POLITECNICO DI TORINO

Corso di Laurea in Physics of Complex Systems

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Freezing and pinning in a Schelling-type system

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Summary

We investigate the behaviour of a standard multi-agent system (the Schelling's segregation model) when a fraction f of agents are frozen (or "pinned"). We consider cases where the frozen agents are uniformly distributed across the system, and cases where they are grouped together either (i) outside a circular region or (ii) inside an annulus. We find that these lead to different consequences on the long-time average contact density in the system.
Different transitions between segregated and mixed phases can be then investigated with the variation of the control parameters of the system.

Contents

List of Tables

List of Figures

Introduction

In 1969 and 1971, Thomas Schelling published his seminal papers on checker-board simulations of a two-community dynamics [Schelling(1969), Schelling(1971)] – incidentally, this was one of the first multi-agent models. This model revealed how, even with relatively high levels of tolerance for mixity, a system of agents located at sites on a grid could eventually evolve towards patterns of segregation, with the two communities "living" in distinct parts of the grid. This surprising, even seemingly paradoxical, observation has given rise to a vast literature on Schelling's model of segregation, across a wide range of disciplines [Clark(1991), Vinković and Kirman(2006)]. In statistical physics, fruitful analogies with certain interacting particle systems have been drawn, allowing to establish more robust and general results [Stauffer and Solomon(2007), Cortez et al.(2015)Cortez, Medina, Goles, Zarama, Rica, et al.].

Here we are interested in the behavior of a Schelling system where one introduces a fraction f of agents that are "frozen" or "pinned"; that is, they do not move throughout the simulation. However they influence the dynamics, as they create a background or a landscape with respect to which the other agents move. To explore the influence of such fixed agents, we run simulations and examine how the average contact density across the system evolves depending on f and depending on the spatial distribution of the fixed agents. We find that a uniform spatial distribution has a substantial de-segregating effect, whereas a concentration of well-mixed, fixed agents (akin to a neighborhood with imposed mixity) outside a disk or inside an annulus has almost no effect on the mixity in the other parts of the system.

In the following, we first recall Schelling's model and reproduce some existing results (chapters 1, 2 and 3) before investigating the proposed variant.

Chapter 1

Schelling's original model

In Schelling's original model [Schelling(1969),Schelling(1971)] the population is divided in two groups, every agent is recognizable and permanently belongs to the same group. The area representing a territory is divided into a fixed number of squares, each of them being either left vacant or occupied by one agent belonging to one of two possible types. On the lattice, each internal square is surrounded by eight neighbors, each along the edge of the board by five and each in the corner by three of those. The starting point is a random distribution of the two types of agents and vacancies on the board. If an individual is discontent with his own-neighborhood, he moves to the nearest vacant space with a neighborhood satisfying his demand. Only horizontal and vertical moves are allowed; the distance between two squares is measured by the number of spaces traversed and an individual is said to be discontent if the percentage of neighbors of opposite color exceeds a limit, called *tolerance*. After each agent's move, not only its dissatisfaction turns into satisfaction: it can happen that someone who was discontent becomes content and/or vice versa. The unsatisfied agents are moved in turn and any originally discontent member who is content when his turn comes will not move anymore, while any member who becomes discontent because of other ones' movements will move after that all the originally dissatisfied agents have had their chance to move. Schelling experimented different orders of moving the dissatisfied agents and observed that the particular outcome depends very much on the order, but its character does not so much. By changing the parameters, i.e. tolerance (which could be or not the same for each type of agents) and number of agents of each type, Schelling saw that the system reached a segregated state even for relatively high values of the tolerance. In his analysis, he also observed that a percentage of vacancies is a good compromise between freedom in agents' movements and abundance of occupied sites.

Chapter 2

A model of standard Schelling dynamics

2.1 System

We consider a square-lattice system, with a number of sites equal to $width \times length$, where each site has unitary length and is either occupied by one agent or vacant. We use periodic boundary conditions so that the internal, the board and the corner sites are all surrounded by eight neighbors.

The variants we introduce with respect to Schelling's original model are the following:

1. both the unsatisfied and the happy agents move, with probabilities p_u and p_h respectively;
2. the unsatisfied agents do not move certainly, i.e $p_u < 1$;
3. agents do not move to the nearest vacancy but to one chosen randomly over all the vacant sites of the lattice;
4. the degree of satisfaction of all the agents is updated after each move, hence there is no a "waiting list" for the newest member of the unsatisfied (or satisfied) group: every individual moves with p_u (or p_h) at the moment of its examination, which happens when he is selected while going through the agents' list.

We fix the agents concentration ρ, hence the total number of individuals N_{TOT}; also the number of agents of each type, *type 1* and *type 2* are fixed and are indicated respectively by N_1 and N_2. the tolerance τ is the same for each agent.
One at a time, agents are examined and moved with the probability corresponding to their own satisfaction; a complete round over all agents corresponds to one time step and the total number of time steps is indicated by T_S.
By starting from a random distribution of agents and vacancies over the lattice, we expect

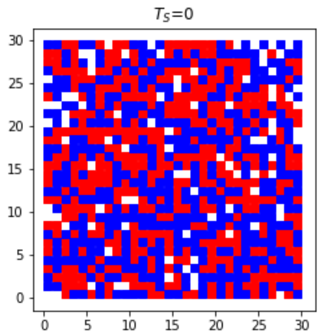
to reach a segregated state, therefore we introduce a variable that quantifies segregation: the contact density $\chi(t)$.
$\chi(t)$ is defined as the average computed over all agents at time t of the quantity

$$\frac{N_d}{N_s + N_d}$$

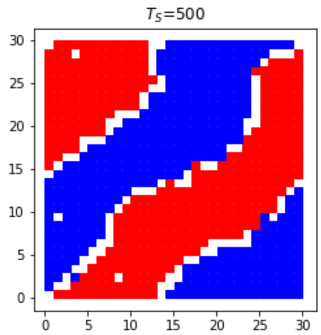
,
where N_d and N_s are respectively the number of different and similar neighbors of each agent.
Given that we expect $\chi(t)$ to hold value around 0.5 in the initial configuration and approximately 0 in the segregated state, we introduce the *normalized contact density* $\chi_n(t)$, in order to use an order parameter varying between 0 and 1. We will perform an average of $\chi_n(t)$ over a number N_E of realizations of the system and will denote as $\langle \chi_n(t) \rangle$ *normalized average contact density* over all ensemble realizations.

2.2 Numerical simulation and results

The values of the parameters used in the simulation are given in Table 4.9.



(a)



(b)

Figure 2.1: Initial and final configurations of the system following standard Schelling dynamics. The red and the blue pixels represent the two types of agents, while white pixels represent vacancies. The initial configuration is random and shows total mixity of the three colors, the final one is completely segregated.

Figure 2.1 shows the initial and final configuration of one over the 20 realizations of the system used to perform the ensemble average. One can observe that in the final configuration there are precisely two clusters, one of red and one of blue agents, considering to be in presence of periodic boundary conditions. The vacancies are mostly interposed between the red and blue agents, in order to minimize the contact between the two different types of individuals and the interfaces are characterized by sharp lines.

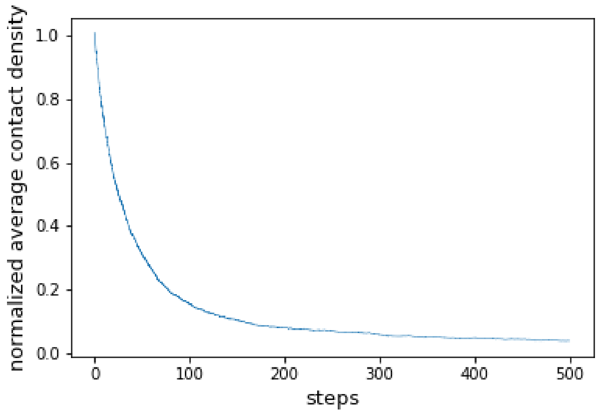


Figure 2.2: Ensemble average of the normalized contact density as a function of steps of time steps, with standard Schelling dynamics

Figure 2.2 shows the time evolution of the normalized average contact density. One can observe that this quantity rapidly converges to zero ($\chi_{n,\infty} \sim 0.04$) and a number of time steps given by $T_S = 500$ is enough to ensure to see the convergence of $\chi_n(t)$ to the stationary value $\chi_{n,\infty}$ of the segregated state. In this regard, we obtain that the standard deviations σ_{50} and σ_{100} of the last 50 and 100 values of $\chi_n(t)$ from their own averages are both less than 5%, in particular $\sigma_{50} \sim 0.002$ and $\sigma_{100} \sim 0.003$.

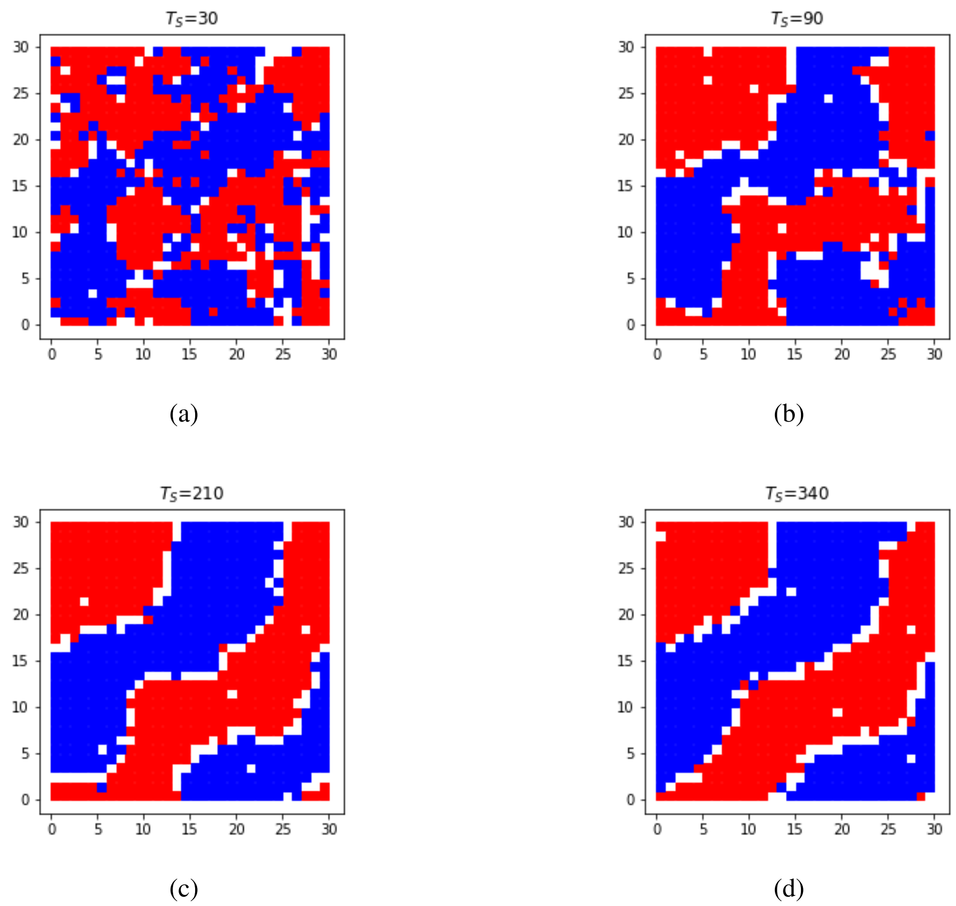


Figure 2.3: Configurations of the system at intermediate steps, with standard Schelling dynamics

Figure 3.3 shows the configuration of the system at (a) $T_S = 30$, (b) $T_S = 90$, (c) $T_S = 210$, (d) $T_S = 340$.
One can gain a picture on how fast the system organizes in two clusters and soon later totally segregates.
The results obtained in this preliminary part of the work are consistent with [Gauvin et al.(2009)Gauvin, Vannimenus, and Nadal, Gauvin et al.(2010)Gauvin, Nadal, and Vannimenus], and the evolution of the system represents the standard Schelling dynamics, which can be found in the literature.

Chapter 3

Variation with respect standard Schelling dynamics: three kinds of agents

We introduce in our system the agents of *type 3*, which can display either color red or blue by switching from one to the other. Agents of *type 1* and *type 2* follow the Schelling dynamics introduced in section 2, while agents of *type 3* do not move in the lattice and, when examined, change from *type 1* to *type 2* or vice versa with a fixed probability, independently of their neighbors and hence of their degree of satisfaction.
Switching agents can be interpreted as sites perpetually occupied and indifferent to neighboring types for various reasons. One of them could be the presence of external institutions, such as housing associations, imposing this kind of policy. Or alternatively, if type represents economic level, it may be that an individual switching from one status to the other avoids, as far as possible, spatial moves in the city, and does not separate from the neighborhood that reflects his past status, because of his difficulty in accepting to leave social habits or because of other plausible social causes.

The first goal is to work in the framework of the model introduced by *Randon-Furling* and *Hazan* in [Hazan and Randon-Furling(2013)], by arranging switching agents in random positions, and this is done in subsection 3.1.
In particular, in 3.1.1 we focus on the evolution of the system and on the analysis of the normalized average contact density as a function of time.

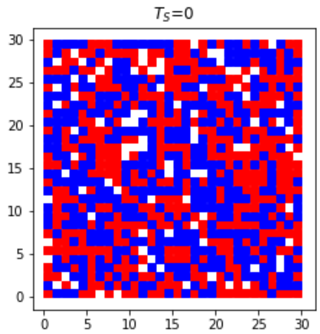
The second purpose is to arrange fixed agents out of a circle surrounding agents of *type 1* and *type 2* which perform Schelling dynamics and to explore if the individuals inside the circle reach the stable segregated state. Our study is based on the *frozen pinning file method*, used in [Cammarota and Cavagna(2017)] in order to evaluate the value of the surface tension in systems with first order phase transition, and is presented in subsection 4.

3.1 Fixed agents randomly arranged in the lattice

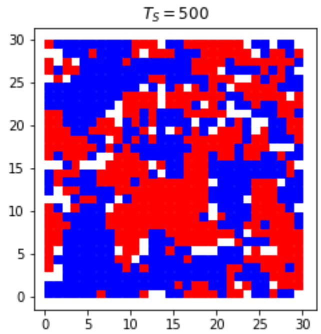
3.1.1 Evolution of the system and of $\chi_n(t)$

The number of agents of each three types are fixed and are given by the parameters N_1, N_2, N_3, where the last one is determined by the density f. Agents of *type 3* are put in randomly chosen locations of the lattice and never change their spatial position but, when examined, switch from *type 1* to *type 2* or vice versa with probability p_s.

The values of the parameters used in the numerical simulations are given in Table4.10.



(a)



(b)

Figure 3.1: Initial and final configurations of the system with a fraction $f = 0.2$ of switching agents.

Figure 3.1 shows the initial and final configuration of the system. One can observe that at $T_S = 500$ red and blue agents organize themselves in clusters of variable size with fuzzy interfaces and the vacancies are positioned at the multiple interfaces between different clusters. This results in a mosaic of relatively small homogeneous areas and one can appreciate the difference between this final configuration and the one in Fig.2.3a, where the system, though still not segregated, displayed vacancies arranged along more defined lines.

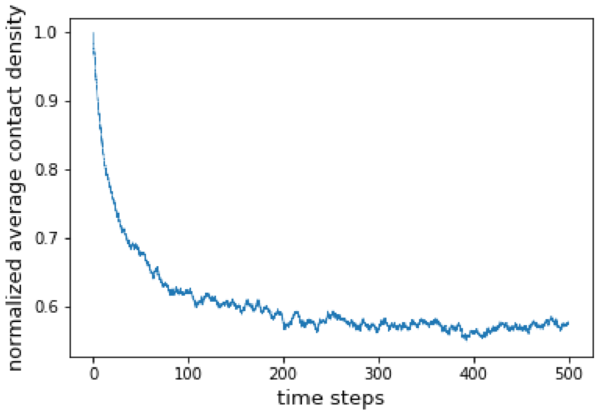
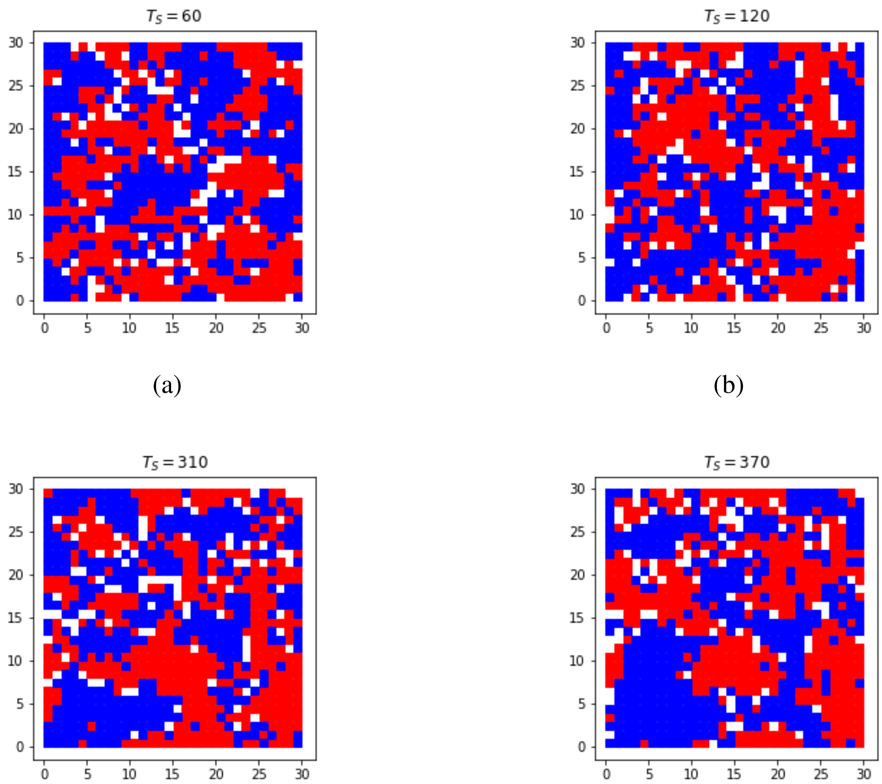


Figure 3.2: Ensemble average $\langle \chi_n(t) \rangle$ of the normalized contact density as a function of time steps, with $f = 0.2$

Figure 3.2 shows the time evolution of the normalized average contact density as a function of time steps. With the presence of switching agents, the system relaxes towards a value of χ_∞ close to 0.6 ($\chi_\infty \sim 0.58$). The behaviour of $\chi_n(t)$ is coherent with the result obtained in [Hazan and Randon-Furling(2013)] and with our expectation: agents of type 3 are a source of noise in the system promoting de-segregation, given that they do not perform Schelling dynamics and switch independently of their degree of satisfaction.



(a) (b) (c) (d)

Figure 3.3: Configurations of the system at intermediate steps, with $f = 0.2$

Figure 3.3 shows the configuration of the system at (a) $T_S = 60$, (b) $T_S = 120$, (c) $T_S = 310$, (d)$T_S = 370$.
One can see the effect of the random noise induced by switching agents: the largest clusters present in (a) or (b) or (c) are not prevailing any more 60 time steps later and there is no convergence to a stable configuration.
This fact is also clear from the more oscillating nature of the average contact density in 3.2, with respect to the one in Fig4.7 for the same number N_E. Though 500 ensemble realizations are sufficient to see enough stabilization of $\chi_n(t)$ over the last 50 and 100 time steps (in this case $\sigma_{50} \sim 0.005$ and $\sigma_{100} \sim 0.006$), $\chi_n(t)$ still displays small fluctuations around a mean "relaxation value" as time goes on. The results are coherent to the ones found in [Hazan and Randon-Furling(2013)] and we can appreciate the identification of an element -switching agents- that mathematically introduces modifications and makes more realistic the dynamics in a city with respect to Schelling original model.

Chapter 4

Fixed agents encircling a Schelling system

4.1 System

In [Cammarota and Cavagna(2017)] it is studied the thermodynamics of a sphere of radius R encircled by a frozen metastable that generates a pinning field at the surface of the sphere and biases its Gibbs distribution. Thanks to the introduction of the frozen pinning field (FPF) method, it is possible to explore the transition of the sphere from metastable to stable phase, by progressively varying its radius R. At a critical value of the radius $R = R_C$, the above-mentioned first-order phase transition occurs: the sphere is thermodinamically favoured to stay in the metastable phase for $R < R_C$, in the stable phase for $R > R_C$.

In Appendix 4.3.1 a more detailed explanation the FPF method and on its derivation from nucleation theory is given.

[1]We introduce a new variant of Schelling model by considering switching agents performing frozen pinning field and exploring the dynamics of *type 1* and *type 2* agents. More precisely, agents of *type 3* are arranged out of a circle, whose origin is positioned in the centre of a square lattice, and they are frozen in the mixed state (corresponding to the metastable phase of [Cammarota and Cavagna(2017)]. We impose $p_s = 0$ in order to set them in an invariant configuration, even though a probability of switching $p_s \neq 0$ would still ensure their non-segregated state. Our purpose is to study whether the agents of *type 1* and *type 2*, performing Schelling dynamics as in 1 and 3 and arranged inside the circle, reach the segregated state (corresponding to the stable phase) or not, for a range of possible values of the radius.

We use in our analysis a new variable quantifying segregation, the overlap Q, an

[1]The notations used in the following are the ones introduced in sections 1 and 3

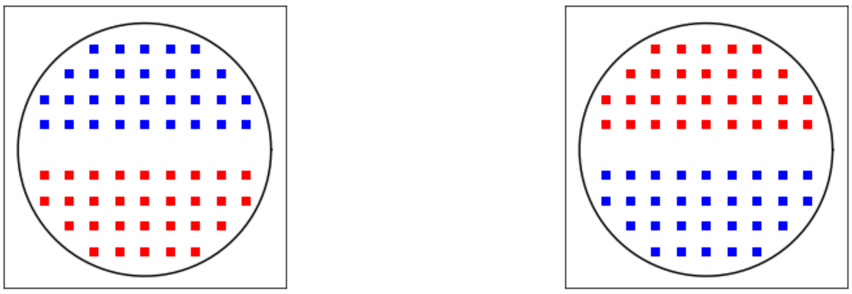
important tool from statistical mechanics also used in [Cammarota and Cavagna(2017)] to evaluate the degree of coincidence of two different configurations of the sphere, in order to control whether the sphere changes phase or not. It is defined as:

$$Q = \frac{1}{n(R)} \left| \sum_{i=1}^{n(R)} \rho_i \tau_i \right| \tag{4.1}$$

where $n(R)$ represents the number of sites inside the sphere, and ρ_i, τ_i are the two different configurations of site i, with values:

$$\rho_i, \tau_i = \begin{cases} -1, & \text{if site } i \text{ is occupied by a blue agent} \\ 0, & \text{if site } i \text{ is occupied by a vacancy} \\ +1, & \text{if site } i \text{ is occupied by a red agent.} \end{cases}$$

The absolute value over the sum in Eq.4.1 is taken in order to obtain $Q \in [0,1]$; in Fig.4.1 [2] one can appreciate the difference between imposing it or not in the definition of Q.



(a) (b)

Figure 4.1: Two segregated configurations red and blue agents inside the circle. The overlap of configuration in 4.1a with itself is $Q = 1$, exactly the same as the one with 4.1b. If absolute value were not taken in Eq.4.1, the overlap between 4.1a and 4.1b would be clearly $Q = -1$.

4.1.1 Numerical simulations and results

We perform numerical simulations on a square lattice, with total number of sites given by $width\ x\ length$ and we vary progressively the radius of the circle, considering integer

[2]In the figure, the density of vacancies (white pixels) inside the circle has been chosen simply to emphasise segregation and it does not correspond to the one fixed in numerical simulations and precised later, in subsection 4.1.1

values of R in the interval $[3{,}6]$. We fix the agents concentration $\rho = 0.9$ both inside and outside the circle, hence the numbers of switch and non-switch agents depend on the value of the radius. The numbers of *type 1* and *type 2*, given respectively by N_1 and N_2, are required to be equal, even though a difference of 1 between the two could occur if the number of non-switch agents is odd. Globally, inside and outside the circle, the number of red agents, N_{red}, is required to equate the number of blue ones, N_{blue}. One time step corresponds to a round over all agents performing Schelling dynamics.

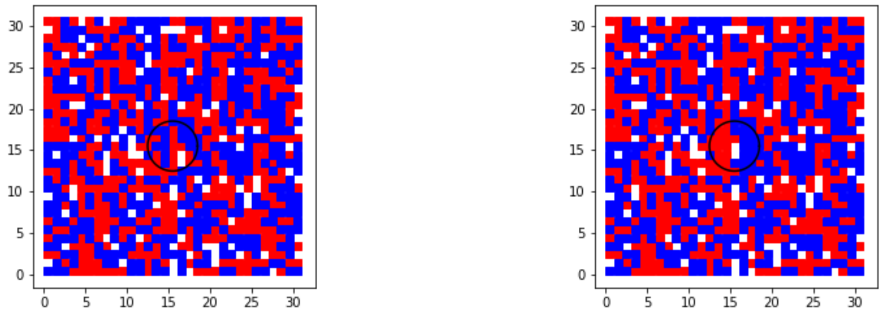
The system is analyzed by observing the lattice configurations, the plot of the average contact density and evaluating the overlap. In particular, for each value of R we evaluate $\langle Q_{initial} \rangle$ and , the $\langle Q_{final} \rangle$ which are respectively the average over N_E overlaps of two random initial configurations and the average over N_E overlaps of two final ones. $\langle Q \rangle$ should hold value close to 0 when considering two different mixed configurations and close to 0.5 in case of segregation: if one considers two identical segregated configurations with equal number of red and blue pixels inside a circle, he can in fact observe that every value of Q in the range $[0{,}1]$ can be obtained, by ideally continuously rotating one of the two circles and keeping the other fixed.

The values of the parameters invariant with R are given in the Table 4.11; parameters depending on R are instead shown in the following subsections, together with the plot of the average contact density for agents performing Schelling dynamics, the values of $\langle Q_{initial} \rangle$ and $\langle Q_{final} \rangle$ and figures of initial and final configurations of the lattice.

4.1.2 R=3

Parameter	Value
$N_1 + N_2$	23
N_3	843
T_S	478

Table 4.1: Simulation parameters, $R = 3$



(a) (b)

Figure 4.2: Initial (a) and final (b) configuration of the system, with switching agents inside the circle of radius R=3. Red and blue pixels represent agents of *type 1* and *type 2*, while white pixels represent vacancies

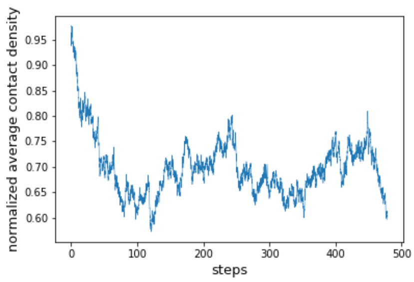


Figure 4.3: Ensemble average of the normalized contact density for agents of *type 1* and *type 2*as a function of steps of time steps, R=3

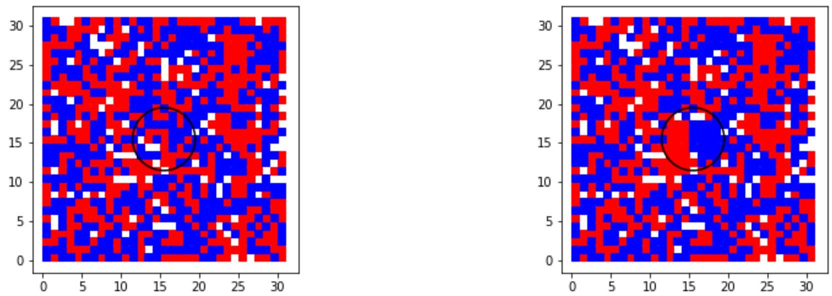
Average overlap	Value
$\langle Q_{initial} \rangle$	0.18 ± 0.12
$\langle Q_{final} \rangle$	0.33 ± 0.21
T_S	478

Table 4.2: Average overlap, $R = 3$

4.1.3 R=4

Parameter	Value
$N_1 + N_2$	41
N_3	825
T_S	478

Table 4.3: Simulation parameters, $R = 4$



(a) (b)

Figure 4.4: Initial (a) and final (b) configuration of the system, with switching agents inside the circle of radius R=4. Red and blue pixels represent agents of *type 1* and *type 2*, while white pixels represent vacancies

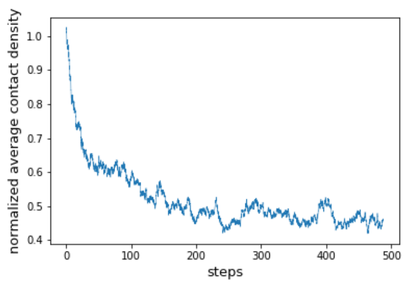


Figure 4.5: Ensemble average of the normalized contact density for agents of *type 1* and *type 2*as a function of steps of time steps, R=4

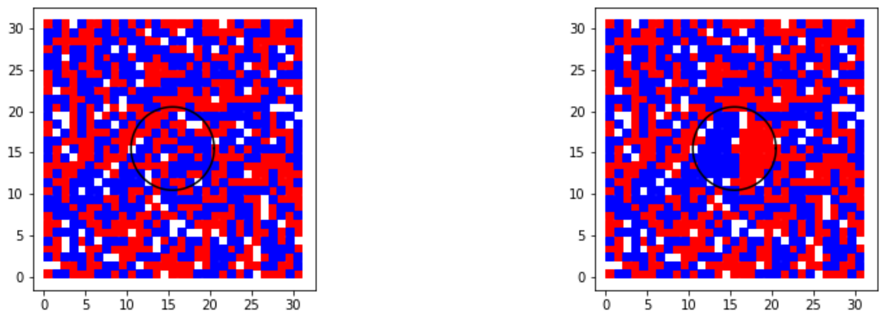
Average overlap	Value
$\langle Q_{initial} \rangle$	0.11 ± 0.08
$\langle Q_{final} \rangle$	0.28 ± 0.19

Table 4.4: Average overlap, $R = 4$

4.1.4 R=5

Parameter	Value
$N_1 + N_2$	63
N_3	803
T_S	492

Table 4.5: Simulation parameters, $R = 5$



(a) (b)

Figure 4.6: Initial (a) and final (b) configuration of the system, with switching agents inside the circle of radius R=5. Red and blue pixels represent agents of *type 1* and *type 2*, while white pixels represent vacancies

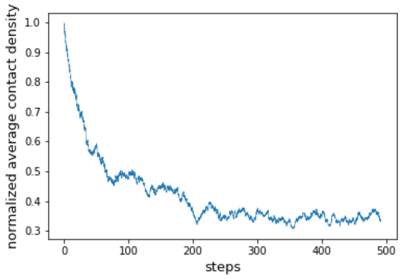


Figure 4.7: Ensemble average of the normalized contact density for agents of *type 1* and *type 2*as a function of steps of time steps, R=5

Average overlap	Value
$\langle Q_{initial} \rangle$	0.09 ± 0.06
$\langle Q_{final} \rangle$	0.37 ± 0.24

Table 4.6: Average overlap, $R = 5$

4.1.5 R=6

Parameter	Value
$N_1 + N_2$	99
N_3	767
T_S	404

Table 4.7: Simulation parameters, $R = 6$

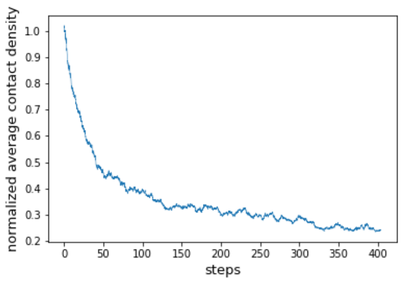
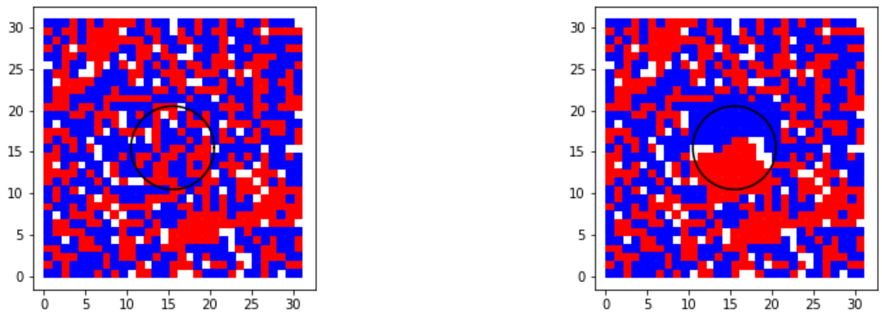


Figure 4.9: Ensemble average of the normalized contact density for agents of *type 1* and *type 2*as a function of steps of time steps, R=6



(a) (b)

Figure 4.8: Initial (a) and final (b) configuration of the system, with switching agents inside the circle of radius R=6. Red and blue pixels represent agents of *type 1* and *type 2*, while white pixels represent vacancies

Average overlap	Value
$\langle Q_{initial} \rangle$	0.07 ± 0.06
$\langle Q_{final} \rangle$	0.38 ± 0.22

Table 4.8: Average overlap, $R = 4$

As expected, the overlap of two random configurations is close to 0, while the one between two segregated configurations is close to 0.5, on average. The value of $\langle Q_{initial} \rangle$ for $R = 3$ is higher with respect to the values of the same quantity for larger R, so that the difference between $\langle Q_{initial} \rangle$ and $\langle Q_{final} \rangle$ is not appreciable. This is due to the fact of focusing on a relative small region of the space and having a local view of the arrangement of agents. In case of small area, also the average contact density does not give good information about the system, as one can see from its fluctuating behaviour in Fig. 4.3.

In general, the overlap shows to be a powerful tool to quantify segregation in the system, giving a good insight of what could be the typical configuration of agents inside the circle.

After having analyzed the dynamics followed by agents inside the circle for $R \in [3,6]$, we can conclude that there is no finite critical radius such that individuals inside the circle can persist in the mixed state or, in other words, that $R_C = 0$. The frozen mixed configuration outside the sphere does not perform an effective pinning field as happens for for systems.

Conclusion

We have worked in the framework of [Hazan and Randon-Furling(2013)], introducing a fraction f of fixed agents arranged randomly on the lattice, within the classical Schelling dynamics with two types of agents. We have firstly analyzed the behaviour of the normalized average contact density $\langle \chi_n(t) \rangle$ in the case $f = 0$, which corresponds to standard Schelling dynamics, and then in the case $f > 0$, representing some of the main results of obtained by *Randon-Furling* and *Hazan*.

We have subsequently explored a new variable quantifying segregation, the overlap Q, an important tool from statistical mechanics also used in [Cammarota and Cavagna(2017)] in order to evaluate the degree of coincidence of two configurations of a sphere of spins embedded in a metastable surrounding. Our evaluation of Q has come after studying the dynamics of Schelling agents included in a portion of the plan, in particular a circle, surrounded by agents frozen in a mixed configuration, taken as a pinning field.

More work is needed to explore thoroughly all the behavior of the system, and to test theoretical models.

Appendices

4.2 Appendix A

4.2.1 Tables

Parameter	Value
τ	0.3
ρ	0.9
width, height	30
N_{TOT}	810
N_1, N_2	405
p_u	0.2
p_h	10^{-4}
T_S	500
N_E	20

Table 4.9: Simulation parameters for the system with two types of agents, standard Schelling dynamycs

Parameter	Value
τ	0.3
ρ	0.9
f	0.2
width, height	30
N_{TOT}	810
N_1, N_2	324
N_3	162
p_u	0.2
p_h	10^{-4}
p_s	0.05
T_S	500
N_E	20

Table 4.10: General simulation parameters for the system with switching agents randomly arranged in the square lattice

Parameter	Value
τ	0.3
ρ	0.9
width, height	31
N_{TOT}	866
N_{red}, N_{blue}	433
p_u	0.2
p_h	10^{-4}
N_E	10

Table 4.11: Simulation parameters for the system with switching agents arranged out of the circle

4.3 Appendix B

4.3.1 The frozen pinning field method

The frozen pinning field (FPF) method derives from nucleation theory: the transition from metastable to stable phase happens in a system through the formation of nuclei of the stable phase. The free-energy cost of a nucleus of size R is given by the following equation:

$$\Delta F(R) = -V_d R^d \delta f + S_d R^{d-1} \sigma \tag{4.2}$$

where the first term represents the volume gain proportional to the bulk free-energy density between the two phases δf (V_d is the volume of the nucleus in d dimension) and the second term is the surface cost proportional to the surface tension σ (S_d is the surface of the nucleus in d dimension). The size R_C of the critical nucleus is obtained by maximizing Eq. 4.2, i.e. by performing:

$$\left.\frac{\partial \Delta F(R)}{\partial R}\right|_{R=R_C} \tag{4.3}$$

Nuclei smaller than the critical radius are unstable and will decrease in size, while larger nuclei are stable and thermodinamically favoured to grow, as can be seen by checking that

$$\frac{\partial \Delta F(R)}{\partial R} > 0 \leftrightarrow R < R_C$$
$$\frac{\partial \Delta F(R)}{\partial R} < 0 \leftrightarrow R > R_C$$

.

The FPF method analyzed in [Cammarota and Cavagna(2017)] consists in giving direct access to the critical nucleus. Briefly, referring to systems whose degrees of freedom are spins on a $2d$ lattice, the system is prepared in an equilibrium configuration belonging to its metastable phase and all the spins outside a sphere of size R are frozen. They produce a pinning filed on the surface of the sphere and the thermodynamics of the "bubble" is explored: it can either remain in the metastable phase with the corresponding free energy density f_m or pay surface energy and switch to the stable phase with free energy density f_s, where $f_m > f_s$. By writing the partition function of the system of the sphere, one can obtain the probability $P_m(R)$ for the bubble to be in the metastable phase and $P_s(R)$ to be in the stable phase, both expressed as a function of R. Given that, as seen in nucleation theory, the transition between the two phases occurs at $R = R_C$, one can impose the equality of the above-mentioned probabilities and find the expression of the critical size of nucleus R_C as a function of the unknown surface tension σ and of other known or numerically obtainable parameters. In the FPF method R_C is measured as the point where $P_m(R) = P_s(R)$, hence the surface tension σ can be found out.

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