POLITECNICO DI TORINO

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Master thesis

Heterogeneous Vicsek model

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Abstract

In this work, I study a variation of the Vicsek model [2] in which a heterogeneity at the noise level is introduced (as proposed by Guisandez et al. [1]). In particular, I investigate the phase diagram of the system and I show that the latter undergoes the usual liquid-gas phase transition of the Vicsek model, instead of the continuous one that, according to the work of Guisanzdez and collaborators, one should see after a tricritical point. These results are just another example of how finite-size effects strongly affect these kinds of models and can lead to wrong results as happened to the original Vicsek model. I conclude studying the properties of the band and liquid phases and introducing an alternative model to implement the heterogeneity.

Contents

1 Active matter systems

Figure 1: Flock of birds, photo by Barn Images on Unsplash

Active matter systems are composed of active particles, agents capable to move dissipating energy. These self-driven units interact among themselves or with the environment. It is through these local interactions that nontrivial collective behaviours can emerge, showing one of the most fascinating and striking features of these systems. It is not difficult to be amazed by these eye-catching shows of coordination, present at very different scales, from molecular motors (e.g actin filaments [3]) to bacterial colonies [4] or a large group of animals, such as flocks of birds [5] or schools of fish [6]. Moreover, similar features can also be observed in non-living systems such as groups of robots [7] or vibrating disks [8].

For the physicist, the collective motion emergence from local interactions between agents, rather than from external forces or geometrical constraints on such a big variety of systems raises the question of the presence of some universal feature shared among all of them.

In order to grasp the core features of these systems, one has to build and study minimal models. In particular, since active matter systems show both ordered and disordered phases, one can inspect the type of transition that arises in such models as well as the kind of parameters associated with them.

One of the simplest models that can show such transition is the celebrated Vicsek model (VM). In the VM identical self-propelled particles move at constant speed v_0, aligning their directions to the average one of the neighbours up to some noise term (see section 2.1 for a more precise definition of the model). It's this competition between alignment and noise that determines a phase transition between an ordered polar phase (all the particle velocity directions are aligned) to a disordered phase in which the velocity directions are randomly distributed. In particular, this is a first-order phase transition, i.e. it shows an abrupt change of the system state, hysteresis and phase coexistence.

Even though in the VM all the agents are identical, in nature local interactions between neighbours particles depend on the individual perception of the world of agents that are never identical. In order to account for this heterogeneity and see how it affects the onset of collective motion, one can associate to each particle a different noise extracted from a given probability distribution, as Guisandez et al. suggested [1].

In particular, they claimed that introducing a normally distributed noise of the particles, the phase transition from disorder to order changes its nature, i.e. for a sufficiently high level of heterogeneity, it becomes continuous.

Continuous phase transitions show universality, long-range correlations, and high susceptibility to external fields at the transition point (better known as *critical point*) and are very different from first-order ones.

In this work, after reviewing the very well-known VM and its possible variations (sections 2.1 and 2.2), I will briefly present their model and results (section 2.3). Then, in section 3 and 4 I will present the results of my simulations, highlighting the differences with theirs. To conclude, I will just briefly mention the intrinsic problems of such implementation of the noise heterogeneity, proposing a different model that can overcome those limitations, opening at the same time the possibility to work analytically on it.

2 Vicsek model

2.1 Original model

The model was introduced by Vicsek and collaborators in the celebrated paper from 1995 [2] and it consists of pointwise particles moving at a constant speed v_0 in a box of size $L_x \cdot L_y$ with periodic boundary conditions.

Each particle aligns its motion direction to the average one of all the particles distant at most r_0 from it. On top of this resulting alignment there is a random angular noise term, whose maximum amplitude is determined by the parameter η. All the particles update their velocity directions θ_k and positions $\mathbf{x_k}$ at discrete time steps, through the so called *backward update rule* for which the position update is done using the velocity of the previous time-step. In particular we have the following update equations:

$$\begin{cases} \mathbf{x_k}(t+1) = \mathbf{x_k}(t) + v_0\, e^{i\theta_k(t)} \\ \theta_k(t+1) = Arg\left[\sum_{j\in\partial k_{t+1}} e^{i\theta_j(t)}\right] + \eta\,\xi_k(t) \end{cases} \tag{1}$$

where ∂k_{t+1} is the set of the neighbours of the k^{th} particle at time $t+1$ (i.e. the particles contained in a circle of radius r_0 centred in $x_k(t+1)$), i is the imaginary unit, and ξ_k is a random number extracted from a uniform distribution between $-\pi$ and π.

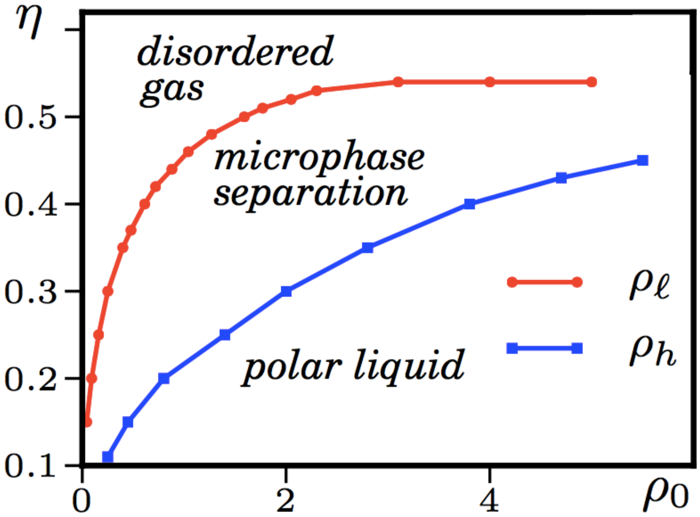


Figure 2: Phase diagram of the original Vicsek model in the noise-density phase space, showing the three possible phases for this simple model: the two homogeneous phases, the disordered gas-like one and the polarized liquid one, and the inhomogeneous band phase for which we have phase separation. The red and blue lines are called respectively superior and inferior binodal (from [10]).

Fixing the interaction radius r_0 as the length-scale of the system and v_0, one has just two independent control parameters: the noise strength η and the particles density ρ_0, reflecting competition between alignment (more effective at high densities) and noise.

To characterize the phase transition usually one defines as order parameter $\phi(t)$, the absolute value of the average polarization $P(t)$:

$$P(t) = \frac{1}{N}\sum_{i=1}^{N} e^{i\theta_k(t)} \qquad \phi(t) = |P(t)| \tag{2}$$

In the limit of high noise or low density, the system displays a disordered phase characterized by exponentially decaying correlations, $\phi \approx 0$, and normal fluctuation of the number of particles, as predicted by the central limit theorem. In particular, computing the variance of the number of particles N in boxes of different sizes and averaging over space and time, one obtains:

$$Var[N] = \langle (N - \langle N \rangle)^2 \rangle \sim \langle N \rangle \tag{3}$$

Increasing the density or decreasing the noise strength the system undergoes a phase transition to a polar ordered state ($\phi > 0$) through a spontaneous symmetry breaking. The polar phase is characterized by giant density fluctuations [11]. In particular, one has:

$$Var[N] \sim \langle N \rangle^{\alpha} \quad \alpha > 1 \tag{4}$$

Even though in the original Vicsek work [2] the transition was described as continuous, later works (such as [9]), have shown that on the contrary, it is discontinuous. In particular, in a region of the parameters close to the transition line one can see the formation of elongated and stable structures of ordered particles that move coherently (in figure [10] one can see the phase diagram).

This structures, known as bands, travel perpendicularly to their elongation axis in a homogeneous and less dense disordered background. The bands span all the system size and have a well defined maximum width. This means that if one takes a system in the band phase and starts to slowly increase its order (e.g decreasing η) the density of the gas will remain constant while the bands will get thicker and thicker until they reach their maximum width. After that, if one keeps increasing the order new bands will arise as far as the ordered phase will not be reached. This phenomenon is known as *micro-phase separation* (in figure 3 one can see the typical smetic alignment of the bands).

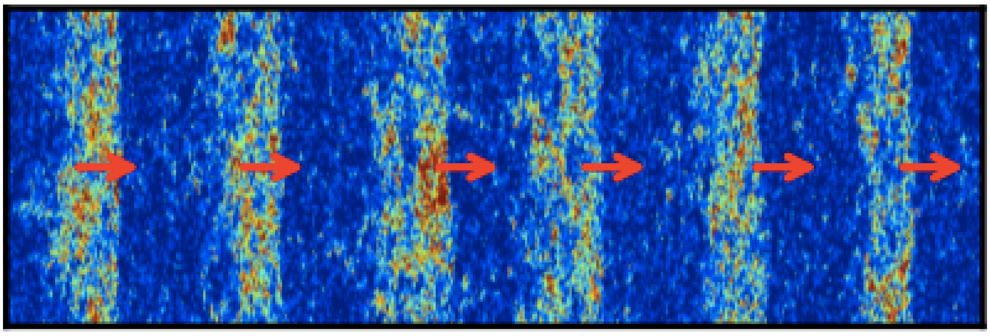


Figure 3: Microphase separation in the normal VM represented through the density of particles (the warmer the color the higher the density). The arrows represent the motion direction of the bands (from [10]).

Due to these properties, the phase transition is actually best understood as a kind of liquid-gas phase transition. In fact, one can identify the gas and the liquid phase with the ordered and disordered one, respectively. These homogeneous phases are separated from the band phase, in which one has phase separation, by two lines called binodals.

As a last remark one can observe that this kind of models is strongly affected by finite size effects. This is the reason why Vicsek and collaborators were, in their first work, not able to see bands and thought that the phase transition was actually continuous.

2.2 Variations

Forward update rule During the years a lot of variations of the simple VM were proposed. Among the most successful there was the introduction of the so-called *forward update rule*, that consist in updating the velocities before the positions resulting in the following update equations:

$$\begin{cases} \theta_k(t+1) = Arg\left[\sum_{j\in\partial k_t} e^{i\theta_j(t)}\right] + \eta\,\xi_k(t) \\ \mathbf{x_k}(t+1) = \mathbf{x_k}(t) + v_0\, e^{i\theta_k(t+1)} \end{cases} \tag{5}$$

As one can easily see the only differences from equations 1 is that $\theta_k(t)$ and ∂k_{t+1} are replaced by $\theta_k(t+1)$ and ∂k_t, respectively. If one thinks of their meaning it is easy to agree that the forward update rule usually makes more sense. Indeed with the backward update rule, one introduces a delay of one time-step in the dynamics. In particular, each particle will align with the velocities of the surrounding particles of the previous time step (as if they were requiring some time to react and to implement their velocity variation). If the two update rules lead to the same results for a continuous model, for the VM the situation is different since it is a discrete time model. Nevertheless, the qualitative behaviour of the system does not change.

In fact, even though the Vicsek-like systems are out of equilibrium systems, strong heuristic evidence suggests that they all belong to the same universality class (i.e. they will share the same qualitative features, at least asymptotically).

Vectorial noise Another possible variation is to use a *vectorial noise* (as suggested by G.Grégoire and H.Chaté [9]) instead of the *angular* one introduced in section 2.1. In this case the update equation for the velocity direction of the k^{th} particle becomes:

$$\theta_k(t+1) = Arg\left[\sum_{j\in\partial k}(e^{i\theta_j(t)} + \eta\, e^{i\xi_k(t)})\right] \tag{6}$$

Once again, in the *thermodynamic* limit, this choice does not affect the qualitative behaviour of the system. However, when we consider systems of a finite size it stabilizes the bands, allowing us to see them more clearly in the smallest systems.

Indeed, one of the reasons why Vicsek could not see bands in [2] was that he used angular noise, but did not reach sufficiently big system sizes.

2.3 Heterogeneous Vicsek model

Guisandez and collaborators used in [1] the backward update scheme with vectorial noise, introducing also a heterogeneity at the noise level. In particular, they chose a normally distributed noise amplitude, with mean μ and variance σ^2:

$$\eta_k \sim \mathcal{N}(\mu, \sigma^2) \tag{7}$$

where $\{\eta_k\}$ are independent and identically distributed random variables.

Their claim are that the heterogeneity strongly affects the VM collective behaviour. Indeed, for a sufficiently high level of heterogeneity, they obtain an order-disorder continuous transition (varying μ), in contrast with the usual first-order transition of VM that still remains when σ is smaller than a threshold value.

Their main result and claim is the phase diagram shown in figure 4. In particular, since they fixed the density of particles ρ_0, they just have two control parameters: μ and σ. Notice that for $\sigma = 0$ one regain the usual VM (μ axis in the figure). On the contrary, when $\sigma \to \infty$ one expects to see just the gas phase since the distribution will tend to a uniform distribution over the real axis.

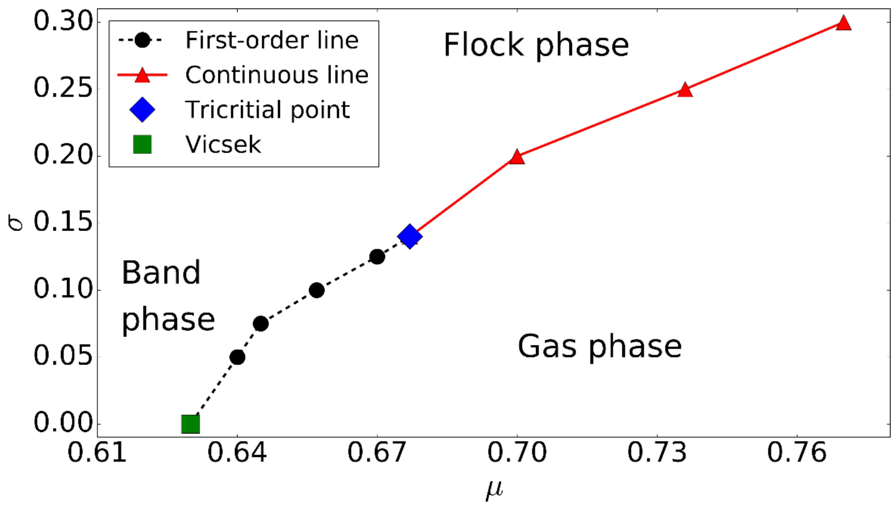


Figure 4: Phase diagram of the heterogeneous Vicsek model proposed in [1]. The claim of the authors was that for σ values smaller than a certain threshold one has the usual transition between gas and bands (dashed line with full circles), but when the noise variance is sufficiently high one has instead a continuous transition (continuous line with triangles) between the gas and a not well defined ”flock phase”. The two lines meet at a tricritical point, marked with a full diamond (from [1]).

Comments on the model Already at this stage one can move some critics to their results and this is the reason why I have tried to reproduce them.

First of all, they have never defined precisely the *flock-phase* that appears in their phase diagram. In particular, they did not explain whether it has some connection to the liquid phase or not. Another problem is that they did not determine a transition line between this last phase and the band one. Finally, we know that the VM is robust to variations and therefore a change in the type of phase transition would be quite surprising.

3 Simulations set-up and results summary

In my simulation I used, first of all, the same set-up of [1] (Vicsek model with vectorial noise) with the only difference that I decided to use the forward update rule. This rule is simpler and is nowadays preferred in most of the published work on Vicsek-like models. Moreover, as I remarked before, it should not affect the qualitative infinite size behaviour of the system.

Being aware that the VM and its variations are usually strongly affected by finite size effects I have started using the same set of parameters as in [1] both to understand what is the flock phase, they mention and what type of transition is actually present between this phase and the band one (not mentioned in their work). Moreover, the aim of my work is to understand if the transition between the gas phase and an ordered phase is indeed continuous or not.

Unless otherwise stated, the density, velocity and interacting radius has been set respectively to $\rho_0 = 1.0$, $v_0 = 0.5$ and $r_0 = 1.0$. For what concerns the dimensions of the system I have started using a square box of size $L = 128$. To reduce the marked finite size effects intrinsically present in this model, I also had to simulate bigger system sizes, up to $L = 2000$. The initial conditions used where either isotropic, i.e. with each particle having a random position and velocity direction (both drawn from a uniform distribution), or taken from a configuration obtained from another simulation.

Simulating systems of size $L = 128$, I obtained a rough sketch of the (finite size) phase diagram (figure 6), obtaining at this stage more or less the same transition line as the one of figure 4.

In particular for σ big enough I could see a phase with non-zero polarization displaying small groups of aligned particles that were not stable in time. This behaviour is different from any other known phase of the VM.

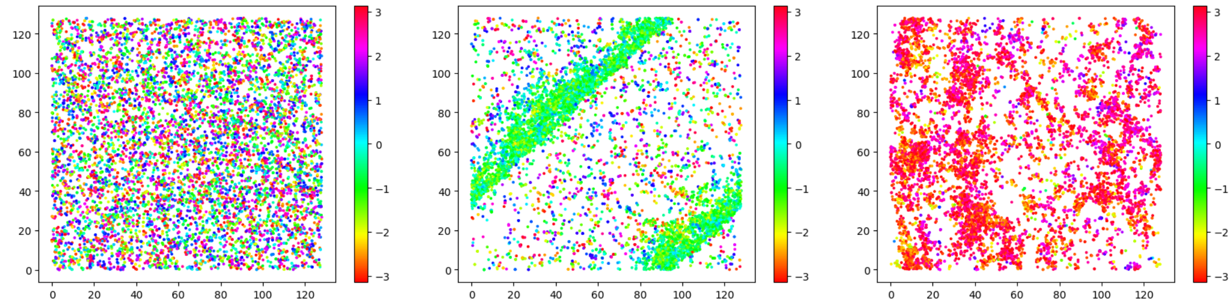
In order to characterize it, I increased the system size and sent longer simulations. These numerical efforts were rewarded when I could see the emergence of bands both with the forward update rule and the backward one (the one used in their work). In other words, the strange system behaviour that is observed at small system sizes is just an artefact of finite size effects.

This results suggested that the VM picture remains the same also in presence of this kind of heterogeneity, showing once again that the model is robust. At this stage, all that was remained to do was to determine if the phases of these model were analogous to the one of the VM.

4 Simulations results

4.1 Finite size phase diagram

In order to create a rough sketch of different areas of the phase space, I have simulated a small system ($L = 128$) for different values of μ and σ, changing little by little one of the two parameters every $N = 10^5$ steps along parallel lines in which the other one was constant. As one can



(a) gas, $\mu = 0.68$ (b) band, $\mu = 0.6$ (c) liquid, $\mu = 0.44$

Figure 5: Examples of the usual VM phases, $L = 128, \sigma = 0.04$. System snapshots: each point represent a particle. The associated color indicate the motion direction of the particle.

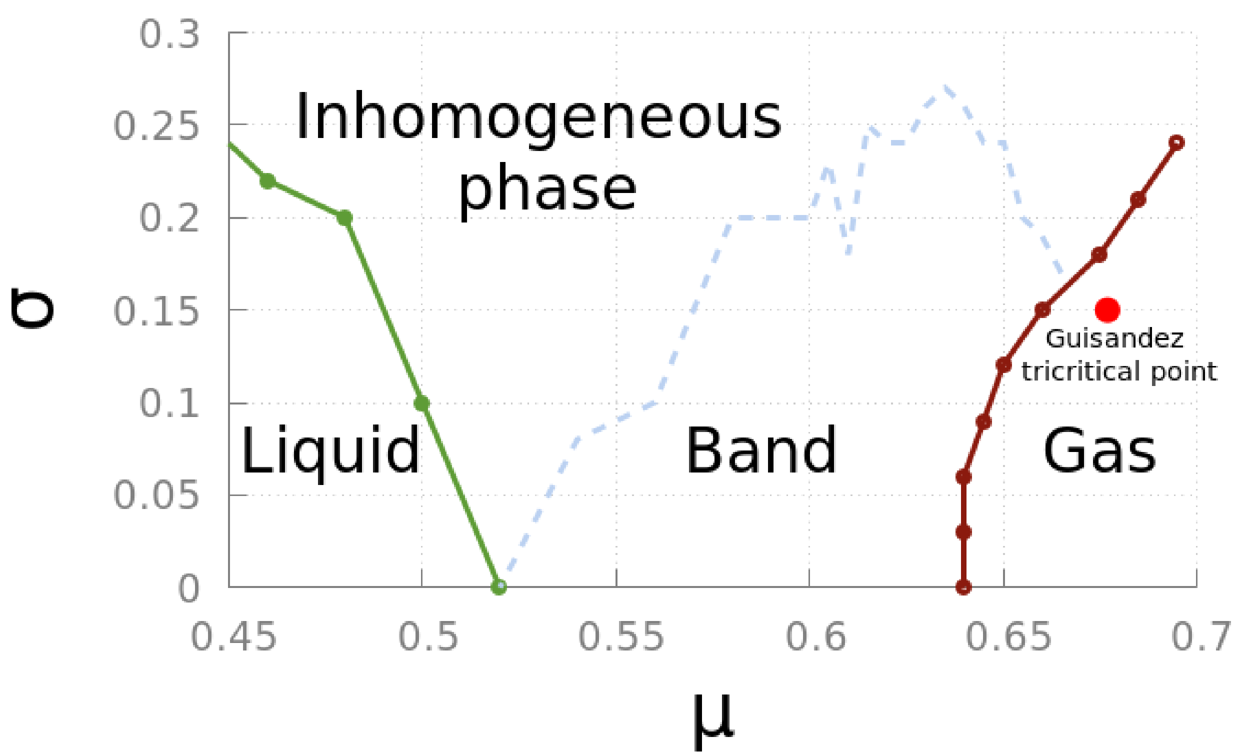


Figure 6: Finite size phase diagram obtained by visual inspection of the configurations snapshots simulations sent for $L = 128$ with the forward update rule. On the right we have the disordered gas-like phase for which the average polarization is almost zero; on the left, there is the highly ordered liquid-like phase. Between them, there is a phase that is not homogeneous. In particular, on the bottom, there is the usual band phase separated by the inhomogeneous, not well defined, phase by the dashed line. At this stage, the phase diagram seems to agree with the results of [1], apart from some slight misalignment of the transition lines. The red circle represents their tri-critical point.

see in figure 6, I have obtained a superior binodal that is just slightly shifted with respect to theirs (so that also their tricritical point does not lay on my line). This little quantitative differences lay on the fact that my resolution was not very high. Moreover, my phase diagram is affected by the system size while the one in figure 4 is obtained using finite-size scaling techniques and is claimed to be asymptotic.

In order to have a better understanding of the underlying physics, I introduce $\overline{\mu} \approx 0.637$, defining it as the value of μ for which I have the binodal line for $\rho_o = 1$ in the usual VM (i.e. on the $\sigma = 0$ axis). For the usual VM, this is the noise level value below which particles can polarize. This value could be in principle used to understand if bands form when the fraction of polarizing particles in the system is big enough (but this is still a work in progress).

Like Guisandez et al., I observe, for a sufficiently big level of heterogeneity (meaning for σ greater than a certain value) a phase transition between the gas phase and a not so well defined phase. This inhomogeneous "new phase", located approximately in the same area as their *flock phase*, seems indeed different from all of the stable phases of the VM (cfr. figure 5). In particular, as one can see in figure 8a the polarization is not zero and while one can see the emergence of ordered patches of particles (figure 7), they are not stable and keep changing direction and shape, being in that different from a band.

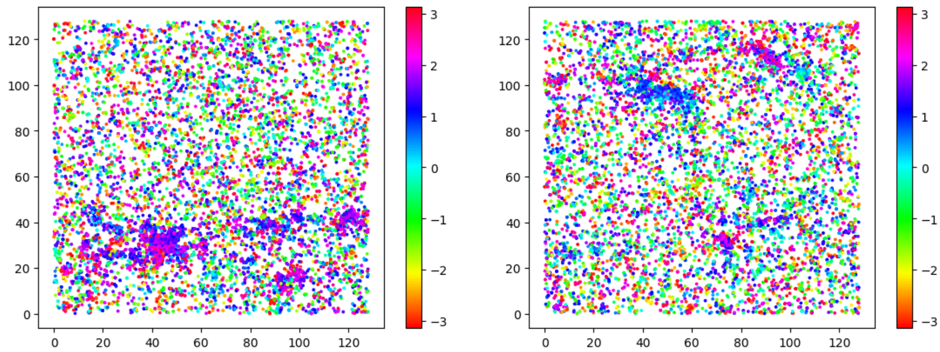
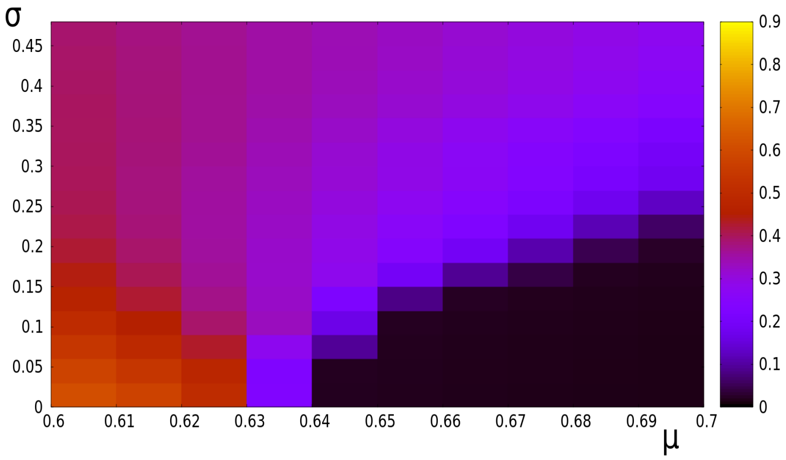


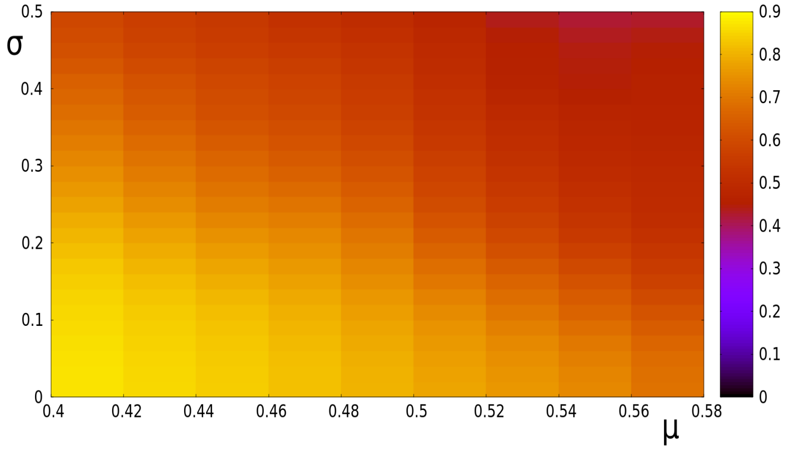
Figure 7: Screenshots of the system at a distance of 2×10^4 time steps in the inhomogeneous phase ($\mu = 0.7, \sigma = 0.29, L = 128$) obtained with the forward update rule. One can see small groups of aligned particles that are not stable in time. The color code represent the particles velocities direction.

In particular, this phase is separated from the usual band phase by the dashed line in the figure 6. Even though the qualitative differences between the two phases are clear looking at the system snapshots with sufficiently different values of σ, the transition between this two phases is actually gradual and not sharp. For that reason the line it is just meant to be a very rough estimation of a point in which we start to see the bands becoming unstable. In particular if one keeps μ constant and slowly increases σ, one can observe the bands starting to change direction more and more frequently, while becoming wider, and eventually disaggregating.

On top of the information gained looking at the system snapshots one can study the value of the polarization averaged over time ($\langle \phi(t) \rangle_t$) in the various point of the phase diagram. For example, in this way,



(a) Picture showing the part of the phase diagram in which there is the transition between the gas and the inhomogeneous phase or the band phase. In this figure, one can see clearly the gas phase in the low-right corner (characterized by an average polarization almost equal to zero)



(b) Picture showing the part of the phase diagram in which there is the transition between the gas and the inhomogeneous phase or the band phase. As one can see the average polarization is not enough to determine clearly the transition, as in both phases the polarization is not zero and it changes gradually between them

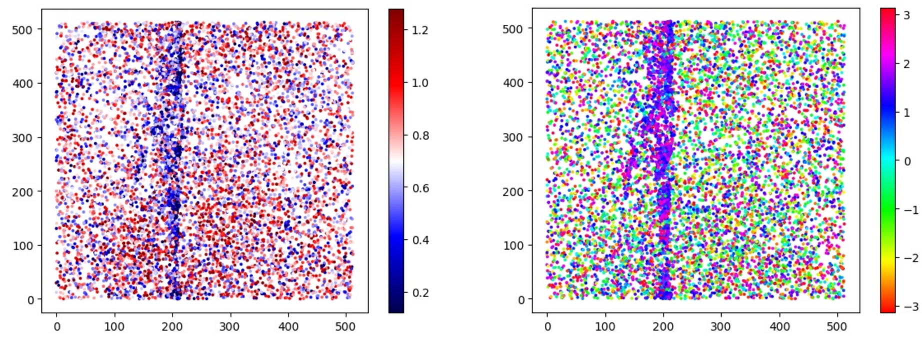
Figure 8: Total polarization average $\langle \phi(t) \rangle_t$ with respect to time (taken discarding the transient) in the phase space $\sigma \times \mu$, obtained with the forward update rule for small systems of size $L = 128$

one can easily see the superior binodal of figure 8a. It is instead more difficult to see the inferior binodal since both the liquid and the nonhomogeneous phases have non-zero polarization and the latter is changing slowly passing through the transition line.

4.2 Flock or Inhomogeneous phase?

In order to characterize the inhomogeneous phase and understand if it really was a new kind of phase, I sent simulations for bigger system sizes. Indeed in [1] was not very clear whether this was the case or on the contrary, the flock phase was related to the liquid one.

Increasing system size In order to be sure of being in the flock phase region, I sent simulation sufficiently far from both mine and their superior binodals (taking into account the shift between them). In particular I have simulated the system for 10^7 time-steps at $\mu = 0.7$ and $\sigma = 0.29$ (from now on point A) at system sizes $L = 256$ and $L = 512$. As shown in figure 10a, even for $L = 256$ I could see the formation of a band that appeared in all the snapshots recorded after its formation.

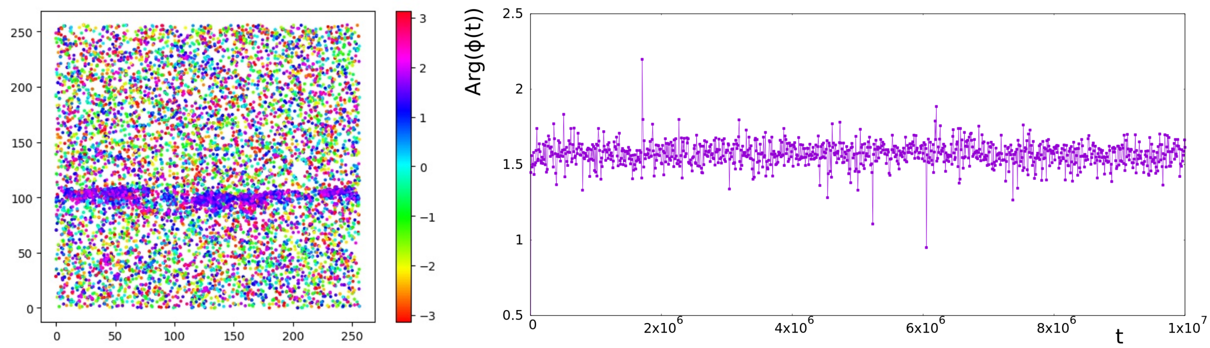


(a) The color code of the particles represents their noise level. **(b)** The color code of the particles represents their velocity direction.

Figure 9: System snapshot after $3.5 \cdot 10^6$ time steps for $\mu = 0.7, \sigma = 0.29, L = 512$, forward update rule, starting from a random initial configuration. One can see clearly a band.

In order to check its actual stability, I have looked at the time series of the total polarization direction (having it much more data-points that the number of snapshots recorded). As one can see in figure 10b this parameter presents just some fluctuations around the mean value but remains steady for 10^7 time-steps.

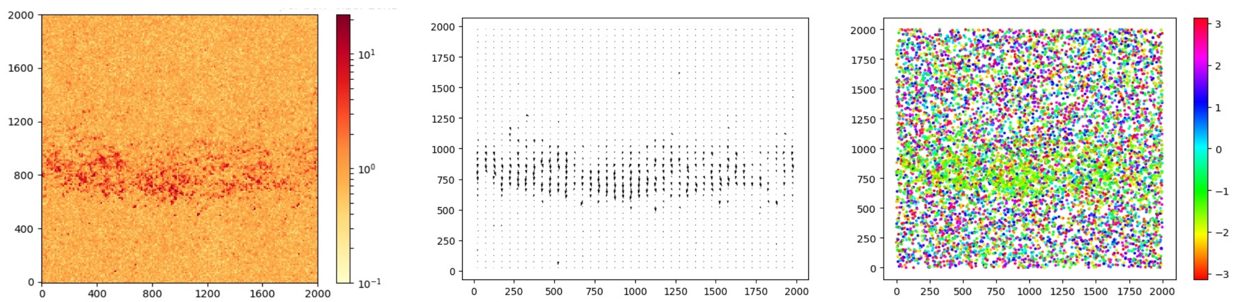
In figure 9 we can see that also for $L = 512$ one has the formation of a band that is visible either plotting the particles coloured by their velocity direction or by their noise. In particular in figure 9a one can see the segregation of the particles based on their noise level, as the less noisy one tend to accumulate into the band.



(a) System snapshot after 7.5×10^6 time steps, the color code of the particles represent their velocity direction. **(b)** Average polarization orientation time series

Figure 10: $\mu = 0.7, \sigma = 0.29, L = 256$, forward update rule, starting from a gas initial configuration. One can see clearly the formation of a stable band

Backward update rule At this point, one can wonder if my results are different from the Guisandez and collaborators ones due to the different update rule used. To understand the influence of the latter on the physics of the system and in particular on the inhomogeneous phase, I sent some simulations in point A with the backward update rule for $L = 256, 512, 1000, 2000$. For $L = 256, 512$ the situation of the inhomogeneous phase remains of undetermined nature, while for the bigger systems I could see the emergence of a stable band.



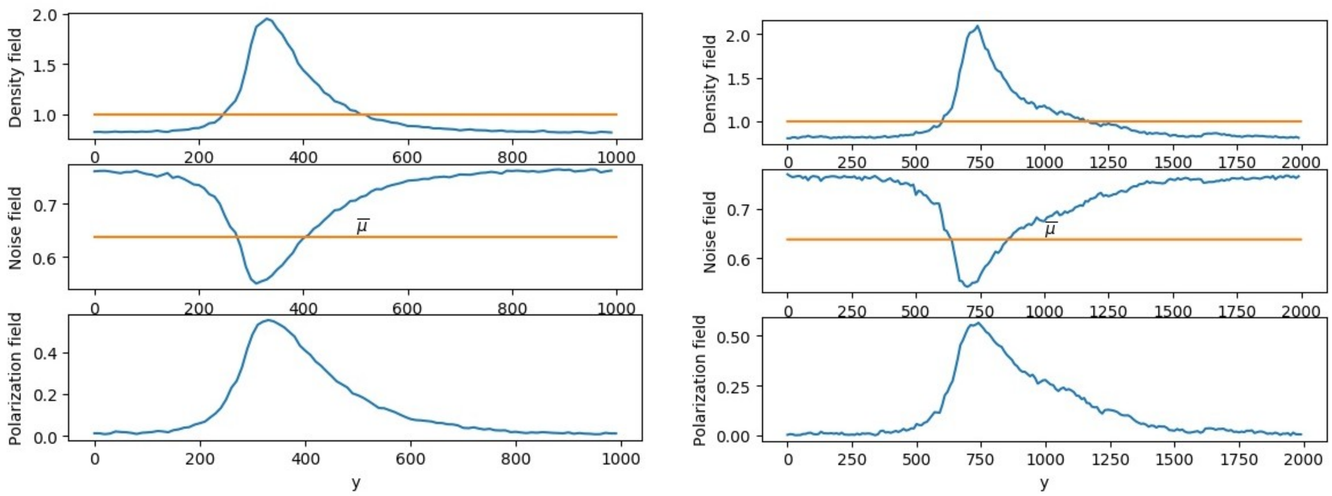
(a) Density of particles, computed in boxes of size 10×10

(b) Average velocity in boxes of size 50×50. In the gas background is almost zero, in the band it's directed downwards

(c) The color code of the particles represents their velocity direction

Figure 11: System snapshot representing a band after $4 \cdot 10^5$ time steps for $\mu = 0.7, \sigma = 0.29, L = 512$, backward update rule, starting from a random initial configuration.

In particular in figure 11 one can see a band formed in the point A for $L = 2000$. Since it is actually weaker and thicker than the bands obtained with the forward update rule, I represented it in various ways. Anyhow, given the difficulty to see clear bands, I have time-averaged their profiles of density, noise and polarization modulus. In this way, I had the confirmation that this structure is actually stable in time (figure 12).



(a) Band profile at $L = 1000$, averaged over 64 system snapshots each taken 10^5 time-steps after the previous one

(b) Band profile at $L = 2000$, averaged over 25 system snapshots each taken 10^5 time-steps after the previous one

Figure 12: Density, noise and polarization profiles of bands (obtained with the backward update rule) in point A, averaged over system snapshot at different times and over space (using slices perpendicular to the y-direction of size 10). The straight line in the density field represents the average density of particles, the one in the noise field $\overline{\mu} = 0.637$ represents the value below which particles can polarize in the normal Vicsek model (i.e. when $\sigma = 0$ in the heterogeneous model). In particular, we can see that particles that can polarize tend to regroup in the band

Looking at the profiles one can notice that these bands are "weaker" and less dense compared to the ones usually obtained with the forward update rule (cf. the profiles of the "forward band" in figure 13). Moreover the width it is comparable with the system size, meaning that probably the bands need even more space to form. This suggests us that the reason why Guisandez and collaborators did not see bands was just that the systems that they simulated were too small (they considered systems with a number of particles up to 131072, i.e. system with a size smaller than $L = 512$ for which indeed I cannot see bands too).

Using the backward update rule makes more difficult to see bands due to the stronger finite size effects but it leads to the same result in the $L \to \infty$ limit. Once I had determined that I kept using the forward update rule, since it is computationally less expensive, allowing me to see bands at smaller system sizes.

4.3 Bands phase properties

Bands profiles Looking at the profiles of the bands (obtained averaging in time and also in space over slices along the whole system) one can see (figure 13) that they are analogous to the ones of the bands of the normal VM, with a very sharp front.

Apparently, the only new features that they present is the segregation of particles based on their noise level, as already observed previously (figure 9a). In particular one can see that the less noisy particles are mainly positioned on the band front, leading all the others.

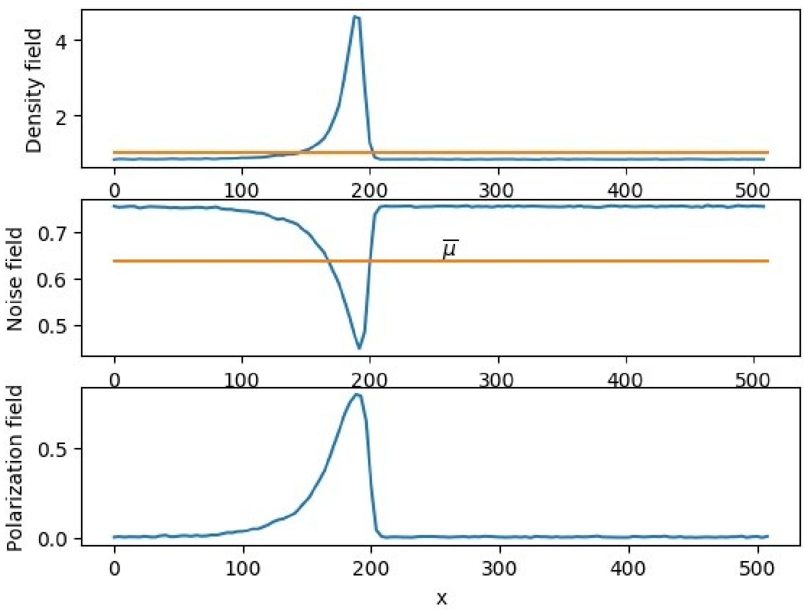
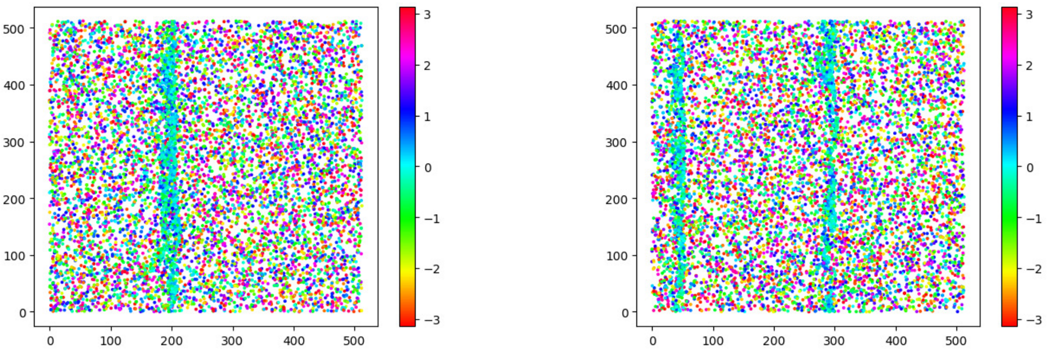


Figure 13: Density, noise and polarization profiles of a band averaged over time and space (over slices along y of size $4 \times L$). The horizontal lines in the density and noise profiles represent respectively the average density of particles and $\overline{\mu}$ (see text or figure 12 for more informations). System in the point A, $L = 512$, forward update rule.

Increasing particles density To investigate the nature of the band phase one can try to understand if there is or not microphase separation or not.

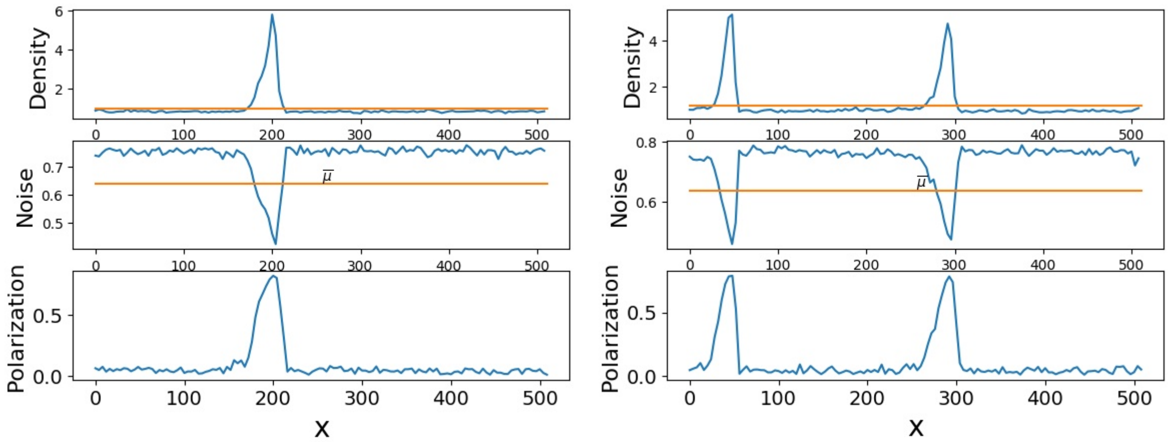
In particular one could increase the system size in a point where we have bands. Since this method is computationally quite demanding I have instead decided to directly increase the particles density in the point A for $L = 512$. Notice that with this second method we are changing point in the phase space and thus one should be careful to remain in the band phase.



(a) $\rho = 1$, stable band obtained in the simulations of section 4.2

(b) $\rho = 1.216$ after $9 \cdot 10^4$ steps from the last density step

Figure 14: Two system snapshots taken at successive time-steps in the process of increasing the density of the particles in point A for $L = 512$. The initial density was set to $\rho = 1$ and then increased of 5% every 10^5 time-steps, starting from a stable band. Between the two snapshots we have an event of band split. The color code of the particles represent their velocity direction.



(a) $\rho = 1$, single band

(b) $\rho = 1.216$ after $9 \cdot 10^4$ steps from the last density step: a second band emerge

Figure 15: Profiles of density, noise and of polarization of the two snapshots of figure 14 averaged over slices normal to the x direction of size 4. The straight line in the density field represents the average density of particles, the one in the noise field $\overline{\mu} = 0.637$ represents the value below which particles can polarize in the normal Vicsek model

Increasing slowly the particles density I could see an event of band splitting that would suggest the presence of a phenomenon of microphase separation also in the heterogeneous model (see figure 14 for details).

To check whether this is actually true I have studied the profiles of the bands before and after the band-splitting event (figure 15). Unexpectedly the density level of the gas is different in the two cases (whereas for the normal VM it is not, at least in the limit of infinite system size).

In particular, increasing the density ρ_o the density of the gas increases non linearly, as one can see in figure 16. This behaviour could derive from the fact that, increasing the particles density, I'm actually adding both particles that can polarize (with $\mu < \overline{\mu}$), and particles that cannot. These particles go directly in the gas phase, increasing the density of the gas in the system.

Anyhow the density range considered it is probably too small to conclude anything. Further investigations are needed.

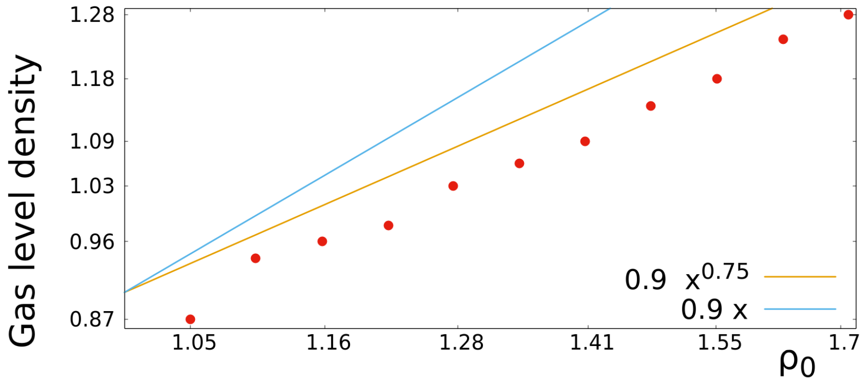


Figure 16: Gas level in the system as a function of the average particles density ρ_0 in the system. In the VM is, on average, constant. Here the behaviour seems to be different from the constant or the linear one but one needs a bigger density variation to really conclude something. The dots represent the data, the top and middle lines are meant just to give a visual reference of the non-linearity of the relation between the two quantities.

4.4 Liquid phase properties

4.4.1 Giant number fluctuations

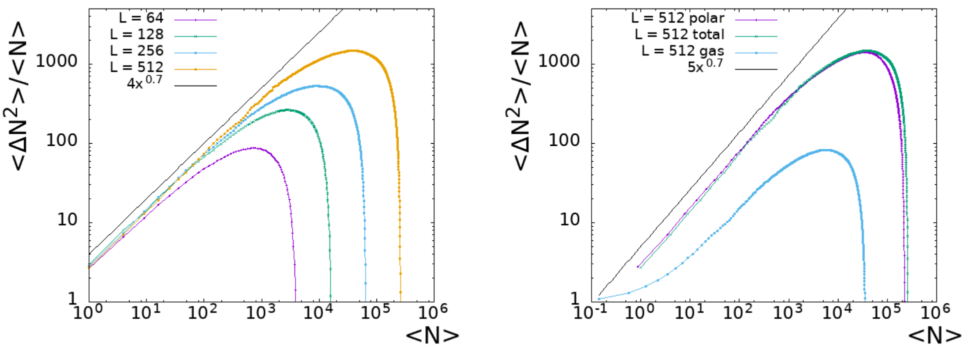
To characterize the liquid phase I sent some simulation for different system sizes, computing the fluctuation of the number of particles in boxes of various sizes. As one can see in figure 17a the fluctuation are not normal. In particular, I get:

$$Var[N] \sim \langle N \rangle^{\alpha} \tag{8}$$

with $\alpha \approx 1.7$, that is compatible with the simulations of Hugues Chaté et al. that have obtained an exponent $\alpha \approx 1.6$ for the usual liquid phase of the VM [12]. At this point one can wonder whether this feature belongs to all the particles or just to the ones with $\mu < \overline{\mu}$. One can see in figure 17b that indeed all the particles show a behaviour that is not normal, even the particles that in the usual VM would be in the gas phase ($\mu > \overline{\mu}$).

4.4.2 Long range order

Another feature of the usual VM liquid phase that we can check is the presence of long-range order. By simulating the system in the liquid



(a) Relative variance fluctuation of the number of particles for different system sizes

(b) Relative variance fluctuation of all the particles, of the particles with $\mu < \overline{\mu}$ (the polar one), and of the particles with $\mu > \overline{\mu}$ (the ones in the gas phase for the normal VM). $L = 512$.

Figure 17: Relative fluctuaction of the variance of the number of particles N in log-scale. Simulations sent for $\sigma = 0.2$ and $\mu = 0.4$

phase (at $\mu = 0.4$ and $\sigma = 0.2$) for different system size I have obtained the plot in figure 18a that shows the average polarization dependence with the system size. We can see that indeed the polarization decreases (as a power law, cfr. figure 18b) to a constant value. As we expected the liquid phase of the heterogeneous gaussian VM seems to share the same features of the normal liquid phase.

4.5 Final comments

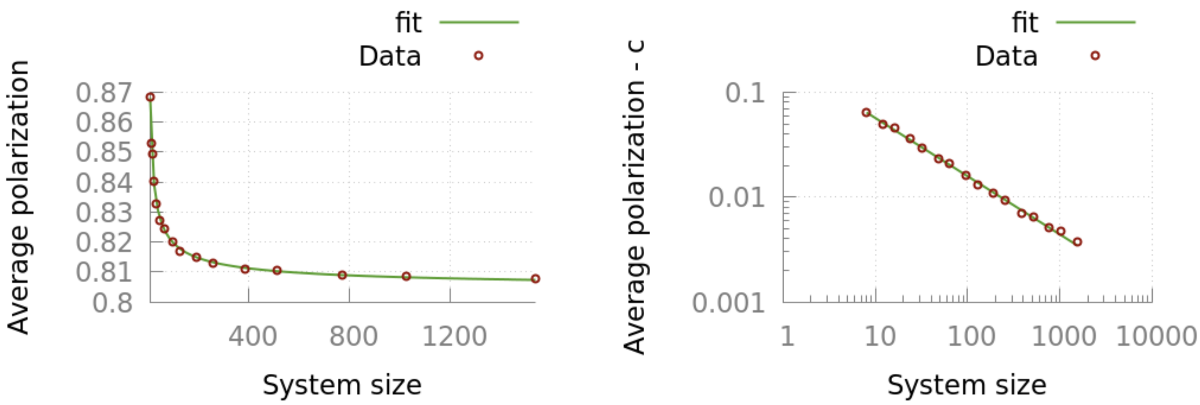
I was able to show with my simulations that the Vicsek picture is not affected that much by the presence of this kind of heterogeneity at the noise level. Indeed even for $\sigma > 0.21$ (tricritical "Guisandez point") the model still displays a liquid-gas phase transition and in particular it is not continuous.

The work of Guisandez and collaborators shows us, once again, that one should be careful about the finite size effects on the VM. Indeed, using the backward update rule makes more difficult to see bands. In particular, if the system size is not big enough, the system cannot fit a band and one can only see some small flocks of particles moving coherently through the system. On the other hand, using the forward update rule, one can simulate smaller systems to investigate their properties and therefore it is preferable to adopt it in this kind of models.

Even though the heterogeneous model does not seem to be that different from the usual VM, we can see anyhow some new feature, as the segregation of the particles according to their noise levels in the bands.

A part from the results one can wonder if the Gaussian distribution was actually the most suited way to introduce the heterogeneity.

One problem is that if one keeps increasing σ beyond the values that I consider in my simulations, the number of particles with a negative noise amplitude will become nonnegligible. This is actually a problem since the distribution actually deviates more and more from



(a) Time averaged total polarization $\langle\phi(t)\rangle_t$ as a function of the system size. We can see that as the size of the system goes to $+\infty$ it tends to saturate.

(b) Difference between the average polarization as a function of the system size and the average polarization value at ∞. We can see that the power law fit is indeed appropriate (log scale).

Figure 18: Plots showing the long range order in the liquid phase. The fitting function is $f(x) = c + \frac{a}{x^b}$, with $c \approx 0.804, a \approx 0.204, b \approx 0.555$. Simulations sent for various system size up to $L = 1536$, in $\mu = 0.4$ and $\sigma = 0.2$. The actual value of the polarization average is done averaging over many different simulation (especially for the smallest system size) in order to increase the accuracy of the plot.

the Gaussian one (since actually, it is only the amplitude of the noise that matters and not its sign).

Another thing to notice is that increasing σ, at least asymptotically, the fraction of particles able to polarize in the VM will decrease, being given by:

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\overline{\mu}}^{\overline{\mu}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \tag{9}$$

which goes to zero for $\sigma \to \infty$. For this reason, we expect a gas phase for a value of σ sufficiently high.

To avoid all this defect of the model one can introduce a bimodal model that is simpler and moreover tractable at the continuous level also from an analytical point of view.

Bimodal model In this model, one has two populations of particles with two different but constant noise amplitudes η_1 and η_2. This model can be described with 3 free parameters: η_1, η_2, and the fractions of particles f of one population with respect to the total number of particles. If one fixes for example $\eta_1 = 1$ (i.e. putting the population 1 in the gas phase) one can have just two free parameters and can get a phase diagram analogous to the $\sigma \times \mu$ of the Gaussian case (since the remaining two parameters can actually be associated to the noise mean and variance of the noise distribution). The advantage of this model is that the phase diagram is bounded so that we can avoid the problems of the Gaussian model mentioned before. Moreover, it is more amenable to analytical treatment. In particular one can write two Boltzmann equations, probably leading to coupled Toner-Tu equations. Studies of the model equations and its phase diagram are on the way.

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