POLITECNICO DI TORINO

Master Course
in Physics of Complex Systems

Master Thesis

Work Statistics in a Quench of the Quantum Spherical Model

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ACADEMIC YEAR 2017-2018

Abstract

We will discuss the statistics of work performed in a quench of the quantum spherical model with a quartic interaction potential. The pre-quench and post-quench hamiltonians are supposed time independent. The moment generating function $G(s)$ of the work can be mapped into that of an effective gaussian model in which a parameter $r_{eff}(t)$ turns out to be time dependent. We will show that $r_{eff}(t)$ does not modify the original time-independence of the probability distribution and we will give an explicit formula to compute $G(s)$.

Introduction

The quantum spherical model is a useful paradigm to study many quantum non-equilibrium problems, from the equilibration after a quench of one or more parameters of the hamiltonian, [1] to the *dynamical phase transition* that may occur in the stationary state reached at $t \to \infty$ [2]. One of the main open questions is, for instance, whether or not the final state approached after the quench could be described by a generalized Gibbs ensable [1]. In the following we will suppose the hamiltonian time independent. In addition it will be shown that the spherical model is equivalent to an $O(N)$ vector model in the limit $N \to \infty$. For $N \to \infty$, the model is exaclty solvable as it can be mapped into a gaussian one with a time-dependent effective parameter $r_{eff}(t)$ [3]. If on the one hand, this effective formulation makes the solution of the model analytically and numerically feasible, on the other it introduces an appearent time dependence in the problem which seemingly implies a time-dependence of relevant quantities such as the work performed upon quenching the system and its statistics. Indeed, the known form of the work-generating function $G(s)$ of a gaussian model includes $r_{eff}(t)$ but the starting hamiltonian is supposed time-independent and this property is expected to be retained also in the statistics of the work. Consequently, we will first illustrate how to prove that $G(s)$ is time independent also in the gaussian formulation of the problem and only then a form of $G(s)$ will be provided.

In the first chapter we will give a brief introduction to the thermodinamics of equilibrium and non-equilibrium transformations. In the second chapter instead, we will define the model and summarize the main tools needed in the third chapter where the solution of the issues outlined will be provided.

Chapter 1

Motivations

1.1 Definitions

When a physical system is composed of a large number of degrees of freedom, the description of any relevant observable can be achieved only via statistical measures. In what follows the word *system*, will designate a large set of interacting degrees of freedom such as particles, fields etc., for which the tools of statistical mechanics are meaningful and must be applied.

1.1.1 Equilibrium states and quasi-static processes

Generally speaking, a system can be found in an equilibrium state or in a non-equilibrium state. In equilbrium, the observables, such as energy, volume, number of particles, are time-independent. The existence of equilibrium states is a postulate of thermodinamics, justified by the empirical evidence that a system left unperturbed for a long time approaches a state where the values of the observables do not change with time and are independent from the history of the external actions exerted on the system [4]. In thermodynamics the concept of equilibrium state is particularly useful to define a class of ideal processes called *quasi-static* which are, by definition, made up by a sequence of equilibrium states. Once the value of few relevant variables is given the knowledge of a system at equilibrium is complete. As a consequence, when such relevent parameters are known for every step of a quasi-static process one can predict, inter alia, the work and heat exchanged in the transformation.

A real process is always a sequence of equilibrium and non-equilibrium states and one defines the *relaxation time* as the time interval needed for a system to reach equilibrium when it is carried out of equilibrium by an external action. If a thermodynamical transformation is sufficiently slow, such as the system can relax to equilibrium at each short time step, then the process is quasi-static.

From a statistical viewpoint, in an equilibrium state, the values of the relevant physical variables are equal to an average done on a suitable probabilty distribution, called statistical ensamble.

1.1.2 Work performed in a quasi static process

In order to compute the work done changing the state of the system let us define a path $\gamma(t)$ in the space Ω of the paramenters that can be controlled, from an initial state A at time $t = 0$, to a final state B at time $t = \infty$. When the system is in contact with an external enviroment we must consider also the positions and coordinates of the particles of the thermal bath otherwise, striclty speaking, an hamiltonian function can not be defined. Since the energy of the system is known at each step, the work performed can be evaluated integrating the differential energy along the path. The work W done along $\gamma(t)$ is [5]

$$W = \int_0^\infty dt \overline{\frac{\partial H(p, q, \gamma(t))}{\partial t}} \tag{1.1}$$

where, by definition, the energy $H(p, q, \gamma(t))$ of the system is istantaneously equal to its ensable average. If the system is isolated, the quasi-static process is called adiabatic with reference to the absence of heat exchanged during the transformation.

Let us note that in a quasi-static process, the work done is not a stochastic variable; it is merely equal to the free energy difference, $\Delta F = F_B - F_A$ between the final and initial state, and is the least possible amount of work needed to carry the system from the state A to the state B.

1.1.3 Non-equilibrium transformations and definition of a Quench

When a control parameter is varied sufficiently fast the system is brought out-of-equilibrium. If the properties of a system in equilibrium — and consequently of a quasi-static transformation — depend only of a few internal variables, on the contrary there are many ways to bring a system out of equilibrium so that the protocol used must be carefully specified since it determines the values of the relevant physical observables.

The most simple protocol is the *quench* of an external parameter r, i.e., an istantaneous variation from its initial value r_i to the final value r_f; a *global* quench means that this change is the same in every part of the system. For an out of equilibrium transformation W becomes a random variable, with an average $\langle W \rangle$ always larger then ΔF, i.e.,

$$\langle W \rangle \geq \Delta F, \tag{1.2}$$

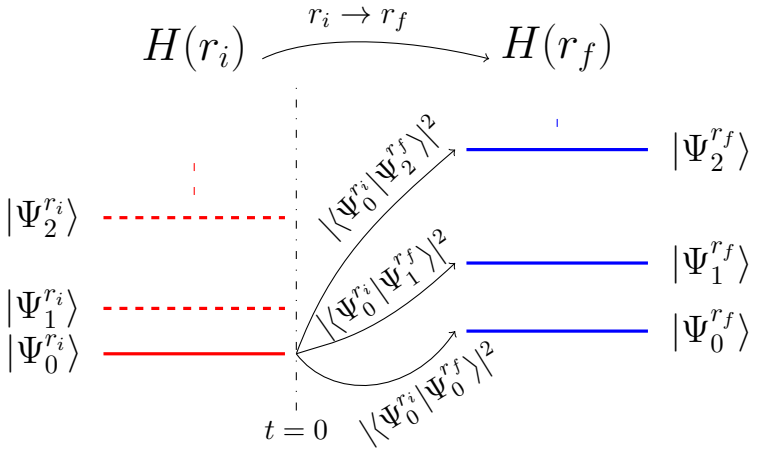

Figure 1.1: Schematic representation of the initial and post-quench energy levels

so that wondering about the form of the probability distribution as well as whether exist some limit where the probability distribution becomes universal is a legitimate and compelling question of statistical mechanics.

1.1.4 Work and Heat produced quenching a quantum system

In the following we will deal with a thermally isolated system prepared in the ground state $|\Psi_0^{r_i}\rangle$ of the initial hamiltonian $H(r_i)$. In Fig. 1.1 the energy levels of the pre-quench and post-quench hamiltonians are illustrated. The heat produced in the quench is the macroscopic phenomenon of microscopic transitions between different energy levels while the word *work* refers to the reversible part of energy change due to the modification of the level structure [6].

In a quasi-static process, the system stays in the ground state of the istantaneous hamiltonian, the energy exchanged is denoted as *work*, and the entropy does not change (in the simple case of the ground state it is always equal to 0). This is coherent with a fundamental result of thermodynamics according to which in an adiabatic process, the entropy does not vary [15].

On the other hand, an abrupt variation of one parameter r of the hamiltonian, $r_i \to r_f$, such that $[H(r_i), H(r_f)] \neq 0$ induces transitions, referred as excitations, from the ground state $|\Psi_0^{r_i}\rangle$ of $H(r_i)$, to any eigenstate of $H(r_f)$ with a probabilty given by $|\langle\Psi_0^{r_i}|\Psi_n^{r_f}\rangle|^2$. Practically speaking, this means that if the system is initially prepared in $|\Psi_0^{r_i}\rangle$, in a series of repeated measurements of the post-quench energy, the empirical probability of the transitions $|\Psi_0^{r_i}\rangle \to |\Psi_n^{r_f}\rangle$ tends to $|\langle\Psi_n^{r_f}|\Psi_0^{r_i}\rangle|^2$ as the number of measurements

grows.

Excitations can survive a lot after the quench as in the case of integrable systems or decay soon after it as for non integrable ones. In both cases they are the microscopic origin of the termodinamic irreversibilty described through the concepts of irreversible heat and irreversible entropy.

1.1.5 Why quantum quenches?

In the past two decades the tecnique of optical lattices applied to cold atoms has made it possible to investigate experimentally the picture we have portrayed in Sec. 1.1.4. Optical traps can simulate with high precision several theoretical models where the system is completely isolated from any external disturbance. Prior to this, speaking about the unitary dynamics after a quench of an isolated quantum system would have been self-referential since the means to build up an experiment to observe the phenomenon where absent.

Important experiments have been conducted to test whether thermalization occurs in presence of integrability for a quasi-one dimensional bosonic gas [7] and show the absence of dumping of the system taken out of equilibrium. In other words in $1D$ no thermalization occurs in the time of the experiment for a system brought out of equilibrium.

It became also possible to observe [8] a *quantum phase trasition* from a superfluid phase to Mott insulator phase, i.e. a phase transition driven only by quantum fluctuations where the relative strengths of kinetic and interaction energies play a role analogous to those of internal energy and entropy in a thermal phase transition.

Other recent experimental studies [9] investigated the phenomenon of prethermalization, another interesting consequence of integrability in the non-equilibrium dynamics of a complex system.

These are just few examples that bear witness to the growing interest towards the field of non-equilibrium physics of interacting quantum systems. The present simple work should be regared part of this wide landscape.

1.2 The spherical model

1.2.1 Introduciton: the Ising Model

One goal of statistical physics is explaining the phase diagram of a system starting from the microscopic interactions between the elementary degrees of freedom. The macroscopic properties of two phases such as the strength of materials, their electrical conductivity, etc. differ deeply one from the other but these differences, as well as the concept of phase itself, arise only in the thermodynamical limit, as a result of a collective behaviour of the

microscopic degrees of freedom when an external temperature-like parameter is varied.

The Ising model has been studied for long time in order to understand the macroscopic phenomenology of the phases, starting from a simple microscopic interaction. The degrees of freedom of the Ising model are scalar spin variables σ_i located on a lattice; here i stands for the i^{th} site and $\sigma = \{-1, 1\}$ is a binary random variable. Setting J as the strength of the interaction between two nearby spins, the Boltzmann weight of a configuration is computed with the hamiltonian

$$H = -J \sum_{<ij>} \sigma_i \sigma_j \tag{1.3}$$

where the sum runs over all the sites of the lattice, and $< ij >$ stands for a pair of nearest neighbour spins. A global configuration of the system can be described by a super spin vector $\boldsymbol{\sigma}$ with a number of components m equal to the number of sites. Geometrically speaking the tip of the super spin vector lies on one of the 2^m corners of an hypercube in an m-dimensional space, the configuration space of the system. Figure 1.2 portrays with red dots all the configurations for a system of two spins.

The fundamental quantity that connects the microscopical description of a system to the thermodynamic variables is the partition function Z defined as

$$Z = \sum_{\boldsymbol{\sigma}} e^{-\beta H(\boldsymbol{\sigma})}, \tag{1.4}$$

where $\sum_{\boldsymbol{\sigma}}$ is a sum over all the different configuration of the system, and β is an inverse temperature parameter. The free energy F of the system is then computed as

$$F = -\frac{1}{\beta} \log Z \tag{1.5}$$

and all the other variables follow from the rules of thermodynamics. Therefore solving a model means compute explicitly the partition function, and for the Ising model this aim can be accoplished just for a set of spin placed on a line, i.e. a dimension $d = 1$, or lying on a plane $d = 2$. Accordingly, many approximations to the Ising model have been proposed in order to understand the equilibrium theromdynamics of complex systems in $d > 2$.

1.2.2 The strict spherical model

The spherical model is an approximation of the Ising model invented and solved by Berlin and Kac in 1952. Its name comes from the geometrical interpretation previously given to the Ising model: the tip of $\boldsymbol{\sigma}$ lies on the surphace of the hypersphere that cricumscribes the 'Ising' hypercube so that each spin σ_i can vary over $[-\sqrt{m}; \sqrt{m}]$ as can be seen from Fig. 1.2. The

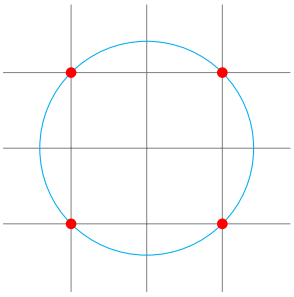

Figure 1.2: Configurations of a system of $m = 2$ spins for the spherical model (blue) and the Ising model (red); the hypercube for $m = 2$ is a square.

model is soluble because now Z can be computed with a integral instead of a discrete sum. Formally the hamiltonian of the spherical model is the same as that of the Ising one, with the constrint

$$\sum_i \sigma_i^2 = m, \tag{1.6}$$

so that

$$H_s = -J \sum_{<ij>} \sigma_i \sigma_j \cdot \delta \left(\sum_i \sigma_i^2 - m \right). \tag{1.7}$$

In order to compute the partition function Z_s it is convenient to use the integral representation of the Dirac delta

$$\delta \left(m - \sum_i \sigma_i^2 \right) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \mathrm{dz}\ e^{z(\sum_i \sigma_i^2 - m)} \tag{1.8}$$

consequently

$$Z_s(m, J) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \mathrm{dz}\ e^{zm} \int_{-\infty}^{+\infty} \prod_i \mathrm{d}\sigma_i\ e^{-z \sum_i \sigma_i^2 + J \sum_{<} ij > \sigma_i \sigma_j} \tag{1.9}$$

and the gaussian integrals in σ_i on the right hand side can be explicitly calculated. The partition function is finally evaluated computing the remaining complex integral in z with the saddle point method [10].

1.2.3 The mean spherical model

The generalization to the quantum spherical model is simpler dealing with the so called it mean spherical model [12]. Instead of using the strict contraint of Eq. (1.6) one can impose the average of the super spin vector to be equal to m

$$\left\langle \sum_i \sigma_i^2 \right\rangle = m. \tag{1.10}$$

In this way both the $\boldsymbol{\sigma} = \sum \sigma_i^2$ and the energy are treated in the canonical ensable, while in Eqs. (1.7) and (1.9) $\boldsymbol{\sigma}$ was understood as a microcanoical variable. In the hamiltonian of Eq. (1.7) we must add the dynamical variable $\sum_i \sigma_i^2$ together with its canonically conjugate parameter λ

$$H_s = -J \sum_{<ij>} \sigma_i \sigma_j + \lambda \sum_i \sigma_i^2, \tag{1.11}$$

thus Z_s becomes

$$Z_s(\lambda, J) = \int_{-\infty}^{+\infty} \prod_i \mathrm{d}\sigma_i \; e^{-\lambda \sum_i \sigma_i^2 + J \sum_{<ij>} \sigma_i \sigma_j}, \tag{1.12}$$

and Eq. (1.10) is satisfied provided

$$-\frac{\partial Z_s(\lambda, J)}{\partial \lambda} = m. \tag{1.13}$$

The energetic term $\lambda \sum_i \sigma_i^2$ in Eq. (1.11) is an harmonic confinement for σ_i around the equilibrium value $\sigma_i = 0$ and curvature of the quadratic binding potential, given by Eq. (1.13), enforces the condition of Eq. (1.10). In conclusion the spins should be now better regarded as harmonic springs rather then rigid rods.

The thermodynamical properties of the strict and mean spherical model are the same, specifically the lower critical dimension d_l is equal to 2 and the upper critial one d_u is 4. For $2 < d < 4$ the critical exponents are non classical due to their dependence on the dimension d [10].

1.2.4 Physical meaning of the model

The global constraints shown in Eqs. (1.6) and (1.10) imply a long range effective interaction among the spins, to which one can hardly assign a direct physical meaning. However, in 1962 H. E. Stanley proved the equivalence between the spherical model and a spin-vector model with nearest neighbour interaction and $O(N)$ symmetry when $N \to \infty$ [11]. Precisely Eq. (1.7) was found equivalent to the $N \to \infty$ limit of the hamiltonian

$$H^{(N)} = -J \sum_{<ij>} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j, \tag{1.14}$$

where $\boldsymbol{\sigma}_i = \{\sigma_1(i), ..., \sigma_N(i)\}$ and

$$\sum_{k=1}^{N} \sigma_k(i)^2 = N \qquad \text{for} \quad i = 1, 2, ..., m. \tag{1.15}$$

Accordingly, a model with $O(N = \infty)$ symmetry can be referred as spherical model and vice-versa.

1.2.5 An application of the spherical model in $d < \infty$

In the family of $O(N)$ models many statistical observables are monotonic functions of N [11]. The spherical model can thus be used to bound from above or below the values of important physical quantities of more realistic models like a vector model with two or three components.

Let us give an example of this approach in the case of the upper critical dimensionality d_u of a model. The correlation length for the family of the $O(N)$ models turns out to be a decreasing function of N [10], in fact when the number of components of a single spin increase the fluctuations are also expected to grow promoting faster decorrelation between the spins. Above the upper critical dimensionality fluctuations are negligible and the spherical model can be used to bound d_u from above for all the $O(N)$ models with finite N. The upper critical diemensionality of the spherical model is four so that for every $N < \infty$, $d_u \leq 4$.

1.2.6 The quantum spherical model

The quantum counterpart of the model can be obtained regarding $\sigma_h(i)$ as position operators and defining their conjugate momenta $\pi_k(j)$ in accordance with the canonical commutation relations

$$[\sigma_h(i), \pi_k(j)] = i\delta_{ij}\delta_{hk}. \tag{1.16}$$

The quantum hamiltonian includes a kinetic part H_{kin} that can be chosen in more ways [12], [13]. Let us follow [12] and define $H_{kin} \doteq \sum_i \boldsymbol{\pi}_i^{(N)} \cdot \boldsymbol{\pi}_i^{(N)}/2$. Then, using Eq. (1.14) the total quantum hamiltonian H_{Qs} is

$$H_{Qs} = \sum_i \frac{\boldsymbol{\pi}_i \cdot \boldsymbol{\pi}_i}{2} - J \sum_{<ij>} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \tag{1.17}$$

and each σ_i must satisfy the contraints given in Eq. (1.15). We could instead fix the average value lenght of each spin and write down the hamiltonian

$$H_{Qs} = \sum_i \frac{\boldsymbol{\pi}_i \cdot \boldsymbol{\pi}_i}{2} - J \sum_{<ij>} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j + \sum_i \lambda_i \boldsymbol{\sigma_i^2}, \tag{1.18}$$

in the same fashion of what was done in Eq. (1.11). All the λ_i's have the same value since the model is supposed omogeneous and isotropic, therefore

$$H_{Qs} = \sum_i \frac{\boldsymbol{\pi}_i \cdot \boldsymbol{\pi}_i}{2} - J \sum_{<ij>} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j + \lambda \sum_i \boldsymbol{\sigma_i^2}. \tag{1.19}$$

The σ operators at two different positions commute with each other therefore when the lattice parameter $\boldsymbol{r}_{i+1} - \boldsymbol{r}_i$ go to zero (suppose for simplicity to have a cubic lattice) we can rewrite Eq. (1.19) as [14]

$$H(r, u) = \int \mathrm{d}^d x \left[\frac{1}{2} \boldsymbol{\pi_x^2} + \frac{c}{2} (\nabla \boldsymbol{\sigma})^2 + \frac{r}{2} \boldsymbol{\sigma}^2 \right] \tag{1.20}$$

Chapter 2

The Model

Let us consider a system that is described by the quantum $O(N)$ hamiltonian in d spatial dimensions with a $(\boldsymbol{\phi}^2)^2$ interaction term

$$H(r,u) = \int \mathrm{d}^d x \left[\frac{1}{2}\boldsymbol{\Pi}^2 + \frac{1}{2}(\nabla\boldsymbol{\phi})^2 + \frac{r}{2}\boldsymbol{\phi}^2 + \frac{u}{4!N}(\boldsymbol{\phi}^2)^2\right] \tag{2.1}$$

where $\boldsymbol{\phi} = (\phi_1, ..., \phi_N)$ is a real bosonic field with N components, $\boldsymbol{\Pi} = (\Pi_1, ..., \Pi_N)$ its conjugate momentum with $[\phi_i(\boldsymbol{x}), \Pi_j(\boldsymbol{x}')] = i\delta^{(d)}(\boldsymbol{x} - \boldsymbol{x}')\delta_{ij}$ and

$$\boldsymbol{\phi}^2 = \sum_{i=1}^{N} \phi_i^2, \tag{2.2}$$

$$\boldsymbol{\Pi}^2 = \sum_{i=1}^{N} \left(\frac{\partial\phi_i}{\partial t}\right)^2, \tag{2.3}$$

$$(\nabla\boldsymbol{\phi})^2 = \sum_{i=1}^{N}\sum_{j=1}^{d} \left(\frac{\partial\phi_i}{\partial x_j}\right)^2, \tag{2.4}$$

$$(\boldsymbol{\phi}^2)^2 = \left(\sum_{i=1}^{N} \phi_i^2\right)^2. \tag{2.5}$$

In addition, suppose that at $t = 0$ the system is prepared in the ground state $|\Psi_0^{r_i}\rangle$ of the pre-quench hamiltonian $H(r_i, u)$ with energy

$$E_0^{r_i} = \langle\Psi_0^{r_i}|H(r_i,u)|\Psi_0^{r_i}\rangle, \tag{2.6}$$

and the parameter r is instantaneously changed from r_i to r_f. As a result, the hamiltonians $H(r_i, u)$ and $H(r_f, u)$ do not commute and when the energy of the final state is measured, the eigenvalue $E_n^{r_f}$ relative to the eigenstate $|\Psi_n^{r_f}\rangle$ of the post-quench hamiltonian $H(r_f, u)$ occurs with a probability $P(E_n)$ given by

$$P(E_n) = |\left\langle\Psi_n^{r_f}|\Psi_0^{r_i}\right\rangle|^2, \tag{2.7}$$

and the work W_n performed in a transition $|\Psi_0^{r_i}\rangle \to |\Psi_n^{r_f}\rangle$ is then

$$W_n = E_n^{r_f} - E_0^{r_i}. \tag{2.8}$$

It is convenient to study the properties of a probability distribution via its moment generating function. The latter is defined as

$$G(s) \doteq \left\langle e^{-sW} \right\rangle, \tag{2.9}$$

Using $G(s)$ paves the way for a fruitful mapping between $G(s)$ and the partition function $Z(s)$ of a classical system in $d+1$ dimensions. This analogy is helpful because of the available literature about the behaviour of a classical system.

2.1 Quantum-to-classical correspondence

It is then worth sheding some light on this quantum-to-classical mapping. From Eqs. (2.7), (2.8) and the definition of $G(s)$ given in Eq. (2.9) one obtains,

$$\begin{aligned} G(s) &= \sum_{n=0}^{\infty} e^{-s(E_n^{r_f} - E_0^{r_i})} |\langle \Psi_n^{r_f} | \Psi_0^{r_i} \rangle|^2 \\ &= \sum_{n=0}^{\infty} \langle \Psi_0^{r_i} | \Psi_n^{r_f} \rangle e^{-s(E_n^{r_f} - E_0^{r_i})} \langle \Psi_n^{r_f} | \Psi_0^{r_i} \rangle \\ &= \langle \Psi_0^{r_i} | e^{-s(H - E_0^{r_i})} | \Psi_0^{r_i} \rangle. \end{aligned} \tag{2.10}$$

If s is an integer, then $\langle \Psi_0^{r_i} | e^{-sH} | \Psi_0^{r_i} \rangle$ can be seen as a partition function $Z(s)$ of a statistical system confined between a slab with boundary states $|\Psi_0^{r_i}\rangle$, thickness s and transfer matrix e^{-H}. Accordingly, the classical system has an extra dimension with respect to the quantum one, namely the thickness s of the slab. More generally, s can be a real number and, as long as it stays positive, the same mapping holds [16]. In summary,

$$G(s) = \langle \Psi_0^{r_i} | e^{-s(H - E_0^{r_i})} | \Psi_0^{r_i} \rangle \doteq Z(s). \tag{2.11}$$

Setting $\beta = 1$ we can now exploit the known structure of the classical free energy of a slab,

$$F(s) = -\log Z(s), \tag{2.12}$$

to understand some features of the physics of the quantum counterpart. $F(s)$ consists of tree contributions

$$F \doteq sAf_b + 2Af_a + Af_c(s), \tag{2.13}$$

where A is the area of one of the two surfaces and f_b, f_a are the bulk, surface free energy densities and $f_c(s)$ includes the other decreasing powers of s.

The mapping shown in Eq. (2.11) is established as follows: when $s \to \infty$, $f_c(s) \to 0$ the contributions to $G(s)$ coming form transitions to excited states can be neglected and — denoting with $|\Psi_0^{r_f}\rangle$ the ground state of the post quench hamiltonian — we have

$$G(s \to \infty) = \langle \Psi_0^{r_i} | e^{-s(H - E^{r_i})} | \Psi_0^{r_i} \rangle = e^{-s(E_0^{r_f} - E_0^{r_i})} |\langle \Psi_0^{r_i} | \Psi_0^{r_f} \rangle|^2, \tag{2.14}$$

so that

$$F(s) = s(E_0^{r_i} - E_0^{r_f}) - 2 \log |\langle \Psi_0^{r_i} | \Psi_0^{r_f} \rangle|. \tag{2.15}$$

A comparison with Eq. (2.13) yields the rather transparent correspondence

$$\begin{aligned} f_b &= \frac{E_0^{r_i} - E_0^{r_f}}{A} \\ f_a &= -\frac{\log |\langle \Psi_0^{r_i} | \Psi_0^{r_f} \rangle|}{A}. \end{aligned} \tag{2.16}$$

Finally — for an arbitrary positive value of s — $f_c(s)$ is derived from Eq. (2.13) as

$$f_c(s) = \frac{-\log G(s)}{A} - s f_b - 2 f_a. \tag{2.17}$$

When r_f vanishes the system becomes critical [6] and from the known scaling behaviour of the classical free energy densities f_s and f_c it is possible to infer some of the universal features of the quantum counterpart [19].

2.2 The limit $N \to \infty$

Let us now focus on with the solution of Eq. (2.1). When the number of components N goes to infinity, $(\boldsymbol{\Phi}^2)^2$ can be written as [20]

$$(\boldsymbol{\phi}^2)^2 = 2(N+2)\langle \phi^2 \rangle \boldsymbol{\phi}^2 - N(N+2)\langle \phi^2 \rangle^2, \tag{2.18}$$

where $\langle \phi^2 \rangle$ stands for

$$\langle \phi^2 \rangle = \int \mathrm{d}^d x \langle \Psi_0^{r_i} | \phi^2(\boldsymbol{x}, t) | \Psi_0^{r_i} \rangle. \tag{2.19}$$

Equation (2.18) can be obtained expanding $\exp\left(\frac{u}{24N}(\boldsymbol{\phi}^2)^2\right)$ in the partition function of the $O(N)$ quantum model. The linear term is proportional to $(\boldsymbol{\phi}^2)^2$ which is, explicitly, a sum of $\phi_i^2 \phi_j^2$ [see Eq. (2.2)]. In order to derive Eq. (2.18) we need to contract two of the four fields and retain just the leading order contributions in N. In Fig. 2.1 we illustrate the Feynman diagrams of $\phi_i^2 \phi_j^2$ and their contraction; they are in number of $N(N-1) \cdot 2$ if $i \neq j$, where the factor 2 comes from the fact that the lines can be

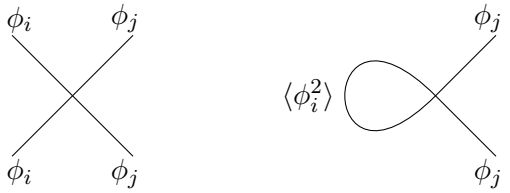

Figure 2.1: First order contractions of $(\boldsymbol{\phi}^2)^2$

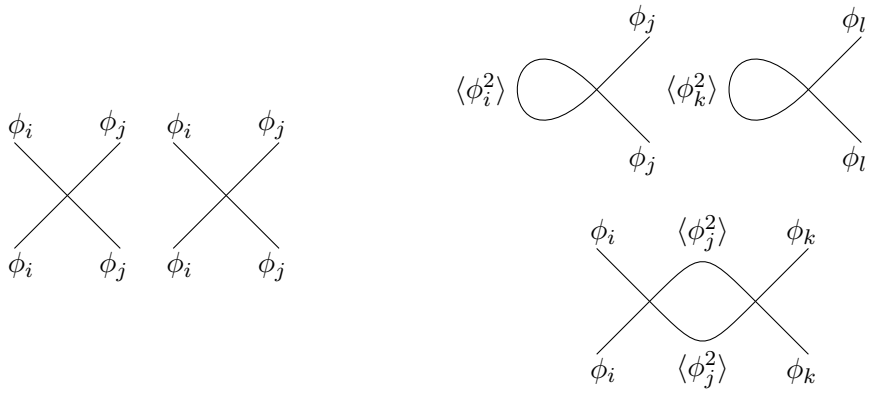

Figure 2.2: Second order contractions of $(\boldsymbol{\phi}^2)^2$

contracted from the left, i-side, or from the right, j-side. If $i = j$, Wick's theorem proves that the contractions are $3N \cdot 2$, so that we obtain an overall contribution of

$$[2N(N-1) + 6N]\langle\phi^2\rangle\boldsymbol{\phi}^2 = 2(N+2)\langle\phi^2\rangle\boldsymbol{\phi}^2. \tag{2.20}$$

The constant in Eq. (2.18) is needed in order to reproduce correctly the average of $(\boldsymbol{\phi}^2)^2$ since $\langle(\boldsymbol{\phi}^2)^2\rangle - \langle 2(N+2)\langle\phi^2\rangle\boldsymbol{\phi}^2\rangle = -N(N+2)\langle\phi^2\rangle^2$.

For the higher-order terms, the number of bubble-like contractions is subleading in N, and for $N \to \infty$ their contribution vanishes. In Fig. 2.2 an example of the second order term is illustrated: the contractions of the upper-right type are $\propto N^4$, while lower candy-like ones are $\propto N^3$.

Replacing the left-hand side of Eq. (2.18) in Eq. (2.1) and taking the limit $N \to \infty$, we obtain, at the leading order,

$$H(r,u) = \int \mathrm{d}^d x \, \left[\frac{1}{2}\boldsymbol{\Pi}^2 + \frac{1}{2}(\nabla\boldsymbol{\phi})^2 + \frac{r}{2}\boldsymbol{\phi}^2 + \frac{u}{12}\langle\phi^2\rangle\boldsymbol{\phi}^2\right] - N\frac{u}{24}\langle\phi^2\rangle^2. \tag{2.21}$$

Setting then

$$r_{eff}(t) \doteq r + \frac{u}{6}\langle\phi^2\rangle(t), \tag{2.22}$$

$$H_{eff}(t) \doteq \int \mathrm{d}^d x \ \left[\frac{1}{2}\boldsymbol{\Pi}^2 + \frac{1}{2}(\nabla\boldsymbol{\phi})^2 + \frac{r_{eff}(t)}{2}\boldsymbol{\phi}^2\right], \tag{2.23}$$

$$c(t) \doteq -N\frac{u}{24}\langle\phi^2\rangle^2, \tag{2.24}$$

the original hamltonian can be written as

$$H(r,u) = H_{eff}(t) + c(t), \tag{2.25}$$

and the dynamics of the field can be studied via the two coupled Eqs. (2.22) and (2.23), where r_{eff} has to be determined self consistently at each time t of the evolution.

2.3 Gaussian generating function

The effective hamiltonian in Eq. (2.23) is quadratic. It is then useful to sum up the known results related to the work generating function of a gaussian model with scalar field ϕ since they will be used in chapter 3.

A model is said to be gaussian if the hamiltonian H takes the quadratic form

$$H = \int \mathrm{d}^d x \ \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{r}{2}\phi^2\right]. \tag{2.26}$$

The quench involves a sudden change of r from r_i to r_f, the latter being time independent. Under these conditions, the generating function can be found passing in Fourier space, where Eq. (2.26) becomes

$$H = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\left(\frac{1}{2}\Pi_{\boldsymbol{k}}\Pi_{-\boldsymbol{k}} + \frac{\omega_{\boldsymbol{k}}^2(r)}{2}\phi_{\boldsymbol{k}}\phi_{-\boldsymbol{k}}\right), \tag{2.27}$$

and $\omega_{\boldsymbol{k}}(r) = \sqrt{|\boldsymbol{k}|^2 + r}$. Introducing the ladder operators

$$\begin{aligned} a_{\boldsymbol{k}} &= \frac{\omega_{\boldsymbol{k}}\phi_{\boldsymbol{k}} + i\Pi_{\boldsymbol{k}}}{\sqrt{2\omega_{\boldsymbol{k}}}}, \\ a_{\boldsymbol{k}}^\dagger &= \frac{\omega_{\boldsymbol{k}}\phi_{\boldsymbol{k}} - i\Pi_{\boldsymbol{k}}}{\sqrt{2\omega_{\boldsymbol{k}}}}, \\ with \quad & [a_{\boldsymbol{k}}, a_{\boldsymbol{k}}^\dagger] = 1, \end{aligned} \tag{2.28}$$

the hamiltonian in Eq. (2.27) takes the form of an integral sum of independent harmonic oscillators with frequencies $\omega_{\boldsymbol{k}}$, i.e.,

$$H = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\omega_{\boldsymbol{k}}\left(a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + \frac{1}{2}\right) \doteq \int \frac{\mathrm{d}^d k}{(2\pi)^d} h_{\boldsymbol{k}}, \tag{2.29}$$

with

$$h_{\boldsymbol{k}} = \omega_{\boldsymbol{k}} \left(a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{1}{2} \right). \tag{2.30}$$

In the following $\int \mathrm{d}^d k/(2\pi)^d$ will be denoted simply by $\int_{\boldsymbol{k}}$. In addition, let us denote the pre-quench and post-quench frequencies $\omega_{\boldsymbol{k}}(r_i)$ and $\omega_{\boldsymbol{k}}(r_f)$ with $\omega_{\boldsymbol{k},i}$ and $\omega_{\boldsymbol{k},f}$ respectively.

If the system is prepared in the ground state of H, which is a product state of the form $|\Psi_0^{r_i}\rangle = \prod_{\boldsymbol{k}} |\psi_{\boldsymbol{k}_0}\rangle$ with $|\psi_{\boldsymbol{k}_0}\rangle$ the ground state of $h_{\boldsymbol{k}}$, then the generating function $G(s)$ defined in Eq. (2.9) is the product of the generating functions $G_{\boldsymbol{k}}(s)$ of all the normal modes

$$G(s) = \prod_{\boldsymbol{k}} G_{\boldsymbol{k}}(s); \tag{2.31}$$

and setting

$$\lambda_{\boldsymbol{k}} = \frac{\omega_{\boldsymbol{k},i} - \omega_{\boldsymbol{k},f}}{\omega_{\boldsymbol{k},i} + \omega_{\boldsymbol{k},f}}, \tag{2.32}$$

it can be shown that $G_{\boldsymbol{k}}(s)$ takes the form [21]

$$G_{\boldsymbol{k}}(s) = e^{-s(\omega_{\boldsymbol{k},i} - \omega_{\boldsymbol{k},f})/2} \sqrt{\frac{1 - \lambda_{\boldsymbol{k}}^2}{1 - \lambda_{\boldsymbol{k}}^2 e^{-2s\omega_{\boldsymbol{k},f}}}}, \tag{2.33}$$

The quantum-mechanical expression of the free energy of Eq. (2.13) is then

$$f(s) = -\frac{\log G(s)}{A} = s \int_{\boldsymbol{k}} \frac{\omega_{\boldsymbol{k},i} - \omega_{\boldsymbol{k},f}}{2} - \int_{\boldsymbol{k}} \frac{1 - \lambda_{\boldsymbol{k}}^2}{2} + \int_{\boldsymbol{k}} \frac{1 - \lambda_{\boldsymbol{k}}^2 e^{-2s\omega_{\boldsymbol{k},f}}}{2}; \tag{2.34}$$

in conclusion

$$f_b(s) = s \int_{\boldsymbol{k}} \frac{\omega_{\boldsymbol{k},i} - \omega_{\boldsymbol{k},f}}{2}, \tag{2.35}$$

$$f_a = -\int_{\boldsymbol{k}} \frac{1 - \lambda_{\boldsymbol{k}}^2}{2}, \tag{2.36}$$

$$f_c(s) = \int_{\boldsymbol{k}} \frac{1 - \lambda_{\boldsymbol{k}}^2 e^{-2s\omega_{\boldsymbol{k},f}}}{2}. \tag{2.37}$$

It has also been proved [22], that if one changes the parameter r continuously as a function of time, keeping the same intitial and final conditions $r(0) = r_i$ and $r(+\infty) = r_f$ respectively, the moments will be also time-dependent and the distribution must approach a delta peak when the time variation of the transition is sufficiently slow.

In our case, $r_{eff}(t)$ changes discontinuously at $t = 0$, from $r_{eff}(0^-)$ to $r_{eff}(0^+)$ and then varies with time according to Eq. (2.22), but the time-dependence of $H_{eff}(t)$ due to the variation of $r_{eff}(t)$ is in apparent contrast with Eq. (2.10) where no time-dependence appear. In order to solve this paradox the time-dependent additive constant $c(t)$, defined in Eq. (2.24), must be included in the computations, as we shall see in the next chapter.

Chapter 3

Results

3.1 Stationarity of the work probabilty distribution function

In addition to the remarks done at the end of chapter 2, also time-translational invariance of the original hamiltonian of the model $H(r,u)$ defined in Eq. (2.1) can be invoked to conclude that $\langle W\rangle$ must be time-independent. Indeed, since external conditions after the quench are held fixed, the average energy of the system can not vary. Neglecting $c(t)$ [compare Eq. (2.23) with Eq. (2.25)], breaks time-translational invariance, thus we must take it into account in the computations in order to restore the time-independence of the work probability distribution. In the following we will show that the time derivative of $c(t)$ compensates for that of $\langle H_{eff}\rangle$, [where H_{eff} is defined in Eq. (2.23)] and this, in turn, will be used to show that all the central moments $\langle(H_{eff}-\langle H_{eff}\rangle)^n\rangle$ are actually time-independent, in spite of the time dependence of H_{eff}.

In order to simplify the notation, we will denote by H_{tot} the hamiltonian $H(m,u)$ of Eq. (2.1), by H_{eff} that in Eq. (2.23) and by $\langle\cdot\rangle$ the average taken with respect to an arbitrary initial pure state.

3.1.1 Stationarity of the average work

To begin with, let us use the Schrödinger picture to compute averages; this choice is done to ease the computation of time derivatives. According to this prescription, the fields are considered as fixed operators and the time dependence of their moments and correlations follows from that of the wave function; for instance denoting ϕ_S and ϕ_H the field operator in the Schrödinger and Heisenberg picture, the following equivalence for the two-point correlation function holds

$$\left\langle\Psi_H|\phi_H(x,t)\phi_H(x',t')|\Psi_H\right\rangle = \left\langle\Psi_S(t)|\phi_S(x)\phi_S(x')|\Psi_S(t')\right\rangle . \tag{3.1}$$

From now on the Schrödinger picture will be used. The time-derivative of the expectation value of a generic observable $A(t)$ is [23]

$$\frac{\mathrm{d}\langle A(t)\rangle}{\mathrm{d}t} = dfraci\hbar\langle[H_{tot}, A(t)]\rangle + \left\langle \frac{\partial A(t)}{\partial t} \right\rangle, \tag{3.2}$$

where the partial derivative is justified since an observable can be a function of time through scalar coefficients, even though the field and momentum operators, ϕ and Π, are fixed. In our case $r_{eff}(t)$ is at the origin of the time dependence of H_{eff}, thus Eq. (3.2), using Eq. (2.25), translates into

$$\frac{\mathrm{d}\langle H_{tot}\rangle}{\mathrm{d}t} = \left\langle \frac{\partial H_{tot}}{\partial t} \right\rangle = \left\langle \frac{\partial H_{eff}}{\partial t} \right\rangle + \frac{\mathrm{d}c(t)}{\mathrm{d}t}. \tag{3.3}$$

By definition $d\langle H_{tot}\rangle/dt = 0$, thus we expect

$$\left\langle \frac{\partial H_{eff}}{\partial t} \right\rangle = -\frac{\mathrm{d}c(t)}{\mathrm{d}t}; \tag{3.4}$$

but let us show this explicitly.

The only time-dependent part in H_{eff} is $r_{eff}(t)$ which, in turn, does not depend on $\boldsymbol{x}$ and can be factored out from the integral

$$\begin{aligned} \frac{\partial H_{eff}}{\partial t} &= \frac{1}{2}\int \mathrm{d}^d x \left[\frac{\partial\left(\boldsymbol{\Pi}^2 + (\nabla\boldsymbol{\phi})^2\right)}{\partial t} + \frac{\partial\left(r_{eff}\boldsymbol{\phi}^2\right)}{\partial t}\right] \\ &= \frac{1}{2}\left(\int \mathrm{d}^d x\ \boldsymbol{\phi}^2\right)\frac{\mathrm{d}r_{eff}}{\mathrm{d}t}. \end{aligned} \tag{3.5}$$

The time derivative of r_{eff} is, according to Eq. (2.22),

$$\frac{\mathrm{d}r_{eff}}{\mathrm{d}t} = \frac{u}{6}\frac{\mathrm{d}\langle\phi^2\rangle}{\mathrm{d}t}. \tag{3.6}$$

that, combined with Eq. (3.5), leads to

$$\left\langle \frac{\partial H_{eff}}{\partial t} \right\rangle = \frac{u}{12}\langle\boldsymbol{\phi}^2\rangle\frac{\mathrm{d}\langle\phi^2\rangle}{\mathrm{d}t}. \tag{3.7}$$

Putting together Eqs. (3.3), (3.7), and the definition of $c(t)$ given in Eq. (2.24), we find

$$\frac{\mathrm{d}\langle H_{tot}\rangle}{\mathrm{d}t} = \frac{u}{12}\langle\boldsymbol{\phi}^2\rangle\frac{\mathrm{d}\langle\phi^2\rangle}{\mathrm{d}t} - \frac{u}{24}\frac{\mathrm{d}(N\langle\phi^2\rangle^2)}{\mathrm{d}t} = 0, \tag{3.8}$$

since $N\langle\phi^2\rangle = \langle\boldsymbol{\phi}^2\rangle$.

3.1.2 Stationarity of the central moments

The *central* moment of order n of a univariate real-valued random variable X is defined as $\mu_{c_n} \doteq \langle (X - \langle X \rangle)^n \rangle$ and, differently from the moment *about the origin*, $\mu_n \doteq \langle X^n \rangle$, measures the deviations of X with respect to its mean $\langle X \rangle$; the variance, for instance, is the central moment of order 2.

The series expansion of $G(s)$ in Eq. (2.10) shows that the n^{th}-moment about the origin of W is $\langle (H - E_0^{r_i})^n \rangle$, which is clearly time-independent because it refers to the original time-independent hamiltonian. Here, instead, we will study the central moments of H_{eff}, that is $\langle (H_{eff} - \langle H_{eff} \rangle)^n \rangle$, and we will show that, in spite of the fact that H_{eff} depends on t via $r_{eff}(t)$, these moments are actually time-independent.

Let us set $E_0^{r_i} = 0$, which simplifies the notation without changing the essence of the proof. First of all, using Eq. (3.2), we can take the derivative under the sign of average since H_{tot} and H_{eff} differ by an additive constant $c(t)$ [see Eq. (2.25)] which implies $[H_{tot}, H_{eff}] = 0$.

Secondly since $r_{eff}(t) \neq r_{eff}(t')$ then $[H_{eff}(t), \frac{\mathrm{d}H_{eff}(t)}{\mathrm{d}t}] \neq 0$ and when one differentiates with respect to time the order of the operators must be kept. Setting $\Delta H_{eff} \doteq H_{eff} - \langle H_{eff} \rangle$ the time derivative of $\langle \Delta H_{eff} \rangle$ becomes

$$\left\langle \frac{\partial (\Delta H_{eff})^n}{\partial t} \right\rangle = \sum_{k=0}^{n-1} \left\langle (\Delta H_{eff})^k \frac{\partial \Delta H_{eff}}{\partial t} (\Delta H_{eff})^{n-1-k} \right\rangle . \tag{3.9}$$

In addition, Eq. (3.4) illustrates that $-\frac{\mathrm{d}\langle H_{eff} \rangle}{\mathrm{d}t} = \frac{\mathrm{d}c(t)}{\mathrm{d}t}$, where the derivative can be taken out of the average, for the same reason given for Eq. (3.9). Then

$$\frac{\partial \Delta H_{eff}}{\partial t} = \frac{\partial (H_{eff} - \langle H_{eff} \rangle)}{\partial t} = \frac{\partial H_{eff}}{\partial t} + \frac{\mathrm{d}c(t)}{\mathrm{d}t}, \tag{3.10}$$

but, according to Eq. (2.25), the left hand side is the time derivative of H_{tot}, which vanishes by hypothesis and therefore

$$\frac{\partial \Delta H_{eff}}{\partial t} = 0 \implies \left\langle \frac{\partial (\Delta H_{eff})^n}{\partial t} \right\rangle = 0. \tag{3.11}$$

3.2 Moment generating function of the quantum $O(N)$ model

Let us fix $t = 0$ when the quench occurs and consider a system prepared in the ground state $|\Psi_0^{r_i}\rangle$ of $H_{tot}(0^-)$.

In Sec. 3.1 we have seen that the work probability distribution and its moment generating function $G(s)$ are time independent. As a consequence, it is enough to compute $G(s)$ for $t = 0^+$ to know all the moments for every

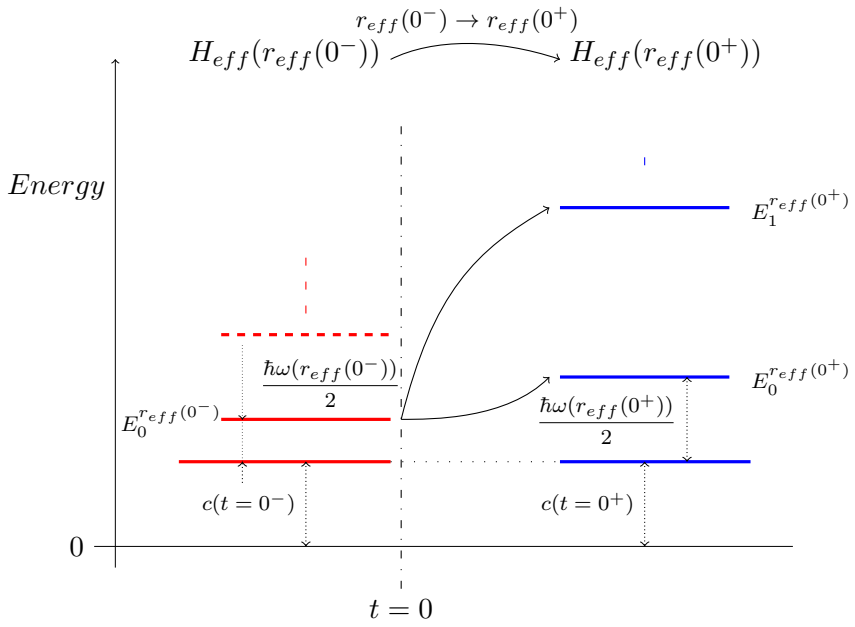

Figure 3.1: Schematic representation of the energy levels at $t = 0^-$ and $t = 0^+$. Notice that the additive constants $c(0^-) = c(0^+)$ are equal as a consequence of the continuity of the energy levels at $t = 0$.

future time. In order to do so, we can use the results illustrated in Sec 2.3 since for $N \to \infty$ the theory is effectively gaussian. This is possible because the continuity of the fields in $t = 0$ allows us to neglect the additive constant, being $c(0^-) = c(0^+)$, and to use Eq. (2.33).

Finally, it is important to note that in a gaussian theory the parameter r shown in Eq. (2.26) must be positive otherwise the hamiltonian is not bounded from below and the model become unstable, therefore one important requirement to fulfill in order to exploit Eq. (2.33) is $r_{eff}(0^+) \geq 0$.

Let us start now the derivation of what has been claimed. From Eq. (2.25) we can keep just H_{eff} which is a sum of N terms, one corresponding to each component of the field $\boldsymbol{\phi}$. We will look at one of them in the momentum representation

$$h_{eff} = \int_{\boldsymbol{k}} \left(\frac{1}{2} \Pi_{\boldsymbol{k}} \Pi_{-\boldsymbol{k}} + \frac{\omega_k^2(r_{eff}(t))}{2} \phi_{\boldsymbol{k}} \phi_{-\boldsymbol{k}} \right), \tag{3.12}$$

where $\omega_{\boldsymbol{k}}(r_{eff}(t)) = \sqrt{r_{eff}(t) + |\boldsymbol{k}|^2}$ is a time-dependent frequency to be determined via Eq. (2.22). The equal-time two-point correlator $\langle \phi^2 \rangle$ of the field ϕ at the same spatial point, which appears on the right hand side, is a function of the $\omega_{\boldsymbol{k}}$'s and we will see that this dependence can be found easily for a system prepared in the ground state at $t = 0$. To this end, it is

convenient to write down the Fourier transform representation of $\langle\phi^2\rangle$, i.e.,

$$\langle\phi^2(t=0^-)\rangle = \int_{\boldsymbol{k}} \langle\Psi_0^{r_{eff}(0^-)}|\phi_{\boldsymbol{k}}\phi_{-\boldsymbol{k}}|\Psi_0^{r_{eff}(0^-)}\rangle, \tag{3.13}$$

which can be evaluated with the aid of the ladder operators of $h_{eff}(0^-)$. In detail we have

$$\phi_{\boldsymbol{k}} = \frac{a_{\boldsymbol{k}} + a^\dagger_{-\boldsymbol{k}}}{\sqrt{2\omega_{\boldsymbol{k}}(r_{eff}(0^-))}}, \tag{3.14}$$

which implies

$$\phi_{\boldsymbol{k}}\phi_{-\boldsymbol{k}} = \frac{1 + a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}} + a^\dagger_{-\boldsymbol{k}}a_{-\boldsymbol{k}} + a_{\boldsymbol{k}}a_{-\boldsymbol{k}} + a^\dagger_{-\boldsymbol{k}}a^\dagger_{\boldsymbol{k}}}{2\omega_{\boldsymbol{k}}(r_{eff}(0^-))}. \tag{3.15}$$

By hypotesis $|\Psi_0^{r_{eff}(0^-)}\rangle$ is the ground state of $h_{eff}(0^-)$ so that

$$\langle\Psi_0^{r_{eff}(0^-)}|\phi_{\boldsymbol{k}}\phi_{-\boldsymbol{k}}|\Psi_0^{r_{eff}(0^-)}\rangle = \frac{1}{2\omega_{\boldsymbol{k}}(r_{eff}(0^-))} \tag{3.16}$$

and Eq. (3.13) becomes

$$\langle\phi^2(t=0^-)\rangle = \int_{\boldsymbol{k}} \frac{1}{2\omega_{\boldsymbol{k}}(r_{eff}(0^-))}. \tag{3.17}$$

Accordingly, Eq. (2.22), referred to the initial state, can be written as

$$r_{eff}(0^-) = r_i + \frac{u}{12}\int_{\boldsymbol{k}} \frac{1}{\sqrt{|\boldsymbol{k}|^2 + r_{eff}(0^-)}}, \tag{3.18}$$

which determines the initial value $r_{eff}(0^-)$ of the effective parameter r_{eff} as a function of r_i. Upon doing the quench $r_i \to r_f$ at time $t = 0$, the fields ϕ and Π vary continuously [25] and therefore

$$\langle\phi^2(t=0^+)\rangle = \langle\phi^2(t=0^-)\rangle. \tag{3.19}$$

Correspondingly, by using Eq. (2.22) one concludes that

$$r_{eff}(0^+) = r_f + \frac{u}{6}\langle\phi^2(t=0^-)\rangle. \tag{3.20}$$

This means that the post-quench effective parameter r_{eff} in $t = 0^+$, is given by

$$r_{eff}(0^+) = r_f + \frac{u}{12}\int_{\boldsymbol{k}} \frac{1}{\sqrt{|\boldsymbol{k}|^2 + r_{eff}(0^-)}}, \tag{3.21}$$

which differs from Eq. (3.18) just for the replacement of r_i by r_f. This also implies that

$$r_{eff}(0^+) - r_{eff}(0^-) = r_f - r_i. \tag{3.22}$$

The moment generating function for $r_{eff}(0^+) \geq 0$ has thus the same form as the gaussian one in Eq. (2.33), with

$$\omega_{\boldsymbol{k}_i} = \omega_{\boldsymbol{k}}(r_{eff}(0^-)), \tag{3.23}$$

$$\omega_{\boldsymbol{k}_f} = \omega_{\boldsymbol{k}}(r_{eff}(0^+)). \tag{3.24}$$

Note that the value $r_{eff}(0^+) = 0$ is reached for

$$r^c_{f,0^+} = -\frac{u}{12}\int_{\boldsymbol{k}} \frac{1}{\sqrt{|\boldsymbol{k}|^2 + r_{eff}(0^-)}} = -\frac{u}{6}\langle\phi^2(t=0^-)\rangle. \tag{3.25}$$

Chapter 4

Conclusions

Summarizing, in the present work we have shown that it is possible to solve the dynamics of the quantum $O(N)$ model in the limit $N \to \infty$ due to its effective gaussian nature. However, when one studies the statistics of the work performed in a quench, it is important to account for the additive constant $c(t)$ which appears in the mapping from the initial, quartic hamiltonian to its gaussian counterpart [see Eq. (2.25)]. This constant makes sure that the statistics of the work one derives on the basis of the effective hamiltonian is actually time-independent, in spite of the fact that H_{eff} carries an explicit dependence on time.

In chapter 3, the statistics of the work has been derived for $r_f \geq r^c_{f,0^+}$ when it is possible to use the results concearning the work performed quenching a pure gaussian model already known in the literature. If $r_f < r^c_{f,0^+}$, $r_{eff}(0^+)$ is negative and a new formula for $G(s)$ — or an extension of the one we provided in our work — should be found.

The numerical solution of the Eqs. (2.22) and (2.23) shows a damped oscillation for $r_{eff}(t)$ as a function of time and as t goes to infinity, $r_{eff}(t)$ approaches an asymptotic value $r_{eff}(t = +\infty)$, always bigger than, or equal to zero. In Fig. 4.1 the time evolution of the effective parameter is illustrated. Unlike the case of a pure gaussian model at equilibrium, $r_{eff}(0^+)$ can be negative without implying an instability for the post quench hamiltonian since the time evolution of H_{eff} will make it grow, become positive and eventually oscillate around its asymptotic value that is always positive or zero. The value of $r_{eff}(+\infty) = 0$ identifies the so-called *dynamical critical point* and occurs at a value of the post quench parameter r_f equal to $r^c_{f,\infty}$. It has been shown [2] that $r_{eff}(+\infty)$ vanishes for

$$r^c_{f,\infty} = -\frac{u}{24} \int_{\boldsymbol{k}} \frac{2|\boldsymbol{k}|^2 + r_{eff}(0^-)}{|\boldsymbol{k}|^2 \sqrt{|\boldsymbol{k}|^2 + r_{eff}(0^-)}}, \tag{4.1}$$

moreover for $r_f \leq r^c_{f,\infty}$, $r_{eff}(+\infty)$ is always equal to zero. Taking the difference between $r^c_{f,0^+}$ and $r^c_{f,\infty}$ it is possible to see that the latter is

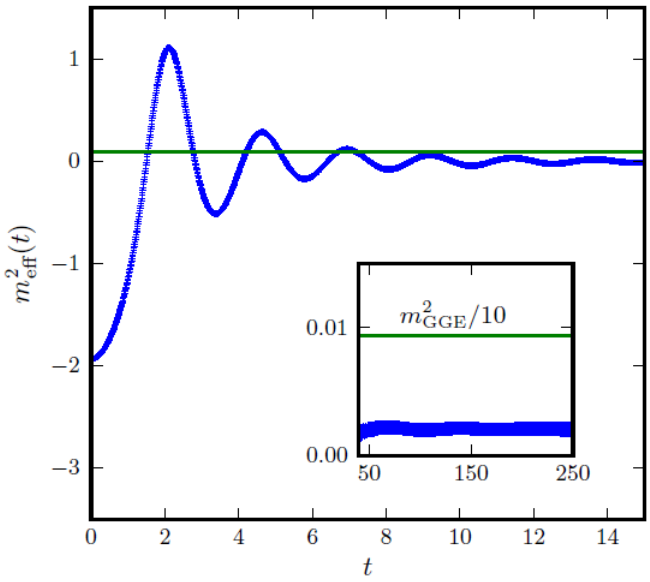

Figure 4.1: Plot of $r_{eff}(t) \doteq m^2_{eff}(t)$ vs time for a sudden quench in $d = 1+\epsilon$ as it appears in [1].

smaller than the former i.e. that

$$r^c_{f,0^+} - r^c_{f,\infty} = \frac{u}{24} \int_{\boldsymbol{k}} \frac{r_{eff}(0^-)}{|\boldsymbol{k}|^2 \sqrt{|\boldsymbol{k}|^2 + r_{eff}(0^-)}} > 0, \tag{4.2}$$

since $r_{eff}(0^-)$ must be positive to have a well defined effective hamiltonian for $t = 0^-$, [see Eq. (2.23)]. Consequently another important issue that needs further investigation is the relation between $r^c_{f,0^+}$ and $r^c_{f,\infty}$, namely whether it is possible to predict the dynamical critical point from the statistics of the work.

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