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Master of Science programme in Physics of Complex Systems

Master Degree Thesis

Non-equilibrium dynamics in the fully connected Bose-Hubbard model

the emergence of dynamical phase transitions and synchronization



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Summary

In the context of out of equilibrium dynamics of isolated interacting quantum many-body systems, we study the mean-field discretized and nonlinear Gross-Pitaevskii equation of motion for a fully connected (or all-to-all coupled) Bose-Hubbard model describing a large population of bosons on a lattice of generic dimension V , with potential experimental applications ranging from ultra-cold atomic gases in optical traps to systems of Josephson junctions. Considering a quench on the system Hamiltonian, we describe the time evolution of a system initialized to the Mott insulating phase and taken to the strong superfluid regime, finding a sharp change in the dynamical behavior when varying the quench intensity across a critical value, that is a dynamical phase transition. Our results generalize the study presented in some previous papers focusing on the particular $V = 2$ case [1–6], $V = 3$ case [7–9] and 1D lattice [10] and place particular emphasis on the long-time relaxation of some dynamical order parameters in the large V limit. On the side of the dynamical phase transition corresponding to a quench towards the strong superfluid regime, we find such relaxation to consist of a π -synchronization of the classical bosonic phases, that we find to compete against the possible site-dependent disorder of the Hamiltonian in what, in analogy with the notorious Kuramoto model for nonlinear coupled oscillators, can be referred to as a synchronization transition. Additionally, in support of the validity of the mean field approximation far from the transition, we report a study beyond mean field based on the Bogoliubov-de Gennes method.

The work is organized as follows. In chapter 1 we introduce the work reviewing the state of the art, motivating the study and outlooking the results. In chapter 2 we introduce the system Hamiltonian and derive the correspondent Heisenberg dynamical equations, sequently obtaining the Gross-Pitaevskii equation within mean-field approximation. In chapter 3 we find the stationary configurations (that is fixed points) in the non-disordered case and describe the topology of the set that they constitute in the phase space for all the possible numbers of sites V . In chapter 4 we study the short-time stability of the stationary configurations linearizing the Gross-Pitaevskii equation and diagonalizing the associated Jacobian matrix. In chapter 5 we numerically observe the long-time chaotic dynamics for small V and the thermalization of some dynamical order parameters for large V , introducing the notion of π -synchronization of the bosonic phases and studying how it can be destroyed by the disorder. In chapter 6 we study the quantum fluctuations beyond mean-field with the Bogoliubov-de Gennes method, arriving to a better comprehension of the limitations of the mean-field approximation. Finally, in chapter 7 we conclude briefly reviewing our results and outlooking possible future directions of study.

Contents

List of Figures	VII
1 Introduction	1
2 Model	5
2.1 Hamiltonian	5
2.2 System initialization: the quench	6
2.3 Heisenberg equations of motion	6
2.4 Mean-field approximation: the Gross-Pitaevskii equation	7
2.5 $V \rightarrow \infty$ limit and continuous equations	10
2.6 Analogue classical system	10
3 Stationary configurations	13
3.1 Stationarity condition	14
3.2 Stationarity in the two-site model	14
3.3 Stationary configurations for $V > 2$	16
3.3.1 $r = 0$ configurations	16
3.3.2 Aligned (superfluid) and π -aligned configurations	17
3.4 Topology of the stationary set	18
4 Stability of the stationary configurations	21
4.1 The effects of conserved quantities	21
4.2 The linearization procedure	22
4.3 Stability in the two-site model	24
4.4 Jacobian for $V \geq 3$	28
4.5 Stability of the uniform configuration	29
4.5.1 Stability of $r = 0$	33
4.5.2 Numerical results	34
4.6 Stability of the superfluid configuration	35
4.7 Stability of the symmetric π -aligned configuration	37
4.7.1 Numerical results	38
4.8 Stability of the delta configuration for $V = 4$	39
4.9 Stability of the uniform configuration in the $V \rightarrow \infty$ limit	40
4.10 The dynamical phase transition as a classical bifurcation	42

4.11	Final remarks	42
5	Long-time nonlinear dynamics	45
5.1	$\tau < 4$ long-time dynamics	45
5.2	$\tau > 4$ long-time dynamics	46
5.3	Competition between hopping strength and disorder	49
5.4	Promising directions for analytical approach	51
6	Quantum fluctuations beyond mean-field	53
6.1	Linearized dynamical equations for the quantum fluctuations	53
6.2	Uniform configuration	55
6.3	Superfluid and symmetric π -aligned configurations	57
6.4	Delta configuration	57
6.5	Stationary configurations with $r = 0$ and $V > 4$	58
6.6	Limitations of the approach and final remarks	59
7	Conclusions	61
	Appendices	63
A	Dynamical equations	65
A.1	Explicitation of the Heisenberg equation of motion	65
A.2	From the dynamical equations for ψ_k and ψ_k^* to the ones for ρ_k and θ_k . . .	66
A.3	$u \rightarrow 0$ limit in non-disordered fully connected Bose-Hubbard model	67
B	π-aligned stationary configurations	69
C	Quantum fluctuations	73
C.1	Quantum fluctuations in the delta configuration	73
C.2	Physically-motivated symmetries of J_{QF}	74

List of Figures

2.1	Graphical representation of the bosonic variables	12
3.1	Graphical representation of some stationary configurations	19
3.2	Schematic representation of the stationary configurations in the phase space	20
4.1	Two-site Bose-Hubbard phase portrait	27
4.2	Dynamical phase transition for the uniform configuration	35
4.3	Dynamical phase transition for the symmetric π -aligned configuration . . .	39
5.1	Long-time nonlinear dynamics for $\tau > 4$	46
5.2	Stationary configurations for $V = 4$	47
5.3	Nonlinear dynamics and the synchronization	48
5.4	Schema of the trajectories in the phase space for $V \gg 1$	50
5.5	Synchronization transition in $V \gg 1$ fully connected Bose-Hubbard model .	52
6.1	Linear stability of the quantum fluctuations	59
B.1	Stationary π -aligned configurations	71

Abbreviations

BH	Bose-Hubbard.
DC	Delta Configuration.
DEs	Dynamical Equations.
DFT	Discrete Fourier Transform.
DOP	Dynamical Order Parameter.
DPT	Dynamical Phase Transition.
DST	Dynamical Systems Theory.
FC	Fully Connected.
FCBHM	Fully Connected Bose-Hubbard Model.
FP	Fixed Point.
GPE	Gross-Pitaevskii Equation.
IC	Initial Condition.
MF	Mean-Field.
MI	Mott Insulator.
PAC	π -Aligned Configuration.
QF	Quantum Fluctuation.
SC	Stationary Configuration.
SF	SuperFluid.
SFC	SuperFluid Configuration.
SPAC	Symmetric π -Aligned Configuration.
SS	Stationary Set.
UC	Uniform Configuration.

Frequent notation

V	Number of lattice sites.
H	Quantum Hamiltonian of the system.
$a_j^\dagger$	Bosonic creation operator for the j -th lattice site.
a_j	Bosonic annihilation operator for the j -th lattice site.
n_j	Bosonic number operator for the j -th lattice site.
$t_{i,j}$	Hopping strength between sites i and j .
u	In-site two-body interaction energy.
μ	Chemical potential.
ω_j	Chemical potential disorder at site j .
σ_ω	Standard deviation of the chemical potential disorder (or disorder strength).
i, j, k	Indexes labeling the lattice sites.
t	Time.
i	Imaginary unit.
$\hat{N}$	Total number of particles operator.
N	Expected total number of particles.
$\langle \bullet \rangle$	Expectation value of the operator $\bullet$.
τ	Hopping strength in the fully connected model.
$\hat{\Psi}$	Average over the sites of a_j .
$\langle \bullet_j \rangle_V$	Average over the sites of the site dependent $\bullet_j$.
ψ_j	Expectation value of a_j .
ρ_j	Squared modulus of ψ_j .
θ_j	Phase of ψ_j .
ρ_0	Expected average number of particles per site.
Ψ	Average over the sites of ψ_j .
r	Modulus of Ψ .
ϕ	Phase of Ψ .
s	Continuous site index in the $V \rightarrow \infty$ limit.
$\rho(s, t)$	$\rho_j(t)$ in the $V \rightarrow \infty$ limit.
$\theta(s, t)$	$\theta_j(t)$ in the $V \rightarrow \infty$ limit.
H_{CL}	Classical mean-field Hamiltonian.
E	Average mean-field energy per site.
Ω	Global (that is common to all sites) phase rotation rate.

θ	Phase difference between ψ_1 and ψ_2 in the two-site model.
δ	Normalized population imbalance in the two-site model.
Δ	Parameter used in the definition of the delta configuration.
α	Fraction of sites with equal phase in a π -aligned stationary configuration.
J	Jacobian matrix of the dynamical equations.
δ_j	Population imbalance at site j with respect to ρ_0 .
r_0	Initial r at time 0.
x_j	Modulus of ψ_j .
$\vec{y}$	$2V$ -dimensional column vector.
$\vec{y}^{(1)}$	First half (or quarter) of the vector $\vec{y}$.
$\vec{y}^{(2)}$	Second half (or quarter) of the vector $\vec{y}$.
$\vec{v}_n$	n -th eigenvector of the Jacobian matrix.
λ_n	n -th eigenvalue of the Jacobian matrix.
τ_c	Critical hopping strength.
$\delta_{k,j}$	Kronecker delta.
$\tilde{v}_q$	Discrete Fourier transform of the function v_j of the site j with respect to the wavenumber q .
q	Fourier wavenumber for the discrete Fourier transform.
$\text{Re}\{\bullet\}$	Real part of the c -number $\bullet$.
$\text{Im}\{\bullet\}$	Imaginary part of the c -number $\bullet$.
$(\bullet_j)_q$	Discrete Fourier transform of the function $\bullet_j$ of the site j with respect to the wavenumber q .
$P(\lambda)$	Characteristic polynomial of the Jacobian matrix.
$\xi(s)$	Perturbation of the phases in the uniform configuration in the $V \rightarrow \infty$ limit.
$\delta(s)$	Perturbation of the bosons number in the uniform configuration in the $V \rightarrow \infty$ limit.
ξ_j	Perturbation of the phase of the j -th site in the uniform configuration.
S	π -alignment dynamical order parameter.
$\sigma_{\omega,c}$	Critical disorder strength destroying the π -alignment.
$\vec{B}$	$2V$ -dimensional column vector describing the state of the quantum fluctuations.
J_{QF}	Jacobian matrix of the dynamical equations for the quantum fluctuations.

Chapter 1

Introduction

The theory of interacting many-body quantum systems at equilibrium is well-established, accounting for the explanation of notorious physical phenomena such as quantum phase transitions, that are sharp changes of the ground state of an Hamiltonian when its parameters are changed across some critical values. Paradigmatic is the Bose-Hubbard (BH) model, describing a system of bosons on a lattice at $T = 0$ temperature and taking into account an in-situ particle repulsion and a site-to-site tunneling (or hopping). Depending on the values of the parameters of its Hamiltonian, the ground state of the system can either be in a Mott Insulator (MI) phase, characterized by integer boson densities, existence of a gap for particle-hole excitation and zero compressibility, or in a SuperFluid (SF) phase, characterized by long range coherence [11, 12].

On the other hand, the behavior of such quantum many-body systems is far less understood when it comes to the out of equilibrium regime, whose relevance has rapidly grown triggered by the significant experimental progresses of the last two decades. In this context, a particularly promising experimental setup to simulate theoretical models is represented by gases of ultra-cold neutral atoms confined in optical traps [13–20] (for a review on ultra-cold gases see reference [21], for a review on cold atoms out of equilibrium see reference [22]). In addition, huge advances in the engineering of the Josephson junctions make them another promising possibility for the simulation of such models [23]. To take these systems to the non-equilibrium regime, the most established protocol (both at the theoretical and at the experimental level) is the so called quantum *quench*, consisting of a sudden change of the Hamiltonian describing the system from H_i to H_f at time $t = 0$ [24–31]. This corresponds to initializing the system to the ground state of the Hamiltonian H_i at $t = 0$ and letting it evolve for $t > 0$ under the Hamiltonian H_f , that is out of equilibrium. In the case of cold atoms in optical traps, this can be achieved tuning the confining lasers whereas for Josephson junctions-based setups it could be allowed by a sudden change of the involved magnetic fluxes. Importantly, thanks to the high degree of isolation from the environment on the experimental timescales, such systems can be considered to be almost isolated. Their study reveals the emergence of interesting behaviours among which we recall the quantum Dynamical Phase Transitions (DPTs), identified by a sharp change of the dynamical behavior for different quench strength [29–32] (for a review on out of equilibrium dynamics of isolated interacting quantum systems see reference

[33]), and synchronization phenomena [34].

A well-established approach to the study of the dynamics of interacting many-bosons systems on a V -dimensional lattice consists in reducing the Heisenberg equation of motion to the discrete nonlinear Gross-Pitaevskii Equation (GPE) via a Mean-Field (MF) substitution of the $2V$ bosonic creation and annihilation operators $a_j^\dagger$ and a_j ($j = 1, 2, \dots, V$ labeling the lattice site) with the $2V$ c -numbers ψ_j and ψ_j^* [2, 4, 5, 8, 10, 34, 35]. Under this approximation, Polkovnikov *et al.* studied the time evolution of bosons on a one-dimensional chain [10] whereas other groups, motivated by analytical tractability [1–9] and experimental viability [36–40], focused on the study of bosonic dimers and trimers (corresponding to $V = 2$ and $V = 3$ respectively). Already for a small number of sites V and at the MF level the dynamics of such systems is particularly rich, revealing for instance the emergence of macroscopic self-trapping [1, 2, 36] and chaos [8, 9, 41].

The primary goal of the present work is to extend the aforementioned MF results to the case of a Fully Connected (FC) (that is all-to-all coupled) BH model with a generic number of sites V , also complementing the study done, without recurring to the GPE, by Sciolla and Biroli on the same model in the $V \rightarrow \infty$ limit [29, 30]. Considering a quantum quench from a small to a large hopping strength, we investigate the out of equilibrium dynamics in the strong SF regime of a system initialized to the MI phase. Focusing on the individuation of the Fixed Points (FPs) of the discrete GPE and on the study of their stability we are able to identify regions of the parameters space leading to qualitatively completely different time evolutions of the initial MI, spotting out the existence of a DPT. For small V we observe chaotic behaviours on both sides of the transition whereas for large V we show some macroscopic variables (namely the Dynamical Order Parameters (DOPs)), to thermalize, that is to relax to some finite value at long-time (unless some residual fluctuations). Indeed, this is a common feature of isolated interacting many-body quantum systems, that we stress to occur even in absence of a thermal bath and a reservoir [42–50]. In particular, on the side of the DPT with large hopping strengths (that is tunneling rates), we observe the tendency of the phases of the MF bosonic variables to acquire a certain degree of π -alignment, such that the phases can be divided into two halves with respective preferred orientation θ_0 and $\theta_0 + \pi$. We refer to such tendency as π -synchronization, and introduce a π -alignment parameter S to quantify it.

Indeed, in a bosonic system more in general the phases of the c -numbers associated within MF to the bosonic variables can be interpreted as phases of classical nonlinearly coupled oscillators that may, in some circumstances, synchronize. For instance, Witthaut *et al.* have recently demonstrated that a particular class of bosonic models can be recasted, in MF, to the notorious Kuramoto model for classically coupled nonlinear oscillators, where the competition between coupling and site dependent disorder drives a *synchronization transition* [34, 51–53]. Inspired by this idea, in analogy with Kuramoto dynamics, we introduce a site-dependent disorder in the model and look at how it affects the aforementioned π -synchronization, remarkably finding evidences of a synchronization transition driven by the competition between the site-to-site coupling (that is hopping strength) and the disorder also in the case of the Bose-Hubbard model. We stress that the occurrence of synchronization among the phases of the MF bosonic variables is somehow an intrinsic property of the isolated quantum system, i.e. it is not due to the presence of

dissipation and external driving as usually considered for populations of quantum dissipative oscillators [54–58]. Additionally, using the Bogoliubov-de Gennes method we study the dynamics of the Quantum Fluctuations (QFs) beyond MF [59], finding that the MF approximation is more reliable for quenches to the strong SF regime, that is far from the MF DPT.

Chapter 2

Model

The goal of this chapter is to derive some approximated and tractable Dynamical Equations (DEs) describing the time evolution of a system of interacting many-body bosons on a FC lattice, that is a lattice where particles can hop from one site to whatever other site with equal tunneling rate. The choice of a FC model is motivated by the analytical tractability allowed by its symmetries and by the fact that it represents an approximate (MF) description of a finite-dimensional system [29, 30]. By means of a MF approximation consisting of the substitution of the bosonic operators with c -numbers, we are able to recast the Heisenberg equations of motion into a classical, nonlinear and discrete GPE of motion for coupled oscillators with variable lengths and phases.

2.1 Hamiltonian

We consider a disordered version of the BH model for bosons on a V -dimensional lattice, described by the following Hamiltonian

$$H = - \sum_{\substack{i,j=1 \\ i \neq j}}^V t_{i,j} a_i^\dagger a_j + \frac{u}{2} \sum_{j=1}^V n_j(n_j - 1) - \mu \sum_{j=1}^V n_j - \sum_{j=1}^V \omega_j n_j \quad (2.1)$$

where $a_j^\dagger$ and a_j are respectively the bosonic creation and annihilation operators at sites j , satisfying the bosonic commutation relation $([a_i, a_j^\dagger] = \delta_{i,j})$, $n_j = a_j^\dagger a_j$ the number operator associated to the j -th site, $t_{i,j}$ the hopping strength for hops between sites i and j , u the energy scale of the on-site two-body interaction and μ the chemical potential setting the average number of particles in the system. The term $-\sum_{j=1}^V \omega_j n_j$ has been added to the usual BH Hamiltonian to take into account a possible disorder of the chemical potential. We assume the ω_j to be independent Gaussian random variables of mean 0 and standard deviation σ_ω (in practice, we will consider explicitly such disorder, that is $\sigma_\omega \neq 0$, only in chapter 5 when studying its competition against the intrinsic tendency of the system to synchronize). To ensure Hermiticity of the Hamiltonian (2.1), we assume of course $t_{i,j} = t_{j,i}$. Importantly, we limit our study to the case of repulsive

in-situ interaction, that is to $u > 0$. Throughout the following we implicitly assume the indexes i, j, k to label the lattice sites and to run over $\{1, 2, \dots, V\}$, unless differently specified.

2.2 System initialization: the quench

It is tautological that, being interested in the study of the non-equilibrium dynamics, we first have to take the system out of equilibrium. A well-established protocol to do it is the quench, that is the sudden change of some parameter of the Hamiltonian with consequent change of the latter from H_i to H_f [24, 26–31]. We assume to prepare the system at time $t = -\infty$, to let it thermalize to the ground state $|\text{GS}_i\rangle$ of H_i , to perform the quench at $t = 0$ and to observe its time evolution for $0 < t < T_{ex}$ where T_{ex} is the typical timescale of the experiment. Thanks to the high degree of isolation achievable in experiments [21], we assume T_{ex} to be much smaller than the timescale over which the system will relax to the new ground state $|\text{GS}_f\rangle$ of H_f . Theoretically, the quench protocol results in considering $|\text{GS}_i\rangle$ as Initial Condition (IC) for the unitary dynamics generated by H_f .

In particular, we are interested in the quench from small to large hopping strengths, that is from a MI regime to a SF regime. The equations considered throughout all the following are associated to $t > 0$ and to H_f .

2.3 Heisenberg equations of motion

The time dynamics of the system can be investigated within Heisenberg formalism. In particular, Heisenberg equations for the creation and annihilation operators at site k reads

$$\begin{cases} \frac{da_k}{dt} = \frac{i}{\hbar} [H, a_k] \\ \frac{da_k^\dagger}{dt} = \frac{i}{\hbar} [H, a_k^\dagger] \end{cases} \quad (2.2)$$

i being the imaginary unit, $\hbar$ the reduced Planck constant and $[A, B] = AB - BA$ the commutator of the operators A and B . In the following, we consider $\hbar = 1$, meaning that frequencies and energies are measured in the same units. With straightforward calculations and after a proper gauge transformation (detailed in appendix A.1) we get

$$\begin{cases} \frac{da_k}{d(it)} = + \sum_{j=1}^V t_{j,k} a_j - u n_k a_k + \omega_k a_k \\ \frac{da_k^\dagger}{d(it)} = - \sum_{j=1}^V t_{j,k} a_j^\dagger + u a_k^\dagger n_k - \omega_k a_k^\dagger \end{cases} \quad (2.3)$$

Total number of particles conservation, system isolation and mixed states

At this point, some important remarks should be done. It is immediate to check that the Hamiltonian (2.1) commutes with the total number of particles operator $\hat{N} = \sum_{j=1}^V n_j$, that is thus conserved in time. The BH Hamiltonian is typically used to describe systems

at equilibrium and in the grand-canonical ensemble [11, 12], where a change of chemical potential μ is assumed to correspond to a change of the total expected number of particles N , thanks to a particle exchange with a reservoir. However, such assumption is not valid when considering the time dynamics generated by the Hamiltonian (2.1), since this is actually not containing any term explicitly describing the exchange of particles with a reservoir, accordingly with the fact that $[H, N] = 0$ and that N is a constant of motion. Our time dynamics will therefore certainly fail in predicting the fluctuations of N that are typical of the grand-canonical ensemble (and of order $1/\sqrt{N}$) or the readaptation of N to a change of μ . Nevertheless, the out of equilibrium dynamics of (2.3) is still meaningful since, as we said, the system can be considered as almost isolated on the experimental timescales. Conversely, on the timescales of the initialization of the system preceding the quench there is no isolation from the environment (thermal bath and particle reservoir), so that the system is in general described by a mixed state [60]. In this context, we denote the expectation value of an operator $\bullet$ with $\langle \bullet \rangle = \text{Tr}[\rho(t)\bullet]$, $\rho(t)$ being the density matrix associated to the system state at time t .

The fully connected model

We will particularly focus on the FC model, that is the one for which

$$t_{i,j} = \tau/V \quad \forall i \neq j \quad (2.4)$$

As detailed in appendix A.1, thanks to a gauge transformation, we can actually safely consider also $t_{i,i} = \tau$. Notice that the Hamiltonian (2.1) for the FC model reads

$$H = -\tau V \hat{\Psi}^\dagger \hat{\Psi} + \frac{u}{2} \sum_{j=1}^V n_j(n_j - 1) - \mu \sum_{j=1}^V n_j - \sum_{j=1}^V \omega_j n_j \quad (2.5)$$

where

$$\hat{\Psi} = \langle a_j \rangle_V = \frac{1}{V} \sum_{j=1}^V a_j \quad (2.6)$$

$\langle \bullet \rangle_V = \frac{1}{V} \sum_{j=1}^V \bullet_j$ denoting (more in general) the average over the sites of the site dependent quantity $\bullet_j$. From 2.5, we clearly see that, at least in the non-disordered case of $\omega_j = 0$, the minimization of the expected energy is associated to the maximization of the expected value $\langle \hat{\Psi}^\dagger \hat{\Psi} \rangle$.

2.4 Mean-field approximation: the Gross-Pitaevskii equation

Being nonlinear and operatorial, the DEs (2.3) reveal to be hardly tractable. To deal with c -numbers rather than with operators we are interested in taking the expectation value of such equations. It is however easy to convince ourselves that, because of nonlinearities, it

is not possible to get closed DEs for a finite set of expectation values. Indeed, the expression of $\frac{d\langle a_k \rangle}{dt}$ depends on $\langle n_k a_k \rangle$, and the expression of $\frac{d\langle n_k a_k \rangle}{dt}$ depends on further other expectation values and so on so forth, in a way that could be referred to as *proliferation*. A paradigmatic and widely accepted workaround for this issue is provided at the MF level when $N/V \gg 1$ by supposing

$$\begin{cases} \langle n_j a_j \rangle & \approx |\langle a_j \rangle|^2 \langle a_j \rangle \\ \langle n_j \rangle & \approx \langle a_j^\dagger \rangle \langle a_j \rangle = |\langle a_j^\dagger \rangle|^2 \end{cases} \quad (2.7)$$

that is considered to hold when in the SF regime and whose validity will be further investigated in chapter 6.

Notice that $\langle a_j^\dagger \rangle$ and $\langle a_j \rangle$ are in general different from zero since the system state is not an eigenstate of $\hat{N}$. In practice, in the following we directly consider specific ICs on $\langle a_j \rangle$ for each $j = 1, 2, \dots, V$, that we assume to correspond to the ground state of the Hamiltonian preceding the quench at $t < 0$. We call

$$\langle a_j \rangle = \psi_j = \sqrt{\rho_j} e^{i\theta_j} \quad (2.8)$$

with $\rho_j \in \mathbb{R}^+$ and $\theta_j \in (0, 2\pi)$ and observe as immediate consequence of the assumption (2.7) that $\frac{N}{V} = \frac{1}{V} \sum_{j=1}^V \langle a_j^\dagger a_j \rangle \approx \frac{1}{V} \sum_{j=1}^V \rho_j = \langle \rho_j \rangle_V = \rho_0$, ρ_0 being the average number of particles per site.

From equation (2.3) and under the approximation (2.7) we obtain the MF DEs

$$\begin{cases} \frac{d\psi_k}{d(it)} = + \sum_{j=1}^V t_{j,k} \psi_j - u |\psi_k|^2 \psi_k + \omega_k \psi_k \\ \frac{d\psi_k^*}{d(it)} = - \sum_{j=1}^V t_{j,k} \psi_j^* + u |\psi_k|^2 \psi_k^* - \omega_k \psi_k^* \end{cases} \quad (2.9)$$

that are the discrete version of the nonlinear Gross-Pitaevskii Equation (GPE).

As detailed in appendix A.2, from equation (2.9) we can derive the following DEs for the squared modulus and the phase of ψ_k (respectively ρ_k and θ_k)

$$\begin{cases} \frac{d\rho_k}{dt} & = 2 \sum_{j=1}^V t_{j,k} \sqrt{\rho_j \rho_k} \sin(\theta_k - \theta_j) \\ \frac{d\theta_k}{dt} & = \sum_{j=1}^V t_{j,k} \sqrt{\frac{\rho_j}{\rho_k}} \cos(\theta_k - \theta_j) - u \rho_k + \omega_k \end{cases} \quad (2.10)$$

From now on we focus only on the FC system, for which it is possible to conveniently compact the DEs similarly to what is commonly done in the Kuramoto model for classical coupled oscillators [53], that is defining the following complex DOP

$$\Psi = \langle \hat{\Psi} \rangle = \frac{1}{V} \sum_{j=1}^V \langle a_j \rangle = \frac{1}{V} \sum_{j=1}^V \sqrt{\rho_j} e^{i\theta_j} \quad (2.11)$$

and writing it in modulus phase form as

$$\Psi = r e^{i\phi} \quad (2.12)$$

with $r \in \mathbb{R}^+$ and $\phi \in (0, 2\pi)$. Considering the real and the imaginary part of $re^{i(\phi-\theta_k)} = \frac{1}{V} \sum_{j=1}^V \sqrt{\rho_j} e^{i(\theta_j-\theta_k)}$, we readily find the following two useful relations

$$\begin{cases} r \cos(\phi - \theta_k) = \frac{1}{V} \sum_{j=1}^V \sqrt{\rho_j} \cos(\theta_j - \theta_k) \\ r \sin(\phi - \theta_k) = \frac{1}{V} \sum_{j=1}^V \sqrt{\rho_j} \sin(\theta_j - \theta_k) \end{cases} \quad (2.13)$$

so that the dynamical system (2.10) can be compactly written as

$$\begin{cases} \frac{d\sqrt{\rho_k}}{dt} = \tau r \sin(\theta_k - \phi) \\ \frac{d\theta_k}{dt} = \frac{\tau r}{\sqrt{\rho_k}} \cos(\theta_k - \phi) - u\rho_k + \omega_k \end{cases} \quad (2.14)$$

where we stress r and ϕ to be in general not constant, since they evolve consistently with all the variables $\{\rho_k, \theta_k\}_{k=1, \dots, V}$. Summing over k the first equation of (2.14) we get

$$\frac{d\langle \rho_k \rangle_V}{dt} = \frac{d\rho_0}{dt} = 0 \quad (2.15)$$

that is we find the expected average number of particles per site (or, equivalently, the expected total number of particles) to be a constant of the GPE.

Unless differently specified, we will consider throughout the text the non-disordered case, that is the one for which $\omega_k = 0 \forall k = 1, 2, \dots, V$, for which 2.14 reads

$$\begin{cases} \frac{d\sqrt{\rho_k}}{dt} = \tau r \sin(\theta_k - \phi) \\ \frac{d\theta_k}{dt} = \frac{\tau r}{\sqrt{\rho_k}} \cos(\theta_k - \phi) - u\rho_k \end{cases} \quad (2.16)$$

We notice that, considering the rescaled time $\tilde{t} = \tau t$, equation (2.16) can be rewritten as

$$\begin{cases} \frac{d\sqrt{\rho_k}}{d\tilde{t}} = r \sin(\theta_k - \phi) \\ \frac{d\theta_k}{d\tilde{t}} = \frac{r}{\sqrt{\rho_k}} \cos(\theta_k - \phi) - \eta \frac{\rho_k}{\rho_0} \end{cases} \quad (2.17)$$

where $\eta = \frac{u\rho_0}{\tau}$. We notice the equations (2.17) to be invariant under a rescaling $\rho_k \rightarrow \alpha\rho_k$ (under which $r \rightarrow \sqrt{\alpha}r$) and to have as only parameter η . Therefore, we conclude that the only relevant quantity for the dynamics is $\eta = \frac{u\rho_0}{\tau}$. It is therefore mathematically convenient to consider only one free parameter, that we decide to be τ . In the following we therefore consider τ in units of $u\rho_0$, setting $u = 1$ and $\rho_0 = 1$ in (2.16) without loss of generality. Notice that this is just a mathematical trick and that, still, we are actually interested in the regime assumed by the MF, that is $\rho_0 \gg 1$. Notice that, being interested in the case of repulsive in-situ interaction only (that is $u > 0$) and being τ written in units of $u\rho_0$, then $\tau > 0$. The assumption of $u, \tau > 0$ is fundamental for the present work, since a negative u, τ would lead to a rather different mathematics and physics.

System initialization

In the case of a quench from the MI to the SF regime, the system at initial time $t = 0$ is in a MI phase. The corresponding MF configuration in this case is [10]

$$\begin{cases} \rho_k = \rho_0 \\ \theta_k = U_k(0, 2\pi) \end{cases} \quad \forall k = 1, 2, \dots, V \quad (2.18)$$

$U_k(0, 2\pi)$ being a random number distributed uniformly between 0 and 2π .

2.5 $V \rightarrow \infty$ limit and continuous equations

For the study of synchronization phenomena it is relevant and instructive to consider the $V \rightarrow \infty$ limit, in which the discrete site index $k = 1, 2, \dots, V$ can be replaced by a continuous variable $s \in (0, 2\pi)$, so that the DEs (2.16) transform to

$$\begin{cases} \frac{\partial \sqrt{\rho(s,t)}}{\partial t} = \tau r \sin(\theta(s,t) - \phi) \\ \frac{\partial \theta(s,t)}{\partial t} = \frac{\tau r}{\sqrt{\rho(s,t)}} \cos(\theta(s,t) - \phi) - \rho(s,t) \end{cases} \quad (2.19)$$

where

$$r e^{i\phi} = \frac{1}{V} \sum_{k=1}^V \sqrt{\rho_k} e^{i\theta_k} \xrightarrow{V \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} \sqrt{\rho(s)} e^{i\theta(s)} \quad (2.20)$$

In this way we passed from a system of $2V$ ordinary differential equations in the $2V$ variables $\{\rho_k, \theta_k\}_{k=1, \dots, V}$, to a system of 2 integro-differential equations in the variables $\rho(s, t)$ and $\theta(s, t)$. Notice that for equation (2.19) to be valid we require as assumption that there exists a permutation of the sites indexes such that the functions $\rho(s, t)$ and $\theta(s, t)$ are continuous, that is such that $\rho_k \xrightarrow{V \rightarrow \infty} \rho_{k+1}$ and $\theta_k \xrightarrow{V \rightarrow \infty} \theta_{k+1} \quad \forall k = 1, \dots, V$ and $\rho_V \xrightarrow{V \rightarrow \infty} \rho_1$ and $\theta_V \xrightarrow{V \rightarrow \infty} \theta_1$.

2.6 Analogue classical system

As a result of having taken the expectation value of equation (2.3) under the approximation (2.7), our quantum model has been recasted into a classical one where the variables are V c -numbers or $2V$ real numbers. Indeed, considering a classical Hamiltonian

$$H_{CL} = - \sum_{i,j}^V t_{j,k} \sqrt{\rho_i \rho_j} \cos(\theta_i - \theta_j) + \frac{u}{2} \sum_i^V \rho_i^2 - \sum_i^V \omega_i \rho_i \quad (2.21)$$

with classical conjugated variables θ_k and ρ_k for $k = 1, \dots, V$ we get that the Hamilton equations of motion $\frac{d\rho_k}{dt} = + \frac{\partial H_{CL}}{\partial \theta_k}$ and $\frac{d\theta_k}{dt} = - \frac{\partial H_{CL}}{\partial \rho_k}$ correspond to the GPE (2.10). At the MF level, studying the time dynamics of the expectation values of the quantum bosonic operators corresponds to studying the time dynamics of V coupled *clocks* (that is

oscillators) of modulus $\sqrt{\rho_k}$ and phase θ_k with $k = 1, \dots, V$ and described by the classical Hamiltonian (2.21). In MF, we can therefore visualize the variables of the problem as clocks, representing them in a polar plot using one marker for each site as in figure (2.1). The k -th blue marker is at distance $\sqrt{\rho_k}$ from the origin and have a phase θ_k , whereas one additional red marker at distance r from the origin and with phase ϕ represents the DOP $\Psi = r e^{i\phi}$. Additionally, as a reference we plot a red circle centered in the origin and of radius $\sqrt{\rho_0}$.

In the case of interest of the FC model, we also notice that it is possible to rewrite the energy as

$$H_{CL} = -V\tau r^2 + \frac{u}{2} \sum_{j=1}^V \rho_j^2 \quad (2.22)$$

The state of the system minimizing H_{CL} under the constraint of $\frac{1}{V} \sum_{j=1}^V \rho_j = \rho_0$ is easily found to be

$$\begin{cases} \rho_k = \rho_0 \\ \theta_k = \phi \end{cases} \quad \forall k = 1, 2, \dots, V \quad (2.23)$$

that is the SuperFluid Configuration (SFC) [10]. We get thus that, in MF, the ground state of the system is always the SFC, consistently with the fact that the GPE holds in the SF regime only. In the present work we are mainly interested in the out of equilibrium dynamics, that is the time evolution of states that are initialized not to the SFC.

We finally observe that, since H_{CL} is conserved, (2.22) gives the following useful relation

$$2\tau r^2 = \frac{u}{V} \sum_{j=1}^V \rho_j^2 + 2E \quad (2.24)$$

E being the average MF energy per site. The value of E is constant in time and determined by the system IC, that is by the state of the system at $t = 0^+$.

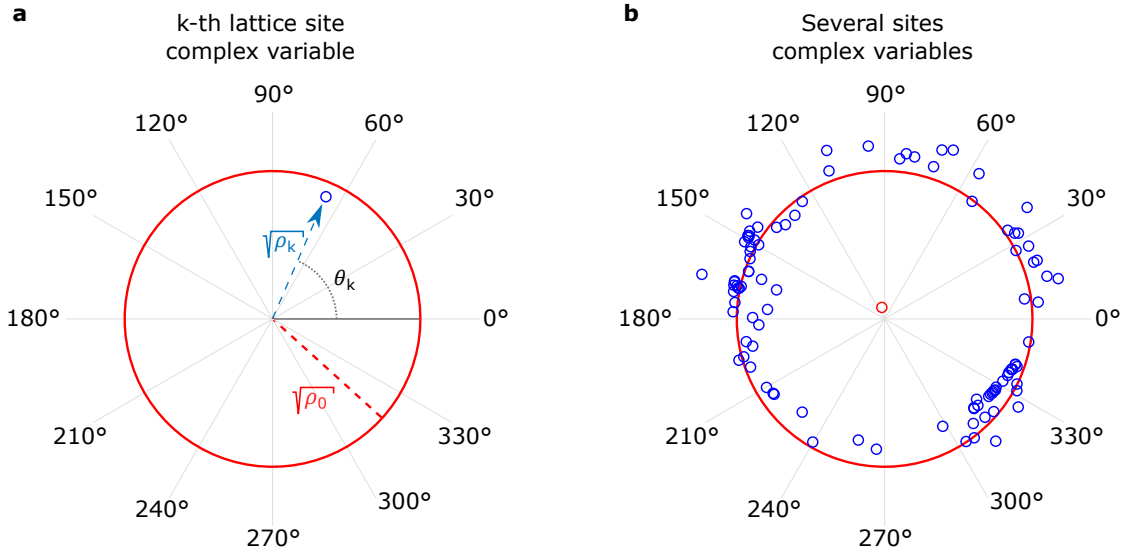


Figure 2.1 | Graphical representation of the bosonic variables. **a**, The k -th site is associated to the complex variable $\psi_k = \sqrt{\rho_k} e^{i\theta_k}$, represented in a polar plot as a blue marker at a distance $\sqrt{\rho_k}$ from the origin and with phase θ_k . **b**, Considering $V = 100$ sites means considering $V = 100$ blue markers plus one red marker representing $\Psi = r e^{i\phi}$, at a distance r from the origin and with phase ϕ . The reference red circle is centered in the origin and of radius $\sqrt{\rho_0}$.

Chapter 3

Stationary configurations

In the previous chapter we introduced, for the Fully Connected Bose-Hubbard Model (FCBHM) and within the MF approximation, the GPE of motion (2.9), that we rewrote as a system of ordinary differential equations (2.16) for the $2V$ real variables $\sqrt{\rho_k}$ and θ_k , with $k = 1, 2, \dots, V$. The first main goal of the present work is to understand the different possible dynamical solutions that such equations can originate for various ICs and in various regions of the parameters space. Naturally, we attempt to do it within the area of mathematics that goes under the name of Dynamical Systems Theory (DST). When an explicit exact solution of a system of DEs is not viable, DST typically approaches the problem by looking at its FPs, that is at that solutions that are constant in time. Actually, we will conveniently define the FPs in a proper corotating frame of reference, that is unless a rotation of all the phases at a constant rate.

We introduce here some terminology that will be widely used throughout the whole text. We call *configuration* the $2V$ -dimensional set of variables $(\theta_1, \dots, \theta_V, \sqrt{\rho_1}, \dots, \sqrt{\rho_V})$ associated to a given state of the system, Stationary Configuration (SC) the configuration corresponding to a FP of the DEs and *phase space* the $2V$ -dimensional space $\Phi = [0, 2\pi]^V \times (\mathbb{R}^+)^V$ in which the configurations live. We define Stationary Set (SS) the set of the SCs and *trajectory* the oriented line that a system follows in the phase space during its time evolution.

The importance of the individuation of the FPs lies in the fact that they are the first key ingredient to obtain information on the dynamics of the system. Indeed, the main features of the dynamics will already emerge in chapter 4 when linearizing the GPE around the SCs, whereas unveiling the topology of the SS will give us considerable intuition on the role of nonlinearities of the GPE in chapter 5. In particular, in the present chapter we observe that the topology of the SS heavily depends on the number of lattice sites V , and that if and only if $V \geq 4$ there exist a continuous, connected manifold of infinitely many SCs in the phase space.

3.1 Stationarity condition

As we already mentioned, it is convenient to adopt a definition for stationarity that allows a *global* phase rotation, that is a common rotation of all the phases $\theta_1, \theta_2, \dots, \theta_V$ at a constant rate. Therefore, we say a configuration to be stationary for the equation (2.16) if it fulfills the condition

$$\begin{cases} \frac{d\sqrt{\rho_k}}{dt} = \tau r \sin(\theta_k - \phi) = 0 \\ \frac{d\theta_k}{dt} = \frac{\tau r}{\sqrt{\rho_k}} \cos(\theta_k - \phi) - \rho_k = \Omega \end{cases} \quad \text{for } k = 1, 2, \dots, V \text{ and } t \geq 0 \quad (3.1)$$

with Ω whatever real number.

For a SC we therefore readily find that the modulus and phase of the complex DOP (2.11) read

$$\begin{cases} r(t) = r(0) \\ \phi(t) = \phi(0) + \Omega t \end{cases} \quad (3.2)$$

As we already stated in chapter 2, r and ϕ have to be consistent with the considered configuration for all times $t > 0$. We understand that for a SC the consistency condition on r and ϕ has to be checked only at initial time $t = 0$, since for later times it is automatically ensured by (3.2). Of course we can always consider to be in a reference frame such that $\phi(0) = 0$. Furthermore, also the consistency condition on the average number of particles per site $\rho_0 = 1$ has to be checked only at initial time, since ρ_0 is a conserved quantity of motion. Summing up, the stationarity condition plus the consistency conditions read

$$\begin{cases} \frac{d\sqrt{\rho_k}}{dt} = \tau r \sin(\theta_k - \phi) = 0 \\ \frac{d\theta_k}{dt} = \frac{\tau r}{\sqrt{\rho_k}} \cos(\theta_k - \phi) - \rho_k = \Omega \\ \frac{1}{V} \sum_{j=1}^V \sqrt{\rho_j} e^{i\theta_j} = r \\ \frac{1}{V} \sum_{j=1}^V \rho_j = 1 \end{cases} \quad \text{for } k = 1, 2, \dots, V \text{ and } t = 0 \quad (3.3)$$

3.2 Stationarity in the two-site model

The easiest case we can consider is of course the one with the lowest V . Excluding the trivial case of $V = 1$, we are therefore lead to start considering a system with just $V = 2$ sites. The $V = 2$ model is relevant in describing the so called bosonic *dimer*, that can be realized in experiments for instance with the so called *bosonic junction*, composed of two Bose-Einstein condensates in a double well potential [36–39]. In the present section we review the main results of the $V = 2$ model, that have already been extensively studied in the past two decades [1–6] and that will reveal to be instructive when considered in relation to the $V > 2$ cases.

Instead of using the stationarity condition (3.3), we prefer in this case to find a new condition from scratch in such a way that we can explicitly enforce the constraint on the

total number of particles conservation and reduce the dimension of the DEs. Equations (2.16) read

$$\begin{cases} \frac{\partial \theta_1}{\partial t} &= -\rho_1 + \frac{\tau}{2} \sqrt{\frac{\rho_2}{\rho_1}} \cos(\theta_2 - \theta_1) \\ \frac{\partial \theta_2}{\partial t} &= -\rho_2 + \frac{\tau}{2} \sqrt{\frac{\rho_1}{\rho_2}} \cos(\theta_1 - \theta_2) \\ \frac{\partial \rho_1}{\partial t} &= -\tau \sqrt{\rho_2 \rho_1} \sin(\theta_2 - \theta_1) \\ \frac{\partial \rho_2}{\partial t} &= -\tau \sqrt{\rho_1 \rho_2} \sin(\theta_1 - \theta_2) \end{cases} \quad (3.4)$$

Where $\frac{\tau}{2} = t_{1,2} = t_{2,1}$. Considering explicitly the conservation of the total number of particles (that reads $\rho_1 + \rho_2 = 2\rho_0$), it is easy to reduce (3.4) to a 2-dimensional system of DEs for the phase difference $\theta = \theta_2 - \theta_1$ and the normalized population imbalance $\delta = \frac{\rho_2 - \rho_1}{\rho_0}$

$$\begin{cases} \frac{\partial \theta}{\partial t} &= -\delta - \frac{\tau}{2} \frac{\delta}{\sqrt{1 - \frac{\delta^2}{4}}} \cos \theta \\ \frac{\partial \delta}{\partial t} &= 2\tau \sqrt{1 - \frac{\delta^2}{4}} \sin \theta \end{cases} \quad (3.5)$$

where as usual we consider without loss of generality $\rho_0 = 1$. Since $|\delta| = \pm 2$ corresponds to the case of all particles in the same site, that is $V = 1$, we consider $|\delta| < 2$. The stationarity condition reads

$$\begin{cases} -\delta = \frac{\tau}{2} \frac{\delta}{\sqrt{1 - \frac{\delta^2}{4}}} \cos \theta \\ 2\tau \sqrt{1 - \frac{\delta^2}{4}} \sin \theta = 0 \end{cases} \quad (3.6)$$

Notice that, with respect to the condition (3.3), here the global phase rotation rate Ω is not appearing since the considered variable is the phase difference. Still, of course, the condition (3.6) allows a global phase rotation.

The second equation of (3.6) is solved by $\theta = 0$ and by $\theta = \pi$. In the two cases, the first equation of (3.6) reads

$$\begin{cases} -\delta = \frac{\tau}{2} \frac{\delta}{\sqrt{1 - \frac{\delta^2}{4}}} & \text{for } \theta = 0 \\ +\delta = \frac{\tau}{2} \frac{\delta}{\sqrt{1 - \frac{\delta^2}{4}}} & \text{for } \theta = \pi \end{cases} \quad (3.7)$$

that is solved by the following four SCs

$$\begin{cases} \theta = 0 & \delta = 0 & \text{SFC} \\ \theta = \pi & \delta = 0 & \text{SPAC} \\ \theta = \pi & \delta = \sqrt{4 - \tau^2} & \text{PAC}_+ \\ \theta = \pi & \delta = -\sqrt{4 - \tau^2} & \text{PAC}_- \end{cases} \quad (3.8)$$

where SFC stays for SuperFluid Configuration, SPAC for Symmetric π -Aligned Configuration and PAC for π -Aligned Configuration. Of course PAC_+ and PAC_- exist only for $\tau < 2$, since δ and θ are real variables. Notice that we are implicitly assuming τ in

units of $u\rho_0$, and that thus the condition of existence of PAC_+ and PAC_- is actually $\tau < 2u\rho_0$, that clearly depends on the interplay between the hopping strength τ and the in-situ interaction $u\rho_0$. Notice as well that in finding the SCs we assumed a repulsive interaction, i.e. $u > 0$ (that is $\tau > 0$, since τ is written in units of $u\rho_0$). In the case of attractive in-situ interaction ($u < 0$), some of the SCs would be analogue to (3.8) and some others would not, leading to new mathematics and physics, that we do not address in the present work.

3.3 Stationary configurations for $V > 2$

From the first equation of (3.3) we find that only two kinds of SCs are possible: the ones with $r = 0$ and the ones with $\sin(\theta_k - \phi) = 0 \ \forall k = 1, \dots, V$. We treat these 2 classes of SCs separately.

3.3.1 $r = 0$ configurations

The first class of SCs is the one with $r = 0$, for which the stationarity condition (3.3) reads

$$\begin{cases} \frac{d\sqrt{\rho_k}}{dt} = 0 \\ \frac{d\theta_k}{dt} = -\rho_k = \Omega \\ \sum_{j=1}^V \sqrt{\rho_j} e^{i\theta_j} = 0 \\ \frac{1}{V} \sum_{j=1}^V \rho_j = 1 \end{cases} \quad \text{for } k = 1, 2, \dots, V \text{ and } t = 0 \quad (3.9)$$

that is readily reduced to

$$\begin{cases} \rho_k = 1 \\ \sum_{j=1}^V e^{i\theta_j} = 0 \end{cases} \quad \text{for } k = 1, 2, \dots, V \text{ and } t = 0 \quad (3.10)$$

Particularly relevant is the one for which the sites are divided in two equal groups with opposite phases, that is clearly present only in the case of even $V > 2$ ($V = 4, 6, \dots$) and that reads

$$\begin{cases} \rho_k = 1 & \forall k = 1, 2, \dots, V \\ \theta_k = 0 & \forall k = 1, 2, \dots, V/2 \\ \theta_k = \pi & \forall k = V/2 + 1, \dots, V \end{cases} \quad \text{at } t = 0 \quad (3.11)$$

Of course, the configuration (3.11) is defined unless a permutation of the indexes k (as obvious for a FC model) and an arbitrary global phase shift. For obvious reasons, we talk in this case about Symmetric π -Aligned Configuration (SPAC). Another relevant configuration is instead defined for whatever $V > 2$ and reads

$$\begin{cases} \rho_k = 1 \\ \theta_k = \frac{2\pi}{V} k \end{cases} \quad \text{for } k = 1, 2, \dots, V \text{ and } t = 0 \quad (3.12)$$

that we call Uniform Configuration (UC) and that is again defined unless a permutation of the indexes k and an arbitrary global phase shift. In this case the word *uniform* is used to stress that the phases are distributed in $(0, 2\pi)$ with uniform spacing $2\pi/V$. Observe that the definition of UC can be extended also to $V = 2$, for which it corresponds to the SPAC, and that for $V = 3$ the UC is the only possible SC with $r = 0$. For $V > 3$ the condition (3.10) is instead generally satisfied by an infinity of SCs, of which the UC and the SPAC are just particular examples. To see this explicitly, we consider the example of $V = 4$, for which the most generic SC with $r = 0$ (unless a global phase rotation and a permutation of the sites) can be written as

$$\begin{cases} \rho_k = 1 & \forall k = 1, 2, 3, 4 \\ \theta_1 = +\frac{\pi-\Delta}{2} \\ \theta_2 = -\frac{\pi-\Delta}{2} \\ \theta_3 = +\frac{\pi-\Delta}{2} + \pi \\ \theta_4 = -\frac{\pi-\Delta}{2} + \pi \end{cases} \quad (3.13)$$

that we call Delta Configuration (DC) and where Δ ranges continuously from $\pi/2$ (corresponding to the UC) to π (corresponding to the SPAC). It is easy to imagine that for $V = 6, 8, 10, \dots$ there is as well a continuum of SCs with $r = 0$ ranging from the UC to the SPAC whereas for $V = 5, 7, 9, \dots$ again there is a continuum of SCs with $r = 0$, of which the UC is part and the SPAC is not, since for an odd V the latter is not even defined. A graphical representation of the SPAC, UC, DC and others $r = 0$ configurations is shown in figure (3.1) whereas a more thorough analysis of the structure of the SS is given in section 3.4 together with its schematic representations in figure (3.2).

3.3.2 Aligned (superfluid) and π -aligned configurations

The second class of SCs is characterized by $\sin(\theta_k - \phi) = 0 \ \forall k = 1, \dots, V$, that is by $\theta_i - \theta_j \in \{0, \pi\} \ \forall i, j$ at time $t = 0$. Observe that the SPAC is the only SC belonging to both the classes of SCs, having $r = 0$ and satisfying $\theta_i - \theta_j \in \{0, \pi\} \ \forall i, j$ at the same time.

The most trivial SC with such property is the one for which all the phases are aligned, that is

$$\begin{cases} \rho_k = 1 & \forall k = 1, 2, \dots, V \\ \theta_k = 0 & \forall k = 1, 2, \dots, V \end{cases} \quad (3.14)$$

that is as always defined unless a global phase rotation and for which we get $r = 1$ and $\Omega = \tau - 1$. We call this SC the SFC. Notice that, maximizing r , the SFC represents the absolute minimum of the classical energy (2.22), that is the SFC is the ground state of the Hamiltonian for $t > 0$. This is in accordance with the assumption that, for $t > 0$, the system is in the SF regime.

Another possible SCs in this class have instead a fraction $\alpha \in (0, 1)$ of the sites with associated phase ϕ and the remaining fraction $1 - \alpha$ with phase $\phi + \pi$ and will be referred to as π -Aligned Configurations (PACs). Since $\alpha \in \{\frac{1}{V}, \frac{2}{V}, \dots, \frac{V-1}{V}\}$, in general several

of these PACs exist. Notice that the study of such SCs heavily depends on assuming a repulsive in-situ interaction, i.e. $u > 0$ (that is $\tau > 0$, since τ is written in units of $u\rho_0$). An attractive in-situ interaction ($u < 0$) would in general lead to different stationary PACs, and is not addressed in the present work. In appendix B a detailed analysis of the condition (3.3) for this kind of SCs is reported for completeness, but here we just focus on the case for which V is even and $\alpha = 1/2$, that is for which half of the sites have phase ϕ and the other half have phase $\phi + \pi$, that is

$$\psi_k = \begin{cases} +\sqrt{1+\delta} & \text{if } k = 1, \dots, V/2 \\ -\sqrt{1-\delta} & \text{if } k = V/2 + 1, \dots, V \end{cases} \quad \text{at } t = 0 \quad (3.15)$$

that is as usual defined unless a global phase rotation. Without loss of generality we can assume $\delta \in (0, 1)$, since a negative δ would actually be reabsorbed in our study just with an initial π rotation of the reference frame. For the configuration (3.15), the DOP (2.11) at initial time reads

$$re^{i\phi} = \frac{1}{V} \sum_{j=1}^V \sqrt{\rho_j} e^{i\theta_j} = \frac{\sqrt{1+\delta} - \sqrt{1-\delta}}{2} \quad \text{at } t = 0 \quad (3.16)$$

that is

$$\begin{cases} r(0) &= \frac{\sqrt{1+\delta} - \sqrt{1-\delta}}{2} \\ \phi(0) &= 0 \end{cases} \quad (3.17)$$

where $r > 0$ consistently with the assumption $\delta > 0$. Enforcing the two groups of phases to rotate at the same rate, the stationarity condition (3.3) reads

$$-(1+\delta) + \tau \frac{r}{\sqrt{1+\delta}} = -(1-\delta) - \tau \frac{r}{\sqrt{1-\delta}} \quad (3.18)$$

from which in few easy steps we get

$$2\delta = \frac{\tau\delta}{\sqrt{1-\delta^2}} \quad (3.19)$$

that is solved either by $\delta = 0$ (corresponding to the SPAC) or by $\delta = \sqrt{1 - (\frac{\tau}{2})^2}$. The former solution is present for whatever τ , the latter only if $\tau < \tau_c = 2$ (remember that τ is in units of $u\rho_0$). The SF and the PAC with $\alpha = 1/2$ are clearly showing a strong analogy with the two-site problem. A graphical representation of the PAC is shown in figure (3.1).

3.4 Topology of the stationary set

We have seen how rich the stationary set can be and we aim in this section at summarizing the previous results systematically, categorizing the SS for all the possible V .

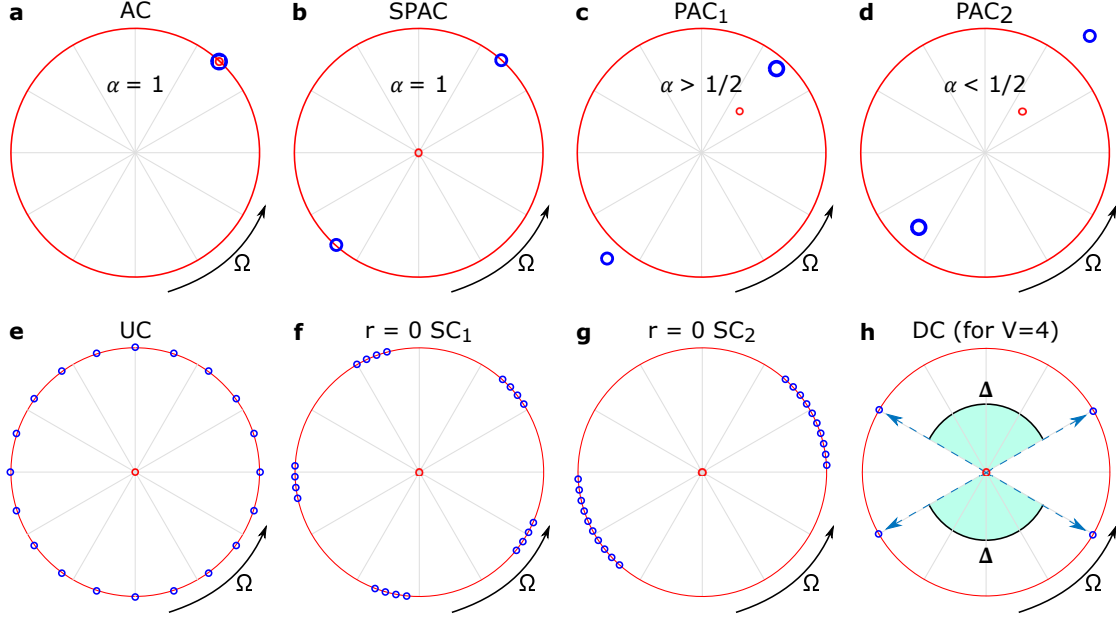


Figure 3.1 | Graphical representation of some stationary configurations. **a**, SuperFluid Configuration (SFC); **b**, Symmetric π -Aligned Configuration (SPAC); **c**, generic π -Aligned Configuration (PAC) with $\alpha > 1/2$; **d**, generic π -Aligned Configuration (PAC) with $\alpha < 1/2$; **e**, Uniform Configuration (UC) for $V = 20$; **f,g**, other examples of configurations with $r = 0$ for $V = 20$; **h**, Delta Configuration (DC) for $V = 4$. In **a-d** we use thicker and larger blue markers to indicate the overlapping of the markers of several different sites. The black arrow reminds that the stationary configurations experience in general a global phase rotation at angular frequency Ω .

- $V = 2$
The SFC is stationary whereas the SPAC coincides with the UC and is always present. The two PACs are present if $\tau < 2$ (that is if the in-situ interaction is large with respect to the hopping strength). Notice that these two PACs are actually equal unless a permutation of the sites and we therefore decide not to distinguish them;
- $V = 3$
The SFC and the UC are stationary, the SPAC is not defined (being V odd) and two PACs, one corresponding to $\alpha = 1/3$ and one to $\alpha = 2/3$ are possibly stationary (depending on the value of τ);
- $V = 4$
The SFC, UC and SPAC are stationary and as usual there can be other stationary PACs. Moreover, the DC corresponds to a continuum of infinitely many SCs ranging from the UC (for $\Delta = \pi/2$) to the SPAC (for $\Delta = \pi$);
- $V = 5, 7, 9, \dots$
The SCs are analogue to the ones of $V = 4$ apart from the fact that the set of $r = 0$ SCs does not include the SPAC, since this is not defined for an odd V ;

- $V = 6, 8, 10, \dots$

The SCs are completely analogue to the ones of $V = 4$.

Notice that, of course, the dimension of the manifold of the $r = 0$ SCs and the number of PACs grow with V . In figure (3.2) we schematically represent the phase space and the SCs for various numbers of sites V . We notice that, depending on V , the structure of the SS changes, with the emergence for $V \geq 4$ of a continuum of SCs with $r = 0$ of which the UC is part and the SPAC is part only if V is even. With an abuse of nomenclature we will often refer to this manifold with the word *line*, that is actually accurate only for $V = 4$. We call *isolated stationary configuration* a SC that is isolated in the geometrical sense from any other SC, that is such that every path in the phase space going to any other SC necessarily contain some non-stationary configuration. We see that the SFC and the PACs (SPAC excluded) are always isolated SCs whereas the UC and the SPAC are isolated for $V < 4$ and non isolated for $V \geq 4$. The distinction of isolated and non isolated SCs will reveal to be fundamental when considering the effects of nonlinearities of the GPE in chapter 5.

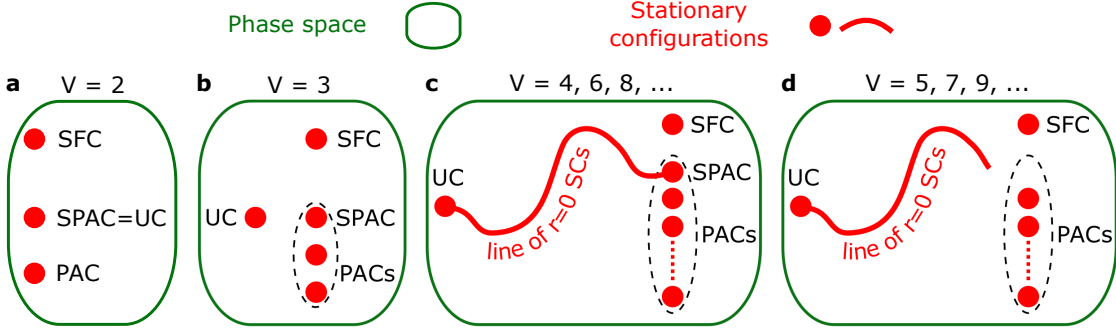


Figure 3.2 | Schematic representation of the stationary configurations in the phase space. The structure of the Stationary Set (SS) depends on the number of sites V . **a**, for $V = 2$ the SS is composed of SPAC, SFC and PAC; **b**, for $V = 3$ also the UC is present; **c**, for $V = 4, 6, 8, \dots$ the SS contains as well a continuous line of $r = 0$ SCs connecting the UC with the SPAC; **d**, for $V = 5, 7, 9, \dots$ the SPAC is missing. We notice all the SCs with $r \neq 0$ to be isolated and the SCs with $r = 0$ to be non isolated if and only if $V \geq 4$.

Chapter 4

Stability of the stationary configurations

In the previous chapter we found the SCs of the DEs (2.16), that is the configurations that are preserved in time unless a global phase rotation at a constant rate. It is natural to wonder about the behavior of a system that is initialized close to a SC but not exactly on it. This is relevant for instance in the case of a quench from the MI to the SF regime, for which the IC is given by a MI phase (2.18). For increasing V , such IC clearly get closer and closer to the UC, possibly coinciding with it only in the $V \rightarrow \infty$ limit. On the other hand, for a finite V it is fundamental to understand the effects on the time dynamics (2.16) of the finiteness of the distance of the MI from the UC and, more in general, of the IC from the closest SC. Such investigation can again be carried out in the context of DST exploiting the linearization of the equations of motion (2.16).

In this chapter, after giving some simple and strong arguments on the role of the conserved quantities in the out of equilibrium dynamics and after briefly reviewing some basic issues concerning the linearization in DST, we diagonalize the Jacobian matrix J associated to the linearized version of the DEs around the most relevant SCs and for all possible V . Looking at the eigenvalues of the Jacobian matrix, we are able to distinguish two regions of the parameter space corresponding to two qualitatively very different behaviours of the system when initialized in the proximity of the UC or of the SPAC. We call *dynamical phase transition* this transition between the two possible regimes and notice that it corresponds to what in DST is commonly referred to as *bifurcation*.

Importantly, we observe that the results of this chapter are obtained considering a repulsive in-situ interaction, that is $u > 0$. Having written τ in units of $u\rho_0$ this reduces to considering $\tau > 0$. A negative u would on the contrary lead to a negative τ , with dramatic changes in the study of the stability of the SCs and in the respective physics.

4.1 The effects of conserved quantities

Before going through the Jacobian diagonalization, it is worth to formulate some simple but powerful arguments based on the energy conservation of the mean field Hamiltonian

(2.22) and on the total number of particles. The presence of two conserved quantities constrains the evolution of the system in a $(2V - 2)$ -dimensional manifold of the phase space, with important consequences that we seek to understand in this section. Considering the conservation of the expected energy (2.24) and the total number of particles we have

$$\begin{cases} -\tau r^2 + \frac{1}{2}\langle \rho_j^2 \rangle_V = E \\ \langle \rho_j \rangle_V = 1 \end{cases} \quad (4.1)$$

where E is a constant depending on the IC and where $\langle \bullet_j \rangle_V = \frac{1}{V} \sum_{j=1}^V \bullet_j$ denotes the average over the sites of the site dependent quantity $\bullet_j$. We can write ρ_j as $\rho_j = 1 + \delta_j$ so that the condition on the total number of particles reads $\langle \delta_j \rangle_V = 0$ and (4.1) becomes

$$\begin{cases} -\tau r^2 + \frac{1}{2} + \frac{1}{2}\langle \delta_j^2 \rangle_V = E \\ \langle \delta_j \rangle_V = 0 \end{cases} \quad (4.2)$$

Initializing the system with $r(t=0) = r_0$ and $\rho_k = 1 \ \forall k = 1, 2, \dots, V$, at time $t = 0$ we get $\langle \delta_j \rangle_V = 0$ and $E = \frac{1}{2} - \tau r_0^2$, so that (4.2) reads

$$\begin{cases} \langle \delta_j^2 \rangle_V = 2\tau(r^2 - r_0^2) \\ \langle \delta_j \rangle_V = 0 \end{cases} \quad (4.3)$$

From (4.3) we observe some important facts. The first is that, since the left member of the first equation is positive definite, for all times $t > 0$ we have

$$r(t) \geq r_0 \quad (4.4)$$

that is r cannot be smaller than the initial r_0 . One consequence of the condition (4.4) is that a system initialized close to the SFC will always stay close to the SFC. Indeed, since r is bounded in $(0, 1)$, if $r_0 = 1 - \epsilon$ with ϵ positive and small then $r(t) \in (1 - \epsilon, 1) \ \forall t > 0$, that is the system stays close to the SFC (corresponding to $r = 1$). This is not surprising since the SF is the ground state of the Hamiltonian after the quench (that is for $t > 0$).

The second implication of (4.3) is that an increase of r must be accompanied by a spread of the ρ_k . For instance, if the system is initialized to the proximity of the UC or of the SPAC (that is $r_0 \approx 0$), then its growth in time is possible provided that there is a spread of the ρ_k around $\rho_0 = 1$. Indeed, the goal of this chapter is to unveil the conditions under which such growth of r with spread of the ρ_k is possible. On the other hand, if r is initially small and if it stays small for all times $t > 0$, then the system can a priori visit the neighborhoods of all the $r = 0$ manifold, as will be discussed in chapter 5.

4.2 The linearization procedure

Since the stationarity condition (3.3) allows a global rotation of the phases at rate Ω , we now prefer to move to a frame rotating exactly at the angular speed Ω associated to the

considered SC. In such frame, the equations of motion (2.16) read

$$\begin{cases} \frac{\partial \theta_j}{\partial t} = -x_j^2 + \tau r \frac{1}{x_j} \cos(\phi - \theta_j) - \Omega \\ \frac{\partial x_j}{\partial t} = -\tau r \sin(\phi - \theta_j) \end{cases} \quad (4.5)$$

where we consider the variable $x_j = \sqrt{\rho_j}$ and where of course also the phase ϕ has been redefined in the new rotating frame, accordingly to all the other phases. In such a frame the phases of a SC are seen not to move at all, that is the definition of SC is the usual one of $\frac{d\theta_k}{dt} = 0$.

Before proceeding, we briefly recall here some very well-known results from DST. We introduce

$$\vec{y} = \begin{pmatrix} \vec{y}^{(1)} \\ \vec{y}^{(2)} \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_V \\ x_1 \\ x_2 \\ \vdots \\ x_V \end{pmatrix} \quad \vec{\nabla} = \begin{pmatrix} \vec{\nabla}^{(1)} \\ \vec{\nabla}^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial \theta_1} \\ \frac{\partial}{\partial \theta_2} \\ \vdots \\ \frac{\partial}{\partial \theta_V} \\ \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_V} \end{pmatrix} \quad (4.6)$$

So that by definition of $\vec{y}^{(1)}, \vec{y}^{(2)}, \vec{\nabla}^{(1)}, \vec{\nabla}^{(2)}$ we have $y_j^{(1)} = \theta_j, y_j^{(2)} = x_j, \nabla_j^{(1)} = \frac{\partial}{\partial \theta_j}, \nabla_j^{(2)} = \frac{\partial}{\partial x_j}$. The Jacobian is the $2V \times 2V$ -dimensional matrix defined by

$$J_{j,k} = \nabla_k \left(\frac{dy_j}{dt} \right) \quad j, k = 1, \dots, 2V \quad (4.7)$$

whereas a SC is identified by a $2V$ -dimensional vector $\vec{y}^{(SC)}$ such that $\left(\frac{d\vec{y}}{dt} \right)_{SC} = 0$. If the IC is $\vec{y}_0 \approx \vec{y}^{(SC)}$, then it is relevant to linearize the DEs around $\vec{y}^{(SC)}$ so that we can write

$$\frac{d\vec{\epsilon}}{dt} \approx J\vec{\epsilon} \quad (4.8)$$

where $\vec{\epsilon} = \vec{y} - \vec{y}^{(SC)}$ and where J is evaluated at the given $\vec{y}^{(SC)}$. Solving the linearized equation (4.8) is extremely easy and gives

$$\vec{\epsilon}(t) = e^{Jt} \vec{\epsilon}(0) \quad (4.9)$$

A basis of Jacobian eigenvectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{2V}\}$, where the eigenvector $\vec{v}_n$ is associated to the n -th eigenvalue λ_n of the Jacobian matrix, can be used to decompose ϵ as $\epsilon(t) = \sum_{n=1}^{2V} \alpha_n(t) \vec{v}_n$ so that its time evolution is readily given from (4.9) by

$$\alpha_n(t) = \alpha_n(0) e^{\lambda_n t} \quad (4.10)$$

We therefore understand that the dynamics of the system in the surroundings of a SC is ruled by the eigenvalues of the Jacobian matrix associated to the linearized equations of motion in that given SC. In this sense, the possible eigenvalues can therefore be classified in three categories: 1) $\lambda_n = 0$, for which $\alpha_n(t) = \alpha_n(0) = cst$; 2) λ_n with $\text{Re}\{\lambda_n\} > 0$, for which $|\alpha_n(t)|$ grows exponentially and to which we refer to as *unstable eigenvalue*; 3) λ_n is such that $\text{Re}\{\lambda_n\} < 0$, for which $|\alpha_n(t)|$ decays to 0 exponentially and to which we refer to as *stable eigenvalue*. In DST it is common to classify the SCs accordingly to the associated Jacobian eigenvalues and, among all the possible cases [61], we report here the one of interest to us, that are

- $i\lambda_n \in \mathbb{R} \ \forall n \in \{1, 2, \dots, 2V\}$
That is all the non-zero eigenvalues are purely imaginary numbers. In this case the SC is said to be a *linear center*, and the solution of the linearized equations is a state cycling periodically and closeby the SC when initialized in its surroundings;
- $\exists n, m \in \{1, 2, \dots, 2V\}$ such that $\text{Re}\{\lambda_n\} > 0, \text{Re}\{\lambda_m\} < 0$
That is there exist at least one eigenvalue with positive real part and at least one eigenvalue with negative real part. In this case the SC is said to be a *saddle*, and the solution of the linearized equations is a state diverging exponentially fast from the SC when initialized in its surroundings (unless in case of initialization exactly along the eigenvectors associated to the stable eigenvalues, that belongs to the realm of mathematics rather than to the one of physics).

Importantly, we talked about *linear* center since the above arguments on the eigenvalues is a priori exact only for the linearized DEs. Analogously, we say a SC to be a *nonlinear* center (or, more simply, a *center*) of the dynamics if, even for the nonlinear DEs, there exist trajectories of the system looping arbitrarily close around the SC. Notice that a nonlinear center is also a linear center whereas the vice versa is in general true (spiral trajectories are a common example of linear centers that are not nonlinear centers).

Since if the SC is a saddle (linear center) the system will (will not) drift away exponentially fast from it, with some abuse of nomenclature we will often refer to it as *stable* (*unstable*) SC. Our goal in this chapter is therefore to linearize the Jacobian matrix associated to the SCs that we have already found in chapter 3. The argument on the conservation of the expected energy and total number of particles of section 4.1 guarantees that the SF configuration is a nonlinear center of the dynamics for all the points of the parameters space (that is all $\tau > 0$), whereas the UC (that is relevant for the initialization to a MI) and the SPAC can a priori either be linear (and possibly nonlinear) centers or saddles, the latter corresponding to the growth in time of r with consequent spread of the ρ_k described in section 4.1.

4.3 Stability in the two-site model

Again we briefly review the well-known results of the $V = 2$ case [1–6]. Continuing on the line followed for the individuation of the SCs in section 3.2, we consider the reduced system of DEs (3.5) and write a 2 dimensional Jacobian with respect to the 2-dimensional

variable $\vec{y} = \begin{pmatrix} \theta \\ \delta \end{pmatrix}$. For the various SCs we find the following Jacobian matrices and associated eigenvalues $\lambda_{1,2}$

- $\vec{y}^{(SFC)} = (0, 0)^T$

$$J = \begin{pmatrix} 0 & -\frac{\tau}{2} - 1 \\ 2\tau & 0 \end{pmatrix} \quad \rightarrow \quad \lambda_{1,2} = \pm 2i \sqrt{\frac{\tau}{2} \left(\frac{\tau}{2} + 1 \right)} \quad (4.11)$$

that is the SFC is always a linear center of the system, since the eigenvalues of the Jacobian matrix are purely imaginary complex conjugated numbers for whatever value of $\tau > 0$, confirming the validity of the argument of section 4.1.

- $\vec{y}^{(SPAC)} = (\pi, 0)^T$

$$J = \begin{pmatrix} 0 & \frac{\tau}{2} - 1 \\ -2\tau & 0 \end{pmatrix} \quad \rightarrow \quad \lambda_{1,2} = \pm 2i \sqrt{\frac{\tau}{2} \left(1 - \frac{\tau}{2} \right)} \quad (4.12)$$

that is the SPAC is a linear center of the system for $\tau > \tau_c = 2$, a saddle for $\tau < \tau_c = 2$, since for $\tau < \tau_c$ one eigenvalue is positive and one is negative.

- $\vec{y}^{(PAC,\pm)} = \left(\pi, \pm \sqrt{4 - \tau^2} \right)^T$, existing only if $\tau < \tau_c = 2$

$$J = \begin{pmatrix} 0 & -\frac{4}{\tau^2} - 1 \\ -2\tau & 0 \end{pmatrix} \quad \rightarrow \quad \lambda_{1,2} = \pm 2i \sqrt{\frac{\tau}{2} \left(1 + \frac{4}{\tau^2} \right)} \quad (4.13)$$

that is the PACs are always linear centers of the system when they exist, i.e. for $\tau > \tau_c = 2$.

Remember that, as explained in section 2.4, we are as always considering τ in units of $u\rho_0$ and exploiting the invariance of the DEs under a rescaling of ρ_0 , considering $u = 1$ and $\rho_0 = 1$ without loss of generality. We therefore recall that $\tau_c = 2$ actually means $\tau_c = 2u\rho_0$ and stress that the dynamics of the system is ruled by the ratio $\eta = u\rho_0/\tau$, that is by the competition between hopping strength τ and in-situ interaction $u\rho_0$.

Notice that considering the 4×4 -dimensional Jacobian (4.7) would just have generated 2 additional non relevant eigenvalues of value 0, as it will be shown in section 4.7.

As already noticed in section 3.4 and figure (3.2), we observe the SCs to be isolated in the phase space, as obvious since they are finite in number. An important consequence of the isolation of the SCs together with the presence of a conserved quantity of the motion (the energy) is that the linear centers are actually also nonlinear centers [61]. This information is powerful and communicating us that the solution of the linearized DEs is accurate when close to a SC, no matter if this is a saddle or a linear center, and that therefore the presence of spirals is therefore excluded even for the whole nonlinear dynamics (3.5). This can be understood clearly looking at figure (4.1.1), where we show the energy landscape in the surroundings of the SPAC for $\tau > \tau_c$ and one possible trajectory (in blue). Since energy is conserved and the SPAC is isolated, the trajectory must necessary be a cycle around the SPAC even in the nonlinear case. To complement the study of the

FPs of the system, we report in figure (4.1a-h) the graphical representation of the flux lines (given by the field $\frac{d\vec{q}}{dt}$ at each point of the phase space) and of the trajectories in the phase space for various τ (and $u = \rho_0 = 1$ as usual). Trajectories are obtained numerically integrating the equations of motion (3.5) for different ICs.

The study of the dynamics would be very much facilitated by the presence of one attractive SC (that is with all the associated Jacobian eigenvalues negative), especially if its basin of attraction is extended to the whole phase space allowing to state with certainty that for long-times the system relaxes to the SC. This is the case for instance in the notorious Kuramoto model, whose solution is indeed starting from this observation and focusing on long-time [53]. However, we have found the phase space of the two-site BH model to be characterized by the presence of centers and saddles only. As we will see in the following sections, this is more in general a distinctive feature of the FCBHM for all $V \geq 2$. Still, as already noticed in section 4.2, the dynamics is dramatically different in the surroundings of a center rather than a saddle, and it is therefore still very interesting to look if and for which values of the model parameters the nature of a SC changes from saddle to center or vice versa. Indeed, we have that

- SPAC is a saddle or a center respectively for $\tau < \tau_c = 2$ and $\tau > \tau_c = 2$. This means that if the system is initialized to be arbitrarily close to SPAC at $t = 0$, then it will evolve in completely different ways depending on $\tau < \tau_c = 2$ or $\tau > \tau_c = 2$.
- SF is always a center. Therefore, a system initialized to be close to SF at $t = 0$ will always stay close to it, no matter of τ .

We anticipate that several of the results obtained for the two-site system will be in close analogy with what observed for the $V > 2$ case treated in the following sections.

Locked and unlocked solutions

Notice that, because of periodic boundary conditions in θ , a trajectory exiting from one side of the θ domain is reentering into the other side. That is, if a trajectory passes through (π, δ^*) and $(-\pi, \delta^*)$, then it is periodic and θ increases (or decreases) of an amount 2π at each period. Let's call such kind of trajectories *unlocked*, in the sense that one of the two oscillators overcomes the other infinitely many times and always in the same direction, i.e. winds around the origin more often than the other one. Conversely, let's say *locked* all the other trajectories, that are the ones forming closed orbits in the phase space (notice that the closure of an orbit can also occur through the domains limit $\theta = \pi$ and $\theta = -\pi$ thanks to periodic boundary conditions). In *locked* trajectories, the phase difference θ is periodic remains therefore bounded. That is there is no oscillator overcoming the other, and they wind around the origin the same number of times and in the same direction.

For $\tau < \tau'_c \approx 1$ there are both locked and unlocked trajectories (separated by a separatrix trajectory) whereas for $\tau > \tau'_c \approx 1$ only locked trajectories are present. We say $\tau'_c \approx 1$ since τ'_c does not emerge from the linear stability analysis but only from numerics.

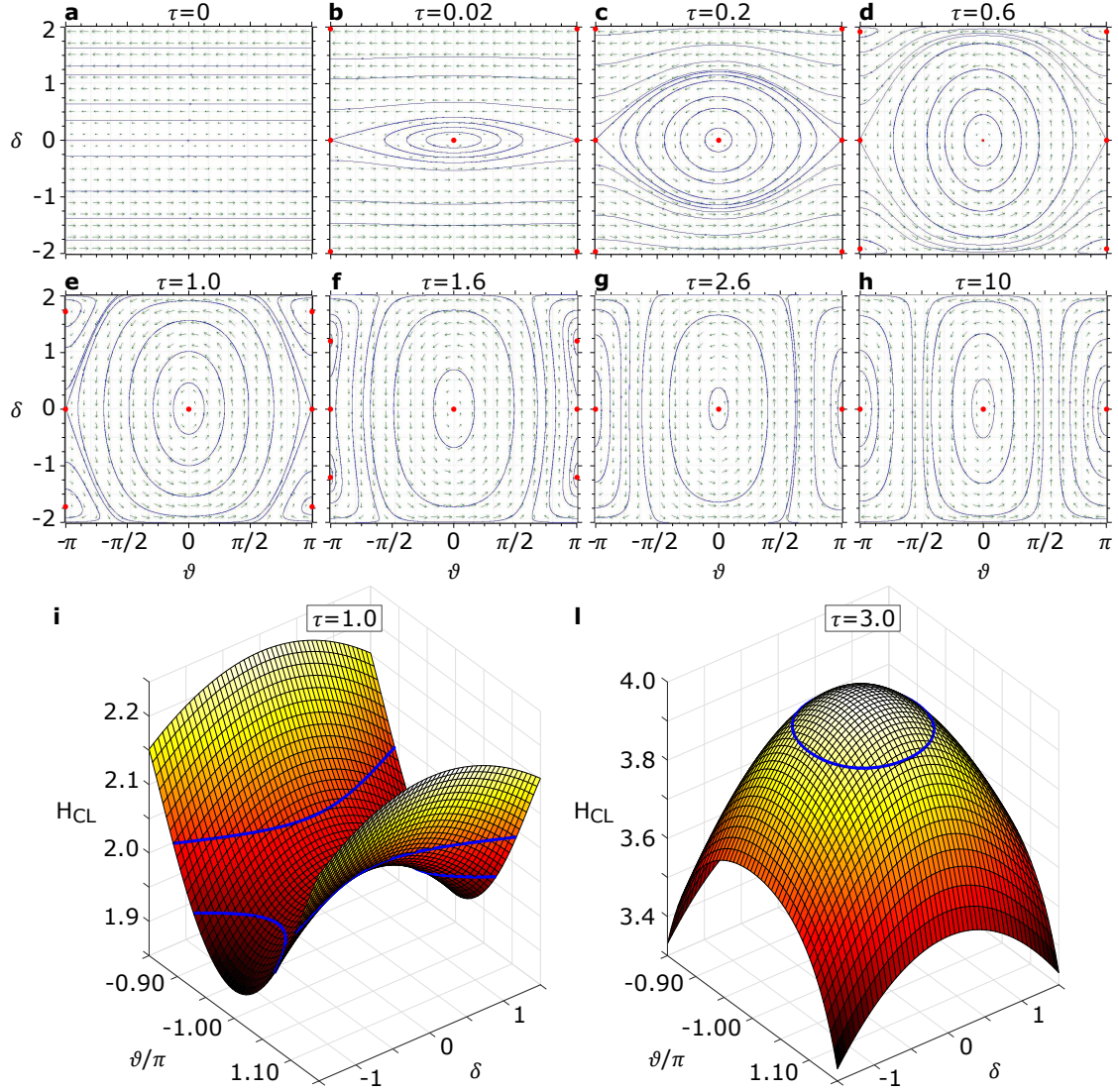


Figure 4.1 | Two-site Bose-Hubbard phase portrait. **a-h**, Representation of trajectories (blue lines), flux (green arrows) and FPs (red dots) in the (θ, δ) phase space associated to the dynamical system (3.5) for $u\rho_0 = 1$ and $\tau = 0, 0.02, 0.2, 0.6, 1.0, 1.6, 2.6, 10$ respectively. Accordingly with the linearization study reported in section 4.3, we observe $FP_0 = (0, 0)$ to be a center for whatever τ , FP_π to be a center for $\tau > \tau_c = 2$ and a saddle for $\tau < \tau_c$, $FP_\pm$ to be present only for $\tau < \tau_c$ and to be centers. Periodic oscillatory solutions correspond to closed blue orbits, where the closure is possibly meant to be through the boundaries $\theta = \pm\pi$ because of periodic boundary conditions on the phase. **i, l**, landscape of the classical energy H_{CL} 2.21 around $FP_\pi = (\pi, 0)$ and representation of some trajectories (in blue) for $\tau = 1$ (**i**) and $\tau = 3$ (**l**). The conservation of the energy and the isolation of FP_π make the latter a nonlinear center.

4.4 Jacobian for $V \geq 3$

Apart from $V = 2$ sites, to our knowledge, stability studies of FCBHM through linearization of the GPE have been done only for $V = 3$ sites [7, 9]. One of the primary goals of the present work is to extend these studies to the generic V case performing an exact diagonalization of the $(2V \times 2V)$ -dimensional Jacobian matrix J associated to the DEs (4.5). Notice that the frame angular speed Ω depends on the particular considered SC, since it has to match the global angular speed of the phases of the configuration. Nevertheless, we do not really care about it since Ω will cancel out when computing the Jacobian matrix, without the need of explicitly finding it. Considering that the first and the second halves of the variable $\vec{y}$ are referring to the phases $\{\theta_k\}_{k=1,\dots,V}$ and to the moduli $\{x_k = \sqrt{\rho_k}\}_{k=1,\dots,V}$ respectively, it is convenient to distinguish the following 4 terms of the Jacobian (4.7)

$$\begin{cases} J_{j,k} &= \frac{\partial}{\partial \theta_k} \left(\frac{d\theta_j}{dt} \right) \\ J_{j+V,k+V} &= \frac{\partial}{\partial x_k} \left(\frac{dx_j}{dt} \right) \\ J_{j+V,k} &= \frac{\partial}{\partial \theta_k} \left(\frac{dx_j}{dt} \right) \\ J_{j,k+V} &= \frac{\partial}{\partial x_k} \left(\frac{d\theta_j}{dt} \right) \end{cases} \quad j, k = 1, \dots, V \quad (4.14)$$

In the following we implicitly assume that the indexes k and j run over $1, 2, \dots, V$. To build the Jacobian matrix we need to evaluate the following partial derivatives

$$\begin{cases} \frac{\partial}{\partial \theta_k} r \cos(\phi - \theta_j) = -\frac{1}{V} x_k \sin(\theta_k - \theta_j) & \text{for } k \neq j \\ \frac{\partial}{\partial \theta_k} r \cos(\phi - \theta_j) = \frac{1}{V} \sum_{i \neq j}^V x_i \sin(\theta_i - \theta_j) = r \sin(\phi - \theta_j) & \text{for } k = j \\ \frac{\partial}{\partial \theta_k} r \sin(\phi - \theta_j) = \frac{1}{V} x_k \cos(\theta_k - \theta_j) & \text{for } k \neq j \\ \frac{\partial}{\partial \theta_k} r \sin(\phi - \theta_j) = -\frac{1}{V} \sum_{i \neq j}^V x_i \cos(\theta_i - \theta_j) = -r \cos(\phi - \theta_j) + \frac{x_j}{V} & \text{for } k = j \\ \frac{\partial}{\partial x_k} r \cos(\phi - \theta_j) = \frac{1}{V} \cos(\theta_k - \theta_j) & \text{for } k \neq j \\ \frac{\partial}{\partial x_k} r \cos(\phi - \theta_j) = \frac{1}{V} & \text{for } k = j \\ \frac{\partial}{\partial x_k} r \sin(\phi - \theta_j) = \frac{1}{V} \sin(\theta_k - \theta_j) & \text{for } k \neq j \\ \frac{\partial}{\partial x_k} r \sin(\phi - \theta_j) = 0 & \text{for } k = j \end{cases} \quad (4.15)$$

that can be compacted in

$$\begin{cases} \frac{\partial}{\partial \theta_k} r \cos(\phi - \theta_j) = -\frac{1}{V} x_k \sin(\theta_k - \theta_j) + \delta_{k,j} r \sin(\phi - \theta_j) \\ \frac{\partial}{\partial \theta_k} r \sin(\phi - \theta_j) = \frac{1}{V} x_k \cos(\theta_k - \theta_j) - \delta_{k,j} r \cos(\phi - \theta_j) \\ \frac{\partial}{\partial x_k} r \cos(\phi - \theta_j) = \frac{1}{V} \cos(\theta_k - \theta_j) \\ \frac{\partial}{\partial x_k} r \sin(\phi - \theta_j) = \frac{1}{V} \sin(\theta_k - \theta_j) \end{cases} \quad (4.16)$$

$\delta_{k,j}$ being the Kronecker delta ($\delta_{k,j} = 1$ if $k = j$, $\delta_{k,j} = 0$ else). Using the expressions (4.16), the Jacobian (4.7) associated to the linearization of the GPE (4.5) is found to be

given by

$$\begin{cases} J_{j,k} &= -\frac{\tau}{V} \left[\frac{x_k}{x_j} \sin(\theta_k - \theta_j) - \delta_{k,j} \frac{rV}{x_j} \sin(\phi - \theta_j) \right] \\ J_{j+V,k+V} &= -\frac{\tau}{V} \sin(\theta_k - \theta_j) \\ J_{j+V,k} &= -\frac{\tau}{V} [x_k \cos(\theta_k - \theta_j) - \delta_{k,j} rV \cos(\phi - \theta_j)] \\ J_{j,k+V} &= -2\delta_{k,j} x_k + \frac{\tau}{V} \frac{1}{x_j} \cos(\theta_k - \theta_j) - \delta_{k,j} \tau r \frac{1}{x_k^2} \cos(\phi - \theta_k) \end{cases} \quad (4.17)$$

In particular, for the case of stationary configurations with $r = 0$ (comprising the UC, the SPAC and the DC) the Jacobian (4.17) simplifies to

$$\begin{cases} J_{j,k} &= -\frac{\tau}{V} \sin(\theta_k - \theta_j) \\ J_{j+V,k+V} &= -\frac{\tau}{V} \sin(\theta_k - \theta_j) \\ J_{j+V,k} &= -\frac{\tau}{V} \cos(\theta_k - \theta_j) \\ J_{j,k+V} &= +\frac{\tau}{V} \cos(\theta_k - \theta_j) - 2\delta_{k,j} \end{cases} \quad (4.18)$$

We now seek to exactly find the $2V$ eigenvalues of the Jacobian matrix (4.17) systematically for all the SCs that we have found in chapter 3 in order to study their stability. Possibly, we will find as well the eigenvectors of the Jacobian, conveying important information on how the state is diverging from the SC in the case the latter is a saddle.

An observation on the neglect of $1/V$ in the Jacobian

It is tempting to consider the large V limit and to neglect the terms scaling as $1/V$ in the Jacobian (4.17), as done for instance in reference [34]. This is however not possible and to understand why we observe that the eigenvalues of the Jacobian matrix are the roots of its characteristic polynomial, that basically consists of a determinant, that for a n -dimensional matrix is given by sums of products with n factors. The implications of this observation are readily understood considering the case of a matrix with entries all equal to $(1 + \frac{1}{n})$. It is easy to convince ourselves that in this trivial case the determinant is a sum of terms of the kind $(1 + \frac{1}{n})^n$, that for $n \rightarrow \infty$ tend to the Neper number $e \approx 2.718$ rather than to 1 (such sum is then equal 0, obviously, but this is not the point of the argument). Therefore, we keep all the terms of the Jacobian.

4.5 Stability of the uniform configuration

We start considering the case of the UC, that is we evaluate the Jacobian matrix (4.17) for $\theta_k = \frac{2\pi}{V}k$ and $x_k = \sqrt{\rho_k} = 1 \forall k = 1, \dots, V$. As it will become clear going through the mathematics, the following treatment for the UC case is valid only for $V \geq 3$. Still, the UC for the $V = 2$ case corresponds to the SPAC and will be, therefore, covered in section 4.7. Moreover, an exhaustive analysis of the $V = 2$ case has already been reported in the study of the two-site problem of sections 3.2 and 4.3. The eigenvalue problem associated to the Jacobian matrix J reads

$$J\vec{y} = \lambda\vec{y} \quad (4.19)$$

and, in order to write it in a handy fashion, we notice that the multiplication of the Jacobian matrix J times a $2V$ -dimensional column vector $\vec{y}$ with $\vec{y} = \begin{pmatrix} \vec{y}^{(1)} \\ \vec{y}^{(2)} \end{pmatrix}$ and $\vec{y}^{(1)}$ and $\vec{y}^{(2)}$ V -dimensional column vectors, reads

$$\begin{cases} (J\vec{y})_j &= -\frac{\tau}{V} \sum_{k=1}^V \sin(\theta_k - \theta_j) y_k^{(1)} + \frac{\tau}{V} \sum_{k=1}^V \cos(\theta_k - \theta_j) y_k^{(2)} - 2y_j^{(2)} \\ (J\vec{y})_{j+V} &= -\frac{\tau}{V} \sum_{k=1}^V \sin(\theta_k - \theta_j) y_k^{(2)} - \frac{\tau}{V} \sum_{k=1}^{V-1} \cos(\theta_k - \theta_j) y_k^{(1)} \end{cases} \quad (4.20)$$

Since the sine and the cosine functions can be written in terms of exponentials and since $\theta_k = \frac{2\pi}{V}k$, the form of (4.20) suggests us to consider the Discrete Fourier Transform (DFT), that for a V -dimensional vector $\vec{v}$ is given by

$$\tilde{v}_q = \frac{1}{V} \sum_{k=1}^V e^{iq\theta_k} v_k \quad q \in \mathbb{Z} \quad (4.21)$$

where we denoted with q the Fourier wavenumber. It is easy to verify that $\tilde{v}_{q_1} = \tilde{v}_{q_2}$ if $\frac{q_1 - q_2}{V} \in \mathbb{Z}$, so that it is possible to restrict, without loss of generality, $q \in \{0, 1, 2, \dots, V-1\}$ and to refer to $q = V-1$ as to $q = -1$. We are thus interested in the evaluation of the following terms

$$\begin{cases} \frac{1}{V} \sum_{i=1}^V \sin(\theta_k - \theta_j) v_i = \frac{1}{V} \sum_{i=1}^V \frac{e^{i(\theta_k - \theta_j)} - c.c.}{2i} v_i = \frac{\tilde{v}_1 e^{-i\theta_j} - c.c.}{2i} = \text{Im}\{\tilde{v}_1 e^{-i\theta_j}\} \\ \frac{1}{V} \sum_{i=1}^V \cos(\theta_k - \theta_j) v_i = \frac{1}{V} \sum_{i=1}^V \frac{e^{i(\theta_k - \theta_j)} + c.c.}{2} v_i = \frac{\tilde{v}_1 e^{-i\theta_j} + c.c.}{2} = \text{Re}\{\tilde{v}_1 e^{-i\theta_j}\} \end{cases} \quad (4.22)$$

where $\text{Re}\{\bullet\}$ and $\text{Im}\{\bullet\}$ denote respectively the real and the imaginary part of the c -number $\bullet$. Having introduced the DFT in (4.21) and having evaluated the terms of (4.22), we can write the expressions of (4.20) in the following compact form

$$\begin{cases} (J\vec{y})_j &= -\tau \text{Im}\{\tilde{y}_1^{(1)} e^{-i\theta_j}\} + \tau \text{Re}\{\tilde{y}_1^{(2)} e^{-i\theta_j}\} - 2y_j^{(2)} \\ (J\vec{y})_{j+V} &= -\tau \text{Im}\{\tilde{y}_1^{(2)} e^{-i\theta_j}\} - \tau \text{Re}\{\tilde{y}_1^{(1)} e^{-i\theta_j}\} \end{cases} \quad (4.23)$$

that allows us to write the eigenvalue problem (4.19) as

$$\begin{cases} \lambda y_j^{(1)} &= -\tau \text{Im}\{\tilde{y}_1^{(1)} e^{-i\theta_j}\} + \tau \text{Re}\{\tilde{y}_1^{(2)} e^{-i\theta_j}\} - 2y_j^{(2)} \\ \lambda y_j^{(2)} &= -\tau \text{Im}\{\tilde{y}_1^{(2)} e^{-i\theta_j}\} - \tau \text{Re}\{\tilde{y}_1^{(1)} e^{-i\theta_j}\} \end{cases} \quad (4.24)$$

The solution of (4.24) will provide us with the Jacobian eigenvalues λ_n (with $n = 1, 2, \dots, 2V$) containing information on the stability of the UC. To approach equation

(4.24) we aim to perform a DFT on it. To do that, we evaluate the following terms

$$\begin{cases} (\text{Im}\{Ae^{-i\theta_j}\})_1 = \frac{1}{V} \sum_{j=1}^V \frac{A-A^*e^{2i\theta_j}}{2i} & = -i\frac{A}{2} \\ (\text{Re}\{Ae^{-i\theta_j}\})_1 = \frac{1}{V} \sum_{j=1}^V \frac{A+A^*e^{2i\theta_j}}{2} & = \frac{A}{2} \\ (\text{Im}\{Ae^{-i\theta_j}\})_{-1} = \frac{1}{V} \sum_{j=1}^V \frac{Ae^{-2i\theta_j}-A^*}{2i} & = +i\frac{A^*}{2} \\ (\text{Re}\{Ae^{-i\theta_j}\})_{-1} = \frac{1}{V} \sum_{j=1}^V \frac{Ae^{-2i\theta_j}+A^*}{2} & = \frac{A^*}{2} \\ (\text{Im}\{Ae^{-i\theta_j}\})_q = \frac{1}{V} \sum_{j=1}^V \frac{Ae^{i(q-1)\theta_j}-A^*e^{i(q+1)\theta_j}}{2i} & = 0 \quad q = 0, 2, 3, \dots, V-2 \\ (\text{Re}\{Ae^{-i\theta_j}\})_q = \frac{1}{V} \sum_{j=1}^V \frac{Ae^{i(q-1)\theta_j}+A^*e^{i(q+1)\theta_j}}{2} & = 0 \quad q = 0, 2, 3, \dots, V-2 \end{cases} \quad (4.25)$$

A being whatever complex number and $(\bullet_j)_q$ being an alternative notation for the DFT of the function $\bullet_j$ with respect to the Fourier wavenumber q (that is $(\bullet_j)_q = \tilde{\bullet}_q$). Importantly, we notice that the expressions for $q = \pm 1$ of (4.25) are valid if and only if $V \geq 3$, since $\sum_{j=1}^V Ae^{\pm 2i\theta_j} \neq 0$ for $V = 2$. We therefore assume for the following treatment that $V \geq 3$. Performing the DFT of (4.24) for $q = \pm 1$ gives

$$\begin{cases} \lambda \tilde{y}_1^{(1)} & = i\frac{\tau}{2} \tilde{y}_1^{(1)} + (\frac{\tau}{2} - 2) \tilde{y}_1^{(2)} \\ \lambda \tilde{y}_1^{(2)} & = i\frac{\tau}{2} \tilde{y}_1^{(2)} - \frac{\tau}{2} \tilde{y}_1^{(1)} \end{cases} \quad (4.26)$$

$$\begin{cases} \lambda \tilde{y}_{-1}^{(1)} & = -i\frac{\tau}{2} \tilde{y}_{-1}^{(1)} + (\frac{\tau}{2} - 2) \tilde{y}_{-1}^{(2)} \\ \lambda \tilde{y}_{-1}^{(2)} & = -i\frac{\tau}{2} \tilde{y}_{-1}^{(2)} - \frac{\tau}{2} \tilde{y}_{-1}^{(1)} \end{cases} \quad (4.27)$$

that are 2-dimensional eigenvalue problems for $\tilde{y}_1^{(1)}, \tilde{y}_1^{(2)}$ and for $\tilde{y}_{-1}^{(1)}, \tilde{y}_{-1}^{(2)}$ respectively and where the subscripts ± 1 refer to the Fourier wavevector q whereas the superscripts 1,2 refer to the bipartition of $\vec{y}$ in the two halves. We can rewrite the problems (4.26) and (4.27) in matricial form as

$$\begin{pmatrix} i\frac{\tau}{2} & (\frac{\tau}{2} - 2) \\ -\frac{\tau}{2} & i\frac{\tau}{2} \end{pmatrix} \begin{pmatrix} \tilde{y}_1^{(1)} \\ \tilde{y}_1^{(2)} \end{pmatrix} = \lambda \begin{pmatrix} \tilde{y}_1^{(1)} \\ \tilde{y}_1^{(2)} \end{pmatrix} \quad (4.28)$$

$$\begin{pmatrix} -i\frac{\tau}{2} & (\frac{\tau}{2} - 2) \\ -\frac{\tau}{2} & -i\frac{\tau}{2} \end{pmatrix} \begin{pmatrix} \tilde{y}_{-1}^{(1)} \\ \tilde{y}_{-1}^{(2)} \end{pmatrix} = \lambda \begin{pmatrix} \tilde{y}_{-1}^{(1)} \\ \tilde{y}_{-1}^{(2)} \end{pmatrix} \quad (4.29)$$

With respective characteristic polynomials $P_{\pm 1}(\lambda)$, eigenvalues $\lambda_{\pm 1}^{\pm}$ and $\lambda_{-1}^{\pm}$ and eigenvectors $v_{\pm 1}^{\pm}$ and $v_{-1}^{\pm}$ equal to

$$P_1(\lambda) = \lambda^2 - i\tau\lambda - \tau \quad P_{-1}(\lambda) = \lambda^2 + i\tau\lambda - \tau \quad (4.30)$$

$$\lambda_1^{\pm} = \frac{i\tau \pm \sqrt{4\tau - \tau^2}}{2} \quad \lambda_{-1}^{\pm} = \frac{-i\tau \pm \sqrt{4\tau - \tau^2}}{2} \quad (4.31)$$

$$v_1^{\pm} = \left(\mp \sqrt{\tau(4 - \tau)}, \tau \right)^T \quad v_{-1}^{\pm} = v_1^{\pm} \quad (4.32)$$

Having found 4 of the $2V$ eigenvalues, we proceed looking for other non-zero eigenvalues, that is for $\lambda \notin \{0, \lambda_1^+, \lambda_1^-, \lambda_{-1}^+, \lambda_{-1}^-\}$. Performing the DFT of equation (4.24) for $q \neq \pm 1$ we get

$$\begin{cases} \lambda \tilde{y}_q^{(1)} &= -2\tilde{y}_q^{(2)} \\ \lambda \tilde{y}_q^{(2)} &= 0 \end{cases} \quad \text{for } q = 0, 2, 3, \dots, V-2 \quad (4.33)$$

that, assuming $\lambda \neq 0$, is solved by $\tilde{y}_q^{(1)} = \tilde{y}_q^{(2)} = 0 \ \forall q = 0, 2, 3, \dots, V-2$. We observe that, if $\lambda \notin \{\lambda_1^+, \lambda_1^-, \lambda_{-1}^+, \lambda_{-1}^-\}$, then $\tilde{y}_1^{(1)} = \tilde{y}_1^{(2)} = \tilde{y}_{-1}^{(1)} = \tilde{y}_{-1}^{(2)} = 0$, since equations (4.28) and (4.29) need still to be satisfied. This implies that $\vec{y} = 0$, being all its Fourier components equal to 0. Thus, we conclude that the only non-zero eigenvalues are $\lambda_1^+, \lambda_1^-, \lambda_{-1}^+, \lambda_{-1}^-$, and that $\lambda_0 = 0$ is an eigenvalue with algebraic multiplicity $m_a = 2V-4$.

Additionally, it is actually possible to explicitly find the eigenvectors associated to $\lambda_0 = 0$. In fact, equation (4.33) for $\lambda = 0$ gives $\tilde{y}_q^{(2)} = 0$ for $q = 0, 2, 3, \dots, V-2$. Moreover, again we can say that because $\lambda_0 \notin \{\lambda_1^+, \lambda_1^-, \lambda_{-1}^+, \lambda_{-1}^-\}$, then $\tilde{y}_1^{(1)} = \tilde{y}_1^{(2)} = \tilde{y}_{-1}^{(1)} = \tilde{y}_{-1}^{(2)} = 0$. Therefore the only constraints on the eigenvectors with eigenvalue λ_0 are that $\vec{y}^{(2)} = 0$ and $\tilde{y}_1^{(1)} = \tilde{y}_{-1}^{(1)} = 0$. The eigenvectors of the Jacobian with eigenvalue $\lambda_0 = 0$ are thus

$$\begin{aligned} \text{readily obtained as } \vec{v}_q &= \begin{pmatrix} \vec{v}^{(1,q)} \\ \vec{v}^{(2)} \end{pmatrix} \text{ with} \\ \begin{cases} v_j^{(1,q)} &\propto e^{iq\frac{2\pi}{V}j} \\ v_j^{(2)} &= 0 \end{cases} &\quad q \in \{0, 2, 3, \dots, V-2\} \end{aligned} \quad (4.34)$$

It is easy to convince ourselves that, because of the above mentioned constraints, there are no other eigenvectors corresponding to λ_0 .

Summing up, the eigenvalues of the Jacobian matrix and their respective algebraic multiplicities for the UC for a number of sites $V \geq 3$ are

$$\lambda_0 = 0 \quad m_a = 2V - 4 \quad (4.35)$$

$$\lambda_1^+ = \frac{i\tau + \sqrt{4\tau - \tau^2}}{2} \quad m_a = 1 \quad (4.36)$$

$$\lambda_1^- = \frac{i\tau - \sqrt{4\tau - \tau^2}}{2} \quad m_a = 1 \quad (4.37)$$

$$\lambda_{-1}^+ = \frac{-i\tau + \sqrt{4\tau - \tau^2}}{2} \quad m_a = 1 \quad (4.38)$$

$$\lambda_{-1}^- = \frac{-i\tau - \sqrt{4\tau - \tau^2}}{2} \quad m_a = 1 \quad (4.39)$$

For completeness we report the characteristic polynomial of the Jacobian matrix, that is easily given by $P(\lambda) = \prod_{i=1}^{2V} (\lambda - \lambda_i)$, reading

$$P(\lambda) = \lambda^{2V-4} \left(\lambda^4 + \tau(\tau - 2)\lambda^2 + \tau^2 \right) \quad (4.40)$$

Having found all the eigenvalues, it is possible to argue on the stability of the UC. We see that for $\tau < \tau_c = 4$ some eigenvalues $(\lambda_1^+, \lambda_2^+)$ have positive real part and some

others $(\lambda_1^-, \lambda_2^-)$ negative real part, making the UC a saddle point of the dynamics. For $\tau > \tau_c = 4$ instead $\lambda_1^+, \lambda_1^-, \lambda_2^+, \lambda_2^-$ are all purely imaginary, making the UC a linear center of the dynamics. We then conclude that for $\tau < \tau_c = 4$, a system initialized closeby the UC (and not exactly on the UC) will drift away from it exponentially fast. The case $\tau > \tau_c$ is actually delicate, since it is not guaranteed that the UC is also a nonlinear center and will require the attention of the chapter 5. As always, we stress that we have considered τ in units of $u\rho_0$, and that therefore $\tau_c = 4$ actually means $\tau_c = 4u\rho_0$, from which we explicitly see that the competition between the hopping strength τ and the in-situ interaction $u\rho_0$ is determinant for the dynamics of the system.

We have therefore found that, depending on the value of the ratio between the hopping strength τ and the in-situ interaction $u\rho_0$, for a system that is initialized in the proximity of the UC (for instance as in the case of MI to SF quench for large V), two qualitatively very different dynamical behaviours are possible. Since the transition between these two regimes is very sharp, and precisely occurring at $\frac{\tau}{u\rho_0} = 4$, such change of behavior is referred to as Dynamical Phase Transition (DPT) [47] and will be further discussed in the following.

4.5.1 Stability of $r = 0$

We have found that for $\tau < \tau_c = 4$ the system will drift away from the UC when initialized close to it. Still, it is not clear how this affects the time evolution of r . Information on such drift are contained into what we call *unstable eigenvectors* of the Jacobian, that is the eigenvectors associated to the eigenvalues with positive real parts, that is $v_{-1}^+ = v_1^+$. Indeed, after an possible short transient, the system aligns along the direction defined by these two eigenvectors. To be explicit, we consider a configuration initialized close to the UC as

$$\begin{cases} \theta_j = \frac{2\pi}{V}j + \delta_{\theta,j} \\ x_j = 1 + \delta_{x,j} \quad \forall j = 1, \dots, V \end{cases} \quad (4.41)$$

being $\delta_{\theta,j}, \delta_{x,j} \ll 1 \quad \forall j = 1, 2, \dots, V$. For such configuration we observe that we can write $re^{i\phi}$ as

$$re^{i\phi} = \frac{1}{V} \sum_{j=1}^V x_j e^{i\theta_j} = \frac{1}{V} \sum_{j=1}^V (1 + \delta_{x,j}) e^{i\delta_{\theta,j}} e^{i\frac{2\pi}{V}j} \quad (4.42)$$

that corresponds to a DFT of the term $(1 + \delta_{x,j})e^{i\delta_{\theta,j}}$. Approximating the exponential at linear order we obtain

$$re^{i\phi} \approx \tilde{\delta}_{x1} + i\tilde{\delta}_{\theta1} \quad (4.43)$$

that is r and ϕ can be written in terms of the unstable Fourier modes, so that it is easy to conclude that for $\tau < \tau_c = 4$

$$re^{i\phi} \sim \left(-i\sqrt{\tau(4-\tau)} + \tau \right) e^{\frac{\sqrt{4\tau-\tau^2}}{2}t} \quad (4.44)$$

where we used the symbol $\sim$ to point out that the exponential divergence will occur after an possible short transient during which the system aligns with the unstable eigenvector. Considering only the exponential dependence, we have therefore that $r \sim e^{t/t_U}$ where the characteristic time scale for the exponential divergence is $t_U = \frac{\sqrt{4\tau - \tau^2}}{2}$ that, close to the transition, gives $t_U \sim \left|1 - \frac{\tau}{\tau_c}\right|^\beta$ with $\beta = 1/2$.

As we observed in section 4.1, because of the conservation of the expected energy and total number of particles, the increase of r will necessarily correspond to an increase of the spread of the number of bosons at each site ρ_k . In a lattice model one could maybe talk about the growth of a boson number wave, but in a FC model, where no metric is defined, this is not possible. In particular, from equation (4.3) we get for a system initialized in the proximity of the UC (relevant for instance for a MI to SF quench when $V \gg 1$) that

$$\begin{cases} \text{Var}[\rho_j] = 2\tau r^2 \\ \langle \rho_j \rangle_V = 1 \end{cases} \quad (4.45)$$

where $\text{Var}[\rho_j]$ denotes the variance over the sites of the number of bosons ρ_j at each site. We thus get that such variance grows exponentially as $\sim e^{2t/t_U}$.

We conclude our analysis by pointing out that the linear stability holds only for short times. Determining until what value r will grow and how it will then behave for $\tau < \tau_c = 4$ is not easy and requires the study of the full, nonlinear GPE (2.9), that will be addressed in chapter 5.

4.5.2 Numerical results

We numerically solve the DEs (2.16) with the MATLAB built in adaptive ordinary differential equations solver `ode45`. We check the stability of the UC by initializing the system to a configuration close to the UC and looking at its time evolution. Since we consider $\text{IC} \approx \text{UC}$ and $\text{IC} \neq \text{UC}$, then $0 < r_0 = r(t=0) \ll 1$ and we are interested in looking whether r grows exponentially or not and for which values of the hopping strength τ .

In particular, we consider as IC the following configuration

$$\begin{cases} \theta_k = \frac{2\pi}{V}k \\ \rho_k = 1 + A \sin(\theta_k) \end{cases} \quad (4.46)$$

$A = 10^{-7}$ being an arbitrarily small value giving $r_0 = 2.50 \times 10^{-8}$. The choice of the sinusoidal noise is due to the fact that the eigenvalues causing instability for the UC are associated with Fourier wavenumber $q = \pm 1$ as seen in subsection 4.5. Of course, the following results hold for a generic random noise and that the sinusoidal noise and the choice of the magnitude of A is just arbitrary. We observe, as expected, a sharp transition at $\tau_c = 4$, with an initial exponential growth of r for $\tau < \tau_c$ in accordance with the predicted time scale $t_U \sim \sqrt{1 - \frac{\tau}{\tau_c}}$ found in subsection 4.5.1. Moreover, we observe that larger r correspond to larger variances of the number of bosons per site ρ_j (**c** in comparison with **b,d,e**), according to the arguments of section 4.1. Notice as well that, since we cannot claim the UC to be a nonlinear center for $\tau > \tau_c$, we cannot a priori

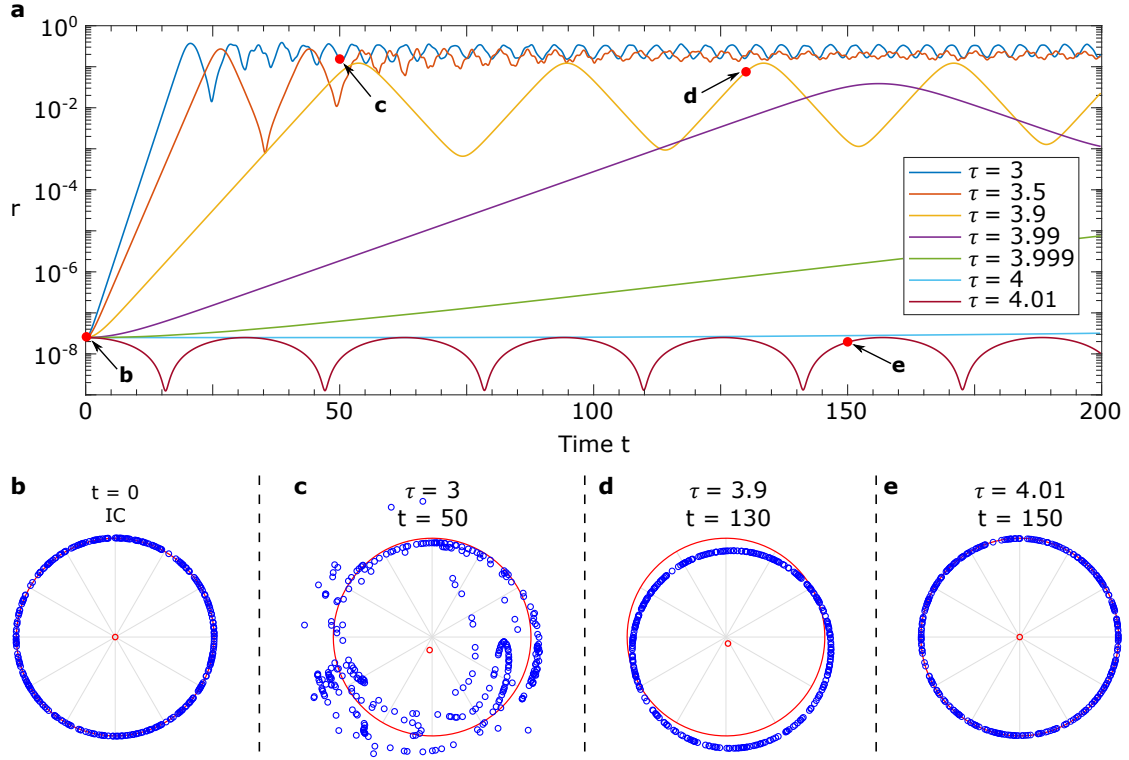


Figure 4.2 | Dynamical phase transition for the uniform configuration. **a**, Time evolution of $r = \left| \frac{1}{V} \sum_{j=1}^V \sqrt{\rho_j} e^{i\theta_j} \right|$ for $V = 1000$ sites, for $\tau = 3, 3.5, 3.9, 3.99, 3.999, 4, 4.01$ and initializing the system in the proximity of the UC as in (4.46). As expected we observe an initial exponential growth of r only for $\tau < \tau_c = 4$, highlighting the DPT due to the emergence of positive real parts in some of the non-zero Jacobian eigenvalues. **b-e**, Graphical representation of the bosonic variables for different τ and times, corresponding to the red dots in **a**. For graphical clarity, only a sample of 300 sites is represented.

consider the system configuration to remain similar to the one of figure (4.2b-e) even for later times. Indeed, we will observe in chapter 5 that at long-times the nonlinearities of the DEs can take the system to other new configurations, such as the partially π -aligned ones.

4.6 Stability of the superfluid configuration

Analogously to what we have done for the UC, we now study the stability of the SFC, that is the configuration for which $\theta_k = 0$, $\rho_k = 1 \forall k = 1, 2, \dots, V$. The argument of section 4.1 is actually sufficient to state that the SFC is a nonlinear center of the dynamics for whatever $\tau > 0$ but, for completeness, we report here a direct study of its stability

diagonalizing the respective Jacobian matrix. For the SF, the Jacobian (4.17) reads

$$\begin{cases} J_{j,k} &= 0 \\ J_{j+V,k+V} &= 0 \\ J_{j+V,k} &= -\frac{\tau}{V} + \tau\delta_{k,j} \\ J_{j,k+V} &= +\frac{\tau}{V} - (2 + \tau)\delta_{k,j} \end{cases} \quad (4.47)$$

and is therefore built up of $V \times V$ -dimensional blocks. It is therefore still natural to write a $2V$ -dimensional column vector $\vec{y}$ as $\vec{y} = \begin{pmatrix} \vec{y}^{(1)} \\ \vec{y}^{(2)} \end{pmatrix}$, $\vec{y}^{(i)}$ being a V -dimensional column vector. The eigenvalue problem (4.19) can therefore be written as

$$J\vec{y} = \begin{pmatrix} +\tau\tilde{y}_0^{(2)} - (2 + \tau)\vec{y}^{(2)} \\ -\tau\tilde{y}_0^{(1)} + \tau\vec{y}^{(1)} \end{pmatrix} = \lambda \begin{pmatrix} \vec{y}^{(1)} \\ \vec{y}^{(2)} \end{pmatrix} \quad (4.48)$$

where $\tilde{y}_0^{(i)} = \frac{1}{V} \sum_{j=1}^V \vec{y}_j^{(i)}$. Looking for non-zero eigenvalues, that is $\lambda \neq 0$, we can multiply by λ the first equation of (4.48) getting

$$\begin{cases} +\tau\lambda\tilde{y}_0^{(2)} - (2 + \tau)\lambda\vec{y}^{(2)} = \lambda^2\vec{y}^{(1)} \\ -\tau\lambda\tilde{y}_0^{(1)} + \tau\vec{y}^{(1)} = \lambda\vec{y}^{(2)} \end{cases} \quad (4.49)$$

From the second equation we readily get $\lambda\tilde{y}_0^{(2)}$ and, plugging it into the first equation we get

$$-(2 + \tau) \left(-\tau\tilde{y}_0^{(1)} + \tau\vec{y}^{(1)} \right) = \lambda^2\vec{y}^{(1)} \quad (4.50)$$

from which we find that $\tilde{y}_0^{(1)} = 0$, so that

$$-\tau(2 + \tau)\vec{y}^{(1)} = \lambda^2\vec{y}^{(1)} \quad (4.51)$$

giving $\lambda_{1,2} = \pm i\sqrt{\tau(2 + \tau)}$. These are the only non-zero eigenvalues and can therefore be used to argue on the stability of the SFC. Since for whatever value of τ the non-zero eigenvalues are purely imaginary complex conjugate numbers, the SFC is a linear center of the dynamics. Since the SFC is an isolated SC (as observed in section 3.4 and figure (3.2)) and since the dynamical system is conserving energy and number of particles, we can actually claim the SFC to be as well a nonlinear center. We conclude thus that initializing the system to a configuration in the neighborhoods of the SFC the former will cycle around the latter for whatever τ . This result generalizes the one shown for the two-site model. As we already noticed in section 4.1, the SFC is actually not only a linear center, but a nonlinear center as well.

4.7 Stability of the symmetric π -aligned configuration

We aim now at studying the stability of the SPAC, that is the one defined for an even V by $\theta_k = 0$ for $k = 1, 2, \dots, V/2$, $\theta_k = \pi$ for $k = V/2 + 1, \dots, V$ and $\rho_k = 1 \quad \forall k = 1, \dots, V$. Again our goal is to diagonalize the Jacobian matrix corresponding to such configuration and to look at its eigenvalues to obtain information on the configuration stability. The Jacobian (4.18) simplifies to

$$\begin{cases} J_{j,k} &= 0 \\ J_{j+V,k+V} &= 0 \\ J_{j+V,k} &= -\frac{\tau}{V} \nu_k \nu_j \\ J_{j,k+V} &= +\frac{\tau}{V} \nu_k \nu_j - 2\delta_{k,j} \end{cases} \quad (4.52)$$

where $\nu_k = 1$ for $k = 1, \dots, V/2$ and $\nu_k = -1$ for $k = V/2 + 1, \dots, V$. The Jacobian matrix is therefore composed of blocks of dimension $V/2 \times V/2$ characterized by equal entries unless the terms given by the Kronecker delta $\delta_{k,j}$. It is therefore natural to write a $2V$ -dimensional column vector $\vec{y}$ as

$$\vec{y} = \begin{pmatrix} \vec{y}^{(1)} \\ \vec{y}^{(2)} \\ \vec{y}^{(3)} \\ \vec{y}^{(4)} \end{pmatrix} \quad (4.53)$$

$\vec{y}^{(i)}$ being a $V/2$ -dimensional column vector. The eigenvalue problem (4.19) reads then

$$J\vec{y} = \begin{pmatrix} +\frac{\tau}{2} \left(\tilde{y}_0^{(3)} - \tilde{y}_0^{(4)} \right) - 2\vec{y}^{(3)} \\ -\frac{\tau}{2} \left(\tilde{y}_0^{(3)} - \tilde{y}_0^{(4)} \right) - 2\vec{y}^{(4)} \\ +\frac{\tau}{2} \left(-\tilde{y}_0^{(1)} + \tilde{y}_0^{(2)} \right) \\ -\frac{\tau}{2} \left(-\tilde{y}_0^{(1)} + \tilde{y}_0^{(2)} \right) \end{pmatrix} = \lambda \begin{pmatrix} \vec{y}^{(1)} \\ \vec{y}^{(2)} \\ \vec{y}^{(3)} \\ \vec{y}^{(4)} \end{pmatrix} \quad (4.54)$$

where we introduced $\tilde{y}_0^{(i)} = \frac{2}{V} \sum_{j=1}^{V/2} \tilde{y}_j^{(i)}$.

Looking for non-zero eigenvalues, that is for $\lambda \neq 0$, we readily obtain $\vec{y}^{(2)} = -\vec{y}^{(1)}$ and $\vec{y}^{(4)} = -\vec{y}^{(3)}$, reducing the problem to

$$\begin{cases} +\tau \tilde{y}_0^{(3)} - 2\vec{y}^{(3)} &= \lambda \vec{y}^{(1)} \\ -\tau \tilde{y}_0^{(1)} &= \lambda \vec{y}^{(3)} \end{cases} \quad (4.55)$$

The second equation of (4.55) implies all the components of $\vec{y}^{(3)}$ to be equal, that is $y_j^{(3)} = \tilde{y}_0^{(3)} \quad \forall j = 1, \dots, V/2$ and $\tau \tilde{y}_0^{(1)} = -\lambda \tilde{y}_0^{(3)}$. From the first equation of (4.55) we get then that also all the components of $\vec{y}^{(1)}$ are equal, and we are thus left with

$$-\tau (\tau - 2) \tilde{y}_0^{(1)} = \lambda^2 \tilde{y}_0^{(1)} \quad (4.56)$$

Since we look for non-trivial solutions (that is with non-zero $\vec{y}$), we consider $\tilde{y}_0^{(1)} \neq 0$ and finally obtain the eigenvalues

$$\lambda^\pm = \pm \sqrt{\tau(2-\tau)} \quad (4.57)$$

Having found all the non-zero eigenvalues, we can state with certainty that the eigenvalues of the Jacobian matrix and their respective algebraic multiplicities m_a for the SPAC for an even number of sites V are

$$\lambda_0 = 0 \quad m_a = 2V - 2 \quad (4.58)$$

$$\lambda^+ = \sqrt{\tau(2-\tau)} \quad m_a = 1 \quad (4.59)$$

$$\lambda^- = -\sqrt{\tau(2-\tau)} \quad m_a = 1 \quad (4.60)$$

corresponding to all the roots of the characteristic polynomial of the Jacobian matrix, that thus turns out to be

$$P(\lambda) = \lambda^{2V-4} (\lambda^2 + \tau(\tau-2)) \quad (4.61)$$

Having found all the eigenvalues, it is possible to study the stability of the SPAC. We see that for $\tau < \tau_c = 2$ $\lambda^+ > 0$ and $\lambda^- < 0$, that is the SPAC is a saddle point of the dynamics. For $\tau > \tau_c = 2$ instead λ^+, λ^- are purely imaginary, making the SPAC a linear center. We then conclude that for $\tau < \tau_c = 2$ a system initialized arbitrarily close to the SPAC drifts away from it exponentially fast. For $\tau > \tau_c$ it is instead non-trivial to guess how the system behaves when initialized close to the SPAC, since being the SPAC non isolated (for $V \geq 4$) we cannot a priori claim it to be a nonlinear center.

Also for a system initialized in the proximity of the SPAC we have therefore spotted out the existence of a DPT occurring at $\tau_c = 2$. Once more, we stress that τ is written in units of $u\rho_0$ and that, therefore, actually $\tau_c = 2u\rho_0$, from which the role of the in-situ interaction, that is competing against the hopping strength, is clearly emerging.

4.7.1 Numerical results

Again we numerically solve the DEs (2.16) initializing the system closeby the SPAC and looking at it at later times, particularly focusing on the time evolution of r .

We consider as IC the following configuration

$$\begin{cases} \theta_k = 0 + \xi_k & \text{for } k = 1, \dots, V/2 \\ \theta_k = \pi + \xi_k & \text{for } k = V/2 + 1, \dots, V \\ \rho_k = 1 + A \sin(\theta_k) \end{cases} \quad (4.62)$$

ξ_k being a Gaussian random number with zero mean and variance $A = 10^{-7}$. The choice of the noise is in this case just arbitrary and leads to $r_0 = 4.95 \times 10^{-9}$. Qualitatively similar results are obtained for whatever IC close to the SPAC. We observe, as expected, a sharp transition at $\tau_c = 2$, with an exponential growth of r at short times and for $\tau < \tau_c$ as shown in figure (4.3).

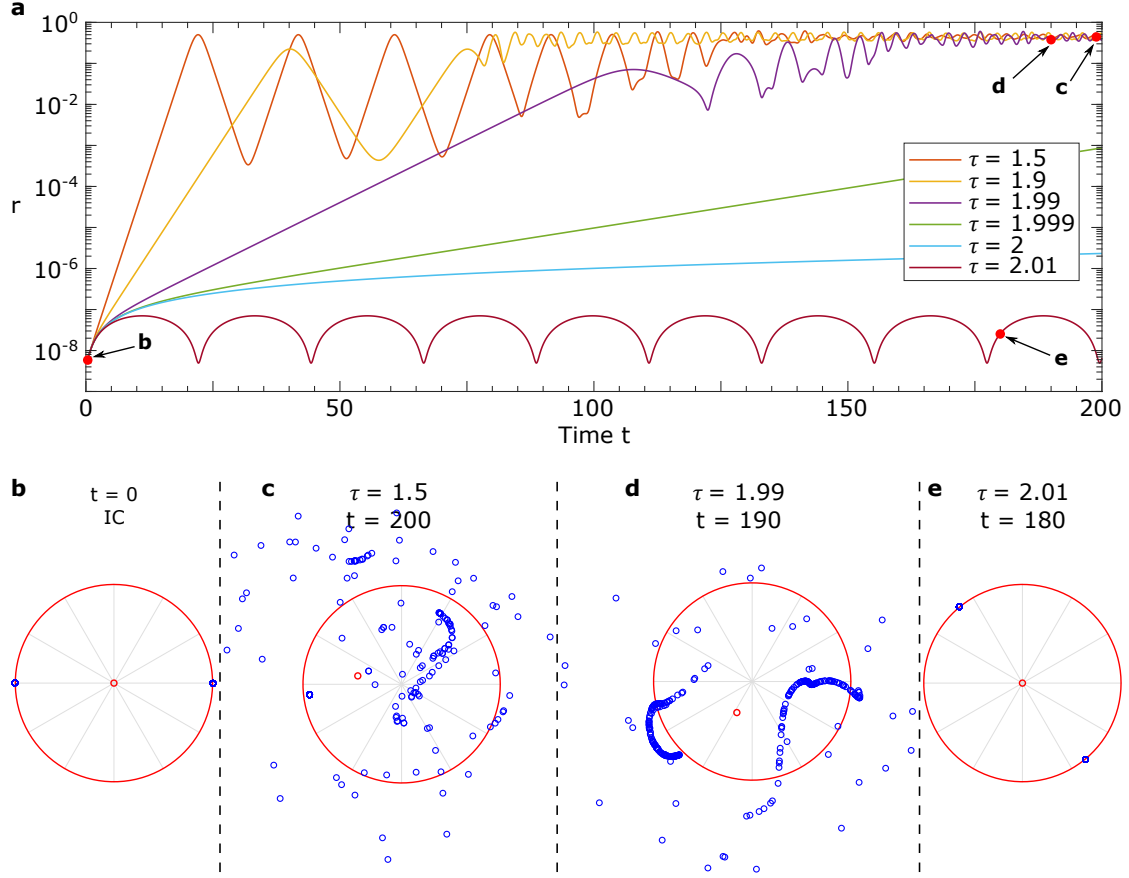


Figure 4.3 | Dynamical phase transition for the symmetric π -aligned configuration. **a**, Time evolution of $r = \left| \frac{1}{V} \sum_{j=1}^V \sqrt{\rho_j} e^{i\theta_j} \right|$ for $V = 1000$ sites, for $\tau = 1.5, 1.9, 1.99, 1.999, 2, 2.01$ and initializing the system in the proximity of the SPAC as in (4.62). As expected we observe an initial exponential growth of r only for $\tau < \tau_c = 2$, highlighting the DPT due to the emergence of positive real parts in some of the non-zero Jacobian eigenvalues. **b-e**, Graphical representation of the bosonic variables for different τ and times, corresponding to the red dots in **a**. For graphical clarity, only a sample of 300 sites is represented.

4.8 Stability of the delta configuration for $V = 4$

In sections 4.5 and 4.7 we studied the stability of the UC and of the SPAC, spotting out the existence of two critical values of the hopping strength, that are $\tau_c = 4$ ($V \geq 3$) and $\tau_c = 2$ ($V \geq 2$) respectively. In section 3.4 and figure (3.2) we also noticed that the UC and the SPAC are connected by a line of SCs with $r = 0$ in the phase space, and it is therefore natural to wonder about how can the τ_c change along this line, expecting it to range from 4 (at the extremity of the line corresponding to the UC) to 2 (at the extremity of the line corresponding to the SPAC) in a continuous manner. To carry on analytical calculations on this line (that we recall to be a proper line only for $V = 4$ and a higher dimensional manifold for $V = 6, 8, \dots$), we consider the DC for $V = 4$.

With the help of a symbolic manipulation software we find the exact characteristic polynomial of the Jacobian matrix (4.18) associated to the considered DC

$$P(\lambda) = \lambda^4 \left(\lambda^4 + \tau(\tau - 2)\lambda^2 + \tau^2 \sin(\Delta) \right) \quad (4.63)$$

From the characteristic polynomial we want as usual to understand if and when the SC is unstable. In analogy with the study of sections 4.5 and 4.7 the non-zero eigenvalues can either be purely imaginary or not. The former case corresponds as usual to a linear center whereas the latter corresponds to a saddle (in fact, since the trace of the Jacobian (4.18) is 0, in this case some of the non-zero eigenvalues have positive real part and some have negative real part). One way of looking at the critical condition for which the stationary configuration passes from being a saddle to being a linear center is therefore to look at when purely imaginary non-zero eigenvalues are roots of the characteristic polynomial $P(\lambda)$. We set therefore $\lambda = i\alpha$ with $\alpha \in \mathbb{R}$ and look for real roots of

$$p(\alpha) = \alpha^4 - \tau(\tau - 2)\alpha^2 + \tau^2 \sin(\Delta) \quad (4.64)$$

For $\tau < 2$ the equation (4.64) has not solutions, meaning that the stationary configuration is a saddle of the dynamics for whatever Δ . For $\tau \geq 2$ the equation reads $\alpha^4 - a\alpha^2 + b = 0$ with $a, b \geq 0$. The minimum of $p(\alpha)$ is in $\alpha^* = \sqrt{\frac{a}{2}}$, so that $p(\alpha^*) = b - \frac{a^2}{4}$ and that (4.64) has real solutions (corresponding to purely imaginary roots of $P(\lambda)$) only for $b \leq \frac{a^2}{4}$. The critical condition reads therefore $b = \frac{a^2}{4}$, finally giving

$$\tau_c = 2(1 + \sin(\Delta)) \quad (4.65)$$

so that the DC corresponding to a given Δ is a saddle if $\tau < \tau_c = 2(1 + \sin(\Delta))$ and a linear center if $\tau > \tau_c = 2(1 + \sin(\Delta))$. Of course, we correctly find that $\tau_c = 2$ for the SPAC ($\Delta = \pi$) and that $\tau_c = 4$ for the UC ($\Delta = \pi/2$).

4.9 Stability of the uniform configuration in the $V \rightarrow \infty$ limit

The UC stability has already been studied for whatever $V > 2$ (including thus the $V \rightarrow \infty$ limit), but it is nevertheless instructive to perform the study passing from the continuous equations (2.19) obtained in the $V \rightarrow \infty$ limit. Indeed, we expect some of the equations that we are about to derive to be potentially useful to analytically approach the nonlinear equations leading to the emergence of thermalization phenomena for large V (details in chapter 5).

Accordingly with the hypothesis of regularity for the functions $\rho(s, t)$ and $\theta(s, t)$, that is needed in the $V \rightarrow \infty$ limit, we focus here on the UC, for which

$$\begin{cases} \theta_{UC}(s, t) = s + \Omega t \\ \rho_{UC}(s, t) = 1 \end{cases} \quad (4.66)$$

As in the finite V case, such configuration is obviously a SC of the DEs (2.19). We move to the frame rotating at angular frequency $\Omega = -1$ and consider a state close to the UC, but different from it, that we write in the form

$$\begin{cases} \theta(s) = s + \xi(s) \\ \sqrt{\rho(s)} = 1 + \delta(s) \end{cases} \quad (4.67)$$

so that the equations of motion (2.19) read

$$\begin{cases} \frac{\partial \delta(s,t)}{\partial t} = \tau r \sin(\theta(s,t) - \phi) \\ \frac{\partial \xi(s,t)}{\partial t} = \frac{\tau r}{1+\delta(s,t)} \cos(\theta(s,t) - \phi) - 2\delta(s,t) - \delta(s,t)^2 \end{cases} \quad (4.68)$$

where $re^{i\phi}$ can be expressed as $re^{i\phi} = \frac{1}{2\pi} \int_0^{2\pi} (1+\delta)e^{i\xi}e^{is} = ((1+\delta)e^{i\xi})_1$, where we denoted $[A(s)]_q = \frac{1}{2\pi} \int_0^{2\pi} A(s)e^{iqs}ds$. Thus, (4.68) reads

$$\begin{cases} \frac{\partial \delta}{\partial t} = -\tau \operatorname{Im}\{((1+\delta)e^{i\xi})_1 e^{-i\theta}\} \\ \frac{\partial \xi}{\partial t} = \frac{\tau}{1+\delta} \operatorname{Re}\{((1+\delta)e^{i\xi})_1 e^{-i\theta}\} - 2\delta - \delta^2 \end{cases} \quad (4.69)$$

We Fourier transform the first equation of (4.69) getting

$$\begin{cases} \frac{\partial \delta_1}{\partial t} = i\frac{\tau}{2} ((1+\delta)e^{i\xi})_1 \\ \frac{\partial \delta_{-1}}{\partial t} = -i\frac{\tau}{2} ((1+\delta)e^{i\xi})_{-1} \\ \frac{\partial \delta_q}{\partial t} = 0 \quad \forall q \in \mathbb{Z} \setminus \{-1, 1\} \\ \frac{\partial \xi}{\partial t} = \frac{\tau}{1+\delta} \operatorname{Re}\{((1+\delta)e^{i\xi})_1 e^{-i\theta}\} - 2\delta - \delta^2 \end{cases} \quad (4.70)$$

To go from the DEs (2.19) to (4.70) we have introduced no approximations. The form of (4.70), with the DFT, is particularly convenient since for $q \neq \pm 1$ we have $\delta_q = cst$. Importantly, this is true for the whole nonlinear dynamics. In this section we are however interested just in the study of the linear dynamics, for which we proceed linearizing (4.70) to get, for $q = \pm 1$

$$\begin{cases} \frac{\partial \delta_1}{\partial t} = \frac{\tau}{2} (i\delta_1 - \xi_1) \\ \frac{\partial \delta_{-1}}{\partial t} = \frac{\tau}{2} (i\delta_{-1} - \xi_{-1}) \\ \frac{\partial \xi}{\partial t} = \frac{\tau}{2} (\delta_1 + i\xi_1) - 2\delta \end{cases} \quad (4.71)$$

Fourier transforming also the equation for θ (that is ξ), we get finally that the problem can be rewritten separately for $q = 1$ and $q = -1$ as

$$\begin{pmatrix} \delta_{\pm 1} \\ \xi_{\pm 1} \end{pmatrix} = J_{\pm 1} \begin{pmatrix} \delta_{\pm 1} \\ \xi_{\pm 1} \end{pmatrix} \quad (4.72)$$

with $J_{\pm 1} = \begin{pmatrix} \pm i\frac{\tau}{2} & -\frac{\tau}{2} \\ \frac{\tau}{2} - 2 & \pm i\frac{\tau}{2} \end{pmatrix}$. As usual, to study the stability of the considered UC we look at the eigenvalues of the Jacobian matrix $J_{\pm 1}$, finding

$$\lambda_1^+ = \frac{i\tau + \sqrt{4\tau - \tau^2}}{2} \quad \lambda_1^- = \frac{i\tau - \sqrt{4\tau - \tau^2}}{2} \quad (4.73)$$

$$\lambda_{-1}^+ = \frac{-i\tau + \sqrt{4\tau - \tau^2}}{2} \quad \lambda_{-1}^- = \frac{-i\tau - \sqrt{4\tau - \tau^2}}{2} \quad (4.74)$$

correctly recasting the problem to the finite V case and spotting the existence of a DPT at $\tau_c = 4$.

4.10 The dynamical phase transition as a classical bifurcation

The study of the eigenvalues of the Jacobian matrix obtained linearizing the DEs shown us that it is possible to individuate, corresponding to each SC, a critical value τ_c of the hopping strength such that

$$\begin{cases} 0 < \tau < \tau_c \Rightarrow \exists \lambda_1, \lambda_2 \text{ s.t. } \operatorname{Re}\{\lambda_1\} > 0, \operatorname{Re}\{\lambda_2\} < 0 \\ 0 < \tau_c < \tau \Rightarrow \forall \lambda : i\lambda \in \mathbb{R} \end{cases} \quad (4.75)$$

or, using the language of DST, such that

$$\begin{cases} 0 < \tau < \tau_c \Rightarrow \text{The stationary configuration is a } \textit{saddle} \\ 0 < \tau_c < \tau \Rightarrow \text{The stationary configuration is a } \textit{linear center} \end{cases} \quad (4.76)$$

We found that

$$\tau_c = \begin{cases} 0 & \text{for the SFC} \\ 2 & \text{for the SPAC} \\ 4 & \text{for the UC} \end{cases} \quad (4.77)$$

In DST these sharp changes of the nature of the FPs, due for instance to the emergence of a positive real part of some of the Jacobian eigenvalues as in our case, are referred to as *bifurcations*. In the context of non-equilibrium quantum dynamics we have instead talked about *dynamical quantum phase transitions*, in order to highlight how a tiny change of a model parameter across a critical value can imply a sharp change in the system dynamics. We therefore understand that in our case the dynamical quantum phase transition corresponds to a bifurcation in the associated classical MF dynamical system. Notice that, differently from equilibrium thermodynamical transitions, the transitions that we found are valid for whatever V , i.e. not necessarily for $V \rightarrow \infty$.

4.11 Final remarks

In this chapter we have studied the stability of the SCs in the context of DST by mean of linearization. When the system state is close to a SC, it is in fact possible to exactly

solve the linearized DEs, finding the time dynamics to be ruled by the eigenvalues of the Jacobian matrix associated to the considered SC. Distinguishing the case in which one or more of the Jacobian eigenvalues have positive real part from the one in which all the non-zero Jacobian eigenvalues are purely imaginary, we were able to correctly discriminate whether $r = 0$ is stable or not for the representative cases of the UC (for whatever $V \geq 3$), the SPAC (for even $V \geq 2$), the SFC (for whatever V) and the DC (for $V = 4$). We found that for each of these SCs there exists a τ_c such that for $0 < \tau < \tau_c$ the SC is a saddle of the dynamics and r grows exponentially in time (until entering in the nonlinear regime) whereas for $\tau > \tau_c$ the SC is a linear center. However, we also pointed out that in the $\tau > \tau_c$ case the true solution of the nonlinear DEs could in general be different from the one of the linearized DEs. Finding explicit solutions of the nonlinear DEs is hard, but it is possible in some cases to claim the solution of the linearized DEs to be accurate. As seen, this is for instance true in the case of the two-site model, for which the fact that the energy is conserved and that the FPs are isolated made us state the linear centers to be also nonlinear centers, that is the oscillatory behavior around a center not to change significantly when considering nonlinear terms in the DEs. Also, for whatever V , the argument of section 4.1 proves that the SF configuration is always a nonlinear center. However, we have seen in section 3.4 and figure (3.2) that for $V > 3$ the phase portrait can be much richer, revealing the presence of several SCs and, among them, a continuum of SCs with $r = 0$. In the proximity of this continuum of SCs, the nonlinearities of the DEs play a fundamental role and lead in particular circumstances to the emergence of synchronization, as we are going to discover in the next chapter.

Chapter 5

Long-time nonlinear dynamics

Aiming to better understand the dynamics generated by the DEs (2.14), in the previous chapters we investigated the structure of the phase space, finding the SCs of the dynamics in chapter 3 and studying their stability in chapter 4. We spotted the existence of DPTs, understanding that the SCs can either be linear centers or saddles of the dynamics depending on the considered SC and on the values of the hopping strength τ (that we wrote in units of $\rho_0 u$) and we have seen that for $V > 3$ the SCs with $r = 0$ constitute a continuum in the phase space that includes the UC and, if V is even, the SPAC. We have remarked that trajectories starting close to a saddle are drifting away from it exponentially rapidly, and we have pointed out that, unless in the case of an isolated SC, a linear center is in general not a nonlinear center. In this chapter we aim to better understand which are the effects of the nonlinearities of the DEs (2.14) on the long-time dynamics of the system. We will discover that for $V \geq 3$ the dynamics is in general chaotic and that, nevertheless, in the $V \rightarrow \infty$ limit, some macroscopic dynamical order parameters may relax to well-defined values (unless some residual fluctuations). Initializing the system in the vicinity of the UC, we will observe it to be characterized by two different DOPs on the two sides of the DPT separated by $\tau_c = 4$, pointing out the tendency of the phases of the system to asymptotically acquire some degree of what we call π -alignment, in particular for $\tau > \tau_c$. Additionally, we investigate the effects of disorder on such π -alignment considering $\sigma_\omega > 0$ in the DEs (2.14) and pointing out the existence of a synchronization transition driven by the competition between disorder and hopping strength τ .

5.1 $\tau < 4$ long-time dynamics

We are interested in numerically studying the long-time behavior of the system when initialized in the neighborhoods of the UC and for $\tau < \tau_c = 4$. We consider as IC the following configuration

$$\begin{cases} \theta_j = \frac{2\pi}{V}j + \xi_j \\ \rho_j = 1 \end{cases} \quad \forall j = 1, 2, \dots, V \quad (5.1)$$

where $\{\xi_j\}_{j=1,\dots,V}$ are independent identically distributed Gaussian random numbers of

mean 0 and standard deviation $\sigma_\theta = 0.06$. Observe that for $\frac{2\pi}{V} \ll 0.06$ (that is $V \gg 100$) the IC (5.1) is statistically analogue to the MI (2.18), that has completely random phases. In figure (5.1) we characterize the configuration with the modulus r of the DOP Ψ (2.12). Starting from approximately 0, since $\tau < 4$, r initially diverges exponentially. For long-

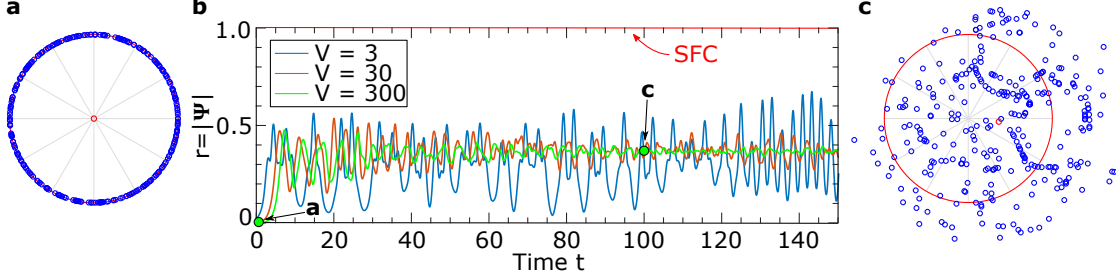


Figure 5.1 | Long-time nonlinear dynamics for $\tau > 4$. **b**, modulus r of the DOP Ψ versus time for $V = 3, 30, 300$ and $\tau = 2 < \tau_c = 4$. The system is initialized in the proximity of the UC as in (5.1), and is observed to drift away from the IC with an initial exponential growth of r followed by subsequent fluctuations of r . For large V the fluctuations reduce and r stabilize around a finite value ~ 0.38 . Notice that $r = 1$ corresponds to the SFC and that even for $V \rightarrow \infty$ there are some residual fluctuations of r . **a**, **c**, graphical representation of all the $V = 300$ bosonic variables at initial time (**a**) and at time $t = 100$ (**c**).

times we observe chaotic oscillations for $V \geq 3$, consistently with the literature [8, 9, 41], and find that the time fluctuations of r at large times are decreasing with V . For $V \rightarrow \infty$ r is still exhibiting some residual fluctuations around a value ~ 0.38 , linked to the variance (with respect to the sites) of the bosons numbers per site $\{\rho_k\}_{k=1,\dots,V}$ by equation (4.3). Notice that the growth of r is not indicating at all a tendency of the system to reach the SFC (for which $r = 1$). This is clear from section 4.1, where we shown that initializing the system to the proximity of the UC it will never reach the SFC because of the conservation of the expected energy and number of particles in the non-equilibrium dynamics. The system can indeed eventually relax to the SFC, but only on much longer timescales and thanks to the interaction with the environment, that is not described by our model.

5.2 $\tau > 4$ long-time dynamics

As we already noticed, for $\tau > 4$ the UC is a linear center but not necessarily a nonlinear center, so that if the system is initialized close to the UC then it can in principle move along the line of $r = 0$ SCs still satisfying the energy conservation (4.3). To track the position of the system in the phase space with respect to the line of $r = 0$ SCs we introduce a π -synchronization DOP S defined at each time $t > 0$ as

$$S(t) = 2 \left(\langle \cos(\theta_j - \phi) \rangle_V - \frac{1}{2} \right) \quad (5.2)$$

The choice of the above parameter is inspired by the order parameter that is typically considered in the study of liquid crystals at equilibrium and considers as ordered all the

configurations where the phase differences between two sites are equal to 0 or to π [62]. Indeed, we see that $S = 0$ for the UC and $S = 1$ for the SPAC (besides the PACs and the SFC). Looking at the evolution of S we are therefore able to quantify the position in time of the system on the line of $r = 0$ SCs that for $V \geq 3$ starts from the UC and possibly reaches the SPAC if V is even. We have a particularly easy intuition of the parameter S and of the line of $r = 0$ SCs in the case of $V = 4$, that is reported in figure (5.2).

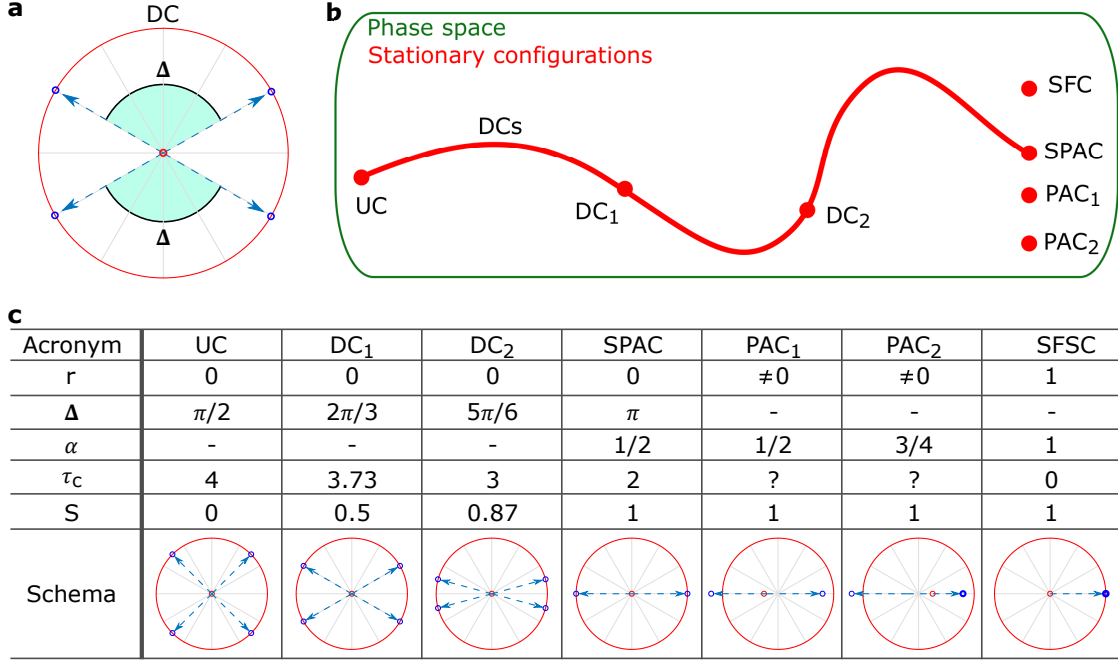


Figure 5.2 | Stationary configurations for $V = 4$. **a**, Graphical representation of the DC, with angle Δ ranging from $\pi/2$ to π . **b**, Schematic representation of the phase portrait with the stationary configurations (in red). Varying Δ from $\pi/2$ to π , the DC moves continuously on a stationary manifold (represented by the red line) from the UC to the SPAC passing through the DC₁ and the DC₂. The isolated red dots represent the SF (that is always stationary) and other possible stationary PACs (present or not depending on the value of τ). **c**, Table with acronyms, r , Δ , α , τ_c , S and graphical representation of some relevant SCs. In the table entries, "-" stays for *undefined* and "?" for *unknown*.

To study the effects of nonlinearities of the DEs (2.16) in terms of position of the system along the line of $r = 0$ SCs, we numerically solve them considering $\tau = 5$, (5.1) as IC and looking at the (long) time evolution of the π -alignment DOP S , that we plot in figure (5.3). In **a-g** the red bottom line ($S = 0$) and top line ($S = 1$) correspond to the system being respectively in the UC and in the SPAC, as explicitly written in **a** and **g** only. For $V = 3$ (**a**) we observe the system to oscillate at $S \approx 0$, in accordance with the phase portrait of figure (3.2), showing the UC to be an isolated SC and, therefore, a nonlinear center. For $V = 4$ (**b**) we observe the system to chaotically oscillate between the UC and the SPAC. For $V = 5$ (**c**) we observe the system not to reach anymore $S = 1$, in accordance with the fact that for odd V the continuum of $r = 0$ SCs is not including

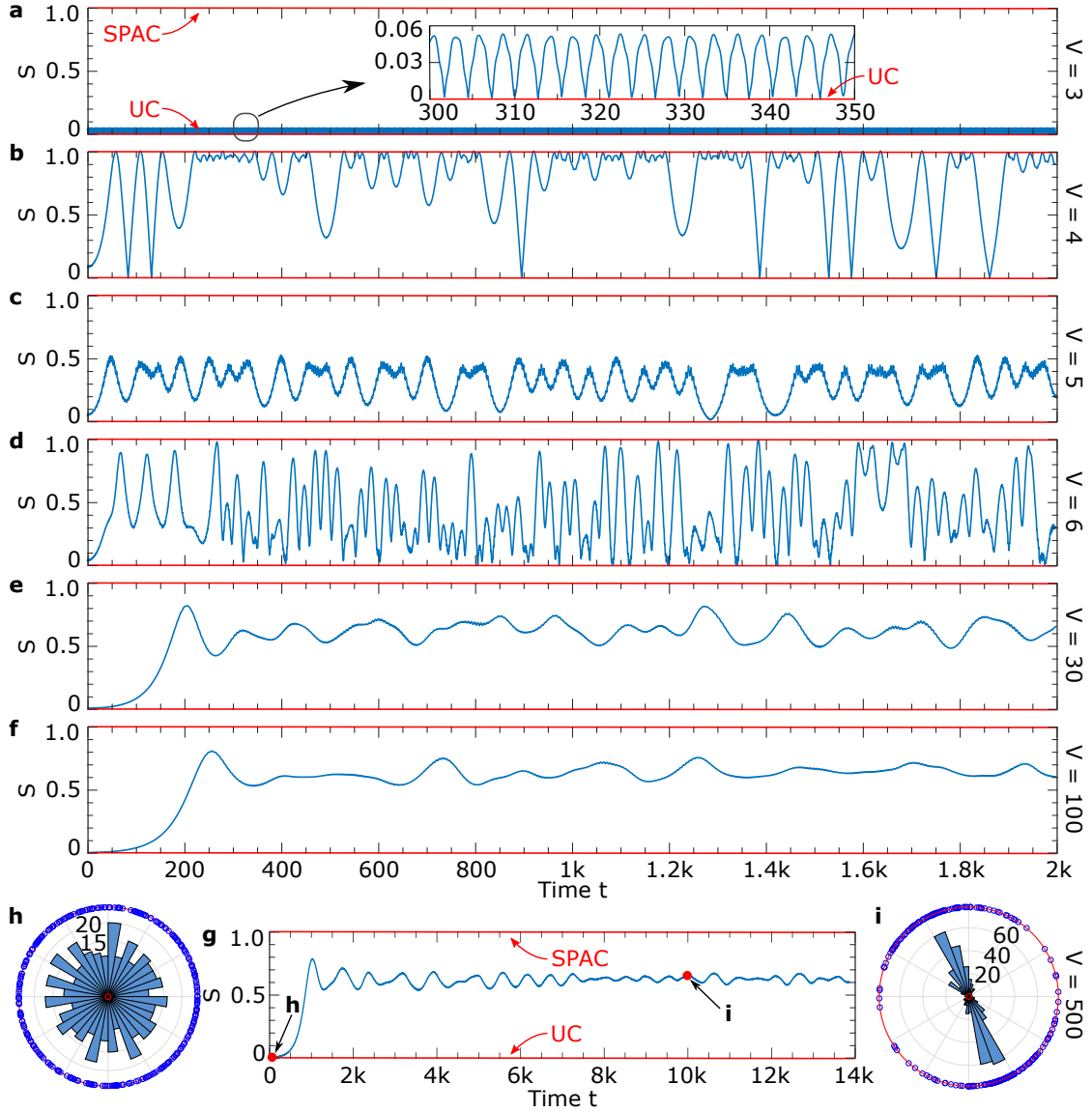


Figure 5.3 | Nonlinear dynamics and the synchronization. **a-g**, π -alignment parameter S versus time for $V = 3, 4, 5, 6, 30, 100, 500$ respectively and for $\tau = 5$. The system is initialized in the proximity of the UC as in (5.1) and for $V > 3$ is observed to move along the continuous line of $r = 0$ SCs ranging from the UC ($S = 0$) to the SPAC ($S = 1$). For $V \geq 4$ we observe a chaotic dynamics and for large V we observe S to stabilize around a finite value ~ 0.6 , unless some residual fluctuation not decaying for $V \rightarrow \infty$. **h, i**, graphical representation of 300 bosonic variables out of the $V = 500$ total ones at initial time $t = 0$ (**h**) and at time $t = 10000$ (**i**), together with a polar histogram for the phases, with 10° wide bins. A finite and stable S at large time indicates stable π -alignment of the phases (**i**).

the SPAC. For $V = 6$ (**d**), analogously to $V = 4$, the system oscillates between the UC

and the SPAC. For large V (30, 100 and 500 in **e**, **f** and **g** respectively) we observe S to fluctuate around a finite value ~ 0.6 . In **h** and **i** we report the graphical representation of the bosonic variables of 300 sites out of the total $V = 500$ ones for $t = 0$ and $t = 10000$ respectively and the corresponding polar histogram for the phases, with 10° wide bins. We point out how a finite S (**i**) corresponds to a certain degree of π -alignment of the phases. Also, we observe fluctuations of S at long-times to decrease with V down to some residual value for $V \rightarrow \infty$. Interestingly and differently from what happens to r , the growth of S in time is found to be not exponential. This is a further confirmation that the emergence of such π -synchronization is intimately linked to the nonlinear terms of the DEs, since exponential divergences are a signature of a linear dynamics. Summing up, we observe that a system initialized in the proximity of the UC has a *robust* shift to a partially π -aligned configuration, in what we refer to as synchronization. We stress that, interestingly, since the system is isolated, the stabilization of S for large V is an intrinsic property and is not due to the presence of driving and dissipation, as typically considered in literature [54–58].

In figure (5.4) we show with a schematic the movement of the system in the phase space for $\tau > 4$, for a system that is initialized in the proximity of the UC and for a large and even V (the schema for a odd V would be very similar though). For $\tau = 2 < \tau_c = 4$ (**a**), the system drifts away from the $r = 0$ SCs and relaxes to a region of the phase space with $r \sim 0.38$ with consequent spread of the number of bosons per site ρ_j (corresponding to figure (5.1)). For $\tau = 5 > 4$ (**b**), the system moves closeby the continuous line of the $r = 0$ SCs while progressively shifting towards the SPAC and increasing the π -alignment parameter S (corresponding to figure (5.3)). For large times, the system reaches an area of partially π -alignment characterized by $S \sim 0.6$. The figure also shows a trajectory around the SFC (**c**), to recall that if the system is initialized in the proximity of the SFC it will then remain close to it, being the SFC isolated and, therefore, a nonlinear center. This is indeed not surprising, since the SFC is the ground state of the Hamiltonian of the system for $t > 0$ (that is after the quench towards the SF regime)

5.3 Competition between hopping strength and disorder

In section 5.2 we have discovered that the nonlinearities of the DEs can lead for long-times to a π -synchronization of the phases of the bosonic variables, when these are treated within the MF approximation and the GPE. This is indeed a more general fact, since whatever quantum model describing a large number of bosons on a lattice can be recasted, within the MF substitution of the bosonic creation and annihilation operators at each site with c -numbers, into a model for nonlinearly coupled classical oscillators (in general of variable phase and length), for which synchronization is a universal and fundamental concept [63]. Recently, this idea has been exploited by Witthaut *et al.* to show that the GPE associated in MF to a particular class of models for bosons on a lattice coincides, under particular circumstances, to the DEs of the notorious Kuramoto model [34]. The latter is a paradigmatic model describing a system of nonlinearly all-to-all coupled oscillators in presence of disorder, that is accounted for by the introduction of random natural

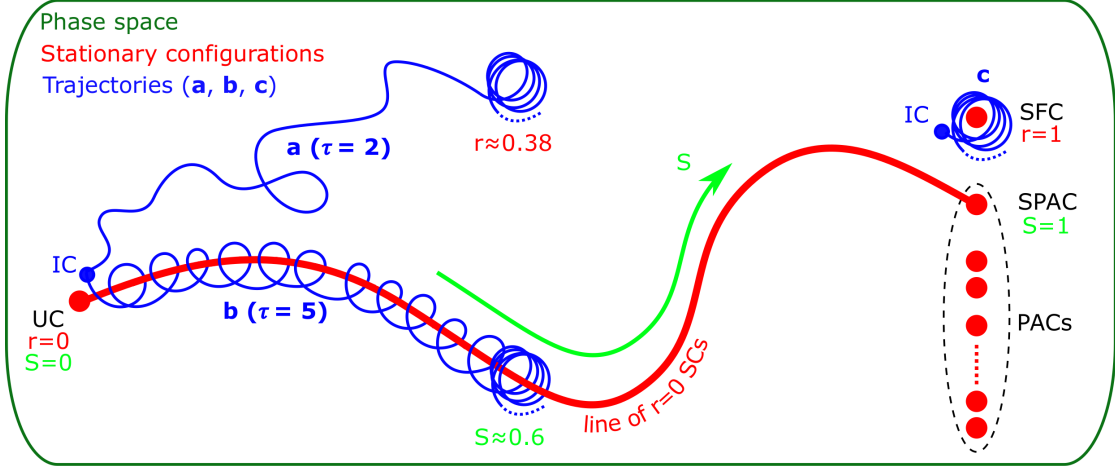


Figure 5.4 | Schema of the trajectories in the phase space for $V \gg 1$. Schematic representation of the motion of the system in the phase space under the nonlinear DEs, up to long-times and for $V \gg 1$. The two blue lines **a** and **b** represent trajectories in the phase space in case of initialization of the system in the proximity of the UC for $\tau = 2$ and $\tau = 5$ respectively. For $\tau < 4$ ($\tau > 4$), the system drifts away from (move close to) the manifold of the $r = 0$ SCs, finally evolving in a region of the phase space characterized by a finite r (S). In the case of same initial configuration of figure (5.1) (figure (5.3)), the state at long-time is characterized by $r \approx 0.38$ ($S \approx 0.6$). Conversely, a system initialized in the vicinity of the SFC never gets far from it (**c**), since the latter is a nonlinear center of the dynamics.

frequencies of the oscillators [51–53]. A sharp synchronization transition separates two regions of small and large disorder, corresponding respectively to the presence and the absence of a finite fraction of the oscillators with locked (that is synchronized) phases at long-times.

Inspired by these works, we naturally wonder about how the introduction of some disorder into the DEs can affect, and possibly destroy, the long-time π -synchronization observed for the non-disordered case in section 5.2. Therefore, for the first time after the introduction of equation (2.16), we now recover the disordered Hamiltonian (2.1) and respective GPE (2.14), relaxing the assumption of $\omega_j = 0$ and considering instead random ω_j with finite spread σ_ω . Considering $V = 4000$ sites and a quench from the MI phase to the SF regime, we initialize the system to the MI configuration (2.18), that is with random phases, and look at the time evolution of the π -synchronization parameter S for various σ_ω . In figure (5.5) we observe that the disorder strength σ_ω competes against the π -synchronization. In (a) we see that the value around which the DOP S fluctuates for long-times decreases with the disorder (that is with σ_ω). In order to average out such fluctuations, we consider a time average of S between $t = 400$ and $t = 600$ and plot it in (b) for $\tau = 5$ and against σ_ω , finding the existence of a crossover between a region of π -synchronized long-time configurations (for small σ_ω) and a region of completely random phases at all times (for large σ_ω). Being such crossover rather sharp (with onset located at $\sigma_\omega \approx 1.5 \times 10^{-2}$), we refer to it as a *synchronization transition*, in analogy with the Kuramoto model. However, differently from the latter, in (c) we notice that the critical

disorder strength $\sigma_{\omega,c}$ needed to destroy the π -synchronization is not growing with the hopping strength τ . A second important difference between our model and the Kuramoto model is of course that we obtain a π -alignment instead of the usual alignment, that is somehow intuitive since the DEs for the phases (2.14) have a cosine instead of a sine. Also, further analysis should clarify whether the critical disorder strength $\sigma_{\omega,c}$ at which the synchronization transition occurs depends on the chosen initial conditions or not.

5.4 Promising directions for analytical approach

Numerical evidence shows that, unless some residual fluctuations, a well-defined value of the DOP S exists at large times for $V \gg 1$ and, at least, for $\tau > \tau_c = 4$. Capturing this fact analytically from the DEs is a crucial challenge to gain further comprehension of the origins of the emergence of the π -synchronization in the FCBHM and, more in general, of synchronization in isolated quantum systems. From one side, the nonlinearities of the DEs are the fundamental ingredient to understand the long-time behavior and the emergence of π -synchronization for a system initialized in the surroundings of the UC. From the other side, as explained in chapter 4, for $\tau > 4$ the number of bosons per site ρ_j remains close to ρ_0 for all $j = 1, \dots, V$, making the continuum limit of sections 2.5 and 4.9 the ideal framework to work in, so that the $V \gg 1$ limit would be implicitly assumed. Additionally, the fact that the system is FC allowed us to write the DEs (2.16) in terms of r and ϕ , in analogy to the notorious Kuramoto model [53]. Possibly, such analogy could be further exploited, using ideas from the Kuramoto model as treating r as a parameter or as considering a function $p(\theta, t)$ describing the density of oscillators at the angle θ at time t (a partial differential equation describing the dynamics of $p(\theta, t)$ would then be the continuity equation). Alternatively, starting from equation (4.70) and keeping the crucially important nonlinear terms, one should try to understand how the even components of the Fourier transforms of the perturbation to the phases ξ (that are associated to π -alignment and to S) are evolving and growing in time.

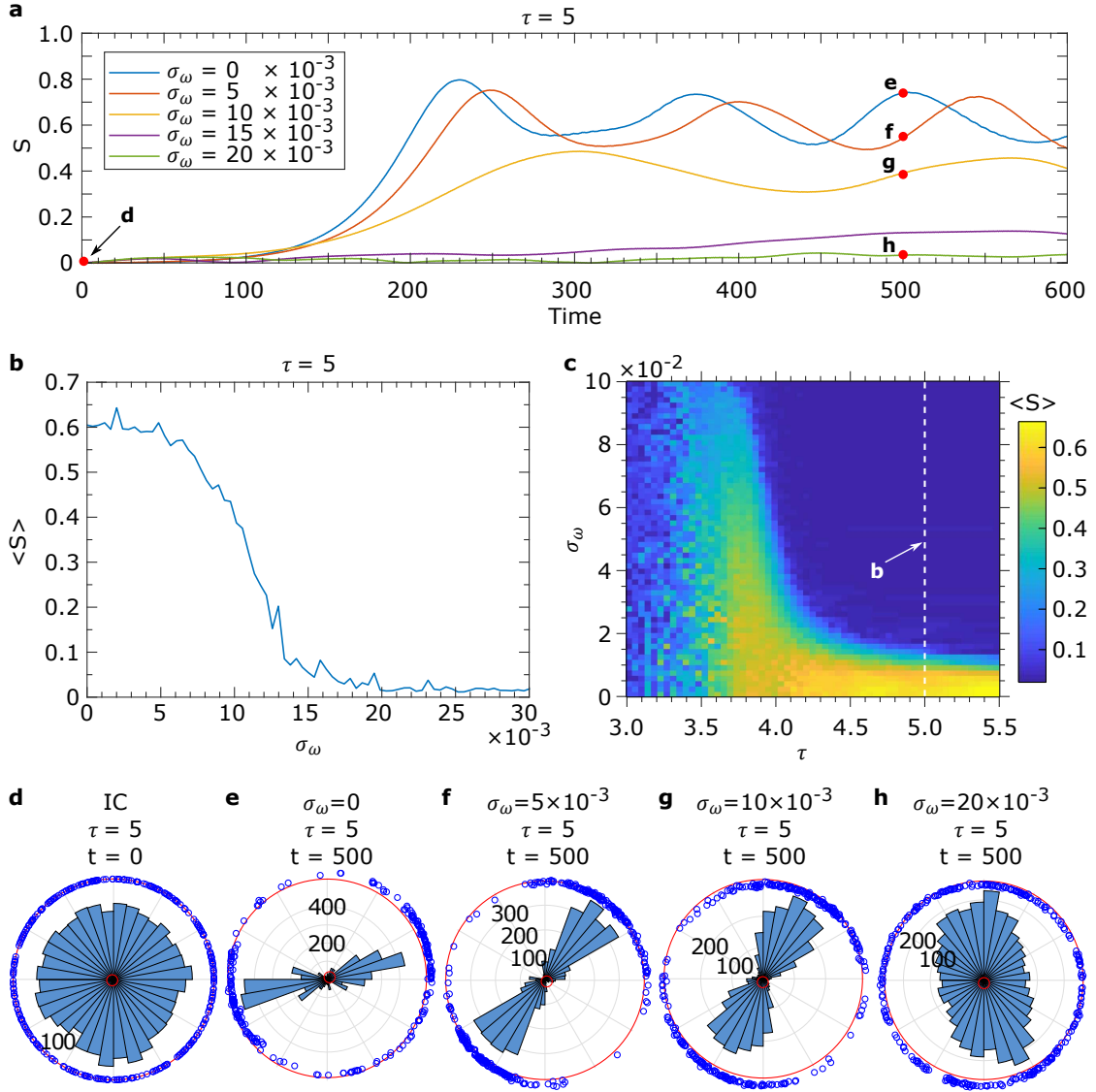


Figure 5.5 | Synchronization transition in $V \gg 1$ fully connected Bose-Hubbard model. a-c, π -Synchronization parameter S for $V = 4000$ sites after a quench from the MI phase (corresponding to $\rho_k = 1$ and to independent random phases θ_k for all $k = 1, 2, \dots, V$ as in (2.18)) to the SF regime. S fluctuates in time around a value that decreases with σ_ω as shown for $\sigma_\omega = 5, 10, 15, 20 \times 10^{-3}$ in (a). To discard the time fluctuations of S , we plot the long-time average $\langle S \rangle_t = \frac{1}{200} \int_{400}^{600} dt S(t)$ versus σ_ω for a given $\tau = 5$ (b) and versus τ and σ_ω (c), highlighting the existence of a sharp crossover that we call *synchronization transition*. d-h, Graphical representation of the complex variables $\psi_k = \sqrt{\rho_k} e^{i\theta_k}$ for 200 sites (out of the total V) at initial time $t = 0$ (d) and at time $t = 500$ for $\tau = 5$ and $\sigma_\omega = 0, 5, 10, 20 \times 10^{-3}$ (from (e) to (h) respectively). The respective phase histograms with bins width 10° enable us to easily visualize the amount of long-time π -synchronization in each case.

Chapter 6

Quantum fluctuations beyond mean-field

To pass from the dynamics (2.3) to (2.9) and possibly to (2.14) we exploited the MF approximation (2.7). In this way we have found that the system undergoes very different time dynamics depending on the value of the hopping strength τ , in what we have referred to as a DPT. However, we have so far not yet investigated whether the MF assumption is valid for all the considered τ or not. By means of the Bogoliubov-de Gennes method, we address such important issue in this chapter, finding that, at least for large τ , QFs are expected not to grow exponentially in time, thus legitimating the MF assumption. Additionally, we conclude arguing on the validity of the Bogoliubov-de Gennes method, claiming it to be particularly accurate for $\tau > \tau_c$.

6.1 Linearized dynamical equations for the quantum fluctuations

We decompose the bosonic annihilation and creation operators at site $k \in \{1, 2, \dots, V\}$ as

$$\begin{cases} a_k = \psi_k + b_k \\ a_k^\dagger = \psi_k^* + b_k^\dagger \end{cases} \quad (6.1)$$

where ψ_k is the MF term (just a c -number) undergoing the dynamics (2.9) and where the operators b_k and $b_k^\dagger$ describe QFs beyond MF associated to the k -th site. Considering the non-disordered ($\omega_k = 0$) case, inserting (6.1) into the exact Heisenberg dynamics (2.3) and subtracting the mean field dynamics (2.9) we find

$$\begin{cases} \frac{db_k}{d(it)} = +\frac{\tau}{V} \sum_{j=1}^V b_j - (\psi_k^* + b_k^\dagger)(\psi_k + b_k)(\psi_k + b_k) + |\psi_k|^2 \psi_k \\ \frac{db_k^\dagger}{d(it)} = -\frac{\tau}{V} \sum_{j=1}^V b_j^\dagger + (\psi_k^* + b_k^\dagger)(\psi_k^* + b_k^\dagger)(\psi_k + b_k) - |\psi_k|^2 \psi_k^* \end{cases} \quad (6.2)$$

that is

$$\begin{cases} \frac{db_k}{d(it)} = \tau \langle b_k \rangle_V - \psi_k^2 b_k^\dagger - 2|\psi_k|^2 b_k - 2\psi_k b_k^\dagger b_k - \psi_k^* b_k^2 - b_k^\dagger b_k b_k \\ \frac{db_k^\dagger}{d(it)} = -\tau \langle b_k^\dagger \rangle_V + (\psi_k^*)^2 b_k + 2|\psi_k|^2 b_k^\dagger + 2\psi_k b_k^\dagger b_k + \psi_k^* (b_k^\dagger)^2 + b_k^\dagger b_k^\dagger b_k \end{cases} \quad (6.3)$$

Assuming b_k and $b_k^\dagger$ to be small, we linearize equation (6.3) finding the Bogoliubov-de Gennes equation [59]

$$\begin{cases} \frac{db_k}{d(it)} = \tau \langle b_k \rangle_V - \psi_k^2 b_k^\dagger - 2|\psi_k|^2 b_k \\ \frac{db_k^\dagger}{d(it)} = -\tau \langle b_k^\dagger \rangle_V + (\psi_k^*)^2 b_k + 2|\psi_k|^2 b_k^\dagger \end{cases} \quad (6.4)$$

that can be written in matricial form as

$$\frac{d}{d(it)} \vec{B}(t) = J_{QF}(t) \vec{B}(t) \quad (6.5)$$

where $\vec{B}(t) = (b_1, b_2, \dots, b_V, b_1^\dagger, b_2^\dagger, \dots, b_V^\dagger)^T$ is a $2V$ -dimensional column vector of operators. Information on the dynamics of the QFs (6.5) can be easily obtained in the case of stationary MF solutions, for which J_{QF} is time independent. In this case, to understand the dynamics of the QFs, we are interested in diagonalizing the $2V$ -dimensional matrix J_{QF} . The eigenvalue problem associated to J_{QF} reads

$$J_{QF} \vec{v}_n = \lambda_n \vec{v}_n \quad (6.6)$$

where $\vec{v}_n$ is a $2V$ -dimensional column vector. Of course, as recalled in section 4.2, the linearized dynamics (6.5) for the QFs is easily solved writing $\vec{B}$ in the basis of the eigenvectors of the matrix J_{QF} , that is

$$\vec{B}(t) = \sum_{n=1}^{2V} c_n \vec{v}_n e^{i\lambda_n t} \quad (6.7)$$

Furthermore, thanks to the symmetries of (6.7) and as explained in appendix C.2, it is possible to reduce (6.7) to

$$\vec{B}(t) = \sum_{n=1}^V \left(c_n \begin{pmatrix} \vec{v}_n^{(1)} \\ \vec{v}_n^{(2)} \end{pmatrix} e^{i\lambda_n t} + c_n^\dagger \begin{pmatrix} \vec{v}_n^{(2)} \\ \vec{v}_n^{(1)} \end{pmatrix}^* e^{-i\lambda_n^* t} \right) \quad (6.8)$$

where the $\{c_n\}_{n=1, \dots, V}$ are time independent operators set by the IC and where we have decomposed $\vec{v}_n$ as $\begin{pmatrix} \vec{v}_n^{(1)} \\ \vec{v}_n^{(2)} \end{pmatrix}$, with $\vec{v}_n^{(1,2)}$ V -dimensional column vectors. We observe that the first and the second halves of the vector $\vec{B}$ have to be one the Hermitian conjugate of the other, that is $B_k = B_{V+k}^\dagger$ for $k = 1, \dots, V$. We refer to this property as *physical consistency*. Of course, the eigenvectors $\vec{v}_n$ by their own can possibly be not physically consistent, and what matters is that still their linear combination (6.8) guarantees the consistency of $\vec{B}$.

From (6.8) we clearly understand that the eigenvalues of J_{QF} with negative imaginary parts correspond to QFs exponentially growing in time (notice the imaginary unit appearing in the left hand side of equation (6.5) and, thus, in the exponents of (6.8)), that is

$$\text{QFs} \sim \exp\left(-\min_{n=1,\dots,2V} \text{Im}\{\lambda_n\}t\right) = \exp\left(\max_{n=1,\dots,2V} \text{Re}\{i\lambda_n\}t\right) \quad (6.9)$$

As SCs we naturally consider the ones treated in the previous chapters, that are the UC, the SFC and the SPAC. For such configurations we have $\psi_k = e^{i\theta_k}$ (that is $\rho_k = \rho_0 = 1$), and the eigenvalue problem (6.6) can therefore be rewritten as

$$\begin{cases} +\tau\langle v_k^{(1)} \rangle_V - e^{+i2\theta_k} v_k^{(2)} - 2v_k^{(1)} & = \lambda v_k^{(1)} \\ -\tau\langle v_k^{(2)} \rangle_V + e^{-i2\theta_k} v_k^{(1)} + 2v_k^{(2)} & = \lambda v_k^{(2)} \end{cases} \quad (6.10)$$

As usual we can assume to be in the frame corotating with the phases of the considered SC, so that the θ_k are constants. Moreover, we consider as usual τ (and the respective critical values) to be written in units of the in-situ interaction $u\rho_0$, and ρ_0 to be set to 1 thanks to the invariance of the DEs under the rescaling of ρ_0 discussed in section 2.4.

6.2 Uniform configuration

In the case of UC one has $\theta_k = \frac{2\pi}{V}k$. We multiply the first equation of (6.10) times $e^{-i\theta_k}$ and the second times $e^{+i\theta_k}$ to get

$$\begin{cases} \tau\langle v_k^{(1)} \rangle_V e^{-i\theta_k} - (\lambda + 2)v_k^{(1)} e^{-i\theta_k} = e^{i\theta_k} v_k^{(2)} \\ \tau\langle v_k^{(2)} \rangle_V e^{+i\theta_k} + (\lambda - 2)v_k^{(2)} e^{i\theta_k} = e^{-i\theta_k} v_k^{(1)} \end{cases} \quad (6.11)$$

Performing the average over k , that is $\langle \bullet \rangle_V$, on (6.11) we get

$$\begin{cases} -(\lambda + 2)\tilde{v}_{-1}^{(1)} = \tilde{v}_{+1}^{(2)} \\ +(\lambda - 2)\tilde{v}_{+1}^{(2)} = \tilde{v}_{-1}^{(1)} \end{cases} \quad (6.12)$$

where $\tilde{v}_q^{(1)}$ and $\tilde{v}_q^{(2)}$ denote respectively the DFT of $v_k^{(1)}$ and $v_k^{(2)}$ with respect to the Fourier wavenumber $q \in \mathbb{Z}$. Equation (6.12) is solved either by $\tilde{v}_{-1}^{(1)} = \tilde{v}_{+1}^{(2)} = 0$ or by $\lambda = \pm\sqrt{3}$ for $\tilde{v}_{-1}^{(1)}, \tilde{v}_{+1}^{(2)} \neq 0$. Since the eigenvalues $\lambda = \pm 3$ are real and, in this sense, do not destabilize the QFs (that is do not make them growing exponentially), we proceed looking for other eigenvalues of J_{QF} . Since (6.12) needs still to be satisfied, the other eigenvalues are associated to eigenvectors such that $\tilde{v}_{-1}^{(1)} = \tilde{v}_{+1}^{(2)} = 0$. In the system (6.10), we isolate $v_k^{(1)}$ from the second equation and substitute it into the first, getting

$$+\tau\tilde{v}_0^{(1)} - e^{+i2\theta_k} v_k^{(2)} = (\lambda + 2) e^{+i2\theta_k} \left(+\tau\tilde{v}_0^{(2)} + (\lambda - 2) v_k^{(2)} \right) \quad (6.13)$$

analogously, again from the system (6.10), we can isolate $v_k^{(2)}$ from the first equation and substitute it into the second, getting

$$-\tau \tilde{v}_0^{(2)} + e^{-i2\theta_k} v_k^{(1)} = (\lambda - 2) e^{-i2\theta_k} \left(+\tau \tilde{v}_0^{(1)} - (\lambda + 2) v_k^{(1)} \right) \quad (6.14)$$

We can rewrite the previous two equations as

$$\begin{cases} +e^{-i2\theta_k} \tau \tilde{v}_0^{(1)} = (\lambda + 2) \tau \tilde{v}_0^{(2)} + (\lambda^2 - 3) v_k^{(2)} \\ -e^{+i2\theta_k} \tau \tilde{v}_0^{(2)} = (\lambda - 2) \tau \tilde{v}_0^{(1)} - (\lambda^2 - 3) v_k^{(1)} \end{cases} \quad (6.15)$$

from which, since we are considering here $\lambda^2 \neq 3$, we finally get

$$\begin{cases} v_k^{(2)} = \frac{e^{-i2\theta_k} \tau \tilde{v}_0^{(1)} - (\lambda + 2) \tau \tilde{v}_0^{(2)}}{\lambda^2 - 3} \\ v_k^{(1)} = \frac{e^{+i2\theta_k} \tau \tilde{v}_0^{(2)} + (\lambda - 2) \tau \tilde{v}_0^{(1)}}{\lambda^2 - 3} \end{cases} \quad (6.16)$$

A consistency condition can be obtained taking the DFT with respect to $q = 0$ of (6.16), reading

$$\begin{cases} \tilde{v}_0^{(2)} = -\frac{\lambda + 2}{\lambda^2 - 3} \tau \tilde{v}_0^{(1)} \\ \tilde{v}_0^{(1)} = +\frac{\lambda - 2}{\lambda^2 - 3} \tau \tilde{v}_0^{(2)} \end{cases} \quad (6.17)$$

that is solved by

$$\begin{cases} \tilde{v}_0^{(2)} = 0 & \text{or} & \lambda^2 + \lambda\tau + 2\tau - 3 = 0 \\ \tilde{v}_0^{(1)} = 0 & \text{or} & \lambda^2 - \lambda\tau + 2\tau - 3 = 0 \end{cases} \quad (6.18)$$

that, since λ cannot in general be a solution of both $\lambda^2 + \lambda\tau + 2\tau - 3 = 0$ and $\lambda^2 - \lambda\tau + 2\tau - 3 = 0$, reduces to

$$\begin{cases} \lambda^2 + \lambda\tau + 2\tau - 3 = 0 \\ \tilde{v}_0^{(1)} = 0 \\ \tilde{v}_0^{(2)} = c_1 \end{cases} \quad \text{or} \quad \begin{cases} \lambda^2 - \lambda\tau + 2\tau - 3 = 0 \\ \tilde{v}_0^{(1)} = c_2 \\ \tilde{v}_0^{(2)} = 0 \end{cases} \quad (6.19)$$

Where c_1, c_2 are constants. We have therefore found the only 4 eigenvalues different from 0 and correspondent eigenvectors (6.16), that are

$$\begin{cases} \lambda_{1,2} = \frac{-\tau \pm \sqrt{(\tau-2)(\tau-6)}}{2} \\ v_k^{(2)} = -c_1(\lambda_{1,2} + 2) \\ v_k^{(1)} = c_1(e^{+i2\theta_k}) \end{cases} \quad \text{or} \quad \begin{cases} \lambda_{3,4} = \frac{+\tau \pm \sqrt{(\tau-2)(\tau-6)}}{2} \\ v_k^{(2)} = c_2(e^{-i2\theta_k}) \\ v_k^{(1)} = c_2(\lambda_{3,4} - 2) \end{cases} \quad (6.20)$$

For completeness, we report the characteristic polynomial of J_{QF} corresponding to the above mentioned eigenvalues, that is

$$P(\lambda) = (\lambda^2 - 3)^{V-2} \left(\lambda^4 - (6 - 4\tau + \tau^2)\lambda^2 + (3 - 2\tau)^2 \right) \quad (6.21)$$

We therefore find that for $\tau \in (2, 6)$ there are eigenvalues (λ_2, λ_4) with negative imaginary part, meaning that the QFs are growing exponentially in time, whereas for $\tau \notin (2, 6)$

there are only real eigenvalues, meaning that the QFs are not exponentially growing and that the MF can be considered to be valid for much longer times (remember that in equation (6.5) we kept an imaginary unit i factorized out of the matrix J_{QF}).

6.3 Superfluid and symmetric π -aligned configurations

In the case of SF and SPAC we have $\psi_k^2 = (\psi_k^*)^2 = 1$ so that the eigenvalue problem (6.10) reads

$$\begin{cases} +\tau\tilde{v}_0^{(1)} - v_k^{(2)} - 2v_k^{(1)} = \lambda v_k^{(1)} \\ -\tau\tilde{v}_0^{(2)} + v_k^{(1)} + 2v_k^{(2)} = \lambda v_k^{(2)} \end{cases} \quad (6.22)$$

Performing the site average $\langle \bullet \rangle_V$ on (6.22) we get

$$\begin{cases} (\tau - 2 - \lambda) \tilde{v}_0^{(1)} = \tilde{v}_0^{(2)} \\ (\tau - 2 + \lambda) \tilde{v}_0^{(2)} = \tilde{v}_0^{(1)} \end{cases} \quad (6.23)$$

that is easily turned into

$$\begin{cases} ((\tau - 2)^2 - \lambda^2) \tilde{v}_0^{(1)} = \tilde{v}_0^{(1)} \\ ((\tau - 2)^2 - \lambda^2) \tilde{v}_0^{(2)} = \tilde{v}_0^{(2)} \end{cases} \quad (6.24)$$

from which we find that either $\lambda^2 = (\tau - 2)^2 - 1 = (\tau - 1)(\tau - 3)$ or $\tilde{v}_0^{(1)} = \tilde{v}_0^{(2)} = 0$. In the second case equation (6.22) reads

$$\begin{pmatrix} -2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} v_k^{(1)} \\ v_k^{(2)} \end{pmatrix} = \lambda \begin{pmatrix} \tilde{v}_0^{(1)} \\ \tilde{v}_0^{(2)} \end{pmatrix} \quad (6.25)$$

that gives eigenvalues $\lambda = \pm\sqrt{3}$. For completeness, we report the characteristic polynomial $P(\lambda)$ of the matrix J_{QF}

$$P(\lambda) = (\lambda^2 - 3)^{V-2} (\lambda^2 - (\tau - 2)^2 + 1) \quad (6.26)$$

We therefore find that for $\tau \in (1, 3)$ there are eigenvalues with negative imaginary part, meaning that the QFs are growing exponentially in time, whereas for $\tau \notin (1, 3)$ there are only real eigenvalues, implying that the QFs are not growing exponentially in time and that the MF can be considered to be valid, at least for longer times.

6.4 Delta configuration

To study analytically the QFs for configurations belonging to the continuum of SCs ranging from the UC to the SPAC, we consider now the DC for $V = 4$. With the help of a software for symbolic manipulation, we get the following exact characteristic polynomial associated to the matrix J_{QF}

$$P(\lambda) = (\lambda^2 - 3)^2 \left[\lambda^4 - (6 - 4\tau + \tau^2)\lambda^2 + (3 - 2\tau)^2 - \tau^2 \cos^2(\Delta) \right] \quad (6.27)$$

with $\Delta \in (\frac{\pi}{2}, \pi)$.

In appendix C.1 we study the roots of (6.27), finding that there exists some eigenvalue of J_{QF} with negative imaginary part if and only if

$$\tau \in \left(\frac{3}{2 - \cos(\Delta)}, \frac{3}{2 + \cos(\Delta)} \right) \cup (4 - 2 \sin(\Delta), 4 + 2 \sin(\Delta)) \quad (6.28)$$

where $\cup$ denotes union between sets. The condition (6.28) represents therefore a condition for the linear instability of the QFs since, as said, the QFs grow as $QFs \sim \exp\left(\max_{n=1, \dots, 2V} \text{Re}\{i\lambda_n\}t\right)$. In figure (6.1a) we plot the exact value of $\max_{n=1, \dots, 2V} \text{Re}\{i\lambda_n\}$ as obtained in appendix C.1. We find that in the region of the parameter space $\frac{3}{2 - \cos(\Delta)} < \tau < 4 + 2 \sin(\Delta)$, that is a range of τ around the DPT at τ_c , the QFs are linearly unstable (apart from a relatively narrow region between $\frac{3}{2 + \cos(\Delta)}$ and $4 + 2 \sin(\Delta)$). Interestingly, we observe that corresponding to the UC ($\Delta = \pi/2$) and the SPAC ($\Delta = \pi$), the rate of increase of the QFs is maximum at the transition, that is at $\tau = \tau_c = 4, 2$ respectively.

6.5 Stationary configurations with $r = 0$ and $V > 4$

In the previous section, we considered $V = 4$ and the DC in order to analytically treat the continuum of SCs ranging from the UC to the SPAC. We show numerically that similar results are observed even in the case of larger V , and in particular of $V = 300$. We consider the following $r = 0$ SC for an even V

$$\begin{cases} \rho_k &= 1 \\ \rho_{k+V} &= 1 \\ \theta_k &= \eta_k \\ \theta_{k+V} &= \theta_k + \pi \end{cases} \quad \forall k = 1, \dots, V/2 \quad (6.29)$$

where η_k is a Gaussian random number with mean 0 and standard deviation σ_θ . Notice that, despite being defined with some random numbers, the configuration (6.29) has by construction $r = 0$ and $\rho_k = 1$, being thus a SC. For $V \rightarrow \infty$ and computing easy Gaussian integrals we get that the π -alignment parameter for such configuration is equal to $S = e^{-2\sigma_\theta}$. Setting $\sigma_\theta = -1/2 \log(S)$ the configuration (6.29) will thus correspond to a SC with $r = 0$ and given S , so that varying S between 0 and 1 it will correspondingly range from the UC to the SPAC, continuously moving on the manifold of $r = 0$ SCs. For each S we can then numerically evaluate and diagonalize the corresponding Jacobian J_{QF} and look at the minimum imaginary part of its eigenvalues. For a better comparison of the results with the case of $V = 4$ (that is the DC, for which $S = \cos(\pi - \Delta)$), we define Δ in the $V > 4$ case as $\Delta = \pi - \arccos(S)$. In figure (6.1b) we plot the numerical value of $-\min_{n=1, \dots, 2V} \text{Im}\{\lambda_n\}$, that is of $\max_{n=1, \dots, 2V} \text{Re}\{i\lambda_n\}$, that is found to be in a stunningly close analogy with the $V = 4$ case of the DC. Considering the quench from the MI phase to the SF regime, for which the IC is in the proximity of the UC, we peculiarly find that the QFs are increasing at the fastest rate in correspondence of the MF DPT, that is for $\tau \approx \tau_c = 4$.

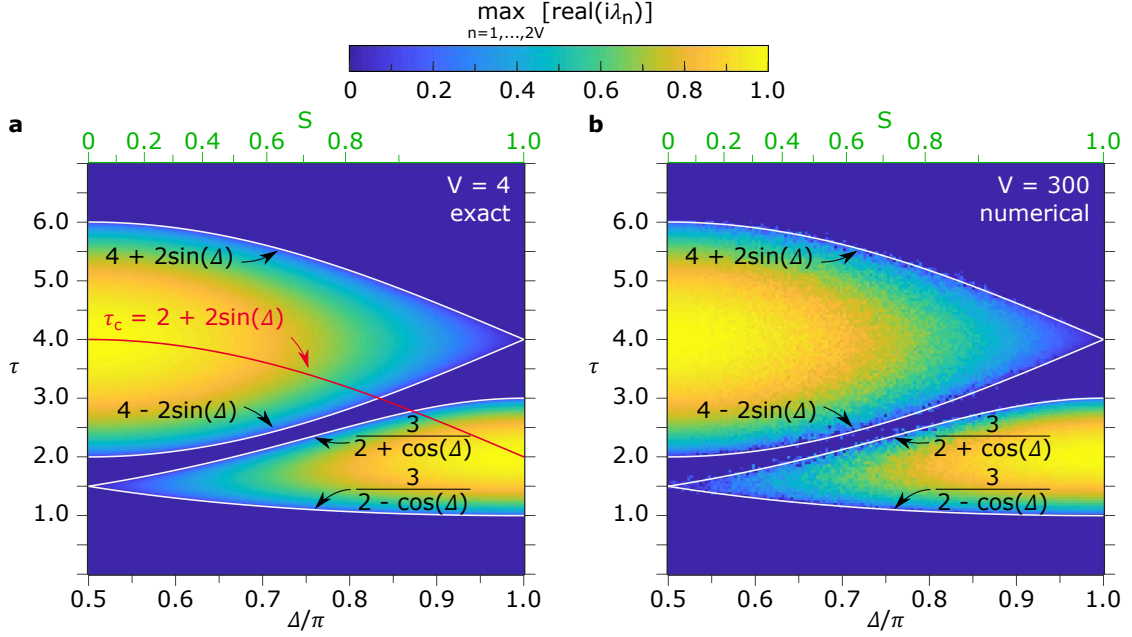


Figure 6.1 | Linear stability of the quantum fluctuations. **a,b,** For different Δ and τ we plot $\max_{n=1,\dots,2V} \text{Re}\{i\lambda_n\}$, where $\{\lambda_n\}_{n=1,\dots,2V}$ are the eigenvalues of the matrix J_{QF} for the considered SC. For $V = 4$ (**a**), Δ is the parameter used in the definition of the DC (3.13) and the eigenvalues λ_n are exactly computed as in section 6.4 and appendix C.1. For $V = 300$ (**b**), Δ is defined as in section 6.5 as $\pi - \arccos(S)$, the considered configuration is (6.29) and the eigenvalues λ_n are computed numerically on a (150×150) -dimensional uniformly spaced grid in the (Δ, τ) plane. As usual, notice that $\Delta = \pi/2$ corresponds to the UC whereas $\Delta = \pi$ corresponds to the SPAC. Interestingly, we observe a stunning similarity between the exact results for $V = 4$ and the numerical results for $V = 300$, finding in both cases that the QFs are linearly unstable for $\tau \in \left(\frac{3}{2-\cos(\Delta)}, \frac{3}{2+\cos(\Delta)}\right)$ and for $\tau \in (4 - 2\sin(\Delta), 4 + 2\sin(\Delta))$ (white lines), linearly stable elsewhere (as analytically proven for $V = 4$ in appendix C.1). For comparison, we report in green in the upper horizontal axis the π -alignment parameter S and, in (**a**) for $V = 4$, the line of critical hopping strength (4.65) $\tau_c = 2 + 2\sin(\Delta)$.

6.6 Limitations of the approach and final remarks

In this chapter, we studied the stability of the QFs thanks to an expansion at linear order of the DEs (2.3) around the MF solution. The linearization led us to spot out the existence of saddles and linear centers of the dynamics of the QFs depending on the considered SC and on the hopping strength τ (written as usual in units of $u\rho_0$). However, we point out some potential limitations of the results of the present chapter. Firstly, we observe as usual that a linear center is not necessarily a nonlinear center, so that a priori the QFs could grow even when $\max_{j=1,\dots,2V} \text{Re}\{i\lambda_j\} = 0$. Conversely, when $\max_{j=1,\dots,2V} \text{Re}\{i\lambda_j\} > 0$, we cannot a priori say how large QFs will be at long-times. In fact, we can just state that they cannot remain arbitrarily small, but only the whole nonlinear dynamics (6.3) can determine to

what extent QFs will grow, so that there could be some SCs with $\max_{j=1,\dots,2V} \text{Re}\{i\lambda_j\} > 0$ and for which MF has still some validity. Furthermore, the results of this chapter have been obtained considering the special case of the SCs. In general, if the configuration of the system is non-stationary, the matrix J_{QF} is time dependent (through the ψ_k and the ψ_k^*) and the dynamics of the QFs is coupled to the MF dynamics (2.9). In chapter 4 we have seen that a SC is unstable for $\tau < \tau_c$, for which the system rapidly drifts away from it. In such case we expect the present study on the linear stability of the QFs to lose relevance, since the coupling with the MF study could a priori have important effects in this case. Conversely, the approach is expected to be particularly relevant for $\tau > \tau_c$ and for large V , for which the system slowly drifts along the manifold of $r = 0$ SCs (figure (5.3)), making the dynamics for the QFs decoupled from the one of the MF variables $\{\theta_k, \rho_k\}_{k=1,\dots,V}$.

Chapter 7

Conclusions

In conclusion, we studied the non-equilibrium dynamics in the Bose-Hubbard (BH) model (2.1) on a Fully Connected (FC) (or all-to-all coupled) V -dimensional lattice (2.4) with potential experimental applications in cold-atoms and systems of Josephson junctions, extending previous works on $V = 2$ (known as the bosonic junction or dimer), $V = 3$ (that is the bosonic trimer) [1–9] and on a 1D lattice [10]. This study allowed us to unveil, at the Mean-Field (MF) level, the presence of a Dynamical Phase Transition (DPT), that is the existence of two qualitatively very different dynamical behaviours depending on the model parameters, and the emergence of a π -synchronization of the phases of the bosonic variables for large V , at long-times and under particular circumstances.

In the context of a quench to the SuperFluid (SF) regime, we performed in chapter 2 the MF substitution of the bosonic creation and annihilation operators at each site with c -numbers ($a_k^\dagger \rightarrow \sqrt{\rho_k} e^{-i\theta_k}$ and $a_k \rightarrow \sqrt{\rho_k} e^{i\theta_k}$). This enabled us to pass from the Heisenberg Dynamical Equations (DEs) for the bosonic operators (2.2) to the discrete and nonlinear Gross-Pitaevskii Equation (GPE) of motion (2.9), semiclassically reformulating the problem as a problem of V classical and nonlinearly coupled oscillators with variable phases and lengths $\{\theta_k, \sqrt{\rho_k}\}_{k=1,\dots,V}$. Thanks to the symmetries proper of FC models, we were able to write these semiclassical DEs in a compact form introducing the complex Dynamical Order Parameter (DOP) (2.11) $\Psi = r e^{i\phi} = \frac{1}{V} \sum_{j=1}^V \sqrt{\rho_j} e^{i\theta_j}$.

In chapter 3 we found the Stationary Configurations (SCs), that is the values of the variables $\{\theta_k, \rho_k\}_{k=1,\dots,V}$ that do not evolve in time (unless a global phase rotation at constant rate affecting all the sites in the same way). Interestingly, we found that the SCs can either have $r = 0$ and $\rho_k = \rho_0 \forall k = 1, \dots, V$ or $\theta_k - \theta_j \in \{0, \pi\} \forall k, j = 1, \dots, V$. Among them, particularly relevant are the SuperFluid Configuration (SFC), for which $\theta_k = 0$ and $\rho_k = \rho_0 \forall k = 1, \dots, V$, the Uniform Configuration (UC), for which $\theta_k = \frac{2\pi}{V}k$ and $\rho_k = \rho_0 \forall k = 1, \dots, V$ and the Symmetric π -Aligned Configuration (SPAC), for which $\rho_k = \rho_0 \forall k = 1, \dots, V$, $\theta_k = 0 \forall k = 1, \dots, V/2$ and $\theta_k = \pi \forall k = V/2 + 1, \dots, V$ (defined only for even V). Of great interest is the UC, since for large V it is close to the initial condition (2.18) that have to be considered when doing a quench from the Mott-insulating phase to the SuperFluid (SF) regime. Interestingly, we found that for $V \geq 4$ there exists a continuum of infinitely many SCs with $r = 0$ of which the UC and (if V is even) the SPAC are part.

Linearizing the GPE around the relevant SCs and diagonalizing the associated Jacobian matrix, in chapter 4 we studied the linear stability of the SCs, finding them to be either saddles or linear centers of the dynamics for $\tau < \tau_c$ and $\tau > \tau_c$ respectively, τ_c being a critical value of the hopping strength depending on the considered SC. In particular, we found that $\tau_c = 0, 2, 4$ (in units of the in-situ interaction $u\rho_0$) for the SFC, the SPAC and the UC respectively. The finiteness of τ_c for the UC and the SPAC determines the presence of two qualitatively different dynamical behaviours when τ is varied across τ_c , in what we called a Dynamical Phase Transition (DPT). Therefore, in the particular case of a quench from the Mott-insulating phase to the SF regime we observe two qualitatively very different behaviours whether $\tau < 4$ or $\tau > 4$.

Thanks to numerical results together with arguments based on the conserved quantities (energy and total number of particles) and on the topology of the phase space, in chapter 5 we investigated the long-time behavior of the nonlinear GPE for a system initialized in the proximity of the UC (as in the case of quench from the Mott-insulating phase to the SF regime). Having stressed the importance of nonlinear terms in the long-time dynamics, we found that, despite the dynamics being in general chaotic for $V \geq 3$, some DOPs thermalize, that is relax to finite values (unless some residual fluctuations). This is the case for r if $\tau < \tau_c = 4$ and for a certain π -synchronization parameter S (5.2) (mainly) if $\tau > \tau_c = 4$. Looking at the time evolution of S we were able to understand that the system can move close to the manifold of the $r = 0$ SCs, acquiring a finite degree of π -alignment at long-times. Inspired by the notorious Kuramoto model for classical coupled oscillators [34, 51–53], we considered the presence of some disorder in the Hamiltonian (2.1), finding that the parameter S vanishes with a rather sharp crossover when increasing the disorder strength, in what we called a synchronization transition.

In order to investigate the validity of the MF approximation, we finally considered in chapter 6 the Bogoliubov-de Gennes equation (6.4) to study the time evolution of the Quantum Fluctuations (QFs), finding that there exists a range of τ around the dynamical phase transition for which the QFs are linearly unstable. Also, we claimed the latter approach to be particularly reliable only for $\tau > \tau_c$, since for $\tau < \tau_c$ the dynamics for the QFs can not be decoupled from the one of the MF variables.

Further investigation on the considered system should aim to better understand the phenomenon of π -synchronization occurring for $\tau > \tau_c$. As noticed in chapter 5, a promising direction for analytical investigation is represented by the continuum limit for $V \rightarrow \infty$. Moreover, as stressed in the main text, fundamental to capture the synchronization phenomenon are the nonlinear terms, so that it is crucial to go beyond the linear equation (4.71), ideally keeping all the nonlinear terms of equation (4.70). Further research should also seek at clarifying the dynamics followed by the phase ϕ of the DOP Ψ , that is apparently non-trivial. Indeed, numerical evidence suggests that for a system initialized with $r \approx 0$, Ψ tends to rotate in the complex plane in the opposite verse with respect to the $\{\psi_k\}_{k=1,\dots,V}$. Finally, as we pointed out in section 5.3, similarly to what we have done in the present work and to what Witthaut *et al.* did in another recent work [34], the MF derivation of the GPE describing the time evolution of a system of classical and nonlinearly coupled oscillators of variable phases and lengths seems a promising approach to address the emergence of synchronization phenomena in the out of equilibrium dynamics of generic isolated quantum systems consisting of a large number of bosons on a lattice.

Appendices

Appendix A

Dynamical equations

A.1 Explication of the Heisenberg equation of motion

To compute the commutators of (2.2) we evaluate the following terms

$$\left\{ \begin{array}{ll} [n_i, a_k] & = -\delta_{i,k} a_i \\ [n_i, a_k^\dagger] & = \delta_{i,k} a_i^\dagger \\ [n_i(n_i - 1), a_k] & = n_i[n_i, a_k] + [n_i, a_k](n_i - 1) = \{-n_i a_i - a_i(n_i - 1)\} \delta_{i,k} \\ & = \{-a_i n_i + a_i - a_i(n_i - 1)\} \delta_{i,k} = -2n_i a_i \delta_{i,k} \\ [n_i(n_i - 1), a_k^\dagger] & = n_i[n_i, a_k^\dagger] + [n_i, a_k^\dagger](n_i - 1) = \{n_i a_i^\dagger + a_i^\dagger(n_i - 1)\} \delta_{i,k} \\ & = \{a_i^\dagger n_i + a_i^\dagger + a_i^\dagger(n_i - 1)\} \delta_{i,k} = +2a_i^\dagger n_i \delta_{i,k} \\ [a_i^\dagger a_j + a_j^\dagger a_i, a_k] & = -\delta_{i,k} a_k - \delta_{j,k} a_k \\ [a_i^\dagger a_j + a_j^\dagger a_i, a_k^\dagger] & = \delta_{i,k} a_k^\dagger + \delta_{j,k} a_k^\dagger \end{array} \right. \quad (\text{A.1})$$

so that it is possible to write (2.2) explicitly as

$$\left\{ \begin{array}{l} \frac{da_k}{d(it)} = + \sum_{j \neq k}^V t_{j,k} a_k - u n_k a_k + (\mu + \omega_k) a_k \\ \frac{da_k^\dagger}{d(it)} = - \sum_{j \neq k}^V t_{j,k} a_k^\dagger + u a_k^\dagger n_k - (\mu + \omega_k) a_k^\dagger \end{array} \right. \quad (\text{A.2})$$

where we considered $\hbar = 1$. Obviously, the two equations of (A.2) are one the Hermitian conjugate of the other. Exploiting the gauge freedom we can safely operate the following substitution for the bosonic creation and annihilation operators

$$\left\{ \begin{array}{l} a_j^\dagger \rightarrow a_j^\dagger e^{-i\Omega_G t} \\ a_j \rightarrow a_j e^{i\Omega_G t} \end{array} \right. \quad (\text{A.3})$$

where Ω_G is whatever real and constant frequency. In fact, the bosonic commutation relations are still preserved, that is $[a_k e^{i\Omega_G t}, a_j^\dagger e^{-i\Omega_G t}] = [a_k, a_j^\dagger] = \delta_{k,j}$, meaning that $a_j e^{i\Omega_G t}$ and $a_j^\dagger e^{-i\Omega_G t}$ are still respectively annihilation and creation bosonic operators at the j -th site. Under the gauge transformation (A.3), (A.2) transforms into

$$\begin{cases} \frac{da_k}{d(it)} = +\Omega_G a_k + \sum_{j \neq k}^V t_{j,k} a_k - u n_k a_k + (\mu + \omega_k) a_k \\ \frac{da_k^\dagger}{d(it)} = -\Omega_G a_k^\dagger - \sum_{j \neq k}^V t_{j,k} a_k^\dagger + u a_k^\dagger n_k - (\mu + \omega_k) a_k^\dagger \end{cases} \quad (\text{A.4})$$

Saying $t_{k,k} = t_{0,0} \forall k$ and considering $\Omega_G = \mu - t_{0,0}$ we finally get

$$\begin{cases} \frac{da_k}{d(it)} = + \sum_{j=1}^V t_{j,k} a_j - u n_k a_k + \omega_k a_k \\ \frac{da_k^\dagger}{d(it)} = - \sum_{j=1}^V t_{j,k} a_j^\dagger + u a_k^\dagger n_k - \omega_k a_k^\dagger \end{cases} \quad (\text{A.5})$$

that is (2.3).

A.2 From the dynamical equations for ψ_k and ψ_k^* to the ones for ρ_k and θ_k

We can derive DEs for $\rho_k = |\psi_k|^2$ and $\theta_k = \angle \psi_k$ starting from the ones for ψ_k and ψ_k^* (2.9). We just have to write $\rho_k = \psi_k \psi_k^*$ and $\theta_k = \frac{1}{2i} \log\left(\frac{\psi_k}{\psi_k^*}\right)$ and proceed with the following straightforward computations

$$\begin{aligned} \frac{d\rho_k}{d(it)} &= \psi_k^* \frac{d\psi_k}{d(it)} + \psi_k \frac{d\psi_k^*}{d(it)} \\ &= + \sum_{j=1}^V t_{j,k} \psi_j \psi_k^* - u |\psi_k|^4 + \omega_k |\psi_k|^2 - \sum_{j=1}^V t_{j,k} \psi_j^* \psi_k + u |\psi_k|^4 - \omega_k |\psi_k|^2 \\ &= + \sum_{j=1}^V t_{j,k} (\psi_j \psi_k^* - \psi_j^* \psi_k) \\ &= + 2i \sum_{j=1}^V t_{j,k} \sqrt{\rho_j \rho_k} \sin(\theta_j - \theta_k) \end{aligned} \quad (\text{A.6})$$

$$\begin{aligned}
\frac{d\theta_k}{d(it)} &= \frac{d}{d(it)} \frac{1}{2i} \log \left(\frac{\psi_k}{\psi_k^*} \right) = \frac{1}{2i} \frac{\frac{\psi_k}{d(it)} \psi_k^* - \frac{\psi_k^*}{d(it)} \psi_k}{|\psi_k|^2} \\
&= \frac{1}{2i|\psi_k|^2} \left(\sum_{j=1}^V t_{j,k} \psi_j \psi_k^* - u|\psi_k|^4 + \omega_k |\psi_k|^2 + \sum_{j=1}^V t_{j,k} \psi_j^* \psi_k - u|\psi_k|^4 + \omega_k |\psi_k|^2 \right) \\
&= \frac{1}{2i|\psi_k|^2} \left(\sum_{j=1}^V t_{j,k} (\psi_j \psi_k^* + \psi_j^* \psi_k) - 2u|\psi_k|^4 + 2\omega_k |\psi_k|^2 \right) \\
&= -i \left(\sum_{j=1}^V t_{j,k} \frac{|\psi_j|}{|\psi_k|} \cos(\theta_j - \theta_k) - u|\psi_k|^2 + \omega_k \right)
\end{aligned} \tag{A.7}$$

that summing up read

$$\begin{cases} \frac{d\rho_k}{dt} &= 2 \sum_{j=1}^V t_{j,k} \sqrt{\rho_j \rho_k} \sin(\theta_k - \theta_j) \\ \frac{d\theta_k}{dt} &= \sum_{j=1}^V t_{j,k} \sqrt{\frac{\rho_j}{\rho_k}} \cos(\theta_k - \theta_j) - u\rho_k + \omega_k \end{cases} \tag{A.8}$$

that is the system of DEs (2.10).

A.3 $u \rightarrow 0$ limit in non-disordered fully connected Bose-Hubbard model

We analyze here the simple case of $u \rightarrow 0$, corresponding to infinite hopping strength $\tau \rightarrow \infty$ if τ is written in units of $u\rho_0$ as in the main text. Since the nonlinear terms of equation (2.3) are not present and there is thus no need of invoking the approximation (2.7), the results we are going to obtain are exact. We consider for simplicity the non-disordered case of $\omega_k = 0 \forall k = 1, 2, \dots, V$. Substituting $u = 0$ in equation (2.9) we easily get

$$\frac{d\psi_k}{d(it)} = +\tau \Psi \tag{A.9}$$

Summing equation (A.9) over $k = 1, 2, \dots, V$ we find

$$\frac{d\Psi}{d(it)} = +\tau \Psi \tag{A.10}$$

that is solved by $\Psi(t) = \Psi_0 e^{i\tau t}$, where Ψ_0 is a constant c -number consistent with the IC $\{\psi_{k,0}\}_{k=1,\dots,V}$ (notice that $r(t=0) = |\Psi_0|$). Putting this result back into (A.9) we get

$$\frac{\partial \psi_k}{\partial t} = i\tau \Psi_0 e^{i\tau t} \tag{A.11}$$

that is finally solved by $\psi_k(t) = \Psi_0(e^{i\tau t} - 1) + \psi_{k,0}$. We can wonder about the behavior of the system for different ICs.

- $\Psi_0 = 0$
 $\Rightarrow \psi_k(t) = \psi_{k,0} \forall k, \forall t > 0$. Importantly, this configuration is robust with respect to perturbations of the IC, that is if $|\Psi_0| = \epsilon \ll 1$ then $|\psi_k(t) - \psi_{k,0}| \sim \epsilon \quad \forall t > 0$, i.e. the configuration remains close to the initial one. This is the case for instance for the quench from the MI to the SF regime for large V , that corresponds to the initialization of the system in the neighborhoods of the UC.
- $\psi_{k,0} = \Psi_0 \forall k = 1, 2, \dots, V$
 $\Rightarrow \psi_k(t) = \Psi_0 e^{i\tau t}$, that is all the phases are rotating at the same angular speed τ . Importantly, again, this configuration is robust with respect to perturbations of the IC, that is if $|\Psi_0 - \psi_{k,0}| = \epsilon_k \ll 1 \forall k = 1, \dots, V$, then $|\psi_k(t) - \Psi_0 e^{i\tau t}| = \epsilon_k \quad \forall t > 0$, i.e. the configuration remains close to the initial one (at least unless a global phase rotation). This result is not surprising since in this case corresponds to the initialization of the system to the neighborhoods of the SFC configuration, that is to the ground state of the system.

Appendix B

π -aligned stationary configurations

We aim here at complementing chapter 3 and in particular section 3.3.2, finding in a comprehensive way other possible π -aligned SCs. As already observed, to satisfy the first equation of the system (3.3) if $r > 0$ we must have $\sin(\theta_k - \phi) = 0 \ \forall k = 1, 2, \dots, V$. This means that $\theta_i - \theta_j \in \{0, \pi\} \ \forall i, j = 1, 2, \dots, V$, that is that $\theta_i \in \{\phi, \phi + \pi\} \ \forall i = 1, 2, \dots, V$. It is therefore convenient to distinguish two sets of sites, that are the ones with phase equal to ϕ and the ones with phase equal to $\phi + \pi$. Since we already treated the case of all phases aligned, we assume both sets to be non empty. We label each site with a binary variable ν_k such that

$$\nu_k = \begin{cases} +1 & \text{if } \theta_k = \phi \\ -1 & \text{if } \theta_k = \phi + \pi \end{cases} \quad (\text{B.1})$$

This labeling is particularly suitable since it allows us to write $\cos(\theta_k - \phi) = \nu_k$. Because of the symmetry under permutation that characterizes the FC models, we can without loss of generality assume the site indexes to be sorted in such a way that

$$\nu_k = \begin{cases} +1 & \text{for } k = 1, \dots, \alpha V \\ -1 & \text{for } k = \alpha V + 1, \dots, V \end{cases} \quad (\text{B.2})$$

where $\alpha V \in \mathbb{N}$. The stationarity and consistency conditions (3.3) read then

$$\begin{cases} \nu_k \frac{\tau r}{\sqrt{\rho_k}} - u \rho_k = \Omega & \forall k = 1, 2, \dots, V \\ r = \frac{1}{V} \sum_{k=1}^V \nu_k \sqrt{\rho_k} \\ r > 0 \\ \frac{1}{V} \sum_{j=1}^V \rho_j = \rho_0 \end{cases} \quad (\text{B.3})$$

Notice that the condition $r > 0$ is equivalent to the condition $\phi = \theta_1$ and that discarding the case in which $\phi = \theta_N$ is not restrictive because we could always recover it with

a π -rotation of the reference frame. For simplicity, we look at the configurations with $\rho_k = \rho_k(\nu_k)$, that is

$$\begin{cases} \sqrt{\rho_k} = x_+ & \text{if } \nu_k = +1 \\ \sqrt{\rho_k} = x_- & \text{if } \nu_k = -1 \end{cases} \quad (\text{B.4})$$

for which, setting $\rho_0 = 1$ and $u = 1$ (without loss of generality as explained in section 2.4), the stationary condition reads

$$\begin{cases} x_+^3 + \Omega x_+ = +\tau r \\ x_-^3 + \Omega x_- = -\tau r \\ \alpha x_+^2 + (1 - \alpha)x_-^2 = \rho_0 \\ r = \alpha x_+ - (1 - \alpha)x_- \end{cases} \quad (\text{B.5})$$

that has to be solved in the unknown $\alpha \in \{\frac{1}{V}, \frac{2}{V}, \dots, \frac{V-2}{V}, \frac{V-1}{V}\}$, $x_+ > 0, x_- > 0, r \in (0, 1), \Omega \in \mathbb{R}$. In general 4 equations in 5 unknowns can be solved by multiple solutions, that is several possible stationary configurations. We therefore decide to treat r as a parameter and the other variables as unknown. In Eq. (B.5) we get from the fourth equation that $\alpha = \frac{x_- + r}{x_+ + x_-}$, we plug it into the third equation and get

$$\frac{x_- + r}{x_+ + x_-} x_+^2 + \frac{x_+ - r}{x_+ + x_-} x_-^2 = 1 \Rightarrow r(x_+ - x_-) + x_+ x_- = 1 \Rightarrow x_- = \frac{1 - r x_+}{x_+ - r} \quad (\text{B.6})$$

where we assumed $x_+ \neq r$ (in fact, the solution with $x_+ = r$ corresponds to the SFC, that has already been discussed). From the first equation we get $\Omega = \frac{\tau r - x_+^3}{x_+}$. Moreover, the conditions on α and x_- can be turned into conditions on x_+ as $x_+ > r$ and $x_+ < 1/r$ respectively.

The stationarity condition therefore finally reads

$$\begin{cases} \left(\frac{\rho_0 - r x_+}{x_+ - r} \right)^3 + \frac{\tau r - x_+^3}{x_+} \frac{\rho_0 - r x_+}{x_+ - r} = -\tau r & x_+ \in (r, \frac{1}{r}) \\ \Omega = \frac{\tau r - x_+^3}{x_+} \\ x_- = \frac{\rho_0 - r x_+}{x_+ - r} \\ \alpha = \frac{x_- + r}{x_+ + x_-} \end{cases} \quad (\text{B.7})$$

that has to be solved in its first equation in the only unknown x_+ for all the possible values of the parameter $r \in (0, 1)$. Having the solution of the first equation of (B.7) we can then plug it into the second, third and fourth equations to get all the other unknowns. In figure (B.1) we show the numerical solution of (B.7). For each τ and r we find Ω (not shown), α and x_+ of the stationary configurations. Importantly, we notice that only the solutions with α multiple of $1/V$ are acceptable and that the plot should therefore be interpreted as follow: we consider a vertical line for a given τ and look at the intercepts with the lines of acceptable α (that is $\alpha = \frac{1}{V}, \frac{2}{V}, \dots, \frac{V-2}{V}, \frac{V-1}{V}$). Each of such intercept corresponds to a SC, to which it is associated a x_+ given by the colormap in that point. For instance, we see that for $\tau > 2$ there is one SC for each acceptable $\alpha < 1/2$ and

there are no SCs with $\alpha > 1/2$ whereas for $\tau < 2$ there can also be SCs with $\alpha \geq 1/2$. We stress once more that the plot is obtained solving (B.7) for all possible α and shows therefore a continuum of solutions, but as said the α to be considered should be discretized. Understanding which are the stationary PACs helps us to acquire a deeper intuition of the structure of the phase space.

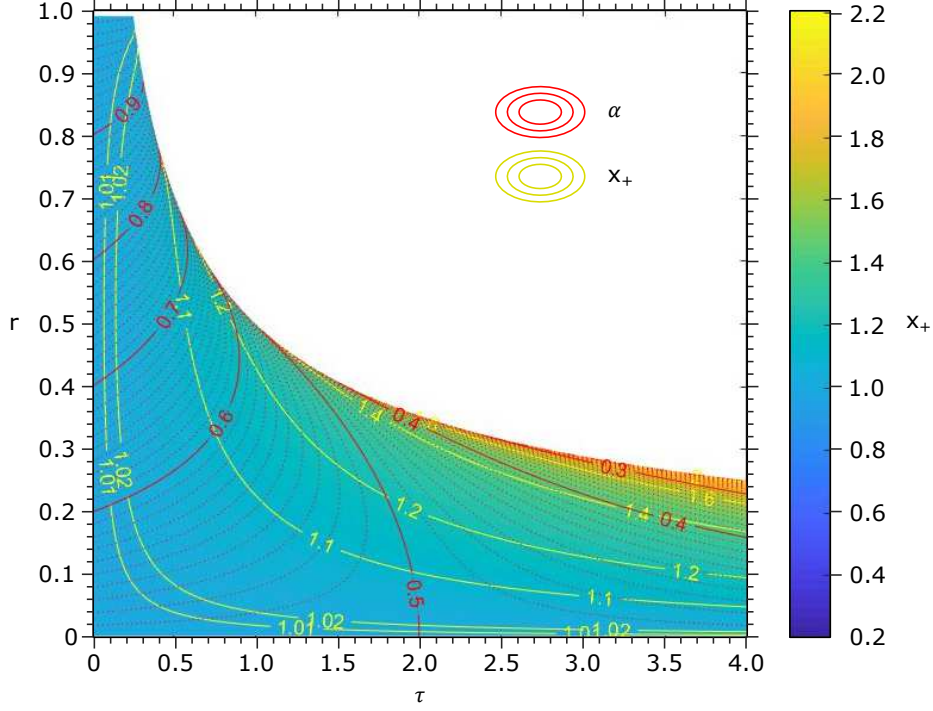


Figure B.1 | Stationary π -aligned configurations. Numerical solution of stationarity equations (B.7). For each r and τ we find the possible SC and plot x_1 (colormap and correspondent yellow contour lines) and α (red contour lines). The region of the (r, τ) plane with no colormap corresponds to a region with no π -aligned SCs.

Appendix C

Quantum fluctuations

C.1 Quantum fluctuations in the delta configuration

We rewrite the characteristic polynomial $P(\lambda)$ (6.27) as

$$P(\lambda) = (\lambda^2 - 3)^2 [\lambda^4 - a\lambda^2 + b] \quad (\text{C.1})$$

where

$$\begin{cases} a = (6 - 4\tau + \tau^2) > 0 \\ b = (3 - 2\tau)^2 - \tau^2 \cos^2(\Delta) \end{cases} \quad (\text{C.2})$$

and where Δ ranges from $\pi/2$ (corresponding to the UC) to π (corresponding to the SPAC).

From the factor $(\lambda^2 - 3)^2$ appearing in (C.1) we readily obtain the eigenvalues $\lambda = \pm\sqrt{3}$ with multiplicity 2, that are real and that therefore do not cause any exponential growth in time of the QFs. The remaining factor $\lambda^4 - a\lambda^2 + b$ gives instead 4 eigenvalues satisfying

$$\lambda^2 = \frac{a \pm \sqrt{a^2 - 4b}}{2} = \frac{a}{2} \left(1 \pm \sqrt{1 - \frac{4b}{a}} \right) \quad (\text{C.3})$$

We observe that complex conjugate imaginary eigenvalues (corresponding to QFs exponentially growing in time) are present in two cases, that are

$$\textcircled{1}: \quad a^2 - 4b < 0 \quad (\text{C.4})$$

$$\textcircled{2}: \quad 1 < \sqrt{1 - \frac{4b}{a^2}} \Rightarrow b < 0 \quad (\text{C.5})$$

We find that $a^2 - 4b = \tau^2 (\tau^2 - 8\tau + 12 + 4\cos^2(\Delta))$, that is a fourth order polynomial in τ with roots $\tau_{1,2} = 4 \pm 2\sin(\Delta)$ (apart from $\tau = 0$, to which we are not interested). As well, we observe that the condition $b < 0$ reads $|3 - 2\tau| < -\tau \cos(\Delta)$. The conditions (C.4) and (C.5), for which QFs will grow exponentially in time, read thus

$$\textcircled{1}: \quad 4 - 2 \sin(\Delta) < \tau < 4 + 2 \sin(\Delta) \quad (\text{C.6})$$

$$\textcircled{2}: \quad \frac{3}{2 - \cos(\Delta)} < \tau < \frac{3}{2 + \cos(\Delta)} \quad (\text{C.7})$$

that for $\Delta = \pi/2$ and $\Delta = \pi$ are in agreement with what we obtained respectively for the UC and the SPAC in chapter 6.

C.2 Physically-motivated symmetries of J_{QF}

As noticed in section 6.1, $\vec{B}$ has to be physically consistent, that is its first and second halves have to be one the Hermitian conjugate of the other at all times under the dynamics (6.3). This directly results in special symmetries of the matrix J_{QF} and, therefore, of its spectrum, that is its eigenvalues $\{\lambda_n\}_{n=1,\dots,2V}$ and eigenvectors $\{\vec{v}_n\}_{n=1,\dots,2V}$. To see this, we write the eigenvalue problem associated to J_{QF} , that is $J_{QF}\vec{v} = \lambda\vec{v}$, for the generic SC. From (6.4) we get

$$\begin{cases} +\tau\langle v_k^{(1)} \rangle_V - \psi_k^2 v_k^{(2)} - 2|\psi_k|^2 v_k^{(1)} & = \lambda v_k^{(1)} \\ -\tau\langle v_k^{(2)} \rangle_V + (\psi_k^*)^2 v_k^{(1)} + 2|\psi_k|^2 v_k^{(2)} & = \lambda v_k^{(2)} \end{cases} \quad (\text{C.8})$$

To unveil the aforementioned symmetries we consider the equations (C.8), change their sign, complex conjugate them, swap them and get

$$\begin{cases} +\tau\langle (v_k^{(2)})^* \rangle_V - \psi_k^2 (v_k^{(1)})^* - 2|\psi_k|^2 (v_k^{(2)})^* & = -\lambda^* (v_k^{(2)})^* \\ -\tau\langle (v_k^{(1)})^* \rangle_V + (\psi_k^*)^2 (v_k^{(2)})^* + 2|\psi_k|^2 (v_k^{(1)})^* & = -\lambda^* (v_k^{(1)})^* \end{cases} \quad (\text{C.9})$$

that is exactly the eigenvalue problem (C.8) when exchanging $\vec{v}^{(1)}, \vec{v}^{(2)}$ and λ with $(\vec{v}^{(2)})^*, (\vec{v}^{(1)})^*$ and $-\lambda^*$ respectively. The immediate consequence is that $\begin{pmatrix} (\vec{v}^{(2)})^* \\ (\vec{v}^{(1)})^* \end{pmatrix}$ is an eigenvector of J_{QF} with associated eigenvalue $-\lambda^*$. It is therefore convenient to sort the indexes of the eigenvalues (and associated eigenvectors) in such a way that we can claim $\lambda_{n+V} = -\lambda_n^* \forall n = 1, 2, \dots, V$, and rewrite the expansion of $\vec{B}$ in the eigenvectors basis (6.7) as

$$\vec{B}(t) = \sum_{n=1}^V \left(c_n \begin{pmatrix} \vec{v}_n^{(1)} \\ \vec{v}_n^{(2)} \end{pmatrix} e^{i\lambda_n t} + c'_n \begin{pmatrix} \vec{v}_n^{(2)} \\ \vec{v}_n^{(1)} \end{pmatrix}^* e^{-i\lambda_n^* t} \right) \quad (\text{C.10})$$

To guarantee the first and the second halves of $\vec{B}$ to be one the Hermitian conjugate of the other, that is the physical consistence of $\vec{B}$, we take $c'_n = c_n^\dagger$, getting finally

$$\vec{B}(t) = \sum_{n=1}^V \left(c_n \begin{pmatrix} \vec{v}_n^{(1)} \\ \vec{v}_n^{(2)} \end{pmatrix} e^{i\lambda_n t} + c_n^\dagger \begin{pmatrix} \vec{v}_n^{(2)} \\ \vec{v}_n^{(1)} \end{pmatrix}^* e^{-i\lambda_n^* t} \right) \quad (\text{C.11})$$

that is (6.8). As explicit example, we observe that for the UC (section 6.2), we have indeed $\{\lambda_1, \lambda_2\} = \{-\lambda_3^*, -\lambda_4^*\}$.

Bibliography

- [1] G. J. Milburn, J. Corney, E. M. Wright, and D. F. Walls, “Quantum dynamics of an atomic Bose-Einstein condensate in a double-well potential,” *Physical Review A*, vol. 55, no. 6, p. 4318, 1997.
- [2] A. Smerzi, S. Fantoni, S. Giovanazzi, and S. R. Shenoy, “Quantum coherent atomic tunneling between two trapped Bose-Einstein condensates,” *Physical review letters*, vol. 79, no. 25, p. 4950, 1997.
- [3] R. Franzosi, V. Penna, and R. Zecchina, “Quantum dynamics of coupled bosonic wells within the bose–hubbard picture,” *International Journal of Modern Physics B*, vol. 14, no. 09, pp. 943–961, 2000.
- [4] R. Gati and M. K. Oberthaler, “A bosonic Josephson junction,” *Journal of Physics B: Atomic, Molecular and Optical Physics*, vol. 40, no. 10, p. R61, 2007.
- [5] E. M. Graefe, H. J. Korsch, and A. E. Niederle, “Mean-field dynamics of a non-hermitian Bose-Hubbard dimer,” *Physical review letters*, vol. 101, no. 15, p. 150408, 2008.
- [6] S. Longhi, “Optical realization of the two-site Bose–Hubbard model in waveguide lattices,” *Journal of Physics B: Atomic, Molecular and Optical Physics*, vol. 44, no. 5, p. 051001, 2011.
- [7] R. Franzosi and V. Penna, “Chaotic behavior, collective modes, and self-trapping in the dynamics of three coupled Bose-Einstein condensates,” *Physical Review E*, vol. 67, no. 4, p. 046227, 2003.
- [8] S. Mossmann and C. Jung, “Semiclassical approach to Bose-Einstein condensates in a triple well potential,” *Physical Review A*, vol. 74, no. 3, p. 033601, 2006.
- [9] B. Liu, L.-B. Fu, S.-P. Yang, and J. Liu, “Josephson oscillation and transition to self-trapping for Bose-Einstein condensates in a triple-well trap,” *Physical Review A*, vol. 75, no. 3, p. 033601, 2007.
- [10] A. Polkovnikov, S. Sachdev, and S. M. Girvin, “Nonequilibrium Gross-Pitaevskii dynamics of boson lattice models,” *Physical Review A*, vol. 66, no. 5, p. 053607, 2002.

- [11] M. P. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, “Boson localization and the superfluid-insulator transition,” *Physical Review B*, vol. 40, no. 1, p. 546, 1989.
- [12] J. Freericks and H. Monien, “Phase diagram of the Bose-Hubbard model,” *EPL (Europhysics Letters)*, vol. 26, no. 7, p. 545, 1994.
- [13] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, “Cold bosonic atoms in optical lattices,” *Physical review letters*, vol. 81, no. 15, p. 3108, 1998.
- [14] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, “Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms,” *Nature*, vol. 415, no. 6867, p. 39, 2002.
- [15] W. Zwerger, “Mott-Hubbard transition of cold atoms in optical lattices,” *Journal of Optics B: Quantum and Semiclassical Optics*, vol. 5, no. 2, p. S9, 2003.
- [16] S. Hofferberth, I. Lesanovsky, B. Fischer, T. Schumm, and J. Schmiedmayer, “Non-equilibrium coherence dynamics in one-dimensional Bose gases,” *Nature*, vol. 449, no. 7160, p. 324, 2007.
- [17] I. Bloch, “Quantum coherence and entanglement with ultracold atoms in optical lattices,” *Nature*, vol. 453, no. 7198, p. 1016, 2008.
- [18] J. Esteve, C. Gross, A. Weller, S. Giovanazzi, and M. K. Oberthaler, “Squeezing and entanglement in a Bose–Einstein condensate,” *Nature*, vol. 455, no. 7217, p. 1216, 2008.
- [19] U. Schneider, L. Hackermüller, S. Will, T. Best, I. Bloch, T. A. Costi, R. W. Helmes, D. Rasch, and A. Rosch, “Metallic and insulating phases of repulsively interacting fermions in a 3D optical lattice,” *Science*, vol. 322, no. 5907, pp. 1520–1525, 2008.
- [20] W. S. Bakr, A. Peng, M. E. Tai, R. Ma, J. Simon, J. I. Gillen, S. Foelling, L. Pollet, and M. Greiner, “Probing the superfluid–to–Mott insulator transition at the single-atom level,” *Science*, vol. 329, no. 5991, pp. 547–550, 2010.
- [21] T. Langen, R. Geiger, and J. Schmiedmayer, “Ultracold atoms out of equilibrium,” 2015.
- [22] I. Bloch, J. Dalibard, and W. Zwerger, “Many-body physics with ultracold gases,” *Reviews of modern physics*, vol. 80, no. 3, p. 885, 2008.
- [23] Y. Makhlin, G. Schön, and A. Shnirman, “Quantum-state engineering with Josephson-junction devices,” *Reviews of modern physics*, vol. 73, no. 2, p. 357, 2001.
- [24] K. Sengupta, S. Powell, and S. Sachdev, “Quench dynamics across quantum critical points,” *Physical Review A*, vol. 69, no. 5, p. 053616, 2004.
- [25] C. Kollath, A. M. Läuchli, and E. Altman, “Quench dynamics and nonequilibrium phase diagram of the Bose-Hubbard model,” *Physical review letters*, vol. 98, no. 18, p. 180601, 2007.

- [26] M. Moeckel and S. Kehrein, “Interaction quench in the Hubbard model,” *Physical review letters*, vol. 100, no. 17, p. 175702, 2008.
- [27] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, “Feshbach resonances in ultracold gases,” *Reviews of Modern Physics*, vol. 82, no. 2, p. 1225, 2010.
- [28] M. Schiró and M. Fabrizio, “Time-dependent mean field theory for quench dynamics in correlated electron systems,” *Physical review letters*, vol. 105, no. 7, p. 076401, 2010.
- [29] B. Sciolla and G. Biroli, “Quantum quenches and off-equilibrium dynamical transition in the infinite-dimensional Bose-Hubbard model,” *Physical review letters*, vol. 105, no. 22, p. 220401, 2010.
- [30] B. Sciolla and G. Biroli, “Dynamical transitions and quantum quenches in mean-field models,” *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2011, no. 11, p. P11003, 2011.
- [31] B. Sciolla and G. Biroli, “Quantum quenches, dynamical transitions, and off-equilibrium quantum criticality,” *Physical Review B*, vol. 88, no. 20, p. 201110, 2013.
- [32] M. Heyl, A. Polkovnikov, and S. Kehrein, “Dynamical quantum phase transitions in the transverse-field Ising model,” *Physical review letters*, vol. 110, no. 13, p. 135704, 2013.
- [33] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, “Colloquium: Nonequilibrium dynamics of closed interacting quantum systems,” *Reviews of Modern Physics*, vol. 83, no. 3, p. 863, 2011.
- [34] D. Witthaut, S. Wimberger, R. Burioni, and M. Timme, “Classical synchronization indicates persistent entanglement in isolated quantum systems,” *Nature communications*, vol. 8, p. 14829, 2017.
- [35] A. R. Kolovsky, H. J. Korsch, and E. M. Graefe, “Bloch oscillations of Bose-Einstein condensates: Quantum counterpart of dynamical instability,” *Physical Review A*, vol. 80, no. 2, p. 023617, 2009.
- [36] M. Albiez, R. Gati, J. Fölling, S. Hunsmann, M. Cristiani, and M. K. Oberthaler, “Direct observation of tunneling and nonlinear self-trapping in a single bosonic Josephson junction,” *Physical review letters*, vol. 95, no. 1, p. 010402, 2005.
- [37] T. Schumm, S. Hofferberth, L. M. Andersson, S. Wildermuth, S. Groth, I. Bar-Joseph, J. Schmiedmayer, and P. Krüger, “Matter-wave interferometry in a double well on an atom chip,” *Nature physics*, vol. 1, no. 1, p. 57, 2005.
- [38] R. Gati, B. Hemmerling, J. Fölling, M. Albiez, and M. K. Oberthaler, “Noise thermometry with two weakly coupled Bose-Einstein condensates,” *Physical review letters*, vol. 96, no. 13, p. 130404, 2006.

- [39] M. Chuchem, K. Smith-Mannschott, M. Hiller, T. Kottos, A. Vardi, and D. Cohen, “Quantum dynamics in the bosonic Josephson junction,” *Physical Review A*, vol. 82, no. 5, p. 053617, 2010.
- [40] J. A. Stickney, D. Z. Anderson, and A. A. Zozulya, “Transistorlike behavior of a Bose-Einstein condensate in a triple-well potential,” *Physical Review A*, vol. 75, no. 1, p. 013608, 2007.
- [41] F. Trimborn, D. Witthaut, and H. J. Korsch, “Beyond mean-field dynamics of small Bose-Hubbard systems based on the number-conserving phase-space approach,” *Physical Review A*, vol. 79, no. 1, p. 013608, 2009.
- [42] A. Flesch, M. Cramer, I. McCulloch, U. Schollwöck, and J. Eisert, “Probing local relaxation of cold atoms in optical superlattices,” *Physical Review A*, vol. 78, no. 3, p. 033608, 2008.
- [43] M. Rigol, V. Dunjko, and M. Olshanii, “Thermalization and its mechanism for generic isolated quantum systems,” *Nature*, vol. 452, no. 7189, p. 854, 2008.
- [44] M. Cramer, A. Flesch, I. P. McCulloch, U. Schollwöck, and J. Eisert, “Exploring local quantum many-body relaxation by atoms in optical superlattices,” *Physical review letters*, vol. 101, no. 6, p. 063001, 2008.
- [45] M. Cramer, C. M. Dawson, J. Eisert, and T. J. Osborne, “Exact relaxation in a class of nonequilibrium quantum lattice systems,” *Physical review letters*, vol. 100, no. 3, p. 030602, 2008.
- [46] M. Rigol, “Breakdown of thermalization in finite one-dimensional systems,” *Physical review letters*, vol. 103, no. 10, p. 100403, 2009.
- [47] M. Eckstein, M. Kollar, and P. Werner, “Thermalization after an interaction quench in the Hubbard model,” *Physical review letters*, vol. 103, no. 5, p. 056403, 2009.
- [48] G. Biroli, C. Kollath, and A. M. Läuchli, “Effect of rare fluctuations on the thermalization of isolated quantum systems,” *Physical review letters*, vol. 105, no. 25, p. 250401, 2010.
- [49] J. Dziarmaga, “Dynamics of a quantum phase transition and relaxation to a steady state,” *Advances in Physics*, vol. 59, no. 6, pp. 1063–1189, 2010.
- [50] M. Kollar, F. A. Wolf, and M. Eckstein, “Generalized Gibbs ensemble prediction of prethermalization plateaus and their relation to nonthermal steady states in integrable systems,” *Physical Review B*, vol. 84, no. 5, p. 054304, 2011.
- [51] Y. Kuramoto, “Self-entrainment of a population of coupled non-linear oscillators,” in *International symposium on mathematical problems in theoretical physics*, pp. 420–422, Springer, 1975.

- [52] S. H. Strogatz, “From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators,” *Physica D: Nonlinear Phenomena*, vol. 143, no. 1-4, pp. 1–20, 2000.
- [53] J. A. Acebrón, L. L. Bonilla, C. J. P. Vicente, F. Ritort, and R. Spigler, “The Kuramoto model: A simple paradigm for synchronization phenomena,” *Reviews of modern physics*, vol. 77, no. 1, p. 137, 2005.
- [54] C. A. Holmes, C. P. Meaney, and G. J. Milburn, “Synchronization of many nanomechanical resonators coupled via a common cavity field,” *Physical Review E*, vol. 85, no. 6, p. 066203, 2012.
- [55] T. E. Lee and H. R. Sadeghpour, “Quantum synchronization of quantum van der Pol oscillators with trapped ions,” *Physical review letters*, vol. 111, no. 23, p. 234101, 2013.
- [56] T. E. Lee, C.-K. Chan, and S. Wang, “Entanglement tongue and quantum synchronization of disordered oscillators,” *Physical Review E*, vol. 89, no. 2, p. 022913, 2014.
- [57] S. Walter, A. Nunnenkamp, and C. Bruder, “Quantum synchronization of a driven self-sustained oscillator,” *Physical review letters*, vol. 112, no. 9, p. 094102, 2014.
- [58] V. M. Bastidas, I. Omelchenko, A. Zakharova, E. Schöll, and T. Brandes, “Quantum signatures of chimera states,” *Physical Review E*, vol. 92, no. 6, p. 062924, 2015.
- [59] Y. Castin and R. Dum, “Instability and depletion of an excited Bose-Einstein condensate in a trap,” *Physical review letters*, vol. 79, no. 19, p. 3553, 1997.
- [60] P. Reimann, “Foundation of statistical mechanics under experimentally realistic conditions,” *Physical review letters*, vol. 101, no. 19, p. 190403, 2008.
- [61] S. H. Strogatz, *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering*. CRC Press, 2018.
- [62] J. Prost, *The physics of liquid crystals*, vol. 83. Oxford university press, 1995.
- [63] A. Pikovsky, M. Rosenblum, J. Kurths, and J. Kurths, *Synchronization: a universal concept in nonlinear sciences*, vol. 12. Cambridge university press, 2003.