POLITECNICO DI TORINO, UNIVERSITÉ PARIS DIDEROT

Master degree course in Physics of Complex Systems

Master Degree Thesis

The effect of a noise trader on the equilibrium in repeated prediction markets

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Academic year 2017 – 2018

Summary

This work aims to study the equilibrium properties of a repeated prediction market when a noise trader comes into play.
Bottazzi and Giachini [1] have completely probed the system where two fractional Kelly traders invest in such a market, finding sufficient conditions for dominance, disappearance or coexistence of the agents.
I have extended these conditions in the case in which there is a noise trader, other then the two Kelly ones.
Then, numerical simulations have been run in order to confirm the analytical results.
It has been found that for some set of parameters the noise trader vanishes, for example when the believes of one or two agents are close to the truth; but there are also set of parameters for which the noise trader dominates or survives, coexisting with a Kelly trader. It is then not trivial to neglect the presence of agents that act irrationally in such a market, and to suppose that the market kicks them out after a transient: it is possible that they survive or even dominate, affecting the prices and consequently their informational efficiency.

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Chapter 1

Introduction

1.1 Description of the problem

Repeated prediction markets in which people can bet repeatedly on a binary event are of large interest because they share many similarities with other financial markets. Bottazzi and Giachini showed the equivalence of a repeated prediction market with an economy where two Arrow securities are exchanged [1].
The probability π^* of the event to occur is considered to be constant over time and unknown to the agents betting in this market.
Each agent has a certain amount of wealth and, at each time step, all of them have to decide how much of it to invest on each of the two possibile outomes; notice that each agent has to bet all of his wealth at every time step. All the investment is then collected, the prices are decided by the *market clearing condition*, and after the outcome has been revealed, the wealth of each agent is updated.
The rules the agents invest according to depend on their believes π^i, which are their personal estimates of the true probability π^*.
In my work, rational agents bet following a variant of the Kelly rule, the *fractional Kelly rule*.
While the former maximizes the expected value of the logarithm of capital at each time step and is risky in the short or medium run, the latter is related to myopic agents that maximize a CRRA utility function with price dependent relative risk aversion coefficient: it is a safer policy, which may let more then one agent survive in the market.
Since the case in which two fractional Kelly traders invest in this market has been completely exploited [1], I focussed to study what happens when a third *noise* (i.e. irrational) trader joins the game. The fraction of wealth the noise trader invest is not deterministic given the parameters, instead it is a random variable.

1.2 The model

Let's now define the quantities of the model and their relationships.

- $s_t = \begin{cases} 1 \; with \; prob \; \pi^* \\ 0 \; with \; prob \; 1 - \pi^* \end{cases}$ is a random variable describing whether the event occurs or not.

- ω_t^i is the wealth of Kelly agent i at time t, $i \in \{1{,}2\}$.
- $\tilde{w}_t$ is the wealth of the irrational agent at time t.
- p_t is the price of the first security at time t, $1 - p_t$ is the price of the second one.
- $\alpha^i(p_t) = c^i\pi^i + (1 - c^i)p_t$, i.e. the fractional Kelly rule, is the fraction of ω_t^i that agent i invests in the first security at time t, where π^i is his belief of π^* and c^i is a "mixing" parameter that can be considered a good approximation of the behavior of a risk averse agent.
- $\eta_t \sim Unif[0{,}1]$ is the random variable describing the fraction of $\tilde{\omega}_t$ that irrational agent invests in the first security at time t.

Then, the dynamic of the wealths is the following:

$$\begin{cases} w_t^1 = \frac{\alpha^1(p_t)}{p_t} w_{t-1}^1 = \frac{c^1\pi^1 + (1-c^1)p_t}{p_t} w_{t-1}^1 \\ w_t^2 = \frac{\alpha^2(p_t)}{p_t} w_{t-1}^2 = \frac{c^2\pi^2 + (1-c^2)p_t}{p_t} w_{t-1}^2 & if\ s_t = 1 \\ \tilde{w}_t = \frac{\eta}{p_t} \tilde{w}_{t-1} \end{cases}$$

$$\begin{cases} w_t^1 = \frac{1-\alpha^1(p_t)}{1-p_t} w_{t-1}^1 = \frac{1-c^1\pi^1 - (1-c^1)p_t}{1-p_t} w_{t-1}^1 \\ w_t^2 = \frac{1-\alpha^2(p_t)}{1-p_t} w_{t-1}^2 = \frac{1-c^2\pi^2 - (1-c^2)p_t}{1-p_t} w_{t-1}^2 & if\ s_t = 0 \\ \tilde{w}_t = \frac{1-\eta}{1-p_t} \tilde{w}_{t-1} \end{cases}$$

Since in this framework the supply equals the demand and the total wealth is conserved (and then normalizable), the number of shares issued for each outcome is equal to 1:

$$w_t^1 + w_t^2 + \tilde{w}_t = 1$$

at every time t.
Solving this implicit equation with respect to p_t, it is easy to obtain the *market clearing condition*, which establishes the prices of the two shares at each time t:

$$p_t = \frac{c^1\pi^1 w_{t-1}^1 + c^2\pi^2 w_{t-1}^2 + \eta\tilde{w}_{t-1}}{c^1 w_{t-1}^1 + c^2 w_{t-1}^2 + \tilde{w}_{t-1}}$$

In this work I will focus on the case in which $c^1 = c^2 = c$, so the equation for the price reads:

$$p_t = \frac{c(\pi^1 w_{t-1}^1 + \pi^2 w_{t-1}^2) + \eta\tilde{w}_{t-1}}{c(w_{t-1}^1 + w_{t-1}^2) + \tilde{w}_{t-1}}$$

Figure 1.1 shows the part of the plane in which the system can move at every time step.
Let's define the following dictionary:

- If an agent's wealth limit $\lim_{t\to\infty} w_t = 1$ almost surely, the agent is said to *dominate*.
- If an agent's wealth limit $\lim_{t\to\infty} w_t = 0$ almost surely, the agent is said to *vanish* or *disappear*.
- If an agent is not going to vanish nor dominate in the long run, he is said to *survive*.
- Agents who survive together are said to *coexist*.

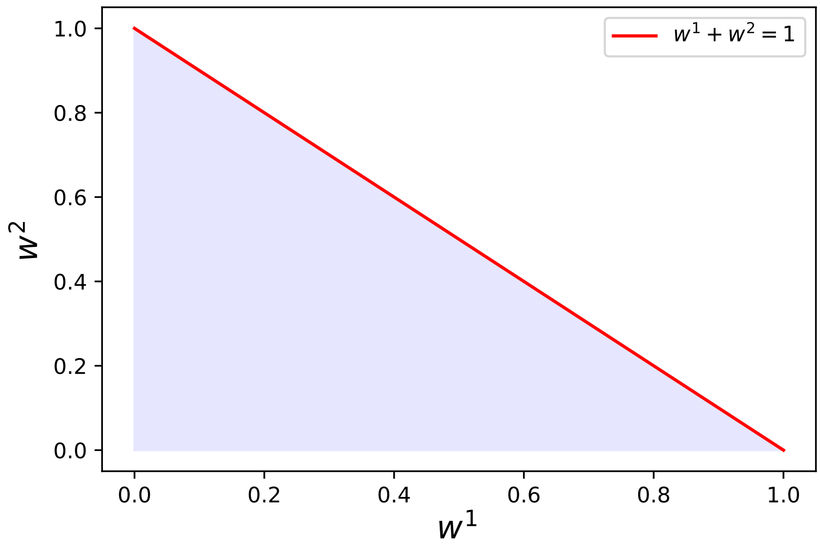

Figure 1.1. Region of the plane that the system can explore

Chapter 2

Theoretical analysis

In this section I will focus on the definiton of the quantities of interest and on their interpretation in order to derive sufficient conditions for determining if an agent dominates or vanishes.

2.1 Quantities of interest

Let's define the conditional expect drift of the log difference of the individual wealth for each of the three agents betting in the market:

$$\mu_N(w^1_{t-1}, w^2_{t-1}) = \mathbf{E}\Bigg[\log\frac{\tilde{w}_t}{\tilde{w}_{t-1}} - \log\frac{1-\tilde{w}_t}{1-\tilde{w}_{t-1}}\Bigg| w^1_{t-1}, w^2_{t-1}\Bigg]$$

$$\mu_i(w^1_{t-1}, w^2_{t-1}) = \mathbf{E}\Bigg[\log\frac{w^i_t}{w^i_{t-1}} - \log\frac{1-w^i_t}{1-w^i_{t-1}}\Bigg| w^1_{t-1}, w^2_{t-1}\Bigg]$$

where $i = 1{,}2$.
The first is related to the irrational agent, while the other to the rational ones.
Their meaning is mainly hidden in the limit where the agent, that the quantity is referred to, has almost $w = 0$ or $w = 1$ at time $t-1$.
One can prove that[1], for the considered system:

$$\lim_{\tilde{w}_{t-1}\to 0}\mu_N(w^1_{t-1}, w^2_{t-1}) = \pi^*\Bigg[\log\frac{k+1}{\pi^1+k\pi^2}-1\Bigg]+(1-\pi^*)\Bigg[\log\frac{k+1}{1-\pi^1+k(1-\pi^2)}-1\Bigg] = \mu^0_N(k)$$

$$\lim_{\tilde{w}_{t-1}\to 1}\mu_N(w^1_{t-1}, w^2_{t-1}) = \pi^*\Bigg[\log\frac{k+1}{(1-c)(k+1)+c(\pi^1+k\pi^2)}+ \\ +\frac{c(\pi^1+k\pi^2)}{(1-c)(k+1)}\log\frac{c(\pi^1+k\pi^2)}{(1-c)(k+1)+c(\pi^1+k\pi^2)}\Bigg]+$$

[1]See the appendix for the derivation

$$+(1-\pi^*)\Bigg[\log\frac{k+1}{k+1-c(\pi^1+k\pi^2)}+\frac{k+1-c(\pi^1+k\pi^2)}{(1-c)(k+1)}\log\frac{k+1-c(\pi^1+k\pi^2)-(1-c)(k+1)}{k+1-c(\pi^1+k\pi^2)}\Bigg]=\mu_N^1(k)$$

where $k=\frac{w_{t-1}^2}{w_{t-1}^1}\in(0,+\infty)$.

And, considering $i,j=1{,}2$ with $i\neq j$ (i.e. if $i=1$ then $j=2$ and vice versa):

$$\lim_{w_{t-1}^i\to 1}\mu_i(w_{t-1}^1,w_{t-1}^2)=\pi^*\Bigg[\log\pi^i\gamma-\Bigg(\frac{c\pi^j+(1-c)\pi^i}{\gamma-1}+1\Bigg)\log(c\pi^j+(1-c)\pi^i+\gamma-1)+$$
$$+\frac{c\pi^j+(1-c)\pi^i}{\gamma-1}\log(c\pi^j+(1-c)\pi^i)+1\Bigg]+(1-\pi^*)\Bigg[\log((1-\pi^i)\gamma)+$$
$$-\frac{\gamma-c\pi^j-(1-c)\pi^i}{\gamma-1}\log(\gamma-c\pi^j-(1-c)\pi^i)+\frac{1-c\pi^j-(1-c)\pi^i}{\gamma-1}\log(1-c\pi^j-(1-c)\pi^i)+1\Bigg]=\mu_i^1(\gamma)$$

where $\gamma=\frac{1-w_{t-1}^i}{w_{t-1}^j}\in(1,+\infty)$.

$$\lim_{w_{t-1}^i\to 0}\mu_i(w_{t-1}^1,w_{t-1}^2)=\pi^*\Bigg[\Bigg(\frac{c\pi^i+\frac{(1-c)c\pi^j}{c+\rho}}{\frac{(1-c)\rho}{c+\rho}}+1\Bigg)\log\Bigg(c\pi^i+\frac{(1-c)c\pi^j}{c+\rho}+\frac{(1-c)\rho}{c+\rho}\Bigg)+$$
$$-\Bigg(\frac{c\pi^i+\frac{(1-c)c\pi^j}{c+\rho}}{\frac{(1-c)\rho}{c+\rho}}\Bigg)\log\Bigg(c\pi^i+\frac{(1-c)c\pi^j}{c+\rho}\Bigg)+$$
$$-\Bigg(\frac{c\pi^i+\frac{(1-c)c\pi^j}{c+\rho}}{\frac{(1-c)\rho}{c+\rho}+\rho}+1\Bigg)\log\Bigg(\frac{1}{1+\rho}\Bigg(c\pi^i+\frac{(1-c)c\pi^j}{c+\rho}+\frac{(1-c)\rho}{c+\rho}+\rho\Bigg)\Bigg)+$$
$$+\Bigg(\frac{c\pi^i+\frac{(1-c)c\pi^j}{c+\rho}}{\frac{(1-c)\rho}{c+\rho}+\rho}\Bigg)\log\Bigg(\frac{1}{1+\rho}\Bigg(c\pi^i+\frac{(1-c)c\pi^j}{c+\rho}\Bigg)\Bigg)\Bigg]+$$
$$+(1-\pi^*)\Bigg[\Bigg(\frac{1-c\pi^i-\frac{(1-c)c\pi^j}{c+\rho}}{\frac{(1-c)\rho}{c+\rho}}\Bigg)\log\Bigg(1-c\pi^i-\frac{(1-c)c\pi^j}{c+\rho}\Bigg)+$$
$$-\Bigg(\frac{1-c\pi^i-\frac{(1-c)c\pi^j}{c+\rho}}{\frac{(1-c)\rho}{c+\rho}}-1\Bigg)\log\Bigg(1-c\pi^i-\frac{(1-c)c\pi^j}{c+\rho}-\frac{(1-c)\rho}{c+\rho}\Bigg)+$$
$$-\Bigg(\frac{1-c\pi^j-\frac{(1-c)c\pi^j}{c+\rho}+\rho}{\frac{(1-c)\rho}{c+\rho}+\rho}\Bigg)\log\Bigg(\frac{1}{1+\rho}\Bigg(1-c\pi^i-\frac{(1-c)c\pi^j}{c+\rho}+\rho\Bigg)\Bigg)+$$
$$+\Bigg(\frac{1-c\pi^j-\frac{(1-c)c\pi^j}{c+\rho}+\rho}{\frac{(1-c)\rho}{c+\rho}+\rho}-1\Bigg)\log\Bigg(\frac{1}{1-\rho}\Bigg(1-c\pi^j-\frac{(1-c)c\pi^j}{c+\rho}-\frac{(1-c)\rho}{c+\rho}\Bigg)\Bigg)\Bigg]=\mu_i^0(\rho)$$

where $\rho=\frac{1-w_{t-1}^j}{w_{t-1}^j}\in(0,+\infty)$.

2.2 Theorems

Now I am going to expose sufficient conditions that guarantee dominance, disappearance or coexistence of agents, based on the functions above defined. This approach has been already applied by Bottazzi and Giachini [1] in the case of absence of the noise trader, i.e. when only two fractional Kelly traders bet in the market: I generalize their results to this 2D system, in the presence of the noise trader.
The following propositions can be proved [2]:

1. If $\mu_N^1(k) > 0$ and $\mu_N^0(k) > 0$ $\forall k \in (0, +\infty)$, then the the noise trader will dominate and agents 1 and 2 vanish with probability 1.

2. If $\mu_i^1(\gamma) > 0$ $\forall \gamma \in (1, +\infty)$ and $\mu_i^0(\rho) > 0$ $\forall \rho \in (0, +\infty)$, then agent i will dominate and agent j and the noise trader will vanish with probability 1.

3. If $\mu_N^1(k) \leq 0$ and $\mu_N^0(k) \geq 0$ $\forall k \in (0, +\infty)$, then the the noise trader will survive with probability 1, and his wealth keeps oscillating over time.

4. If $\mu_i^1(\gamma) \leq 0$ $\forall \gamma \in (1, +\infty)$ and $\mu_i^0(\rho) \geq 0$ $\forall \rho \in (0, +\infty)$, then agent i will survive with probability 1, and his wealth keeps oscillating over time.

5. If $\mu_N^1(k) < 0$ and $\mu_N^0(k) < 0$ $\forall k \in (0, +\infty)$, then the noise trader will vanish with probability 1. The system collapses then on the line $w^1 + w^2 = 1$, and its behavior is the one completely analyzed in [1].

6. If $\mu_i^1(\gamma) < 0$ $\forall \gamma \in (1, +\infty)$ and $\mu_i^0(\rho) < 0$ $\forall \rho \in (0, +\infty)$, then agent i will vanish with probability 1 and the system collapses then on the line $w^i = 0$.
   The behavior of the collapsed system can be predicted considering the propositions 1., 3., 5. and evaluating μ_N^1 and μ_N^0 along the direction the system collapsed to, i.e. $k \to 0$ if $i = 2$ and $k \to +\infty$ if $i = 1$.

In the next section I will provide examples and simulations that show the validity of the above propositions.
Moreover, examples when these propositions can not be applied will be given and explained by means of heuristic arguments.

[2] See the appendix for the derivation

Chapter 3

Numerical simulations

The purpose of this chapter is to explore the system, to show the possibility of different asymptotic results, and to verify their coherence with the analytical results.

3.1 Dominance of a Kelly trader

Let's fix the parameters of the problem as follows:

- $\pi^* = 0.5$
- $c = 0.9$
- $\pi^1 = 0.9$
- $\pi^2 = 0.7$

Now it is possible to study the sign of each μ_i^j (with $i = \{1,2,N\}$ and $j = \{0,1\}$) in order to see if some proposition could be applied to understand the system behavior at equilibrium.
It is evident in figure 3.1 that both $\mu_1^0(\rho)$ and $\mu_1^1(\gamma)$ are negative for every value of ρ and γ. Then, we expect that agent 1 will vanish as a consequence of proposition 6. Moreover, we know that the system collapses on the vertical axis (where $w^1 = 0$), so we can study the sign of $\mu_N^0(k)$ and $\mu_N^1(k)$ along $k = \frac{w_{t-1}^2}{w_{t-1}^1} \to +\infty$.
We can see from figure 3.2 that the expect drift of the log difference of noise trader's wealth for large k is negative in both its limits, which means that the noise trader is expected to vanish, and then agent 2 to dominate.
Notice that one could have looked at $\mu_2^0(\rho)$ and $\mu_2^1(\gamma)$ from the beginning: it is indeed clear from figure 3.3 that they are positive for every value of ρ and γ, so one would have got the same prediction as a consequence of proposition 2. Numerical simulation of the system confirms these results.
I have initialized the system with $M = 1000$ different initial conditions, uniformly distributed in the simplex in which the system is confined (i.e. $w^1 + w^2 + \tilde{w} = 1$, with $0 < w^1, w^2, \tilde{w} < 1$). Then I let them evolve until the equilibrium has been reached [1].

[1] See the appendix for the method used to be sure that equilibrium has been reached

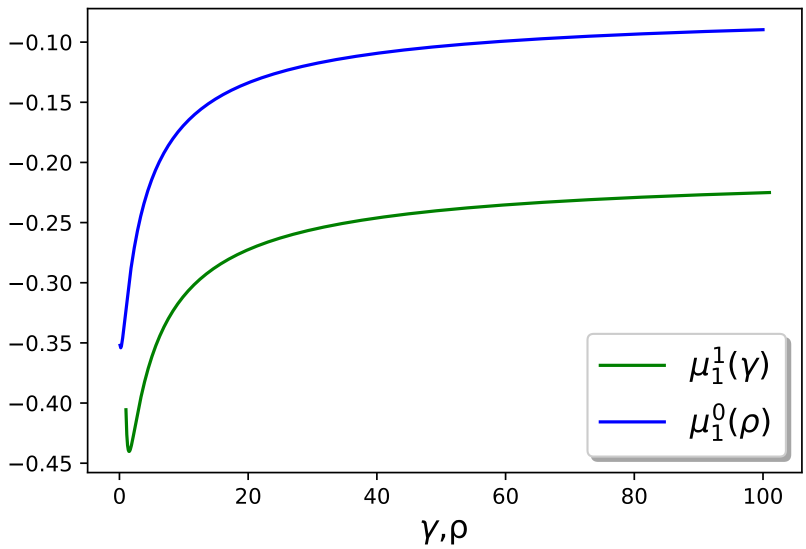

Figure 3.1. Expect drift of the log difference of agent 1's wealth

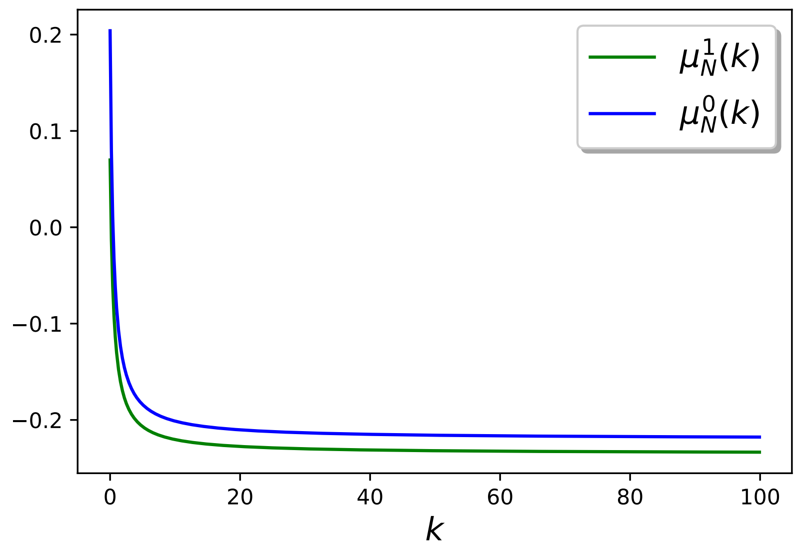

Figure 3.2. Expect drift of the log difference of noise trader's wealth

The equilibrium distribution of w^2 is shown in 3.4, and the average over the M realizations is $\mathbf{E}^{eq}[w^2] = 1$, while, obviously, $\mathbf{E}^{eq}[w^1] = \mathbf{E}^{eq}[\tilde{w}] = 0$.

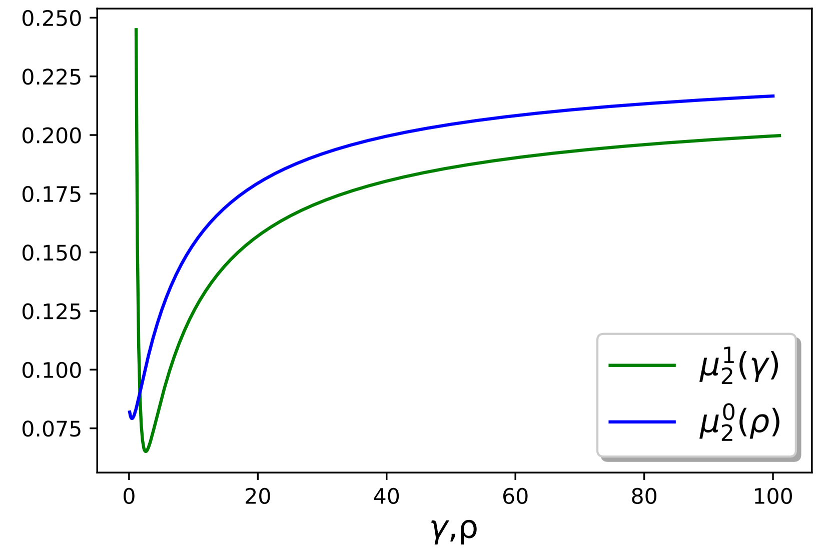

Figure 3.3. Expect drift of the log difference of agent 2's wealth

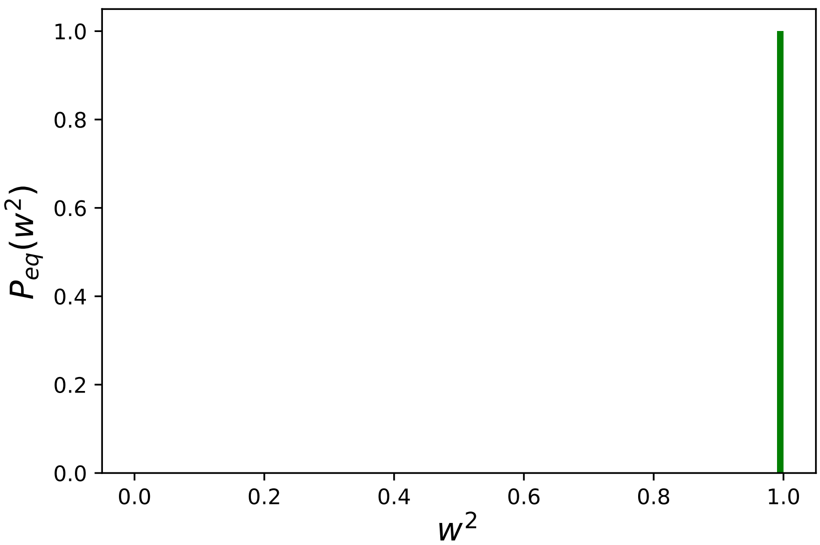

Figure 3.4. Distributon of w^2 at equilibium

3.2 Dominance of the noise trader

One way for the noise trader to dominate is the one in which both the Kelly traders have their believes sufficiently far from the truth and a coefficient c big enough, in order to have a much stronger dependence of $\alpha^i(p_t) = c\pi^i + (1-c)p_t$ on π^i instead on p_t. Then let's set:

- $\pi^* = 0.5$
- $c = 0.9$
- $\pi^1 = 0.95$
- $\pi^2 = 0.99$

In figure 3.5 it is possible to see that $\mu_N^0(k)$ and $\mu_N^1(k)$ are positive $\forall k$, so proposition 1 can be applied: we expect the noise trader to dominate while both Kelly traders vanish. Plot 3.7 confirms that agent 2 is going to vanish (proposition 5). Instead, from figure 3.6 no information can be gathered, since no hypothesis of the propositions are satisfied; anyway, this is not in contradiction with what it has been found previously.
Numerical simulations confirm this result: equilibrium distribution of $\tilde{w}$ is shown in 3.8 and $\mathbf{E}^{eq}[\tilde{w}] = 1$.

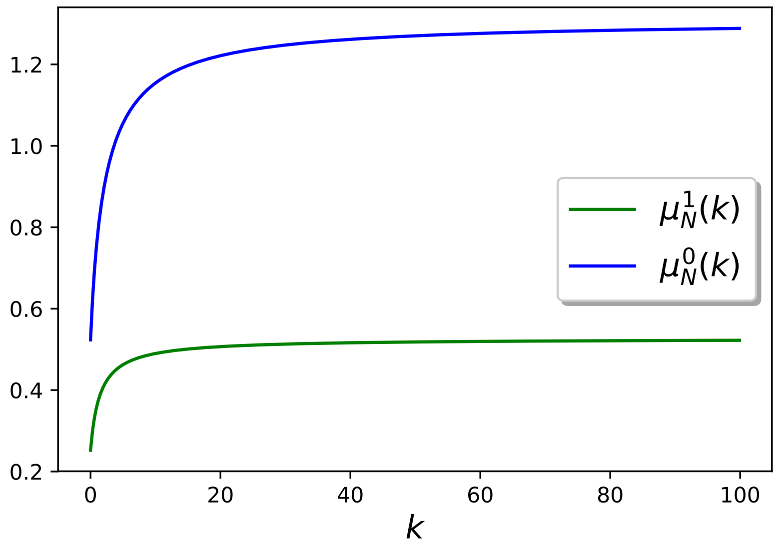

Figure 3.5. Expect drift of the log difference of noise trader's wealth

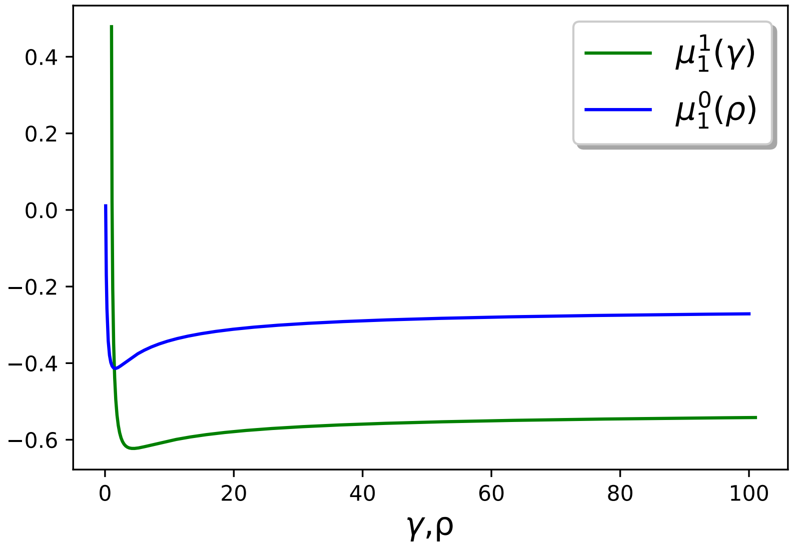

Figure 3.6. Expect drift of the log difference of agent 1's wealth

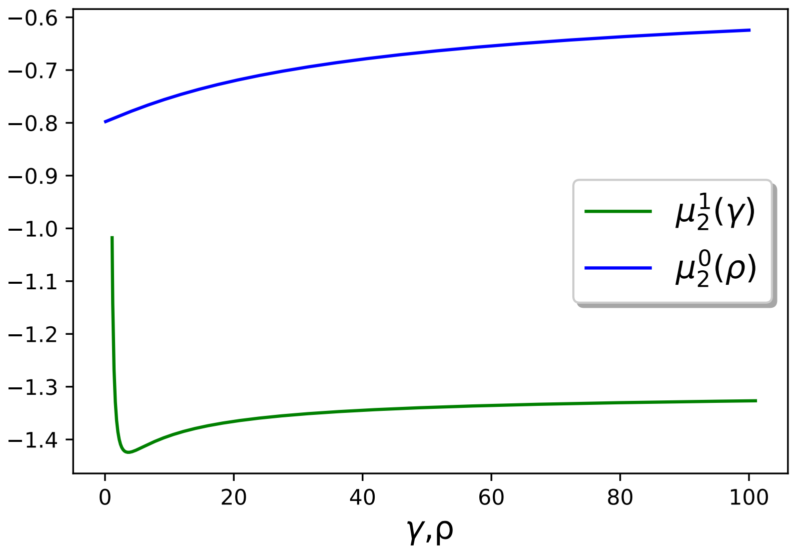

Figure 3.7. Expect drift of the log difference of agent 2's wealth

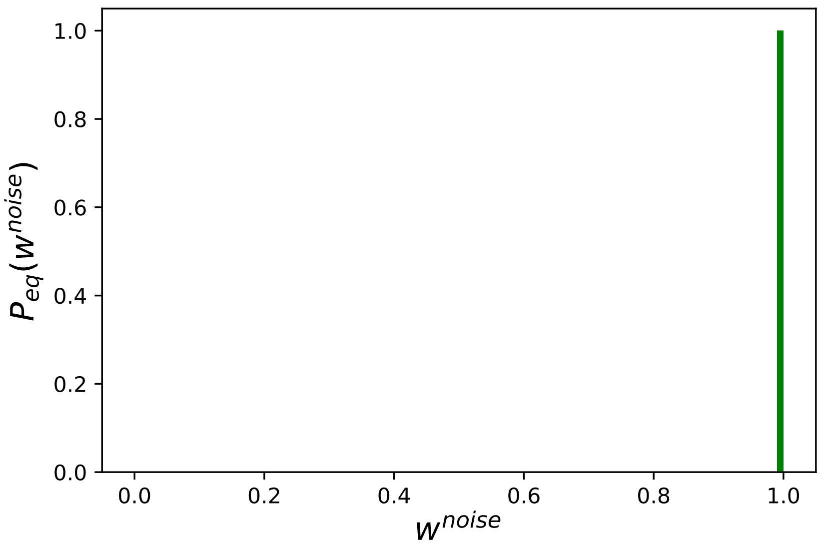

Figure 3.8. Distributon of $\tilde{w}$ at equilibium

3.3 Coexistence between the noise trader and a Kelly trader

A different and very interesting case is the one which concerns the coexistence of the noise trader with a Kelly trader. Such a situation can be realized setting the parameters as follows:

- $\pi^* = 0.1$
- $c = 0.001$
- $\pi^1 = 0.99$
- $\pi^2 = 0.95$

Notice first that, differently from the above examples, π^* is not the mean value of the fraction of wealth invested by the noise trader η anymore, and that it is way farer from π^1 and π^2. Moreover, c has been set very small with respect to others parameters: in this way $\alpha^i(p_t)$ has a very strong dependence on p_t, and a very weak one on π^i.
As usual, let's check if some proposition is applicable.
Figure 3.9 shows that $\mu_1^0(\rho)$ and $\mu_1^1(\gamma)$ are negative $\forall \rho, \gamma$: agent 1 is then expected to lose all of his wealth and disappear as a consequence of proposition 6, and the system is going to collapse on the axis $w^1 = 0$ which corresponds to $k \to +\infty$, $\rho \to +\infty$ and $\gamma \to +\infty$.
At this point, looking at 3.10 it is possible to see that $\mu_N^0(k \to +\infty) > 0$ while $\mu_N^1(k \to +\infty) < 0$: then the noise trader is expected to coexist with agent 2 and their wealths $\tilde{w}$

and w^2 are expected to oscillate together (they always sum to 1) as a consequence of proposition 3. Figure 3.11 confirms this results, since $\mu_2^0(\rho \to +\infty) > 0$ and $\mu_2^1(\gamma \to +\infty) < 0$ and proposition 4 could have been applied.
I have been simulating the process in order to compute equilibrium distributions of the wealths and to check if what has been predicted so far actually holds.
It turned out that agent 1 vanishes with probability 1, as expected. In figure 3.12 the equilibrium distribution of w^2 and $\tilde{w}$ are shown; their expected values have been numerically computed to be $\mathbf{E}^{eq}[w^2] \approx 0.59$ and $\mathbf{E}^{eq}[\tilde{w}] \approx 0.41$, while their variance and standard deviation (which are obviously the same for both the agents) are: $V^{eq}[w^2] = V^{eq}[\tilde{w}] \approx 0.04$ and $StdDev^{eq}[w^2] = StdDev^{eq}[\tilde{w}] \approx 0.20$.

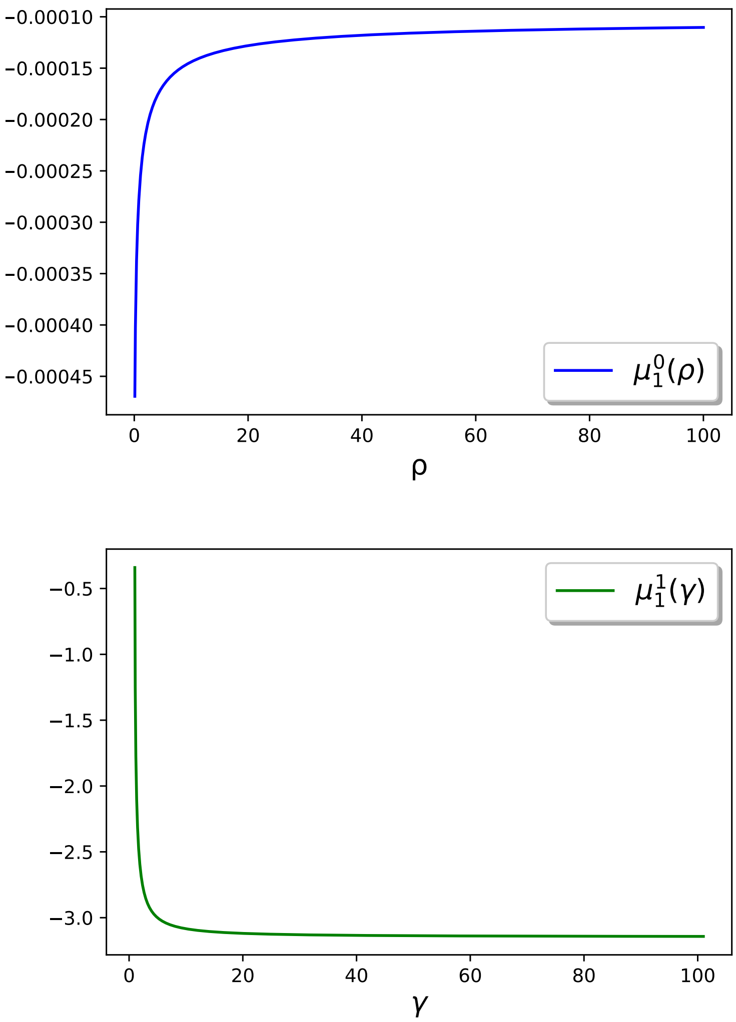

Figure 3.9. Expect drift of the log difference of agent 1's wealth

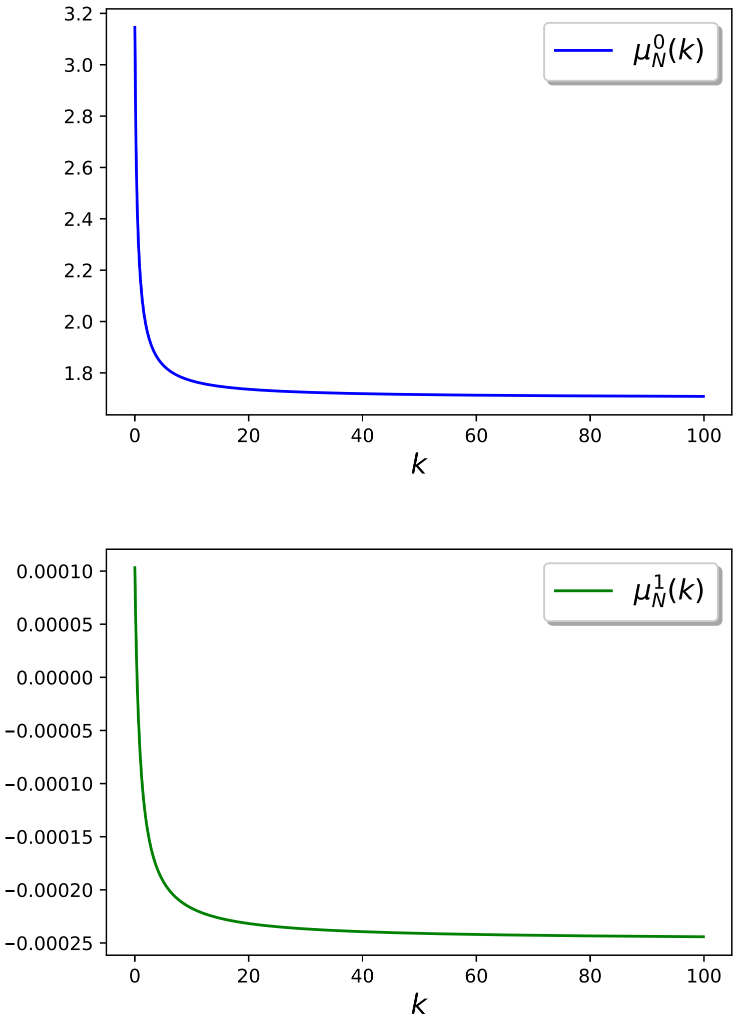

Figure 3.10. Expect drift of the log difference of noise trader's wealth

3.4 Not applicability of the propositions related to dominance or vanishing

The last example I am going to present enlightens that there are sets of parameters for which the propositions studied until now are not enough to determine uniquely the equilibrium properties of the system: indeed, as said, they are sufficient but not necessary. As usual, let's fix the parameters:

- $\pi^* = 0.5$
- $c = 0.1$

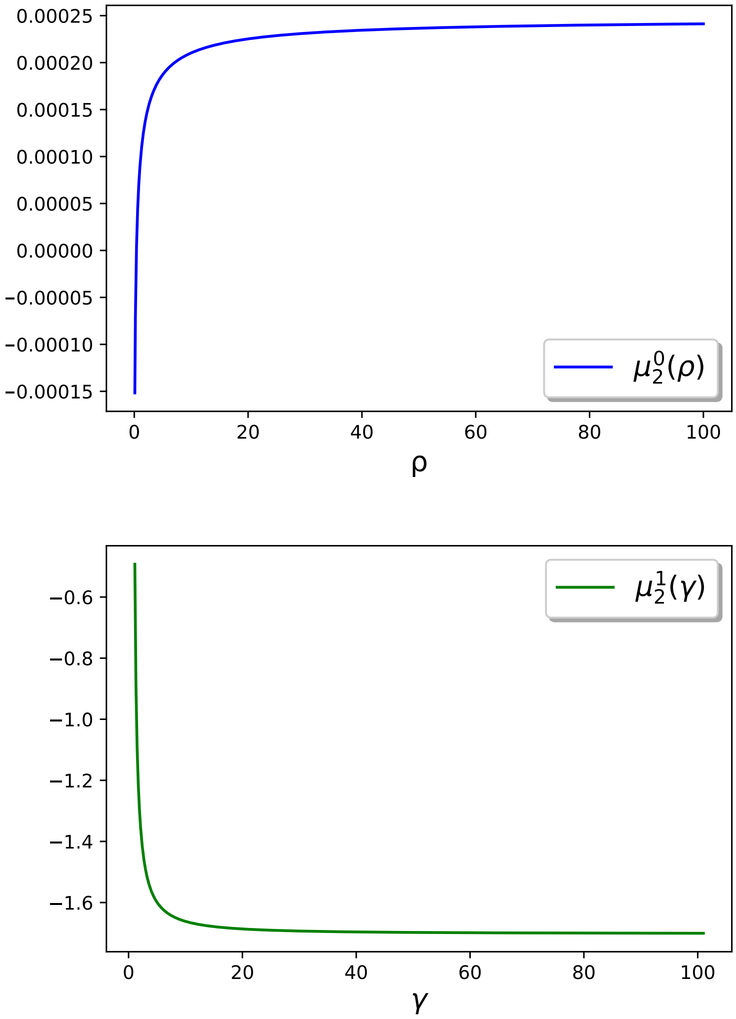

Figure 3.11. Expect drift of the log difference of agent 2's wealth

- $\pi^1 = 0.95$
- $\pi^2 = 0.05$

Proposition 4 can be applied to both agents, as shown in figures 3.13 and 3.14: both the Kelly traders are expected to survive with probability 1.
At this point, two ways are left opened for the system to reach the equilibrium: the noise trader could disappear or coexist with both the Kelly traders. Anyway, in this case, no hypothesis of the propositions are satisfied by $\mu_N^0(k)$ nor $\mu_N^1(k)$ (figure 3.15): while $\mu_N^1(k)$ is negative $\forall k$, $\mu_N^0(k)$ is negative for values of k inside the interval $(k_1; k_2)$ and positive for values of k outside that interval, with $k_1 \approx 0.14$ and $k_2 \approx 7.08$. Notice that these two angular coefficients are symmetric with respect to the bisector $k = 1$: this follows from the

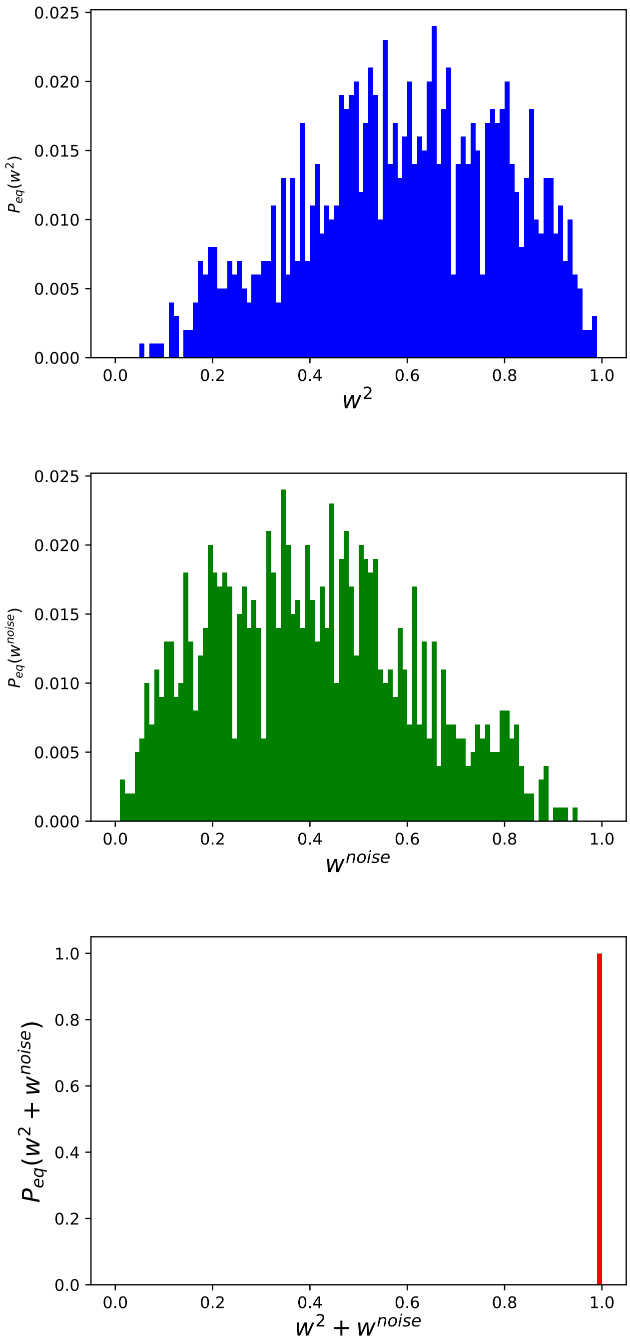

Figure 3.12. Distributon of w^2 and $\tilde{w}$ at equilibium

fact that π^1 and π^2 are symmetric with respect to π^*.
Figure 3.16 represents the equilibrium distributions of the wealths obtained by numerical

simulations, and it is evident that it confirms that the two Kelly agents coexist, as predicted, and that the noise trader disappears. Here $\mathbf{E}^{eq}[w^1] = \mathbf{E}^{eq}[w^2] \approx 0.50$, $V^{eq}[w^1] = V^{eq}[w^2] \approx 0.01$ and $StdDev^{eq}[w^1] = StdDev^{eq}[w^2] \approx 0.11$.
The heuristic argument I am going to propose to explain this behavior is the following.
It has been seen by means of proposition 4 that the system will not collapse on the axis $w^1 = 0$ nor $w^2 = 0$, and that the agents bet according to the same, but symmetric, rule: $1 - \alpha^i(p_t) = \alpha^j(p_t)$, $i, j = \{1,2\}$, $i \neq j$. It is then reasonable to expect that the system will spend most of the time in the region of the plane in which w^1 and w^2 are "similar", or, at least, have the same order of magnitude, i.e. $k = \frac{w^2}{w^1} = O(1)$. But this region it's just the one in which $\mu_N^0(k)$ is negative: in this slice of plane, $\mu_N^1(k)$ and $\mu_N^0(k)$ share the same negative sign, which means that the noise trader is going to vanish.

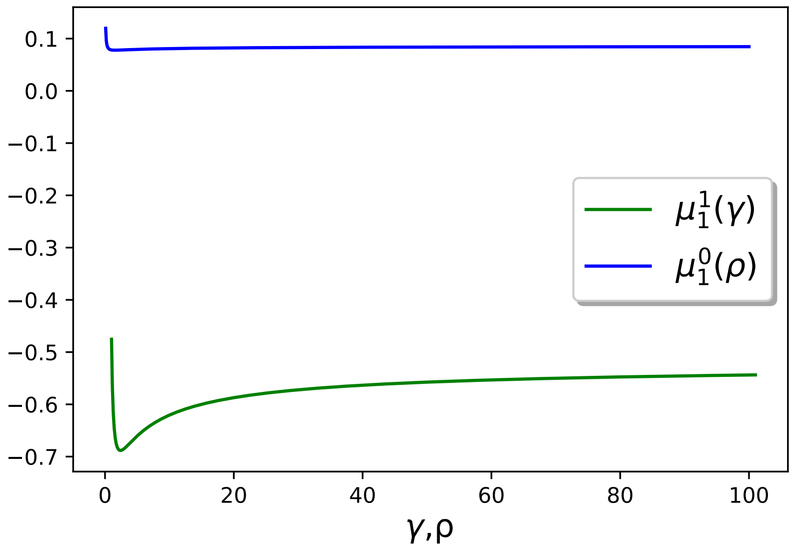

Figure 3.13. Expect drift of the log difference of agent 1's wealth

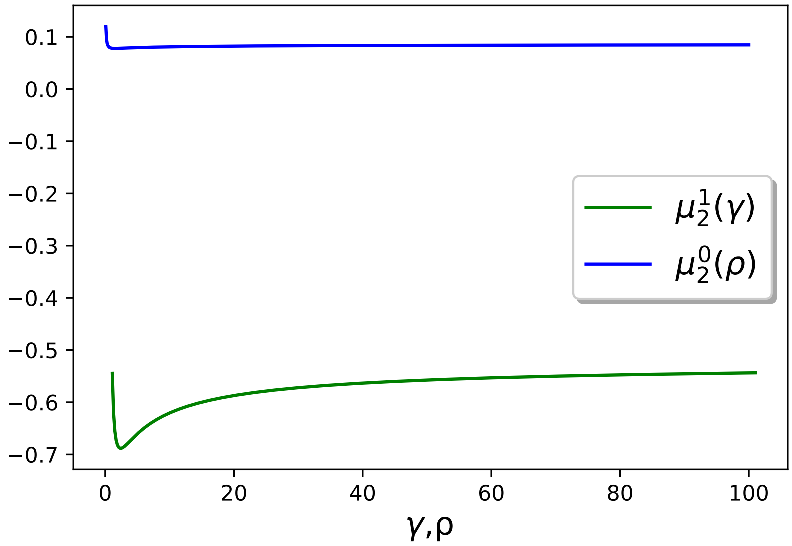

Figure 3.14. Expect drift of the log difference of agent 2's wealth

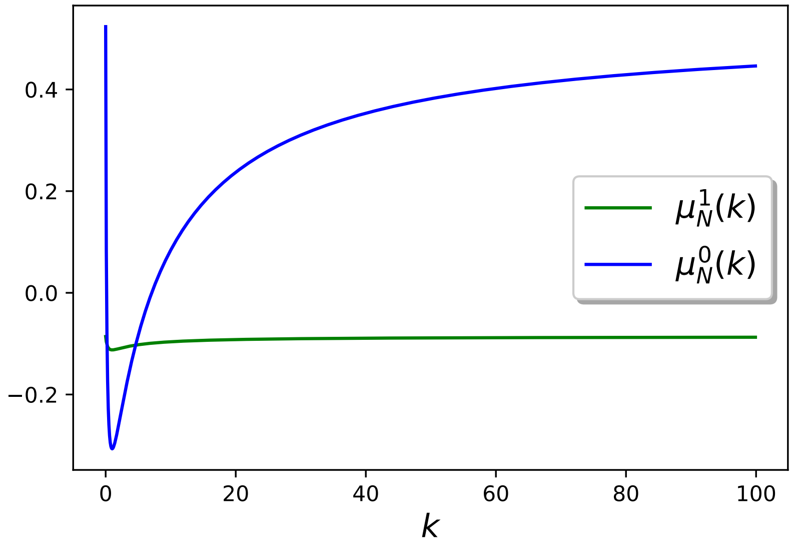

Figure 3.15. Expect drift of the log difference of noise trader's wealth

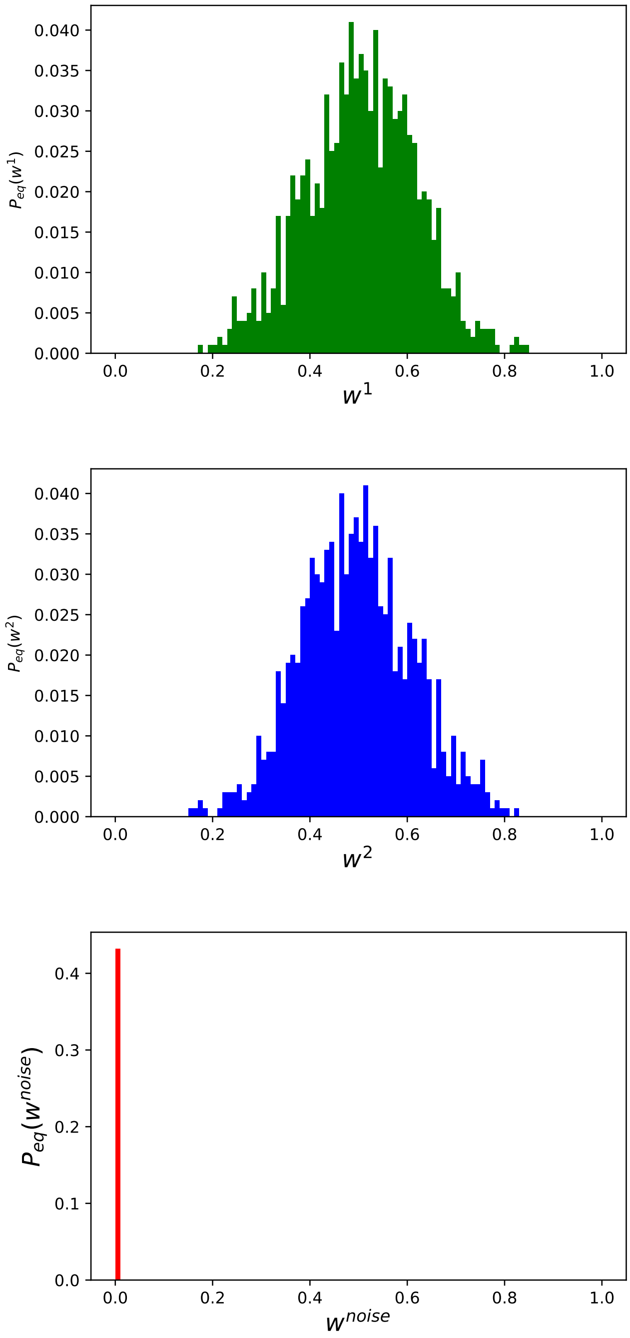

Figure 3.16. Distributon of w^1, w^2 and $\tilde{w}$ at equilibium

Conclusion

In this work I have studied the equilibrium properties of a repeated prediction market in which two fractional Kelly agents and a noise trader invest.
The peculiarity of a noise trader is that he bets according to a rule which has a random component: I have focussed on the most entropic agent, i.e. the one who invests according to a uniform random variable in (0,1).
It has been shown that there are situations (which translate into sets of parameters) for which the noise trader is not trivially thrown out of the market, as someone could intuitively think: instead, he could coexist or even dominate.
The choice of the most entropic agent as the irrational one makes sense because, if I would have found that his presence in the market is not negligible, then the same could be true for other wiser irrational ones. Indeed, what it has been shown in this work is that noise traders can heavly influence the market, the stock prices and their informational efficiency, and they can not safely neglected.

Appendix

3.5 Derivation of the limits of μ_N

$$\mu_N(w^1_{t-1}, w^2_{t-1}) = \mathbf{E}_{s_t,\eta}\left[\log \frac{\tilde{w}_t}{\tilde{w}_{t-1}} - \log \frac{1-\tilde{w}_t}{1-\tilde{w}_{t-1}}\middle| w^1_{t-1}, w^2_{t-1}\right] =$$

$$= \mathbf{E}_\eta\left[\pi^* \log\left(\frac{\eta(w^1_{t-1}+w^2_{t-1})}{\alpha^1(p_t)w^1_{t-1}+\alpha^2(p_t)w^2_{t-1}}\right) + (1-\pi^*)\log\left(\frac{(1-\eta)(w^1_{t-1}+w^2_{t-1})}{(1-\alpha^1(p_t))w^1_{t-1}+(1-\alpha^2(p_t))w^2_{t-1}}\right)\right]$$

$$k = \frac{w^2_{t-1}}{w^1_{t-1}}$$

3.5.1 $\mu^1_N(k)$

In this case $\tilde{w}_{t-1} \to 1$. Then,

$$p_t = \frac{c(\pi^1 w^1_{t-1} + \pi^2 w^2_{t-1}) + \eta(1 - w^1_{t-1} - w^2_{t-1})}{c(w^1_{t-1} + w^2_{t-1}) + (1 - w^1_{t-1} - w^2_{t-1})} \approx \eta$$

And

$$\alpha^1(p_t \to \eta) \approx c\pi^1 + (1-c)\eta$$

$$\alpha^2(p_t \to \eta) \approx c\pi^2 + (1-c)\eta$$

Substituting these expressions one obtains:

$$\mu^1_N(k) = \mathbf{E}_\eta\left[\pi^* \log\left(\frac{\eta(1+k)}{c(\pi^1+k\pi^2)+\eta(1-c)(1+k)}\right) + (1-\pi^*)\log\left(\frac{(1-\eta)(1+k)}{(1+k)-c(\pi^1+k\pi^2)-\eta(1-c)(1+k)}\right)\right]$$

Now the integral over η has to been performed to compute the expected value.
One can separate the logarithms of fractions into differences of logarithms in order to simplify the calculations. Here three types of integral appear:

1.
$$\int_0^1 \log(a\eta + b)\, d\eta = -\frac{b}{a}\log b + \left(\frac{b}{a} - 1\right)\log(b+a) - 1$$

2.
$$\int_0^1 \log(a_1(1-\eta) + b_1)\, d\eta = -\frac{b_1}{a_1}\log b_1 + \left(\frac{b_1}{a_1} - 1\right)\log(b_1+a_1) - 1$$

3.
$$\int_0^1 \log(-a_2\eta + b_2)\, d\eta = \frac{b_2}{a_2}\log b_2 - \left(\frac{b_2}{a_2} - 1\right)\log(b_2 - a_2) - 1$$

The first "π^* term" is of the first kind, with $a = (1+k)$ and $b = 0$; the second one is of the first kind too, with $a = (1-c)(1+k)$ and $b = c(\pi^1 + k\pi^2)$.
The first "$1-\pi^*$ term" is of the second type, with $a_1 = (1+k)$ and $b_1 = 0$; the second one is of the third type, with $a_2 = (1-c)(1+k)$ and $b_2 = (1+k) - c(\pi^1 + k\pi^2)$.
Merging these partial results, one gets the final one.

3.5.2 $\mu_N^0(k)$

In this case $\tilde{w}_{t-1} \to 0$. Then,
$$p_t \approx \frac{\pi^1 + k\pi^2}{1+k}$$
And
$$\alpha^1\left(p_t \to \frac{\pi^1 + k\pi^2}{1+k}\right) \approx c\pi^1 + (1-c)\frac{\pi^1 + k\pi^2}{1+k}$$
$$\alpha^2\left(p_t \to \frac{\pi^1 + k\pi^2}{1+k}\right) \approx c\pi^2 + (1-c)\frac{\pi^1 + k\pi^2}{1+k}$$
Substituting these expressions and manipulating it is possible to get:
$$\mu_N^0(k) = \mathbf{E}_\eta\left[\pi^*\log\left(\frac{\eta(1+k)}{\pi^1 + k\pi^2}\right) + (1-\pi^*)\log\left(\frac{(1-\eta)(1+k)}{1+k-\pi^1 - k\pi^2}\right)\right]$$
Two kinds of integral appear here.
The first logarithm gives an integral of the first type, with $a = \frac{1+k}{\pi^1 + k\pi^2}$ and $b = 0$.
The second one gives an integral of the second type, with $a_1 = \frac{1+k}{1+k-\pi^1-k\pi^2}$ and $b_1 = 0$.
The final result is then given by merging these results.

3.6 Derivation of the limits of μ_i

$$\mu_i(w_{t-1}^1, w_{t-1}^2) = \mathbf{E}_{s_t,\eta}\left[\log\frac{w_t^i}{w_{t-1}^i} - \log\frac{1-w_t^i}{1-w_{t-1}^i}\middle| w_{t-1}^1, w_{t-1}^2\right] =$$
$$= \mathbf{E}_\eta\left[\pi^*\log\left(\frac{\alpha^i(p_t)(1-w_{t-1}^i)}{\alpha^j(p_t)w_{t-1}^j + \eta(1-w_{t-1}^i - w_{t-1}^j)}\right) + (1-\pi^*)\log\left(\frac{(1-\alpha^i(p_t))(1-w_{t-1}^i)}{(1-\alpha^j(p_t))w_{t-1}^j + (1-\eta)(1-w_{t-1}^i - w_{t-1}^j)}\right)\right]$$

3.6.1 $\mu_i^1(\gamma)$

In this case $w_{t-1}^i \to 1$. Then,
$$p_t = \frac{c(\pi^1 w_{t-1}^1 + \pi^2 w_{t-1}^2) + \eta(1 - w_{t-1}^1 - w_{t-1}^2)}{c(w_{t-1}^1 + w_{t-1}^2) + (1 - w_{t-1}^1 - w_{t-1}^2)} \approx \pi^i$$

And,

$$\alpha^i(p_t \to \pi^i) \approx \pi^i$$

$$\alpha^j(p_t \to \pi^i) \approx c\pi^j + (1-c)\pi^i$$

Let's now define

$$\gamma = \frac{1 - w^i_{t-1}}{w^j_{t-1}} \in (1, +\infty)$$

Substituting these expressions in the expected value:

$$\mu^1_i(\gamma) = \mathbf{E}_\eta\Big[\pi^* \log\Big(\frac{\pi^i\gamma}{(c\pi^j + (1-c)\pi^i) + \eta(\gamma-1)}\Big) + (1-\pi^*)\log\Big(\frac{(1-\pi^i)\gamma}{(1 - c\pi^j - (1-c)\pi^i) + (1-\eta)(\gamma-1)}\Big)\Big]$$

Now the integral over η has to be performed in order to evaluate the average.
The first term is an integral of the first type (referring to the previous section), where $a = \frac{\gamma-1}{\pi^i\gamma}$ and $b = \frac{(c\pi^j+(1-c)\pi^i)+\eta(\gamma-1)}{\pi^i\gamma}$.
The second term is instead an integral of the second type, where $a_1 = \frac{\gamma-1}{(1-\pi^i)\gamma}$ and $b_1 = \frac{1-c\pi^j-(1-c)\pi^i}{(1-\pi^i)\gamma}$.
As usual, merging this integrals gives the final result.

3.6.2 $\mu^0_i(\rho)$

In this case $w^i_{t-1} \to 0$.
Define $\rho = \frac{1-w^j_{t-1}}{w^j_{t-1}} \in (0, +\infty)$. Then:

$$p_t = \frac{c(\pi^1 w^1_{t-1} + \pi^2 w^2_{t-1}) + \eta(1 - w^1_{t-1} - w^2_{t-1})}{c(w^1_{t-1} + w^2_{t-1}) + (1 - w^1_{t-1} - w^2_{t-1})} \approx \frac{c\pi^j w^j_{t-1} + \eta(1 - w^j_{t-1})}{c w^j_{t-1} + 1 - w^j_{t-1}} = \frac{c\pi^j + \eta\rho}{c+\rho}$$

And,

$$\alpha^i\Big(p_t \to \frac{c\pi^j + \eta\rho}{c+\rho}\Big) \approx c\pi^i + (1-c)\frac{c\pi^j + \eta\rho}{c+\rho}$$

$$\alpha^j\Big(p_t \to \frac{c\pi^j + \eta\rho}{c+\rho}\Big) \approx c\pi^j + (1-c)\frac{c\pi^j + \eta\rho}{c+\rho}$$

Substituting, one gets:

$$\mu^0_i(\rho) = \mathbf{E}_\eta\Big[\pi^* \log\Big(\frac{(c\pi^i + (1-c)\frac{c\pi^j+\eta\rho}{c+\rho})(\rho+1)}{c\pi^j + (1-c)\frac{c\pi^j+\eta\rho}{c+\rho} + \eta\rho}\Big) + (1-\pi^*)\log\Big(\frac{(1 - c\pi^i - (1-c)\frac{c\pi^j+\eta\rho}{c+\rho})(\rho+1)}{1 - c\pi^j - (1-c)\frac{c\pi^j+\eta\rho}{c+\rho} + (1-\eta)\rho}\Big)\Big]$$

Let's start with the first addend: both the numerator and the denominator give integrals of the first type.
For the numerator, we have $a = (\rho+1)(1-c)\frac{\rho}{c+\rho}$ and $b = c\pi^i + \frac{(1-c)c\pi^j}{c+\rho}$.
For the denominator, we have $a = \rho + \frac{\rho(1-c)}{c+\rho}$ and $b = c\pi^j + \frac{(1-c)c\pi^j}{c+\rho}$.
The difference of this two integrals gives the term which is multiplied by π^*.
In the second addend, the numerator of the logarithm gives an integral of the third type,

while the denominator of the second type.
For the numerator, we have $a_2 = \frac{\rho(1-c)(\rho+1)}{c+\rho}$ and $b_2 = (\rho+1)(1-c\pi^i - \frac{c\pi^j(1-c)}{c+\rho})$. For the denominator, we have $a_1 = \rho + \frac{\rho(1-c)}{c+\rho}$ and $b_1 = 1 - c\pi^j - \frac{c\pi^j(1-c)}{c+\rho} + \rho$.
Now the term multiplied by $1-\pi^*$ has been obtained.
Summing these two terms, one gets the final result.

3.7 Derivation of the Propositions

Since

$$\frac{w_t^i}{w_{t-1}^i} \neq 1 \;\; \forall s_t$$

and

$$\frac{\tilde{w}_t}{\tilde{w}_{t-1}} \neq 1 \;\; \forall s_t$$

there is no possibility for a deterministic fixed point to exist.
Let's define now $z_t^i = \log \frac{w_t^i}{1-w_t^i}$ and $\tilde{z}_t = \log \frac{\tilde{w}_t}{1-\tilde{w}_t}$. Notice that there are no set of w_{t-1}^1 and w_{t-1}^2 for which the values $z_t^i - z_{t-1}^i$ or $\tilde{z}_t - \tilde{z}_{t-1}$ diverge, i.e. these are bounded increments processes.
Moreover, notice that

$$\lim_{z\to+\infty} \mathbf{E}\left[z_t^i - z_{t-1}^i \middle| z_{t-1}^i = z,\; \frac{1-w_{t-1}^i}{w_{t-1}^j} = \gamma \right] = \mu_i^1(\gamma)$$

$$\lim_{z\to-\infty} \mathbf{E}\left[z_t^i - z_{t-1}^i \middle| z_{t-1}^i = z,\; \frac{1-w_{t-1}^j}{w_{t-1}^j} = \rho \right] = \mu_i^0(\rho)$$

$$\lim_{z\to+\infty} \mathbf{E}\left[\tilde{z}_t - \tilde{z}_{t-1} \middle| \tilde{z}_{t-1} = z,\; \frac{w_{t-1}^2}{w_{t-1}^1} = k \right] = \mu_N^1(k)$$

$$\lim_{z\to-\infty} \mathbf{E}\left[\tilde{z}_t - \tilde{z}_{t-1} \middle| \tilde{z}_{t-1} = z,\; \frac{w_{t-1}^2}{w_{t-1}^1} = k \right] = \mu_N^0(k)$$

Then, if $\mu_i^1(\gamma) > 0 \; \forall\gamma$ and $\mu_i^0(\rho) > 0 \; \forall\rho$, Theorem 3.1 of [2] assures that z_t^i diverges to $+\infty$, and so proposition 2 is proved.
The same is true for the noise trader: if $\mu_N^1(k) > 0$ and $\mu_N^0(k) > 0 \; \forall k$, $\tilde{z}_t$ diverges to $+\infty$, and so proposition 1 is proved.
If $\mu_i^1(\gamma) < 0 \; \forall\gamma$ and $\mu_i^0(\rho) > 0 \; \forall\rho$, Theorem 2.2 of [2] assures that z_t^i is persistent, and, if $\pi^* \in (0,1)$, z_t^i can both increase or decrease. Proposition 4 is then proved.
For what regards the noise trader, if $\mu_N^1(k) < 0$ and $\mu_N^0(k) > 0 \; \forall k$, $\tilde{z}_t$ is persistent. Moreover, $\tilde{z}_t$ can always increase or decrease, so proposition 3 is proved.
At the end, if $\mu_i^1(\gamma) < 0 \; \forall\gamma$ and $\mu_i^0(\rho) < 0 \; \forall\rho$ Corollary 3.1 of [2] tells us that z_t^i diverges to $-\infty$. Moreover, in this situation, the 2D system reduces to a 1D one, in which there is only a Kelly trader and the noise trader. Then, other propositions can be applied considering only values of γ, ρ, k corresponding to the new system with reduced dimensionality. Proposition 6 is then proved.
For the same reason, if $\mu_N^1(k) < 0$ and $\mu_N^0(k) < 0 \; \forall k$, then $\tilde{z}_t$ diverges to $-\infty$, and the system collapses to the one studied in [1], i.e. where only two Kelly traders invest.

3.8 Reaching equilibrium

In all my numerical simulations, I have applied this method to be sure that the system had reached the equilibrium.
First, I let the simulation run for T time steps. Then, I save the updated w^1_T, w^2_T, $\tilde{w}_T$ and let the simulation run for T steps more. After this passage, the system has evolved for $2T$ time steps in total.
I compare the distributions of the updated wealths w^1_{2T}, w^2_{2T}, $\tilde{w}_{2T}$ with the previous ones: if they are very close, I stop the simulation and I say that the equilibrium has been reached. If they are not close, I save w^1_{2T}, w^2_{2T}, $\tilde{w}_{2T}$ and let the simulation run for $2T$ steps more, and so on.

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