$$\begin{aligned} \dot{x} &= A(\mu)x + B(\mu)u + R(\mu) \\ e &= C(\mu)x + Q(\mu)w \end{aligned} \tag{5.21}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}$, regulation error $e \in \mathbb{R}$ and $w(t) \in \mathcal{W} \subset \mathbb{R}^m$. The matrices describing the system are considered dependent on a vector μ of uncertainty parameters, with $\mu \in \mathcal{K}_\mu$.

It is possible to rewrite Lemma 5.2.1, for this specific case.

Lemma 5.3.1. *If there exists a matrix $P(\mu) = P(\mu)^\top > 0$ such that*

$$\begin{aligned} P(\mu)A(\mu) + A^\top(\mu)P(\mu) &\leq -I \\ P(\mu)B(\mu) &= C^\top(\mu) \end{aligned} \tag{5.22}$$

then the system (5.21) is incrementally passive with a regular storage function.

Proof. Consider the storage function $V(x_1, x_2) = 1/2\,(x_1 - x_2)^\top P(\mu)(x_1 - x_2)$. Its derivative along any two solutions $x_1(t)$ and $x_2(t)$, corresponding to the inputs $u_1(t)$ and $u_2(t)$ is

$$\dot{V} = (x_1 - x_2)^\top P(\mu)A(\mu)(x_1 - x_2) + (x_1 - x_2)^\top P(\mu)B(\mu)(u_1 - u_2) \tag{5.23}$$

The first component can be written as follows

$$(x_1 - x_2)^\top P(\mu)A(\mu)(x_1 - x_2) = \frac{1}{2}(x_1 - x_2)^\top J(\mu)(x_1 - x_2) \tag{5.24}$$

where $J(\mu) = P(\mu)A(\mu) + A^\top(\mu)P(\mu)$

Considering Lemma 5.3.1, then is possible to say that the first component of (5.23) is lower that zero, and the second component can be rewritten as

$$\begin{aligned} (x_1 - x_2)^\top P(\mu)B(\mu)(u_1 - u_2) &= (C(\mu)x_1 - C(\mu)x_2)^\top (u_1 - u_2) \\ &= (e_1 - e_2)^\top (u_1 - u_2) \end{aligned} \tag{5.25}$$

Then we obtain $\dot{V} \leq (e_1 - e_2)^\top (u_1 - u_2)$, i.e. the system (5.21) is incrementally passive. $\square$

Consider the system (5.21) and the following assumptions

Assumption 5.3.1. *The system described in (5.21) is minimum phase for any* $\mu \in \mathcal{K}_\mu$.

Assumption 5.3.2. *The sign of the high-frequency gain defined by* $b(\mu) = C(\mu)\, B(\mu)$ *is known and constant for any* $\mu \in \mathcal{K}_\mu$ *and* $b(\mu) \neq 0$.

It is possible to apply a change of coordinates and described the system (5.21) in its normal form

$$\begin{aligned} \dot{z} &= A_{11}(\mu)\; z + A_{12}(\mu)\; e + R_1(\mu)\; w \\ \dot{e} &= A_{21}(\mu)\; z + A_{22}(\mu)\; e + b(\mu)\; u + R_2(\mu)\; w \end{aligned} \tag{5.26}$$

where e,$u \in \mathbb{R}$, $z \in \mathbb{R}^{n-1}$ and $w \in \mathcal{W} \subset \mathbb{R}^m$, and to consider the IMU describe in (5.6).

Suppose that the system described in (5.26) is not incrementally passive, then there exists a stabilizer that is able to make it incrementally passive. Consider the control input described in (5.7): it can be rewritten, considering the change of coordiantes, as follows

$$u = b^{-1}\left(-L^*\, e + \tilde{v} - K\, e\right) \tag{5.27}$$

where L^* is the nominal value of the matrix L shown in (5.17), that makes the system (5.26) incrementally passive, $\tilde{v}$ is the output of the internal model and $-Ke$ is the interconnection that close the external loop in Figure 5.3, where $K \geq 0$.

In the case of the presence of uncertainties in the plant to be controlled, it is possible to implement an adaptive law in order to compute the estimate of the matrix $\hat{L}$. We choose an adaptive law of the form

$$\dot{\hat{L}} = \varphi(e) \tag{5.28}$$

with the control law rewritten as

$$u = b^{-1}(\mu)\left(-\hat{L}\, e + \tilde{v} - K\, e\right) \tag{5.29}$$

where $\hat{L}$ is defined as $\hat{L} = \tilde{L} + L^*$.

The demonstration of incremental passivity is similar to the one done in the previous section.

The system (5.26) can be rewritten in the form of (5.21), where $x = \left(z^\top, e^\top\right)^\top$, $C = (0, 1)$, $B = (0, 1)^\top$, $Q = 0$ and

$$A(\mu) = \begin{bmatrix} A_{11}(\mu) & A_{12}(\mu) \\ A_{21}(\mu) & A_{22}(\mu) \end{bmatrix}$$

Replacing (5.27), the matrix $A(\mu)$ becomes

$$A(\mu) = \begin{bmatrix} A_{11}(\mu) & A_{12}(\mu) \\ A_{21}(\mu) & A_{22}(\mu) - L^* \end{bmatrix}$$

Consider the matrix P as

$$P(\mu) = \begin{bmatrix} Q(\mu) & 0 \\ 0 & 1 \end{bmatrix} \tag{5.30}$$

where $Q(\mu) = Q^\top(\mu) > 0$. Choosing P of this form satisfies the second condition of (5.22).

In order to satisfy the first condition of (5.22), it is necessary to compute the Jacobian and it must be negative definite (omitting the parameter vector μ to simplify the notation)

$$\begin{aligned} J &= \begin{bmatrix} Q\,A_{11} + A_{11}^\top Q & Q\,A_{12} + A_{21}^\top \\ A_{21} + A_{12}^\top Q & A_{22} + A_{22}^\top - L^* - L^{*\top} \end{bmatrix} \\ &= -\begin{bmatrix} F & M \\ M^\top & N \end{bmatrix} \end{aligned} \tag{5.31}$$

Thank to the Schur complement, we can say that the Jacobian is negative definite if $F > 0$ and $N - M^\top F^{-1} M > 0$. For the first condition is sufficient that the following relation holds

$$QA_{11} + A_{11}^\top Q \leq -\alpha_1\, I_{n-1} \tag{5.32}$$

For the second condition, as done before, it is possible to write

$$\begin{aligned} L^* + L^{*\top} > &\left(A_{22} + A_{22}^\top\right) \\ &- \Big(\left(A_{21} + A_{12}^\top Q\right)\left(Q\,A_{11} + A_{11}^\top Q\right)^{-1}\left(Q\,A_{12} + A_{21}^\top\right)\Big) \end{aligned} \tag{5.33}$$

or

$$L^* + L^{*\top} > \|A_{22} + A_{22}^\top\| + \frac{1}{\alpha_1}\|A_{21} + A_{12}^\top Q\|\,\|Q\,A_{12} + A_{21}^\top\| \tag{5.34}$$

Due to the fact that it is not possible to compute L^*, the adaptive controller is introduced. The aim is to study the storage function in order to find a suitable adaptive law and guarantee the incremental passivity property.

Consider the storage function

$$V\left(x_1, x_2, \tilde{L}_1, \tilde{L}_2\right) = \frac{1}{2}\left(x_1 - x_2\right)^\top P(\mu)\,\left(x_1 - x_2\right) + \frac{\left(\tilde{L}_1 - \tilde{L}_2\right)^2}{2\,\gamma} \tag{5.35}$$

The time derivative is

$$\dot{V}\left(x_1, x_2, \tilde{L}_1, \tilde{L}_2\right) = \left(x_1 - x_2\right)^\top P(\mu)\left(\dot{x}_1 - \dot{x}_2\right) + \frac{\left(\tilde{L}_1 - \tilde{L}_2\right)}{\gamma}\left(\dot{\tilde{L}}_1 - \dot{\tilde{L}}_2\right) \tag{5.36}$$

The first component can be rewritten, remembering that $x = \left(z^\top, e^\top\right)^\top$ and

$$P(\mu) = \begin{bmatrix} Q(\mu) & 0 \\ 0 & 1 \end{bmatrix}$$

as follows

$$\begin{aligned}(x_1 - x_2)^\top P(\mu)(\dot{x}_1 - \dot{x}_2) = & (z_1 - z_2)^\top Q(\mu)(\dot{z}_1 - \dot{z}_2) \\ & + (e_1 - e_2)(\dot{e}_1 - \dot{e}_2)\end{aligned} \tag{5.37}$$

Replacing (5.26) and (5.29) in the previous equation, the first component becomes

$$\begin{aligned}(z_1 - z_2)^\top Q(\mu)(\dot{z}_1 - \dot{z}_2) = & (z_1 - z_2)^\top Q(\mu) A_{11}(\mu)(z_1 - z_2) \\ & + (z_1 - z_2)^\top Q(\mu) A_{12}(\mu)(e_1 - e_2)\end{aligned} \tag{5.38}$$

and the second

$$\begin{aligned}(e_1 - e_2)(\dot{e}_1 - \dot{e}_2) = & (e_1 - e_2) A_{21}(\mu)(z_1 - z_2) \\ & + (e_1 - e_2) A_{22}(\mu)(e_1 - e_2) \\ & - (e_1 - e_2)\hat{L}_1 e_1 + (e_1 - e_2)\hat{L}_2 e_2\end{aligned}$$

The second component can be rewritten remembering that $\hat{L} = L^* + \tilde{L}$, and that $L_1^* = L_2^*$, as follows

$$\begin{aligned}(e_1 - e_2)(\dot{e}_1 - \dot{e}_2) = & (e_1 - e_2) A_{21}(\mu)(z_1 - z_2) \\ & + (e_1 - e_2) A_{22}(\mu)(\mu)(e_1 - e_2) \\ & - (e_1 - e_2) L^* (e_1 - e_2) \\ & - (e_1 - e_2)\tilde{L}_1 e_1 + (e_1 - e_2)\tilde{L}_2 e_2\end{aligned} \tag{5.39}$$

The resulting time derivative of the storage function is given by (omitting μ)

$$\begin{aligned}\dot{V}\left(x_1, x_2, \tilde{L}_1, \tilde{L}_2\right) = & \begin{bmatrix} (z_1 - z_2) \\ (e_1 - e_2) \end{bmatrix}^\top \begin{bmatrix} Q A_{11} & Q A_{12} \\ A_{21} & A_{22} - L^* \end{bmatrix} \begin{bmatrix} (z_1 - z_2) \\ (e_1 - e_2) \end{bmatrix} \\ & - (e_1 - e_2)\tilde{L}_1 e_1 + (e_1 - e_2)\tilde{L}_2 e_2 \\ & + \frac{\left(\tilde{L}_1 - \tilde{L}_2\right)}{\gamma}\left(\dot{\tilde{L}}_1 - \dot{\tilde{L}}_2\right)\end{aligned} \tag{5.40}$$

Considering the following relations

$$\begin{aligned}\tilde{e} &= e_1 - e_2 \\ e_1 &= \tilde{e} + e_2 \\ e_2 &= e_1 - \tilde{e}\end{aligned} \tag{5.41}$$

is possible to rewrite the second component of (5.40) as follows

$$
\begin{aligned}
&-(e_1-e_2)\tilde{L}_1\,e_1+(e_1-e_2)\,\tilde{L}_2\,e_2=-\tilde{e}\,\tilde{L}_1\,e_1+\tilde{e}\,\tilde{L}_2\,e_2\\
&\qquad=-\frac{1}{2}\,\tilde{e}\,e_1\left(\tilde{L}_1-\tilde{L}_2\right)-\frac{1}{2}\,\tilde{e}\,e_2\left(\tilde{L}_1-\tilde{L}_2\right)-\frac{1}{2}\,\tilde{e}^2\left(\tilde{L}_1+\tilde{L}_2\right)
\end{aligned}
\tag{5.42}
$$

Suppose the following adaptive law

$$
\dot{\tilde{L}}=\dot{\hat{L}}=\gamma\,e^2 \tag{5.43}
$$

with $\gamma>0$. Replacing it in (5.40), the last component can be written as

$$
\frac{\left(\tilde{L}_1-\tilde{L}_2\right)}{\gamma}\left(\dot{\tilde{L}}_1-\dot{\tilde{L}}_2\right)=\left(\tilde{L}_1-\tilde{L}_2\right)\left(e_1^2-e_2^2\right) \tag{5.44}
$$

Replacing (5.41) in the previous equation

$$
\begin{aligned}
&\tilde{L}_1\,e_1^2-\tilde{L}_1\,e_2^2-\tilde{L}_2\,e_1^2+\tilde{L}_2\,e_2^2=\\
&\quad=\tilde{L}_1\,\tilde{e}\,e_1+\tilde{L}_1\,\tilde{e}\,e_2-\tilde{L}_2\,\tilde{e}\,e_1-\tilde{L}_2\,\tilde{e}\,e_2\\
&\quad=\tilde{e}\,e_1\left(\tilde{L}_1-\tilde{L}_2\right)+\tilde{e}\,e_2\left(\tilde{L}_1-\tilde{L}_2\right)\\
&\quad=\frac{1}{2}\,\tilde{e}\,e_1\left(\tilde{L}_1-\tilde{L}_2\right)+\frac{1}{2}\,\tilde{e}\,e_2\left(\tilde{L}_1-\tilde{L}_2\right)\\
&\qquad+\frac{1}{2}\,\tilde{e}\,e_1\left(\tilde{L}_1-\tilde{L}_2\right)+\frac{1}{2}\,\tilde{e}\,e_2\left(\tilde{L}_1-\tilde{L}_2\right)
\end{aligned}
\tag{5.45}
$$

Then, the last two terms in (5.45) can be rewritten as

$$
\begin{aligned}
&\frac{1}{2}\,\tilde{e}\,e_1\left(\tilde{L}_1-\tilde{L}_2\right)+\frac{1}{2}\,\tilde{e}\,e_2\left(\tilde{L}_1-\tilde{L}_2\right)\\
&\quad=\frac{1}{2}\,\tilde{e}\left(\tilde{L}_1-\tilde{L}_2\right)(e_1-e_2)+\tilde{e}\,e_2\left(\tilde{L}_1-\tilde{L}_2\right)
\end{aligned}
\tag{5.46}
$$

The last component of (5.40), considering (5.43), can be then written as

$$
\begin{aligned}
&\frac{\left(\tilde{L}_1-\tilde{L}_2\right)}{\gamma}\left(\dot{\tilde{L}}_1-\dot{\tilde{L}}_2\right)=\\
&+\frac{1}{2}\,\tilde{e}\,e_1\left(\tilde{L}_1-\tilde{L}_2\right)+\frac{1}{2}\,\tilde{e}\,e_2\left(\tilde{L}_1-\tilde{L}_2\right)+\frac{1}{2}\,\tilde{e}^2\left(\tilde{L}_1-\tilde{L}_2\right)+\tilde{e}\,e_2\left(\tilde{L}_1-\tilde{L}_2\right)
\end{aligned}
\tag{5.47}
$$

Replacing (5.42) and (5.47) in (5.40)

$$\dot{V}\left(x_1, x_2, \tilde{L}_1, \tilde{L}_2\right) = \begin{bmatrix} (z_1 - z_2) \\ (e_1 - e_2) \end{bmatrix}^\top \begin{bmatrix} Q\,A_{11} & Q\,A_{12} \\ A_{21} & A_{22} - L^* \end{bmatrix} \begin{bmatrix} (z_1 - z_2) \\ (e_1 - e_2) \end{bmatrix}$$
$$- \frac{1}{2}\,\tilde{e}\,e_1\left(\tilde{L}_1 - \tilde{L}_2\right) - \frac{1}{2}\,\tilde{e}\,e_2\left(\tilde{L}_1 - \tilde{L}_2\right) - \frac{1}{2}\,\tilde{e}^2\left(\tilde{L}_1 + \tilde{L}_2\right)$$
$$+ \frac{1}{2}\,\tilde{e}\,e_1\left(\tilde{L}_1 - \tilde{L}_2\right) + \frac{1}{2}\,\tilde{e}\,e_2\left(\tilde{L}_1 - \tilde{L}_2\right) + \frac{1}{2}\,\tilde{e}^2\left(\tilde{L}_1 - \tilde{L}_2\right) + \tilde{e}\,e_2\left(\tilde{L}_1 - \tilde{L}_2\right) \tag{5.48}$$

The resulting storage function is

$$\dot{V}\left(x_1, x_2, \tilde{L}_1, \tilde{L}_2\right) = \begin{bmatrix} (z_1 - z_2) \\ (e_1 - e_2) \end{bmatrix}^\top \begin{bmatrix} Q\,A_{11} & Q\,A_{12} \\ A_{21} & A_{22} - L^* - \tilde{L}_2 \end{bmatrix} \begin{bmatrix} (z_1 - z_2) \\ (e_1 - e_2) \end{bmatrix}$$
$$+ \tilde{e}\,e_2\left(\tilde{L}_1 - \tilde{L}_2\right) \tag{5.49}$$

It is shown how the time derivative of the storage function has inside the variable errors of $\tilde{L}_1$ and $\tilde{L}_2$, and it is shown how it was not possible to eliminate these terms from the function.

5.4 Numerical Examples

In order to demonstrate the previous conclusions on the design of the stabilizer and the the design of an adaptive law, a numerical example is taken into account. We consider a system of three rooms: in the first room we have the control action and the interested temperature, in the third we have the influence of the external temperature.

5.4.1 Case 1

Consider the system described in Figure 5.4.

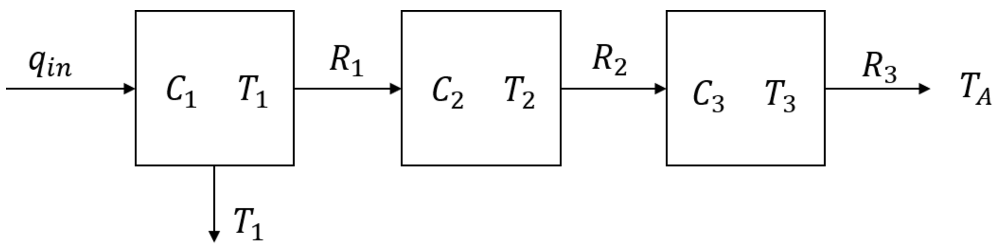


Figure 5.4: Thermal system.

The equations describing the system are the following

$$\begin{aligned}
\underbrace{q_{in}}_{\text{Heat in}} &= \underbrace{C_1\frac{dT_1}{dt}}_{\text{Heat stored}} + \underbrace{\frac{T_1 - T_2}{R_1}}_{\text{Heat out}} \\
\frac{T_1 - T_2}{R_1} &= C_2\frac{dT_2}{dt} + \frac{T_2 - T_3}{R_2} \\
\frac{T_2 - T_3}{R_2} &= C_3\frac{dT_3}{dt} + \frac{T_3 - T_A}{R_3}
\end{aligned} \tag{5.50}$$

and can be rewritten as follows

$$\begin{aligned}
\frac{dT_1}{dt} &= \frac{q_{in}}{C_1} - \frac{T_1}{R_1\,C_1} + \frac{T_2}{R_1\,C_1} \\
\frac{dT_2}{dt} &= \frac{T_1}{R_1\,C_2} - \frac{T_2}{R_1\,C_2} - \frac{T_2}{R_2\,C_2} + \frac{T_3}{R_2\,C_2} \\
\frac{dT_3}{dt} &= \frac{T_2}{R_2\,C_3} - \frac{T_3}{R_2\,C_3} - \frac{T_3}{R_3\,C_3} + \frac{T_A}{R_3\,C_3}
\end{aligned} \tag{5.51}$$

The desired temperature and the external temperature, are respectively the reference and the disturbance generated by an exosystem

$$\dot{w} = S\,w \tag{5.52}$$

where S is a block diagonal matrix of the form

$$S = \begin{bmatrix} 0 & \omega_1 & 0 & 0 & 0 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 & 0 & 0 & 0 \\ 0 & 0 & -\omega_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_3 & 0 \\ 0 & 0 & 0 & 0 & -\omega_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.53}$$

with initial conditions $w_0 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}^\top$, and

$$\begin{aligned}
\omega_1 &= 0.000218 rad/s \\
\omega_2 &= 0.000436 rad/s \\
\omega_3 &= 0.0000727 rad/s
\end{aligned} \tag{5.54}$$

The frequencies are chosen in order to study a realistic case, considering an entire day as time of simulation.

Choosing the states $x = \begin{bmatrix} T_3 \\ T_2 \\ T_1 \end{bmatrix}$, the state space representation can be written as

$$\begin{aligned} \dot{x} &= A\,x + B\,u + P\,w \\ e &= C\,x + Q\,w \end{aligned} \tag{5.55}$$

where

$$A = \begin{bmatrix} -\frac{1}{R_2\,C_3} - \frac{1}{R_3\,C_3} & \frac{1}{R_2\,C_3} & 0 \\ \frac{1}{R_2\,C_2} & -\frac{1}{R_1\,C_2} - \frac{1}{R_2\,C_2} & \frac{1}{R_1\,C_2} \\ 0 & \frac{1}{R_1\,C_1} & -\frac{1}{R_1\,C_1} \end{bmatrix} \tag{5.56}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{C_1} \end{bmatrix}, \qquad P = \begin{bmatrix} 0 & 0 & 0 & 0 & -2.5 & 0 & 12.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{1}{R_3\,C_3} \tag{5.57}$$

and $u = q_{in}$.
The values of the capacitances and resistances are chosen as follows

$$\begin{array}{ll} R_1 = 1.5 & C_1 = 10 \\ R_2 = 0.72 & C_2 = 50 \\ R_3 = 0.60 & C_3 = 45 \end{array}$$

and the resulting matrices A, B and P become

$$A = \begin{bmatrix} -0.0679 & 0.0309 & 0 \\ 0.0278 & -0.0411 & 0.0133 \\ 0 & 0.0667 & -0.0667 \end{bmatrix} \tag{5.58}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}, \qquad P = \begin{bmatrix} 0 & 0 & 0 & 0 & -0.0926 & 0 & 0.463 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.59}$$

We are interested in the state T_1, so in order to have the regulation error on the variable of interest, C and Q are defined as follows

$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 2.5 & 0 & 0.6 & 0 & 0 & 0 & -18 \end{bmatrix} \tag{5.60}$$

The two matrices P and Q are built in order to have as reference and disturbance, the waveforms in Figure 5.5.

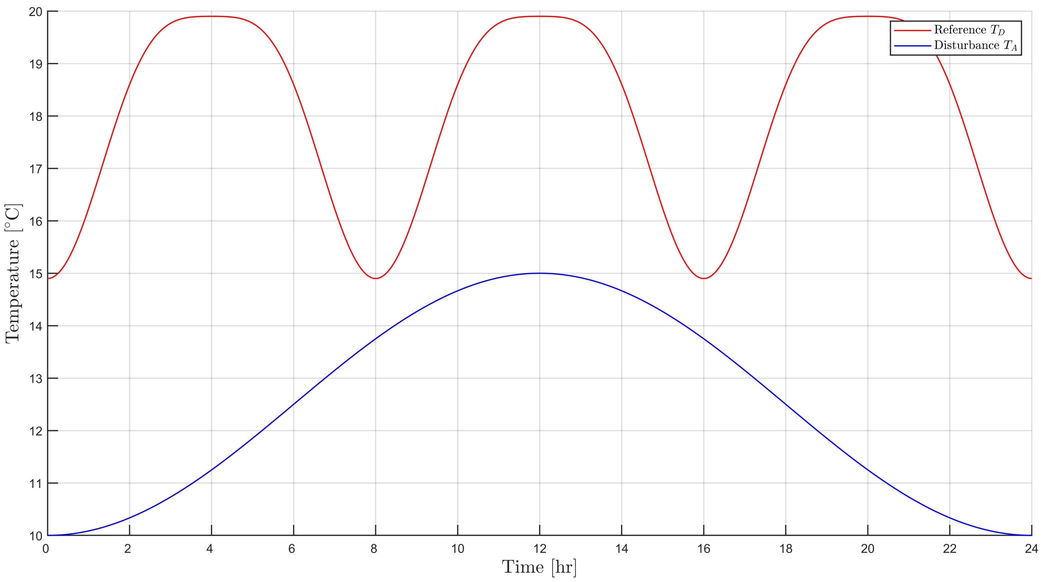


Figure 5.5: Reference signals generated by the exosystem.

The matrix A is stable and its eigenvalues are

$$\begin{aligned} \lambda_1 &= -0.0104 \\ \lambda_2 &= -0.0673 \\ \lambda_3 &= -0.0980 \end{aligned} \tag{5.61}$$

One of the condition for the solvability of the OR problem is that the pair matrices (A, B) is stabilizable. So it is computed the controllability matrix and it is shown that is full rank.

$$M_c = \begin{bmatrix} B & A\,B & A^2\,B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.00004 \\ 0 & 0.0013 & -0.00014 \\ 0.1 & -0.0067 & 0.0005 \end{bmatrix} \tag{5.62}$$

In order to verify if the system is incrementally passive, consider the storage function $V(x_1, x_2) = \dfrac{1}{2}(x_1 - x_2)^\top \mathcal{P}(x_1 - x_2)$, and the matrix $\mathcal{P}$ defined as

$$\mathcal{P} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 45 & 0 \\ 0 & 0 & 10 \end{bmatrix} \tag{5.63}$$

The inequality (2.22) and the conditions (5.10) are satisfied, i.e. the plant is incrementally passive.

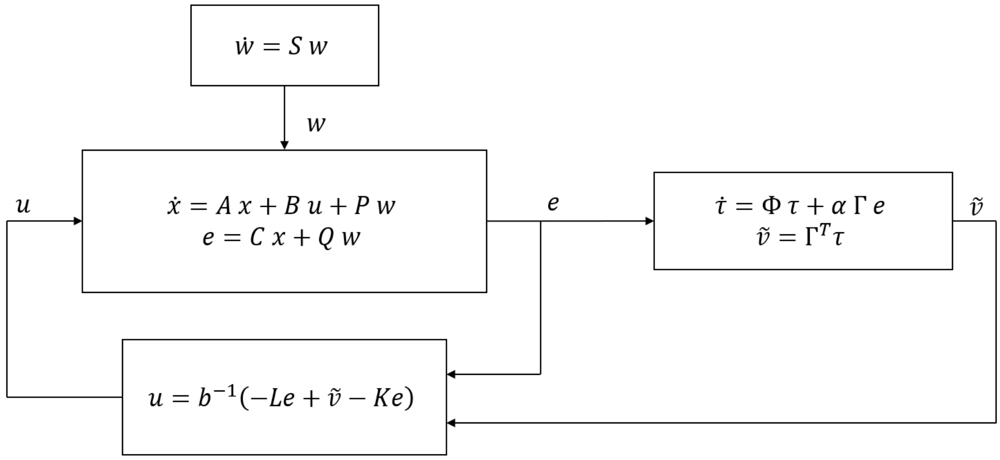


Figure 5.6: Overall block diagram.

As shown in the precious Section, it is chosen a control law of the form

$$u = b^{-1}\,(-L\,e + \tilde{v} - K\,e) \tag{5.64}$$

where e is the regulation error and $\tilde{v}$ is the output of the internal model

$$\begin{aligned} \dot{\tau} &= \Phi\,\tau + \alpha\,\Gamma\,e \\ \tilde{v} &= \Gamma^\top\,\tau \end{aligned} \tag{5.65}$$

where $\Phi = S$, $\Gamma = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$, $\alpha = 1$ and initial condition $\tau_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top$. The reachability matrix related to the IMU

$$M_r = \begin{bmatrix} \Gamma & \Phi\,\Gamma & \Phi^2\,\Gamma & \Phi^3\,\Gamma & \Phi^4\,\Gamma & \Phi^5\,\Gamma & \Phi^6\,\Gamma \end{bmatrix} \tag{5.66}$$

is full rank.

The block scheme of the system is shown in Figure 5.6.

Applying a change of coordinates, it is possible to rewrite the system (5.55) as follows

$$\begin{aligned} \dot{z} &= A_{11}\,z + A_{12}\,e + P_1\,w \\ \dot{e} &= A_{21}\,z + A_{22}\,e + b\,u + P_2\,w \end{aligned} \tag{5.67}$$

where $b = 10$ and

$$A_{11} = \begin{bmatrix} -0.0679 & 0.0309 \\ 0.0278 & -0.0411 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} 0 \\ 0.0133 \end{bmatrix}$$
$$A_{21} = \begin{bmatrix} 0 & 0.0667 \end{bmatrix}, \qquad A_{22} = \begin{bmatrix} -0.0677 \end{bmatrix}$$

The system (5.67) is in normal form, so it is possible to apply Theorem 5.2.1. The first condition (5.15) is satisfied considering a lower and upper bound to the resistance $R_1 = [0.01, 100]$. The matrix A_{11} at lower and upper bound can be written as

$$A_1 = \begin{bmatrix} -0.0679 & 0.0309 \\ 0.0278 & -2.0278 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -0.0679 & 0.0309 \\ 0.0278 & -0.0280 \end{bmatrix} \tag{5.68}$$

and remembering that the matrix $\mathcal{P}$ is

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_1 & 0_{2\times 1} \\ 0_{1\times 2} & \mathcal{P}_2 \end{bmatrix} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 45 & 0 \\ 0 & 0 & 10 \end{bmatrix} \tag{5.69}$$

the following inequalities hold

$$\mathcal{P}_1 A_1 + A_1^\top \mathcal{P}_1 < 0, \qquad \mathcal{P}_1 A_2 + A_2^\top \mathcal{P}_1 < 0 \tag{5.70}$$

and the condition (5.15) is satisfied for $\mathcal{P}_1$.

Taking into account the matrix A with the nominal value of R_1 and replacing the control law (5.64), it is possible to evaluate L able to satisfy condition (5.2.1). The Jacobian can be written as follows

$$J = \begin{bmatrix} \mathcal{P}_1 A_{11} + A_{11}^\top \mathcal{P}_1 & \mathcal{P}_1 A_{12} + A_{21}^\top \mathcal{P}_2 \\ \mathcal{P}_2 A_{21} + A_{12}^\top \mathcal{P}_1 & 2\,\mathcal{P}_2 (A_{22} - L) \end{bmatrix} = - \begin{bmatrix} F & M \\ M^\top & N \end{bmatrix} \tag{5.71}$$

In order to have the Jacobian negative definite, it is necessary to satisfy the two inequalities: $F > 0$ and $N - M^\top F^{-1} M > 0$.

The first condition is satisfied with $\mathcal{P}_1$ defined before. The second one is satisfied for $L > -0.0355$. This means that even if $L = 0$ the system is incrementally passive and the regulation error will decay to zero.

Simulating the system on Simulink, we highlight the comparison between the state and the reference, the behaviour of $\tilde{v}$, the input u and the regulation error that goes to zero.

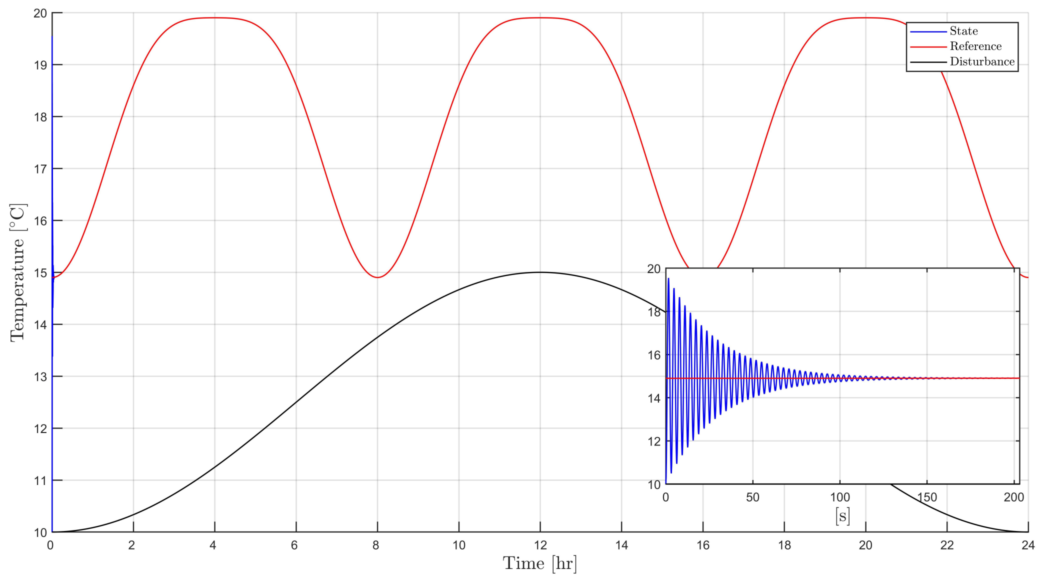


Figure 5.7: Case 1 - Comparison between the state and the reference.

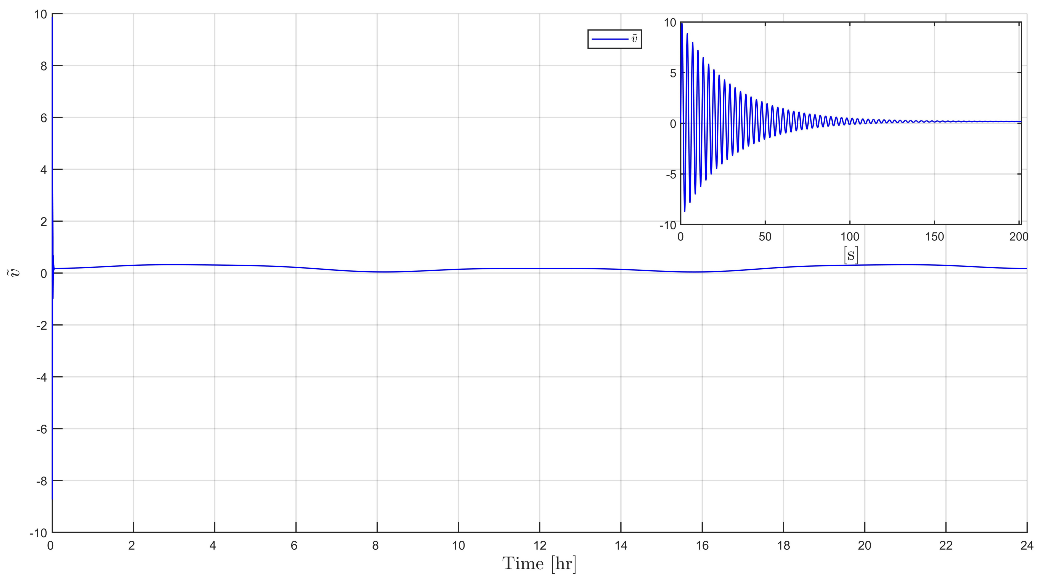


Figure 5.8: Case 1 - Output of the Internal Model Unit.

For each images a zoom of the transient is done, in order to shown the high oscillating behaviour of the signals.

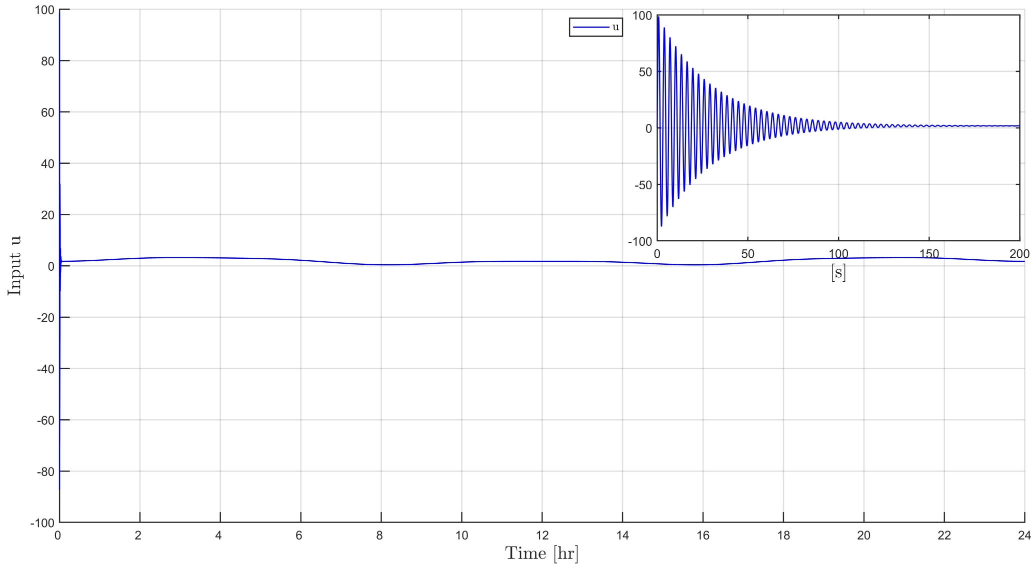


Figure 5.9: Case 1 - Control input.

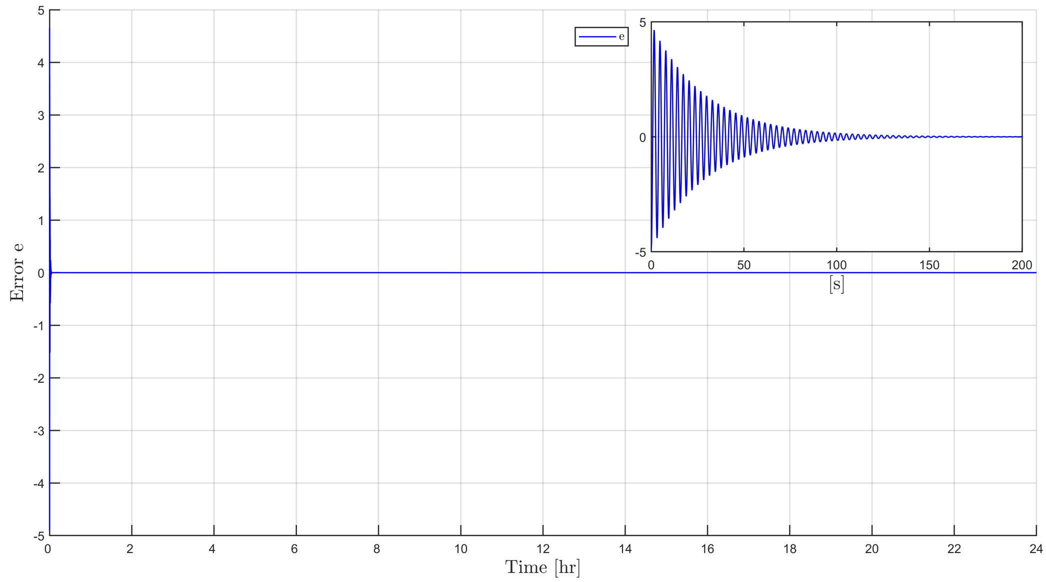


Figure 5.10: Case 1 - Regulation error.

5.4.2 Case 2

It was shown how the previous system does not require the presence of the matrix L in order to make the plant incrementally passive and to obtain regulation of the error to zero. For academical purpose we consider again the same general system, replacing $R_1 = -10$. The resulting matrix A is the following

$$A = \begin{bmatrix} -0.0679 & 0.0309 & 0 \\ 0.0278 & -0.0144 & -0.0133 \\ 0 & -0.0667 & 0.0667 \end{bmatrix} \tag{5.72}$$

All the other matrices are the same as before. Consider again the change of coordinates in (5.67), where $b = 10$ and

$$A_{11} = \begin{bmatrix} -0.0679 & 0.0309 \\ 0.0278 & -0.0144 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} 0 \\ -0.0133 \end{bmatrix}$$
$$A_{21} = \begin{bmatrix} 0 & -0.0667 \end{bmatrix}, \qquad A_{22} = \begin{bmatrix} 0.0677 \end{bmatrix}$$

The problem is solvable if and only if the plant is minimum phase: this means that the zero dynamic is stable and the eigenvalues of A_{11} have negative real part. In the case the plant is not minimum phase, then is not possible to obtain a stabilizer able to solve the OR problem and make the plant incrementally passive.

The eigenvalues of A_{11} are

$$\lambda_1 = -0.0808$$
$$\lambda_2 = -0.0015$$

The eigenvalue of A_{22} is instead positive and its value is $\lambda_3 = 0.0667$. The plant described by these matrices is not incrementally passive, so differently from before, we known that a stabilizer is necessary to make the plant incrementally passive and that $L \neq 0$.

As before, the condition of Theorem 5.2.1 must be always satisfied, in case of bounds on parameters.

Consider the matrix $\mathcal{P}$ as

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_1 & 0_{2\times 1} \\ 0_{1\times 2} & \mathcal{P}_2 \end{bmatrix} = \begin{bmatrix} 45 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 10 \end{bmatrix} \tag{5.73}$$

Again we consider a lower and upper bound on R_1 as $R_1 = [-100, -2]$, obtaining the matrices

$$A_1 = \begin{bmatrix} -0.0679 & 0.0309 \\ 0.0278 & -2.0276 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -0.0679 & 0.0309 \\ 0.0278 & -0.0178 \end{bmatrix} \tag{5.74}$$

and is verified that the following inequalities hold

$$\mathcal{P}_1 A_1 + A_1^\top \mathcal{P}_1 < 0, \qquad \mathcal{P}_1 A_2 + A_2^\top \mathcal{P}_1 < 0 \tag{5.75}$$

In order to make the plant incrementally passive, we compute the nominal value of L^* by solving the inequality (5.33).

The closed-loop system is incrementally passive for $L^* > 4.2341$.

Considering only the plant and the stabilizer, with $L = 4.5$, it is possible to verify that for the matrix A written as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} - L \end{bmatrix}$$

and the matrix $\mathcal{P}$, the conditions (5.10) are satisfied and the system (plant + stabilizer) is incrementally passive.

The same simulation done in the previous case are done, considerint $L = 4.5$, and are shown in the following figures.

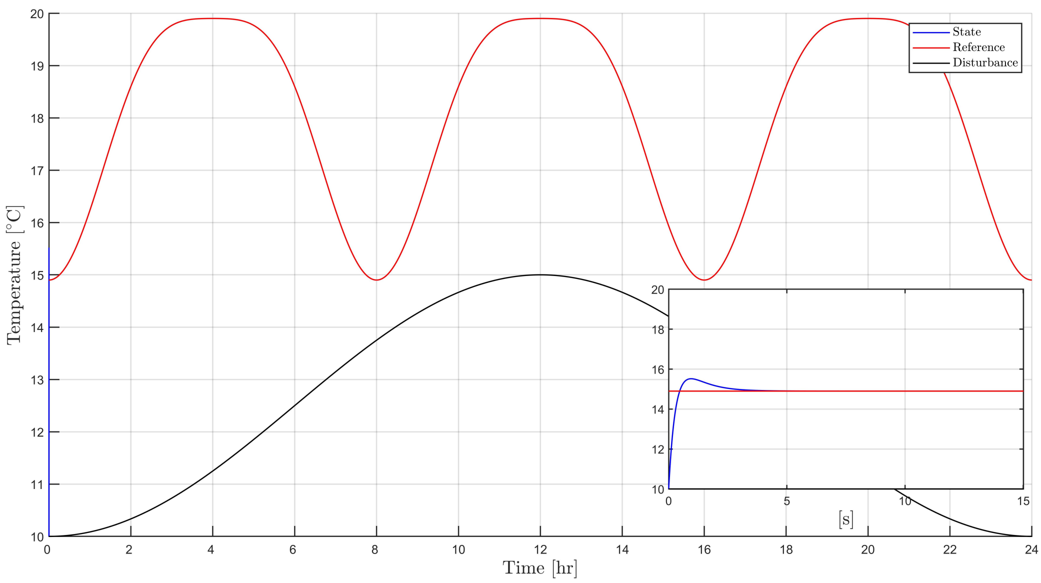


Figure 5.11: Case 2 - Comparison between the state and the reference.

Again a zoom is done in order to highlight the transient.

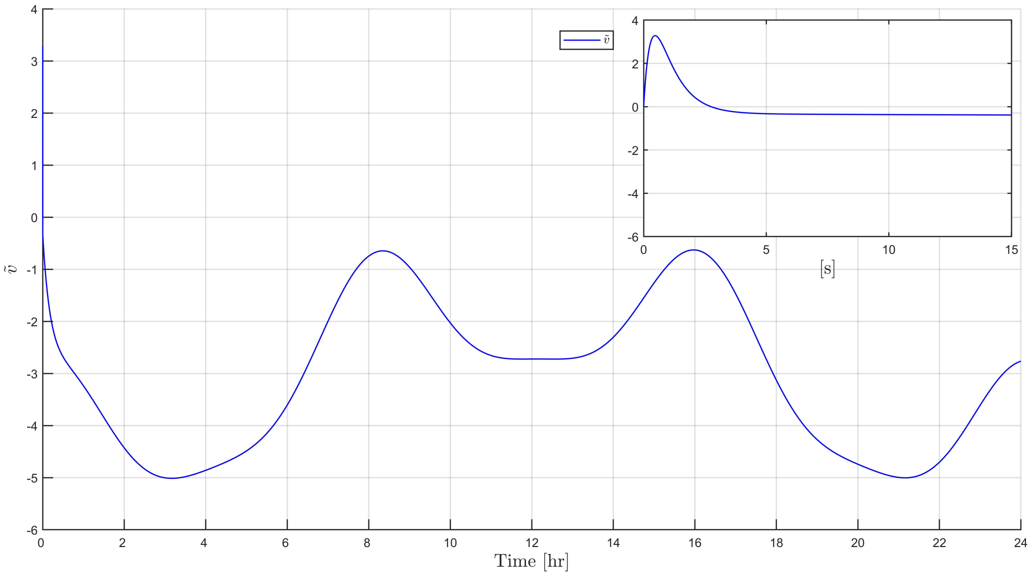


Figure 5.12: Case 2 - Output of the Internal Model Unit.

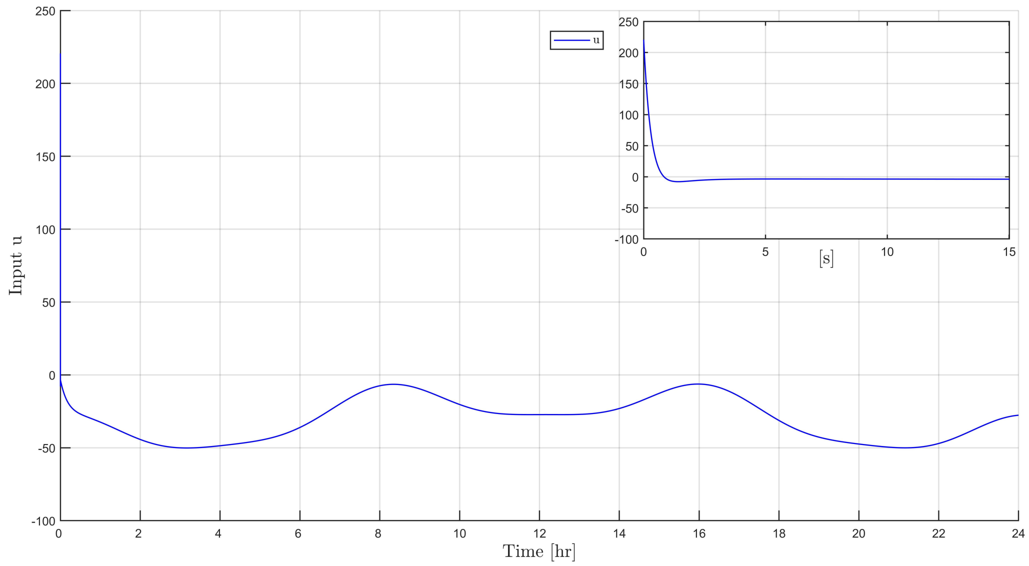


Figure 5.13: Case 2 - Control input.

Adaptive approach

The matrix L^* is the nominal value of the matrix in the case we know everything about the plant. In the presence of uncertainties in the parameters of the system,

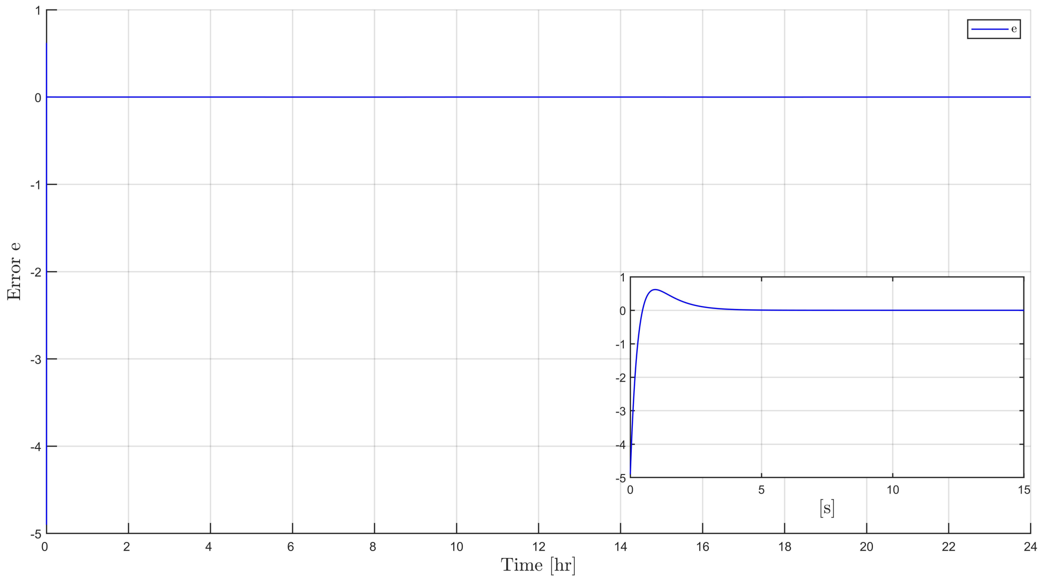


Figure 5.14: Case 2 - Regulation error.

solving (5.33) is not possible. For this problem it is necessary to implement an adaptive law as shown in Section 5.3. The chosen adaptive law has the form

$$\dot{\hat{L}} = \gamma\, e^2 \tag{5.76}$$

where $\gamma > 0$, with initial condition $\hat{L}_0 = 0.0001$.

The simulations are carried out considering $K = 0.001$ and $\gamma = 0.1$.

The comparison between the state and the reference and the resulting input are shown in Figure 5.15 and Figure 5.16.

In Figure 5.17 it is shown the comparison between the nominal value of L^* found in the case we do not have uncertainties on the parameters and the estimated value $\hat{L}$ obtained by the adaptive law (5.76).

It is evident how the estimate $\hat{L}$ reach a value that is lower then $L^* = 4.2341$. This means that the system can be made incrementally passive with a lower value, e.g. $L^* = 1.62$. It is shown also the transient of the regulation error e and the output of the IMU $\tilde{v}$.

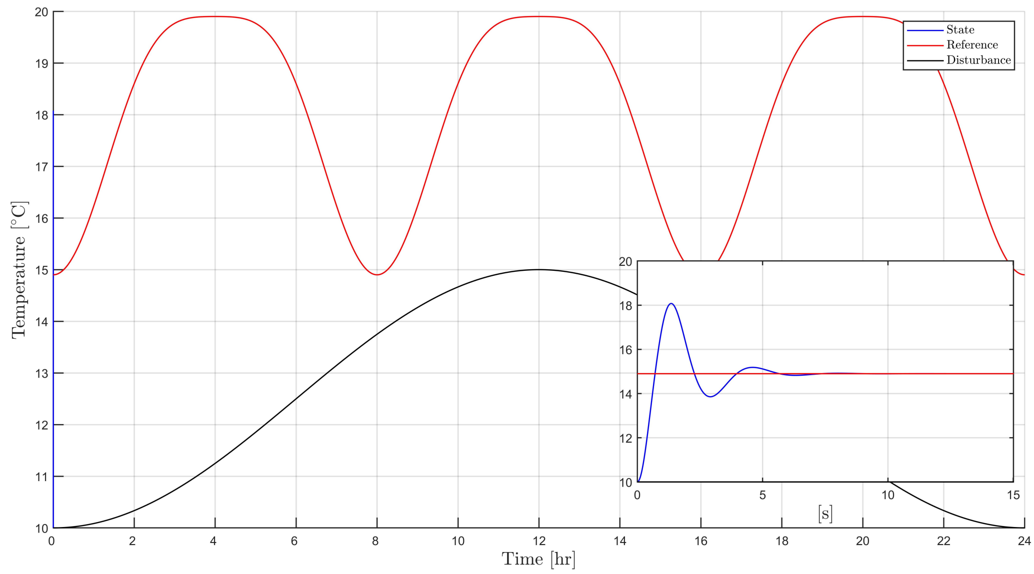


Figure 5.15: Adaptive case - Comparison between the state and the reference.

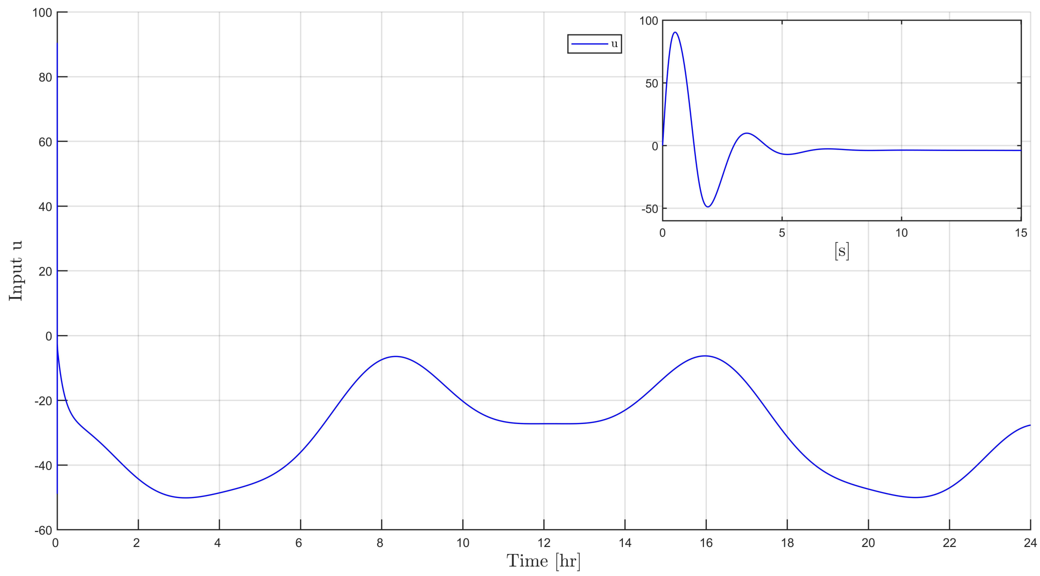


Figure 5.16: Adaptive case - Control input.

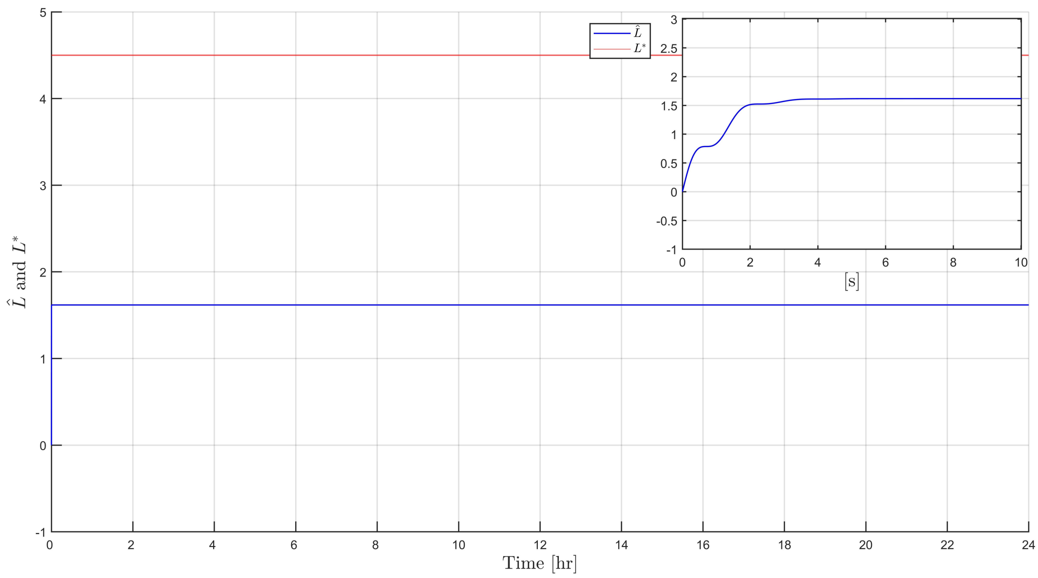


Figure 5.17: Adaptive case - Comparison between the nominal value L^* and the estimated one $\hat{L}$.

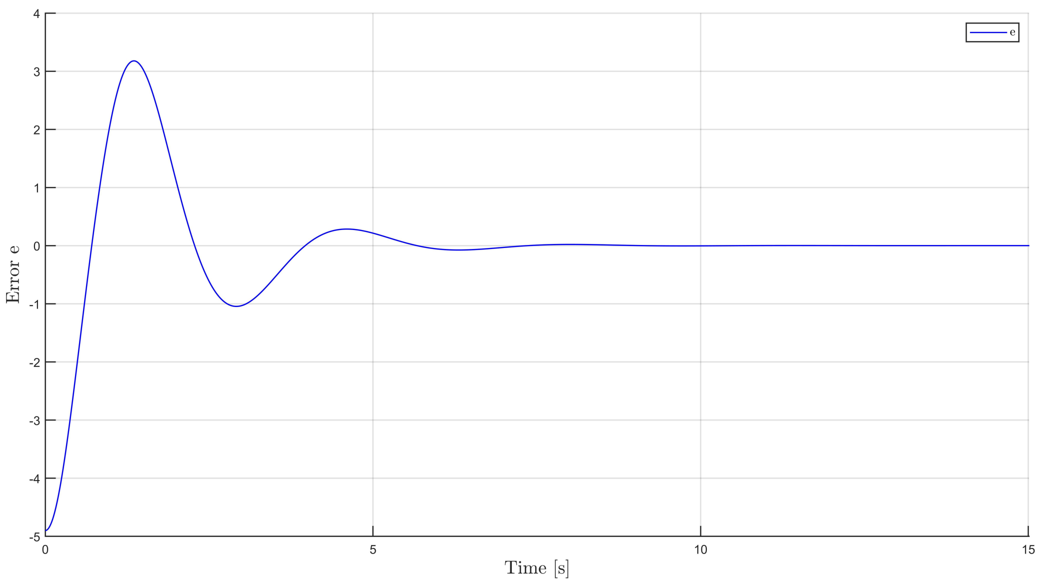


Figure 5.18: Adaptive case - Regulation error.

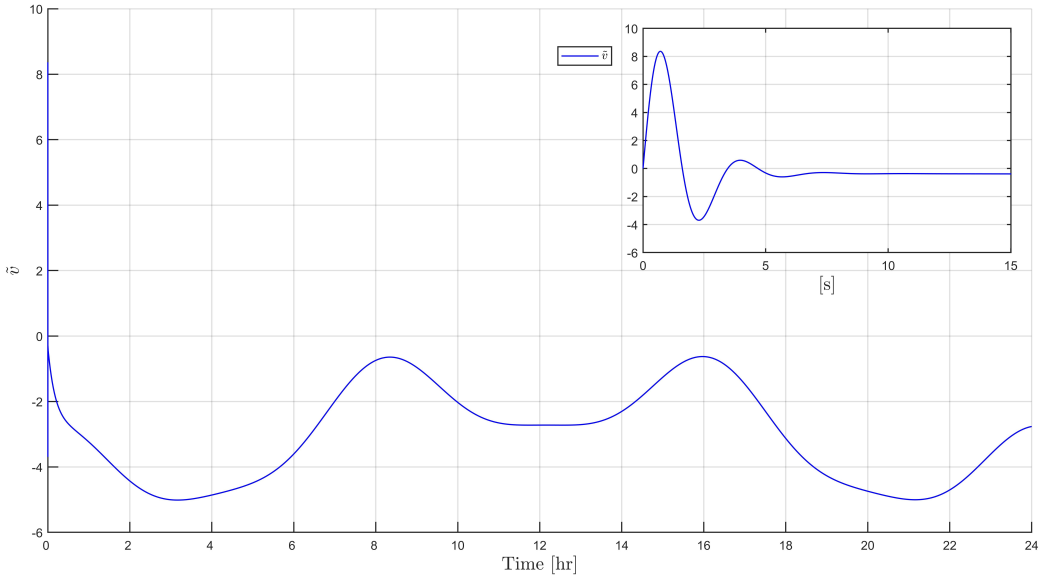


Figure 5.19: Adaptive case - Output of the Internal Model Unit.

Chapter 6

Conclusion

6.1 Comparison between OR and MRAC

The two problems were studied in order to highlight the common points and assumption, and the differences between them. For the comparison the two cases taken into account are: the Error-Feedback problem for the Output Regulation and the Model Reference Adaptive Control shown in Section 4.3.

The main characteristics of the Output Regulation problem can be summarized as follows:

- The main objective of the OR problem is to completely reject the disturbances of a specific family of signals, while tracking the desired reference. In particular the main objective is to have the regulation error going asymptotically to zero.

- The family of signals, including both disturbances and references, is generated by an autonomous system called *exosystem*.

- There is the internal model unit inside the controller: due to the unavailability of the signals generated by the exosystem and the state of the plant, it is necessary to embed a copy of the exosystem in a suitable post-processor.

- The OR problem is robust against uncertainties due to the fact that the stabilizer does not rely on the knowledge of the matrices affected by uncertainties.

For the Model Reference Adaptive Control the main features are:

- The main problem is to make the closed-loop system behave as the reference model in terms of input-output behaviour, and not to follow precisely the reference.

- The input is assumed to be a bounded piecewise differentiable function.
- The system can be considered driven by a non-autonomous exosystem: in the stabilizer there is a copy of the exosystem.
- As the OR problem, the solution given in Section (4.3) is robust against uncertainties: the adaptive control is effective even if the parameters of the plant are unknown.

6.2 Incremental Passivity

In the analysis of the thermal system in Section 5.4 it was proven how the definition and lemmas regarding the incremental passivity for non-linear are satisfied in the case of linear system. It was also shown how the interconnection between different incrementally passive systems makes the closed-loop system incrementally passive. In the second case a stabilizer for the plant was designed, in order to make the plant passive.

The hypothesis in which are considered the plant matrices affected by uncertainties has shown how the property of incremental passivity can be guaranteed even with values of L different from the computer nominal value L^*.

Further researches can be done considering the exosystem affected by uncertainties and implement and adaptive law in order to estimate the parameters of the internal model.

Appendix A

A.1 System representation

It is possible to use different formulations in order to represent a system; one of the most used is the State-Space representation:

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned} \tag{A.1}$$

with

$$A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$$

where $x(t)$, $u(t)$ and $y(t)$ are vectors defined as

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \in \mathbb{R}^n \quad u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix} \in \mathbb{R}^m \quad y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix} \in \mathbb{R}^p \tag{A.2}$$

The system is proper if $D \neq 0$ and strictly proper if $D = 0$: this means that the input does not affect the output directly but through the state.

Another representation is the ARMA one (auto regressive moving average: in this case we have only one equation but with an higher derivative.

$$y^{(n)} + \alpha_1\,y^{(n-1)} + \cdots + \alpha_n\,y = \beta_0\,u^{(n)} + \beta_1\,u^{(n-1)} + \cdots + \beta\,u \tag{A.3}$$

The equation can be rewritten as

$$D(p)\,y(t) = N(p)\,u(t)$$

with $D(p) = p^n + \alpha_1\,p^{n-1} + \cdots + \alpha_n$ and $N(p) = \beta_0\,p^n + \beta_1\,p^{n-1} + \cdots + \beta_n$, with the operator $p^k\,y = \dfrac{dy^{(k)}}{dt^{(k)}}$

Given any transfer function $H(s)$, any state-space model that is both controllable and observable and has the same input-output behaviour as the transfer function, is said to be a minimal realization of the transfer function.

A.2 Coordinate change

Given the following SISO system

$$
\begin{aligned}
\dot{x} &= A\,x + B\,u \\
y &= C\,x
\end{aligned}
\tag{A.4}
$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$. Let denote with r the so called *relative degree*, that is the least integer for which $C\,A^{r-1}B \neq 0$.

Proposition A.2.1. *The r rows of the $r \times n$ matrix*

$$
T_1 = \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{r-1} \end{pmatrix}
\tag{A.5}
$$

are linearly independent. As a consequence, $r \leq n$.

In the case where r is strictly less than n, there exists a matrix $T_0 \in \mathbb{R}^{(n-r)\times n}$ such that

$$
T = \begin{pmatrix} T_0 \\ T_1 \end{pmatrix} = \begin{pmatrix} T_0 \\ C \\ CA \\ \cdots \\ CA^{r-1} \end{pmatrix}
\tag{A.6}
$$

is *nonsingular*.

Proposition A.2.2. *It is always possible to pick T_0 in such a way that the matrix (A.6) is non singular and $T_0\,B = 0$.*

Through the matrix T is possible to define a change of variable as follows

$$
z = T_0\,x, \quad \dot{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \cdots \\ \xi_r \end{pmatrix} = \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{r-1} \end{pmatrix} x
$$

Deriving in time the two previous equation, the resulting equations describing the system are

$$
\begin{aligned}
\dot{z} &= T_0\,\dot{x} = T_0\,(A\,x + B\,u) = T_0\,A\,x + T_0\,B\,u \\
\dot{\xi} &= \hat{A}\,\xi + \hat{B}\left(C\,A^r\,x + C\,A^{r-1}\,B\,u\right) \\
y &= C\,x = \xi_1 = \hat{C}\,\xi
\end{aligned}
\tag{A.7}
$$

The previous equation are still function of the variable x, so it is necessary to express x as a (linear) function of the new variables z and ξ

$$x = M_0\, z + M_1\, \xi$$

where M_0 and M_1 are partition of the inverse of T

$$\begin{pmatrix} M_0 & M_1 \end{pmatrix} \begin{pmatrix} T_0 \\ T_1 \end{pmatrix} = I$$

Setting

$$A_{00} = T_0\, A\, M_0, \quad A_{01} = T_0\, A\, M_1, \quad B_0 = T_0\, B$$
$$A_{10} = C\, A^r\, M_0, \quad A_{11} = C\, A^r\, M_1, \quad b = C\, A^{r-1}\, B$$

the equations in (A.7) becomes

$$\begin{aligned} \dot{z} &= A_{00}\, z + A_{01}\, \xi + B_0\, u \\ \dot{\xi} &= \hat{A}\, \xi + \hat{B}\, (A_{10}\, z + A_{11}\, \xi + b\, u) \\ y &= \hat{C}\, \xi \end{aligned} \tag{A.8}$$

The equations written in (A.8) are referred to as the *normal form* of the system. The term B_0 can be made equal to 0 if is valid the Proposition A.2.2, and the normal form is said to be *strict*. The term b, differently, is always nonzero, and it is referred as the *high-frequency gain* of the system.

A.2.1 Minimum-Phase

Consider now the system described by

$$\begin{aligned} \dot{x} &= A\,(\mu)\, x + B\,(\mu)\, u \\ y &= C\,(\mu)\, x \end{aligned} \tag{A.9}$$

where μ is a vector of uncertain parameters, assumed to be constant and to range over a fixed, and known, compact set $\mathbb{M}$.

Assumption A.2.1. *$A(\mu)$,$B(\mu)$ and $C(\mu)$ are matrices function of the uncertain parameter μ. For every $\mu \in \mathbb{M}$, the pair $(A(\mu), B(\mu))$ is reachable and the pair $(A(\mu), C(\mu))$ is observable. Moreover:*

(i) The relative degree of the system is the same for all $\mu \in \mathbb{M}$.

(ii) The zeros of the transfer function $C(\mu)(s\, I - A(\mu))^{-1}\, B(\mu)$ have negative real part for all $\mu \in \mathbb{M}$.

Generally, if the second point of Assumption A.2.1 is valid for a system, the system is referred as a *minimum-phase system*. As before, by means of a change of coordinates

$$\tilde{x} = \begin{pmatrix} z \\ \xi \end{pmatrix} = \begin{pmatrix} T_0(\mu) \\ T_1(\mu) \end{pmatrix} x$$

where

$$T_1(\mu) = \begin{pmatrix} C(\mu) \\ C(\mu)\,A(\mu) \\ \cdots \\ C(\mu)\,A^{r-1}(\mu) \end{pmatrix}$$

The matrix $T_0(\mu)$ is chosen to satisfy $T_0(\mu)\,B(\mu) = 0$. The system (A.8) becomes

$$\begin{aligned} \dot{z} &= A_{00}(\mu)\,z + A_{01}(\mu)\,\xi \\ \dot{\xi} &= \hat{A}\,\xi + \hat{B}\left[A_{10}(\mu)\,z + A_{11}(\mu)\,\xi + b(\mu)\,u\right] \\ y &= \hat{C}\xi \end{aligned} \tag{A.10}$$

In the end there are two consequences to the Assumption A.2.1:

(i) The high-frequency gain $b(\mu)$ will be always nonzero for all $\mu \in \mathbb{M}$.

(ii) Thanks to the converse Lyapunov Theorem, the eigenvalues of $A_{00}(\mu)$ have negative real part for all $\mu \in \mathbb{M}$. The theorem tells that exists a positive definite symmetric matrix $P(\mu)$, such that

$$P(\mu)\,A_{00}(\mu) + {A_{00}}^T(\mu)\,P(\mu) = -I \quad \text{for all } \mu \in \mathbb{M} \tag{A.11}$$

Relative degree 1

In the case in which $r = 1$ and the system is minimum phase, the (A.8) can be written as follows

$$\begin{aligned} \dot{z} &= A_{00}(\mu)\,z + A_{10}(\mu)\,\xi \\ \dot{\xi} &= A_{10}(\mu)\,z + A_{11}(\mu)\,\xi + b(\mu)\,u \\ y &= \xi \end{aligned} \tag{A.12}$$

A.3 Reachability and Controllability

A.3.1 Reachability

Let consider the system

$$\dot{x}(t) = A\,x(t) + B\,u(t) \tag{A.13}$$

and recall the notation

$$x\left(t\right)=\Phi\left(t\right)x\left(0\right)+\Psi\left(t\right)u_{\left[0,t\right)}\left(\cdot\right) \tag{A.14}$$

where the first element can be considerate equal to zero.

Definition A.3.1. *The reachability set is the set of points (states in $\mathbb{R}^n$) that is possible to reach modifying the input in a certain time t.*

$$\mathbb{R}^+\left(t\right)=\left\{x\in\mathbb{R}^n : x=\Psi\left(t\right)u_{\left[0,t\right)}\left(\cdot\right)\quad \textit{for some}\quad u_{\left[0,t\right)}\left(\cdot\right)\in\mathbb{R}^p\right\} \tag{A.15}$$

with $\mathbb{R}^+\subseteq\mathbb{R}^n,\quad \forall t\geq 0$, where $\mathbb{R}^+=\cup_{t\geq 0}\mathbb{R}^+\left(t\right)$.

Definition A.3.2. *The system described by the pair (A,B) is completely reachable if $\mathbb{R}^+=\mathbb{R}^n$. The set $\mathbb{R}^+$ is defined as*

$$\mathbb{R}^+=Im\left[R\right] \tag{A.16}$$

where R is the reachability matrix

$$R=\left[\begin{array}{ccccc} B & AB & A^2B & \cdots & A^{n-1}B \end{array}\right] \tag{A.17}$$

In order to check the complete reachability , without computing the reachability matrix R, is possible to use the PHB test (Popov-Belevitch-Hautus).

Definition A.3.3 (PHB test). *The system described by the pair (A,B) is completely reachable if and only if*

$$rank\left[\begin{array}{cc} A-\lambda I & B \end{array}\right]=\#rows\ n,\quad \forall\lambda\in\sigma\left(A\right) \tag{A.18}$$

A.3.2 Controllability

Considering the same system of (A.13), it is possible to define the controllability set as follows

Definition A.3.4. *The controllability set is the set of points (states in $\mathbb{R}^n$) that can be controlled to the origin in $[0,t)$.*

$$\mathbb{R}^-\left(t\right)=\left\{x\in\mathbb{R}^n : \Phi\left(t\right)x+\Psi\left(t\right)u_{\left[0,t\right)}\left(\cdot\right)=0\quad \textit{for some}\quad u_{\left[0,t\right)}\left(\cdot\right)\right\} \tag{A.19}$$

with $\mathbb{R}^-\subseteq\mathbb{R}^n,\quad \forall t\geq 0$, where $\mathbb{R}^-=\cup_{t\geq 0}\mathbb{R}^-\left(t\right)$.

Definition A.3.5. *The system is completely controllable if $\mathbb{R}^-=\mathbb{R}^n$. The complete reachability implies the complete controllability, but not the inverse.*

Consider the reachability matrix (A.17), and apply a coordinate change

$$z = T\,x \rightarrow \left(\tilde{A}, \tilde{B}\right) = \left(T\,A\,T^{-1}, T\,B\right) \tag{A.20}$$

obtaining

$$\tilde{R} = T\,R \tag{A.21}$$

where $rank\tilde{R} = rankR, \quad \forall T$ not singular.

Definition A.3.6 (Controllability canonical form). *The system (A, B) is completely reachable if and only if $\exists T_c \in \mathbb{R}^{n\times m}$, not singular, such that*

$$A_c = T_c\,A\,T_c^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 \end{bmatrix} \quad B_c = T_c\,B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{A.22}$$

where the parameters in the last row of A_c are obtained from

$$\det\left(\lambda I - A\right) = \lambda^n + \alpha_1\,\lambda^{n-1} + \cdots + \alpha_n \tag{A.23}$$

The (A.22) is referred to as the ***controllability canonical form****.*

A.4 Observability

A.4.1 Observers

Consider the system

$$\begin{aligned} \dot{x} &= A\,x \\ y &= C\,x \end{aligned} \tag{A.24}$$

with state $x \in \mathbb{R}^n$ and output $y \in \mathbb{R}^p$. The idea behind the observability is to how obtain information on the system from the output. Defining the observable set as

$$\epsilon^+ = \left(\epsilon_{NO}^+\right)_\perp$$

where ϵ_{NO} is the not-observable set defined as follows

$$\epsilon_{NO}^+\left(t_1\right) = \left\{x \in \mathbb{R}^n : c\,\Phi\left(t\right)x \equiv 0, \forall t \in [0, t_1)\right\} \tag{A.25}$$

The not-observable set is the set of state such that simulating the system from that initial state, the observed output in the interval $[0, t_1)$ is identically 0.

Definition A.4.1. *The system described by the pair (A, C) is completely observable if and only*

$$rank\mathcal{O} = n \tag{A.26}$$

where $\mathcal{O}$ is the observability matrix described by

$$\mathcal{O} = \begin{bmatrix} C \\ C\,A \\ \vdots \\ C\,A^{n-1} \end{bmatrix} \tag{A.27}$$

Reduced-order Observer

Consider the following system:

$$\begin{aligned} \dot{x} &= A\,x + B\,u \\ y &= C\,x \end{aligned} \tag{A.28}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$. In the case of the Luemberger observer, we have the dimension of the observer equal to n. In this case we can measure the output y; due to the fact that y is a linear combination of the state, we have at least p information about it. Considering $C = \begin{bmatrix} I_p & 0 \end{bmatrix}$ it means that $y_1, \ldots, y_p$ are coincident with the first p state variables $x_1, \ldots, x_p$. This allows to design an observer with a reduced-order given by $n - p$. It is possible to rewrite the state equation as follows:

$$\dot{x} = \begin{pmatrix} \dot{y} \\ \dot{x}_2 \end{pmatrix} = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \begin{pmatrix} y \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \tag{A.29}$$

It is possible to rewrite the state equations as

$$\begin{aligned} \dot{y} &= A_{11}\,y + A_{12}\,x_2 + B_1\,u \\ \dot{x}_2 &= A_{21}\,y + A_{22}\,x_2 + B_2\,u \end{aligned}$$

and then from the first one define:

$$\dot{y} - A_{11}\,y - B_1\,u = A_{12}\,x_2 := \bar{y} \tag{A.30}$$

We can that write the equation of the observer equation as

$$\dot{\hat{x}}_2 = A_{21}\,y + A_{22}\,\hat{x}_2 + B_2\,u + L\,(\bar{y} - A_{12}\,\hat{x}_2) \tag{A.31}$$

The error dynamics is defined:

$$e = x_2 - \hat{x}_2$$

$$\dot{e} = (A_{22} - L\,A_{12})\,e \tag{A.32}$$

if the matrix inside the parenthesis is Hurwitz, then the error goes to zero and the estimate of the state goes to the real value:

$$e\,(t) \to_{t\to\infty} = 0 \quad \Rightarrow \quad \hat{x}_2\,(t) \to x_2\,(t) \tag{A.33}$$

In order to have this matrix Hurwitz we need to have the pair (A_{22}, A_{12}) observable and detectable: this is valid only if the pair (A, C) is observable and detectable.

This kind of result is obtainable only if $\dot{y}$ is known; in the case we can not measure the value, we need to do a change of variable of the following type:

$$\begin{aligned} \hat{z} &= \hat{x}_2 - L\,y \\ \dot{\hat{z}} &= f\,(\hat{z}, u, y) \end{aligned} \tag{A.34}$$

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