POLITECNICO DI TORINO

Master of Science in Mathematical Engineering

*Master Thesis*

# Chirp-type solutions to Schrödinger Equations


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Academic Year 2018–2019

*To my beloved parents and boyfriend.*
*To all my wonderful grandparents,*
*who are looking out for me from a different,*
*360-degree perspective.*

# Contents

# Acknowledgements

I should like to express my warmest thanks to my Supervisor Fabio Nicola and Co-Supervisor Luigi Giacomo Rodino. They are such very well prepared and worthy Professors, always willing to help me out, give their support and pieces of wise advices. They both are great persons in the insight. They are not the only Professors I am grateful to: I do wish to thank the brilliant Professor Alfio Grillo for being the first to really comprehend my path and figure out my skills and abilities, as well. He went deep into my mind and gave pieces of honest, constructive advices. A special thanking goes to the great Professor Paolo Tilli, too: he helped sorting out the right kind of thesis, showing great care.

As it concerns my family, I would at first like to offer my heartfelt thanks to my precious and devoted parents, Amina and Augusto, just from the bottom of my heart! They have been so very supporting and helping in all sort of feasible ways, all throughout these tough years. I do love them, both. There are no best parents I could ever have. Besides, I wanna thank from the heart my boyfriend Marco for all the joy he brought into my life, all the wrong that he is making right and every dream he will make come true. Wishing us many other of these seven years together. I would finally like to convey my heartfelt thanking to my caring and loving grandparents, Luciana, Alma, Eugenio, Francesco (my gramps "Cicco"), Angela, who are always with me and taking care of me, from another, parallel Dimension. If I am everything I am, I just have to say thanks to them: my life would be nothing without my grammy and gramps standing near.

# 1 Introduction

This thesis is devoted to the study of some aspects of the linear Schrödinger equation. This equation, which was introduced by Schrödinger in 1926, describes the wave function of a quantum-mechanical system. For an historical introduction in quantum mechanics and the physical aspects, we refer to Section 4.

In our work, we focus on the search of solutions to the Schrödinger equation of a particular form: solutions that are, for any fixed time, the so-called "chirp" functions. This family of functions are very important in Signal Theory (cf. for example [1]) and are strictly linked to the Schrödinger equation for the free particle. Indeed, the expression of the fundamental solution to this equation in one dimension involves chirp functions, as explained in Section 3.1, cf. [6], [17].

We restrict our study to Schrödinger equations in one spatial dimension. The system describes the wave function for a single material particle that just moves in one dimension, then. In general, the system for a single particle is governed by the Hamiltonian, $a = T+V$, which is the sum of kinetic energy $T$ and potential energy $V$. In particular, we deal with two instances: the first one is the equation for a free particle, which corresponds to the potential energy $V = 0$, while the second one is the equation for the quantum harmonic oscillator, which corresponds to the potential energy $V(x) = \frac{1}{2}x^2$.

For each case, we have at first studied the homogeneous Cauchy problem for an initial datum $\bar{u}_0$ that is an absolutely convergent series of "chirp functions" of the form

$$\bar{u}_0(x) = \sum_{n=-\infty}^{+\infty} \alpha_n e^{inx^2},$$

for $\alpha_n \in \mathbb{C}$. So, the two problems are

$$\begin{cases} \partial_t u(t,x) - i\partial_{xx} u(t,x) = 0, & (t,x) \in \mathbb{R} \times \mathbb{R} \\ u(0,x) = \bar{u}_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

for the free particle and

$$\begin{cases} \partial_t u(t,x) + \frac{i}{2}(-\partial_{xx} u(t,x) + x^2 u(t,x)) = 0, & (t,x) \in \mathbb{R} \times \mathbb{R} \\ u(0,x) = \bar{u}_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

for the quantum harmonic oscillator. In both cases, we proved the existence of solutions of the form

$$u(t,x) := \sum_{n=-\infty}^{+\infty} \alpha_n u_n(t,x),$$

where $u_n(t,\cdot)$ is a "chirp function", as well. We proved that this solution is classical in a set $\tilde{A} \times \mathbb{R}$, where $\tilde{A}$ is a set of times that excludes the singularities of the functions $u_n(\cdot, x)$. We have also proved and computed the limits, in the sense of distributions, of the solution

$u(t,\cdot)$ as $t \to t_0$, for $t_0 \in \mathbb{R} \setminus \tilde{A}$. Thanks to this result, the solution $u$ can be extended to a continuous function $u : \mathbb{R} \to \mathcal{D}'(\mathbb{R})$. For a detailed statement on these results, we refer to Theorem 5.1, Theorem 5.9, Corollary 5.2 and Corollary 5.10.

The second problem we studied is the Cauchy problem for the non-homogeneuos equation with zero initial datum. Precisely, given a function $v$ that is a uniformly convergent (on compact intervals of time) series of "chirp" functions $v_n(t,x)$, see equation (5.17) and (5.29) in the sequel for the precise form, we considered the problems

$$\begin{cases} \partial_t u(t,x) - i\partial_{xx} u(t,x) = v(t,x), & (t,x) \in \mathbb{R} \times \mathbb{R} \\ u(0,x) = 0, & x \in \mathbb{R}, \end{cases} \tag{1.3}$$

for the free particle equation, and

$$\begin{cases} \partial_t u(t,x) + \frac{i}{2}(-\partial_{xx} u(t,x) + x^2 u(t,x)) = v(t,x), & (t,x) \in \mathbb{R} \times \mathbb{R} \\ u(0,x) = 0, & x \in \mathbb{R}, \end{cases} \tag{1.4}$$

for the quantum harmonic oscillator. In each case, we proved the existence of solutions of the form

$$u(t,x) := \sum_{n=-\infty}^{+\infty} u_n(t,x),$$

where $u_n(t,\cdot)$ is again a suitable "chirp function" that can be calculated in terms of $v_n$. Again, we proved that this solution is classical in a set $\tilde{A} \times \mathbb{R}$, where $\tilde{A}$ is a set of times that excludes the singularities of the functions $u_n(\cdot,x)$. The results are then similar to the ones of the homogeneous case. For a precise statement, we refer to Theorem 5.5, Theorem 5.11, Corollary 5.7 and Corollary 5.13.

Finally, for each of the two dealt Problems, we more generally have what follows below. Taking the sum of the solutions obtained from the Cauchy Problem (1.1) and the Cauchy Problem (1.3), we found a solution for the more general Free particle Cauchy Problem:

$$\begin{cases} \partial_t u(t,x) - i\partial_{xx} u(t,x) = v(t,x), & (t,x) \in \mathbb{R} \times \mathbb{R} \\ u(0,x) = \bar{u}_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.5}$$

thanks to the linearity property of the Cauchy Problem (1.5) itself.

Again, by taking the sum of the solutions obtained from the Cauchy Problem (1.2) and the Cauchy Problem (1.4), we found a solution to the more general Quantum harmonic oscillator Cauchy Problem:

$$\begin{cases} \partial_t u(t,x) + \frac{i}{2}(-\partial_{xx} u(t,x) + x^2 u(t,x)) = v(t,x), & (t,x) \in \mathbb{R} \times \mathbb{R} \\ u(0,x) = \bar{u}_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.6}$$

thanks, again, to the linearity of the Cauchy Problem (1.6) itself.

For precise statements on these results, we refer to Theorem 5.8, as regards the Free particle case, and Theorem 5.14, as it concerns the Quantum harmonic oscillator one.

All the above illustrated results are proved in Section 6. As it concerns their proofs, we just made use of techniques from classical differential calculus. Apart from the elementary theory of distributions, in this thesis we have not used tools from Functional Analysis. It would be interesting to give a structure of Hilbert space to the sums of chirps in the preceeding theorems and, in this context, deduce continuity results. This is left to future investigations.

Moreover, in Section 5.4, we showed how the solution to the free particle problems, given by Theorems 5.1 and 5.5, can be even attained using the Fourier transform representation in $\mathcal{S}'(\mathbb{R})$. Let us point out this approach cannot be carried out for the quantum harmonic oscillator problem, where the pure differential methods, that we here made use of, turned out to be very useful.

Let's stress that our results, attained via classical differential calculus, match with the recent ones on the propagation of the Gabor wave front set for the Schrödinger equation, proved in [5] and [14], achieved using pseudo-differential theory for more general problems also involving a nonlinear term. See Section 5.3. We hope our results will be applied in future works to identify phenomena of anomalous propagation of singularities for nonlinear Schrodinger equations. This depends on the fact that, as observed in [14], powers of chirps generate new Gabor wave front sets.

# 2 Schrödinger equation

The Schrödinger equation is a linear partial differential equation that describes the wave function or state function of a quantum-mechanical system. It is a key result in quantum mechanics and its discovery was a significant landmark in the development of the subject. The equation is named after Erwin Schrödinger, who derived the equation in 1925 and published it in 1926 [20], laying the basis for the work that ended up with his Nobel Prize in Physics in 1933.

In the space of positions $\mathbb{R}^d$, $d = 1, 2, 3$, the Schrödinger equation for the wave function of a single particle of mass $m$, subject to a potential $V : \mathbb{R}^d \to \mathbb{R}$, is

$$i\hbar \frac{\partial}{\partial t} u(t,x) = -\frac{\hbar^2}{2m} \Delta u(t,x) + V(x)u(t,x), \tag{2.1}$$

where $\hbar$ denotes the reduced Planck constant: $\hbar = h/(2\pi)$ and $h$ is the Planck constant (see Section 4.2). In equation (2.1), $\Delta$ is the Laplacian operator defined by

$$\Delta u(t,x) = \sum_{n=1}^{d} \frac{\partial^2}{\partial x_n^2} u(t,x).$$

From the mathematical point of view, the equation (2.1) is widely studied (see [2]).

After a time-rescaling and a redefinition of the potential $V$, the equation (2.1) can be studied in the following form

$$\frac{\partial}{\partial t} u(t,x) = -i\left(-\Delta u(t,x) + V(x)u(t,x)\right). \tag{2.2}$$

## 2.1 Solutions by separation of variables

Among the methods used to construct solutions to the Schrödinger equation, we have the one of separation of variables. This method consists in looking for solutions of the form $u(t,x) = g(t)h(x)$. The equation (2.2) in $\mathbb{R} \times \mathbb{R}^d$, $d = 1, 2, 3$, can be rewritten as

$$g'(t)h(x) = ig(t)\left(\Delta h(x) - V(x)h(x)\right). \tag{2.3}$$

Equation (2.3) has nontrivial solutions if there exist $E \in \mathbb{R}$ such that the following equation

$$-\Delta h(x) + V(x)h(x) = Eh(x) \tag{2.4}$$

has nontrivial solutions $h : \mathbb{R}^d \to \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{C}$ satisfies

$$g'(t) = -iEg(t). \tag{2.5}$$

The solutions to the ordinary differential equation (2.5) are of the form

$$g(t) = Ce^{-iEt}, \qquad \text{for } C \in \mathbb{C}. \tag{2.6}$$

We introduce the operator $H : D(H) \subset L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defined by

$$Hh = -\Delta h + Vh,$$

where $D(H)$ is the domain of $H$. This domain should depend on the form of $V$ and it is a suitable subspace of the Sobolev space $H^2(\mathbb{R}^d)$. The problem of searching for nontrivial solutions of (2.4) has thus been reduced into finding the eigenvalues and the eigenfunctions of the operator $H$, and (2.4) it can be rewritten as

$$Hh - Eh = 0.$$

Then, the general solution of (2.2) can be written as

$$u(t,x) = \sum_{E\in\sigma(H)} C(E) h_E(x) e^{-iEt}, \tag{2.7}$$

where $\sigma(H)$ denotes the spectrum of $H$, $h_E$ is an eigenfunction of $H$ related to the eigenvalue $E$ and $C(E)$ are complex constants as in (2.6).

In this thesis, we do not look into this theory that is broadly studied, indeed. We will just lay out the particular case of the one dimensional quantum harmonic oscillator, right in the following paragraph.

### 2.1.1 Eigenvalues and eigenfunctions of the quantum harmonic operator

In this section, we look at the eigenvalues and eigenfunctions of the one dimensional quantum harmonic operator.

As in the previous paragraph, we consider, for $d = 1$, the operator

$$H : D(H) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

defined by

$$(Hu)(x) = -u''(x) + x^2 u(x),$$

where the domain of the operator $H$ is the space

$$D(H) = H^2_\Gamma := \{u \in L^2(\mathbb{R}) : xu \in L^2(\mathbb{R}), x^2 u \in L^2(\mathbb{R}), xu' \in L^2(\mathbb{R})\},$$

(see, for instance, the book [15] for the definition of these kind of spaces in a general framework).

In order to compute the eigenvalues and eigenfunctions of $H$, we introduce the operators $\Psi_+$ and $\Psi_-$ defined by

$$\Psi_+ u(x) = -u'(x) + xu(x) \quad \text{and} \quad \Psi_- u(x) = u'(x) + xu(x).$$

It is not difficult to show that

$$\Psi_+\Psi_- u = Hu - u. \tag{2.8}$$

Then, from (2.8), it follows that 1 is an eigenvalue of $H$ with associated eigenfunctions the solutions of the equation $\Psi_- u_0 = 0$, which are the scalar multiples of

$$\tilde{u}_0(x) := e^{-x^2/2}.$$

Reasoning by induction, it can be proved that the numbers

$$\lambda_n := 2n + 1, \quad n = 0, 1, 2, \dots$$

are eigenvalues of $H$ with respective eigenfunctions

$$\tilde{u}_n = \Psi_+^n \tilde{u}_0. \tag{2.9}$$

The $L^2(\mathbb{R})$ norm of $\tilde{u}_n$ can be computed. Namely, it holds the following recursive relation

$$\|\tilde{u}_n\|^2_{L^2(\mathbb{R})} = 2n\|\tilde{u}_{n-1}\|^2_{L^2(\mathbb{R})}. \tag{2.10}$$

Since

$$\|\tilde{u}_0\|^2_{L^2(\mathbb{R})} = \sqrt{\pi},$$

it follows that

$$\|\tilde{u}_n\|^2_{L^2(\mathbb{R})} = 2^n n! \sqrt{\pi}.$$

In a similar way, it can be shown that

$$(\tilde{u}_m, \tilde{u}_n)_{L^2(\mathbb{R})} = 0, \qquad \text{for } m \neq n.$$

Then the system $\{\tilde{u}_n : n \in \mathbb{N}\}$ is orthogonal.

The system $\{\tilde{u}_n : n \in \mathbb{N}\}$ can be normalized and we can define the orthonormal system $\{u_n : n \in \mathbb{N}\}$, where $u_n = \tilde{u}_n / \|\tilde{u}_n\|_{L^2(\mathbb{R})}$. From the definition of the operator $\Psi_+$ and (2.9), it follows that the form of $u_n$ is

$$u_n(x) = P_n(x) e^{-x^2/2},$$

where $P_n$ are polynomials that are called Hermite polynomials. Besides, it is not difficult to show that the system $\{u_n : n \in \mathbb{N}\}$ is also complete (see [15] Section 2.2). From the completeness of the system and the Hilbert space theory, it follows that the full spectrum of $H$ is constituted just by the numbers $\lambda_n = 2n + 1$, $n = 0, 1, 2, \dots$.

According to the formula (2.7), we conclude that all the solutions of the equation

$$\frac{\partial}{\partial t} u(t, x) = -i \left( -\frac{\partial^2}{\partial x^2} u(t, x) + x^2 u(t, x) \right), \tag{2.11}$$

obtained by separation of variables, are of the form

$$u(t, x) = \sum_{n=0}^{+\infty} C_n u_n(x) e^{-i(2n+1)t}, \tag{2.12}$$

where $C_n$ are suitable scalar constants. More precisely $C_n$, $n = 0, 1, 2, \ldots$, are the Fourier coefficients with respect to the orthonormal basis $\{u_n : n \in \mathbb{N}\}$ of the initial datum $u(0, \cdot)$. In particular, if $\bar{u}_0 \in L^2(\mathbb{R})$ is given, denoting by $C_n$ its Fourier coefficients (with respect to the orthonormal basis $\{u_n : n \in \mathbb{N}\}$), the formula (2.12) provides a solution of the Cauchy problem for the equation (2.11), with initial datum $\bar{u}_0$.

However, we are interested in studying another type of solutions, namely the ones that are called "chirp solutions". The previous theory cannot be applied since chirp functions do not belong to $L^2(\mathbb{R})$ (see next Section 3.1).

## 2.2 Fundamental solution

### 2.2.1 Fourier transform

For $u \in L^1(\mathbb{R}^d)$, we define its Fourier transform by

$$\mathscr{F}u(\xi) = \hat{u}(\xi) := \int_{\mathbb{R}^d} u(x) e^{-ix\cdot\xi} \, \mathrm{d}x,$$

where $x \cdot \xi$ denotes the standard scalar product in $\mathbb{R}^d$. Since $|\hat{u}(\xi)| \leq \|u\|_{L^1(\mathbb{R}^d)}$, it follows that $\mathscr{F} : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ is a continuous linear map.

The Fourier transform can be extended to a large class of distributions in $\mathbb{R}^d$. To this purpose, we introduce the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

Let $\alpha \in \mathbb{N}^d$ by a $d$-multi-index, the lenght of $\alpha$ is defined by $|\alpha| = \alpha_1 + \cdots + \alpha_d$. If $u \in C^\infty(\mathbb{R}^d)$, we define $D^\alpha u := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$ and $x^\alpha = {x_1}^{\alpha_1} \cdots {x_d}^{\alpha_d}$.

For $k \in \mathbb{N}$ and $u \in C^\infty(\mathbb{R}^d)$, we define the seminorm

$$[u]_k := \sup_{|\alpha| \leq k,\, x \in \mathbb{R}^d} (1 + |x|)^k |D^\alpha u(x)|, \qquad \text{and } |u|_k := \sup_{|\alpha| \leq k,\, x \in \mathbb{R}^d} |D^\alpha u(x)|.$$

The Schwartz space is defined by

$$\mathcal{S}(\mathbb{R}^d) = \{u \in C^\infty(\mathbb{R}^d) : [u]_k < +\infty, \, \forall \, k \in \mathbb{N}\},$$

endowed with the topology induced by the family of seminorms $\{[\cdot]_k : k \in \mathbb{N}\}$. Similarly, we define the space

$$\mathcal{D}(\mathbb{R}^d) = \{u \in C^\infty(\mathbb{R}^d) : |u|_k < +\infty, \, \forall \, k \in \mathbb{N}\},$$

endowed with the topology induced by the family of seminorms $\{|\cdot|_k : k \in \mathbb{N}\}$.

$\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{D}(\mathbb{R}^d)$ are Fréchet space and $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$.

The space $\mathcal{D}'(\mathbb{R}^d)$ of the continuous linear functionals on $\mathcal{D}(\mathbb{R}^d)$ is called space of distributions. The space $\mathcal{S}'(\mathbb{R}^d)$ of the continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$ is called space of temperated distributions. The space $\mathcal{S}'(\mathbb{R}^d)$ can be identified with a subspace of $\mathcal{D}'(\mathbb{R}^d)$: to be precise, the set of distributions $T \in \mathcal{D}'(\mathbb{R}^d)$ such that

$$|\langle T, \phi \rangle| \leq C[\phi]_k, \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^d),$$

for some $C > 0$ and some $k \in \mathbb{N}$.

We define the Dirac delta distribution by $\langle \delta, \phi \rangle := \phi(0)$, $\forall \phi \in \mathcal{D}(\mathbb{R}^d)$. Clearly $\delta \in \mathcal{S}'(\mathbb{R}^d)$.

The following result holds:

**Theorem 2.1** *The Fourier transform $\mathscr{F}$ maps $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$. The map $\mathscr{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is linear, continuous, invertible with continuous inverse. Furthermore, the map $\mathscr{F}^{-1} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is represented by*

$$\mathscr{F}^{-1}\phi(x) = (2\pi)^{-d}\mathscr{F}\phi(-x) = (2\pi)^{-d}\int_{\mathbb{R}^d}\phi(\xi)e^{i\xi\cdot x}\,\mathrm{d}\xi.$$

We here recall the notion of transposition of linear operators. If $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is linear and continuous, then the operator ${}^tA : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, defined by

$$\langle {}^tAu, \phi \rangle := \langle u, A\phi \rangle,$$

is linear and continuous.

The transpose ${}^t\mathscr{F}$ of the operator $\mathscr{F}$ is still called Fourier transform operator. Since it is easy to check that ${}^t\mathscr{F}u = \mathscr{F}u$ for any $u \in L^1(\mathbb{R}^d)$, in order to denote ${}^t\mathscr{F}u$, we still use the notation $\mathscr{F}u$ for $u \in \mathcal{S}'(\mathbb{R}^d)$ .

The following properties hold:

$$(iD)^\alpha \mathscr{F}u = \mathscr{F}(x^\alpha u(x)), \qquad (i\xi)^\alpha \mathscr{F}u = \mathscr{F}(D^\alpha u). \tag{2.13}$$

In particular, it holds that $-\xi_i^2\mathscr{F}u(\xi) = \mathscr{F}(\partial^2_{x^i x^i}u)(\xi)$, so

$$-|\xi|^2\mathscr{F}u(\xi) = \mathscr{F}(\Delta u)(\xi). \tag{2.14}$$

Moreover, it is true that

$$\mathscr{F}(u * \psi) = \mathscr{F}(u)\mathscr{F}(\psi), \qquad \forall u \in \mathcal{S}'(\mathbb{R}^d),\ \forall \psi \in \mathcal{S}(\mathbb{R}^d). \tag{2.15}$$

Finally, we recall the Fourier transform of the function $x \mapsto e^{-z|x|^2}$ for $z \in \mathbb{C}$, such that $\mathcal{R}e(z) \geq 0$, where $\mathcal{R}e(z)$ denotes the real part of $z$ (notice that this function belongs to $\mathcal{S}'(\mathbb{R}^d)$). We have

$$\mathscr{F}(e^{-z|x|^2})(\xi) = \left(\frac{\pi}{z}\right)^{d/2} e^{-\frac{|\xi|^2}{4z}}, \tag{2.16}$$

where

$$z^{-d/2} = |z|^{-d/2}e^{-i\theta d/2} \quad \text{if } z = |z|e^{i\theta}, \quad \text{for } \theta \in [-\pi/2, \pi/2]. \tag{2.17}$$

### 2.2.2 Fourier transform, free particle equation and fundamental solution.

In this Subsection, we describe how to obtain the so-called fundamental solution to the free particle Schrödinger equation in $\mathbb{R}^d$. We consider the Cauchy problem

$$\frac{\partial}{\partial t}u(t,x) = i\Delta u(t,x), \qquad u(0,\cdot) = u_0(\cdot), \tag{2.18}$$

for a given initial datum $u_0 \in \mathcal{S}(\mathbb{R}^d)$. Formally assuming that $u(t,\cdot) \in \mathcal{S}'(\mathbb{R}^d)$ for any $t \in \mathbb{R}$, and denoting by $\hat{u}(t,\xi)$ the Fourier transform of $u(t,\cdot)$, by (2.14) the problem (2.18) can be rewritten in the following form:

$$\frac{\partial}{\partial t}\hat{u}(t,\xi) + i|\xi|^2\hat{u}(t,\xi) = 0, \qquad \hat{u}(0,\xi) = \hat{u}_0(\xi). \tag{2.19}$$

For any fixed $\xi \in \mathbb{R}^d$, the problem (2.19) is a Cauchy problem for an ordinary differential equation, which has the unique solution

$$\hat{u}(t,\xi) = e^{-i|\xi|^2 t}\hat{u}_0(\xi).$$

By (2.16) with $z = \frac{-i}{4t}$, we have that

$$e^{-i|\xi|^2 t} = \hat{k}(t,\xi), \qquad \text{where} \qquad k(t,x) := (4\pi i t)^{-d/2} e^{i|x|^2/(4t)}.$$

Using the convolution formula for the Fourier transform (2.15), we get that

$$u(t,x) = k(t,x) * u_0(x). \tag{2.20}$$

The representation formula for the solution still holds for more general initial data: it is sufficient that the convolution in (2.20) makes sense (for instance if $u_0 \in L^1(\mathbb{R}^d)$). More generally, for $u_0 \in \mathcal{S}'(\mathbb{R}^d)$, we can take by definition $u(t,x) = \mathscr{F}^{-1}(e^{-i|\xi|^2 t}\hat{u}_0(\xi))$.

The function $k : (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^d \to \mathbb{C}$ is called *Fundamental solution* of the Schrödinger equation for the free particle.

We conclude this Section by proving that $\lim_{t\to 0} k(t,\cdot) = \delta$ in $\mathcal{S}'(\mathbb{R}^d)$ and then in $\mathcal{D}'(\mathbb{R}^d)$, as well. To this end, we have to prove that

$$\lim_{t\to 0}\int_{\mathbb{R}^d} k(t,x)\varphi(x)\,\mathrm{d}x = \varphi(0), \qquad \forall\, \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{2.21}$$

Using the Fourier transform, we have

$$\int_{\mathbb{R}^d} k(t,x)\varphi(x)\,\mathrm{d}x = (2\pi)^{-d}\int_{\mathbb{R}^d}\hat{k}(t,\xi)\hat{\varphi}(\xi)\,\mathrm{d}\xi = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-it\xi^2}\hat{\varphi}(\xi)\,\mathrm{d}\xi. \tag{2.22}$$

By dominated convergence, we get

$$\lim_{t\to 0}\int_{\mathbb{R}^d} e^{-it\xi^2}\hat{\varphi}(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^d}\hat{\varphi}(\xi)\,\mathrm{d}\xi. \tag{2.23}$$

By the inversion formula for the Fourier transform, we have

$$\int_{\mathbb{R}^d}\hat{\varphi}(\xi)\,\mathrm{d}\xi = (2\pi)^d\varphi(0)$$

and we conclude.

# 3 Frequency modulated signals: Chirp

## 3.1 Chirp functions

We call chirp function an application $v : \mathbb{R} \to \mathbb{C}$ of the form $v(x) = \alpha e^{i\beta x^2}$, for $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{R}$. Recalling the Euler's formula, it holds that $v(x) = \alpha \cos(\beta x^2) + i\alpha \sin(\beta x^2)$ and then

$$v(x) = \alpha \cos((\beta x)x) + i\alpha \sin((\beta x)x). \tag{3.1}$$

From the representation (3.1), it follows that a chirp function is a wave function of amplitude $|\alpha|$, with frequency that linearly increases as $|x|$ grows up. Precisely, the angular frequency of $v$ at the position $x$ is $|\beta x|$.

We observe that $v$ is not in $L^1(\mathbb{R}^d)$ but it it can be identified with an element of $\mathcal{S}'(\mathbb{R})$.

In Signal Theory, the variable $x$ stands for time and it is denoted by $t$. The "chirp" terminology comes from the very Signal Theory and it reminds to a bird's twitter that is short and its frequency is increasing with respect to time. Actually, common signals are bounded in time, whereas we here deal with signals defined for all the times.

Chirp signals are very helpful and used in lasers, radars, sonars and other physical applications.

As we have seen in Subsection 2.2.2, the fundamental solution to the Schrödinger equation for the one dimensional free particle is of the form

$$k(t,x) := (4\pi i t)^{-1/2} e^{\frac{ix^2}{4t}}, \qquad t \in \mathbb{R} \setminus \{0\}, \quad x \in \mathbb{R}. \tag{3.2}$$

That is, for any fixed $t \neq 0$, the function $x \mapsto k(t,x)$ is a chirp function.

Moreover, for an initial datum $u_0 \in L^1(\mathbb{R})$, the solution to the Cauchy problem of the Schrödinger equation for the one dimensional free particle is given by

$$u(t,x) = (k(t,\cdot) * u_0(\cdot))(x), \qquad t \in \mathbb{R} \setminus \{0\}, \quad x \in \mathbb{R}. \tag{3.3}$$

Since

$$(k(t,\cdot) * u_0(\cdot))(x) = \int_{\mathbb{R}} k(t, x-y) u_0(y)\, \mathrm{d}y, \tag{3.4}$$

the solution $x \mapsto u(t,x)$ can be seen as a superposition of chirp functions, as well. Chirp functions are a good class of functions where the Schrödinger equation can be cast, then.

Since the importance of these Chirp functions, let us just spend few words of interest as it concerns their usage in applications.

## 3.2 Basics of signal processing and transmission

From an engineering point of view, the variation of a physical quantity in time $x(t)$ can be seen as a signal. In this regards, the most common taxonomy (see [12], [11]) for the signal features is to distinguish two domains for analyzing a signal: the most obvious is

the time domain, which deals with the features and the changes of the waveform in time, but one can also analyze the signal in the frequency domain.
The frequency domain is the dual of the time domain and the bridging tool used to pass from one domain to another one is the Fourier transform (and its dual, the inverse Fourier transform or anti-transform): the idea that justifies such a dual relation is the fact that every signal, in time, can be reconstructed as a linear combination of an infinite number of fixed-frequency harmonics.
In the practice of numerical signal processing, one can only use a finite number of harmonics to reconstruct the signal and that introduces some spurious effects (e.g. ripples nearby the zones where an abrupt change of time derivative occurs) known as Gibbs phenomenon. That happens because of the deep entanglement between the support of the signal-function in time and its spectral content (e.g. harmonics) in the frequency domain: when limiting the spectrum of the frequency used for the reconstruction of the signal (namely, limiting the signal bandwidth), the richness of the reconstructed signal is lowered. Thus, the inverse Fourier transform is not able to grasp the higher frequencies (for example) that are responsible for the representation, in the frequency domain, of the abrupt time variation.

When the signal is hence transmitted, beside the distortion introduced by a reconstruction based on a limited number of harmonics, the propagation channel that is needed for transmission introduces a further bias in the original signal (together with the perturbation introduced by the usage of the intermediate blocks of transmitting apparatus) and that impacts on the quality of the received signal, introducing noise. In general, said $x(t)$ the transmitted signal, the received one is expressed by:

$$x(t) + n(t),$$

where $n(t)$ is the noise introduced (usually by the transmission channel, but not solely by it) which is usually modeled as a random signal with a given distribution. The most general assumption (zero-knowledge assumption) for the noise is to be uniformly distributed over all the frequencies, so that all the frequencies are affected in the same (random) way by the channel distorsions. In signal processing, such a kind of noise modeling (it is not the only one) is called AWGN (Additive White Gaussian Noise), which also includes the assumption on $n(t)$ to be additive with respect to $x(t)$ (see [16]).

The elective way to measure the impact of noise on the original signal is the Signal-to-Noise Ratio (SNR). So, to evaluate the efficiency of a transmission device/channel, or, more generally, the prevalence of the noise with respect to the signal, one can evaluate the ratio of powers between $x(t)$ and $n(t)$:

$$SNR = \frac{P_{signal}}{P_{noise}} = \frac{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, \mathrm{d}t}{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |n(t)|^2 \, \mathrm{d}t},$$

where $T$ is the span of the time window where the power is measured.

However, outside the ideal framework and except few selected applications, thinking of just transmitting the signal over a channel with no further processing is pretty naive and

it generates a number of issues on the effectiveness of the transmission. Above all, the description of the dispersion operated by the channel is estimated in a statistical manner and it is subject to a number of real-world parameters tunings, which are often difficult to measure and out of control (for example, just think of the way rain or humidity can warp a broadcast signal).
In order to increase the SNR and reduce the impact of the fading effects during the transmission, a way to fortify the signal and make it more resilient to the noise is necessary.

### 3.2.1 Chirps in signal processing

In order to reduce the impact of the channel noise over the signal and being able to optimize the power used in transmission, the modulation of the broadcast signal over the channel is required. To modulate a signal, one needs a carrier signal that is modified in shape, according to the information content of the actual signal: in other words, the modulation changes the shape of the carrier as to encode the information one is interested in transmitting.
For the sake of simplicity, one can generally think of encoding the information, that needs to be carried, in the amplitude of the carrier signal, thus obtaining an amplitude modulation (AM). Otherwise, one can deal with the frequency of the carrier waveform, performing a frequency modulation (FM).

In applications like remote sensing or satellite communication, there is the need to have a good SNR for broadcasting a signal across very long distances. However, in order to avoid very costly transmitters with high peak power, the need for a specific frequency modulation has arisen. The elective modulation used in these applications is the one of having a sine wave carrier with a linearly increasing frequency. Such a linear frequency modulation (LFM) requires a carrier which takes the name of chirp, in analogy with the chirping of birds, that produce a sound which progressively increases in frequency.

### 3.2.2 Chirps and Fourier optics

One of the applications of Chirp signals shows up in Optic theory (see [7]). For instance, Chirps are used in optics when calculating the focal lenght in the case of image formation through a thin lens.

Let $t(x)$ be the illumination field just before the light is conveyed into a thin lens (a lens is defined to be thin when its thickness, compared to the curvature of the surface, is negligible). One can take into account the effect of the lens via the factor

$$e^{\frac{-i\pi}{\lambda f}x^2},$$

where $\lambda$ is the wave length and $f$ is the lens focal lenght. We thus obtain the field of the distribution of the light after the lens as a product function, $U(x)$, where $x$ accounts for the transverse coordinate with respect to the travelling direction,

$$U(x) = t(x)\beta(x)e^{\frac{-i\pi}{\lambda f}x^2}.$$

The function $\beta(x)$ is a piecewise constant function which controls whether the focus is inside the lens aperture: if the image is inside the aperture, therefore in good focus, $\beta(x) = 1$, otherwise, as the image does not pass the aperture, $\beta(x) = 0$. Here, the Fresnel transform (see [10]) comes in the calculation of the light distribution (which stands for the "image formation" concept) at a given distance $z$ after the lens.
The Fresnel transform of the $U(x)$ distribution is expressed by:

$$U_f(u) = \frac{1}{\sqrt{i\lambda z}} \int_{-\infty}^{+\infty} U(x) e^{\frac{i\pi}{\lambda z}(u-x)^2} \, \mathrm{d}x. \tag{3.5}$$

If one observes (3.5), it can be seen that the Fresnel transform can be expressed as a convolution, by

$$U_f(u) = \left( \frac{1}{\sqrt{i\lambda z}} U(x) * e^{\frac{i\pi}{\lambda z}x^2} \right)(u). \tag{3.6}$$

As we know from Section 2.2, the Fourier transform of a chirp is still a chirp; plus, as we can observe from the same Section, there is a strong connection between the convolution of chirp-like functions and the Fourier transform itself. Bearing this in mind and assuming an infinite aperture ($\beta(x) = 1$ everywhere), a close relation between the Fresnel and the Fourier transform can be seen, too. From (3.6), one can highlight that the Fourier transform just differs from a chirp convolution by a phase factor. This last one is not relevant for many applications, since the intensity of the incident light (in the optics example) is the only aspect that is really measured.

# 4 Introduction to quantum mechanics

## 4.1 The rise of quanta

A classical way to approach the problem of motion for a system of particles is to make use of Newton's laws as to determine the behaviour of the particles in the future: once the initial conditions are known, one can predict the trajectory of all the particles with an (ideally) arbitrary accuracy. The idea, which was dominating until the early nineteenth century, was challenged by a vigorous debate in the scientific community originated from empirical experiments that provided a set of seemingly contradictory outcomes.

The major breakthrough in the arising atomic theory was the discovery of the atoms and the electrons, which experimentally supported the Rutherford model (the planet-like model) for the atom: it was in accordance with the experiences carried out by the physicists. However, the rotation of a charge around the central nucleus required the emission of radiation according to Maxwell's equations, which would lower the electron's energy and finally drive it to fall over the positive-charged nucleus.
That would have meant that the matter itself is unstable and it has to quickly collapse on itself. But that was clearly not happening while Maxwell's equation and Newtons's laws appeared to be still rock solid in their respective realms.

However, the major trouble for the classical mechanics happened when the black body problem also threatened the long-time established thermodynamic principles.
Let us imagine a closed domain which stands in thermal equilibrium with the external environment at a temperature $T$: such an object is known as a black-body since it absorbs electromagnetic radiation of any frequency. As to preserve the thermal equilibrium, a black body also emits electromagnetic radiation and the energy of this radiation is experimentally known to follow the law:

$$E \propto T^4. \tag{4.1}$$

Thinking classically, the Stefan-Boltzmann equation in eq. (4.1) is justified by thinking of the effect of the field in a close cavity being equivalent to a system made of an infinite number of independent harmonic oscillators.
The relevant ones among all the modes of the electromagnetic field are called normal modes and are involved in the calculation of the density of the energy $u$ emitted by a black body, which is expressed by the Rayleg-Jeans equation:

$$u(\nu, T) = \frac{8\pi k_B T \nu^2}{c^3}, \tag{4.2}$$

where $\nu$ is the frequency of the radiation, $k_B$ the Boltzmann constant, $T$ indicates the temperature of the body and $c$ is the light speed in vacuum.
When the frequencies are low, eq. (4.2) is in good agreement with the experimental results but when one tries to calculate the average energy $E$ within the range of ultraviolet rays

and raises the frequencies, the integral of (4.2),

$$\bar{E} = \int_0^\infty u(\nu, T) d\nu = \infty, \tag{4.3}$$

delivers an infinite value for the energy, that is severely nonsensical from the perspective of the physical interpretation and experimental validation. Clearly, there were clues that a deep flaw in classical physics existed.
Trying to fix that flaw, the German physicist Max Planck, early in the 20th century, focused on the so-called catastrophe of the ultraviolet, attacking the issue about an infinite density of energy for the black body from another perspective. He introduced a fixed fictitious amount, the quantum, which he assumed to be the base-block of the exchange of energies between electromagnetic radiation and matter, at a given frequency $\nu : \epsilon(\nu) = h\nu$.

With such a simple mathematical reformulation, the very heart of the classical physics was shaken : while the energy of a single oscillator, standing for a mode of the electromagnetic field, can continuously vary its value, Planck's perspective restricts the possible values just to integer multiples of a finite quantity and its consequences are spreading over the way the average energy is calculated.
Assuming discrete values, the spectral radiance $u$ turns its form into:

$$u(\nu, T) = \frac{h\nu}{e^{\frac{h\nu}{k_B T}} - 1} \tag{4.4}$$

and the calculation of the average energy for the equivalent harmonic oscillators avoids the ultraviolet catastrophe, showing:

$$\bar{E} = \frac{h\nu}{e^{\frac{h\nu}{k_B T}} - 1}. \tag{4.5}$$

The curve depicted by eq. (4.4) is not diverging for high frequencies and, by fitting the formula to experimental results, the value of the Planck constant $h$ was calculated with extremely high accuracy: $h = 6.626 \times 10^{-34} Js$.

## 4.2 Waves and particles

To Planck's understanding, the idea of the quanta was aimed at solving the black body problem, being the quantum mechanism a mere property of the atoms rather than the radiation. Nevertheless, Einstein, in its work regarding the photoelectric effect, was more inclined to think of the light emission/absorption as a property of the electromagnetic radiation, proposing the light with a given frequency $\omega$ to be exchanged only in "bundles" of energy $h\omega$, called photons.

A certain idea of duality between radiation and particles began its journey exactly at that moment: if absorption/emission occurs in a discrete way, also energies of particles must be quantized, therefore. Since the quantum of energy is characterized by the

frequency, all the undulatory phenomena associated with an electromagnetic field can be thought as flows of particles, indeed.
Afterwards, even Niels Bohr tried to use the idea of quantization to solve the issue about the non self-destruction of atoms (predicted classically). This was aimed at building a model of the atom which involved fixed energy jumps and allowed quantized circular orbits, based on the value $\hbar = \frac{h}{2\pi}$.
The hydrogen atoms, the simplest ones, allowed straightforward calculations: the angular momentum $L$ of an electron of mass $m$, radius $r$, rotating around the nucleus with velocity $v$, is expressed as $L = mvr = \hbar n$, where $n$ is the integer value called (principal) quantum number.

In order to have a stable atom, the orbiting electron must compensate both the centripetal acceleration and the coulombian attraction:

$$\frac{e^2}{r^2} = \frac{mv^2}{r}, \tag{4.6}$$

where $e$ is the electron charge. Including the quantized orbits, eq. (4.6) leads to the calculation of the Bohr radius $r_n$, which provided the first estimated measure of the size of an hydrogen atom, and the energy $E_n$ associated with the electron on that orbit:

$$r_n = \frac{\hbar^2 n^2}{me^2}, \tag{4.7}$$

$$E_n = -\frac{me^4}{2\hbar^2 n^2}. \tag{4.8}$$

Quantization, now, even entered the atomic model, prescribing specific energies and orbits, that are discrete and dependent on $n$: by confirming Einstein's intuition, Bohr proposed that the atoms emit/absorb a quantum of light, a photon, when jumping downwards/upwards between two allowed energy levels.

The quantum framework brought concepts from wave physics as to describe particle-based phenomena. Hence, it suggested a kind of deep interplay between undulatory and corpuscular nature of the world. A radical idea, that was not even solidly backed by experimental clues, was carried out by Louis De Broglie, when he proposed the hypothesis that every particle with a mass can behave as a wave as well, associating a wave to every particle, whose wavelength $\lambda$ was:

$$\lambda = \frac{h}{mv}. \tag{4.9}$$

What such a wave would actually represent in the physical reality was pretty obscure, indeed. Nevertheless, the idea of De Broglie was perfectly compatible with Bohr's atom: if the orbiting electron behaves like a wave, the length of the orbit must be an integer multiple of the wavelength, just in order to avoid interferences that would cancel out the wave-electron.

Later on, by accelerating the electrons and measuring their scattering (in angle) over a metallic surface, the experimental ground was provided (Davisson-Germer experiment) to demonstrate the validity of the wave-particle dualism introduced by De Broglie.

The matching of waves with particles leads to several interesting consequences, when one tries to provide more mathematical insight into the world of quantum physics: a particle with energy $E$ and momentum $p = mv$ is associated to a wave with a frequency $f$ and wavelength $\lambda$. More formally, a simple real-valued single frequency, one-dimensional wave function can be represented as:

$$\Psi(x,t) = \alpha \sin(kx - \omega t), \tag{4.10}$$

where the angular frequency $\omega = 2\pi f$ and the wavenumber $k = 2\pi\lambda$ are used.
Knowing the frequency and the wavelength, the phase velocity $v_p$ is the velocity of the wave crests (or troughs):

$$v_p = \frac{\omega}{k} \tag{4.11}$$

and can be thought as the speed of the particle associated to the wave. It is important to highlight that the phase velocity depends on the value of the wavenumber $k$. In general $\omega$ is a function of $k$.

The locating of the associated particle position in the space is a challenging task when nothing but the harmonic wave function is known: it is a standing wave, repeating itself periodically over the space, with the same amplitude. There is no distinguishing feature that can suggest which position can take the particle.
However, one would like to be able to alterate the amplitude of the wave in order to have a highly localized region in space (ideally a point) where the particle is located, while having a zero wave elsewhere, as the particle is not supposed to be there. To achieve that, one must take advantage of the interference phenomena occurring when two or more waves of different wavelength are being summed up. One thus builds a localized version of the wave function (the so-called wave packet) by summing and appropriately weighting the contribution of the wave-numbers, just in order to translate the information about the position into the wave shape.

Referring to the expression adopted for the wave function in eq. (4.10), it would rather be extended into a sum of (infinite) waves with different wavenumber, as follows:

$$\Psi(x,t) = \int_{-\infty}^{+\infty} A(k) \cos(kx - \omega(k)t)\, \mathrm{d}k, \tag{4.12}$$

where all the waves are interfering, constructively and destructively according to the weights managed by the $A(k)$ kernel, so to have zero amplitude everywhere except a localized region where the particle is located.
Once the wave packet has been localized, one would even try to calculate the speed which the packet itself will move at (and also spread, within a dispersive media). That is the

group velocity $v_g$ and it stands for the speed of the wave packet envelope.
The relation between the angular frequency $\omega$ and the wavenumber $k$ is defined as the dispersion relation and its form influences the actual group velocity $v_g$. Assuming the most general form for the dispersion, $\omega = \omega(k)$, the centre of the wave packet can be defined as:

$$\langle X(t) \rangle = \frac{\int x|\Psi(x,t)|^2 \, \mathrm{d}x}{\int |\Psi(x,t)|^2 \, \mathrm{d}x}, \tag{4.13}$$

where the intensity associated with $\Psi$ , $I = \int |\Psi(x,t)|^2 \, \mathrm{d}x$, is constant in time $t$, and there is no issue in normalizing it to one: $\int |\Psi(x,t)|^2 \, \mathrm{d}x = 1$.
The double nature of quantum particles, let us include the De Broglie intuition on the wavelength associated to an energy content, provides a less general adapted dispersion relation. From the Planck's relation, in fact, we can deduce the dispersion relation:

$$\omega(k) = \frac{E}{\hbar} = \frac{p^2}{2m}\frac{1}{\hbar} = \frac{\hbar^2 k^2}{2m}\frac{1}{\hbar} = \frac{\hbar k^2}{2m}. \tag{4.14}$$

## 4.3 Time evolution: Schrödinger's equation

The expression in eq. (4.12) was built based on physical considerations about the interference of waves and it is indeed one possible form for the solution of the well known wave equation:

$$\frac{\partial^2 \Psi(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \Psi(x,t)}{\partial t^2}, \tag{4.15}$$

for which the plane wave elementary solution has the form:

$$\Psi(x,t) = Ae^{ikx - i\omega t}, \tag{4.16}$$

if $c = \omega/k$. In fact, if one comes to derive eq. (4.16) with respect to time,

$$\frac{\partial \Psi(x,t)}{\partial t} = -i\omega A e^{ikx - i\omega t} = -i\omega A \Psi(x,t), \tag{4.17}$$

by calculating the second partial derivative with respect to $x$, one obtains:

$$\frac{\partial^2 \Psi(x,t)}{\partial x^2} = i^2 k^2 \Psi(x,t). \tag{4.18}$$

Substituting eq. (4.17) and eq. (4.18) into the dispersion relation recalled in eq. (4.14), we obtain

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x,t)}{\partial x^2}, \tag{4.19}$$

which, assuming a zero external potential $V(x)$, is the (non-relativistic) Schrödinger equation for the free particle in one dimension.

As pointed out when interpreting the wave-function as a phenomena of interference of multiple waves, the localizing of a particle both in space and time is a task which turns out to be challenging, or better, intrinsically unsolvable in quantum mechanics.
The intrinsic uncertainty at the core of the quantum mechanics is condensed into the inequality:

$$\langle x^2 \rangle \langle p^2 \rangle \geq \frac{\hbar^2}{4}, \tag{4.20}$$

where $x$ and $p$ represent the position and the momentum of a particle, respectively.
The result depicted above in the eq. (4.20) is known as the “Heisenberg uncertainty principle” and it exposes one of the most counterintuitive aspects of the quantum mechanics: it is not possible to achieve any arbitrary precision when measuring two conjugate variables, i.e. position and momentum.

Historically, Heisenberg himself proposed its inequality to be interpreted as the effect of the interaction between the photon used for a measurement and the system itself, so that the measure would result in investigating the perturbed system.
However, it was later proved that a class of uncertainty relations is always obtainable for wave-like systems and that such a principle was a necessary implication of the postulates adopted in quantum mechanics.
The essence of uncertainty in quantum physics is therefore not quite a matter of a lack of adequate experimental tools but it is rather an intrinsic property of the very fabric of the physical reality.
Schrödinger’s equation is central not only for historical reasons but also because it is assumed as a given postulate of quantum mechanics.

## 4.4 Probability and postulates

The idea of bridging the realm of waves and particles triggered the introduction of a whole mathematical structure governing both the form and the time evolution of the wave-function associated to a particle. While, in classical mechanics, a motion of a particle was determined by the equation based on forces and momenta, in quantum mechanics the wave function associated to a system and its evolution is determined by the Schroedinger’s equation.
However, historically, there was a lack of understanding on how the wave function (and in particular it’s amplitude) had to be interpreted from a physical point of view.

As the behaviour of the quantum particles exhibits both corpuscular and undulatory features (eg. double-slit experiment), it can not be described classically. Therefore, concepts like “trajectories” are not applicable.
Although opposing factions were struggling to catch the physical meaning of such a dual behaviour, even questioning the concept of “reality” itself, Max Born’s probabilistic interpretation came out as an useful tool enabling physicists to make calculations and predic-

tions (see [3]).

Mathematically rooted into the probability theory, Born's approach allowed to put aside the discussion on the probabilistic nature of quantum mechanics, by starting from some postulates.

### 4.4.1 Quantum states

The state of a system at a given instant $t$ is described by a complex-valued function $\Psi(x,t)$ and its squared 2-norm $|\Psi(x,t)|^2$defines the probability distribution (*pdf*) of the values the state can have. Therefore, the probability $P(\Sigma_1)$ of finding a particle in a given subset $\Sigma_1$ of the (measurable) phase space $\mathbb{R}^d$, is:

$$P(\Sigma_1) = \int_{\Sigma_1} |\Psi(x,t)|^2|, \mathrm{d}x. \tag{4.21}$$

Being $P(\mathbb{R}^d)$ a probability, it must be normalized to 1, all over the states:

$$\int_{\mathbb{R}^d} |\Psi(x,t)|^2 \, \mathrm{d}x = 1, \tag{4.22}$$

irrespectively of the chosen instant $t$.
In order to have a meaningful integral in (4.21), a further requirement is for $\Psi(x,t)$ to be square-integrable over all the phase space $\mathbb{R}^d$ . That makes the Hilbert space $H = L^2(\mathbb{R}^d)$ the elective space for these functions. Being the probability just dependent on the squared norm $|\Psi(x,t)|^2$, the phase of the function is not relevant when it comes to the probability calculation.
The use of this very Hilbert space associated to physical systems is one of the basic postulates of the quantum mechanics.
In fact, we ourselves, as mathematicians, recall $L^2(\mathbb{R}^d)$ has so many "beautiful" properties. First of all, as a Hilbert space, it is a vector space endowed with a scalar product and it is complete with respect to the norm induced by the scalar product itself. Then, it is a separable space, which means $L^2(\mathbb{R}^d)$ has at least an orthonormal countable basis and this latest property is so very crucial to us. Moreover, this space has so many other useful properties that directly follow from the Functional Analysis theory, (see, for example, [4]).

### 4.4.2 Measurements and observables

The introduction of the quantization operated by Planck changed the spectrum of values the physical quantities could have. In classical mechanics, a quantity $f$ (an observable) can vary in a continuous range of values, while in the quantum perspective the set of the admissible values for an observable (eigenvalues) is not necessarily continuous.

In quantum mechanics, an observable $f$ is a linear, limited operator $F$ on $L^2(\mathbb{R}^d)$, namely, $F$ belongs to $\mathscr{L}(L^2(\mathbb{R}^d))$. Bearing this identification in mind, and even according to what we explained in Section 2.1, we can always deal with the eigenvalues-eigevectors problem related to such an operator $F$.
In particular, as in quantum mechanics we are just interested in having real eigenvalues, another constraint is for $F$ to be selfadjoint (or, more generally, hermitian).

So, let us turn to the eigenvalues-eigevectors problem related to the operator $F$:

$$F\psi_j = f_j\psi_j, \tag{4.23}$$

where $f_j$ are the eigenvalues with their respective eigenfunctions $\psi_j$, $j = 1, 2, 3, \ldots$.

In quantum mechanics, the eigenvalues $f_j$ are the activating energies of a system and the corresponding eigenvectors $\psi_j$ are called "fundamental states". They describe the system when is still steady and they have a probabilistic interpretation: they stand for the probability densities of the position of a particle (or a point), hence accounting for the concept of area underlying a certain function. It follows that knowing the probability that a given particle is exactly at a given point is a pure mirage: the area underlying a point is simply zero. That makes not possible getting deterministic pieces of information from a measurement.

As explained in the previous paragraph, thanks to the separability property of $L^2(\mathbb{R}^d)$, the following "Fundamental Theorem on Spectral Theory" for the quantum mechanics holds.

**Theorem 4.1** *The eigenfunctions $\psi_1$, $\psi_2$, $\psi_3, \ldots$ constitute a complete system, namely, every function $\psi \in L^2(\mathbb{R}^d)$, $d = 1, 2, 3, \ldots$, can be written as*

$$\psi(x) = \sum_{j=1}^{+\infty} \alpha_j \psi_j(x), \quad x \in \mathbb{R}^d. \tag{4.24}$$

This is a very powerful tool in quantum mechanics: it tells that, once we fix a basis (actually, a basis made of fundamental states), every function in $L^2(\mathbb{R}^d)$ can be expressed as a superposition of fundamental states. That is to say we are in some way able to depict "the whole universe" (namely, $L^2(\mathbb{R}^d)$, for us). (Of course, the same observable (operator) can also be described making different choices of a basis; nevertheless, with this very choice of a basis, we have that the matrix associated to the selfadjoint operator $F$ takes a particular form: the diagonal one).

The association between observables and operators, together with the intrinsic statistical interpretation of the process of measure, are both assumed as quantum mechanics postulates.

For example, the momentum operator $P$ reads:

$$P\psi(x,t) := -i\hbar\nabla\psi(x,t), \tag{4.25}$$

which can be proved to be a symmetric operator and dimensionally coherent with the observable "momentum".
Therefore, the total energy of a particle, the Hamiltonian , has its own operator $H$, too:

$$H\psi(x,t) := \frac{P^2\psi(x,t)}{2m}. \tag{4.26}$$

Interesting enough, as a consequence of the equation (4.23), choosing two operators, $F$ and $G$, associated to two observable $f$ and $g$ that can be simultaneously measured, it is true that they have the same set of eigenfunctions $\psi_i$ and their products, $FG$ and $GF$, lead exactly to the same decomposition:

$$FG\psi = FG\sum_i a_i\psi_i = \sum f_i g_i a_i \psi_i, \tag{4.27}$$

$$GF\psi = GF\sum_i a_i\psi_i = \sum g_i f_i a_i \psi_i. \tag{4.28}$$

Using the Poisson notation $[\cdot]$, then we can state (see [8]) that $F$ and $G$ are two commuting operators when:

$$[F, G] := FG - GF = 0. \tag{4.29}$$

From a physical perspective, that means that two variables are simultaneously measurable if and only if their respective operators are commuting.
The position along the x-direction and the momentum along same direction, for example, are incompatible variables. In fact, by applying the commutation operator to $X \to x$ and $P_x \to -i\hbar\frac{\partial}{\partial x}$, we obtain:

$$[X, P_x] = [x, -i\hbar\frac{\partial}{\partial x}] = i\hbar \neq 0 \tag{4.30}$$

So, the commutativity of the operator is not there. All this again states the Heisenberg's uncertainty principle (recall the equation (4.20)), just using the basic assumptions so far adopted.

Namely, considering two symmetric operators $A$ and $B$ acting on a state $\psi$, for which $||\psi||^2 = 1$ with no loss of generality, one can remind the Cauchy-Schwarz inequality and write:

$$|(\psi, AB\psi)| \leq ||A\psi||||B\psi|| \tag{4.31}$$

$$|(\psi, BA\psi)| \leq ||A\psi||||B\psi||, \tag{4.32}$$

which directly comes from the symmetry assumptions.
From (4.27) and (4.31), we are allowed to write down the inequality:

$$||A\psi||^2||B\psi||^2 \geq \frac{1}{4}|(\psi, [A, B]\psi)|^2. \tag{4.33}$$

If one builds the operators $A$ and $B$, such that:

$$A = X - \langle X \rangle I \tag{4.34}$$

$$B = P_x - \langle P_x \rangle I, \tag{4.35}$$

where $I$ is the identity operator, the commutator $[A, B]$ can be derived as:

$$[A, B] = [X, P_x] = i\hbar, \tag{4.36}$$

which delivers the result in the equation (4.30).

It is also straightforward to verify that, if one changes the axes along which the momentum is measured, switching to $P_z$,

$$[X, P_z] = [x, -i\hbar\frac{\partial}{\partial z}] = 0, \tag{4.37}$$

the variables then happen to be compatible for simultaneous measurements.

Nevertheless, the problem of measurement in quantum mechanics have deep implications and it is still controversial and far from being generally accepted. Indeed, several non-orthodox interpretations are struggling over the peculiarity of quantum mechanics, in its Born interpretation (also known as the "Copenhagen interpretation"), just to provide prediction solely based on statistical approaches. The superposition of different states (mixed states), for example, cannot provide an exact result for a given measure, but just the outcome probability associated to the admissible eigenvalues for an observable.
That remains true even in ideal experiments and deals with the intrinsic probabilistic nature of the phenomena at the quantum scale.

### 4.4.3 Collapsing wave-functions

What happens right after the measurement process has taken place is surely the most debated postulate in the Copenhagen interpretation. In order to grasp the formal framework behind the measurement process, it is useful to recall the bra-ket notation introduced by Paul Dirac [19].
The idea behind the bra-ket notation is that of having a more abstract representation of a vector which is not dependent on the specific orthonormal basis chosen. The "ket" $|v\rangle$ is defined in analogy with a column vector, while a "bra" $\langle v|$ is a row vector. Their composition provides the inner product $\langle v||v'\rangle$.
Every element of the Hilbert space (e.g. a wave-funciton) is therefore associated with a ket vector,

$$\Psi(x, t) \iff |\psi\rangle, \tag{4.38}$$

for which the linearity property holds:

$$|\lambda_1\psi_1 + \lambda_2\psi_2\rangle = \lambda_1|\psi_1\rangle + \lambda_2|\psi_2\rangle, \tag{4.39}$$

for every $\psi_1, \psi_2$ chosen in the Hilbert space $H = L^2(\Sigma)$.
Additionally, also a co-vector, which is related to a functional operator $\chi$ in the Hilbert

space, is represented by a bra and the action of the functional $\chi$ over $|\psi\rangle$ is written as $\langle\chi|\psi\rangle$. Now it is easier to check what happens when a measurement event takes place: before measuring a given observable, the state of the system in the bracket notation is $|\psi\rangle$:

$$|\psi\rangle = \sum_i |\alpha_i\rangle\langle\alpha_i|\psi\rangle, \tag{4.40}$$

where $\alpha_i$ are the eigenvalues (namely the possible measurement outcomes) associated to the state. This means that, before the measurement, all the eigenvalues are possible and the state of the system is a superposition of all those possible outcomes. When the measurement occurs, the state in (4.40) is instantly projected over the eigenvector associated to the measured eigenvalue $\alpha_i$:

$$|\psi\rangle \implies |\alpha_i\rangle. \tag{4.41}$$

It is important to stress that it would happen even with an ideal measurement.
Therefore, a further postulate states that, if one makes iterated measurements over a system, all the measurements after the first would deliver the same result. That is because the system instantly collapses into an eigenstate right after the first measurement, so that the system is already influenced by it and keeps trace in the next measurements. What happens during the collapse of the state into the measured eigenstate is still debated. It does constitute one of the hottest points on the controversial among the different interpretations of quantum mechanics. Historically, it had many adversaries as it appeared to challenge relativity (due to the idea of the instant collapse) and also introduced non-locality, which led to the rise of a number of issues (e.g. Einstein-Podolski-Rosen paradox).
However, the Copenhagen interpretation prefers not to enter the details of what happens during the collapse of the wave-function: it just takes it as a postulate and goes further.

# 5 Setting of the problem and main results

## 5.1 Free particle problem

We consider the problem

$$\begin{cases} \partial_t u(t,x) - i\partial_{xx} u(t,x) = v(t,x), & (t,x) \in \mathbb{R}\times\mathbb{R} \\ u(0,x) = \bar{u}_0(x), & x \in \mathbb{R}. \end{cases} \tag{5.1}$$

We would like to solve the problem (5.1) when $\bar{u}_0$ is a sum of chirp functions of the form $e^{inx^2}$, $n \in \mathbb{Z}$, and $v$ is a sum of chirp functions of the form $\tilde{g}_n(t)e^{i\tilde{f}_n(t)x^2}$. We are looking for solutions which are sum of chirp functions of the form $g_n(t)e^{f_n(t)x^2}$, too.

We start dealing with the homogeneous problem

$$\begin{cases} \partial_t u(t,x) - i\partial_{xx} u(t,x) = 0, & (t,x) \in \mathbb{R}\times\mathbb{R} \\ u(0,x) = \bar{u}_0(x), & x \in \mathbb{R}. \end{cases} \tag{5.2}$$

As it concerns the Cauchy problem (5.2) for the Schrödinger equation,

$$\partial_t u(t,x) - i\partial_{xx} u(t,x) = 0, \qquad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{5.3}$$

we have the following result.

**Theorem 5.1** *Let $\alpha_n \in \mathbb{C}$, $n \in \mathbb{Z}$, be a sequence such that*

$$\sum_{n=-\infty}^{+\infty} |\alpha_n| < +\infty \tag{5.4}$$

*and*

$$\bar{u}_0(x) = \sum_{n=-\infty}^{+\infty} \alpha_n e^{inx^2}. \tag{5.5}$$

*Denoting by $k(t,x) = (4\pi i t)^{-1/2} e^{\frac{ix^2}{4t}}$ the fundamental solution of the equation (5.3), we define $u_n : A_n \times \mathbb{R} \to \mathbb{C}$, for $n \neq 0$, by*

$$u_n(t,x) := \left(\frac{\pi i}{n}\right)^{1/2} k(t + \frac{1}{4n}, x),$$

*where $A_n := \mathbb{R} \setminus \{-1/(4n)\}$, and $u_0 : A_0 \times \mathbb{R} \to \mathbb{C}$, where $A_0 = \mathbb{R}$, by $u_0(t,x) := 1$. Then, the function $u : A \times \mathbb{R}$ defined by*

$$u(t,x) := \sum_{n=-\infty}^{+\infty} \alpha_n u_n(t,x), \tag{5.6}$$

*where*

$$A := \bigcap_{n\in\mathbb{Z}} A_n = \{t \in \mathbb{R} : t \neq 1/(4n), n \in \mathbb{Z}\setminus\{0\}\},$$

*is a solution of the problem (5.2) in the following sense:*

- *the series in* (5.11) *absolutely converges for any* $(t,x) \in A \times \mathbb{R}$, *hence* $u(t,x)$ *is well defined for* $(t,x) \in A \times \mathbb{R}$;

- *defining* $\tilde{A} = A \setminus \{0\}$, *we have* $u \in C^\infty(\tilde{A} \times \mathbb{R})$;

- $u$ *is a classical solution in* $\tilde{A} \times \mathbb{R}$ *of the equation in* (5.2);

- $$\lim_{t \to 0} \int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}} \bar{u}_0(x)\varphi(x)\,\mathrm{d}x,$$

  *for any* $\varphi \in C_c^\infty(\mathbb{R})$;

- *for any* $n \in \mathbb{Z} \setminus \{0\}$, *we have*

  $$\lim_{t \to -1/(4n)} \int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = \alpha_n \left(\frac{\pi i}{n}\right)^{1/2} \varphi(0) + \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}\setminus\{n\}} \alpha_k u_k(-1/(4n), x)\varphi(x)\,\mathrm{d}x,$$

  *for any* $\varphi \in C_c^\infty(\mathbb{R})$.

In this thesis, we use the convention (2.17) as regards the 1/2 and −1/2 powers of complex numbers.

The solution that we constructed in Theorem 5.1 can be extended to a generalized solution that enjoys the property of continuity, with respect to time, and takes values in the space of distributions. More precisely, we are going to state this property in the following Corollary.

**Corollary 5.2** *Under the assumptions of Theorem 5.1, let* $u$ *be the solution given by Theorem 5.1. Let* $\tilde{u} : \mathbb{R} \to \mathcal{D}'(\mathbb{R})$ *be the function of distributions defined in the following way:*

$$\tilde{u}(t) := u(t, \cdot), \qquad \text{for } t \in A,$$

$$u(-1/(4n), \cdot) := \alpha_n \left(\frac{\pi i}{n}\right)^{1/2} \delta + \sum_{k \in \mathbb{Z}\setminus\{n\}} \alpha_k u_k(-1/(4n), \cdot),$$

*for* $n \in \mathbb{Z} \setminus \{0\}$. *Then* $\tilde{u}$ *is continuous.*

*Proof.* The last two points of Theorem 5.1 imply the continuity of $\tilde{u}$ at 0 and at the points $1/(4n)$, for $n \in \mathbb{Z} \setminus \{0\}$. Moreover, the continuity of $u$ at the other points $t_0 \in \tilde{A}$ easily follows by the continuity of $t \in \tilde{A} \to u(t,x)$, for any $x \in \mathbb{R}$, and the Lebesgue dominated convergence theorem. □

**Remark 5.3 (The class of initial data $\bar{u}_0$)** We observe that the class of initial data of Theorem 5.1 coincides with the class of functions of this form: $\bar{u}_0 : \mathbb{R} \to \mathbb{C}$ such that

$\bar{u}_0(x) = \bar{u}_0(-x)$ for any $x \in \mathbb{R}$, $h_{\bar{u}_0} : \mathbb{R} \to \mathbb{C}$ defined by $h_{\bar{u}_0}(y) := \bar{u}_0(\sqrt{|y|})$ is $2\pi$-periodic and the Fourier coefficients $\alpha_n$ of $h_{\bar{u}_0}$, defined by

$$\alpha_n = \frac{1}{2\pi}\int_0^{2\pi} h_{\bar{u}_0}(y)e^{-iny}\,\mathrm{d}y, \qquad n \in \mathbb{Z}, \tag{5.7}$$

satisfy (5.4).

Indeed, if $\alpha_n \in \mathbb{C}$, $n \in \mathbb{Z}$, is given and it satisfies (5.4), then $\bar{u}_0$, defined by

$$\bar{u}_0(x) = \sum_{n=-\infty}^{+\infty} \alpha_n e^{inx^2}, \tag{5.8}$$

should be even, $h_{\bar{u}_0}$ should be $2\pi$-periodic and its Fourier coefficients coincide with $\alpha_n$.

Viceversa, if $\bar{u}_0 : \mathbb{R} \to \mathbb{C}$ satisfies the above conditions, then $h_{\bar{u}_0}$ has the Fourier series representation

$$h_{\bar{u}_0}(y) = \sum_{n=-\infty}^{+\infty} \alpha_n e^{iny}, \tag{5.9}$$

where $\alpha_n$ are defined in (5.7) and then $\bar{u}_0$ has the representation (5.8).

We observe that the convergence of the series (5.8) is uniform and consequently $\bar{u}_0$ is continuous.

**Remark 5.4** We would like to remark that Problem (5.2) admits infinite other solutions. Namely, these solutions can be constructed as follows. Given a sequence $\beta_n \in \mathbb{C}$ such that

$$\sum_{n=-\infty}^{+\infty} |\alpha_n||\beta_n| < +\infty, \tag{5.10}$$

we define the sequence $\tilde{u}_n : A_n \times \mathbb{R} \to \mathbb{C}$, for $n \neq 0$, in this way:

$$\tilde{u}_n(t,x) = \begin{cases} \beta_n u_n(t,x) & \text{if } t < -1/(4n) \\ u_n(t,x) & \text{if } t > -1/(4n) \end{cases} \qquad \text{for } n > 0,$$

and

$$\tilde{u}_n(t,x) = \begin{cases} u_n(t,x) & \text{if } t < -1/(4n) \\ \beta_n u_n(t,x) & \text{if } t > -1/(4n) \end{cases} \qquad \text{for } n < 0,$$

and $\tilde{u}_0 = u_0$. Then, the function $\bar{u} : A \times \mathbb{R}$, defined by

$$\bar{u}(t,x) := \sum_{n=-\infty}^{+\infty} \alpha_n \tilde{u}_n(t,x), \tag{5.11}$$

satisfies all the points of Theorem 5.1 except the last one, when $\beta_n \neq 1$ for almost one $n \in \mathbb{Z}$. In particular, the function $\bar{u}$ cannot be extended to a continuous function with values in $\mathcal{D}'(\mathbb{R})$.

We now take into account the non-homogeneous problem with zero initial datum:

$$\begin{cases} \partial_t u(t,x) - i\partial_{xx} u(t,x) = v(t,x), & (t,x) \in \mathbb{R}\times\mathbb{R} \\ u(0,x) = 0, & x \in \mathbb{R}. \end{cases} \tag{5.12}$$

Before stating the problem, we define the piecewise constant function $c : \mathbb{R}\setminus\{0\} \to \mathbb{C}$ by

$$c(t) = \begin{cases} i^{-1/2} & \text{if } t > 0 \\ (-i)^{-1/2} & \text{if } t < 0 \end{cases} = \begin{cases} e^{-i\pi/4} & \text{if } t > 0 \\ e^{i\pi/4} & \text{if } t < 0. \end{cases} \tag{5.13}$$

With this function, the fundamental solution can be written as

$$k(t,x) = c(t)|4\pi t|^{-1/2} e^{\frac{ix^2}{4t}},$$

and we point out that the functions $u_n$ of Theorem 5.1 are multiple of the following translations

$$k(t+1/(4n),x) = c_n(t)\left|\frac{n}{\pi}\right|^{1/2} |4nt+1|^{-1/2} e^{\frac{inx^2}{4nt+1}}, \tag{5.14}$$

where

$$c_n(t) := c(4nt+1), \tag{5.15}$$

observing that $|c_n| = 1$.

Willing to find a solution of (5.12) as a sum of chirp functions, we are led to consider a right hand side, $v$, as a sum of chirp functions $v_n$ of a form comparable to the one of $u_n$. More precisely, similarly to (5.14), we consider $v_n : A_n \times \mathbb{R} \to \mathbb{C}$, where $A_n := \mathbb{R}\setminus\{-1/(4n)\}$, defined by

$$v_n(t,x) := \tilde{g}_n(t)c_n(t)e^{\frac{inx^2}{4nt+1}},$$

for $\tilde{g}_n : \mathbb{R} \to \mathbb{C}$ continuous. We remark that the pointwise limit, $\lim_{t\to -1/(4n)} v_n(t,x)$, does not exist for any $x \in \mathbb{R}$, whereas $\lim_{t\to -1/(4n)} v_n(t,\cdot) = 0$ in $\mathcal{D}'(\mathbb{R})$.

See the next Theorem 5.5 and the Proposition 6.7.

**Theorem 5.5** *Let $\tilde{g}_n : \mathbb{R} \to \mathbb{C}$, $n \in \mathbb{Z}$, be a sequence of continuous functions such that*

$$\sum_{n=-\infty}^{+\infty} \sup_{t\in[a,b]} |\tilde{g}_n(t)| < +\infty, \qquad \textit{for any } [a,b] \subset \mathbb{R}. \tag{5.16}$$

*Let $v_n : A_n \times \mathbb{R} \to \mathbb{C}$, where $A_n := \mathbb{R}\setminus\{-1/(4n)\}$, defined by*

$$v_n(t,x) := \tilde{g}_n(t)c_n(t)e^{\frac{inx^2}{4nt+1}},$$

*and $v_0 : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ defined by $v_0(t,x) = \tilde{g}_0(t)$. Defining $v : A \times \mathbb{R} \to \mathbb{C}$ by*

$$v(t,x) := \sum_{n=-\infty}^{+\infty} v_n(t,x), \tag{5.17}$$

*where* $A := \bigcap_{n\in\mathbb{Z}} A_n = \{t \in \mathbb{R} : t \neq 1/(4n), n \in \mathbb{Z}\setminus\{0\}\}$, *and* $F[\tilde{g}_n] : A_n \times \mathbb{R} \to \mathbb{C}$ *by*

$$F[\tilde{g}_n](t) := |4nt+1|^{-1/2} \int_0^t |4ns+1|^{1/2} \tilde{g}_n(s)\,\mathrm{d}s,$$

*then the function* $u : A \times \mathbb{R}$, *defined by*

$$u(t,x) := \sum_{n=-\infty}^{+\infty} F[\tilde{g}_n](t) c_n(t) e^{\frac{inx^2}{4nt+1}}, \tag{5.18}$$

*is a solution to the problem* (5.12) *in the following sense:*

- *the series in* (5.18) *absolutely converges for any* $(t,x) \in A \times \mathbb{R}$, *hence* $u(t,x)$ *is well defined for* $(t,x) \in A \times \mathbb{R}$;
- *defining* $\tilde{A} = A \setminus \{0\}$, *we have* $u(\cdot,x) \in C^1(\tilde{A})$ *for any* $x \in \mathbb{R}$ *and* $u(t,\cdot) \in C^\infty(\mathbb{R})$ *for any* $t \in \tilde{A}$;
- $u$ *is a classical solution in* $\tilde{A} \times \mathbb{R}$ *of the equation in* (5.12);
- $$\lim_{t\to 0} \int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = 0,$$

  *for any* $\varphi \in C_c^\infty(\mathbb{R})$.
- *for any* $n \in \mathbb{Z}\setminus\{0\}$, *we have*

  $$\lim_{t\to -1/(4n)} \int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = (\pi)^{1/2}|n|^{-1/2} \int_0^{-1/(4n)} |4ns+1|^{1/2}\tilde{g}_n(s)\,\mathrm{d}s\,\varphi(0) + \\ + \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}\setminus\{n\}} F[\tilde{g}_k](-1/(4n)) c_k(-1/(4n)) e^{\frac{ikx^2}{-k/n+1}} \varphi(x)\,\mathrm{d}x,$$

  *for any* $\varphi \in C_c^\infty(\mathbb{R})$.

We remark that the condition (5.16) guarantees the uniform convergence of the series (5.17) in $([a,b]\cap A) \times \mathbb{R}$, for each $[a,b] \subset \mathbb{R}$, and then the continuity of $v$ in $A \times \mathbb{R}$.

**Remark 5.6 (An example of $v$)** A simple example of functions $\tilde{g}_n$ satisfying (5.16) is given by constants: $\tilde{g}_n(t) = \alpha_n \in \mathbb{C}$, such that (5.4) holds. In this case, the functions $F[\alpha_n]$ can be computed. Since

$$\int_0^t |4ns+1|^{1/2}\,\mathrm{d}s = \frac{1}{6n}(\mathrm{sign}(4nt+1)|4nt+1|^{3/2} - 1),$$

then we have

$$F[\alpha_n](t) = \alpha_n |4nt+1|^{-1/2} \frac{1}{6n}(\mathrm{sign}(4nt+1)|4nt+1|^{3/2} - 1).$$

We point out that we used the piecewise constant functions, $c_n$, in the definition of $u$ in (5.18), just in order to obtain the existence of the limits, for $t \to -1/(4n)$, in the last point of the Theorem. To this end, see the Proposition 6.6 and the limit (6.23). This choice is crucial in order to have the continuous extension in $\mathcal{D}'(\mathbb{R})$. Indeed, even in this case, it holds the following Corollary.

**Corollary 5.7** *Under the assumptions of Theorem 5.5, let $u$ be the solution given by Theorem 5.5. Let $\tilde{u} : \mathbb{R} \to \mathcal{D}'(\mathbb{R})$ be the function of distributions defined in the following way:*

$$\tilde{u}(t) := u(t, \cdot), \qquad \text{for } t \in A,$$

$$u(-1/(4n), \cdot) := \Big((\pi)^{1/2}|n|^{-1/2} \int_0^{-1/(4n)} |4ns+1|^{1/2} \tilde{g}_n(s)\, \mathrm{d}s\Big)\delta + \sum_{k \in \mathbb{Z}\setminus\{n\}} F[\tilde{g}_k](-1/(4n)) c_k(-1/(4n)) e^{\frac{ik(\cdot)^2}{-k/n+1}},$$

*for $n \in \mathbb{Z} \setminus \{0\}$. Then $\tilde{u}$ is continuous.*

*Proof.* The last two points of Theorem 5.5 imply the continuity of $\tilde{u}$ at 0 and at the points $1/(4n)$, for $n \in \mathbb{Z} \setminus \{0\}$. Moreover, the continuity of $u$ at the other points, $t_0 \in \tilde{A}$, easily follows by the continuity of $t \in \tilde{A} \to u(t, x)$, for any $x \in \mathbb{R}$, and the Lebesgue dominated convergence theorem. □

Thanks to the linearity of the problem (5.1), by taking the sum of the solutions obtained in Theorems 5.1 and 5.5, we do obtain a solution to the problem (5.1), with $\bar{u}_0$ and $v$ of the form (5.5) and (5.17).

**Theorem 5.8** *Let the assumptions of Theorems 5.1 and 5.5 hold. Then, there exists a solution $u$ of the Cauchy problem* (5.1) *satisfying all the properties given in Theorem 5.1. Moreover, $u$ can be extended to a continuous function $\tilde{u} : \mathbb{R} \to \mathcal{D}'(\mathbb{R})$.*

## 5.2 Quantum harmonic oscillator problem

We consider the problem

$$\begin{cases} \partial_t u(t,x) + \frac{i}{2}(-\partial_{xx} u(t,x) + x^2 u(t,x)) = v(t,x), & (t,x) \in \mathbb{R} \times \mathbb{R} \\ u(0,x) = \bar{u}_0(x), & x \in \mathbb{R}. \end{cases} \tag{5.19}$$

We again would like to solve the problem (5.19) when $\bar{u}_0$ is a sum of chirp functions of the form $e^{inx^2}$, $n \in \mathbb{Z}$, and $v$ is a sum of chirp functions of the form $\tilde{g}_n(t) e^{i\tilde{f}_n(t)x^2}$. We are looking for solutions which are sum of chirp functions of the form $g_n(t) e^{f_n(t)x^2}$.

We start looking at the homogeneous problem

$$\begin{cases} \partial_t u(t,x) + \frac{i}{2}(-\partial_{xx} u(t,x) + x^2 u(t,x)) = 0, & (t,x) \in \mathbb{R} \times \mathbb{R} \\ u(0,x) = \bar{u}_0(x), & x \in \mathbb{R}. \end{cases} \tag{5.20}$$

As it concerns the Cauchy problem (5.20) for the Schrödinger equation,

$$\partial_t u(t,x) + \frac{i}{2}(-\partial_{xx}u(t,x) + x^2 u(t,x)) = 0, \qquad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{5.21}$$

we have the following result.

**Theorem 5.9** *Let $\alpha_n \in \mathbb{C}$, $n \in \mathbb{Z}$, be a sequence such that*

$$\sum_{n=-\infty}^{+\infty} |\alpha_n| < +\infty \tag{5.22}$$

*and*

$$\bar{u}_0(x) = \sum_{n=-\infty}^{+\infty} \alpha_n e^{inx^2}. \tag{5.23}$$

*Let $u_0 : A_0 \times \mathbb{R} \to \mathbb{C}$ be the function defined by*

$$u_0(t,x) := q(t)|\cos(t)|^{-1/2} e^{\frac{i}{2}\tan(-t)x^2}, \tag{5.24}$$

*where $A_0 := \{t \in \mathbb{R} : t \neq \pi/2 + k\pi, k \in \mathbb{Z}\}$, and $q$ is the piecewise constant function*

$$q(t) := e^{-ki\pi/2} \qquad \text{if } t \in (\pi/2 + (k-1)\pi, \pi/2 + k\pi), \quad k \in \mathbb{Z}.$$

*Let us define $u_n : A_n \times \mathbb{R} \to \mathbb{C}$, for $n \in \mathbb{Z}$, by*

$$u_n(t,x) := |\cos(\arctan(2n))|^{1/2} u_0(t - \arctan(2n), x),$$

*where $A_n := \{t \in \mathbb{R} : t \neq \pi/2 + \arctan(2n) + k\pi, k \in \mathbb{Z}\}$.*

*Then the function $u : A \times \mathbb{R} \to \mathbb{C}$, where $A := \bigcap_{n\in\mathbb{Z}} A_n$, defined by*

$$u(t,x) := \sum_{n=-\infty}^{+\infty} \alpha_n u_n(t,x), \tag{5.25}$$

*is a solution of the problem (5.20) in the following sense:*

- *the series in (5.25) absolutely converges for any $(t,x) \in A \times \mathbb{R}$, hence $u(t,x)$ is well defined for $(t,x) \in A \times \mathbb{R}$;*
- *defining $\tilde{A} = A \setminus \{k\pi, k \in \mathbb{Z}\}$, we have $u \in C^\infty(\tilde{A} \times \mathbb{R})$;*
- *$u$ is a classical solution in $\tilde{A} \times \mathbb{R}$ of the equation (5.21);*
- $$\lim_{t\to 0} \int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}} \bar{u}_0(x)\varphi(x)\,\mathrm{d}x,$$
  *for any $\varphi \in C_c^\infty(\mathbb{R})$;*

- *for any $k \in \mathbb{Z}$, it holds*

$$\lim_{t\to k\pi} \int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = q(k\pi)\int_{\mathbb{R}} \bar{u}_0(x)\varphi(x)\,\mathrm{d}x,$$

  *for any $\varphi \in C_c^\infty(\mathbb{R})$;*

- *for any $n \in \mathbb{Z}\setminus\{0\}$ and $k \in \mathbb{Z}$, we have*

$$\lim_{t\to\pi/2+\arctan(2n)+k\pi} \int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = \alpha_n(2\pi)^{1/2}\varphi(0)+ \\ + \int_{\mathbb{R}} \sum_{h\in\mathbb{Z}\setminus\{n\}} \alpha_h u_h(\pi/2+\arctan(2n)+k\pi, x)\varphi(x)\,\mathrm{d}x,$$

  *for any $\varphi \in C_c^\infty(\mathbb{R})$.*

- *Moreover, $u$ is $4\pi$-periodic in time: $u(t+k4\pi, x) = u(t,x)$, for any $t \in \tilde{A}$ and $x \in \mathbb{R}$.*

As we noticed after the statement of Theorem 5.1 and Theorem 5.5, even in this case, the solution $u(t,\cdot)$ defined by (5.25), for $t \in A$, can be extended to a continuous function, $\tilde{u} : \mathbb{R} \to \mathcal{D}'(\mathbb{R})$. This fact is possible thanks to the appropriate choice of the piecewise constant function $q$ in (6.50) that makes $t \mapsto u_0(t,\cdot)$ a function that can be extended to a continuous function, $\tilde{u}_0 : \mathbb{R} \to \mathcal{D}'(\mathbb{R})$.

More precisely, we have the following Corollary.

**Corollary 5.10** *Under the assumptions of Theorem 5.9, let $u$ be the solution given by Theorem 5.9. Let $\tilde{u} : \mathbb{R} \to \mathcal{D}'(\mathbb{R})$ be the function of distributions defined in the following way:*

$$\tilde{u}(t) := u(t,\cdot), \qquad \text{for } t \in A,$$

*for $n \in \mathbb{Z}\setminus\{0\}$, and $k \in \mathbb{Z}$,*

$$u(\pi/2+\arctan(2n)+k\pi,\cdot) := \alpha_n(2\pi)^{1/2}\delta + \sum_{h\in\mathbb{Z}\setminus\{n\}} \alpha_h u_h(\pi/2+\arctan(2n)+k\pi,\cdot).$$

*Then $\tilde{u}$ is continuous.*

*Proof.* The forth, fifth and sixth points of Theorem 5.9 imply the continuity of $\tilde{u}$ at points $k\pi$, for $k \in \mathbb{Z}$, and at the points $\pi/2+\arctan(2n)+k\pi$, for $n \in \mathbb{Z}\setminus\{0\}$ and $k \in \mathbb{Z}$ . Moreover, the continuity of $u$ at the other points, $t_0 \in \tilde{A}$, easily follows by the continuity of $t \in \tilde{A} \to u(t,x)$, for any $x \in \mathbb{R}$, and the Lebesgue dominated convergence theorem. □

We now consider the non-homogeneous problem with zero initial datum:

$$\begin{cases} \partial_t u(t,x) + \frac{i}{2}(-\partial_{xx}u(t,x) + x^2 u(t,x)) = v(t,x), & (t,x) \in \mathbb{R}\times\mathbb{R} \\ u(0,x) = 0, & x \in \mathbb{R}. \end{cases} \tag{5.26}$$

We note that the functions $u_n$ of Theorem 5.9 can be written as

$$u_n(t,x) = |\cos(\arctan(2n))|^{1/2} |\cos(t - \arctan(2n))|^{-1/2} q_n(t) e^{\frac{i}{2}\tan(\arctan(2n)-t)x^2}, \quad (5.27)$$

where $q_n$ is the piecewise constant function

$$q_n(t) := q(t - \arctan(2n)).$$

As we did for the free particle problem, even for the quantum harmonic oscillator one, we are willing to find a solution of (5.26) as a sum of chirp functions, and thus led to consider a right hand side, $v$, as a sum of chirp functions $v_n$ of a form resembling the one of $u_n$. More precisely, similarly to (5.27), we consider

$$v_n(t,x) := \tilde{g}_n(t) q_n(t) e^{\frac{i}{2}\tan(\arctan(2n)-t)x^2},$$

for $\tilde{g}_n : \mathbb{R} \to \mathbb{C}$ continuous. See the next Theorem 5.11 and the Proposition 6.14.

**Theorem 5.11** *Let $\tilde{g}_n : \mathbb{R} \to \mathbb{C}$, $n \in \mathbb{Z}$, be a sequence of continuous functions such that*

$$\sum_{n=-\infty}^{+\infty} |n|^{1/2} \sup_{t\in[a,b]} |\tilde{g}_n(t)| < +\infty, \qquad \text{for any } [a,b] \subset \mathbb{R}. \quad (5.28)$$

*Let $v_n : A_n \times \mathbb{R} \to \mathbb{C}$, where $A_n := \{t \in \mathbb{R} : t \neq \pi/2 + \arctan(2n) + k\pi, k \in \mathbb{Z}\}$, defined by*

$$v_n(t,x) := \tilde{g}_n(t) q_n(t) e^{\frac{i}{2}\tan(\arctan(2n)-t)x^2}, \quad n \neq 0,$$

*and $v_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ defined by $v_0(t,x) = \tilde{g}_0(t)$. Defining $v : A \times \mathbb{R} \to \mathbb{C}$ by*

$$v(t,x) := \sum_{n=-\infty}^{+\infty} v_n(t,x), \quad (5.29)$$

*where $A := \bigcap_{n\in\mathbb{Z}} A_n$, and $F[\tilde{g}_n] : A_n \times \mathbb{R} \to \mathbb{C}$ by*

$$F[\tilde{g}_n](t) := |\cos(t - \arctan(2n))|^{-1/2} \int_0^t |\cos(s - \arctan(2n))|^{1/2} \tilde{g}_n(s)\, \mathrm{d}s,$$

*then the function $u : A \times \mathbb{R}$, defined by*

$$u(t,x) := \sum_{n=-\infty}^{+\infty} F[\tilde{g}_n](t) q_n(t) e^{\frac{i}{2}\tan(\arctan(2n)-t)x^2}, \quad (5.30)$$

*is a solution of the problem (5.26) in the following sense:*

- *the series in (5.30) absolutely converges for any $(t,x) \in A \times \mathbb{R}$, hence $u(t,x)$ is well defined for $(t,x) \in A \times \mathbb{R}$;*

- *defining* $\tilde{A} = A \setminus \{k\pi, k \in \mathbb{Z}\}$, *we have* $u(\cdot, x) \in C^1(\tilde{A})$ *for any* $x \in \mathbb{R}$ *and* $u(t, \cdot) \in C^\infty(\mathbb{R})$ *for any* $t \in \tilde{A}$;
- $u$ *is a classical solution in* $\tilde{A} \times \mathbb{R}$ *of the equation in* (5.26);
- $\lim_{t\to 0} \int_{\mathbb{R}} u(t, x)\varphi(x)\, \mathrm{d}x = 0$, *for any* $\varphi \in C_c^\infty(\mathbb{R})$;
- *for any* $k \in \mathbb{Z}$, *we have*

$$\lim_{t\to k\pi} \int_{\mathbb{R}} u(t, x)\varphi(x)\, \mathrm{d}x = \int_{\mathbb{R}} u(k\pi, x)\varphi(x)\, \mathrm{d}x,$$

  *for any* $\varphi \in C_c^\infty(\mathbb{R})$;
- *for any* $n \in \mathbb{Z} \setminus \{0\}$ *and* $k \in \mathbb{Z}$, *let* $t_{n,k} := \pi/2 + \arctan(2n) + k\pi$. *We have*

$$\begin{aligned}
&\lim_{t\to t_{n,k}} \int_{\mathbb{R}} u(t, x)\varphi(x)\, \mathrm{d}x = \\
&(2\pi)^{1/2} \int_0^{t_{n,k}} |\cos(s - \arctan(2n))|^{1/2} \tilde{g}_n(s)\, \mathrm{d}s\, \varphi(0) + \\
&+ \int_{\mathbb{R}} \sum_{h\in\mathbb{Z}\setminus\{n\}} F[\tilde{g}_h](t_{n,k}) q_h(t_{n,k}) e^{\frac{i}{2}\tan(\arctan(2h)-t_{n,k})x^2} \varphi(x)\, \mathrm{d}x,
\end{aligned}$$

  *for any* $\varphi \in C_c^\infty(\mathbb{R})$.

The condition (5.28) is imposed in order to have the convergence of the series (5.30), for $t = k\pi$ and $k \in \mathbb{Z}$, $k \neq 0$.

**Remark 5.12** As observed for the free particle problem, a simple example of functions $\tilde{g}_n$ satisfying (5.16) is given by constants: $\tilde{g}_n(t) = \alpha_n \in \mathbb{C}$, such that (5.4) holds. In this case, the functions $F[\alpha_n]$ cannot be computed in terms of elementary functions. Indeed, the function

$$E(t) := \int_0^t |\cos(s)|^{1/2}\, \mathrm{d}s$$

can be computed in terms of elliptic integrals of second kind, that cannot be expressed by elementary functions (see for instance [13]).

Even in this case, the solution $u(t, \cdot)$ defined by (5.30), for $t \in A$, can be extended to a continuous function $u : \mathbb{R} \to \mathcal{D}'(\mathbb{R})$. Indeed, the following Corollary holds.

**Corollary 5.13** *Under the assumptions of Theorem 5.11, let* $u$ *be the solution given by Theorem 5.11. Let* $\tilde{u} : \mathbb{R} \to \mathcal{D}'(\mathbb{R})$ *be the function of distributions defined in the following way:*

$$\tilde{u}(t) := u(t, \cdot), \qquad \textit{for } t \in A,$$

*for* $n \in \mathbb{Z} \setminus \{0\}$, $k \in \mathbb{Z}$ *and* $t_{n,k} = \pi/2 + \arctan(2n) + k\pi$,

$$u(t_{n,k}, \cdot) := \left( (2\pi)^{1/2} \int_0^{t_{n,k}} |\cos(s - \arctan(2n))|^{1/2} \tilde{g}_n(s)\, \mathrm{d}s \right) \delta + \sum_{h \in \mathbb{Z} \setminus \{n\}} F[\tilde{g}_h](t_{n,k}) q_h(t_{n,k}) e^{\frac{i}{2} \tan(\arctan(2h) - t_{n,k})(\cdot)^2}.$$

*Then* $\tilde{u}$ *is continuous.*

*Proof.* The last three points of Theorem 5.11 imply the continuity of $\tilde{u}$ at points $k\pi$, for $k \in \mathbb{Z}$, and at the points $\pi/2 + \arctan(2n) + k\pi$, for $n \in \mathbb{Z} \setminus \{0\}$ and $k \in \mathbb{Z}$ . Moreover, the continuity of $u$ at the other points, $t_0 \in \tilde{A}$, easily follows by the continuity of $t \in \tilde{A} \to u(t, x)$, for any $x \in \mathbb{R}$, and the Lebesgue dominated convergence theorem. □

As we did in the case of the free particle problem, thanks to the linearity of the problem (5.19), by taking the sum of the solutions obtained in Theorems 5.9 and 5.11, we do obtain a solution to problem (5.19), with $\bar{u}_0$ and $v$ of the form (5.23) and (5.29). We are going to state the final result.

**Theorem 5.14** *Let the assumptions of Theorems 5.9 and 5.11 hold. Then, there exists a solution* $u$ *of the Cauchy problem* (5.19) *satisfying all the properties given in Theorem 5.9. Moreover,* $u$ *can be extended to a continuous function* $\tilde{u} : \mathbb{R} \to \mathcal{D}'(\mathbb{R})$.

## 5.3 Applications to the Gabor wave front set

Let us show how the results obtained in Theorem 5.1 match with the results on the propagation of the Gabor wave front set for the Schrödinger equation, see [5] and [14].

Let us recall the definition of the Gabor wave front set for a Schwartz distribution (see [9] and [18]). For $z = (x, \xi) \in \mathbb{R}^2$ and $u \in \mathscr{S}'(\mathbb{R})$, we define

$$Tu(z) := 2^{-1/2} \pi^{-3/4} \int_{\mathbb{R}} e^{-iy\xi} e^{-|x-y|^2/2} u(y)\, \mathrm{d}y. \tag{5.31}$$

The map $T$ is called Bargman transform of $u$. Notice that, up to a coefficient, for any $x \in \mathbb{R}$, the map $\xi \mapsto Tu(x, \xi)$ is the Fourier transform of the distribution $y \mapsto e^{-|x-y|^2/2} u(y)$. In (5.31), the integral denotes a duality when $u$ is not a function. We define the Gabor wave front set of $u \in \mathcal{S}'(\mathbb{R})$, denoted by $WF_G(u)$, in this way: for $z_0 \in \mathbb{R}^2 \setminus \{(0,0)\}$, $z_0 \notin WF_G(u)$ if there exists a conic neighborhood

$$V_{z_0,\varepsilon} := \{z \in \mathbb{R}^2 \setminus \{(0,0)\} : \left| \frac{z}{z} - \frac{z_0}{z_0} \right| < \varepsilon, |z| > 1/\varepsilon\}$$

such that

$$|z|^N |Tu(z)| \leq C_N, \qquad \forall\, z \in V_{z_0,\varepsilon}, \quad \forall N \in \mathscr{N}, \tag{5.32}$$

for some constants $C_N > 0$.

As it concerns our applications, it would be sufficient to examine, from [14], the following example. For $\lambda \in \mathbb{R} \setminus \{0\}$, we consider the so called chirp function $u(x) = e^{i\lambda x^2/2}$. For this kind of function, we have

$$WF_G(u) = \{(x, \xi) \in \mathbb{R}^2 : x \neq 0, \xi = \lambda x\}. \tag{5.33}$$

We even notice that

$$WF_G(1) = \{(x, \xi) \in \mathbb{R}^2 : x \neq 0, \xi = 0\}, \qquad WF_G(\delta) = \{(x, \xi) \in \mathbb{R}^2 : x = 0, \xi \neq 0\}. \tag{5.34}$$

Let us recall, from [14], the following basic results on the propagation of singularities for the Schrödinger equation

$$-i\partial_t u + a(x, -i\partial_x)u = 0, \tag{5.35}$$

where $a(x, \xi)$ is a real quadratic form. In our examples, we have the two particular cases: $a(x, \xi) = \xi^2$ for the free particle equation in (5.2), and $a(x, \xi) = \frac{1}{2}(x^2 + \xi^2)$ for the quantum harmonic oscillator equation in (5.20).

We take into account the corresponding Hamiltonian system:

$$\dot{x} = a_\xi(x, \xi), \quad \dot{\xi} = -a_x(x, \xi). \tag{5.36}$$

We denote by $\chi_t : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ the flow of system (5.36), namely $\chi_t(y, \eta) = (x(t), \xi(t))$, where $(x(t), \xi(t))$ is the unique solution of system (5.36) such that $(x(0), \xi(0)) = (y, \eta)$.

For solutions of equation (5.35), we have

$$WF_G(u(t, \cdot)) = \chi_t(WF_G(u(0, \cdot))), \qquad \forall\, t \in \mathbb{R}. \tag{5.37}$$

In this perspective, we revisit our preceding results.

**Free particle equation.** In this case, we have $a(x, \xi) = \xi^2$ and the system (5.36) is

$$\dot{x} = 2\xi, \quad \dot{\xi} = 0. \tag{5.38}$$

Integrating the system (5.38), we obtain that

$$\chi_t(y, \eta) = (y + 2\eta t, \eta), \qquad \forall\, (y, \eta) \in \mathbb{R}^2.$$

Let $n \in \mathbb{Z}$. From (5.33), we have

$$WF_G(e^{inx^2}) = \{(x, \xi) \in \mathbb{R}^2 : x \neq 0, \xi = 2nx\}. \tag{5.39}$$

We compute

$$\chi_t(WF_G(e^{inx^2})) = \{(x(1 + 4nt), 2nx) : x \neq 0\}.$$

So, we have that
$$\chi_{-1/(4n)}(WF_G(e^{inx^2})) = \{(0,\xi) : \xi \neq 0\},$$
and
$$\chi_t(WF_G(e^{inx^2})) = \{(x, 2nx/(1+4nt)) : x \neq 0\}, \qquad \text{for } t \neq -1/(4n).$$
This matches with
$$WF_G(u_n(t,\cdot)),$$
where $u_n$ is the solution in (5.14), as a consequence of (5.33).

Considering the initial datum (5.5), we have
$$WF_G(\bar{u}_0) = \bigcup_{n\in\mathbb{Z},\alpha_n\neq 0} \{(x,\xi) \in \mathbb{R}^2 : x \neq 0, \xi = 2nx\} \tag{5.40}$$
and
$$\chi_t(WF_G(\bar{u}_0)) = \bigcup_{n\in\mathbb{Z},\alpha_n\neq 0} \{(x(1+4nt), 2nx) : x \neq 0\}.$$
Because of the previous observation, we have that
$$\chi_t(WF_G(\bar{u}_0)) = WF_G(\tilde{u}(t)), \qquad \forall\, t \in \mathbb{R},$$
where $\tilde{u}(t)$is the generalized solution given in Corollary 5.2.

**Quantum harmonic oscillator equation.** In this case, we have $a(x,\xi) = \frac{1}{2}(x^2+\xi^2)$ and the system (5.36) is
$$\dot{x} = \xi, \quad \dot{\xi} = -x. \tag{5.41}$$
Integrating the system (5.41), we get that
$$\chi_t(y,\eta) = (y\cos t + \eta \sin t, \eta\cos t - y \sin t), \qquad \forall\,(y,\eta) \in \mathbb{R}^2.$$
Let $n \in \mathbb{Z}$, by (5.39) and the expression of the flow $\chi_t$, we get
$$\chi_t(WF_G(e^{inx^2})) = \{(x(\cos t + 2n\sin t), x(2n\cos t - \sin t)) : x \neq 0\}.$$
Then, denoting by $t_{n,k} := \pi/2 + \arctan(2n) + k\pi$, $k \in \mathbb{Z}$, we have that
$$\chi_{t_{n,k}}(WF_G(e^{inx^2})) = \{(0,\xi) : \xi \neq 0\}.$$
For $t \neq t_{n,k}$,
$$\chi_t(WF_G(e^{inx^2})) = \{(x, x\frac{(2n\cos t - \sin t))}{(\cos t + 2n \sin t)} : x \neq 0\} = \{(x, x\frac{2n - \tan t}{1+2n\tan t} : x \neq 0\}.$$
Using the addition formula for the tangent, we have
$$\frac{2n - \tan t}{1 + 2n\tan t} = \tan(\arctan(2n) - t).$$

Since, denoting by $u_n$ the function (5.27), by (5.33),

$$WF_G(u_n(t,\cdot)) = \{(x, x\tan(\arctan(2n) - t) : x \neq 0\},$$

we obtain the matching

$$\chi_t(WF_G(e^{inx^2})) = WF_G(u_n(t,\cdot)).$$

Reasoning as in the previous paragraph, we finally have that

$$\chi_t(WF_G(\bar{u}_0)) = WF_G(\tilde{u}(t)), \qquad \forall\, t \in \mathbb{R},$$

where $\tilde{u}(t)$is the generalized solution given in Corollary 5.10.

## 5.4 Fourier transform and chirp solutions

In this Section, we show that the solution to the free particle problems, given in Theorems 5.1 and 5.5, can be found using the Fourier transform representation in $\mathcal{S}'(\mathbb{R})$, as well. Let us stress that this approach cannot be carried out for the quantum harmonic oscillator problem.

We assume that $\bar{u}_0$ is of the form

$$\bar{u}_0(x) = \sum_{n=-\infty}^{+\infty} \alpha_n e^{inx^2}, \tag{5.42}$$

for $\alpha_n \in \mathbb{C}$, $n \in \mathbb{Z}$, such that

$$\sum_{n=-\infty}^{+\infty} |\alpha_n| < +\infty. \tag{5.43}$$

First of all, for $n \in \mathbb{Z} \setminus \{0\}$, using (2.16) with $z = -ni$, we have that

$$\mathscr{F}(e^{inx^2})(\xi) = \left(\frac{\pi i}{n}\right)^{1/2} e^{\frac{-i\xi^2}{4n}}. \tag{5.44}$$

Moreover, it holds that

$$\mathscr{F}(\bar{u}_0)(\xi) = \sum_{n=-\infty}^{+\infty} \alpha_n \mathscr{F}(e^{inx^2})(\xi). \tag{5.45}$$

Indeed, the sequence $s_N$ of the partial sum

$$s_N(x) := \sum_{n=-N}^{+N} \alpha_n e^{inx^2} \tag{5.46}$$

converges to $\bar{u}_0$ in $\mathcal{S}'(\mathbb{R})$. For the continuity of $\mathscr{F}$ as an operator in $\mathcal{S}'(\mathbb{R})$, it follows (5.45).

We recall the formulation of the problem (5.2) in Fourier variables:

$$\frac{\partial}{\partial t}\hat{u}(t,\xi) + i\xi^2 \hat{u}(t,\xi) = 0, \qquad \hat{u}(0,\xi) = \mathscr{F}(\bar{u}_0)(\xi), \tag{5.47}$$

which has solution

$$\hat{u}(t,\xi) = e^{-i\xi^2 t}\mathscr{F}(\bar{u}_0)(\xi).$$

From the representation (5.45) and (5.44), it follows that

$$\begin{aligned}\hat{u}(t,\xi) &= e^{-i\xi^2 t}\sum_{n=-\infty}^{+\infty}\alpha_n\mathscr{F}(e^{inx^2})(\xi)\\ &= \alpha_0\delta + \sum_{n\in\mathbb{Z},n\neq 0}\alpha_n\left(\frac{\pi i}{n}\right)^{1/2} e^{\frac{-i(1+4nt)\xi^2}{4n}}.\end{aligned} \tag{5.48}$$

On the other hand, still using (2.16), it holds that

$$\mathscr{F}(e^{\frac{inx^2}{1+4nt}})(\xi) = \left(\frac{\pi i(1+4nt)}{n}\right)^{1/2} e^{\frac{-i(1+4nt)\xi^2}{4n}}, \tag{5.49}$$

for $t \neq -1/(4n)$. Then, we have

$$\hat{u}(t,\xi) = \alpha_0\delta + \sum_{n\in\mathbb{Z},n\neq 0}\alpha_n\left(\frac{\pi i}{n}\right)^{1/2}\left(\frac{\pi i(1+4nt)}{n}\right)^{-1/2}\mathscr{F}(e^{\frac{inx^2}{1+4nt}})(\xi), \tag{5.50}$$

for $t \neq -1/(4n)$. Moreover,

$$\begin{aligned}\hat{u}(-1/(4k),\xi) = \alpha_0\delta + \alpha_k\left(\frac{\pi i}{k}\right)^{1/2} &+ \\ \sum_{n\in\mathbb{Z},n\neq 0,n\neq k}\alpha_n\left(\frac{i(1-n/k)}{n}\right)^{-1/2}&\mathscr{F}(e^{\frac{inx^2}{1-n/k}})(\xi).\end{aligned} \tag{5.51}$$

From the last two equations, we can define

$$u(t,x) = \sum_{n\in\mathbb{Z}}\alpha_n\left(\frac{\pi i}{n}\right)^{1/2}\left(\frac{\pi i(1+4nt)}{n}\right)^{-1/2} e^{\frac{inx^2}{1+4nt}}, \tag{5.52}$$

and

$$\begin{aligned}u(-1/(4k),x) = \alpha_k\left(\frac{\pi i}{k}\right)^{1/2}&\delta + \\ \sum_{n\in\mathbb{Z},n\neq k}\alpha_n\left(\frac{\pi i}{n}\right)^{1/2}\left(\frac{\pi i(1-n/k)}{n}\right)^{-1/2}& e^{\frac{inx^2}{1-n/k}}.\end{aligned} \tag{5.53}$$

The last two equations trace out a function, $u(t,\cdot) \in \mathcal{S}'(\mathbb{R})$, for any $t \in \mathbb{R}$, which coincides with the continuous extension of the solution given by Theorem 5.1.

With similar computations, even the solution of the non-homogeneous problem (5.12) can be computed, having $v$ satisfying the same assumptions as the ones of Theorem 5.5.

Finally, we observe that the quantum harmonic oscillator problem

$$\begin{cases} \partial_t u(t,x) + \frac{i}{2}(-\partial_{xx} u(t,x) + x^2 u(t,x)) = 0, & (t,x) \in \mathbb{R} \times \mathbb{R} \\ u(0,x) = \bar{u}_0(x), & x \in \mathbb{R}, \end{cases} \tag{5.54}$$

via Fourier variables, becomes

$$\begin{cases} \partial_t \hat{u}(t,\xi) + \frac{i}{2}(-\partial_{\xi\xi} \hat{u}(t,\xi) + \xi^2 \hat{u}(t,\xi)) = 0, & (t,\xi) \in \mathbb{R} \times \mathbb{R} \\ u(0,x) = \bar{u}_0(x), & x \in \mathbb{R}, \end{cases} \tag{5.55}$$

which is still a problem with the same level of complexity as the initial one. So, in this case, the Fourier transform method doesn't give any help and the pure differential calculus, earlier carried out in this thesis, proves to be so very useful to us.

# 6 Proofs of main results

## 6.1 Proof of Theorem 5.1

**Lemma 6.1** *If $u$ is of the form $u(t,x) = g(t)e^{if(t)x^2}$, for some real differentiable function $f$ and complex valued function $g$, such that it solves the equation*

$$\partial_t u(t,x) - i\partial_{xx} u(t,x) = 0, \tag{6.1}$$

*then $f$ and $g$ cannot be defined on all $\mathbb{R}$, but on $\mathbb{R} \setminus \{-b/4\}$, for some $b \in \mathbb{R}$,*

$$f(t) = \frac{1}{4t+b}$$

*and*

$$g(t) = \begin{cases} C_1|4t+b|^{-1/2} & \text{if } t \in (-\infty, -b/4) \\ C_2|4t+b|^{-1/2} & \text{if } t \in (-b/4, +\infty) \end{cases}$$

*for $C_1, C_2 \in \mathbb{C}$. Moreover, $\lim_{t\to -b/4} |u(t,x)| = +\infty$, for any $x \in \mathbb{R}$, when $C_1$ and $C_2$ are not 0.*

*Moreover, if $C_1 = iC_2$, then*

$$\lim_{t\to -b/4} \int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = C_2(\pi i)^{1/2}\varphi(0), \tag{6.2}$$

*for any $\varphi \in C_c^\infty(\mathbb{R})$.*

*Proof.* Let $u(t,x) = g(t)e^{if(t)x^2}$, for some differentiable functions $f : D \subset \mathbb{R} \to \mathbb{R}$ and $g : D \subset \mathbb{R} \to \mathbb{C}$ defined on a open domain $D \subset \mathbb{R}$. We compute

$$\partial_t u(t,x) = (g'(t) + ig(t)f'(t)x^2)e^{if(t)x^2} \tag{6.3}$$

and

$$\partial_{xx} u(t,x) = (-4g(t)f^2(t)x^2 + i2g(t)f(t))e^{if(t)x^2}. \tag{6.4}$$

Then,

$$\partial_t u(t,x) - i\partial_{xx} u(t,x) = (g'(t) + 2g(t)f(t) + i[g(t)f'(t) + 4g(t)f^2(t)]x^2)e^{if(t)x^2}. \tag{6.5}$$

If (6.1) holds, then

$$g'(t) + 2g(t)f(t) = 0, \tag{6.6}$$
$$g(t)f'(t) + 4g(t)f^2(t) = 0. \tag{6.7}$$

Denoting by $F$ a primitive of $f$, all the solutions of (6.6) are of the form $g(t) = Ke^{-2F(t)}$, where the constant $K \in \mathbb{C}$ could be different on every connected component of $D$. In particular, on the connected components where $K \neq 0$, the equation (6.7) is then equivalent to

$$f'(t) + 4f^2(t) = 0.$$

All the solutions to this equation are of the form $f(t) = \frac{1}{4t+b}$, for some $b \in \mathbb{R}$, with maximal domain $\mathbb{R} \setminus \{-b/4\}$. By a direct computation and the above observation, we obtain that $g(t) = K|4t + b|^{-1/2}$, for some constants $K$ on the two components $(-\infty, -b/4)$ and $(-b/4, +\infty)$.

Finally, we prove (6.2). It is sufficient to prove the result for $b = 0$. From the expression of the fundamental solution for $d = 1$ in Subsection 2.2.2, we observe that $k(t, x) = (\pi i)^{-1/2}|4t|^{-1/2}e^{ix^2/(4t)}$, for $t > 0$ and $k(t, x) = (-\pi i)^{-1/2}|4t|^{-1/2}e^{ix^2/(4t)}$, for $t < 0$. Applying (2.21), it follows that

$$\lim_{t\to 0^-} \int_{\mathbb{R}} C_1|4t|^{-1/2}e^{ix^2/(4t)}\varphi(x)\,\mathrm{d}x = C_1e^{-i\pi/4}(\pi)^{1/2}\varphi(0) \tag{6.8}$$

and

$$\lim_{t\to 0^+} \int_{\mathbb{R}} C_2|4t|^{-1/2}e^{ix^2/(4t)}\varphi(x)\,\mathrm{d}x = C_2e^{i\pi/4}(\pi)^{1/2}\varphi(0). \tag{6.9}$$

Finally, (6.2) follows from (6.8) and (6.9) if $C_1 = iC_2$.

For arbitrary $b \in \mathbb{R}$, it is sufficient to change $t$ into $t + b/4$ and take the limit for $t \to -b/4$.

□

**Proposition 6.2** *The function* $k(t, x) = (4\pi i t)^{-1/2}e^{\frac{ix^2}{4t}}$ *belongs to* $C^\infty((\mathbb{R} \setminus \{0\}) \times \mathbb{R})$*, satisfies*

$$\partial_t u(t, x) - i\partial_{xx}u(t, x) = 0, \qquad \text{for } (t, x) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}, \tag{6.10}$$

*and* $\lim_{t\to 0} \int_{\mathbb{R}} k(t, x)\varphi(x)\,\mathrm{d}x = \varphi(0)$*, for any* $\varphi \in C_c^\infty(\mathbb{R})$*.*

*Proof.* The proof follows by application of Lemma 6.1 with $b = 0$. The regularity of $k$ is obvious. The last limit is (2.21). □

**Proposition 6.3** *Let* $\beta \in \mathbb{R}$*,* $\beta \neq 0$*. The function* $u_\beta(t, x) = (\frac{\pi i}{\beta})^{1/2}k(t+1/(4\beta), x)$*, where* $k$ *is given by Proposition 6.2, belongs to* $C^\infty((\mathbb{R} \setminus \{-1/(4\beta)\}) \times \mathbb{R})$*, satisfies*

$$\partial_t u(t, x) - i\partial_{xx}u(t, x) = 0, \qquad \text{for } (t, x) \in (\mathbb{R} \setminus \{-1/(4\beta)\}) \times \mathbb{R}, \tag{6.11}$$

*and* $u_\beta(0, x) = e^{i\beta x^2}$*.*
*Moreover,* $\lim_{t\to -1/(4\beta)} \int_{\mathbb{R}} u_\beta(t, x)\varphi(x)\,\mathrm{d}x = (\frac{\pi i}{\beta})^{1/2}\varphi(0)$*, for any* $\varphi \in C_c^\infty(\mathbb{R})$*.*

*Proof.* The result follows by Proposition 6.2, observing that $u_\beta$ is obtained from $k$ by a translation in time and a multiplication by a constant. □

**Proposition 6.4** *Let $\alpha_n \in \mathbb{C}$, $n \in \mathbb{Z}$, be a sequence such that $\sum_{n=-\infty}^{+\infty} |\alpha_n| < +\infty$. Denoting by $u_0(t,x) = 1$ and $u_n(t,x)$, for $n \neq 0$, the functions given by Proposition 6.3 for $\beta = n$, then the series,*

$$\sum_{n=-\infty}^{+\infty} \alpha_n u_n(t,x), \tag{6.12}$$

*absolutely converges for any $(t,x) \in A \times \mathbb{R}$, where $A := \{t \in \mathbb{R} : t \neq 1/(4n), n \in \mathbb{Z} \setminus \{0\}\}$. We define*

$$u(t,x) := \sum_{n=-\infty}^{+\infty} \alpha_n u_n(t,x), \qquad (t,x) \in A \times \mathbb{R}, \tag{6.13}$$

*and $\tilde{A} = A \setminus \{0\}$. Then $u \in C^\infty(\tilde{A} \times \mathbb{R})$ and $u$ is a classical solution in $\tilde{A} \times \mathbb{R}$ of the equation in (5.2).*

*Proof.* First of all, we observe that $u_n(t,x)$ can be written in the following form

$$u_n(t,x) = c_n(t)|4nt+1|^{-1/2} e^{\frac{inx^2}{4nt+1}},$$

where $c_n$ is defined in (5.15) and $|c_n| = 1$.

Let $\varepsilon > 0$, $\varepsilon \neq 1/(4n)$, $n \in \mathbb{N}$. We define $C_\varepsilon = \mathbb{R} \setminus (\cup_{|n|>1/(4\varepsilon)} (1/(4n) - \varepsilon, 1/(4n) + \varepsilon) \cup (-\varepsilon, \varepsilon))$.

Since

$$|\alpha_n u_n(t,x)| \leq |\alpha_n||4nt+1|^{-1/2},$$

we have that

$$\sup_{t \in C_\varepsilon} |\alpha_n u_n(t,x)| \leq |\alpha_n||4n\varepsilon|^{-1/2}.$$

By Weierstrass criterion, the series in (6.13) uniformly converges in $C_\varepsilon \times \mathbb{R}$, for every $\varepsilon > 0$. In particular, $u(t,x)$ is well defined in $\tilde{A} \times \mathbb{R}$. Moreover, it follows that $u$ is continuous in $\tilde{A} \times \mathbb{R}$.

Since, for $k \in \mathbb{N}$,

$$\frac{\partial^k}{\partial x^k} u_n(t,x) = c_n(t)|4nt+1|^{-1/2}(2inx)^k(4nt+1)^{-k} e^{\frac{inx^2}{4nt+1}},$$

we have

$$\left|\alpha_n \frac{\partial^k}{\partial x^k} u_n(t,x)\right| \leq |\alpha_n||4nt+1|^{-k-1/2}|2nx|^k$$

and

$$\sup_{(t,x) \in C_\varepsilon \times [-1/\varepsilon, 1/\varepsilon]} \left|\alpha_n \frac{\partial^k}{\partial x^k} u_n(t,x)\right| \leq |\alpha_n||4n\varepsilon|^{-k-1/2}|2n1/\varepsilon|^k.$$

By Weierstrass criterion, the series

$$\sum_{n=-\infty}^{+\infty} \alpha_n \frac{\partial^k}{\partial x^k} u_n(t,x)$$

uniformly converges in $C_\varepsilon \times [-1/\varepsilon, 1/\varepsilon]$ for every $\varepsilon > 0$.

Since

$$\frac{\partial}{\partial t}u_n(t,x) = c_n(t)(-2n|4nt+1|^{-3/2}\,\mathrm{sign}\,(4nt+1) - 4in^2x^2(4nt+1)^{-2}|4nt+1|^{-1/2})e^{\frac{inx^2}{4nt+1}},$$

we have that

$$\left|\alpha_n \frac{\partial}{\partial t}u_n(t,x)\right| \le |\alpha_n|(2|n||4nt+1|^{-3/2} + |4nx|^2||4nt+1|^{-5/2})$$

and

$$\sup_{(t,x)\in C_\varepsilon\times[-1/\varepsilon,1/\varepsilon]} \left|\alpha_n \frac{\partial}{\partial t}u_n(t,x)\right| \le |\alpha_n|(2|n||4n\varepsilon|^{-3/2} + |4n1/\varepsilon|^2||4n\varepsilon|^{-5/2}).$$

By Weierstrass criterion, the series

$$\sum_{n=-\infty}^{+\infty} \alpha_n \frac{\partial}{\partial t}u_n(t,x)$$

uniformly converges in $C_\varepsilon \times [-1/\varepsilon, 1/\varepsilon]$ for every $\varepsilon > 0$. One can analogously prove that the series of the $k$-derivatives with respect to $t$ of $\alpha_n u_n$ uniformly converges in $C_\varepsilon \times [-1/\varepsilon, 1/\varepsilon]$, for every $\varepsilon > 0$.

As a consequence of the uniform convergence of the derivatives, we obtain the regularity $u \in C^\infty(\tilde{A} \times \mathbb{R})$.

Moreover, since we can exchange the series with the derivatives, by Proposition 6.3, we obtain that $u$ is a classical solution of the equation in $\tilde{A} \times \mathbb{R}$.

□

**Proposition 6.5** *The function $u$ defined in* (6.13) *satisifes*

$$\lim_{t\to 0} \int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}} \bar{u}_0(x)\varphi(x)\,\mathrm{d}x,$$

*for any $\varphi \in C_c^\infty(\mathbb{R})$.*

*Proof.* Let $\varphi \in C_c^\infty(\mathbb{R})$. First of all, we prove that

$$\int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = \sum_{n=-\infty}^{+\infty} \int_{\mathbb{R}} \alpha_n u_n(t,x)\varphi(x)\,\mathrm{d}x, \qquad \forall t \in A. \tag{6.14}$$

Indeed, as proved in Proposition 6.4, the convergence of the series $\sum_{n=-\infty}^{+\infty} \alpha_n u_n(t,x)\varphi(x) = u(t,x)\varphi(x)$ is uniform on $C_\varepsilon \times \mathbb{R}$, for any $\varepsilon > 0$. The identity (6.14) follows by exchanging the integral with the series, because of the uniform convergence of the series and the compactness of the support of $\varphi$.

We prove that there exist $C$ independent of $n \in \mathbb{Z}$, and $t \in A$ such that

$$\left| \int_{\mathbb{R}} \alpha_n u_n(t,x)\varphi(x)\,\mathrm{d}x \right| \le C|\alpha_n||n|^{-1/2}. \tag{6.15}$$

Indeed, by Proposition 6.2, there exists $\delta > 0$ such that

$$\left| \int_{\mathbb{R}} k(t,x)\varphi(x)\,\mathrm{d}x \right| \le |\varphi(0)| + 1 \qquad \forall t \in [-\delta, 0) \cup (0, \delta], \tag{6.16}$$

and

$$\begin{aligned} \left| \int_{\mathbb{R}} k(t,x)\varphi(x)\,\mathrm{d}x \right| &\le \sup_{x\in\mathbb{R}} |\varphi(x)| \int_{-R}^{R} |k(t,x)|\,\mathrm{d}x \\ &\le \sup_{x\in\mathbb{R}} |\varphi(x)| 2R(4\pi)^{-1/2}|\delta|^{-1/2} \qquad \forall t \in (-\infty, -\delta) \cup (\delta, +\infty). \end{aligned} \tag{6.17}$$

By (6.16) and (6.17), there exists $C > 0$ such that

$$\left| \int_{\mathbb{R}} k(t,x)\varphi(x)\,\mathrm{d}x \right| \le C \qquad \forall t \in (-\infty, 0) \cup (0, +\infty). \tag{6.18}$$

Since $u_n(t,x) = (\frac{\pi i}{n})^{1/2} k(t + \frac{1}{4n}, x)$, by (6.18), we obtain (6.15).

By (6.15), it follows that

$$\lim_{t\to 0} \sum_{n=-\infty}^{+\infty} \int_{\mathbb{R}} \alpha_n u_n(t,x)\varphi(x)\,\mathrm{d}x = \sum_{n=-\infty}^{+\infty} \lim_{t\to 0} \int_{\mathbb{R}} \alpha_n u_n(t,x)\varphi(x)\,\mathrm{d}x. \tag{6.19}$$

Since $u_n(\cdot, x)$ is continuous at 0, we get

$$\lim_{t\to 0} \int_{\mathbb{R}} \alpha_n u_n(t,x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}} \alpha_n u_n(0,x)\varphi(x)\,\mathrm{d}x,$$

and by (6.14) we conclude. □

## 6.2 Proof of Theorem 5.5

**Proposition 6.6** *Let $\tilde{g} : \mathbb{R} \to \mathbb{C}$ be a continuous function not identically 0. Let*

$$v(t,x) := \tilde{g}(t) e^{i\tilde{f}(t)x^2}. \tag{6.20}$$

*If $\tilde{f}$ is not of the form $\tilde{f}(t) = \frac{1}{4t+b}$ for some $b \in \mathbb{R}$, then the equation*

$$\partial_t u(t,x) - i\partial_{xx} u(t,x) = v(t,x) \tag{6.21}$$

*has no solution of the form $u(t,x) = g(t)e^{if(t)x^2}$.*

*If* $\tilde f(t) = \frac{1}{4t+b}$ *for some* $b \in \mathbb{R}$*, for* $t \neq -b/4$*, then*

$$u(t,x) := |4t+b|^{-1/2} \int_0^t |4s+b|^{1/2} \tilde g(s)\,\mathrm{d}s \;\; e^{\frac{ix^2}{4t+b}} \tag{6.22}$$

*is a classical solution of equation* (6.21) *in* $(\mathbb{R} \setminus \{-b/4\}) \times \mathbb{R}$ *and* $u(0,x) = 0$ *for every* $x \in \mathbb{R}$*.*

*Moreover,* $u(\cdot, x) \in C^1(\mathbb{R} \setminus \{-b/4\})$ *for any* $x \in \mathbb{R}$ *and* $u(t, \cdot) \in C^\infty(\mathbb{R})$ *for any* $t \in \mathbb{R} \setminus \{-b/4\}$*.*

*Finally, if* $c$ *is the piecewise constant function defined in* (5.13)*, we have*

$$\lim_{t \to -b/4} c(4t+b) \int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = (\pi)^{1/2} \int_0^{-b/4} |4s+b|^{1/2} \tilde g(s)\,\mathrm{d}s\,\varphi(0), \tag{6.23}$$

*for any* $\varphi \in C_c^\infty(\mathbb{R})$*.*

*Proof.* Let us assume that $u(t,x) = g(t)e^{if(t)x^2}$ for some real differentiable functions $f : D \subset \mathbb{R} \to \mathbb{R}$ and $g : D \subset \mathbb{R} \to \mathbb{C}$, defined on a open domain $D \subset \mathbb{R}$. Using the computations done in Lemma 6.1, by (6.5), if (6.21) holds for $v$ of the form (6.27), then

$$g'(t) + 2g(t)f(t) = \tilde g(t), \tag{6.24}$$

$$g(t)f'(t) + 4g(t)f^2(t) = 0, \tag{6.25}$$

$$e^{if(t)x^2} = e^{i\tilde f(t)x^2} \tag{6.26}$$

The last equality holds if and only if

$$f(t)x^2 = \tilde f(t)x^2 + 2k\pi,$$

for some $k \in \mathbb{Z}$ and for any $x \in \mathbb{R}$. This implies that $f(t) = \tilde f(t)$ for any $t \in D$.

If $\tilde f(t)$ is not of the form $\tilde f(t) = \frac{1}{4t+b}$ for some $b \in \mathbb{R}$, $t \neq -c/4$, then equation (6.25) does not hold on the set where $g(t) \neq 0$. Then, denoting by $F$ a primitive of $f$, the solutions of equation (6.24) are of the form

$$g(t) = e^{-2F(t)}e^{2F(0)}g(0) + e^{-2F(t)} \int_0^t e^{2F(s)} \tilde g(s)\,\mathrm{d}s.$$

By a direct computation and by imposing the condition $g(0) = 0$, we get to the solution (6.22).

By the continuity of the function $t \mapsto \int_0^t |4s+b|^{1/2}\tilde g(s)\,\mathrm{d}s$, and the limits (6.8) and (6.9), we obtain (6.23).

□

**Proposition 6.7** *Let* $\tilde g_n : \mathbb{R} \to \mathbb{C}$*,* $n \in \mathbb{Z}$*, be a sequence of continuous functions such that* $\sum_{n=-\infty}^{+\infty} \sup_{t \in [a,b]} |\tilde g_n(t)| < +\infty$ *for any* $[a,b] \subset \mathbb{R}$*, and*

$$v(t,x) := \sum_{n=-\infty}^{+\infty} \tilde g_n(t) c_n(t) e^{\frac{inx^2}{4nt+1}}. \tag{6.27}$$

*Defining* $F[\tilde{g}_n](t) := |4nt+1|^{-1/2}\int_0^t |4ns+1|^{1/2}\tilde{g}_n(s)\,\mathrm{d}s$, *then the series*

$$\sum_{n=-\infty}^{+\infty} F[\tilde{g}_n](t)c_n(t)e^{\frac{inx^2}{4nt+1}} \tag{6.28}$$

*absolutely converges for any* $(t,x)\in A\times\mathbb{R}$, *where* $A := \{t\in\mathbb{R} : t\neq 1/(4n), n\in\mathbb{Z}\setminus\{0\}\}$. *We define*

$$u(t,x) := \sum_{n=-\infty}^{+\infty} F[\tilde{g}_n](t)c_n(t)e^{\frac{inx^2}{4nt+1}}. \tag{6.29}$$

*We define* $\tilde{A} = A\setminus\{0\}$. *Then* $u(\cdot,x)\in C^1(\tilde{A})$ *for any* $x\in\mathbb{R}$, $u(t,\cdot)\in C^\infty(\mathbb{R})$ *for any* $t\in\tilde{A}$, $u$ *is a classical solution in* $\tilde{A}\times\mathbb{R}$ *of the equation* (6.21) *and* $u(0,x)=0$ *for every* $x\in\mathbb{R}$.

*Proof.* We at first define the functions $u_n(t,x)$ by

$$u_n(t,x) = c_n(t)|4nt+1|^{-1/2}\int_0^t |4ns+1|^{1/2}\tilde{g}_n(s)\,\mathrm{d}s\, e^{\frac{inx^2}{4nt+1}}. \tag{6.30}$$

Let $\varepsilon>0$, $\varepsilon\neq 1/(4n)$, $n\in\mathbb{N}$. We define $C_\varepsilon = \mathbb{R}\setminus(\cup_{|n|>1/(4\varepsilon)}(1/(4n)-\varepsilon, 1/(4n)+\varepsilon)\cup(-\varepsilon,\varepsilon))$ and $D_\varepsilon = C_\varepsilon\cap[-1/\varepsilon,1/\varepsilon]$.

Since

$$|u_n(t,x)| \leq |4nt+1|^{-1/2}|t|\sup_{s\in[-|t|,|t|]}|4ns+1|^{1/2}|\tilde{g}_n(s)|,$$

we have that

$$\sup_{t\in D_\varepsilon}|u_n(t,x)| \leq |4n\varepsilon|^{-1/2}|1/\varepsilon||4|n|1/\varepsilon+1|^{1/2}\sup_{t\in[-1/\varepsilon,1/\varepsilon]}|\tilde{g}_n(t)|.$$

By Weierstrass criterion, the series in (6.13) uniformly converges in $D_\varepsilon\times\mathbb{R}$ for every $\varepsilon>0$. Namely, $u(t,x)$ is well defined in $\tilde{A}\times\mathbb{R}$. Moreover, it follows that $u$ is continuous in $\tilde{A}\times\mathbb{R}$.

Since, for $k\in\mathbb{N}$,

$$\frac{\partial^k}{\partial x^k}u_n(t,x) = c_n(t)|4nt+1|^{-1/2}(2inx)^k(4nt+1)^{-k}\int_0^t |4ns+1|^{1/2}\tilde{g}_n(s)\,\mathrm{d}s\ e^{\frac{inx^2}{4nt+1}},$$

we have

$$\left|\frac{\partial^k}{\partial x^k}u_n(t,x)\right| \leq |4nt+1|^{-k-1/2}|2nx|^k|t|\sup_{s\in[-|t|,|t|]}|4ns+1|^{1/2}|\tilde{g}_n(s)|$$

and

$$\sup_{(t,x)\in D_\varepsilon\times[-1/\varepsilon,1/\varepsilon]}\left|\frac{\partial^k}{\partial x^k}u_n(t,x)\right| \leq |4n\varepsilon|^{-k-1/2}|2n1/\varepsilon|^k|1/\varepsilon||4|n|1/\varepsilon+1|^{1/2}\sup_{t\in[-1/\varepsilon,1/\varepsilon]}|\tilde{g}_n(t)|.$$

By Weierstrass criterion, the series

$$\sum_{n=-\infty}^{+\infty} \frac{\partial^k}{\partial x^k} u_n(t,x)$$

uniformly converges in $C_\varepsilon \times [-1/\varepsilon, 1/\varepsilon]$, for every $\varepsilon > 0$.

Since

$$\frac{\partial}{\partial t} u_n(t,x) = c_n(t)(-2n|4nt+1|^{-3/2}\,\mathrm{sign}\,(4nt+1)\int_0^t |4ns+1|^{1/2}\tilde{g}_n(s)\,\mathrm{d}s + \tilde{g}_n(t)$$
$$- 4in^2x^2(4nt+1)^{-2}|4nt+1|^{-1/2}\int_0^t |4ns+1|^{1/2}\tilde{g}_n(s)\,\mathrm{d}s)e^{\frac{inx^2}{4nt+1}},$$

we have that

$$\left|\frac{\partial}{\partial t} u_n(t,x)\right| \le (2|n||4nt+1|^{-3/2} + |4nx|^2||4nt+1|^{-5/2})|t| \sup_{s\in[-|t|,|t|]} |4ns+1|^{1/2}|\tilde{g}_n(s)| + |\tilde{g}_n(t)|$$

and

$$\sup_{(t,x)\in D_\varepsilon\times[-1/\varepsilon,1/\varepsilon]} \left|\frac{\partial}{\partial t} u_n(t,x)\right| \le (2|n||4n\varepsilon|^{-3/2} + |4n1/\varepsilon|^2||4n\varepsilon|^{-5/2})\cdot$$
$$\cdot |1/\varepsilon||4|n|1/\varepsilon + 1|^{1/2} \sup_{t\in[-1/\varepsilon,1/\varepsilon]} |\tilde{g}_n(t)| + \sup_{t\in[-1/\varepsilon,1/\varepsilon]} |\tilde{g}_n(t)|.$$

By Weierstrass criterion, the series

$$\sum_{n=-\infty}^{+\infty} \frac{\partial}{\partial t} u_n(t,x)$$

uniformly converges in $D_\varepsilon \times [-1/\varepsilon, 1/\varepsilon]$ for every $\varepsilon > 0$.

As a consequence of the uniform convergence of the derivatives, we obtain the regularity $u(\cdot, x) \in C^1(\tilde{A})$ for any $x \in \mathbb{R}$, and $u(t,\cdot) \in C^\infty(\mathbb{R})$ for any $t \in \tilde{A}$.

Besides, since we can exchange the series with the derivatives, by Proposition 6.6, we obtain that $u$ is a classical solution of the equation in $\tilde{A} \times \mathbb{R}$. □

**Proposition 6.8** *The function $u$ defined in* (6.29) *satisifes*

$$\lim_{t\to 0} \int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = 0,$$

*for any $\varphi \in C_c^\infty(\mathbb{R})$.*

*Proof.* Let $\varphi \in C_c^\infty(\mathbb{R})$. In the first place, we prove that, for the sequence $u_n$ defined in (6.30), it holds

$$\int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = \sum_{n=-\infty}^{+\infty} \int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x, \qquad \forall t \in A. \tag{6.31}$$

Indeed, as proved in Proposition 6.7, the convergence of the series $\sum_{n=-\infty}^{+\infty} u_n(t,x)\varphi(x) = u(t,x)\varphi(x)$ is uniform on $D_\varepsilon \times \mathbb{R}$ for any $\varepsilon > 0$. The identity (6.31) follows by exchanging the integral with the series, because of the uniform convergence of the series and the compactness of the support of $\varphi$.

We prove that there exist $C$ independent of $n \in \mathbb{Z}$ such that

$$\left| \int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x \right| \leq C \sup_{s\in[-1,1]} |\tilde{g}_n(s)| \qquad \forall t \in [-1,-1/(4n)) \cup (-1/(4n),1]. \tag{6.32}$$

In order to do this, we have to refine

$$\lim_{t\to -1/(4n)} \int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x = (\pi)^{1/2}|n|^{-1/2} \int_0^{-1/(4n)} |4ns+1|^{1/2}\tilde{g}_n(s)\,\mathrm{d}s\;\varphi(0), \tag{6.33}$$

already proved in Proposition 6.6, by showing that the limit is uniform with respect to $n$. Since

$$\int_0^t |4ns+1|^{1/2}\,\mathrm{d}s = \frac{1}{6n}(\mathrm{sign}(4nt+1)|4nt+1|^{3/2}-1),$$

we have

$$\int_0^{-1/(4n)} |4ns+1|^{1/2}\,\mathrm{d}s = -\frac{1}{6n}.$$

We prove that there exists $\delta > 0$ independent of $n$, such that

$$\begin{aligned} &\left| \int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x \right| \leq (\pi)^{1/2}|n|^{-1/2}6^{-1}|n|^{-1} \sup_{s\in[-1,1]} |\tilde{g}_n(s)||\varphi(0)| + 1 \\ &\qquad \forall t \in [-1/(4n)-\delta,-1/(4n)) \cup (-1/(4n),-1/(4n)+\delta]. \end{aligned} \tag{6.34}$$

$$\begin{aligned} &\left| \int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x - (\pi)^{1/2}|n|^{-1/2} \int_0^{-1/(4n)} |4ns+1|^{1/2}\tilde{g}_n(s)\,\mathrm{d}s\;\varphi(0) \right| \\ &\leq \left| \int_{\mathbb{R}} |4nt+1|^{-1/2} e^{\frac{inx^2}{4nt+1}}\varphi(x)\,\mathrm{d}x - (\pi)^{1/2}|n|^{-1/2}\varphi(0) \right| \left| \int_0^t |4ns+1|^{1/2}\tilde{g}_n(s)\,\mathrm{d}s \right| \\ &+ |(\pi)^{1/2}|n|^{-1/2}\varphi(0)| \left| \int_0^t |4ns+1|^{1/2}\tilde{g}_n(s)\,\mathrm{d}s - \int_0^{-1/(4n)} |4ns+1|^{1/2}\tilde{g}_n(s)\,\mathrm{d}s \right| \\ &= \left| (\pi i)^{1/2} \int_{\mathbb{R}} \bar{u}(t+1/(4n),x)\varphi(x)\,\mathrm{d}x - (\pi)^{1/2}\varphi(0) \right| |n|^{-1/2} \left| \int_0^t |4ns+1|^{1/2}\tilde{g}_n(s)\,\mathrm{d}s \right| \\ &+ |(\pi)^{1/2}\varphi(0)| \left| |n|^{-1/2} \int_{-1/(4n)}^t |4ns+1|^{1/2}\tilde{g}_n(s)\,\mathrm{d}s \right| \end{aligned} \tag{6.35}$$

Since

$$|n|^{-1/2} \left| \int_0^t |4ns+1|^{1/2}\tilde{g}_n(s)\,\mathrm{d}s \right| \leq |n|^{-1/2} 2|4|n|+1|^{1/2} \sup_{s\in[-1,1]} |\tilde{g}_n(s)| \leq 6 \sup_{s\in[-1,1]} |\tilde{g}_n(s)| \tag{6.36}$$

and

$$\left| |n|^{-1/2} \int_{-1/(4n)}^{t} |4ns+1|^{1/2} \tilde{g}_n(s)\,\mathrm{d}s \right| \leq \sup_{s\in[-1,1]} |\tilde{g}_n(s)| |n|^{-1/2} 6^{-1} |n|^{-1} |4nt+1|^{3/2}$$
$$\leq \sup_{s\in[-1,1]} |\tilde{g}_n(s)| 6^{-1} |4t+1/n|^{3/2}, \tag{6.37}$$

by (5.16), we have that $\sup_{s\in[-1,1]} |\tilde{g}_n(s)|$ is bounded with respect to $n$. Then, by (6.35), (6.36) and (6.37), there exist $\delta > 0$ independent of $n$ such that

$$\left| \int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x - (\pi)^{1/2}|n|^{-1/2} \int_0^{-1/(4n)} |4ns+1|^{1/2}\tilde{g}_n(s)\,\mathrm{d}s\ \varphi(0) \right| < 1$$
$$\forall t \in [-1/(4n)-\delta, -1/(4n)) \cup (-1/(4n), -1/(4n)+\delta]. \tag{6.38}$$

If $t \in [-1, -1/(4n)-\delta) \cup (-1/(4n)+\delta, 1]$

$$\left| \int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x \right| \leq (4|n|\delta)^{-1/2} \int_{\mathbb{R}} |\varphi(x)|\,\mathrm{d}x \int_0^t |4ns+1|^{1/2} |\tilde{g}_n(s)|\,\mathrm{d}s$$
$$\leq (4|n|\delta)^{-1/2} \int_{\mathbb{R}} |\varphi(x)|\,\mathrm{d}x\, 6^{-1}|n|^{-1}(|4|n|+1|^{3/2}+1) \sup_{s\in[-1,1]} |\tilde{g}_n(s)|. \tag{6.39}$$

By (6.34) and (6.39), there exists $C > 0$ independent of $n$ such that (6.32) holds.

By (6.15), it follows that

$$\lim_{t\to 0} \sum_{n=-\infty}^{+\infty} \int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x = \sum_{n=-\infty}^{+\infty} \lim_{t\to 0} \int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x. \tag{6.40}$$

Since $u_n(\cdot, x)$ is continuous at 0, we have

$$\lim_{t\to 0} \int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x = 0,$$

and by (6.31) we conclude.

□

## 6.3 Proof of Theorem 5.9

**Lemma 6.9** *If $u$ is of the form $u(t,x) = g(t)e^{if(t)x^2}$, for some real differentiable function $f$ and complex valued function $g$, and it solves the equation*

$$\partial_t u(t,x) + \frac{i}{2}(-\partial_{xx}u(t,x) + x^2 u(t,x)) = 0, \tag{6.41}$$

*then $f$ and $g$ cannot be defined on all $\mathbb{R}$, but on $\mathbb{R}\setminus\{c+\pi/2+k\pi : k\in\mathbb{Z}\}$, for some $c\in\mathbb{R}$,*

$$f(t)=\frac{1}{2}\tan(c-t)$$

*and*

$$g(t)=C_k|\cos(t-c)|^{-1/2}\quad \text{if } t\in(k\pi+c-\pi/2,k\pi+c-\pi/2),\qquad k\in\mathbb{Z}$$

*for $C_k\in\mathbb{C}$. Moreover, $\lim_{t\to c+\pi/2+k\pi}|u(t,x)|=+\infty$, for any $x\in\mathbb{R}$, when $C_k$ and $C_{k+1}$ are not 0.*

*Besides, if $C_k=iC_{k+1}$, then*

$$\lim_{t\to c+\pi/2+k\pi}\int_{\mathbb{R}}u(t,x)\varphi(x)\,\mathrm{d}x=C_k(2\pi)^{1/2}i^{1/2}\varphi(0), \tag{6.42}$$

*for any $\varphi\in C_c^\infty(\mathbb{R})$.*

*Proof.* Let $u(t,x)=g(t)e^{if(t)x^2}$ for some real differentiable function $f:D\subset\mathbb{R}\to\mathbb{R}$ and complex valued function $g:D\subset\mathbb{R}\to\mathbb{C}$ defined on a open domain $D\subset\mathbb{R}$. We compute

$$\partial_t u(t,x)=(g'(t)+ig(t)f'(t)x^2)e^{if(t)x^2} \tag{6.43}$$

and

$$\partial_{xx}u(t,x)=(-4g(t)f^2(t)x^2+i2g(t)f(t))e^{if(t)x^2}. \tag{6.44}$$

Then,

$$\partial_t u(t,x)+\frac{i}{2}(-\partial_{xx}u(t,x)+x^2u(t,x))=(g'(t)+g(t)f(t)+i[g(t)f'(t)+2g(t)f^2(t)+g(t)/2]x^2)e^{if(t)x^2}. \tag{6.45}$$

If (6.41) holds, then

$$g'(t)+g(t)f(t)=0, \tag{6.46}$$
$$g(t)f'(t)+2g(t)f^2(t)+g(t)/2=0. \tag{6.47}$$

Denoting by $F$ a primitive of $f$, all the solutions of (6.46) are of the form $g(t)=Ke^{-F(t)}$, where the constant $K\in\mathbb{C}$ could be different on every connected component of $D$. In particular, on the connected components where $K\neq 0$, the equation (6.47) is equivalent to

$$f'(t)+2f^2(t)+1/2=0.$$

All the solutions to this equation are of the form $f(t)=\frac{1}{2}\tan(-t+c)$ for some $c\in\mathbb{R}$, with maximal domain $\mathbb{R}\setminus\{c+\pi/2+k\pi : k\in\mathbb{Z}\}$. By a direct computation, we get that $F(t)=\ln|\cos(t-c)|^{1/2}$ and, by the above observation, we obtain that $g(t)=C_k|\cos(t-c)|^{-1/2}$, for some constants $K_k$ on the components $(k\pi+c-\pi/2,k\pi+c-\pi/2)$.

We finally prove (6.42). It is sufficient to prove the result for $c=0$ and $k=0$. By (6.8), (6.9) and the change of variable

$$s=\frac{-1}{2\tan t}=\frac{-\cos t}{2\sin t}$$

we have

$$\lim_{t\to(\pi/2)^-}\int_{\mathbb{R}} C_0|\cos t|^{-1/2}e^{-i\tan t x^2/2}\varphi(x)\,\mathrm{d}x = \lim_{s\to 0^-}\int_{\mathbb{R}} C_0|2s|^{-1/2}e^{ix^2/(4s)}\varphi(x)\,\mathrm{d}x \\ = C_0e^{-i\pi/4}(2\pi)^{1/2}\varphi(0) \tag{6.48}$$

and

$$\lim_{t\to(\pi/2)^+}\int_{\mathbb{R}} C_1|\cos t|^{-1/2}e^{-i\tan t x^2/2}\varphi(x)\,\mathrm{d}x = \lim_{s\to 0^+}\int_{\mathbb{R}} C_1|2s|^{-1/2}e^{ix^2/(4s)}\varphi(x)\,\mathrm{d}x \\ = C_1e^{i\pi/4}(2\pi)^{1/2}\varphi(0). \tag{6.49}$$

Then (6.42) follows from (6.48) and (6.49), if $C_0 = iC_1$.

□

**Proposition 6.10** *Let $u_0 : A_0\times\mathbb{R}\to\mathbb{C}$ be the function defined by*

$$u_0(t,x) := q(t)|\cos(t)|^{-1/2}e^{\frac{i}{2}\tan(-t)x^2}, \tag{6.50}$$

*where $A_0 := \{t\in\mathbb{R} : t\neq\pi/2+k\pi, k\in\mathbb{Z}\}$, and $q$ is the piecewise constant function*

$$q(t) := e^{-ki\pi/2} \qquad \text{if } t\in(\pi/2+(k-1)\pi, \pi/2+k\pi), \quad k\in\mathbb{Z}.$$

*Then, $u_0$ belongs to $C^\infty(A_0\times\mathbb{R})$ and it is a classical solution of*

$$\partial_t u(t,x) + \frac{i}{2}(-\partial_{xx}u(t,x) + x^2u(t,x)) = 0, \qquad \text{for } (t,x)\in A_0\times\mathbb{R}, \tag{6.51}$$

*such that*

$$\lim_{t\to\pi/2+k\pi}\int_{\mathbb{R}} u_0(t,x)\varphi(x)\,\mathrm{d}x = (2\pi)^{1/2}\varphi(0), \tag{6.52}$$

*for any $\varphi\in C_c^\infty(\mathbb{R})$ and any $k\in\mathbb{Z}$.*

*Proof.* The proof follows by application of Lemma 6.9 with $c = 0$. The regularity of $u$ is obvious. The choice of the function $q$ has been made to obtain (6.52) with the same constant, $(2\pi)^{1/2}$, for any $k\in\mathbb{Z}$. □

The functions

$$u_n(t,x) := |\cos(-\arctan(2n))|^{1/2}u_0(t-\arctan(2n), x), \tag{6.53}$$

for $n\in\mathbb{Z}$, defined in $A_n\times\mathbb{R}$, where $A_n := \{t\in\mathbb{R} : t\neq\pi/2+\arctan(2n)+k\pi, k\in\mathbb{Z}\}$, satisfy

$$\partial_t u(t,x) + \frac{i}{2}(-\partial_{xx}u(t,x) + x^2u(t,x)) = 0, \qquad \text{for } (t,x)\in A_n\times\mathbb{R}, \tag{6.54}$$

$u_n(0,x) = e^{inx^2}$, $u_n\in C^\infty(A_n\times\mathbb{R})$,

$$\lim_{t\to\pi/2+\arctan(2n)+k\pi}\int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x = (2\pi)^{1/2}\varphi(0),$$

for any $\varphi\in C_c^\infty(\mathbb{R})$ and any $k\in\mathbb{Z}$.

**Proposition 6.11** *Let $\alpha_n \in \mathbb{C}$, $n \in \mathbb{Z}$, be a sequence such that $\sum_{n=-\infty}^{+\infty} |\alpha_n| < +\infty$. Let $u_n : A_n \to \mathbb{C}$ be the sequence defined by* (6.50) *and* (6.53)*. Then the series,*

$$\sum_{n=-\infty}^{+\infty} \alpha_n u_n(t,x), \tag{6.55}$$

*absolutely converges for any $(t,x) \in A \times \mathbb{R}$, where $A := \cap_{n\in\mathbb{Z}} A_n$ . We define*

$$u(t,x) := \sum_{n=-\infty}^{+\infty} \alpha_n u_n(t,x), \qquad (t,x) \in A \times \mathbb{R}, \tag{6.56}$$

*and $\tilde{A} = A \setminus \{k\pi, k \in \mathbb{Z}\}$. Then $u \in C^\infty(\tilde{A} \times \mathbb{R})$ and $u$ is a classical solution in $\tilde{A} \times \mathbb{R}$ of the equation in* (5.2)*.*

*Proof.* We at first observe that $u_n$ can be written (see (5.27)) in the following form

$$u_n(t,x) = |\cos(-\arctan(2n))|^{1/2} |\cos(t - \arctan(2n))|^{-1/2} q_n(t) e^{\frac{i}{2}\tan(\arctan(2n)-t)x^2},$$

where $q_n$ is the piecewise constant function

$$q_n(t) := q(t - \arctan(2n)).$$

Let us observe that $u_n(\cdot, x)$ is $4\pi$-periodic. Since $q$ is constant in each interval of the form $(\pi/2 + (k-1)\pi, \pi/2 + k\pi)$, it is sufficient to prove the result for $t \in (-\pi/2, \pi/2)$.

Let $\varepsilon \in (0, \pi/2)$, $\varepsilon \neq \pi/2 - \arctan(2|n|)$, $n \in \mathbb{Z}$. Let $t_n := \pi/2 + \arctan(2n)$ if $n \leq 0$ and $t_n := -\pi/2 + \arctan(2n)$ if $n > 0$. We define

$$C_\varepsilon = (-\pi/2 + \varepsilon, \pi/2 - \varepsilon) \setminus (\cup_{|n|<1/2\tan(\pi/2-\varepsilon)} (t_n - \varepsilon, t_n + \varepsilon) \cup (-\varepsilon, \varepsilon)).$$

Since

$$|\alpha_n u_n(t,x)| \leq |\alpha_n| |\cos(-\arctan(2n))|^{1/2} |\cos(t - \arctan(2n))|^{-1/2},$$

we have that

$$\sup_{t\in C_\varepsilon} |\alpha_n u_n(t,x)| \leq |\alpha_n| |\cos(-\arctan(2n))|^{1/2} |\cos(\pi/2 - \varepsilon)|^{-1/2}.$$

By Weierstrass criterion, the series in (6.56) uniformly converges in $C_\varepsilon \times \mathbb{R}$, for every $\varepsilon > 0$. In particular, $u(t,x)$ is well defined in $\tilde{A} \times \mathbb{R}$. Furthermore, it follows that $u$ is continuous in $\tilde{A} \times \mathbb{R}$.

Since, for $k \in \mathbb{N}$,

$$\frac{\partial^k}{\partial x^k} u_n(t,x) = q_n(t) |\cos(-\arctan(2n))|^{1/2} |\cos(t - \arctan(2n))|^{-1/2} \cdot$$
$$\cdot\, (ix \tan(\arctan(2n) - t))^k e^{\frac{i}{2}\tan(\arctan(2n)-t)x^2},$$

we have

$$\left|\alpha_n \frac{\partial^k}{\partial x^k} u_n(t,x)\right| \leq |\alpha_n| |\cos(-\arctan(2n))|^{1/2} |\cos(t-\arctan(2n))|^{-1/2} \cdot \\ \cdot |x|^k |\tan(\arctan(2n)-t)|^k,$$

and

$$\sup_{(t,x)\in C_\varepsilon \times [-1/\varepsilon, 1/\varepsilon]} \left|\alpha_n \frac{\partial^k}{\partial x^k} u_n(t,x)\right| \leq |\alpha_n| |\cos(-\arctan(2n))|^{1/2} |\cos(\pi/2-\varepsilon)|^{-1/2} \cdot \\ \cdot |\varepsilon|^{-k} |\tan(\pi/2-\varepsilon)|^k.$$

By Weierstrass criterion, the series

$$\sum_{n=-\infty}^{+\infty} \alpha_n \frac{\partial^k}{\partial x^k} u_n(t,x)$$

uniformly converges in $C_\varepsilon \times [-1/\varepsilon, 1/\varepsilon]$, for every $\varepsilon > 0$.

Since

$$\frac{\partial}{\partial t} u_n(t,x) = q_n(t) |\cos(-\arctan(2n))|^{1/2} \cdot \\ \cdot \left( \frac{1}{2} |\cos(t-\arctan(2n))|^{-3/2} \operatorname{sign}(\cos(t-\arctan(2n))) \sin(t-\arctan(2n)) \right. \\ \left. -\frac{i}{2} |\cos(t-\arctan(2n))|^{-1/2} (\cos(t-\arctan(2n)))^{-2} x^2 \right) e^{\frac{i}{2}\tan(\arctan(2n)-t)x^2}, \tag{6.57}$$

we have that

$$\left|\alpha_n \frac{\partial}{\partial t} u_n(t,x)\right| \leq |\alpha_n| |\cos(-\arctan(2n))|^{1/2} \cdot \\ \cdot \left( \frac{1}{2} |\cos(t-\arctan(2n))|^{-3/2} + \frac{1}{2} |\cos(t-\arctan(2n))|^{-5/2} |x|^2 \right) \tag{6.58}$$

and

$$\sup_{(t,x)\in C_\varepsilon \times [-1/\varepsilon, 1/\varepsilon]} \left|\alpha_n \frac{\partial}{\partial t} u_n(t,x)\right| \\ \leq |\alpha_n| |\cos(-\arctan(2n))|^{1/2} \left( \frac{1}{2} |\cos(\pi/2-\varepsilon)|^{-3/2} + \frac{1}{2} |\cos(\pi/2-\varepsilon)|^{-5/2} \varepsilon^{-2} \right). \tag{6.59}$$

By Weierstrass criterion, the series

$$\sum_{n=-\infty}^{+\infty} \alpha_n \frac{\partial}{\partial t} u_n(t,x)$$

uniformly converges in $C_\varepsilon \times [-1/\varepsilon, 1/\varepsilon]$, for every $\varepsilon > 0$. Similarly, one can prove that the series of the $k$-derivatives with respect to $t$ of $\alpha_n u_n$ uniformly converges in $C_\varepsilon \times [-1/\varepsilon, 1/\varepsilon]$ for every $\varepsilon > 0$.

As a consequence of the uniform convergence of the derivatives, we attain the regularity $u \in C^\infty(\tilde{A} \times \mathbb{R})$.

Moreover, since we can exchange the series with the derivatives, by Proposition 6.10, we have that $u$ is a classical solution of the equation in $\tilde{A} \times \mathbb{R}$.

□

**Proposition 6.12** *The function $u$ defined in* (6.56) *satisifes*

$$\lim_{t \to k\pi} \int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}} \bar{u}_0(x)\varphi(x)\,\mathrm{d}x,$$

*for any $\varphi \in C_c^\infty(\mathbb{R})$ and for any $k \in \mathbb{Z}$, where $\bar{u}_0(x) = \sum_{n=-\infty}^{+\infty} \alpha_n e^{inx^2}$.*

*Proof.* Let $\varphi \in C_c^\infty(\mathbb{R})$. First of all, we prove that

$$\int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = \sum_{n=-\infty}^{+\infty} \int_{\mathbb{R}} \alpha_n u_n(t,x)\varphi(x)\,\mathrm{d}x, \qquad \forall t \in A. \tag{6.60}$$

As already proved in Proposition 6.11, the convergence of the series $\sum_{n=-\infty}^{+\infty} \alpha_n u_n(t,x)\varphi(x) = u(t,x)\varphi(x)$ is uniform on $C_\varepsilon \times \mathbb{R}$ for any $\varepsilon > 0$. The identity (6.60) follows by exchanging the integral with the series, because of the uniform convergence of the series and the compactness of the support of $\varphi$.

We prove that there exist $C$ independent of $n \in \mathbb{Z}$, and $t \in A$, such that

$$\left| \int_{\mathbb{R}} \alpha_n u_n(t,x)\varphi(x)\,\mathrm{d}x \right| \le C|\alpha_n||\cos(-\arctan(2n))|^{1/2}. \tag{6.61}$$

Indeed, by Proposition 6.10, there exists $\delta > 0$ such that

$$\left| \int_{\mathbb{R}} u_0(t,x)\varphi(x)\,\mathrm{d}x \right| \le |\varphi(0)| + 1 \qquad \forall t \in [\pi/2 - \delta, \pi/2) \cup (-\pi/2, -\pi/2 + \delta], \tag{6.62}$$

and

$$\begin{aligned} \left| \int_{\mathbb{R}} u_0(t,x)\varphi(x)\,\mathrm{d}x \right| &\le \sup_{x \in \mathbb{R}} |\varphi(x)| \int_{-R}^{R} |u_0(t,x)|\,\mathrm{d}x \\ &\le \sup_{x \in \mathbb{R}} |\varphi(x)| 2R |\cos(\pi/2 - \delta)|^{-1/2} \qquad \forall t \in (-\pi/2 + \delta, \pi/2 - \delta). \end{aligned} \tag{6.63}$$

By (6.62) and (6.63), there exists $C > 0$ such that

$$\left| \int_{\mathbb{R}} u_0(t,x)\varphi(x)\,\mathrm{d}x \right| \le C \qquad \forall t \in A_0, \tag{6.64}$$

Since $u_n(t,x) = |\cos(-\arctan(2n))|^{1/2} u_0(t - \arctan(2n), x)$, by (6.64) we obtain (6.61).

By (6.61), it follows that

$$\lim_{t\to 0} \sum_{n=-\infty}^{+\infty} \int_{\mathbb{R}} \alpha_n u_n(t,x)\varphi(x)\,\mathrm{d}x = \sum_{n=-\infty}^{+\infty} \lim_{t\to 0} \int_{\mathbb{R}} \alpha_n u_n(t,x)\varphi(x)\,\mathrm{d}x. \tag{6.65}$$

Since $u_n(\cdot, x)$ is continuous at 0, we have

$$\lim_{t\to 0} \int_{\mathbb{R}} \alpha_n u_n(t,x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}} \alpha_n u_n(0,x)\varphi(x)\,\mathrm{d}x,$$

and, by (6.60), we conclude. □

## 6.4 Proof of Theorem 5.11

**Proposition 6.13** *Let $\tilde{g} : \mathbb{R} \to \mathbb{C}$ be a continuous function not identically* 0. *Let*

$$v(t,x) := \tilde{g}(t) e^{i\tilde{f}(t)x^2}. \tag{6.66}$$

*If $\tilde{f}$ is not of the form $\tilde{f}(t) = f(t) = \frac{1}{2}\tan(c-t)$ for some $c \in \mathbb{R}$, then the equation*

$$\partial_t u(t,x) + \frac{i}{2}(-\partial_{xx} u(t,x) + x^2 u(t,x)) = v(t,x) \tag{6.67}$$

*has no solution of the form $u(t,x) = g(t)e^{if(t)x^2}$.*

*If $\tilde{f}(t) = f(t) = \frac{1}{2}\tan(c-t)$ for some $c \in \mathbb{R}$, for $t \neq c + \pi/2 + k\pi$, then*

$$u(t,x) := |\cos(c-t)|^{-1/2} \int_0^t |\cos(c-s)|^{1/2} \tilde{g}(s)\,\mathrm{d}s \;\; e^{\frac{ix^2}{2}\tan(c-t)} \tag{6.68}$$

*is a classical solution of the equation* (6.67) *in $(\mathbb{R} \setminus \{c + \pi/2 + k\pi : k \in \mathbb{Z}\}) \times \mathbb{R}$ and $u(0,x) = 0$, for every $x \in \mathbb{R}$.*

*Moreover, $u(\cdot, x) \in C^1(\mathbb{R} \setminus \{c + \pi/2 + k\pi : k \in \mathbb{Z}\})$ for any $x \in \mathbb{R}$, $u(t, \cdot) \in C^\infty(\mathbb{R})$ for any $t \in \mathbb{R} \setminus \{c + \pi/2 + k\pi : k \in \mathbb{Z}\}$, and*

$$\lim_{t\to c+\pi/2+k\pi} q(t-c) \int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = (2\pi)^{1/2} \int_0^{c+\pi/2+k\pi} |\cos(c-s)|^{1/2} \tilde{g}(s)\,\mathrm{d}s\, \varphi(0), \tag{6.69}$$

*for any $\varphi \in C_c^\infty(\mathbb{R})$.*

*Proof.* Let us assume that $u(t,x) = g(t)e^{if(t)x^2}$ for some real differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ and complex valued function $g : D \subset \mathbb{R} \to \mathbb{C}$ defined on a open domain $D \subset \mathbb{R}$. Using the computations made in Lemma 6.1, by (6.5), if (6.67) holds for $v$ of the form (6.66), then

$$g'(t) + g(t)f(t) = \tilde{g}(t), \tag{6.70}$$

$$g(t)f'(t) + 2g(t)f^2(t) + g(t)/2 = 0, \tag{6.71}$$

$$e^{if(t)x^2} = e^{i\tilde{f}(t)x^2} \tag{6.72}$$

The last equality holds if and only if

$$f(t)x^2 = \tilde{f}(t)x^2 + 2k\pi,$$

for some $k \in \mathbb{Z}$ and for any $x \in \mathbb{R}$. This implies that $f(t) = \tilde{f}(t)$ for any $t \in D$.

If $\tilde{f}$ is not of the form $\tilde{f}(t) = \frac{1}{2}\tan(c-t)$ for some $c \in \mathbb{R}$, $t \neq \pi/2 + c + k\pi$, for any $k \in \mathbb{Z}$, then the equation (6.71) does not hold on the set where $g(t) \neq 0$. Denoting by $F$ a primitive of $f$, the solutions to the equation (6.70) are of the form

$$g(t) = e^{-2F(t)}e^{2F(0)}g(0) + e^{-2F(t)}\int_0^t e^{2F(s)}\tilde{g}(s)\,\mathrm{d}s.$$

By a direct computation and by imposing the condition $g(0) = 0$, we obtain the solution (6.68).

By (6.42) and the continuity of the function $t \mapsto \int_0^t |\cos(c-s)|^{1/2}\tilde{g}(s)\,\mathrm{d}s$, we get to (6.69).

□

**Proposition 6.14** *Let $\tilde{g}_n : \mathbb{R} \to \mathbb{C}$, $n \in \mathbb{Z}$, be a sequence of continuous functions such that $\sum_{n=-\infty}^{+\infty} \sup_{t\in[a,b]} |\tilde{g}_n(t)| < +\infty$, for any $[a,b] \subset \mathbb{R}$ and*

$$v(t,x) := \sum_{n=-\infty}^{+\infty} \tilde{g}_n(t)q_n(t)e^{\frac{i}{2}\tan(\arctan(2n)-t)x^2}. \tag{6.73}$$

*Defining*

$$F[\tilde{g}_n](t) := |\cos(t-\arctan(2n))|^{-1/2}\int_0^t |\cos(s-\arctan(2n))|^{1/2}\tilde{g}_n(s)\,\mathrm{d}s,$$

*then the series*

$$\sum_{n=-\infty}^{+\infty} F[\tilde{g}_n](t)q_n(t)e^{\frac{i}{2}\tan(\arctan(2n)-t)x^2} \tag{6.74}$$

*absolutely converges for any $(t,x) \in A \times \mathbb{R}$, where $A := \cap_{n\in\mathbb{Z}}A_n$. For $(t,x) \in A \times \mathbb{R}$, we define*

$$u(t,x) := \sum_{n=-\infty}^{+\infty} F[\tilde{g}_n](t)q_n(t)e^{\frac{i}{2}\tan(\arctan(2n)-t)x^2}. \tag{6.75}$$

*We define $\tilde{A} = A \setminus \{k\pi, k \in \mathbb{Z}\}$. Then $u(\cdot,x) \in C^1(\tilde{A})$ for any $x \in \mathbb{R}$, $u(t,\cdot) \in C^\infty(\mathbb{R})$ for any $t \in \tilde{A}$, $u$ is a classical solution in $\tilde{A} \times \mathbb{R}$ of the equation (6.67) and $u(0,x) = 0$, for every $x \in \mathbb{R}$.*

*Proof.* We at first define the functions $u_n(t,x)$ by

$$u_n(t,x) = |\cos(t-\arctan(2n))|^{-1/2}\int_0^t |\cos(s-\arctan(2n))|^{1/2}\tilde{g}_n(s)\,\mathrm{d}s\, q_n(t)e^{\frac{i}{2}\tan(\arctan(2n)-t)x^2}.$$

Let $\varepsilon \in (0, \pi/2)$, $\varepsilon \neq \pi/2 - \arctan(2|n|)$, $n \in \mathbb{Z}$. Let $t_n := \pi/2 + \arctan(2n)$ if $n \leq 0$ and $t_n := -\pi/2 + \arctan(2n)$ if $n > 0$. We define

$$C_\varepsilon = (-\pi/2 + \varepsilon, \pi/2 - \varepsilon) \setminus (\cup_{|n|<1/2\tan(\pi/2-\varepsilon)}(t_n - \varepsilon, t_n + \varepsilon) \cup (-\varepsilon, \varepsilon)),$$

and

$$\tilde{C}_\varepsilon = \cup_{k \in \mathbb{Z}}(C_\varepsilon + k\pi).$$

Finally, for $R > 0$, we define $D_{\varepsilon,R} = \tilde{C}_\varepsilon \cap [R, R]$.

Since

$$|u_n(t,x)| \leq |\cos(t - \arctan(2n))|^{-1/2}|t| \sup_{s \in [-|t|,|t|]} |\tilde{g}_n(s)|,$$

we have that

$$\sup_{t \in D_{\varepsilon,R}} |u_n(t,x)| \leq |\cos(\pi/2 - \varepsilon)|^{-1/2} R \sup_{t \in [-R,R]} |\tilde{g}_n(t)|.$$

By Weierstrass criterion, the series in (6.74) uniformly converges in $D_{\varepsilon,R} \times \mathbb{R}$ for every $\varepsilon > 0$ and $R > 0$. In particular, $u(t,x)$ is well defined in $\tilde{A} \times \mathbb{R}$. Moreover, since $u_n$ are continuous in $\tilde{A} \times \mathbb{R}$, it follows that $u$ is continuous in $\tilde{A} \times \mathbb{R}$.

Since, for $k \in \mathbb{N}$,

$$\frac{\partial^k}{\partial x^k} u_n(t,x) = u_n(t,x)(ix \tan(\arctan(2n) - t))^k,$$

we have

$$\left|\frac{\partial^k}{\partial x^k} u_n(t,x)\right| \leq |u_n(t,x)||x|^k|\tan(\arctan(2n) - t)|^k$$

and then

$$\sup_{(t,x) \in D_{\varepsilon,R} \times [-R,R]} \left|\frac{\partial^k}{\partial x^k} u_n(t,x)\right| \leq |\cos(\pi/2 - \varepsilon)|^{-1/2-k} R^{k+1} \sup_{t \in [-R,R]} |\tilde{g}_n(t)|.$$

By Weierstrass criterion, the series

$$\sum_{n=-\infty}^{+\infty} \frac{\partial^k}{\partial x^k} u_n(t,x)$$

uniformly converges in $D_{\varepsilon,R} \times [-R,R]$ for every $\varepsilon > 0$ and $R > 0$.

Since

$$\begin{aligned}
\frac{\partial}{\partial t} u_n(t,x) = q_n(t) e^{\frac{i}{2}\tan(\arctan(2n)-t)x^2} \int_0^t |\cos(s - \arctan(2n))|^{1/2} \tilde{g}_n(s)\, \mathrm{d}s \cdot \\
\cdot \left(\frac{1}{2}|\cos(t - \arctan(2n))|^{-3/2} \operatorname{sign}(\cos(t - \arctan(2n))) \sin(t - \arctan(2n)) \right. \\
\left. - \frac{i}{2}|\cos(t - \arctan(2n))|^{-1/2} (\cos(t - \arctan(2n)))^{-2} x^2 \right) + \\
+ \tilde{g}_n(t) q_n(t) e^{\frac{i}{2}\tan(\arctan(2n)-t)x^2},
\end{aligned}$$

we have that

$$\left|\frac{\partial}{\partial t}u_n(t,x)\right| \leq |\cos(-\arctan(2n))|^{1/2}\frac{1}{2}\left(|\cos(t-\arctan(2n))|^{-3/2} + |\cos(t-\arctan(2n))|^{-5/2}|x|^2\right)\cdot$$
$$\cdot |t| \sup_{s\in[-|t|,|t|]} \tilde{g}_n(s)| + |\tilde{g}_n(t)|$$

and

$$\sup_{(t,x)\in D_{\varepsilon,R}\times[-R,R]}\left|\frac{\partial}{\partial t}u_n(t,x)\right| \leq |\cos(-\arctan(2n))|^{1/2}\cdot$$
$$\cdot\frac{1}{2}\left(|\cos(\pi/2-\varepsilon)|^{-3/2} + |\cos(\pi/2-\varepsilon)|^{-5/2}R^2\right)\cdot$$
$$\cdot R \sup_{t\in[R,R]}|\tilde{g}_n(t)| + \sup_{t\in[-R,R]}|\tilde{g}_n(t)|.$$

By Weierstrass criterion, the series

$$\sum_{n=-\infty}^{+\infty}\frac{\partial}{\partial t}u_n(t,x)$$

uniformly converges in $D_{\varepsilon,R}\times[-R,R]$ for every $\varepsilon>0$ and $R>0$.

As a consequence of the uniform convergence of the derivatives, we achieve the regularity $u(\cdot,x)\in C^1(\tilde{A})$ for any $x\in\mathbb{R}$, and $u(t,\cdot)\in C^\infty(\mathbb{R})$, for any $t\in\tilde{A}$.

Moreover, since we can exchange the series with the derivatives, by Proposition 6.13, we obtain that $u$ is a classical solution of the equation (6.67) in $\tilde{A}\times\mathbb{R}$. □

**Proposition 6.15** *The function $u$ defined in* (6.75) *satisifes*

$$\lim_{t\to 0}\int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = 0,$$

*for any $\varphi\in C_c^\infty(\mathbb{R})$.*

*Proof.* Let $\varphi\in C_c^\infty(\mathbb{R})$. In the first place, we prove that

$$\int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = \sum_{n=-\infty}^{+\infty}\int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x, \qquad \forall t\in A. \tag{6.76}$$

As proved in Proposition 6.14, then, the convergence of the series $\sum_{n=-\infty}^{+\infty}u_n(t,x)\varphi(x) = u(t,x)\varphi(x)$ is uniform on $D_{\varepsilon,R}\times\mathbb{R}$ for any $\varepsilon>0$ and $R>0$. The identity (6.76) follows by the exchanging of the integral with the series, because of the uniform convergence of the series and the compactness of the support of $\varphi$.

We prove that there exist $C$ independent of $n\in\mathbb{Z}$ such that

$$\left|\int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x\right| \leq C\sup_{s\in[-1,1]}|\tilde{g}_n(s)| \qquad \forall t\in(-\pi/2,\pi/2)\setminus\{t_n\}, \tag{6.77}$$

where $t_n := \pi/2 + \arctan(2n)$ if $n \leq 0$ and $t_n := -\pi/2 + \arctan(2n)$ if $n > 0$.

In order to do this, we have to refine

$$\lim_{t \to t_n} \int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x = (2\pi)^{1/2} \int_0^{t_n} |\cos(\arctan(2n) - s)|^{1/2} \tilde{g}_n(s)\,\mathrm{d}s\, \varphi(0), \qquad (6.78)$$

already proved in Proposition 6.13, by showing that the limit is uniform with respect to $n$. We prove that there exists $\delta > 0$ independent of $n$, such that

$$\begin{aligned} \left| \int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x \right| &\leq (2\pi)^{1/2}\pi/2 \sup_{s \in [-\pi/2,\pi/2]} |\tilde{g}_n(s)||\varphi(0)| + 1 \\ &\forall t \in [t_n - \delta, t_n + \delta] \setminus \{t_n\}. \end{aligned} \qquad (6.79)$$

If $t \in [-\pi/2, t_n - \delta) \cup (t_n + \delta, \pi/2]$

$$\begin{aligned} \left| \int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x \right| &\leq |\cos(t - \arctan(2n))|^{-1/2} \int_0^t |\cos(s - \arctan(2n))|^{1/2} \tilde{g}_n(s)\,\mathrm{d}s \int_{\mathbb{R}} |\varphi(x)|\,\mathrm{d}x \\ &\leq |\cos(\pi/2 + \delta)|^{-1/2} \int_{\mathbb{R}} |\varphi(x)|\,\mathrm{d}x\, \pi \sup_{s \in [-1,1]} |\tilde{g}_n(s)|. \end{aligned} \qquad (6.80)$$

By (6.79) and (6.80), there exists $C > 0$ independent of $n$, such that (6.77) holds.

By (6.77), it follows that

$$\lim_{t \to 0} \sum_{n=-\infty}^{+\infty} \int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x = \sum_{n=-\infty}^{+\infty} \lim_{t \to 0} \int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x. \qquad (6.81)$$

Since $u_n(\cdot, x)$ is continuous at 0, we have

$$\lim_{t \to 0} \int_{\mathbb{R}} u_n(t,x)\varphi(x)\,\mathrm{d}x = 0,$$

and, by (6.31), we conclude.

□

Using the same technique, and using the condition (5.28) that ensures the convergence of the series (5.30), for $t = k\pi$, $k \in \mathbb{Z}$, $k \neq 0$, we can prove the following

**Proposition 6.16** *The function $u$ defined in* (6.75) *satisifes*

$$\lim_{t \to k\pi} \int_{\mathbb{R}} u(t,x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}} u(k\pi, x)\varphi(x)\,\mathrm{d}x,$$

*for any $k \in \mathbb{Z}$ and any $\varphi \in C_c^\infty(\mathbb{R})$.*

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