POLITECNICO DI TORINO

Master of Science in Mathematical Engineering

Existence results in shape optimization and applications to spectral problems

Master Thesis

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Introduction

This work is mainly devoted to the presentation of some classical results about existence of solutions for shape optimization problems, i.e. problems formulated as

$$\min\{F(A) \mid A \in \mathcal{A}\}$$

where F is a functional and $\mathcal{A}$ is a suitable class of subsets of $\mathbb{R}^n$.
The main result is a theorem proved by Buttazzo and Dal Maso (1993), [10]. Indeed, some other results were previously available, but they required some constraints on ∂A that are sometimes unnatural in the applications, and in the work of Buttazzo and Dal Maso only a volume constraint is imposed, i.e. $|A| = c$. The theorem, assuming that F is decreasing with respect to inclusion and semicontinuous with respect to a suitable topology, guarantees that for every $c \in [0, |\Omega|]$ there exists a solution for the optimization problem

$$\min\Big\{F(A) \mid A \in \mathcal{A}(\Omega), |A| = c\Big\},$$

where Ω is open and bounded in $\mathbb{R}^n$ and $\mathcal{A}(\Omega)$ is a suitable family of subsets of Ω. This class of problems includes the minimization of spectral functionals

$$\min\Big\{\Phi(\lambda_1(A), ..., \lambda_k(A)) \mid A \in \mathcal{A}(\Omega), |A| = c\Big\},$$

where $\{\lambda_i(A)\}$ are the eigenvalues of the Dirichlet-Laplacian on A, and $\Phi : \mathbb{R}^k \longrightarrow \mathbb{R}$ is non-decreasing. We observe that in this theorem it is required that all subsets are contained in a bounded open subset Ω (the "box"), which is fixed. This hypothesis is used, for example, to guarantee the compactness of $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, but this seemed to be only a technical request. So, for years, many efforts were made to prove the existence of solutions for the problem

$$\min\Big\{\Phi(\lambda_1(A), ..., \lambda_k(A)) \mid A \in \mathcal{A}(\mathbb{R}^n), |A| = c\Big\},$$

without any a priori confinement for the admissible sets. Twenty years after the work of Buttazzo and Dal Maso a positive answer is finally available, due to D. Bucur [5] (2012) and A. Pratelli-D. Mazzoleni [27] (2013) who, using different techniques, solved this open problem.

The structure of the thesis is the following:

- In Chapter 1 we develop all the necessary tools (C_p-capacity, quasiopen sets, $\Gamma-$convergence) to work on this kind of problems.
- In Chapter 2 we prove the theorem of Buttazzo and Dal Maso with all details.
- In Chapter 3 we focus our attention on the spectral case and we discuss the main ideas (with no technical details) behind the theorem of Bucur, Pratelli and Mazzoleni.

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List of symbols

- $B_r(x_0)$ Open ball in x_0 with radius r.
- $\mathbf{1}_A(x)$ Indicator function of set A.
- $|E|$ Lebesgue measure of E in $\mathbb{R}^n$.
- $C_c^\infty(\Omega)$ C^∞ function with compact support in Ω.
- $C_{1,p}(A,\Omega)$ Capacity of A with respect to Ω.
- $C_{1,p}(A)$ Capacity of A with respect to $\mathbb{R}^n$.
- $\mathcal{A}(\Omega)$ Family of quasiopen subsets of Ω.
- $u \wedge v$ Minimum between u and v.
- $u \vee v$ Maximum between u and v.
- $E \subset\subset \Omega$ Compact embedding of subsets of $\mathbb{R}^n$.
- $P(A,\Omega)$ Perimeter of A with respect to Ω.
- $\mathcal{H}^k(S)$ k-dimensional Hausdorff measure of S in $\mathbb{R}^n$.
- $W^{k,p}(U) \hookrightarrow L^q(U)$ Continuous embedding of $W^{k,p}(U)$ in $L^q(U)$.
- $W^{k,p}(U) \subset\subset L^q(U)$ Compact embedding of $W^{k,p}(U)$ in $L^q(U)$.

Contents

Chapter 1

Preliminary tools

In this chapter we introduce all the mathematical tools that are necessary to treat the optimization problems in the rest of the thesis. In particular, we will develop in detail the theory of capacity and Γ-convergence. Finally, some remarks will be made regarding spectral theory of elliptical operators.

1.1 Capacity

One of the key tools for treating optimization problems governed by elliptic equations is the notion of capacity. The main references for this part are [12, 22, 7, 18].

1.1.1 Basic results

Definition 1.1. Let Ω be an open and bounded subset of $\mathbb{R}^n$ and let A be an open subset contained in Ω. For each $p \in [1, +\infty)$ we define the *$p-$Capacity of an open subset A with respect to Ω* as

$$C_{1,p}(A, \Omega) := \inf \left\{ \int_\Omega |Du|^p \, dx \;\Big|\; u \in W_0^{1,p}(\Omega),\ u \geq 1 \text{ a.e. on } A \right\}. \tag{1.1}$$

If it is not ambiguous we use the equivalent notations $C_{1,p}(A)$ or $C_p(A)$ and we call the $p-$capacity simply "capacity".

Remark 1.2. We observe that if $A = \Omega$ then the *$p-$Capacity of A* is $+\infty$ because $\inf \emptyset = +\infty$.

Proposition 1.3. *If $p > 1$ then the infimum in* (1.1) *is a minimum, i.e.*

$$C_{1,p}(A,\Omega) = \min\left\{ \int_\Omega |Du|^p\,dx \;\Big|\; u \in W_0^{1,p}(\Omega),\ u \geq 1 \ a.e.\ on\ A \right\}$$

To prove this we will use the Direct Method of the Calculus of Variations. First we recall the following result:

Theorem 1.4. *Let $(X, ||\cdot||)$ be a reflexive Banach space and let C be a non-empty closed and convex subset of X. Let $F : C \to \mathbb{R} \cup \{+\infty\}$ be a coercive functional that is lower semi-continuous with respect to the weak convergence. Then there exists $u \in C$ such that $F(u) = \inf_C F = \min_C F$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset C$ be a minimizing sequence, i.e. $\lim_{n\to+\infty} F(u_n) = \inf_C F$. In particular the sequence $\{F(u_n)\}$ is bounded and, by coerciveness, we have $||u_n|| \leq K$, where K is a constant independent of n. Since X is reflexive there exists a subsequence (still denoted by $\{u_n\}$) and an element $u \in X$ such that $u_n \rightharpoonup u$. The subset C is closed and convex so it is weakly closed, and this implies that $u \in C$. Now using the semi-continuity we finally obtain

$$\inf_C F \leq F(u) \leq \liminf_{n\to\infty} F(u_n) = \lim_{n\to\infty} F(u_n) = \inf_C F.$$

$\square$

We also recall the following result

Theorem 1.5. *Let $(X, ||\cdot||)$ be a reflexive Banach space and let C be a closed and convex subset of X. If $F : C \to \mathbb{R} \cup \{+\infty\}$ is convex and continuous then, it is lower semi-continuous with respect to the weak convergence.*

For a proof see, e.g., [4]. Now we can prove Proposition 1.3.

Proof of Proposition 1.3. Let $X = W_0^{1,p}(\Omega)$, $F(v) = \int_\Omega |Dv|^p\,dx$ and $C = \{u \in W_0^{1,p}(\Omega) \mid u \geq 1$ a.e. on $A\}$. Since $p > 1$ the space X is reflexive. The functional F is continuous and convex, so it lower semi-continuous. The subset C is obviously convex and, to prove that it is closed, it is sufficient to remember that from a strong convergent sequence we can extract a subsequence convergent a.e.. The claim follows from Theorem 1.4. $\square$

A very important propriety of $W^{1,p}(\Omega)$ is the following:

Proposition 1.6. *Let $u, v \in W^{1,p}(\Omega)$ (resp. in $W_0^{1,p}(\Omega)$) and define $\psi := \max\{u, v\}$ and $\phi := \min\{u, v\}$. Then $\psi, \phi \in W^{1,p}(\Omega)$ (resp. in $W_0^{1,p}(\Omega)$) and for a.e. $x \in \Omega$*

$$D\psi(x) = \begin{cases} Du(x) & \text{if } u(x) \geq v(x) \\ Dv(x) & \text{if } u(x) \leq v(x), \end{cases} \qquad D\phi(x) = \begin{cases} Du(x) & \text{if } u(x) \leq v(x) \\ Dv(x) & \text{if } u(x) \geq v(x). \end{cases}$$

We can prove the following:

Proposition 1.7. *For all $p \in [1, +\infty)$ one has*

$$C_{1,p}(A, \Omega) = \inf \Big\{ \int_\Omega |Du|^p \, dx \;\Big|\; u \in W_0^{1,p}(\Omega),\ \mathbf{1}_A \leq u \leq 1 \text{ a.e. on } \Omega \Big\}.$$

Proof. We define

$$\gamma_{1,p}(A, \Omega) := \inf \Big\{ \int_\Omega |Du|^p \, dx \;\Big|\; u \in W_0^{1,p}(\Omega),\ \mathbf{1}_A \leq u \leq 1 \text{ a.e. on } \Omega \Big\}.$$

Obviously $\gamma_{1,p}(A, \Omega) \geq C_{1,p}(A, \Omega)$ so we have to prove the reverse inequality. By definition of infimum, for every $\varepsilon > 0$ there exists $u_\varepsilon \in W_0^{1,p}(\Omega)$, $u_\varepsilon \geq 1$ a.e. on Ω such that

$$C_{1,p}(A, \Omega) + \varepsilon \geq \int_\Omega |Du_\varepsilon|^p \, dx.$$

If we define $v_\epsilon := (u_\varepsilon \wedge 1) \vee 0$ then $v_\varepsilon \in W_0^{1,p}(\Omega)$ and $\mathbf{1}_A \leq v_\varepsilon \leq 1$ a.e. on Ω. By definition of v_ε we have

$$C_{1,p}(A, \Omega) + \varepsilon \geq \int_\Omega |Du_\varepsilon|^p \, dx \geq \int_\Omega |Dv_\varepsilon|^p \, dx \geq \gamma_{1,p}(A, \Omega)$$

i.e.

$$C_{1,p}(A, \Omega) + \varepsilon \geq \gamma_{1,p}(A, \Omega) \qquad \forall \varepsilon > 0.$$

□

Now we extend the notion of capacity to a general subset of Ω.

Definition 1.8. Let Ω be an open and bounded subset of $\mathbb{R}^n$ and let E be a subset of Ω. For each $p \in [1, +\infty)$ we define the *$p-$Capacity of E with respect to Ω* as

$$C_{1,p}(E, \Omega) := \inf \Big\{ C_{1,p}(A, \Omega) \,|\, A \text{ open, } E \subseteq A \Big\}. \tag{1.2}$$

We observe that the last definition can be rewritten in the following way

$$C_{1,p}(E,\Omega) = \inf \left\{ \int_\Omega |Du|^p \, dx \;\Big|\; u \in W_0^{1,p}(\Omega),\; u \geq 1 \text{ a.e. on } U_E \in \mathcal{U}_E \right\}$$

where $\mathcal{U}_E$ is the family of open neighborhoods of E.

We recall the following result

Theorem 1.9 (Dunford-Pettis criterion)**.** *Let Ω be a open and bounded subset of $\mathbb{R}^n$. A subset $\mathcal{H}$ of $L^1(\Omega)$ is relatively weakly compact if and only if*

i) *$\mathcal{H}$ is bounded,*

ii) *$\mathcal{H}$ is uniform integrable, i.e. $\forall \varepsilon > 0$ there exists $\delta > 0$ such that*

$$\int_E |f| \, dx < \varepsilon$$

whenever $f \in \mathcal{H}$ and $|E| < \delta$.

Proposition 1.10. *The $p-$Capacity in Definition 1.8 induces a set function*

$$C_{1,p}(\,\cdot\,,\Omega) : \mathcal{P}(\Omega) \to [0,+\infty]$$

that satisfies the following proprieties:

i) *(Monotonicity with respect to inclusion) if $E_1 \subseteq E_2 \subseteq \Omega$ then*

$$C_{1,p}(E_1,\Omega) \leq C_{1,p}(E_1,\Omega),$$

ii) *(Strong subadditivity) if $E_1, E_2 \subseteq \Omega$ then*

$$C_{1,p}(E_1 \cup E_2,\Omega) + C_{1,p}(E_1 \cap E_2,\Omega) \leq C_{1,p}(E_1,\Omega) + C_{1,p}(E_2,\Omega),$$

iii) *(Countable subadditivity) if $\{E_h\} \subseteq \Omega$ then*

$$C_{1,p}\Big(\bigcup_h E_h,\Omega\Big) \leq \sum_{h=1}^{+\infty} C_{1,p}(E_h,\Omega).$$

Proof.

i) Obvious by definition.

ii) Let $u_1, u_2 \in W_0^{1,p}(\Omega)$ such that $u_1 \geq 1$ a.e. on $U_{E_1} \in \mathfrak{U}_{E_1}$, and $u_2 \geq 1$ a.e. on $U_{E_2} \in \mathfrak{U}_{E_2}$ and define $\psi := \max\{u, v\}$ and $\phi := \min\{u, v\}$. From Proposition 1.6 it follows that

$$\int_\Omega |D\psi|^p \, dx + \int_\Omega |D\phi|^p \, dx = \int_\Omega |Du_1|^p \, dx + \int_\Omega |Du_2|^p \, dx.$$

Observing that $\psi, \phi \in W_0^{1,p}(\Omega)$ and $\psi \geq 1$ a.e. on on a neighborhood of $E_1 \cup E_2$ and $\phi \geq 1$ a.e. on on a neighborhood of $E_1 \cap E_2$ it follows that

$$C_{1,p}(E_1 \cup E_2, \Omega) + C_{1,p}(E_1 \cap E_2, \Omega) \leq \int_\Omega |Du_1|^p \, dx + \int_\Omega |Du_2|^p \, dx$$

and this is true for all u_1, u_2.

iii) For all $\varepsilon > 0$ and for all $h \in \mathbb{N}$ there exists $v_{h,\varepsilon} \in W_0^{1,p}(\Omega)$, $v_{h,\varepsilon} \geq 1$ on A_h, open neighborhood of E_h, and such that

$$C_{1,p}(E_h, \Omega) + \frac{\varepsilon}{2^h} \geq \int_\Omega |Dv_{h,\varepsilon}|^p \, dx.$$

Defined the function $v_\varepsilon := \sup_h v_{h,\varepsilon}$, it follows that $v_\varepsilon \geq 1$ a.e. on $\bigcup_h A_h \supseteq \bigcup_h E_h$. We have to prove that v_ε is in $W_0^{1,p}(\Omega)$. We split the proof in two cases.

- (Case $p > 1$) We put $u_{h,\varepsilon} := v_{1,\varepsilon} \vee v_{2,\varepsilon} \vee ... \vee v_{h,\varepsilon}$ so $u_{h,\varepsilon} \to v_\varepsilon$ a.e. on Ω, $u_{h,\varepsilon} \in W_0^{1,p}(\Omega)$ and

$$\begin{aligned} ||u_{h,\varepsilon}||^p_{W_0^{1,p}(\Omega)} \leq \sum_{i=1}^{h} \int_\Omega |Dv_{i,\varepsilon}|^p \, dx &\leq \sum_{i=1}^{+\infty} \int_\Omega |Dv_{i,\varepsilon}|^p \, dx \\ &\leq \sum_{i=1}^{+\infty} C_{1,p}(E_i, \Omega) + \varepsilon. \end{aligned} \tag{1.3}$$

We can suppose that $\sum_{i=1}^{+\infty} C_{1,p}(E_i, \Omega) < +\infty$ (otherwise the problem is trivial), so there exists a subsequence (still denoted by $\{u_{h,\varepsilon}\}$) and a function $u \in W_0^{1,p}(\Omega)$ such that $u_{h,\varepsilon} \rightharpoonup u$ and, from compact embedding theorems it follows that $u_{h,\varepsilon} \to u$ a.e. on Ω. This implies that $u = v_\varepsilon$. In particular $v_\varepsilon \in W_0^{1,p}(\Omega)$ and

$$C_{1,p}\Big(\bigcup_h E_h, \Omega\Big) \leq \int_\Omega |Dv_\varepsilon|^p \, dx,$$

but using the semicontinuity of the norm and the bound in (1.3) we get

$$||v_\varepsilon||^p_{W_0^{1,p}(\Omega)} \le \liminf_{h\to+\infty} ||u_{h,\varepsilon}||^p_{W_0^{1,p}(\Omega)} \le \sum_{i=1}^{+\infty} C_{1,p}(E_i,\Omega)+\varepsilon$$

so

$$C_{1,p}\Big(\bigcup_h E_h,\Omega\Big) \le \sum_{h=1}^{+\infty} C_{1,p}(E_h,\Omega)+\varepsilon \quad \forall\varepsilon>0.$$

- (Case $p=1$) We will prove only that $v_\varepsilon \in W_0^{1,p}(\Omega)$ because the other passages are equal to the case $p>1$. Defining

$$f=\sum_{h=1}^{+\infty}|Dv_{h,\varepsilon}|$$

we get

$$\sum_{h=1}^{\infty}\int_\Omega |Dv_h|\,dx \le \sum_{h=1}^{+\infty} C_{1,p}(E_h)+\varepsilon<+\infty$$

so $f\in L^1(\Omega)$. We define $u_{h,\varepsilon} := v_{1,\varepsilon}\vee v_{2,\varepsilon}\vee ...\vee v_{h,\varepsilon}$ and obviously $|Du_h|\le f$. Then, using Theorem 1.9, we obtain the existence of a subsequence $\{u_{h_k}\}$ and a function φ such that $Du_{h_k}\rightharpoonup\varphi$ weakly in $L^1(\Omega)$. Thanks to compact embedding theorems $u_{h_k}\longrightarrow u$ in $L^1(\Omega)$, $\varphi=Du$ and $u\in W_0^{1,1}(\Omega)$. Moreover $u_h\rightharpoonup u$ in $W_0^{1,1}(\Omega)$ and $u_h\longrightarrow u$ in $L^1(\Omega)$. Then, recalling that $u_h\longrightarrow v$ a.e. on Ω, it follows $v=u$ and $v\in W_0^{1,1}(\Omega)$.

□

Let define

$$\mathcal{E}_p(\Omega) := \{E\subseteq\Omega \,|\, C_{1,p}(E,\Omega)=0\}.$$

As a consequence of Proposition 1.10, it follows the following result.

Proposition 1.11. *Let Ω be an open and bounded subset of $\mathbb{R}^n$. Then:*

- *if $\{E_h\}\subseteq\mathcal{E}_p(\Omega)$ then $\bigcup_h E_h\in\mathcal{E}_p(\Omega)$,*
- *if $E\subseteq F\in\mathcal{E}_p(\Omega)$ then $E\in\mathcal{E}_p(\Omega)$.*

Proposition 1.12. *Let $\Omega \subseteq \Omega'$ be two open and bounded subsets of $\mathbb{R}^n$. Then for all subsets $E \subseteq \Omega$ the following inequality holds*

$$C_{1,p}(E, \Omega') \le C_{1,p}(E, \Omega).$$

Proof. It is sufficient to prove the inequality for A open. If $C_{1,p}(A, \Omega)$ is infinite then we have nothing to prove. If it is finite then for any $\varepsilon > 0$ there exists $u_\varepsilon \in W_0^{1,p}(\Omega)$, $u_\varepsilon \ge 1$ a.e. on A such that

$$\int_\Omega |Du_\varepsilon|^p \, dx < C_{1,p}(A, \Omega) + \varepsilon.$$

We define the extension of u_ε to Ω' as

$$v_\varepsilon(x) := \begin{cases} u_\varepsilon(x) & x \in \Omega, \\ 0 & x \in \Omega' \setminus \Omega. \end{cases}$$

Then v_ε belongs to $W_0^{1,p}(\Omega')$ and

$$\int_{\Omega'} |Dv_\varepsilon|^p \, dx = \int_\Omega |Du_\varepsilon|^p \, dx.$$

Obviously $v_\varepsilon \ge 1$ a.e. on A, and then

$$C_{1,p}(A, \Omega') \le \int_{\Omega'} |Dv_\varepsilon|^p \, dx = \int_\Omega |Du_\varepsilon|^p \, dx < C_{1,p}(A, \Omega) + \varepsilon \quad \forall \varepsilon > 0.$$

The inequality follows since ε is arbitrary. □

Proposition 1.13. *Let $\Omega \subseteq \Omega'$ be two open and bounded subsets of $\mathbb{R}^n$. Then for all subsets $E \subseteq \Omega$*

$$C_{1,p}(E, \Omega') = 0 \iff C_{1,p}(E, \Omega) = 0.$$

Proof.
($\Leftarrow$) Follows from the previous inequality.
($\Rightarrow$) Suppose $C_{1,p}(E, \Omega') = 0$ and let $\{E_h\}$ be a sequence such that $E = \bigcup_h E_h$ and $E_h \subset\subset \Omega$. If we prove that $C_{1,p}(E_h, \Omega) = 0$ for each h then $C_{1,p}(E, \Omega) = 0$. In order to avoid indices, it is sufficient to prove that

$$\forall E \subset\subset \Omega \subset \Omega', \; C_{1,p}(E, \Omega') = 0 \implies C_{1,p}(E, \Omega) = 0.$$

By definition of $C_{1,p}(E,\Omega') = 0$, there exists $\{A_h\}$ such that A_h is open, $E \subseteq A_h \subseteq \Omega'$ and

$$C_{1,p}(A_h,\Omega') < \frac{1}{h} \quad \forall\, h.$$

Moreover, there exists $u_h \in W_0^{1,p}(\Omega')$ such that $u_h \geq 1$ a.e. on A_h and

$$\int_{\Omega'} |Du_h|^p\, dx < \frac{1}{h}.$$

Consider a cut-off function $\varphi \in C_c^\infty(\Omega)$ such that $\varphi = 1$ on a neighborhood A of E and consider the function $v_h := \varphi u_h \in W_0^{1,p}(\Omega)$, $v_h \geq 1$ a.e. on the open neighborhood $A_h \cap A \supseteq E$. By definition

$$\begin{aligned} C_{1,p}(A\cap A_h,\Omega) \leq \int_\Omega |Dv_h|^p\, dx = & \int_\Omega |Du_h\varphi + u_h D\varphi|^p dx \\ \leq & C\left[\int_\Omega |Du_h|^p\, dx + \int_\Omega |u_h|^p dx\right] \\ \leq & C\left[\int_{\Omega'} |Du_h|^p\, dx + \int_{\Omega'} |u_h|^p dx\right] \end{aligned}$$

where $C = C(||\varphi||_\infty, ||\,|D\varphi|\,||_\infty)$ is independent of h. Using the Poincaré inequality and the fact that

$$\int_{\Omega'} |Du_h|^p\, dx \longrightarrow 0 \quad \text{as } h \to +\infty,$$

it follows that

$$C_{1,p}(E,\Omega) \leq C_{1,p}(A\cap A_h,\Omega) \longrightarrow 0 \quad \text{as } h \to +\infty,$$

because $A_h \cap A$ is a open neighborhood of E.

□

In other words the null capacity sets are independent of the ambient space Ω. Now we compute explicitly the capacity of some particular set.

Proposition 1.14. *Let $A := (\alpha,\beta) \subset \Omega := (a,b) \subseteq \mathbb{R}$ be two open intervals with $-\infty < a < \alpha < \beta < b < +\infty$. Then the $2-$Capacity is*

$$C_{1,2}(A,\Omega) := C_2((\alpha,\beta),(a,b)) = \frac{1}{\alpha - a} + \frac{1}{b-\beta} \tag{1.4}$$

Proof. By definition

$$C_2(A,\Omega) := \min_{K_\psi} \int_a^b |u'|^2 \, dx$$

where $K_\psi := \{u \in W_0^{1,2}((a,b)) \,|\, u \geq \psi \text{ a.e.}\}$ and

$$\psi(x) := \begin{cases} 1 & \text{if } x \in (\alpha,\beta), \\ -\infty & \text{if } x \notin (\alpha,\beta). \end{cases}$$

From the theory of Obstacle Problems (see the second Chapter) there is existence and uniqueness of the minimizer. Moreover the minimizer satisfies the variational inequality

$$\begin{cases} u \in K_\psi, \\ \displaystyle\int_a^b u'(v'-u')\, dx \geq 0, \quad \forall\, v \in K_\psi. \end{cases}$$

so the function is piecewise linear, precisely it is

$$u(x) := \begin{cases} \dfrac{x-a}{\alpha-a} & \text{if } x \in (a,\alpha), \\ 1 & \text{if } x \in (\alpha,\beta), \\ \dfrac{b-x}{b-\beta} & \text{if } x \in (\beta,b). \end{cases}$$

Finally, it is possible to compute the value of the capacity, i.e.

$$C_2((\alpha,\beta),(a,b)) = \frac{1}{\alpha-a} + \frac{1}{b-\beta}.$$

□

Corollary 1.15. *Let $\Omega := (a,b) \subseteq \mathbb{R}$ be an interval, let $t \in \Omega$ be a point in the interval and $E := \{t\}$. Then the 2−Capacity is*

$$C_{1,2}(E,\Omega) := C_2(\{t\},(a,b)) = \frac{1}{t-a} + \frac{1}{b-t}.$$

Proof.

$$\begin{aligned} C_2(\{t\},(a,b)) &= \inf_{\varepsilon>0} C_2((t-\varepsilon,t+\varepsilon),(a,b)) \\ &= \inf_{\varepsilon>0} \frac{1}{t-\varepsilon-a} + \frac{1}{b-t-\varepsilon} = \frac{1}{t-a} + \frac{1}{b-t}. \end{aligned}$$

□

Using polar coordinates it is possible to compute $C_p\big(B_r(x_0), B_R(x_0)\big)$ with $0 < r < R$. First we note that the problem is invariant under translations, i.e.

Remark 1.16. Let Ω be an open and bounded subset of $\mathbb{R}^n$. Let E be a subset of Ω and $z \in \mathbb{R}^n$, then

$$C_{1,p}(E, \Omega) = C_{1,p}(E + z, \Omega + z)$$

where

$$\Omega + z := \{y \in \mathbb{R}^n \,|\, \exists\, x \in \Omega \text{ s.t. } y = x + z\},$$

$$E + z := \{y \in \mathbb{R}^n \,|\, \exists\, x \in E \text{ s.t. } y = x + z\}.$$

In particular, let $B_r(x_0)$, $B_R(x_0)$ be two open ball in $\mathbb{R}^n$ where $0 < r < R$ and $x_0 \in \mathbb{R}^n$. Then

$$C_{1,p}\big(B_r(x_0), B_R(x_0)\big) = C_{1,p}\big(B_r(\mathbf{0}), B_R(\mathbf{0})\big) \tag{1.5}$$

Example 1.17 (C_p-capacity of a ball for $p \in (1, +\infty)$)**.** We compute the $C_{1,p}\big(B_r(x_0), B_R(x_0)\big)$ for $p \in (1, +\infty)$ and $0 < r < R$. We can assume $x_0 = 0$. We denote $B_r = B_r(0)$ and $B_R = B_R(0)$.
We have to solve the minimization problem

$$C_p(B_r, B_R) = \min_{\substack{u \in W_0^{1,p}(B_R) \\ u \geq 1 \text{ a.e. in } B_r}} \int_{B_R} |Dv|^p \, dx \tag{1.6}$$

We put $K_{r,R} = \{u \in W_0^{1,p}(B_R) \,|, u \geq 1 \text{ a.e. in } B_r\}$. We split the problem into several steps.

1) Proposition 1.3 ensures the existence of a minimizer.

2) The minimum is unique. Seeking a contradiction, we suppose that there exist two different minimum point u and v. We put $w := \frac{1}{2}u + \frac{1}{2}v$ then $w \in K_{r,R}$ and we note that $u \neq v$ implies that $Du \neq Dv$[1], otherwise if $Du = Dv$ a.e. on B_R, by definition of weak derivative it follows that $u - v = c \in \mathbb{R}$ a.e. on B_R but $u - v \in W_0^{1,p}(B_R)$ so $c = 0$, but we

[1] i.e. $|\{x \in \Omega \,|\, Du(x) \neq Dv(x)\}| > 0$

have assumed that $u \neq v$. We observe that $z \longmapsto |z|^p$ is a strict convex function so

$$|Dw(x)|^p = \left|\frac{1}{2}Du(x) + \frac{1}{2}Dv(x)\right|^p < \frac{1}{2}|Du|^p + \frac{1}{2}|Dv|^p$$

on $\{x \in \Omega \,|\, Du(x) \neq Dv(x)\}$ and we have proved that this subset has positive measure so

$$C_p(B_r, B_R) \leq \int_{B_R} |Dw|^p\, dx < \frac{1}{2}\int_{B_R} |Du|^p\, dx + \frac{1}{2}\int_{B_R} |Dv|^p\, dx,$$

so $C_p(B_r, B_R) < C_p(B_r, B_R)$.

3) The minimum is a radial function, i.e. $u \circ T = u$ for all rotation maps T. In fact $|\det(T)| = 1$ so, defined

$$F(v) := \int_\Omega |Dv|^p\, dx,$$

it follows that

$$F(v) = F(v \circ T) \quad \forall v \in K_{r,R},$$

but if we choose $v = u$, where u is the minimum point, then $u \circ T = u$ because from point **2**) it is unique. So there exists a function $v : [0, R] \to \mathbb{R}$ such that $u(x) = v(|x|)$ a.e. on B_R.

4) It follows that $v \in W^{1,p}((\varepsilon, R))$ for all $\varepsilon \in (0, R)$. Moreover v is continuous on (ε, R) and u is continuous on $B_R \setminus \overline{B}_\varepsilon$.

5) Moreover $u \leq 1$ a.e. on B_r.

$$\begin{aligned} C_p(B_r, B_R) &\leq \int_{B_R} |D(u \wedge 1)|^p\, dx \leq \int_{\{u \leq 1\}} |Du|^p\, dx \\ &\leq \int_{B_R} |Du|^p\, dx = C_p(B_r, B_R). \end{aligned}$$

So also $u \wedge 1$ is a minimum point but the minimum is unique then $u \wedge 1 = u$, i.e. $u \leq 1$ and $v \leq 1$ on $(0, R)$. In particular $u = 1$ a.e. on B_r and $v = 1$ on $(0, r]$.

Now we can compute the minimum. We have

$$\int_{B_R} |Du|^p\,dx = \int_{B_R\setminus B_r} |Du|^p\,dx = \int_{B_R\setminus B_r} |v'(|x|)|^p\,dx$$
$$= \int_r^R |v'(\rho)|^p\,\rho^{n-1}\sigma_{n-1}d\rho,$$

where $\sigma_{n-1} := \mathcal{H}^{n-1}(\partial B_1))$ and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. We note that $v \in W^{1,p}((r,R))$, $v(r) = 1$ and $v(R) = 0$. Moreover v is minimum point of the one dimensional problem

$$\min_{\substack{w\in W^{1,p}((r,R))\\ w(r)=1,\, w(R)=0}} \int_r^R |w'(\rho)|^p\,\rho^{n-1}\sigma_{n-1}\,d\rho.$$

Indeed, if we consider $w \in W^{1,p}((r,R))$, $w(r) = 1$, $w(R) = 0$ we can define $z(x) := w(|x|)$, $z \in W_0^{1,p}(B_R)$ and

$$\int_{B_R} |Dz|^p\,dx = \int_r^R |w'(\rho)|^p\,\rho^{n-1}\sigma_{n-1}\,d\rho,$$

but u is the minimum point. We have reduced the problem to the one-dimensional case. We consider $\varphi \in C_c^\infty((r,R))$, then $w := v + t\varphi$ is an admissible function for all $t \in \mathbb{R}$. Then we can compute the Euler–Lagrange equation, i.e.

$$\int_r^R |v'|^{p-2}v'\varphi'\,\rho^{n-1}\sigma_{n-1}\,d\rho = 0 \quad \forall\varphi \in C_c^\infty((r,R)).$$

Then there exists a constant $c \in \mathbb{R}$ such that

$$|v'(\rho)|^{p-2}v'(\rho)\rho^{n-1} = c \qquad \forall\rho \in (r,R).$$

As a consequence of the boundary condition it follows that $v' < 0$, $c < 0$ and

$$(-v'(\rho))^{p-2}v'(\rho)\rho^{n-1} = c < 0 \qquad \forall\rho \in (r,R),$$

$$(-v'(\rho))^{p-1}\rho^{n-1} = -c > 0 \qquad \forall\rho \in (r,R),$$

$$v'(\rho)\rho^{\frac{n-1}{p-1}} = -k \qquad \forall\rho \in (r,R),$$

where $k = (-c)^{\frac{1}{p-1}} > 0$ and finally

$$v'(\rho) = -k\rho^{-\frac{n-1}{p-1}} \qquad \forall \rho \in (r, R).$$

We can fix the constant k using the boundary conditions

$$1 = v(r) - v(R) = -\int_r^R v'(\rho)\,d\rho = k\int_r^R \rho^{-\frac{n-1}{p-1}}\,d\rho,$$

so

$$k = \left[\int_r^R \rho^{\frac{1-n}{p-1}}\,d\rho\right]^{-1}.$$

We have two cases:

- (Case $p \neq n$)

$$k = \frac{p-n}{p-1}\frac{1}{R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}}}$$

  and

$$C_p(B_r, B_R) = \begin{cases} \sigma_{n-1}\left(\frac{p-n}{p-1}\right)^{p-1} \dfrac{1}{\left[R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}}\right]^{p-1}} & \text{if } p > n, \\ \sigma_{n-1}\left(\frac{n-p}{p-1}\right)^{p-1} \dfrac{1}{\left[r^{\frac{p-n}{p-1}} - R^{\frac{p-n}{p-1}}\right]^{p-1}} & \text{if } p < n. \end{cases}$$

- (Case $p = n$)

$$k = \left[\log\left(\frac{R}{r}\right)\right]^{-1}$$

  and

$$C_p(B_r, B_R) = \sigma_{n-1}\frac{1}{\left[\log\left(\frac{R}{r}\right)\right]^{p-1}}.$$

Remark 1.18.

- In the case where $n = 3$ and $p = 2$, we get the "physical" capacity of a capacitor with two spherical concentric armors.

- In the case where $n = p = 2$, we get the "physical" capacity of a cylindrical capacitor with infinite length.

Example 1.19. Taking the limit $r \to 0^+$ in the expression of $C_p(B_r, B_R)$, we obtain

$$C_p(\{\mathbf{0}\}, B_R) = \begin{cases} \sigma_{n-1}\left(\dfrac{p-n}{p-1}\right)^{p-1} \dfrac{1}{R^{p-n}} & \text{if } p > n, \\ 0 & \text{if } p \leq n. \end{cases}$$

These results can be generalized by the following theorem

Theorem 1.20. *Let $\Omega_n \subseteq \mathbb{R}^n$ be an open subset and let $M_k \subset \Omega_n$ be a C^1 $k-$dimensional manifold and $p \in (1, +\infty)$. Then*

i) *if $k \leq n-p$, then $C_{1,p}(M_k, \Omega_n) = 0$*

ii) *if $k > n-p$, then $C_{1,p}(M_k, \Omega_n) > 0$.*

The proof, based on the Lindelöf Covering Theorem, is long and technical. Several lemmas are necessary so we prefer to refer to the bibliography (see, e.g., [18]).

1.1.2 Quasi-continuous functions

Definition 1.21. If a property $P(x)$ holds for all $x \in \Omega$ except for the elements of a set $Z \subset \Omega$ with $C_{1,p}(Z) = 0$, we say that $P(x)$ holds C_p-*quasi-everywhere* (shortly q.e.) on Ω, whereas the expression almost everywhere (shortly a.e.) refers, as usual, to the Lebesgue measure.

Proposition 1.22. *If a subset has null C_p-capacity then its Lebesgue measure is zero.*

Proof. It follows from the Poincaré inequality. Indeed,

$$|E| = \int_\Omega \mathbf{1}_E \, dx \leq \int_\Omega |u|^p \, dx \leq C(\Omega) \int_\Omega |Du|^p \, dx,$$

for all $u \in W_0^{1,p}(\Omega)$ such that $u \geq 1$ a.e. on an open neighborhood of E, so we find

$$|E| \leq C(\Omega) \, C_{1,p}(E, \Omega).$$

$\square$

Definition 1.23. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $f_h : \Omega \to \overline{\mathbb{R}}$, $f : \Omega \to \overline{\mathbb{R}}$. Then we say that

i) $\{f_h\}$ *converges* to f C_p-*quasi everywhere* if $f_h(x) \to f(x)$ as $h \to +\infty$ C_p-*quasi-everywhere* on Ω. We write $f_h \to f$ C_p-q.e..

ii) $\{f_h\}$ *converges* to f in C_p-*capacity* if for every $t > 0$

$$C_{1,p}(\{x \in \Omega \,|\, |f_h(x) - f(x)| > t\}, \Omega) \to 0 \quad \text{as } h \to +\infty.$$

We write $f_h \to f$ C_p-capacity.

iii) $\{f_h\}$ *converges* to f C_p-*quasi uniformly* if for every $\varepsilon > 0$ there exists $A \subseteq \Omega$ with $C_{1,p}(A, \Omega) < \varepsilon$ such that $f_h \to f$ uniformly on $\Omega \setminus A$. We write $f_h \to f$ C_p-q.u..

Proposition 1.24. *Let $f_h, g_h : \Omega \to \overline{\mathbb{R}}$ and $f, g : \Omega \to \overline{\mathbb{R}}$. If $f_h = g_h$ C_p-q.e. for any h and $f = g$ C_p-q.e. then*

i) *$f_h \to f$ C_p-q.e. if and only if $g_h \to g$ C_p-q.e.,*

ii) *$f_h \to f$ C_p-capacity if and only if $g_h \to g$ C_p-capacity,*

iii) *$f_h \to f$ C_p-q.u. if and only if $g_h \to g$ C_p-q.u..*

The proof follows directly from the definitions. Now we study the implications between these types of convergence.

Proposition 1.25. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded subset, $\{f_h\}$ and f functions defined on Ω, i.e. $f_h : \Omega \to \overline{\mathbb{R}}$, $f : \Omega \to \overline{\mathbb{R}}$. If $f_h \to f$ C_p-q.u. then $f_h \to f$ C_p-q.o.*

Proof. For all $k \in \mathbb{N}$ consider $\varepsilon = \frac{1}{2^k}$. Then there exists A_k such that $C_{1,p}(A_k, \Omega) < \frac{1}{2^k}$ and $f_h \to f$ uniformly on $\Omega \setminus A_k$. We define $E_k := \cup_{i \geq k} A_i$ then $E_{k+1} \subseteq E_k$. Setting $E := \cap_k E_k$ we claim that $C_{1,p}(E, \Omega) = 0$ and $f_h \to f$ pointwise on $\Omega \setminus E$. By countable subadditivity it follows that $C_{1,p}(E_k) < \frac{1}{2^{k-1}}$. The capacity is monotonic with respect to the inclusion, then $C_{1,p}(E, \Omega) \leq C_{1,p}(E_k, \Omega) < \frac{1}{2^{k-1}}$. Taking the limit for $k \to +\infty$ it follows that $C_{1,p}(E, \Omega) = 0$. We have to prove the convergence. For each $x \in \Omega \setminus E$ there exists k such that $x \in \Omega \setminus A_k$ but on A_k we have uniform convergence to f, in particular we have pointwise convergence. □

Proposition 1.26. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and consider $f_h : \Omega \to \overline{\mathbb{R}}$, $f : \Omega \to \overline{\mathbb{R}}$. If $f_h \to f$ C_p-q.u. then $f_h \to f$ C_p-capacity.*

Proof. We have to prove that $\forall t > 0$

$$C_{1,p}(\{x \in \Omega \,|\, |f_h(x) - f(x)| > t\}, \Omega) \to 0 \quad \text{as } h \to +\infty. \tag{1.7}$$

From hypothesis for any $\varepsilon > 0$ there exists $A \subseteq \Omega$ with $C_{1,p}(A, \Omega) < \varepsilon$ such that $f_h \to f$ uniformly on $\Omega \setminus A$. Then $\forall t > 0 \ \exists \ k$ such that $\forall h \geq k$ $|f_h(x) - f(x)| < t \ \forall x \in \Omega \setminus A$. We observe that

$$\left\{x \in \Omega \;\middle|\; |f_h(x) - f(x)| > t\right\} \subseteq A$$

and

$$C_{1,p}(\{x \in \Omega \big| \, |f_h(x) - f(x)| > t\}, \Omega) \leq C_{1,p}(A, \Omega) < \varepsilon.$$

In conclusion for any $t > 0$ and for any $\varepsilon > 0$ there exists k such that $\forall h \geq k$

$$C_{1,p}(\{x \in \Omega \,|\, |f_h(x) - f(x)| > t\}, \Omega) < \varepsilon$$

i.e. we have proved (1.7). □

Definition 1.27. We say that a sequence $\{f_h\}$ is a *Cauchy sequence with respect to the Capacity* (shortly C_p-Cauchy) if for every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that for every $h, k \geq m$

$$C_{1,p}(\{x \in \Omega \,|\, |f_h(x) - f_k(x)| > \varepsilon\}, \Omega) < \varepsilon.$$

Proposition 1.28. *If $\{f_h\}$ is a C_p-Cauchy sequence then*

i) *there exists a subsequence $f_{h_k} \to f$ $C_p-q.u.$*

ii) *$f_h \to f$ in C_p-capacity.*

Proof.

i) For all $i \in \mathbb{N}$ we choose $\varepsilon = \dfrac{1}{2^i}$, so there exist $m_i \in \mathbb{N}$ such that for any $h, k \geq m_i$

$$C_{1,p}\left(\left\{x \in \Omega \,|\, |f_h(x) - f_k(x)| > \frac{1}{2^i}\right\}, \Omega\right) < \frac{1}{2^i}.$$

We can suppose $m_i < m_{i+1}$, then it follows that

$$C_{1,p}\left(\left\{x \in \Omega \,|\, |f_{m_{i+1}}(x) - f_{m_i}(x)| > \frac{1}{2^i}\right\}, \Omega\right) < \frac{1}{2^i}.$$

We define

$$E_i := \Big\{x \in \Omega \,|\, |f_{m_{i+1}}(x) - f_{m_i}(x)| > \frac{1}{2^i}\Big\}$$

and by definition $C_{1,p}(E_i, \Omega) < \dfrac{1}{2^i}$. For all $j \in \mathbb{N}$ we put

$$A_j := \bigcup_{i=j}^{+\infty} E_i$$

and from countable subadditivity

$$C_p(A_j) \leq \sum_{i=j}^{+\infty} C_p(E_j) < \sum_{i=j}^{+\infty} \frac{1}{2^i} = \frac{1}{2^{j-1}}.$$

We choose $x \in \Omega \setminus A_j$ and $j \leq r \leq s$ then

$$|f_{m_s}(x) - f_{m_r}(x)| \leq |f_{m_s}(x) - f_{m_{s-1}}(x)| + ... + |f_{m_{r+1}}(x) - f_{m_r}(x)|.$$

Since $x \notin A_j$ then $x \notin E_i$ $r \leq i \leq s-1$ and

$$|f_{m_s}(x) - f_{m_r}(x)| \leq \sum_{i=r}^{s-1} \frac{1}{2^i} < \sum_{i=r}^{+\infty} \frac{1}{2^i} = \frac{1}{2^{r-1}}.$$

Moreover

$$\sup_{x \in \Omega \setminus A_j} |f_{m_s}(x) - f_{m_r}(x)| \leq \frac{1}{2^{r-1}}.$$

From this it follows that f_{m_i} is a Cauchy sequence with respect to the uniform norm on $\Omega \setminus A_j$. Then for all j there exists a function f_j such that $f_{m_i} \longrightarrow f_j$ uniformly on $\Omega \setminus A_j$ but $\Omega \setminus A_j \subset \Omega \setminus A_{j+1}$ so there exists a function[2] $f : \Omega \setminus \cap_{j=1}^{+\infty} A_j \longrightarrow \overline{\mathbb{R}}$ not depending from j such that for all $j \in \mathbb{N}$ we have $f_{m_i} \longrightarrow f$ uniformly on $\Omega \setminus A_j$.

ii) We fix $t > 0$ and we consider

$$\Big\{x \in \Omega \,\Big|\, |f_h(x) - f(x)| > t\Big\}$$

where f is the function defined in **i)**. We fix $\varepsilon > 0$ and we want to prove that there exists $\overline{h}_\varepsilon \in \mathbb{N}$ such that for any $h \geq \overline{h}_\varepsilon$

$$C_{1,p}\Big(\Big\{x \in \Omega \,\Big|\, |f_h(x) - f(x)| > t\Big\}, \Omega\Big) < \varepsilon.$$

[2]observe that $C_p(\cap_{j=1}^{+\infty} A_j) = 0$

We choose $i \in \mathbb{N}$ such that $\frac{1}{2^i} < \frac{t}{2}$ and $\frac{1}{2^i} < \frac{\varepsilon}{2}$. Then

$$\{x \in \Omega \,\big|\, |f_h(x) - f(x)| > t\} \subseteq$$

$$\{x \in \Omega \,\big|\, |f_h(x) - f_{m_i}(x)| > \frac{t}{2}\} \cup \{x \in \Omega \,\big|\, |f_{m_i}(x) - f(x)| > \frac{t}{2}\}$$

from the triangular inequality. We choose $h \geq m_i$ and, since $\frac{1}{2^i} < \frac{t}{2}$, it follows that

$$\{x \in \Omega \,\big|\, |f_h(x) - f(x)| > t\} \subseteq$$

$$\{x \in \Omega \,\big|\, |f_h(x) - f_{m_i}(x)| > \frac{1}{2^i}\} \cup \{x \in \Omega \,\big|\, |f_{m_i}(x) - f(x)| > \frac{t}{2}\}.$$

It follows that

$$C_p(\{x \in \Omega \,\big|\, |f_h(x) - f(x)| > t\}) \leq$$

$$C_p(\{x \in \Omega \,\big|\, |f_h(x) - f_{m_i}(x)| > \frac{1}{2^i}\}) + C_p(\{x \in \Omega \,\big|\, |f_{m_i}(x) - f(x)| > \frac{t}{2}\})$$

$$< \frac{1}{2^i} + C_p(\{x \in \Omega \,\big|\, |f_{m_i}(x) - f(x)| > \frac{t}{2}\})$$

$$< \frac{\varepsilon}{2} + C_p(\{x \in \Omega \,\big|\, |f_{m_i}(x) - f(x)| > \frac{t}{2}\}).$$

But using Proposition 1.26 we can suppose that

$$C_p\Big(\{x \in \Omega \,\big|\, |f_{m_i}(x) - f(x)| > \frac{t}{2}\}\Big) < \frac{\varepsilon}{2}$$

for i large enough. Then

$$C_p(\{x \in \Omega \,\big|\, |f_h(x) - f(x)| > t\}) <$$

$$\frac{\varepsilon}{2} + C_p\Big(\{x \in \Omega \,\big|\, |f_{m_i}(x) - f(x)| > \frac{t}{2}\}\Big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

In conclusion for any $t > 0$ and for any $\varepsilon > 0$ there exists $i \in \mathbb{N}$ such that if $h \geq m_i$ then

$$C_p(\{x \in \Omega \,\big|\, |f_h(x) - f(x)| > t\}) < \varepsilon.$$

$\square$

Proposition 1.29. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded subset and $f_h : \Omega \to \overline{\mathbb{R}}$, $f : \Omega \to \overline{\mathbb{R}}$. If $f_h \to f$ in C_p-capacity then $\{f_h\}$ is a C_p-Cauchy sequence.*

Proof. From the triangular inequality it follows that

$$\{x \in \Omega \,|\, |f_h(x) - f_k(x)| > \varepsilon\} \subseteq$$
$$\{x \in \Omega \,|\, |f_h(x) - f(x)| > \frac{\varepsilon}{2}\} \cup \{x \in \Omega \,|\, |f_k(x) - f(x)| > \frac{\varepsilon}{2}\}$$

-by subadditivity and the hypothesis of convergence follows that

$$C_{1,p}(\{x \in \Omega \,|\, |f_h(x) - f_k(x)| > \varepsilon\}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

□

Corollary 1.30. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $f_h : \Omega \to \overline{\mathbb{R}}$, $f : \Omega \to \overline{\mathbb{R}}$. If $f_h \to f$ in C_p-capacity then there exists a subsequence $f_{h_k} \to f$ C_p-q.u..*

We can summarize the convergence results in the following diagram. In the case of a dashed arrow one has only the convergence of a subsequence.

$$\begin{tikzcd}
& C_p - \text{q.u.} \arrow[dl, "\text{Prop. } 1.25"'] \arrow[dr, shift left, "\text{Prop. } 1.26"] & \\
C_p - \text{q.o.} & & C_p - \text{capacity} \arrow[ll, dashed] \arrow[ul, dashed, shift left, "\text{Cor. } 1.30"]
\end{tikzcd}$$

Remark 1.31. In the one dimensional case $C_{1,p}(\{x\}, \Omega) \geq K_p(\Omega) > 0$. Then the C_p-q.o. convergence is equivalent to the pointwise one (obvious by the fact that a point has positive capacity). The C_p-capacity convergence is equivalent to the uniform one. In fact one implication is Proposition 1.26. For the other consider a sequence such that for any $t > 0$ and for any $\varepsilon > 0$ there exists k such that if $h \geq k$ then

$$C_{1,p}(\{x \in \Omega \,|\, |f_h(x) - f(x)| > t\}, \Omega) < \varepsilon$$

If we chose $\varepsilon < K_p(\Omega)$ then $\{x \in \Omega \,|\, |f_h(x) - f(x)| > t\} = \emptyset$ and we obtain that for any$t > 0$ there exists k such that if $h \geq k$ then $|f_h(x) - f(x)| \leq t$ $\forall\, x \in \Omega$ and this is the uniform convergence.

Example 1.32. We show with an example that in general C_p-q.o. convergence does not implies C_p-capacity convergence. Let $\Omega := (0, 1)$ be the domain of interest. We consider the sequence

$$f_h(x) := \begin{cases} hx + 1 - \frac{h}{2} & \text{if } x \in [\frac{1}{2} - \frac{1}{h}, \frac{1}{2}] \\ -hx + 1 + \frac{h}{2} & \text{if } x \in (\frac{1}{2}, \frac{1}{2} + \frac{1}{h}] \\ 0 & \text{if } x \in \Omega \setminus [\frac{1}{2} - \frac{1}{h}, \frac{1}{2} + \frac{1}{h}] \end{cases}$$

and the function

$$f(x) := \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \in \Omega \setminus \{0\}. \end{cases}$$

Then, thanks to Remark 1.31, $f_h \to f$ C_p-q.e. on Ω but $\{f_h\}$ does not converge in C_p-capacity.

Definition 1.33. A function $f : \Omega \to \overline{\mathbb{R}}$ is C_p-*quasicontinuous* if for every $\varepsilon > 0$ there exists $A = A_\varepsilon \subseteq \Omega$ such that $C_{1,p}(A, \Omega) < \varepsilon$ and $f|_{\Omega \setminus A}$ is continuous on $\Omega \setminus A$.

Remark 1.34. We are not saying that f is continuous on $\Omega \setminus A$ but only its restriction $f|_{\Omega \setminus A}$. For example consider the function

$$f(x) := \begin{cases} 1 & \text{if } \mathbb{Q} \cap (0, 1) \\ 0 & \text{if } (0, 1) \setminus \mathbb{Q} \end{cases}$$

The function $f|_{(0,1)\setminus\mathbb{Q}}$ is continuous (because it is constant) but f is not continuous on $(0, 1) \setminus \mathbb{Q}$.

Remark 1.35. In Definition 1.33 we can choose A open. Suppose that for every $\varepsilon > 0$ there exists $A_\varepsilon \subseteq \Omega$ such that $C_{1,p}(A_\varepsilon, \Omega) < \varepsilon$ and $f|_{\Omega \setminus A_\varepsilon}$ is continuous on $\Omega \setminus A_\varepsilon$. By definition, $C_{1,p}(A_\varepsilon, \Omega) = \inf\{C_{1,p}(U, \Omega) \,|\, U \text{ open}, U \supseteq A_\varepsilon\}$. So for any $\alpha > 0$ there exists an open set $U_{\alpha,\varepsilon}$ such that $C_{1,p}(U_{\alpha,\varepsilon}, \Omega) < C_{1,p}(A, \Omega) + \alpha < \varepsilon + \alpha$ and f is again continuous on $\Omega \setminus U_{\alpha,\varepsilon} \subseteq \Omega \setminus A_\varepsilon$. We have proved that for every $r > 0$ there exists U_r, open subset of Ω, such that $f|_{\Omega \setminus U_r}$ is continuous on $\Omega \setminus U_r$.

From now on, we suppose that A is open in Definition 1.33.

Example 1.36. Let $\Omega = B_1((0,0)) \subset \mathbb{R}^2$. If f is continuous in $\Omega \setminus \{(0,0)\}$, then f is C_p-quasicontinuous.

Example 1.37. If $\Omega = (a,b) \subseteq \mathbb{R}$, then f is continuous if and only if it is quasicontinuous.

Example 1.38. Let $\Omega := (-1,1)^2 \subseteq \mathbb{R}^2$ be the domain of the function

$$f(x_1,x_2) = \begin{cases} 1 & \text{if } x_1 \geq 0, \\ 0 & \text{if } x_1 < 0. \end{cases}$$

For any p this function is not C_p-quasicontinuous.

Proposition 1.39. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and consider $f_h : \Omega \to \overline{\mathbb{R}}$, $f : \Omega \to \overline{\mathbb{R}}$. If $f_h \to f$ in C_p-capacity and f_h is C_p-quasicontinuous for every $h \in \mathbb{N}$, then f is C_p-quasicontinuous.*

Proof. By Proposition 1.30, there exists a subsequence f_{h_k} converging C_p-q.u. on Ω. By the quasicontinuity of f_h for all $\varepsilon > 0$ there exists a family of subsets $\{E_k\}$ such that $C_{1,p}(E_k,\Omega) < \frac{\varepsilon}{2^{k+1}}$ and f_{h_k}, restricted to $\Omega \setminus E_k$, is continuous on $\Omega \setminus E_k$. Let E be a subset with $C_{1,p}(E,\Omega) < \frac{\varepsilon}{2}$ such that $f_{h_k} \to f$ uniformly on $\Omega \setminus E$. If $A = E \cup \bigcup_k E_k$, we have that $C_{1,p}(A) < \varepsilon$, $f_{h_k}|_{\Omega \setminus A}$ is continuous and on $\Omega \setminus A$ we have uniform convergence. Then f is continuous. □

The purpose of this section is to prove that for every $u \in W_0^{1,p}(\Omega)$ there exists a C_p-quasicontinuous function $\tilde{u} : \Omega \to \mathbb{R}$ such that $\tilde{u} = u$ a.e. on Ω. First we need the following result.

Lemma 1.40. *Let $u \in C^0(\Omega) \cap W_0^{1,p}(\Omega)$ and let $t > 0$. Then*

$$C_{1,p}(\{x \,|\, |u| > t\}, \Omega) \leq \frac{\int_\Omega |Du|^p \, dx}{t^p}.$$

Proof. For all $t > 0$, we have $\dfrac{|u|}{t} > 1$ a.e. on $\{x \in \Omega \,|\, |u| > t\}$. By continuity, $\{x \in \Omega \,|\, |u| > t\}$ is open so

$$C_{1,p}(\{x \mid |u| > t\}, \Omega) \leq \int_\Omega \left| D\Big(\frac{|u|}{t}\Big) \right|^p dx = \frac{\int_\Omega |D|u||^p \, dx}{t^p}.$$

Moreover, $D(|u|) = Du\,\mathbf{1}_{\{u>0\}} - Du\,\mathbf{1}_{\{u<0\}}$ a.e. on Ω and $|D(|u|)| = |Du|$ a.e. on Ω. Then

$$C_{1,p}(\{x \,|\, |u| > t\}, \Omega) \leq \frac{\int_\Omega |Du|^p\,dx}{t^p}.$$

□

Lemma 1.41. *Let $u : \Omega \to \mathbb{R}$ be a C_p-quasicontinuous function. If $u \in W_0^{1,p}(\Omega)$ and $t > 0$, then*

$$C_{1,p}(\{x \in \Omega \,|\, |u| > t\}) \leq \frac{\int_\Omega |Du|^p\,dx}{t^p}. \tag{1.8}$$

Proof. From the hypothesis of C_p-quasicontinuity, it follows that for every $\varepsilon > 0$ there exists A, open subset of Ω, with $C_p(A) < \varepsilon$ such that $u|_{\Omega\setminus A}$ is continuous on $\Omega \setminus A$. In general, $\{x \in \Omega \,|\, |u| > t\}$ is not open, but it is possible to prove that $\{x \in \Omega \,|\, |u| > t\} \cup A$ is open. First we observe that

$$\{x \in \Omega \,|\, |u(x)| > t\} \cup A = \{x \in \Omega \setminus A \,|\, |u|_{\Omega\setminus A}(x)| > t\} \cup A.$$

By the continuity of $u|_{\Omega\setminus A}$, we obtain that $\{x \in \Omega \setminus A \,|\, |u|_{\Omega\setminus A}(x)| > t\}$ is open in $\Omega \setminus A$ with respect to the relative topology, i.e. there exists an open subset V of Ω such that

$$\{x \in \Omega \setminus A \,|\, |u|_{\Omega\setminus A}(x)| > t\} \cup A = ((\Omega \setminus A) \cap V) \cup A.$$

On the other hand $((\Omega \setminus A) \cap V) \cup A = V \cup A$ and, substituting this identity in the previous equation, we obtain

$$\{x \in \Omega \,\big|\, |u(x)| > t\} \cup A = \{x \in \Omega \setminus A \,\big|\, |u|_{\Omega\setminus A}(x)| > t\} \cup A = V \cup A.$$

Since $V \cup A$ is open then $U := \{x \in \Omega \,\big|\, |u(x)| > t\} \cup A$ is open and, by definition of capacity,

$$C_{1,p}(\{x \in \Omega \,\big|\, |u(x)| > t\}, \Omega) \leq C_{1,p}(U, \Omega).$$

Since $C_p(A) < \varepsilon$ there exists $w \in W_0^{1,p}(\Omega)$, $w \geq 1$ a.e. on A such that

$$\int_\Omega |Dw|^p\,dx < \varepsilon.$$

We set $v := \dfrac{|u|}{t} \vee w$. Obviously $v \geq 1$ a.e. on U. Since U is open and v is an admissible function,

$$C_{1,p}(U,\Omega) \leq \int_\Omega |Dv|^p\,dx.$$

Since

$$|Dv|^p = \begin{cases} \dfrac{|D|u||^p}{t^p} = \dfrac{|Du|^p}{t^p} & \text{if } \dfrac{|u|}{t} > w, \\[2ex] |Dw|^p & \text{if } \dfrac{|u|}{t} \leq w, \end{cases} \qquad \text{a.e. on } \Omega$$

we get

$$|Dv|^p \leq \frac{|Du|^p}{t^p} + |Dw|^p$$

and

$$\begin{aligned} C_{1,p}(\{x \in \Omega \,\big|\, |u(x)| > t\}) \leq C_{1,p}(U) \leq & \frac{1}{t^p}\int_\Omega |Du|^p\,dx + \int_\Omega |Dw|^p\,dx \\ < & \frac{1}{t^p}\int_\Omega |Du|^p\,dx + \varepsilon, \end{aligned}$$

$$C_{1,p}(\{x \in \Omega \,\big|\, |u(x)| > t\}) < \frac{1}{t^p}\int_\Omega |Du|^p\,dx + \varepsilon, \quad \forall \varepsilon > 0.$$

From the arbitrariness of ε, it is possible to conclude that

$$C_{1,p}(\{x \in \Omega \,\big|\, |u(x)| > t\}) \leq \frac{1}{t^p}\int_\Omega |Du|^p\,dx.$$

□

Theorem 1.42 (Existence of a quasicontinuous representative)**.** *For every $u \in W_0^{1,p}(\Omega)$ there exists a C_p-quasicontinuous function $\widetilde{u} : \Omega \to \mathbb{R}$ such that $\widetilde{u} = u$ a.e. on Ω.*

Proof. Consider $p \in [1, +\infty)$ and $u \in W_0^{1,p}(\Omega)$. Then there exist a sequence $\{u_h\} \subset C_c^\infty(\Omega)$ such that $||u_h - u||_{W_0^{1,p}(\Omega)} \longrightarrow 0$. We claim that $\{u_h\}$ is a Cauchy sequence with respect to the C_p-capacity. We have to prove that for any $\varepsilon > 0$ there exists $\overline{h}_\varepsilon \in \mathbb{N}$ such that if $h, k \geq \overline{h}_\varepsilon$ then

$$C_{1,p}(\{x \in \Omega \,\big|\, |u_h(x) - u_k(x)| > \varepsilon\}, \Omega) < \varepsilon.$$

Using Lemma 1.40,

$$C_{1,p}(\{x \mid |u_h - u_k| > \varepsilon\}, \Omega) \leq \frac{\int_\Omega |D(u_h - u_k)|^p\,dx}{\varepsilon^p}.$$

On the other hand $\{u_h\}$ is Cauchy sequence in $W_0^{1,p}(\Omega)$, so there exists m such that

$$\int_\Omega |D(u_h - u_k)|^p\,dx < \varepsilon^{p+1}, \quad \forall\, h, k \geq m.$$

and then

$$C_{1,p}(\{x \,|\, |u_h - u_k| > \varepsilon\}, \Omega) < \varepsilon.$$

Thanks to Proposition 1.28, there exists a function $\widetilde{u}$ such that $u_h \longrightarrow \widetilde{u}$ in C_p-capacity. Since u_h is continuous (and in particular C_p-quasicontinuous) then, by Proposition 1.39, the limit function $\widetilde{u}$ is quasicontinuous. The proof is complete if we show that $\widetilde{u} = u$ a.e. on Ω. By Corollary 1.30, there exists a subsequence $\{u_{h_k}\}$ such that $u_{h_k} \longrightarrow u$ C_p-q.u. on Ω, and in particular C_p-q.e.. This implies the convergence a.e. on Ω. Moreover, the fact that $u_{h_k} \longrightarrow u$ in $W_0^{1,p}(\Omega)$ implies that there exists a subsequence $u_{h_{k_j}} \longrightarrow u$ a.e. on Ω. Therefore $u = \widetilde{u}$ a.e. on Ω.

□

Remark 1.43. When $p > n$, one has $C_p(\{x\}) \geq K_p(\Omega) > 0$ for every $x \in \Omega$. This is a different way to prove that for $p > n$ every function in $W_0^{1,p}(\Omega)$ has a continuous representative.

Moreover we have the following result.

Lemma 1.44. *Let $u, v : \Omega \to \mathbb{R}$ be C_p-quasicontinuous functions. If $u, v \in W_0^{1,p}(\Omega)$ and $u = v$ a.e. on Ω, then $u = v$ C_p-q.e.*

Proof. We have

$$\begin{aligned} C_{1,p}(\{x \in \Omega \,|\, u(x) \neq v(x)\}) &= C_{1,p}(\{x \in \Omega \,|\, |u(x) - v(x)| > 0\}) \\ &\leq \sum_{h=1}^{+\infty} C_{1,p}(\{x \in \Omega \,|\, |u(x) - v(x)| > \frac{1}{h}\}) \\ &\leq \sum_{h=1}^{+\infty} h^p \int_\Omega |D(u - v)|^p\,dx = 0. \end{aligned}$$

The last equality is due to the hypothesis that $u = v$ a.e. □

Remark 1.45. Without the hypothesis of C_p-quasicontinuity, the statement is not true. Indeed, consider the domain $\Omega = (-1,1) \subset \mathbb{R}$ and the functions $u \equiv 0$ on Ω, $v \equiv 0$ on $\Omega \setminus \{0\}$ and $v(0) = 1$. Then $u = v$ a.e. on Ω but $u \neq v$ C_p-q.e.

Corollary 1.46. *If $u \in W_0^{1,p}(\Omega)$, then the C_p-quasicontinuous representative is uniquely defined up to a set of null p-capacity.*

Lemma 1.47. *Let $u \in W_0^{1,p}(\Omega)$ be a Sobolev function and let $\tilde{u} : \Omega \to \overline{\mathbb{R}}$ be its C_p-quasicontinuous representative. Then*

$$C_{1,p}(\{x \in \Omega \,\big|\, |\tilde{u}| = +\infty\}, \Omega) = 0.$$

Proof. For every $h > 0$

$$C_{1,p}(\{x \in \Omega \,|\, |\tilde{u}| = +\infty\}) \leq C_{1,p}(\{x \in \Omega \,|\, |\tilde{u}| \geq h\}) \leq \frac{\int_\Omega |Du|^p \, dx}{h^p}.$$

Taking the limit $h \to +\infty$ we obtain the claim. □

As a consequence, from now on we consider the C_p-quasicontinuous representative $\tilde{u}$ as a function with values in $\mathbb{R}$ instead of $\overline{\mathbb{R}}$.

Theorem 1.48. *Let $\{u_h\}$ and u be a sequence and a function in $W_0^{1,p}(\Omega)$. If $u_h \to u$ in $W_0^{1,p}(\Omega)$, then $\widetilde{u}_h \to \widetilde{u}$ in C_p-capacity and there exists a subsequence $\{u_{h_k}\}$ such that $\widetilde{u}_{h_k} \to u$ C_p-uniformly (in particular, C_p-a.e.).*

Proof. We have

$$C_{1,p}(\{x \in \Omega \,|\, |\tilde{u}_h - \tilde{u}| > t\}) \leq \frac{\int_\Omega |D(u_h - u)|^p \, dx}{t^p}.$$

Since $u_h \to u$ in $W_0^{1,p}(\Omega)$, we obtain that the sequence is convergent in C_p-capacity. The existence of a subsequence converging C_p-uniformly and C_p-q.e. is ensured by Corollary 1.30. □

Theorem 1.49. *If A is an open subset of Ω and $u, v \in W_0^{1,p}(\Omega)$, then*

$$u = v \quad a.e. \ on \ A \quad \Longrightarrow \quad \tilde{u} = \tilde{v} \quad C_p - a.e. \ on \ A.$$

Proof. Let $A' \subset\subset A$ be an open subset. We consider a cut-off function $\phi \in C_c^\infty(A)$ such that $\phi = 1$ on A'. Then ϕu, $\phi v \in W_0^{1,p}(A)$ and $\phi u = \phi v$ on Ω. Using Lemma 1.44 we obtain $\widetilde{\phi u} = \widetilde{\phi v}$. Moreover $\phi \tilde{u} = \widetilde{\phi u}$ and $\phi \tilde{v} = \widetilde{\phi v}$ C_p-q.e. on Ω (again by Lemma 1.44). Since $\phi|_{A'} = 1$, then $\tilde{u} = \tilde{v}$ on C_p-q.e. on A'. Finally, using $A = \bigcup_i A_i$ with $A_i \subset\subset A$ (A_i open for every i) and the subadditivity of C_p, the proof is complete. □

Theorem 1.50. *Let A be an open subset of Ω and let u and v be two functions in $W_0^{1,p}(\Omega)$. If $u \leq v$ a.e. on A, then $\widetilde{u} \leq \widetilde{v}$ C_p-q.e. on A.*

Proof. We note that $u = u \wedge v \in W_0^{1,p}(\Omega)$ a.e. on A. By previous theorem, we get $\widetilde{u} = \widetilde{u \wedge v}$ C_p-q.e.. Moreover $\widetilde{u} \wedge \widetilde{v} = \widetilde{u \wedge v}$ C_p-q.e. on Ω, so $\widetilde{u} = \widetilde{u} \wedge \widetilde{v}$ C_p-q.e. on A, i.e. $\widetilde{u} \leq \widetilde{v}$ C_p-q.e. on A. □

Theorem 1.51. *If $p \in [1, +\infty)$ and $u \in W^{1,p}(\Omega)$, then there exists a C_p-quasicontinuous function $\widetilde{u} : \Omega \to \mathbb{R}$ such that $\tilde{u} = u$ a.e. on Ω.*

Proof. Let $A \subset\subset \Omega$ and consider $\varphi \in C_c^\infty(\Omega)$ such that $\varphi|_A = 1$. Then $\varphi u \in W_0^{1,p}(\Omega)$ and there exists a C_p-quasicontinuous representative. Let $\{A_i\}$ be an exhaustion by compact sets ($A = \bigcup_i A_i$, with $A_i \subset\subset A$). We apply previous argument to A_i for every i, and the proof is complete. □

Theorem 1.52. *Let Ω be an open bounded subset of $\mathbb{R}^n$ and let $E \subseteq \Omega$. If $p \in [1, +\infty)$, then*

$$C_{1,p}(A, \Omega) = \inf \Big\{ \int_\Omega |Du|^p \, dx \,|\, u \in W_0^{1,p}(\Omega),\, \widetilde{u} \geq 1 \ C_p\text{-}q.e. \ on\, E \Big\}. \tag{1.9}$$

If $p > 1$, then

$$C_{1,p}(E, \Omega) = \min \Big\{ \int_\Omega |Du|^p \, dx \,|\, u \in W_0^{1,p}(\Omega),\, \widetilde{u} \geq 1 \ C_p\text{-}q.e. \ on\, E \Big\}.$$

Proof. We denote by $C'_p(E, \Omega)$ the right-hand side of (1.9). First of all we prove that $C'_p(E, \Omega) \leq C_{1,p}(E, \Omega)$. By definition of capacity, for every $\varepsilon > 0$ there exists A, open subset, such that $E \subseteq A \subseteq \Omega$ and

$$C_p(A, \Omega) < C_p(E, \Omega) + \varepsilon$$

By definition of $C_{1,p}(A,\Omega)$, there exists a function $u \in W_0^{1,p}(\Omega)$ such that $u \geq 1$ a.e. on A and

$$\int_\Omega |Du|^p \, dx < C_p(E,\Omega) + \varepsilon.$$

By Theorem 1.50, it follows that $\widetilde{u} \geq 1$ C_p-q.e. on $A \supseteq E$. Then u is an admissible competitor for $C_p'(E,\Omega)$, and so

$$C_p'(E,\Omega) < C_p(E,\Omega) + \varepsilon \quad \forall \varepsilon > 0.$$

From the arbitrariness of $\varepsilon > 0$

$$C_p'(E,\Omega) \leq C_p(E,\Omega).$$

We have to prove the reverse inequality. If $C_p'(E,\Omega) = +\infty$ the inequality is clear so we can suppose $C_p'(E,\Omega) < +\infty$. Then for every $\varepsilon > 0$ there exists $u \in W_0^{1,p}(\Omega)$ such that $\widetilde{u} \geq 1$, C_p-q.e. on E and

$$\int_\Omega |Du|^p \, dx < C_p'(E,\Omega) + \varepsilon.$$

We can suppose that $\widetilde{u} > 1$ (otherwise we consider $(1+\eta)\widetilde{u}$). The representative $\widetilde{u}$ is quasicontinuous, then for every $\varepsilon > 0$ there exists $A \subseteq \Omega$ open such that $C_{1,p}(A) < \varepsilon$, and $\widetilde{u}|_{\Omega \setminus A}$ is continuous on $\Omega \setminus A$. Moreover

$$U := \{x \in \Omega \,|\, \widetilde{u}(x) > 1\} \cup A$$

is open (the proof is similar to the one in Lemma 1.41). Since $C_{1,p}(A) < \varepsilon$, then there exists a function $w \in W_0^{1,p}(\Omega)$, $w \geq 1$ a.e. on A such that

$$\int_\Omega |Dw|^p \, dx < \varepsilon.$$

We define $v := u \vee w \in W_0^{1,p}(\Omega)$, $v \geq 1$ a.e. on U and, since U is open, we have that

$$C_p(U) \leq \int_\Omega |Dv|^p \, dx.$$

Since $E \subseteq N \cup \{x \in \Omega \,|\, \widetilde{u}(x) > 1\} \subseteq N \cup U$, where

$$N := \{x \in E \,|\, \widetilde{u}(x) \leq 1\},$$

we have that

$$C_p(E) \leq C_p(U) + C_p(N) = C_p(U) \leq \int_\Omega |Dv|^p \, dx,$$

(the capacity of N is zero because $\widetilde{u} \geq 1$ C_p-q.e. on E). We note that

$$\int_\Omega |Dv|^p \, dx \leq \int_\Omega |Du|^p \, dx + \int_\Omega |Dw|^p \, dx$$

and so

$$C_p(E) \leq C'_p(E) + \varepsilon + \varepsilon \quad \forall \varepsilon > 0.$$

By the arbitrariness of ε we have $C_p(E) = C'_p(E)$.
We have to prove the last part of the theorem, i.e. the existence of a minim-minimizer for $p > 1$. We define the set

$$K := \{u \in W_0^{1,p}(\Omega) \mid \widetilde{u} \geq 1 \ C_p\text{-q.e. on } E\} \subset W_0^{1,p}(\Omega)$$

and the functional $F : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$

$$F(u) := ||u||^p_{W_0^{1,p}(\Omega)} = \int_\Omega |Du|^p \, dx,$$

and we have to study the minimization problem

$$\min_{u \in K} F(u).$$

We can apply the direct method of the calculus of variations. The functional is weakly lower semi-continuous (as observed in previously). The set K is obviously convex. If we prove that it is closed then it is weakly closed and the direct method ensures the existence of a minimizer. Let $\{u_h\} \subset K$ be a sequence such that $u_h \to u$ in $W_0^{1,p}(\Omega)$. As a consequence of Theorem 1.48 there exists of a subsequence $\widetilde{u}_{h_k} \to \widetilde{u}$ C_p-q.e. on Ω. By assumption $\widetilde{u}_{h_k} \geq 1$ C_p-q.e. on E, and using the countable subadditivity of the capacity, $\widetilde{u} \geq 1$ C_p-q.e. on E. In conclusion, $\widetilde{u} \in K$. $\square$

Using the convexity of $z \longmapsto |z|^p$, the following theorem can be proved

Theorem 1.53. *For every $E \subseteq \Omega$ and $p \in (1, +\infty)$, there exists a unique minimizer for $C_p(E)$.*

In the case of compact subset we have the following result.

Theorem 1.54. *Let $K \subset \Omega$ be a compact subset and consider $p \in (1, +\infty)$, then*

$$C_{1,p}(K, \Omega) = \inf \Big\{ \int_\Omega |Du|^p \, dx \;\Big|\; u \in C_c^\infty(\Omega),\ u \geq 1 \ \textit{on}\, K \Big\}.$$

Proof. We define

$$\widetilde{C_{1,p}}(K, \Omega) := \inf \Big\{ \int_\Omega |Du|^p \, dx \,|\, u \in C_c^\infty(\Omega),\ u \geq 1 \text{ on } K \Big\}.$$

It can be easily seen that it is possible to assume $u > 1$ in $\widetilde{C_{1,p}}$ (it is sufficient to consider $(1+\eta)u$, with $\eta > 0$). First we prove that $C_p(K) \leq \widetilde{C_p}(K)$. If $\widetilde{C_p}(K) = +\infty$ the inequality is obvious and can suppose that $\widetilde{C_p}(K) < +\infty$. Then for every $\varepsilon > 0$ there exists $u \in C_c^\infty(\Omega)$, $u > 1$ on K such that

$$\int_\Omega |Du|^p \, dx < \widetilde{C_{1,p}}(K, \Omega) + \varepsilon.$$

Since $\{u > 1\} \supset K$ and $u \in C_c^\infty(\Omega)$, it follows

$$C_p(K) \leq C_p(\{u > 1\}) \leq \int_\Omega |Du|^p \, dx < \widetilde{C_{1,p}}(K, \Omega) + \varepsilon,$$

that is

$$C_p(K) < \widetilde{C_{1,p}}(K, \Omega) + \varepsilon, \quad \forall \varepsilon > 0.$$

This inequality implies

$$C_p(K) \leq \widetilde{C_{1,p}}(K, \Omega).$$

Now we prove that $\widetilde{C_p}(K) \leq C_p(K)$. We can suppose $C_p(K) < +\infty$. For any $\varepsilon > 0$ there exists A open, $K \subset A \subset\subset \Omega$ such that

$$C_p(A) < C_p(K) + \frac{\varepsilon}{2}.$$

There exists a function $u \in W_0^{1,p}(\Omega)$, $u \geq 1$ a.e. on A such that

$$\int_\Omega |Du|^p \, dx < C_p(A) + \varepsilon.$$

It is possible to suppose that u as compact support in Ω. Otherwise it is possible to choose a cut off function $\varphi \in C_c^\infty(\Omega)$, $\varphi|_A = 1$ and we define

$v_h := \varphi u + (1-\varphi)u_h$, where $\{u_h\} \subset C_c^\infty(\Omega)$ is a sequence convergent to u in $W_0^{1,p}(\Omega)$ norm. Then for any $h \in \mathbb{N}$ we have $supp(v_h) \subset\subset \Omega$, $v_h \geq 1$ a.e. on A and $v_h \to u$ in $W_0^{1,p}(\Omega)$. Then, for h big enough,

$$\int_\Omega |Dv_h|^p \, dx < C_p(A) + \varepsilon.$$

We assume that u has compact support in Ω. Let $\{\rho_h\}$ a family of mollifiers, i.e. $\rho_h \in C_c^\infty(\Omega)$, $\rho_h \geq 0$, $\int_\Omega \rho_h \, dx = 1$, $supp(\rho_h) \subseteq B_{1/h}(\mathbf{0})$. Then $u * \rho_h \in C_c^\infty(\Omega)$, $u * \rho_h \geq 1$ on K, for h large enough. Moreover $u * \rho_h \to u$ in $W_0^{1,p}(\Omega)$ so

$$\int_\Omega |D(u*\rho_h)|^p \, dx < C_p(K) + \varepsilon$$

and

$$\widetilde{C_p}(K) \leq \int_\Omega |D(u*\rho_h)|^p \, dx < C_p(K) + \varepsilon \quad \forall \varepsilon > 0.$$

In particular, using the arbitrariness of ε, we get

$$\widetilde{C_p}(K) \leq C_p(K).$$

□

We show now other proprieties of Capacity as set function:

Theorem 1.55 (Continuity with respect to increasing sequences). *Let $\{E_h\}$ be an increasing sequence of subsets of Ω. For every $p \in (1, +\infty)$*

$$C_{1,p}(E, \Omega) = \sup_h C_{1,p}(E_h, \Omega),$$

where $E := \bigcup_h E_h$.

Proof. By monotonicity of capacity we obtain the following inequality

$$C_{1,p}(E) \geq \sup_h C_{1,p}(E_h),$$

and the proof is complete if we prove the reverse inequality, that is

$$C_{1,p}(E) \leq \sup_h C_{1,p}(E_h).$$

We can suppose $M := \sup_n C_{1,p}(E_h) < +\infty$ (otherwise the inequality is trivial). Using Theorem 1.52 we obtain

$$C_{1,p}(E_h, \Omega) = \min\Big\{ \int_\Omega |Du|^p\,dx \mid u \in W_0^{1,p}(\Omega),\ \widetilde{u} \geq 1\ C_p\text{-q.e. on } E \Big\}$$

for every h, i.e. there exists $u_h \in W_0^{1,p}(\Omega)$ such that $\widetilde{u}_h \geq 1\ C_p$-q.e. on E_h and

$$C_p(E_h) = \int_\Omega |Du_h|^p\,dx,$$

so

$$C_p(E_h) = \int_\Omega |Du_h|^p\,dx \leq M.$$

Therefore the sequence $\{u_h\} = \{\widetilde{u}_h\}$ is bounded in $W_0^{1,p}(\Omega)$. Since $p \neq 1$ there exists a subsequence (still denoted by $\{u_h\}$) and a function $u \in W_0^{1,p}(\Omega)$ such that $u_h \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$. As a consequence of the semicontinuity of the norm and the fact that $\{C_p(E_h)\}$ is increasing, it follows that

$$\int_\Omega |Du|^p\,dx \leq \liminf_{h\to+\infty} \int_\Omega |Du_h|^p\,dx = \lim_{h\to+\infty} C_p(E_h).$$

If we prove that $\widetilde{u} \geq 1\ C_p-$q.e. in E then

$$C_p(E) \leq \int_\Omega |Du|^p\,dx \leq \lim_{h\to+\infty} C_p(E_h) = \sup_{h\in\mathbb{N}} C_p(E_h).$$

We set

$$K_{E_h} := \Big\{ v \in W_0^{1,p}(\Omega) \mid \widetilde{v} \geq 1, C_p - \text{q.e. in } E_h \Big\}.$$

that is convex and closed for every h (this follows from Theorem 1.48). Moreover $K_{E_{h+1}} \subseteq K_{E_h}$, so if we fix k it follows that $u_h \in K_{E_h} \subseteq K_{E_k}$ for every $h \geq k$. The set K_{E_k} is closed and convex so $u \in K_{E_k}$ and can do this for every $k \in \mathbb{N}$. Therefore $u \in K_{E_k}$ for every k, i.e. $\widetilde{u} \geq 1, C_p -$ q.e. in E_k for every k. From the countable subadditivity of capacity we get

$$\widetilde{u} \geq 1,\ \ C_p - \text{q.e. in } E.$$

□

Remark 1.56. The previous statement is true also in the case $p = 1$, but the proof is more difficult.

Theorem 1.57 (Continuity with respect to decreasing sequences of compact sets)**.** *Let $\{K_h\}$ be a decreasing sequence of compact subsets of Ω, and let $K := \bigcap_h K_h$. Then, for any $p \in (1, +\infty)$,*

$$C_{1,p}(K, \Omega) = \inf_h C_{1,p}(K_h, \Omega).$$

Proof. We have proved that for compact sets

$$C_{1,p}(K, \Omega) = \inf \Big\{ \int_\Omega |D\varphi|^p \, dx \,|\, \varphi \in C_c^\infty(\Omega),\, \varphi \geq 1 \text{ on } K \Big\}.$$

It is sufficient to prove that $C_{1,p}(K, \Omega) \geq \inf_h C_{1,p}(K_h, \Omega)$ (the reverse inequality follows from monotonicity of the capacity). We suppose that $C_p(K) < +\infty$ (otherwise the inequality is trivial), then for any $\varepsilon > 0$ there exists a function $\varphi \in C_c^\infty(\Omega)$, such that $\varphi > 1$ on K and

$$\int_\Omega |D\varphi|^p \, dx < C_p(K) + \varepsilon.$$

We define $A := \{\varphi > 1\}$, which is open and contains the subset K. Then there exists h_0 such that $K_h \subset A$ for every $h \geq h_0$. It follows that $\varphi > 1$ on K_h and

$$C_p(K_h) \leq \int_\Omega |D\varphi|^p \, dx < C_p(K) + \varepsilon \quad \forall\, \varepsilon > 0, \forall\, h \geq h_0.$$

From the arbitrariness of ε, it is possible to conclude that

$$\inf_h C_p(K_h) \leq C_p(K).$$

□

Remark 1.58. The hypothesis of compactness is essential and a possible counterexample is the following. Let $\Omega := (-1, 1)$, $E_h := (0, \frac{1}{h})$. It follows that $E = \bigcap_h E_h = \emptyset$ and $C_{1,p}(E) = 0$, but $C_{1,p}(E_h) \geq c > 0$.

Remark 1.59. For $p > n$ every Sobolev function has a continuous representative. Then, a property which holds C_p-quasieverywhere holds in fact everywhere. For this reason from the shape optimization point of view the most interesting case is when $p \in (1, n]$.

From now on we identify the function u with its quasicontinuous representative $\widetilde{u}$. In this way a pointwise condition can be imposed on $u(x)$ for C_p-q.e. $x \in \Omega$.

1.1.3 Sobolev spaces for quasiopen subsets

Definition 1.60. Let Ω be an open and bounded subset of $\mathbb{R}^n$. A subset A of Ω is said to be C_p-*quasiopen* or C_p-*quasiclosed* if for every $\varepsilon > 0$ there respectively exists an open or closed subset A_ε of Ω, such that $C_p(A_\varepsilon \Delta A) < \varepsilon$. The *family of all quasiopen* subsets of Ω is denoted by $\mathcal{A}_p(\Omega)$ or simply $\mathcal{A}(\Omega)$.

Definition 1.61. For every $A \in \mathcal{A}_p(\Omega)$ we denote by $H_0^1(A)$ the space of all functions $u \in H_0^1(\Omega)$ such that $u = 0$ q.e. on $\Omega \setminus A$.

Remark 1.62. Functions in $H_0^1(A)$ are however defined on the entire Ω.

Proposition 1.63. *The space $H_0^1(A)$ is a closed subspace of $H_0^1(\Omega)$. In particular it has the Hilbert space structure inherited from $H_0^1(\Omega)$.*

Proof. It's a consequence of Theorem 1.48. □

Proposition 1.64. *If A is open, the previous definition of $H_0^1(A)$ is equivalent to the usual one.*

For a proof see [22, Chapter 3, Proposition 3.3.42].

Theorem 1.65. *Let $A \subset \Omega$ be quasiopen, where Ω is an open and bounded subset of $\mathbb{R}^n$. Then $H_0^1(A)$ is dense in $L^2(A)$ and $\{u_A^f \,|\, f \in L^\infty(\Omega)\}$ is dense in $H_0^1(A)$, where u_A^f is the unique weak solution of $-\Delta u = f$ in A.*

For a proof see [22, Chpater 3, Proposition 3.3.44].

Proposition 1.66. *Given a C_p-quasicontinuous function $u : \Omega \longrightarrow \mathbb{R}$, the set $\{u > \alpha\}$ is C_p-quasiopen for every $\alpha \in \mathbb{R}$.*

Proof. There exists, for all $\varepsilon > 0$, an open set A_ε, with $C_p(A_\varepsilon, \Omega) < \varepsilon$, such that $u|_{\Omega \setminus A_\varepsilon}$ is continuous. It is possible to prove that $\{x \in \Omega \,|\, u > \alpha\} \cup A_\varepsilon$ is open. First we observe that

$$\{x \in \Omega \,|\, u(x) > \alpha\} \cup A = \{x \in \Omega \setminus A \,|\, u\big|_{\Omega \setminus A}(x) > \alpha\} \cup A.$$

From the continuity of $u|_{\Omega \setminus A_\varepsilon}$ we obtain that $\{x \in \Omega \setminus A_\varepsilon \mid u\big|_{\Omega \setminus A_\varepsilon}(x) > \alpha\}$ is open in $\Omega \setminus A_\varepsilon$ with respect to the relative topology, that is there exists an open subset V_ε of Ω such that

$$\{x \in \Omega \setminus A_\varepsilon \mid u\big|_{\Omega \setminus A_\varepsilon}(x) > \alpha\} \cup A_\varepsilon = ((\Omega \setminus A_\varepsilon) \cap V_\varepsilon) \cup A_\varepsilon,$$

on the other hand $((\Omega \setminus A_\varepsilon) \cap V_\varepsilon) \cup A_\varepsilon = V_\varepsilon \cup A_\varepsilon$, and substituting this identity in the previous equation we obtain

$$\{x \in \Omega \,\big|\, u(x) > \alpha\} \cup A_\varepsilon = \{x \in \Omega \setminus A_\varepsilon \,\big|\, u|_{\Omega \setminus A}(x) > \alpha\} \cup A_\varepsilon = V_\varepsilon \cup A_\varepsilon.$$

Since $V_\varepsilon \cup A_\varepsilon$ is open, then $U_\varepsilon := \{x \in \Omega \,\big|\, u(x) > \alpha\} \cup A_\varepsilon$ is open and, by definition, we get $C_{1,p}(U_\varepsilon \,\Delta\, \{u > \alpha\}, \Omega) \leq C_{1,p}(A_\varepsilon, \Omega) < \varepsilon$.

□

1.2 Gamma-convergence

This notion of convergence was introduced in a paper by E. De Giorgi and T. Franzoni in 1975, [15]. The main references that we follow for this topic are the book by Dal Maso [13] and the lecture notes by Braides [3].

1.2.1 The case of metric spaces

Definition 1.67. Let X be a metric space and $F_n : X \to \overline{\mathbb{R}}$ a sequence of functionals on X. Then, we say that $\{F_n\}$ $\Gamma-$*converge* to $F : X \to \overline{\mathbb{R}}$, and we write $F = \Gamma - \lim_{n \to +\infty} F_n$, if the following two conditions hold.

- *Lower bound inequality*: For every sequence $\{x_n\} \subset X$ such that $x_n \to x$ as $n \to +\infty$,
$$F(x) \leq \liminf_{n \to \infty} F_n(x_n).$$
- *Upper bound inequality*: For every $x \in X$, there exists a sequence $\{\overline{x}_n\}$ converging to x such that
$$F(x) \geq \limsup_{n \to \infty} F_n(\overline{x}_n).$$

  The functional F is called the $\Gamma-$*limit* of $\{F_n\}$.

Remark 1.68. The second condition in the previous definition is equivalent to
$$F(x) = \lim_{n \to \infty} F_n(\overline{x}_n).$$

Remark 1.69. The definition of $\Gamma-$convergence can be given on a generic topological space. The sequential definition given here is equivalent to the topological one in the case of metrizable spaces (see [13] or [3]).

Definition 1.70. Let X be a metric space and $F_n : X \to \overline{\mathbb{R}}$ a sequence of functionals on X. Then we define the $\Gamma-$*liminf* and $\Gamma-$*limsup* of F_n as

$$\Gamma - \liminf_{n\to+\infty} F_n(x) := \inf\Big\{ \liminf_{n\to+\infty} F_n(x_n) \,|\, x_n \to x \Big\},$$

$$\Gamma - \limsup_{n\to+\infty} F_n(x) := \inf\Big\{ \limsup_{n\to+\infty} F_n(x_n) \,|\, x_n \to x \Big\}.$$

The $\Gamma-$liminf and $\Gamma-$limsup always exist and

$$\Gamma - \liminf_{n\to+\infty} F_n(x) \leq \Gamma - \limsup_{n\to+\infty} F_n(x).$$

They are equal if and only if the $\Gamma - \lim$ exists, and in this case

$$\Gamma - \lim_{n\to+\infty} F_n(x) = \Gamma - \liminf_{n\to+\infty} F_n(x) = \Gamma - \limsup_{n\to+\infty} F_n(x).$$

Remark 1.71. It can easily be seen that the $\Gamma-$convergence of a sequence $\{F_n\}$ is independent from its pointwise convergence.

Proposition 1.72. *In **v)**,**vi)** and **vii)** we suppose that X is a normed vector space.*

i) *If $F = \Gamma - \lim_{n\to+\infty} F_n$, then F is a lower semicontinuous function with respect to the distance.*

ii) *A constant sequence $F_n = F$ $\Gamma-$converges to its constant value F if and only if the functional F is lower semicontinuous with respect to the distance. In general $\Gamma - \lim\, F_n = \overline{F}$, where F is the lower semicontinuous envelope or relaxation of F (see Chapter 2 for the definition).*

iii) *Consider a subsequence of functionals $\{F_n\}$ and a subsequence $\{F_{n_k}\}$. Then*

$$\Gamma - \liminf_{n\to+\infty} F_n \leq \Gamma - \liminf_{k\to+\infty} F_{n_k} \leq \Gamma - \limsup_{k\to+\infty} F_{n_k} \leq \Gamma - \limsup_{n\to+\infty} F_n.$$

In particular if a sequence $\Gamma-$converges, then every subsequence converges to the same limit.

iv) *The $\Gamma-$convergence is stable under continuous perturbations. Precisely, if $\{F_n\}$ $\Gamma-$converges to F and $G : X \to \overline{\mathbb{R}}$ is continuous, then $\{F_n+G\}$ $\Gamma-$converges to $F+G$.*

v) *The $\Gamma-$limit of a sequence of convex functionals is convex.*

vi) *The $\Gamma-$limit of a sequence of quadratic forms*[3] *is a quadratic form.*

vii) *Let $\alpha > 0$ then the $\Gamma-$limit of a sequence of positively $\alpha-$homogeneous functionals*[4] *is positively $\alpha-$homogeneous.*

For the proofs of previous facts we refer to [13].
One of the most interesting property of $\Gamma-$convergence is the convergence of minimizers of the functionals $\{F_n\}$ to minimizers of the $\Gamma-$limit F. This is true only under suitable hypotheses. For example we can consider the case of constant sequence $\{F_n\}$ that converges to the relaxed functional. An other possible example is the sequence $F_n : \mathbb{R} \to \mathbb{R}$, $F_n(x) = \mathbf{1}_{\mathbb{R}\setminus\{n\}}(x)$. We give here two basic results.

Proposition 1.73. *Let X be a metric space and $F_n : X \to \overline{\mathbb{R}}$ a sequence of functionals on X. Assume that:*

- *$\{F_n\}$ $\Gamma-$converge to the $\Gamma-$limit $F : X \to \overline{\mathbb{R}}$,*
- *there exists a minimum point x_n of F_n on X for all $n \in \mathbb{N}$,*
- *there exists $\overline{x} \in X$ such that $x_n \to \overline{x}$.*

Then $F(\overline{x}) = \min_X F$.

Proof. Let z be an arbitrary point of X. By the second condition in the definition of $\Gamma-$convergence there exists a sequence $\{z_n\}$ such that $\lim_{n\to+\infty} F_n(z_n) = F(z)$. Then

$$F(\overline{x}) \leq \liminf_{n\to+\infty} F_n(x_n) \leq \liminf_{n\to+\infty} F_n(z_n) = \lim_{n\to+\infty} F_n(z_n) = F(z).$$

The first inequality is due to the first condition in the definition of $\Gamma-$convergence. The second one is due to the fact that x_n is a minimum point for F_n. □

[3] $F_n(x+y) + F_n(x-y) = 2F_n(x) + 2F_n(y)$
[4] $F_n(tx) = t^\alpha F_n(x)$ for all $t > 0$

We need the following lemma.

Lemma 1.74. *Let X be a metric space, let $K \subseteq X$ be a compact set and $F_n : X \to \overline{\mathbb{R}}$ a sequence of functionals on X. If $F = \Gamma - \liminf_{n\to+\infty} F_n$, then*

$$\liminf_{n\to+\infty} \inf_{x\in K} F_n(x) \geq \inf_{x\in K} F(x) = \inf_{x\in K} \Gamma - \liminf_{n\to+\infty} F_n(x)$$

Proof. For any $n \in \mathbb{N}$ there exists $x_n \in K$ such that

$$F_n(x_n) < \inf_{x\in K} F_n(x) + \frac{1}{n}.$$

Defined $I_n := \inf_{x\in K} F_n(x)$ there exists a subsequence $\{I_{n_k}\}$ and an element $x_0 \in K$ such that $\liminf_{n\to+\infty} I_n = \lim_{k\to+\infty} I_{n_k}$ and $x_{n_k} \longrightarrow x_0$ (this is due to the compactness of K). Then

$$\begin{aligned}\liminf_{n\to+\infty} I_n = \lim_{k\to+\infty} I_{n_k} &= \liminf_{k\to+\infty} I_{n_k} \\ &\geq \liminf_{k\to+\infty} F_{n_k}(x_{n_k}) - \frac{1}{n_k} = \liminf_{k\to+\infty} F_{n_k}(x_{n_k}).\end{aligned}$$

By definition of $\Gamma - \liminf$ we have

$$\liminf_{k\to+\infty} F_{n_k}(x_{n_k}) \geq \Gamma - \liminf_{k\to+\infty} F_{n_k}(x_0)$$

on the other hand $\Gamma - \liminf_{k\to+\infty} F_{n_k}(x_0) \geq \Gamma - \liminf_{n\to+\infty} F_n(x_0) = F(x_0)$, so

$$\liminf_{n\to+\infty} I_n \geq \Gamma - \liminf_{n\to+\infty} F_n(x_0) = F(x_0) \geq \inf_{x\in K} F(x).$$

□

Theorem 1.75 (Convergence of minima and minimizers)**.** *Let X be a metric space and $F_n : X \to \overline{\mathbb{R}}$ a sequence of functionals on X. Suppose that:*

- *there exists a compact set K (independent of n) such that*

$$\inf\{F_n(x) \,|\, x \in X\} = \inf\{F_n(x) \,|\, x \in K\} \quad \forall\, n \in \mathbb{N},$$

- *$\{F_n\}$ $\Gamma-$converge to the $\Gamma-$limit $F : X \to \overline{\mathbb{R}}$.*

Then F admits minimum on X and we have the convergence of minima

$$\min\{F(x)\,|\,x \in X\} = \lim_{n\to+\infty} \inf\{F_n(x)\,|\,x \in X\}.$$

Moreover, if we consider a convergent sequence $x_n \longrightarrow x$ such that

$$\lim_{n\to+\infty} F_n(x_n) - \inf_X F_n = 0,$$

then x is a minimizer for F.

Proof. Since the $\Gamma-$limit F is lower semicontinuous it admits a minimum on K. Given $x_0 \in \arg\min_K F$ we have to prove that it is also a minimum point on X. Let $y \in X$ be an arbitrary point. From the $\Gamma-$convergence hypothesis there exists a sequence $\{y_n\}$ converging to y such that

$$F(y) \geq \limsup_{n\to+\infty} F_n(y_n).$$

Then

$$\begin{aligned} F(y) &\geq \limsup_{n\to+\infty} F_n(y_n) \geq \liminf_{n\to+\infty} F_n(y_n) \\ &\geq \liminf_{n\to\infty} \inf_X F_n = \liminf_{n\to\infty} \inf_K F_n \geq \inf_K F = F(x_0) \end{aligned}$$

where the last inequality is due to the previous Lemma. Then $F(y) \geq F(x_0)$ for any $y \in X$.

We prove now the convergence of minima. From the previous computations

$$\liminf_{n\to\infty} \inf_X F_n \geq \inf_X F.$$

Given $x_0 \in \arg\min_K F$, there exists a sequence $\{x_n\}$ converging to x_0 such that

$$\min_X F = F(x_0) \geq \limsup_{n\to+\infty} F_n(x_n)$$

so

$$\min_X F \geq \limsup_{n\to+\infty} F_n(x_n) \geq \limsup_{n\to+\infty} \inf_X F_n \geq \liminf_{n\to\infty} \inf_X F_n \geq \min_X F$$

and we conclude that $\lim_{n\to+\infty} \inf_X F_n = \min_X F$, i.e. the sequence of minima converges to the infimum (minimum) of the $\Gamma-$limit.

Now we consider a convergent sequence $x_n \longrightarrow x$ such that $\lim_{n\to+\infty} F_n(x_n) - \inf_X F_n = 0$ and we have to prove that x is a minimizer for F. We have that

$$\begin{aligned}\limsup_{n\to+\infty} F_n(x_n) &= \limsup_{n\to+\infty} F_n(x_n) + \inf_X F_n - \inf_X F_n = \limsup_{n\to+\infty} \inf_X F_n \\ &= \lim_{n\to\infty} \inf_X F_n = \min_X F\end{aligned}$$

on the other hand

$$\limsup_{n\to+\infty} F_n(x_n) \geq \liminf_{n\to+\infty} F_n(x_n) \geq F(x),$$

therefore $F(x) \leq \min_X F$, i.e. x is a minimum point for F. □

Remark 1.76. If F_n is lower semicontinuous for every n, then there exists a minimum point x_n of F_n for every n and obviously

$$\lim_{n\to+\infty} F_n(x_n) - \inf_X F_n = 0.$$

Since $\{x_n\} \subset K$ that is compact, then, up to subsequence there exists $x \in K$ such that $x_n \longrightarrow x$. For this reason in previous theorem we say that there is "convergence of minimizers".

Using the compactness of $\overline{\mathbb{R}}$ and a diagonal argument, it is possible to prove the following useful property of $\Gamma-$convergence.

Theorem 1.77 (Compactness)**.** *Assume that X has a countable base. Let $\{F_n\}$ be a sequence of functionals from X into $\overline{\mathbb{R}}$, then there exists a subsequence $\Gamma-$convergent to a functional $F : X \longrightarrow \overline{\mathbb{R}}$.*

A possible reference for a proof is [13, Chapter 8, Theorem 8.6]

1.2.2 The case of the weak topology

It is well know that in general the weak topology is not metrizable but, under suitable additional assumptions, the previous results can be extended to the case of the weak topology of a Banach space with a separable dual.
We recall the following classical result:

Theorem 1.78. *Assume that X is a Banach space and that the dual X' of X is separable. Then there exists a metric d on X such that the weak topology on every norm bounded subset*[5] *B of X coincides with the topology induced on B by the metric d.*

Corollary 1.79. *Assume that X is a Banach space with a separable dual and let d be a metric on X. The following conditions are equivalent:*

i) *on every norm bounded subset B of X the weak topology coincides with the topology induced on B by the metric d;*

ii) *a sequence $\{x_n\}$ in X converges weakly to a point $x \in X$ if and only if $\{x_n\}$ is norm bounded and converges to x in the metric d.*

A possible reference for a proof of the theorem and corollary above is [13, Chapter 8, Proposition 8.7, Corollary 8.8].

Proposition 1.80. *Let X be a reflexive Banach space that is compactly imbedded in a Banach space W. Let d be the distance on X induced by the norm of W. Then the condition **ii)** of Corollary above is satisfied.*
If, in addition, X is separable, then the weak topology on each norm bounded subset of X is induced by the metric of W.

Proof. We have to prove that a sequence $\{x_h\}$ in X converges weakly to a point $x \in X$ if and only if $\{x_h\}$ is norm bounded and converges to x in the metric d. We remember that a weakly convergent sequence is bounded and that, from compact embedding, is convergent in d and one implication is proved. On other hand, if a sequence is bounded, we can extract a subsequence weakly convergent to an element $z \in X$. In particular we can do this for all subsequences of $\{x_h\}$. From the compact embedding we conclude that the weak limit of each subsequence is x. This proves the opposite implication. Finally, the last statement is a consequence of separability and corollary above. □

Example 1.81. A frequent case is when $X = H_0^1(\Omega)$ and $W = L^2(\Omega)$, where Ω is a bounded open subset of $\mathbb{R}^n$. By the last proposition, on each bounded

[5] the weak topology on a subset B of X is, by definition, the topology on B induced by the weak topology of X.

subset B of $H_0^1(\Omega)$ the weak topology of $H_0^1(\Omega)$ is induced by the metric of $L^2(\Omega)$.

1.3 Spectral Theory

Definition 1.82. Let $T : H \to H$ be a linear operator on Hilbert space H. We say that $\lambda \in \mathbb{C}$ is an eigenvalue of T if $\ker(T - \lambda I) \neq \{\mathbf{0}\}$.

Definition 1.83. Given a Hilbert space H with scalar product denoted by $\langle \cdot, \cdot \rangle$ and a linear operator $T : H \to H$, we say that

- T is *strictly positive* (resp. *positive*) if $\langle Tx, x \rangle > 0$ (resp. $\langle Tx, x \rangle \geq 0$) for all $x \in H \setminus \{\mathbf{0}\}$,
- T is *self-adjoint* if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in H$,
- T is *compact* if the image of every bounded set has compact closure in H.

Remark 1.84. Let $T : H \to H$ be a linear operator on a Hilbert space H.

- If T is compact then it is continuous.
- If T is self-adjoint then the eigenvalues are real.

We recall the following classical result.

Theorem 1.85. *Let $T : H \to H$ be a compact and self-adjoint linear operator defined on a Hilbert space H. Then*

i) *there exists a set of positive eigenvalues $\{\lambda_n^+\}_{n \in I^+}$ (with eigenvectors $\{v_n^+\}_{n \in I^+}$) and a set of negative eigenvalues $\{\lambda_n^-\}_{n \in I^-}$ (with eigenvectors $\{v_n^-\}_{n \in I^-}$) where I^+ and I^- are finite or countable sets of indices (eventually empty). Moreover $0 < \lambda_{n+1}^+ \leq \lambda_n^+$ and $\lambda_n^- \leq \lambda_{n+1}^- < 0$.*

ii) *The elements of $\{v_n^+\}_{n \in I^+} \cup \{v_n^-\}_{n \in I^-}$ are orthonormal, i.e.*

$$\langle v, w \rangle = \begin{cases} 0 & \text{if } v \neq w, \\ 1 & \text{if } v = w, \end{cases}$$

for any $v, w \in \{v_n^+\}_{n \in I^+} \cup \{v_n^-\}_{n \in I^-}$.

iii) *Let*

$$H_+ := \overline{\operatorname{span}\{v_n^+\}_{n\in I^+}}, \quad H_- := \overline{\operatorname{span}\{v_n^-\}_{n\in I^-}},$$

$$H_0 := \ker(T),$$

it follows

$$H = H_+ \oplus H_- \oplus H_0.$$

iv) *If $\mu \neq 0$ is an eigenvalue of T, then $\mu \in \{\lambda_n^+\}_{n\in I^+} \cup \{\lambda_n^-\}_{n\in I^-}$.*

v) *If I^+ is infinite (resp. I^- is finite) then $\lambda_n^+ \longrightarrow 0$ as $n \to +\infty$ (resp. $\lambda_n^- \longrightarrow 0$).*

Remark 1.86. The multiplicity of each eigenvalue is finite.

A possible reference for a proof of the last theorem is [24].

Corollary 1.87. *Let $T : H \to H$ be a compact and self-adjoint linear operator defined on an infinite dimensional Hilbert space H. If T is strictly positive then there exists a sequence of positive eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$. The set of corresponding eigenvectors $\{v_n^+\}_{n\in\mathbb{N}}$ is a orthonormal basis of H. Moreover $0 < \lambda_{n+1} \leq \lambda_n$, $\lim_{n\to+\infty}\lambda_n = 0$ and if μ is an eigenvalue then $\mu \in \{\lambda_n\}$.*

Chapter 2

Existence results for monotone functionals

Now we are able to set shape optimization problems in the correct form and solve it using tools of previous chapter. We shall follows mainly the paper of Buttazzo and Dal Maso, [10], and the book of Bucur and Buttazzo, [7].

2.1 Problem setting

In this section we begin recalling some well known facts about variational formulations for elliptic operators. Then we give the definition of $\gamma-$convergence for sequence of quasiopen sets. Finally we expose the main theorem and some examples.

2.1.1 Variational formulations

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We are interested in elliptic operators

$$L : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$$

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{i,j}(x)\frac{\partial u}{\partial x_j}\Big) + a(x)u \tag{2.1}$$

where $a \in L^\infty(\Omega)$ is non negative a.e. on Ω and $a_{i,j} \in L^\infty(\Omega)$, $a_{i,j} = a_{j,i}$ a.e. on Ω. Moreover we suppose that there exists a positive constant $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} a_{i,j}(x)\,\zeta_i\,\zeta_j \geq \alpha|\zeta|^2 \qquad \forall\,\zeta = (\zeta_i)_{i=1}^n \in \mathbb{R}^n,\ \text{a.e. } x \in \Omega. \tag{2.2}$$

We are interested in the study of the Dirichlet problem

$$\text{find } u \in H_0^1(A) \quad \text{such that} \quad Lu = f \quad \text{in } A \tag{2.3}$$

for a given $A \in \mathcal{A}(\Omega)$ and $f \in H^{-1}(\Omega)$, where the equation above is satisfied in the weak sense, namely

$$\int_A A(x)Du \cdot Dv\,dx + \int_A a(x)uv\,dx = \,<f, v> \qquad \forall\, v \in H_0^1(A) \tag{2.4}$$

where the matrix $A(x)$ is defined as $A(x) := \big(a_{i,j}(x)\big)_{i,j=1,\dots,n}$ and $<f, v>$ is the duality bracket. The left-hand side of (2.4) is a bilinear form β on $H_0^1(A)$, namely

$$\beta : H_0^1(A) \times H_0^1(A) \longrightarrow \mathbb{R}$$

that, under the previous hypothesis is continuous, coercive and symmetric. The restriction of f to $H_0^1(A)$ is continuous and linear so we can apply Lax-Milgram theorem[1]. Then there exists a unique solution to (2.3) that is also the unique minimum point of the functional

$$I : H_0^1(A) \longrightarrow \mathbb{R},$$

$$I(u) = \int_A A(x)|Du|^2\,dx + \int_A a(x)u^2\,dx - 2<f, u> .$$

Definition 2.1. Under the previous hypotheses on L we define the *resolvent operator* $R_A^L : H^{-1}(\Omega) \to H_0^1(A)$ which associates to every $f \in H^{-1}(\Omega)$ the unique solution of (2.3).

When L is the Laplacian the corresponding resolvent operator is denoted by R_A.

Theorem 2.2. *The resolvent operator $R_A^L : H^{-1}(\Omega) \longrightarrow H_0^1(A)$ is continuous and symmetric, i.e.*

$$<f, R_A^L(g)> \,=\, <R_A^L(f), g>$$

[1] since the bilinear form is symmetric the Riesz Theorem is sufficient

for any $f, g \in H^{-1}(\Omega)$. Moreover, if we consider the injections $i_1 : L^2(A) \to H^{-1}(\Omega)$ and $i_2 : H_0^1(A) \to L^2(A)$, then

$$T_A^L := i_2 \circ R_A^L \circ i_1 : L^2(A) \longrightarrow L^2(A)$$

is compact, self-adjoint and strictly positive.

Proof. As a consequence of the Lax-Milgram Theorem, the resolvent operator is continuous. Moreover it is symmetric, i.e. $< f, R_A^L(g) > = < R_A^L(f), g >$ for all $f, g \in H^{-1}(\Omega)$. Indeed

$$\int_A A(x) D R_A^L(f) \cdot Dv \, dx + \int_A a(x) R_A^L(f) v \, dx = < f, v > \qquad \forall\, v \in H_0^1(A),$$

$$\int_A A(x) D R_A^L(g) \cdot Dw \, dx + \int_A a(x) R_A^L(g) w \, dx = < g, w > \qquad \forall\, w \in H_0^1(A).$$

If we choose $v = R_A^L(g)$ and $w = R_A^L(f)$, using $a_{i,j} = a_{j,i}$ a.e., we obtain $< f, R_A^L(g) > = < R_A^L(f), g >$. This implies also that T_A^L is self-adjoint on L^2. Indeed if $f, g \in L^2(A)$ then

$$\langle f, T_A^L g \rangle_{L^2} = < f, R_A^L(g) > = < R_A^L(f), g > = \langle T_A^L f, g \rangle_{L^2}.$$

Moreover we note that i_1 is continuous and, for Rellich Theorem, the embedding $i_2 : H_0^1(A) \hookrightarrow L^2(A)$ is compact. Then T_A^L is compact because it is the composition of a continuous operator and a compact one. It remains to show that it is strictly positive on $L^2(A)$. If we choose $f \in L^2(A)$ and $v = u = R_A^L(f)$ in Equation 2.4 then

$$0 \leq \int_A A(x) Du \cdot Du \, dx + \int_A a(x) u^2 \, dx = < f, R_A^L(f) > = \langle f, T_A^L(f) \rangle_{L^2}$$

We suppose $\langle f, T_A^L(f) \rangle_{L^2} = 0$ and we set $u = R_A^L(f)$,

$$\int_A A(x) Du \cdot Du \, dx + \int_A a(x) u^2 \, dx = 0$$

and by the coerciveness assumption we get that $u = 0$. Then $< f, v >= \langle f, v \rangle_{L^2} = 0$ for any $v \in H_0^1(A)$. Using Theorem 1.65 we get $< f, v >= \langle f, v \rangle_{L^2} = 0$ for any $v \in L^2(A)$, i.e. $f = 0$ in $L^2(A)$. $\square$

As a consequence of the previous proposition and Theorem 1.85, we have the following result.

Corollary 2.3. *There exists a decreasing sequence of positive eigenvalues* $\{\mu_n\}$ *of* T_A^L. *Moreover* $0 < \mu_{n+1} \leq \mu_n$, $\lim_{n\to+\infty} \mu_n = 0$ *and if* ω *is an eigenvalue then* $\omega \in \{\mu_n\}$.

2.1.2 Convergence of quasiopen subsets

Definition 2.4. Let Ω be an open bounded subset of $\mathbb{R}^n$. We say that a sequence $\{A_h\}$ in $\mathcal{A}(\Omega)$ $\gamma-$*converges* to $A \in \mathcal{A}(\Omega)$ if $\{R_{A_h}(f)\}$ converges to $R_A(f)$ strongly in $H_0^1(A)$ for every $f \in H^{-1}(\Omega)$.

Definition 2.5. Let $\Omega \subset \mathbb{R}^n$ be bounded and let $\{A_h\}$ be a sequence of quasi-open subsets of Ω. The sequence of spaces $H_0^1(A_h)$ converges in the sense of *Mosco* to the space $H_0^1(A)$ if the following conditions are satisfied:

M1) For all $\phi \in H_0^1(A)$ there exists a sequence $\phi_h \in H_0^1(A_h)$ strongly convergent in $H_0^1(\Omega)$ to ϕ.

M2) For every sequence $\phi_{n_h} \in H_0^1(A_{n_h})$ weakly convergent in $H_0^1(\Omega)$ to a function ϕ, we have $\phi \in H_0^1(\Omega)$.

For every open set $A \subset \Omega$ we denote by $P_{H_0^1(A)}$ the orthogonal projection of $H_0^1(\Omega)$ onto $H_0^1(A)$ with respect to the scalar product of $H_0^1(\Omega)$.

Proposition 2.6. *Let* $\{A_h\}$ *and* A *be open subsets of* Ω. *The following assertions are equivalent.*

1) *For every* $f \in H^{-1}(\Omega)$ *we have* $u_{A_h,f} \longrightarrow u_{A,f}$ *strongly in* $H_0^1(\Omega)$ *(i.e.,* A_h $\gamma-$*converges to* A*).*

2) *We have* $u_{A_h,1} \longrightarrow u_{A,1}$ *strongly in* $H_0^1(\Omega)$.

3) *The spaces* $H_0^1(A_h)$ *converges in the sense of Mosco to* $H_0^1(\Omega)$.

4) *The sequence* $\Phi_{A_h} : L^2(\Omega) \longrightarrow \overline{\mathbb{R}}$ *defined as*

$$\Phi_{A_h}(u) := \begin{cases} \int_{A_h} |Du|^2\,dx & \text{if } u \in H_0^1(\Omega), \\ +\infty & \text{if } u \notin H_0^1(\Omega), \end{cases}$$

$\Gamma-$converges in $L^2(\Omega)$ to the functional $\Phi_A : L^2(\Omega) \longrightarrow \overline{\mathbb{R}}$ defined as

$$\Phi_A(u) := \begin{cases} \int_A |Du|^2\,dx & \text{if } u \in H^1_0(\Omega), \\ +\infty & \text{if } u \notin H^1_0(\Omega). \end{cases}$$

5) *For every $u \in H^1_0(\Omega)$ the sequence $\{P_{H^1_0(A_h)}u\}$ converges strongly in $H^1_0(\Omega)$ to $P_{H^1_0(A)}u$.*

6) *The sequence R_{A_h} converges*[2] *in the operator norm of $\mathcal{L}(L^2(\Omega))$ to R_Ω, i.e.*

$$\lim_{n\to+\infty} \sup_{||f||_{L^2(\Omega)}\le 1} ||R_{A_h}(f) - R_A(f)||_{L^2(\Omega)} = 0.$$

We refer to [7, Proposition 4.5.3, Pag. 98] for the proof. Moreover we have to following useful characterization, due to Sveràk:

Proposition 2.7. *Let $\{A_h\}$ and A be open subsets of Ω. Then*

$$P_{H^1_0(A_h)}u \xrightarrow{H^1_0(\Omega)} P_{H^1_0(A)}u \quad \forall u \in H^1_0(\Omega) \;\Leftrightarrow\; w_{A_h} \rightharpoonup w_A$$

where w_A is the weak solution of $-\Delta w = 1$ on A and w_{A_h} is the weak solution of $-\Delta w = 1$ on A_h.

See [28, Corollary 3.3] for a proof.

Remark 2.8. The results of previous propositions are true also for quasi-open sets. We have only to change the proof of implication 2) $\Rightarrow$ 1). In that case one can use the following density result:

$$\overline{\{\phi u_{A,1} \,|\, \phi \in C^\infty_c(\Omega)\}}^{H^1_0} = H^1_0(A).$$

For the proof see [14] and [7, Remark 4.5.5, Pag. 100].
Moreover we have the following result, that is a direct consequence of Mosco convergence:

Proposition 2.9. *A sequence $\{A_h\} \subset \mathcal{A}(\Omega)$ $\gamma-$converges to $A \in \mathcal{A}(\Omega)$ if and only if $\{R^L_{A_h}(f)\}$ converges to $R^L_A(f)$ strongly in $H^1_0(\Omega)$ for every $f \in H^{-1}(\Omega)$, where L is an elliptic operator that satisfies the assumption at the beginning of the section.*

[2] precisely the sequence T_{A_h} defined in Theorem 2.2

See also [22, Chapter 3, Proposition 3.5.9].

Proposition 2.10. *The operators $R^L_{A_h}$ are equicontinuous from $H^{-1}(\Omega)$ to $H^1_0(\Omega)$.*

Proof. From Lax-Milgram Lemma we have the inequality $(*)$

$$\|R^L_{A_h}(f)\|_{H^1_0(\Omega)} = \|R^L_{A_h}(f)\|_{H^1_0(A)} \underset{(*)}{\leq} \frac{\|f\|_{(H^1_0(A_h))^*}}{\alpha} \leq \frac{\|f\|_{H^{-1}(\Omega)}}{\alpha} \quad \forall\, h.$$

Then

$$\left\|R^L_{A_h}\right\|_{\mathcal{L}(H^{-1}(\Omega),H^1_0(\Omega))} \leq \frac{1}{\alpha} \quad \forall h$$

and then the family $\{R^L_{A_h}\}$ is equicontinuous. □

Now we can state the following result that is an existence theorem for shape optimization problems with volume constraints, [10].

Theorem 2.11 (**Buttazzo-Dal Maso**, 1993)**.** *Let $F : \mathcal{A}(\Omega) \longrightarrow \mathbb{R}$ be a mapping satisfying the following assumptions:*

i) *F is lower semicontinuous with respect to $\gamma-$convergence;*

ii) *F is decreasing, namely*

$$A \subset B \Longrightarrow F(A) \geq F(B).$$

Then for every $c \in [0, |\Omega|]$ there exists a minimizer for the optimization problem

$$\min\Big\{F(A) \,|\, A \in \mathcal{A}(\Omega), |A| = c\Big\}. \tag{2.5}$$

We give here an example of functional of interest. In Chapter 3 we focus our attention on spectral functionals.

Example 2.12. Let $g : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ be a Borel function such that:

- $g(x, \cdot, \cdot)$ is lower semicontinuous on $\mathbb{R} \times \mathbb{R}^n$ for a.e. $x \in \Omega$,

- $g(x,s,\zeta) \geq -\alpha(x) - \beta(s^2 + |\zeta|^2)$ for a.e. $x \in \Omega$, for a suitable function $\alpha \in L^1(\Omega)$ and for a suitable constant $\beta \in \mathbb{R}$.

For every $A \in \mathcal{A}(\Omega)$ let $u_A = R_A^L(f)$, where $f \in H^{-1}(\Omega)$ is given. Then the map

$$F(A) = \int_\Omega g(x, u_A(x), Du_A(x))\, dx$$

is lower semicontinuous with respect to $\gamma-$convergence. First we have to prove that the functional

$$G : H_0^1(\Omega) \longrightarrow \mathbb{R}$$

$$G(u) = \int_\Omega g(x, u(x), Du(x))\, dx$$

is lower semicontinuous on $H_0^1(\Omega)$. To prove this consider a sequence $u_n \to u$ in $H_0^1(\Omega)$ and the auxiliary functional

$$H \ : H_0^1(\Omega) \longrightarrow \mathbb{R}$$

$$H(u) = \int_\Omega h(x, u(x), Du(x))\, dx$$

where $h(x,s,\zeta) := g(x,s,\zeta) + \alpha(x) + \beta(s^2 + |\zeta|^2) \geq 0$ a.e. in Ω. There exists a subsequence u_{n_k} such that $u_{n_k}(x) \to u(x)$ a.e., $Du_{n_k}(x) \to Du(x)$ a.e. and $\liminf_{n\to+\infty} H(u_n) = \lim_{k\to+\infty} H(u_{n_k})$. By semicontinuity of h and by the Fatou Lemma

$$\begin{aligned} H(u) = \int_\Omega h(x, u(x), Du(x))\, dx &\leq \int_\Omega \liminf_{k\to+\infty} h(x, u_{n_k}(x), Du_{n_k}(x))\, dx \\ &\leq \liminf_{k\to+\infty} \int_\Omega h(x, u_{n_k}(x), Du_{n_k}(x))\, dx \\ &= \liminf_{k\to+\infty} H(u_{n_k}) = \liminf_{n\to+\infty} H(u_n) \end{aligned}$$

so the functional H is lower semicontinuous. Since

$$\int_\Omega \alpha(x) + \beta(u^2 + |Du|^2) - \alpha(x) - \beta(u_n^2 + |Du_n|^2)\, dx \longrightarrow 0 \text{ as } n \to +\infty$$

we get also that $G(u) \leq \liminf_{n\to+\infty} G(u_n)$, i.e. G is lower semicontinuous with respect to the strong convergence.
Now the conclusion is easy:

$$A_n \xrightarrow{\gamma} A \quad \Longrightarrow \quad u_{A_n} \xrightarrow{H_0^1} u_A \quad \Longrightarrow$$

$$F(A) = G(u_A) \leq \liminf_{n\to+\infty} G(u_{A_n}) = \liminf_{n\to+\infty} F(A_n)$$

i.e. F is lower semicontinuous with respect to $\gamma-$convergence.

Example 2.13. Let $f \in H^{-1}(\Omega)$, with $f \geq 0$ in the sense of distributions, and let $g : \Omega \times \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ be a Borel function such that:

- $g(x, \cdot)$ is lower semicontinuous and decreasing on $\mathbb{R}$ for a.e. $x \in \Omega$,
- $g(x, s) \geq -\alpha(x) - \beta s^2$ a.e. $x \in \Omega$, where $\alpha \in L^1(\Omega)$ and $\beta \in \mathbb{R}$.

As done in the previous example, for every $A \in \mathcal{A}(\Omega)$ we define $u_A := R_A^L(f)$. Then the map

$$F(A) = \int_\Omega g(x, u_A(x))\, dx$$

is decreasing with respect to the set inclusion. To prove this first we observe that $u_A \geq 0$ a.e. on A. Indeed we have that

$$\int_A A(x) Du_A \cdot Dv\, dx + \int_A a(x) u_A v\, dx \geq 0 \quad \forall\, v \in H_0^1(A)$$

In particular we can choose $v = u_A^-$. Moreover we recall that

$$u^- := \begin{cases} -u & \text{on } \{u < 0\} \\ 0 & \text{on } \{u \geq 0\} \end{cases} \quad \text{and} \quad D(u^-) := \begin{cases} -Du & \text{on } \{u < 0\} \\ 0 & \text{on } \{u \geq 0\} \end{cases}$$

so we obtain

$$\int_A A(x) Du_A \cdot Du_A^-\, dx + \int_A a(x) u_A u_A^-\, dx \geq 0$$

that is

$$-\int_A A(x) Du_A^- \cdot Du_A^-\, dx - \int_A a(x)(u_A^-)^2\, dx \geq 0$$

and, by the coerciveness assumption,

$$\alpha ||u_A^-||^2_{H_0^1(A)} = 0$$

which implies $u_A^- = 0$ a.e.

Now we consider A_1, $A_2 \in \mathcal{A}(\Omega)$ with $A_1 \subset A_2$. If we prove that $u_{A_1} \leq u_{A_2}$ a.e. on Ω then, using the monotonicity of g, the proof is complete. First we note that $u_{A_1} \leq u_{A_2}$ on $\Omega \setminus A_1$ since they are positive and $u_{A_1} = 0$ on $\Omega \setminus A_1$. Moreover

$$\int_{A_1} A(x)D(u_{A_2} - u_{A_1}) \cdot Dv\, dx + \int_{A_1} a(x)(u_{A_2} - u_{A_1})v\, dx = 0, \quad \forall\, v \in H_0^1(A_1)$$

and $(u_{A_2} - u_{A_1})^- \in H_0^1(A_1)$ so

$$\int_{A_1} A(x)D(u_{A_2} - u_{A_1})^- \cdot D(u_{A_2} - u_{A_1})^-\, dx + \int_{A_1} a(x)((u_{A_2} - u_{A_1})^-)^2\, dx = 0$$

which implies

$$\alpha||(u_{A_2} - u_{A_1})^-||^2_{H_0^1(A_1)} = 0$$

that is

$$(u_{A_2} - u_{A_1})^- = 0 \quad \text{a.e. in } A_1.$$

This implies $u_{A_2} \geq u_{A_1}$ a.e. on Ω.

As a consequence of these two examples we obtain the following result.

Proposition 2.14. *Let $f \in H^{-1}(\Omega)$, with $f \geq 0$ in the sense of distributions, and let $g : \Omega \times \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ be a Borel function such that:*

- *$g(x, \cdot)$ is lower semicontinuous and decreasing on $\mathbb{R}$ for a.e. $x \in \Omega$,*
- *$g(x, s) \geq -\alpha(x) - \beta s^2$ a.e. $x \in \Omega$, where $\alpha \in L^1(\Omega)$ and $\beta \in \mathbb{R}$.*

Let $c > 0$ be fixed and define $u_A := R_A^L(f)$. Then the minimization problem

$$\min\Big\{ \int_\Omega g(x, u_A(x))\, dx \mid A \in \mathcal{A}(\Omega), |A| = c\Big\}$$

admits a solution.

The last functional can be studied in the general framework of Optimal Control Problems (where the space of states is $H_0^1(\Omega)$ and the space of controls is $\mathcal{A}(\Omega)$). A possible reference for other details and examples is [7, Chapter 3, Section 3.6].

2.2 Proof of the main result

For every $A \in \mathcal{A}(\Omega)$ we denote by $w_A \in H_0^1(A)$ the unique weak solution of

$$-\Delta w_A = 1 \quad \text{in } A. \tag{2.6}$$

We want prove that

$$1 + \Delta w_A \geq 0 \quad \text{in } \Omega \tag{2.7}$$

and that

$$w_A \geq w \quad \text{q.e. in } \Omega, \forall\, w \in Z_A \tag{2.8}$$

where

$$Z_A := \Big\{ w \in H_0^1(\Omega) \,\Big|\, w \leq 0 \text{ q.e. on } \Omega \setminus A,\, 1 + \Delta w \geq 0 \text{ in } \Omega \Big\}.$$

In order to prove this we consider the solution $u_A \in H_0^1(\Omega)$ of the variational inequality

$$u_A \in K_A, \quad \int_\Omega Du_A \cdot D(v - u_A)\, dx \geq \int_\Omega (v - u_A)\, dx \quad \forall v \in K_A, \tag{2.9}$$

where

$$K_A := \Big\{ u \in H_0^1(\Omega) \,|\, u \leq 0 \text{ q.e. on } \Omega \setminus A \Big\}.$$

Since K_A is closed and convex, the existence and uniqueness of u_A is ensured by the result below.

2.2.1 Variational inequalities and supersolutions

Theorem 2.15. *Let H be a real Hilbert space, let $K \subset H$ be a non empty convex and closed subset, let $F \in H'$ be a fixed functional. Then, there exists a unique solution to the problem*

$$\min_{u \in K} \left\{ \frac{1}{2} \|u\|_H^2 - <F, u> \right\}.$$

Moreover u is the solution of the previous minimization problem if and only if it satisfies the following variational inequality

$$\begin{cases} u \in K \\ \langle u, v - u \rangle_H \geq <F, v - u> \qquad \forall v \in K. \end{cases}$$

Proof. By the Riesz Representation Theorem for any $F \in H'$ there exists a unique element $w_F \in H$ such that

$$< F, v > = \langle w_F, v \rangle_H \quad \forall v \in H,$$

so the minimization problem can be rewritten as

$$\min_{u \in K} \left\{ \frac{1}{2}\|u\|_H^2 - \langle w_F, u \rangle_H + \frac{1}{2}\|w_F\|_H^2 \right\} - \frac{1}{2}\|w_F\|_H^2.$$

so we can study the problem

$$\min_{u \in K} \left\{ \frac{1}{2}\|u\|_H^2 - \langle w_F, u \rangle_H + \frac{1}{2}\|w_F\|_H^2 \right\} = \min_{u \in K} \frac{1}{2} \|u - w_F\|_H^2.$$

From the Projection Theorem in Hilbert Spaces [4, Chapter V, Theorem V.2] we can conclude the proof. □

Definition 2.16. Let $A : H^1(\Omega) \longrightarrow H^{-1}(\Omega)$ be the operator defined as

$$Au := - \sum_{i,j=1}^{n} D_i(a_{i,j}(x) D_j u)$$

where $a_{i,j} \in L^\infty(\Omega)$, $a_{i,j} = a_{j,i}$ a.e. and suppose there exists two constants $0 < \lambda \leq \Lambda < +\infty$ such that

$$\lambda |\zeta|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x) \zeta_i \zeta_j \leq \Lambda |\zeta|^2 \qquad \text{a.e. } x \in \Omega, \forall\, \zeta \in \mathbb{R}^n$$

Given $f \in H^{-1}(\Omega)$ we say that a function $u \in H^1(\Omega)$ is a *subsolution* for the equation $Au - f = 0$ if

$$< Au - f, \varphi > \ \leq 0 \quad \forall \varphi \in H_0^1(\Omega),\ \widetilde{\varphi} \geq 0 \text{ q.e.}$$

and we say that a function $u \in H_0^1(\Omega)$ is a *supersolution* if

$$< Au - f, \varphi > \ \geq 0 \quad \forall \varphi \in H_0^1(\Omega),\ \widetilde{\varphi} \geq 0 \text{ q.e.}$$

Remark 2.17. We observe that the operator $A = -\Delta$ satisfies the conditions above.

Proposition 2.18. *Let $\psi : \Omega \longrightarrow \overline{\mathbb{R}}$ be a measurable function. Let u be solution of the variational inequality*

$$u \in K_\psi, \quad \int_\Omega Du \cdot D(v-u)\,dx \geq \,<f, v-u> \qquad \forall v \in K_\psi,$$

where $K_\psi := \{v \in H_0^1(\Omega) \,|\, \tilde{v} \geq \psi \ \ C_2 - q.e. \ in \ \Omega\}$ then u is a supersolution of $-\Delta u = f$.

Proof. We choose $v := u+\varphi$, then $\widetilde{v} = \widetilde{u}+\widetilde{\varphi}$. Indeed $\widetilde{u}+\widetilde{\varphi}$ is quasicontinuous since it is the sum of two quasicontinuous functions. Moreover $u = \tilde{u}$ and $\varphi = \tilde{\varphi}$ a.e.. So $u+\varphi = \widetilde{u}+\widetilde{\varphi}$ a.e. and $\tilde{v} = v$ a.e. so $\widetilde{v} = \widetilde{u}+\widetilde{\varphi}$ a.e. and, using Lemma 1.44, we have $\widetilde{v} = \widetilde{u}+\widetilde{\varphi}$ C_2-q.e.. So $v = u+\varphi \in K_\psi$ and this is true for any $\varphi \in H_0^1(\Omega)$, $\varphi \geq 0$ q.e.. □

Theorem 2.19. *Let $\psi : \Omega \longrightarrow \overline{\mathbb{R}}$ be a measurable function. Let u be solution of the variational inequality*

$$u \in K_\psi, \quad \int_\Omega Du \cdot D(v-u)\,dx \geq \,<f, v-u> \qquad \forall v \in K_\psi,$$

where $K_\psi := \{v \in H_0^1(\Omega) \,|\, \widetilde{v} \geq \psi \ \ C_2 - q.e. \ in \ \Omega\}$. Let w be a supersolution of $-\Delta w = f$ such that $\widetilde{w} \geq \psi$ $C_2-q.e.$ on Ω, and suppose that $\widetilde{w} \geq 0$ $C_2-q.e.$ on $\partial\Omega$. Then

$$u \leq w \quad C_2 - q.e. \ in \ \Omega.$$

Proof. Letting $A := -\Delta$, we have that $< Aw-f, z > \geq 0 \,\forall\, z \in H_0^1(\Omega)$, $z \geq 0$ q.e. and $< Au-f, v-u > \geq 0 \,\forall\, v \in K_\psi$. Since $u \in H_0^1(\Omega)$ and $w \geq 0$ q.e. on $\partial\Omega$, then $(u-w)^+ \in H_0^1(\Omega)$. We can choose $z := (u-w) \vee 0 = (u-w)^+$ a.e. and $v = w \wedge u$. Indeed $\widetilde{z} := (\widetilde{u}-\widetilde{w}) \vee 0 \geq 0$ q.e. and $\widetilde{v} = \widetilde{w} \wedge \widetilde{u} \geq \psi$ q.e., i.e. z and v are admissible functions. Moreover $w \wedge u - u = (w-u) \wedge 0$ so

$$<Aw-f, (u-w)^+> \,\geq 0 \quad \text{and} \quad - \,<Au-f, (u-w)^+> \,\geq 0$$

which implies

$$<A(w-u), (u-w)^+> \,\geq 0 \quad \text{i.e.,} \qquad <A(u-w), (u-w)^+> \,\leq 0$$

and, from the ellipticity condition,

$$\lambda \int_\Omega |D[(u-w)^+]|^2\,dx = 0$$

This implies $u \leq w$ a.e. on Ω and, using Theorem 1.50, we obtain $\widetilde{u} \leq \widetilde{w}$ C_2-q.e. on Ω. $\square$

Now we consider the problem stated in (2.9). Let u_A be the solution and consider the closed and convex subset

$$\widehat{K} := \Big\{u \in H_0^1(\Omega) \,|\, u \geq 0 \text{ q.e. on } \Omega \setminus A\Big\}.$$

Then $\widehat{u} := -u_A \in \widehat{K}$ is solution of the variational inequality

$$\widehat{u} \in \widehat{K}, \quad \int_\Omega D\widehat{u} \cdot D(\widehat{v} - \widehat{u})\, dx \geq -\int_\Omega (\widehat{v} - \widehat{u})\, dx \quad \forall \widehat{v} \in \widehat{K}.$$

We also note that $w = 0$ is a supersolution of $-\Delta u + 1 = 0$, then using Theorem 2.19 we get $-u_A = \widehat{u} \leq 0$ and in particular $u_A = 0$ q.e. on $\Omega \setminus A$, namely $u_A \in H_0^1(A) \subset K_A$. We obtain that w_A is a solution of variational inequality in $H_0^1(A)$ and from uniqueness $w_A = u_A$ q.e.. Moreover, using Proposition 2.18, we get

$$\Delta w_A + 1 \geq 0$$

and the inequality

$$w_A \geq w \quad \text{ q.e. in } \Omega, \forall w \in Z_A \tag{2.10}$$

follows again from Theorem 2.19.
Let $\mathcal{K}$ be the closed and convex subset of $H_0^1(\Omega)$ defined as

$$\mathcal{K} := \Big\{w \in H_0^1(\Omega) \,\Big|\, w \geq 0 \text{ q.e.}, 1 + \Delta w \geq 0\Big\}.$$

Lemma 2.20. *Under the assumptions and notations above one has $w_A \in \mathcal{K}$ for all $A \in \mathcal{A}(\Omega)$.*

Proof. It is a consequence of (2.7) and (2.8). $\square$

Proposition 2.21. *The subset $\mathcal{K}$ is bounded in $H_0^1(\Omega)$.*

Proof. If we choose w as a test function in $1 + \Delta w \geq 0$ we obtain

$$\int_\Omega |Dw|^2\, dx \leq \int_\Omega w\, dx = \int_\Omega |w|\, dx$$

moreover, by Hölder and Poincarè inequalities, there exists a positive constant C_Ω such that

$$\int_\Omega |Dw|^2\,dx \leq C_\Omega \Big(\int_\Omega |Dw|^2\,dx \Big)^{\frac{1}{2}},$$

which is

$$\Big(\int_\Omega |Dw|^2\,dx \Big)^{\frac{1}{2}} \leq C_\Omega,$$

i.e. $\mathcal{K}$ is bounded in $H_0^1(\Omega)$. $\square$

Corollary 2.22. *The subset $\mathcal{K}$ is compact in $L^2(\Omega)$.*

Proof. Compactness follows from previous proposition and the Rellich's Theorem. $\square$

Now we summarize the main steps necessary to prove the existence of a solution to problem stated in (2.5).

First we shall define a functional $G : \mathcal{K} \longrightarrow \mathbb{R}$ such that

- G is decreasing on $\mathcal{K}$, i.e. $G(u) \geq G(v)$ whenever $u \leq v$ q.e. on Ω,
- G is lower semicontinuous on $\mathcal{K}$ with respect to the strong topology of $L^2(\Omega)$,
- $G(w_A) = F(A)$ for every $A \in \mathcal{A}(\Omega)$.

Then we shall prove that the minimization problem

$$\min \Big\{ G(w) \,|\, w \in \mathcal{K},\ |\{w > 0\}| \leq c \Big\}$$

has a solution w^*, and that $A^* = \{w^* > 0\}$ is a minimum point of

$$\min \Big\{ F(A) \,|\, A \in \mathcal{A}(\Omega), |A| \leq c \Big\}.$$

Then a solution of (2.5) is obtained choosing a set $\tilde{A}$ such that $\tilde{A} \supset A^*$, $|\tilde{A}| = c$ and using the monotonicity of F.

2.2.2 The relaxation of a functional

We need to recall the notion of relaxation of a functional. For a basic reference see [19].

Definition 2.23. Let $F : X \longrightarrow \overline{\mathbb{R}}$ be a functional on a metric space (X, d). The *relaxation* of F is the functional

$$\overline{F}(x) := \inf\Big\{ \liminf_{h\to+\infty} F(x_h) \,|\, \{x_h\} \subset X,\, x_h \xrightarrow{d} x \Big\}.$$

It's clear that $\overline{F}(x) \leq F(x)$ for any $x \in X$ (consider $x_h = x$). Moreover we have the following lemma.

Lemma 2.24. *Let $\overline{F}$ be the relaxation of F. Then for any $x \in X$, $\varepsilon > 0$, $r > 0$, there exists $y \in X$ such that, if $\overline{F}(x) \in \mathbb{R}$, then*

$$d(x, y) \leq r \quad \text{and} \quad F(y) \leq \overline{F}(x) + \varepsilon.$$

Proof. By definition of relaxation there exists a sequence $x_h \longrightarrow x$ such that

$$\liminf_{h\to+\infty} F(x_h) \leq \overline{F}(x) + \frac{\varepsilon}{2}$$

and in particular

$$F(x_h) \leq \overline{F}(x) + \varepsilon$$

for a subsequence (still denoted by $\{x_h\}$). Now we can take $y = x_h$ for h large enough. □

Lemma 2.25. *For every $x \in X$ there exists a sequence $\{x_h\}$ converging to x such that*

$$\overline{F}(x) := \lim_{h\to+\infty} F(x_h)$$

In other words, the infimum in the definition of relaxation is in fact a minimum.

Proof. We assume $F(x) \in \mathbb{R}$, the other cases being similar. Applying the previous lemma with $r = \frac{1}{h}$ and $\varepsilon = \frac{1}{h}$, we obtain a point $x_h \in X$ such that $d(x_h, x) \leq \frac{1}{h}$ and

$$F(x_h) \leq \overline{F}(x) + \frac{1}{h}.$$

Then $x_h \longrightarrow x$ and $\limsup_{h\to+\infty} F(x_h) \leq \overline{F}(x)$. Moreover, by definition of relaxation $\overline{F}(x) \leq \liminf_{h\to+\infty} F(x_h)$, so we conclude

$$\overline{F}(x) = \lim_{h\to+\infty} F(x_h).$$

□

Proposition 2.26. *The relaxation $\overline{F}$ is lower semicontinuous.*

Proof. Let $x_h \longrightarrow x$ be a convergent sequence. Applying the previous lemma to the point x_h with $r = \frac{1}{h}$ and $\varepsilon = \frac{1}{h}$ we obtain a point $y_h \in X$ such that

$$d(y_h, x_h) \leq \frac{1}{h} \qquad F(y_h) \leq \overline{F}(x_h) + \frac{1}{h}.$$

Then $d(y_h, x) \leq d(y_h, x_h) + d(x_h, x) \longrightarrow 0$ as $h \to +\infty$ and

$$\liminf_{h\to+\infty} F(y_h) \leq \liminf_{h\to+\infty} \overline{F}(x_h).$$

Moreover, by definition of relaxation, $\overline{F}(x) \leq \liminf_{h\to+\infty} F(y_h)$ so we find

$$\overline{F}(x) \leq \liminf_{h\to+\infty} F(y_h) \leq \liminf_{h\to+\infty} \overline{F}(x_h).$$

□

2.2.3 Construction of the auxiliary functional

In order to construct G, for every $w \in \mathcal{K}$ we set

$$J(w) = \inf\{F(A) \,|\, A \in \mathcal{A}(\Omega),\, w_A \leq w\},$$

and we define G as the relaxation of J on $\mathcal{K}$, with respect to the strong topology of $L^2(\Omega)$:

$$G(w) := \overline{J}(w) = \inf\left\{ \liminf_{h\to+\infty} J(w_h) \,|\, \{w_h\} \subset \mathcal{K},\, w_h \xrightarrow{L^2} w \right\}, \quad \forall\, w \in \mathcal{K}.$$

Lemma 2.27. *If $u, v \in \mathcal{K}$, then $w := \max\{u, v\} \in \mathcal{K}$.*

Proof. We have that $u \geq 0$ q.e., $1 + \Delta u \geq 0$ and $v \geq 0$ q.e. and $1 + \Delta v \geq 0$. Then letting

$$f(\varphi) = \int_\Omega -\varphi\, dx \quad \forall \varphi \in H_0^1(\Omega)$$

and $A = -\Delta$ we know that

$$< A(-u) - f, \varphi > \; \geq 0 \quad \forall \varphi \in H_0^1(\Omega),\ \widetilde{\varphi} \geq 0 \ \text{ q.e.,}$$

$$< A(-v) - f, \varphi > \; \geq 0 \quad \forall \varphi \in H_0^1(\Omega),\ \widetilde{\varphi} \geq 0 \ \text{ q.e.,}$$

we have to prove that $w := \max\{u, v\} \in \mathcal{K}$, i.e. $w \geq 0$ q.e. and

$$< A(-w) - f, \varphi > \; \geq 0 \quad \forall \varphi \in H_0^1(\Omega),\ \widetilde{\varphi} \geq 0 \ \text{ q.e.}$$

Obviously $-w = -\max\{u, v\} = \min\{-u, -v\}$. From now on we set $\widehat{w} = -w$, $\widehat{v} = -v$, $\widehat{u} = -u$ for simplicity of the notation and we follow [23, Theorem 6.6, Chpater II]. We define the convex

$$K_{\widehat{w}} = \left\{ \eta \in H_0^1(\Omega) \,|\, \eta \geq \widehat{w} \ \text{ q.e. in } \Omega \right\}$$

and we denote by $\zeta \in K_{\widehat{w}}$ the solution of the variational inequality

$$\int_\Omega D\zeta \cdot D(\eta - \zeta)\, dx \geq \; < f, \eta - \zeta > \qquad \forall \eta \in K_{\widehat{w}},$$

Using Theorem 2.19 we get

$$\zeta \leq \widehat{u} \quad \text{ and } \quad \zeta \leq \widehat{v} \quad \text{ q.e in } \Omega$$

which implies $\zeta \leq \min\{\widehat{u}, \widehat{v}\} = \widehat{w}$. Since $\zeta \in K_{\widehat{w}}$, it follows that $\zeta = \min\{\widehat{u}, \widehat{v}\} = \widehat{w}$. In conclusion, $\widehat{w}$ is solution of the variational inequality and then it is a supersolution. □

Proposition 2.28. *Let $G : \mathcal{K} \longrightarrow \mathbb{R}$ be the functional defined above. Then*

- *G is decreasing on $\mathcal{K}$ i.e., $G(u) \geq G(v)$ whenever $u \leq v$ q.e. on Ω,*

- *G is lower semicontinuous on $\mathcal{K}$ with respect to the strong topology of $L^2(\Omega)$.*

Proof. The lower semicontinuty follows by Proposition 2.26. Let $u, v \in \mathcal{K}$ with $u \leq v$ q.e. on Ω. By Lemma 2.25, there exists a sequence $\{u_h\} \subset \mathcal{K}$ converging to u in $L^2(\Omega)$ such that

$$G(u) = \lim_{h \to +\infty} J(u_h).$$

If $v_h = v \vee u_h$, then $\{v_h\} \subset \mathcal{K}$ as a consequence of Lemma 2.27. The functional J is obviously decreasing and since $\{v_h\} \subset \mathcal{K}$ converges to v in $L^2(\Omega)$ we find

$$G(v) \leq \liminf_{h \to +\infty} J(v_h) \leq \lim_{h \to +\infty} J(u_h) = G(u).$$

□

Lemma 2.29. *Let $\{A_h\}$ be a sequence of quasi-open subsets of Ω such that w_{A_h} converges weakly in $H_0^1(\Omega)$ to a function w, and let $\{u_h\}$ be a sequence in $H_0^1(\Omega)$ such that $u_h = 0$ q.e. on $\Omega \setminus A_h$. Assume that u_h converges weakly in $H_0^1(\Omega)$ to a function u. Then $u = 0$ q.e. on $\{w = 0\}$.*

Proof. We divide the proof into several steps.

- We consider the funcionals

$$\Phi_h : L^2(\Omega) \longrightarrow \overline{\mathbb{R}}$$

$$\Phi_h(v) := \begin{cases} \int_{A_h} |Dv|^2\,dx & \text{if } v \in H_0^1(A_h) \\ +\infty & \text{if } v \notin H_0^1(A_h) \end{cases}$$

Using Theorem 1.77, it is possible to extract a subsequence (still called $\{\Phi_h\}$) that $\Gamma-$converges to a functional Φ in $L^2(\Omega)$. The limit functional is a quadratic form on $L^2(\Omega)$ with domain $D(\Phi)$ contained in $H_0^1(\Omega)$. Since $\{u_h\}$ is weakly convergent it is also strongly convergent in $L^2(\Omega)$ and bounded in $H_0^1(\Omega)$, then

$$\Phi(u) \leq \liminf_{h \to +\infty} \Phi_h(u_h) \leq \liminf_{h \to +\infty} \int_\Omega |Du_h|^2\,dx < +\infty$$

that is $u \in D(\Phi)$.

- Let V be the closure of $D(\Phi)$ in $L^2(\Omega)$ and let

$$B : D(\Phi) \times D(\Phi) \longrightarrow \mathbb{R}$$

be the bilinear form defined by

$$B(v,w) = \frac{1}{4}(\Phi(v+w) - \Phi(v-w)).$$

We define the linear operator $A : D(A) \longrightarrow L^2(\Omega)$ as $Av = f$,

$$D(A) = \Big\{ v \in D(\Phi) \,|\, \exists f \in V \text{ s.t. } B(v,w) = \langle f, w \rangle_{L^2(\Omega)} \, \forall w \in D(\Phi) \Big\}.$$

From [13, Chapter 12, Theorem 12.17] it follows that $D(A)$ is dense in $D(\Phi)$ with respect to the graph norm associated to Φ, namely

$$||v||_\Phi := \big(||v||^2_{L^2(\Omega)} + \Phi(v) \big)^{\frac{1}{2}}.$$

From the definition of $\Gamma-$convergence it is straightforward that if $F_h \xrightarrow{\Gamma} F$, $G_h \xrightarrow{\Gamma} G$ and $F_h \le G_h$ then $F \le G$. Using this fact we prove that $||v||_\Phi \ge ||v||_{H^1_0(\Omega)}$, so we obtain that $D(A)$ is dense in $D(\Phi)$ with respect to the strong topology of $H^1_0(\Omega)$. We have proved above that $u \in D(\Phi)$, so it is enough to show that for every $v \in D(A)$, we have $v = 0$ q.e. on $\{w = 0\}$ because, by a density argument, we obtain the result also for u.

- Let v be a fixed function in $D(A)$ and let $f = Av$, then v is the minimizer for the functional

$$\Psi(z) = \frac{1}{2}\Phi(z) - \langle f, z \rangle_{L^2}$$

(for a proof of this claim see [13, Chapter 12, Proposition 12.12]). Let v_h be the minimizer of the functional

$$\Psi_h(z) = \frac{1}{2}\Phi_h(z) - \langle f, z \rangle_{L^2}$$

i.e. the solution of

$$-\Delta v_h = f \quad \text{in } A_h, \; v_h \in H^1_0(A_h).$$

From the definition of Φ_h, it follows that

$$\Psi_h(z) \geq \frac{1}{2}||z||^2_{H^1_0(\Omega)} - \langle f, z\rangle \quad \forall\, z \in H^1_0(\Omega).$$

Then the sequence is equicoercive and by the stability under continuous perturbations, Theorem 1.75 and 1.80, we obtain that $v_h \rightharpoonup v$ weakly in $H^1_0(\Omega)$.

- For every $\varepsilon > 0$ there exists $f^\varepsilon \in L^\infty(\Omega)$ such that $||f^\varepsilon - f||_{L^2(\Omega)} < \varepsilon$. Let $v^\varepsilon_h \in H^1_0(A_h)$ be the solution of $-\Delta v^\varepsilon_h = f^\varepsilon$ in A_h, there exists $c > 0$ such that
$$||v^\varepsilon_h - v_h||_{H^1_0(\Omega)} \leq c||f^\varepsilon - f||_{L^2(\Omega)} < c\varepsilon$$
and there exists a subsequence (still denoted by $\{v^\varepsilon_h\}_h$) convergent to a function v^ε weakly in $H^1_0(\Omega)$. Moreover, using the weak lower semicontinuity of the norm and the inequality above, we obtain
$$||v^\varepsilon - v||_{H^1_0(\Omega)} < c\varepsilon.$$
Then it is enough to show that $v^\varepsilon = 0$ q.e. on $\{w = 0\}$ for every $\varepsilon > 0$. Since $-\Delta v^\varepsilon_h = f^\varepsilon$ and $f^\varepsilon \in L^\infty(\Omega)$, we have
$$0 = \frac{f^\varepsilon}{||f^\varepsilon||_{L^\infty}} + \Delta\Big(\frac{v^\varepsilon_h}{||f^\varepsilon||_{L^\infty}}\Big) \leq 1 + \Delta\Big(\frac{v^\varepsilon_h}{||f^\varepsilon||_{L^\infty}}\Big).$$
By (2.10) we get $|v^\varepsilon_h| \leq ||f^\varepsilon||_{L^\infty} w_{A_h}$ q.e. on Ω and, in particular, $v^\varepsilon = 0$ q.e. on $\{w = 0\}$.

$\square$

Lemma 2.30. *Let A and $\{A_h\}$ be quasi-open subsets of Ω such that w_{A_h} converges weakly in $H^1_0(\Omega)$ to a function w, with $w \leq w_A$ q.e. in Ω. Let $A^\varepsilon = \{w_A > \varepsilon\}$ and suppose that $w_{A_h \cup A^\varepsilon}$ converges to some function w^ε weakly in $H^1_0(\Omega)$. Then $w^\varepsilon \leq w_A$ q.e. in Ω.*

Proof. For every $\varepsilon > 0$ we define

$$v^\varepsilon := 1 - \frac{1}{\varepsilon}(w_A \wedge \varepsilon) \;\in H^1_0(\Omega).$$

Then $0 \leq v^\varepsilon \leq 1$ q.e. on Ω, $v^\varepsilon = 0$ q.e. on A^ε, and $v^\varepsilon = 1$ q.e. on $\Omega \setminus A$. We put

$$u_h := v^\varepsilon \wedge w_{A_h \cup A^\varepsilon}$$

then $u_h = 0$ q.e. on A^ε and $u_h = 0$ q.e. on $\Omega \setminus (A_h \cup A^\varepsilon)$, so that $u_h = 0$ q.e. on $\Omega \setminus A_h$. Since $\{u_h\}$ converges to $v^\varepsilon \wedge w^\varepsilon$ weakly in $H^1_0(\Omega)$, we can apply previous lemma, obtaining $v^\varepsilon \wedge w^\varepsilon = 0$ q.e. on $\{w = 0\}$ and in particular $v^\varepsilon \wedge w^\varepsilon = 0$ q.e. on $\Omega \setminus A$. Since $v^\varepsilon = 1$ q.e. on $\Omega \setminus A$, we have $w^\varepsilon = 0$ q.e. on $\Omega \setminus A$. Since $-\Delta w_{A_h \cup A^\varepsilon} \leq 1$ on Ω, we get $-\Delta w^\varepsilon \leq 1$ on Ω, and this give us

$$w^\varepsilon \leq w_A \quad \text{q.e. on } \Omega.$$

□

Lemma 2.31. *Let $A \in \mathcal{A}(\Omega)$ and let $\{w_h\}$ be a sequence in $\mathcal{K}$ converging to w_A strongly in $L^2(\Omega)$. Then*

$$F(A) \leq \liminf_{h \to +\infty} J(w_h).$$

Proof. By the definition of J, for every h there exists $A_h \in \mathcal{A}(\Omega)$ such that $w_{A_h} \leq w_h$ and

$$F(A_h) < J(w_h) + \frac{1}{h}.$$

The sequence $\{w_{A_h}\}$ is bounded in $H^1_0(\Omega)$, as ensured by Proposition 2.21. Then there exists a subsequence (still denoted by $\{w_{A_h}\}$) that converges weakly in $H^1_0(\Omega)$ (and strongly in $L^2(\Omega)$) to a function $w \in \mathcal{K}$. Since w_h converges to w_A strongly in $L^2(\Omega)$ and $w_{A_h} \leq w_h$ then $w \leq w_A$ a.e. and, using Theorem 1.50, $w \leq w_A$ q.e. in Ω. Now, letting $A^\varepsilon := \{w_A > \varepsilon\}$, up to subsequence we can assume that $\{w_{A_h \cup A^\varepsilon}\} \subset \mathcal{K}$ is weakly convergent to a function w^ε. Using the previous lemma, we get $w^\varepsilon \leq w_A$ q.e.. One can easily seen that $w_{A^\varepsilon} = (w_A - \varepsilon)^+$. The inequality $w_{A^\varepsilon} \leq w_{A_h \cup A^\varepsilon}$ is a direct consequence of the inclusion $A^\varepsilon \subset A_h \cup A^\varepsilon$, and this implies $(w_A - \varepsilon)^+ \leq w^\varepsilon$. Since $\{w^\varepsilon\}_{\varepsilon > 0}$ is uniformly bounded in $H^1_0(\Omega)$ and since we have the inequality

$$(w_A - \varepsilon)^+ \leq w^\varepsilon \leq w_A,$$

we get that, up to subsequence, $\{w^\varepsilon\}$ weakly converges to w_A in $H^1_0(\Omega)$. Using a diagonalization argument we can find a sequence $\varepsilon_h \to 0$ such that

$$w_{A_h \cup A^{\varepsilon_h}} \rightharpoonup w_A \quad \text{weakly in } H^1_0(\Omega).$$

Using Propositions 2.6 and 2.7 we get that $\{A_h \cup A^{\varepsilon_h}\}$ $\gamma-$converges to A and, by the lower semicontinuity of F, we have

$$F(A) \leq \liminf_{h\to+\infty} F(A_h \cup A^{\varepsilon_h}) \leq \liminf_{h\to+\infty} F(A_h) = \liminf_{h\to+\infty} J(w_h).$$

□

Proposition 2.32. *If $A \in \mathcal{A}(\Omega)$, then*

$$G(w_A) = F(A).$$

Proof. Let $A \in \mathcal{A}(\Omega)$, by definition of G we get

$$G(w_A) \leq J(w_A) \leq F(A).$$

On the other hand, by Lemma 2.25 we can find a sequence $\{w_h\}$ converging to w_A in $L^2(\Omega)$ such that $G(w_A) = \lim_{h\to+\infty} J(w_h)$. Finally, by previous lemma, we obtain

$$F(A) \leq \liminf_{h\to+\infty} J(w_h) = \lim_{h\to+\infty} J(w_h) = G(w_A)$$

□

2.2.4 Existence of optimal shape

Now we are in a position to prove Theorem 2.11.

***Proof of Theorem** 2.11.* We divide the proof into several steps. Let $c > 0$ be fixed.

- First observe that in order to prove the existence of a solution of problem

  $$(\mathbf{P_c}) \quad \min\{F(A) \,|\, A \in \mathcal{A}(\Omega),\ |A| = c\},$$

  it is enough to study the problem

  $$(\widetilde{\mathbf{P}}_\mathbf{c}) \quad \min\{F(A) \,|\, A \in \mathcal{A}(\Omega),\ |A| \leq c\}.$$

  Indeed, if $\widetilde{A}$ is a minimum point for $(\widetilde{\mathbf{P}}_\mathbf{c})$, then there exists $A \in \mathcal{A}(\Omega)$ such that $A \supseteq \widetilde{A}$ and $|A| = c$. If B is an arbitrary element of $\mathcal{A}(\Omega)$ with $|B| = c$, then

  $$F(A) \leq F(\widetilde{A}) \leq F(B),$$

  where the first inequality is due to the fact that F is decreasing.

- In order to prove the existence of a solution of problem $(\widetilde{\mathbf{P}}_{\mathbf{c}})$ we consider the problem
$$\min\{G(w) \,|\, w \in \mathcal{K},\ |\{w > 0\}| \le c\}. \tag{2.11}$$
We remember that $\mathcal{K}$ is compact with respect to the strong topology of $L^2(\Omega)$ and that G is lower semicontinuous with respect to the same topology. Moreover the subset $\{w \in \mathcal{K} \,\big|\, |\{w > 0\}| \le c\}$ is closed. Indeed, let $\{w_h\} \subset \mathcal{K}$ be a sequence converging in L^2 to $w \in \mathcal{K}$ and such that $|\{w_h > 0\}| \le c$ for every h. Then, up to subsequence, one has $w_h \to w$ a.e. and
$$\mathbf{1}_{\{w>0\}} \le \liminf_{h \to +\infty} \mathbf{1}_{\{w_h>0\}}.$$
Using the Fatou's Lemma, we obtain $|\{w > 0\}| \le c$. In conclusion, by the direct method, there exists a solution w^* for (2.11).

- Now we claim that $\widetilde{A} := \{w^* > 0\}$ is a minimizer for $(\widetilde{\mathbf{P}}_{\mathbf{c}})$. Indeed, from 2.10 it follows that $w^* \le w_{\widetilde{A}}$ q.e. in Ω. From the monotonicity of G and the previous proposition, we get
$$F(\widetilde{A}) = G(w_{\widetilde{A}}) \le G(w^*).$$
Let $A \in \mathcal{A}(\Omega)$ be arbitrary. By Lemma 2.20, it follows that $w_A \in \mathcal{K}$. If $|A| \le c$, then $|\{w_A > 0\}| \le c$ and $G(w^*) \le G(w_A) = F(A)$. Finally,
$$F(\widetilde{A}) \le F(A).$$
From the arbitrariness of A, it follows that $\widetilde{A}$ is a minimizer for $(\widetilde{\mathbf{P}}_{\mathbf{c}})$.

$\square$

Chapter 3

The case of spectral functionals

In this chapter, following [21], we focus our attention on spectral functionals. We consider $\{\lambda_j^L(\Omega)\}$, the eigenvalues of an elliptic and coercive operator L with Dirichlet boundary condition. If $L = -\Delta$, we simply write $\{\lambda_j(\Omega)\}$. Let $c > 0$ and $k \in \mathbb{N}$ be fixed. The generic spectral optimization problem can be formulated as

$$\min\{F(\lambda_1^L(\Omega), ..., \lambda_k^L(\Omega)) \,|\, \Omega \subseteq \mathbb{R}^n, \Omega \text{ quasiopen }, |\Omega| = c\}.$$

where $F : \mathbb{R}^k \longrightarrow \mathbb{R}$ is a given function. We recall some relevant results.

- The Faber-Krahn inequality asserts that the solution of

  $$\min\{\lambda_1(\Omega) \,|\, \Omega \subseteq \mathbb{R}^n, \Omega \text{ open }, |\Omega| = c\}$$

  is the ball of volume c.

- The solution of

  $$\min\{\lambda_2(\Omega) \,|\, \Omega \subseteq \mathbb{R}^n, \Omega \text{ open }, |\Omega| = c\}$$

  consists of two equal and disjoint balls of volume $\frac{c}{2}$. This is the Krahn-Szegö inequality.

- The solution of

  $$\min\left\{\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \,|\, \Omega \subseteq \mathbb{R}^n, \Omega \text{ open and with finite measure}\right\}$$

  is the ball. This result is due to Ashbaugh and Benguria.

For proofs and other examples we refer to [20]. It is interesting to study the problem

$$\min\{\lambda_k(\Omega) \,|\, \Omega \subseteq \mathbb{R}^n, \Omega \text{ open }, |\Omega| = c\}.$$

For $k \geq 3$ very few answers are available. We summarize the main results:

- when $n = 2$ and $k = 3$ it is conjectured that the minimizer is the disc;
- when $n = k = 3$ it has been observed numerically that the minimizer is not the ball, [21, Chapter 11];
- when $n = 2$ and $k \geq 5$, then the ball cannot be minimizer (see [2]).

From these examples we can understand that it is not possible to compute explicitly the solution of these problems, and it is necessary to prove the existence in an abstract way. Moreover, we can investigate some qualitative properties of the optimal set, e.g.:

- Is its perimeter finite?
- Is it bounded?
- Is its boundary smooth ?
- Does it have any symmetry ?

One expects to have smooth open sets as minimizers but this is a regularity problem that we will not consider.

3.1 Bounded design region

Let $\Omega \subset \mathbb{R}^n$ be bounded and open, and let $c \in [0, |\Omega|]$ be fixed. In this section we obtain a local existence result for the problem

$$\min\{F(\lambda_1^L(A), ..., \lambda_k^L(A)) \,|\, A \subseteq \Omega,\ A \text{ quasiopen}, |A| = c\}. \tag{3.1}$$

We say that the result is local for the presence of the bounded region Ω. Let us recall some facts about eigenvalue theory of elliptic operators.

Definition 3.1. Given $A \in \mathcal{A}(\Omega)$, we say that $\lambda = \lambda^L(A)$ is an *eigenvalue* of L in A if there exists a non-trivial solution $u \in H^1_0(A)$ of the problem

$$Lu = \lambda u \quad \text{in } A$$

in the sense of (2.3), where L is an elliptic operator that satisfies the assumptions stated at the beginning of Chapter 2, namely (2.1) and (2.2).

Also in the case of quasiopen subsets we have the following classical result:

Proposition 3.2. *Let L be an elliptic operator that satisfies* (2.1) *and* (2.2). *Then*

$$\lambda_k^L(A) := \min_{E_k} \left\{ \max_{v \in E_k \setminus \{0\}} \left\{ \frac{\int_A A(x)|Du|^2\,dx + \int_A a(x)u^2\,dx}{\int_A v^2\,dx} \right\} \right\}$$

where the minimum is over all the k-dimensional subspaces of $H^1_0(A)$.

For a proof see [22, Corollary 4.7.4].

Lemma 3.3. *Let $\{T_n\}$ be a sequence of compact operators on a Hilbert space converging in the operator norm to an operator T. Let $\lambda_m(T)$ be an enumeration of non-zero eigenvalues of the T, each repeated with its multiplicity. There exists an enumeration $\lambda_m(T_n)$ of the eigenvalues of T_n, each repeated with its multiplicity, such that*

$$\lim_{n \to +\infty} \lambda_m(T_n) = \lambda_m(T),$$

and the limit is uniform in m.

A possible reference for a proof is [16, Chapter XI, Lemma XI.9.5].

Proposition 3.4 (Eigenvalues of an elliptic operator)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded and open, and consider $A \in \mathcal{A}(\Omega)$. Let L be an elliptic operator that satisfies* (2.1) *and* (2.2).

- *If $C_p(A, \Omega) > 0$, using Corollary 2.3, we get that the eigenvalues can be arranged in a non-decreasing sequence $\lambda_k^L(A) = 1/\mu_k(A)$, where each eigenvalue is repeated according to its multiplicity. Moreover, $\lambda_k^L(A) > 0$ for every k and $\lim_{k \to +\infty} \lambda_k^L(A) = +\infty$.*

- *If $C_p(A) = 0$, then $H^1_0(A) = \{0\}$. In this case we put $\lambda^L_k(A) = +\infty$ for every k.*

For every k the map

$$\lambda^L_k : \mathcal{A}(\Omega) \longrightarrow \overline{\mathbb{R}}$$

is continuous with respect to $\gamma-$convergence.

Proof. Consider a sequence $\{A_h\} \subset \mathcal{A}(\Omega)$ that is $\gamma-$convergent to a set $A \in \mathcal{A}(\Omega)$. If $C_p(A, \Omega) > 0$, by Lemma 3.3 and Proposition 2.6, we have that each eigenvalue of $T^L_{A_h}$ converges to the corresponding eigenvalue of T^L_A. If $C_p(A, \Omega) = 0$, then $R^L_A = 0$ and each eigenvalue of T^L_A converges to 0, and this implies that $\lim_{h\to+\infty} \lambda^L_k(A_h) = +\infty = \lambda^L_k(A)$. □

Theorem 3.5 (Domains with minimal *k-th* eigenvalue)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded and open, and consider $A \in \mathcal{A}(\Omega)$. Let L be an elliptic operator that satisfies* (2.1) *and* (2.2). *Then, for every $c \in [0, |\Omega|]$ there exists a minimizer for problems*

$$\min\Big\{\lambda^L_k(A) \,\Big|\, A \in \mathcal{A}(\Omega), |A| = c\Big\}, \tag{3.2}$$

$$\min\Big\{\Phi(\lambda^L(A)) \,\Big|\, A \in \mathcal{A}(\Omega), |A| = c\Big\}, \tag{3.3}$$

where $\lambda^L(A)$ denotes the sequence $\left\{\lambda^L_k(A)\right\}$ and the function $\Phi : \mathbb{R}^{\mathbb{N}} \longrightarrow \overline{\mathbb{R}}$ satisfies the following conditions:

$$\lambda^h_k \to \lambda_k \;\; \forall k \in \mathbb{N} \Rightarrow \Phi(\lambda) \le \liminf_{h\to+\infty} \Phi(\lambda^h),$$

$$\lambda_k \le \mu_k \;\; \forall k \in \mathbb{N} \Rightarrow \Phi(\lambda) \le \Phi(\mu).$$

Proof. The mappings $A \mapsto \lambda^L_k(A)$ are decreasing with respect to set inclusion (this follows from the min-max formula given by Proposition 3.2). Moreover, they are continuous with respect to $\gamma-$convergence, as proved in the previous proposition. Therefore we can apply Theorem 2.11 and, for every k and $c \in [0, |\Omega|]$, we obtain the existence of a minimizer for (3.2). In a similar way we prove the existence for (3.3). □

In particular, we have the following

Corollary 3.6. *Let $F : \mathbb{R}^k \longrightarrow \mathbb{R}$ be lower semicontinuous and non-decreasing in each variable. Then problem* (3.1) *has at least one solution.*

This theorem can be proved by a direct proof that uses results of eigenvalue theory. For a proof see [26] or [21].

3.2 Global existence results

In this section we deal with the problem

$$\min\{F(\lambda_1(\Omega), ..., \lambda_k(\Omega)) \,|\, \Omega \subseteq \mathbb{R}^n, \Omega \text{ quasiopen }, |\Omega| = c\}, \tag{3.4}$$

where $\{\lambda_j(\Omega)\}$ are the eigenvalues of Dirichlet-Laplacian. The passage from a bounded design region to the whole $\mathbb{R}^n$ is not trivial because we can not use the compact embeddings for Sobolev spaces. A possible way to study this problem is based on the concentration-compactness principle of Pierre-Louis Lions, as done in [8] (see also [6] or [7]). The concentration-compactness principle (see [25]) tries to investigates on "how" the embedding $H^1(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ can be non compact describing the behaviour in $L^2(\mathbb{R}^n)$ of a bounded sequence $\{u_h\} \subset H^1(\mathbb{R}^n)$. Precisely, three situations may occur for a subsequence: compactness (possibly making some translations), vanishing or dichotomy. First we need some definitions.

Definition 3.7. Let E be a subset of $\mathbb{R}^n$. We define the 2-*capacity* of E as

$$C_{1,2}(E) = \inf\Big\{ \int_{\mathbb{R}^n} |Du|^2 + u^2\, dx \,\Big|\, u \in W^{1,2}(\mathbb{R}^n),\ u \geq 1 \text{ a.e. on } U \in \mathcal{U}_E \Big\}$$

where $\mathcal{U}_E$ is the family of open neighborhoods of E. If it is not ambiguous we use also the notation $C_2(E)$.

It is possible to give the definition also for $p \neq 2$, but the Hilbert case is more interesting for our purpose. The capacity here defined has the same properties of the one defined for bounded subset (Chapter 1), but for proof and details we refer to classical bibliography (e.g., [22, 7, 18]). We stress that also in this case we have the following results.

Theorem 3.8. *Every function $u \in H^1(\mathbb{R}^n)$ has a quasicontinuous representative, which is unique up to null capacity sets.*

Proposition 3.9. *Let $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a quasicontinuous function. Then the set $\{u > \alpha\}$ is quasiopen for every $\alpha \in \mathbb{R}$.*

Definition 3.10. For every $A \in \mathcal{A}(\mathbb{R}^n)$ we denote by $H^1_0(A)$ the space of all functions $u \in H^1(\mathbb{R}^n)$ such that $u = 0$ q.e. on $\mathbb{R}^n \setminus A$.

Proposition 3.11. *If A is open, then the previous definition of $H^1_0(A)$ is equivalent to the classical one, that is the closure of $C^\infty_c(A)$ with respect to the H^1 norm.*

Definition 3.12. A positive Borel measure μ is called *capacitary* if it is absolutely continuous with respect to the capacity, i.e. $C_2(A) = 0 \Rightarrow \mu(A) = 0$. We define $\mathcal{M}_0(\mathbb{R}^n)$ as the set of capacitary measure on $\mathbb{R}^n$.

Definition 3.13. Given $\mu \in \mathcal{M}_0(\mathbb{R}^n)$, we define its regular set A_μ as the union of all open sets $A \subset \mathbb{R}^n$ such that $\mu(A) < +\infty$.

Definition 3.14. Given $\mu \in \mathcal{M}_0(\mathbb{R}^n)$, if A_μ has finite measure, we define

$$R_\mu : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

the *resolvent operator* associated to μ as $R_\mu(f) = u$, where u is the unique solution of

$$\int_{\mathbb{R}^n} Du \cdot Dv \, dx + \int_{\mathbb{R}^n} uv \, d\mu = \int_{\mathbb{R}^n} fv \, dx \quad \forall \, v \in H^1(\mathbb{R}^n) \cap L^2_\mu(\mathbb{R}^n).$$

Since $|A_\mu| < +\infty$, we have the following.

Proposition 3.15. *The space $H^1_0(A_\mu)$ is compactly embedded in $L^2(A_\mu)$. Moreover R_μ is compact, positive and self-adjoint.*

Corollary 3.16. *We can define the eigenvalues associated to the measure μ as the eigenvalues of the elliptic operator $-\Delta + \mu I$, i.e. $\lambda_h(\mu) := \frac{1}{\omega_h(R_\mu)}$ where $\{\omega_h(R_\mu)\}$ are the eigenvalues of R_μ.*

Remark 3.17. The definition of resolvent operator given in Chapter 2 for quasi open subsets is a particular case of the one given for capacitary measure. For every quasiopen set A such that $|A| < +\infty$, we define the measure

$$\mu_A(E) := \begin{cases} 0 & \text{if } C_2(E \setminus A) = 0 \\ +\infty & \text{if } C_2(E \setminus A) > 0 \end{cases}$$

In this case $H^1(\mathbb{R}^n) \cap L^2_{\mu_A}(\mathbb{R}^n) = H^1_0(A)$, $R_{\mu_A} = R_A$ (the resolvent defined in the second chapter) and $\lambda_h(\mu) = \lambda_h(A)$.

For other details see [7, Section 4.3, Section 6.1] and [11]. Now we explain the concentration-compactness principle in the case of subsets of $\mathbb{R}^n$ (see [6]).

Theorem 3.18. *Let $\{A_h\}$ be a sequence of quasiopen sets of $\mathbb{R}^n$, each of measure equal to c. Then one of the following situations occurs:*

- *(Compactness) There exists a subsequence $\{A_{h_k}\}$, a sequence of vectors $\{y_k\} \subset \mathbb{R}^n$ and a positive capacitary measure μ, such that*

$$\lim_{k\to+\infty} ||R_{y_k+A_{h_k}} - R_\mu||_{\mathcal{L}(L^2(\mathbb{R}^n))} = 0$$

- *(Dichotomy) There exists a subsequence $\{A_{h_k}\}$ and a sequence of subsets $\widetilde{A}_k \subseteq A_{h_k}$ such that*

$$\lim_{k\to+\infty} ||R_{A_{h_k}} - R_{\widetilde{A}_k}||_{\mathcal{L}(L^2(\mathbb{R}^n))} = 0 \quad \textit{and} \quad \widetilde{A}_k = A_k^1 \cup A_k^2$$

  with $d(A_k^1, A_k^2) \to +\infty$ and $\liminf_{k\to+\infty} |A_k^i| > 0$ for $i = 1, 2$.

In [8] Bucur and Henrot use this result to prove the existence of a minimizer of the third eigenvalue of the Dirichlet Laplacian. Precisely, a minimizing sequence $\{A_h\}$ can be either in the compactness case, in which we get existence (the proof is similar to the one of Bucur and Dal Maso), or in the dichotomy case. In this situation, the problem can be reduced to finding the minimizers of first and second eigenvalue, that are solved problems (see the beginning of the chapter). The theorem proved by Bucur and Henrot is the following.

Theorem 3.19. *For $k \geq 2$ if there exists a bounded minimizer of $\lambda_1, ..., \lambda_k$ in the class $\mathcal{A}(\mathbb{R}^n)$, then there exists at least a minimizer of λ_{k+1} in $\mathcal{A}(\mathbb{R}^n)$.*

Since the minimizers for λ_1 and λ_2 are bounded, we get the following corollary.

Corollary 3.20. *There exists at least a minimizer for*

$$\min\{\lambda_3(\Omega) \,|\, \Omega \subseteq \mathbb{R}^n, \Omega \textit{ quasiopen }, |\Omega| = c\}.$$

Since it is not known if minimizers of λ_3 are bounded, Bucur studied this problem in a different way. More precisely, in [5], Bucur give the definition of *shape subsolution for the torsion energy*, and he proved that each subsolution is a bounded set with finite perimeter. On the other hand, Bucur proved that minimizers for

$$\min\{\lambda_k(\Omega) \,|\, \Omega \subseteq \mathbb{R}^n, \Omega \text{ quasiopen }, |\Omega| = c\}$$

are shape subsolutions, so they are bounded. In this way it is possible to use the previous theorem.
The global existence result was proved independently also by Mazzoleni and Pratelli in [27] using a different technique: a *surgery* result.
Now, we follow [21] giving the proof of the global existence result, using Lemma 3.22 as a fundamental tool, for which we give only an idea of the proof in the following section. This Lemma is based on a combination of the two methods, shape subsolutions and surgery, as done in [9]. For the complete proof and details we refer to the bibliography. First we recall the definition of perimeter.

Definition 3.21. Let Ω be an open subset of $\mathbb{R}^n$ and let $A \subset \Omega$ be a measurable set. The *perimeter* of A in Ω is defined as follows

$$P(A, \Omega) := \sup\left\{\int_\Omega \chi_A(x)\,\mathrm{div}\boldsymbol{\phi}(x)\,\mathrm{d}x \colon \boldsymbol{\phi} \in C_c^\infty(\Omega, \mathbb{R}^n),\ \|\boldsymbol{\phi}\|_{L^\infty(\Omega)} \leq 1\right\}$$

In the case $\Omega = \mathbb{R}^n$ we use the notation $P(A)$ instead of $P(A, \Omega)$.

Lemma 3.22 (**Surgery**). *For every K, $c > 0$, there exist $D, C > 0$ depending only on K, c and the dimension n such that, for every quasiopen set $A \subset \mathbb{R}^n$ with $|A| = c$, there exists a quasiopen set $\widetilde{A}$ with $|\widetilde{A}| = c$,* $\mathrm{diam}(\widetilde{A}) \leq D$, *$P(\widetilde{A}) \leq C$ and, if for some k we have $\lambda_k(A) \leq K$, then*

$$\lambda_i(\widetilde{A}) \leq \lambda_i(A) \quad 1 \leq i \leq k.$$

Moreover, if $P(A) > C$, the inequalities are strict.

Theorem 3.23 (**Bucur, Mazzoleni, Pratelli**). *Let $F : \mathbb{R}^k \longrightarrow \mathbb{R}$ be nondecreasing in each variable and lower semicontinuous. Then the problem*

$$\min\{F(\lambda_1(A), ..., \lambda_k(A)) \,|\, A \subseteq \mathbb{R}^n, A \text{ quasiopen }, |A| = c\} \qquad (3.5)$$

has at least one solution. If F is strictly increasing in at least one variable, then every solution is a bounded set with finite perimeter.

Proof. Let $\{A_h\}$ be a minimizing sequence. One can choose A_h such that $\lambda_i(A_h) \leq K$ for all i and h and some suitable value K. Indeed, we recall that

$$0 < \lambda_1(A_h) \leq \lambda_2(A_h) \leq ... \leq \lambda_k(A_h).$$

Suppose there exists $i_0 \in \{1, ..., k\}$ such that $\lambda_{i_0}(A_h) \to +\infty$ as $h \to +\infty$, then $\lambda_i(A_h) \to +\infty$ as $h \to +\infty$ for $i = i_0+1, ..., k$. If $i_0 = 1$, by monotonicity assumption, F must be constant and the problem is trivial. If $i_0 > 1$ then the sequences $\{\lambda_1(A_h)\}, ..., \{\lambda_{i_0-1}(A_h)\}$ are bounded by a constant $K > 0$, so, up to subsequence, they converge to the real numbers $l_1, ..., l_{i_0-1}$. By monotonicity assumption, F must be constant in the last $k-i_0+1$ variables, so we can study the problem only in the first $i_0 - 1$ (we have formally a new problem with $k = i_0 - 1$). In conclusion, we can assume that $\lambda_i(A_h) \leq K$ for all i and h and some suitable value $K > 0$. Then, we can use the Surgery Lemma obtaining a new sequence $\{\widetilde{A}_h\}$ with uniformly bounded diameter, that satisfies the measure constraint and

$$\lambda_i(\widetilde{A}_h) \leq \lambda_i(A_h), \quad 1 \leq i \leq k.$$

The monotonicity of F implies that this new sequence is also minimizing. Since the sequence is equibounded in diameter, i.e. $\operatorname{diam}(\widetilde{A}_h) \leq D$, we can study the problem in a ball of diameter D where we can apply the result of Bucur-Dal Maso.
Assume now that A is an optimal set and F is strictly increasing in at least one variable. If $P(A) > C$, then the inequality in the Surgery lemma is strict and $\widetilde{A}$ is a new minimizer. Since F is strictly increasing, this is a contradiction since A was optimal. $\square$

3.2.1 Some remarks on the Surgery Lemma

In this subsection we describe the main ideas of the proof of Surgery Lemma without going into details. For the complete proof we refer to [9].

Definition 3.24. We define the *torsion energy* functional $E : \mathcal{A}(\mathbb{R}^n) \longrightarrow \mathbb{R}$ as

$$E(A) := \min_{u \in H_0^1(A)} \frac{1}{2} \int_{\mathbb{R}^n} |Du|^2 \, dx - \int_{\mathbb{R}^n} u \, dx$$

The unique function that minimizes $E(A)$ is called *torsion function.* It is the function $w_A \in H_0^1(A)$ that satisfies $-\Delta w_A = 1$ in A in the weak sense.

Definition 3.25. We say that a quasiopen set $\Omega \subset \mathbb{R}^n$ is a *local shape subsolution* for the torsion energy if there exists $\varepsilon, \delta > 0$ such that, for all quasiopen sets $A \subseteq \Omega$ with the property that $\int_{\mathbb{R}^n} (w_\Omega - w_A) \, dx < \delta$, we have

$$E(\Omega) + \varepsilon|\Omega| \leq E(A) + \varepsilon|A|.$$

Lemma 3.26. *Let Ω be a local shape subsolution for the torsion energy. There exists $r_0 > 0$, $C_0 > 0$ such that for every $x_0 \in \mathbb{R}^n$ and $r \in (0, r_0)$,*

$$\sup_{x \in B_r(x_0)} w_\Omega(x) \leq C_0 r \;\; \Rightarrow \;\; w_\Omega = 0 \;\; on \; B_{r/2}(x_0).$$

The proof can be found in [5, Lemma 1], which is a particular case of [1, Lemma 3.4]. The main result proved by Bucur in [5] is the following.

Theorem 3.27. *Assume Ω is a local shape subsolution for the torsion energy. Then Ω is bounded and has finite perimeter.*

The proof of the boundedness is a consequence of the previous lemma. The proof of the finiteness of the perimeter is based on some direct computations where the co-area formula is used.

Sketch of the proof of Surgery Lemma. Let $\Omega \subseteq \mathbb{R}^n$ be a quasiopen set of volume c. We solve first the problem

$$(\mathrm{P}_\eta) \qquad \min \left\{ E(A) + \eta|A| \;\middle|\; A \text{ quasiopen, } A \subseteq \Omega \right\}$$

where $\eta > 0$ will be fixed later. For every $\eta > 0$, problem (P_η) has a solution. The existence can be proved by the direct method of the calculus of variations, as a consequence of the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$. Let us denote by Ω_η a solution and

$$\widetilde{\Omega} = \left(\frac{|\Omega|}{|\Omega_\eta|} \right)^{\frac{1}{n}} \Omega_\eta.$$

Since Ω_η is a subsolution for the torsion energy, the diameter and perimeter are controlled by constants depending only on η, c, n, as a consequence of the previous theorem. The same holds for $\widetilde{\Omega}$, up to rescaling factors. We recall that, if $\Omega_1 \subseteq \Omega_2$, for every k then

$$\Big|\frac{1}{\lambda_k(\Omega_1)} - \frac{1}{\lambda_k(\Omega_2)}\Big| \leq 4k^2 e^{1/4\pi} \lambda_k(\Omega_2)^{n/2}(E(\Omega_1) - E(\Omega_2)),$$

(for a proof see [5]). From this, it is possible to obtain

$$\lambda_k(\Omega_\eta) - \lambda_k(\Omega) \leq C(|\Omega|^{\frac{2}{n}} - |\Omega_\eta|^{\frac{2}{n}})$$

that is

$$\lambda_k(\Omega_\eta) + C|\Omega_\eta|^{\frac{2}{n}} \leq \lambda_k(\Omega) + C|\Omega|^{\frac{2}{n}}.$$

The constant C depends on $\eta, \lambda_k(\Omega), c$ and n and it is smaller when η and $\lambda_k(\Omega)$ are smaller. It is possible to choose η small enough to obtain the following inequality

$$\lambda_k(\Omega_\eta)|\Omega_\eta|^{\frac{2}{n}} \leq \lambda_k(\Omega)|\Omega|^{\frac{2}{n}}.$$

We recall now the rescaling property of eigenvalue of the Dirichlet Laplacian

$$\lambda_k(t\Omega) = \frac{1}{t^2}\lambda_k(\Omega), \quad \forall\, t > 0,$$

and, using the definition of $\widetilde{\Omega}$, we get $\lambda_k(\widetilde{\Omega}) \leq \lambda_k(\Omega)$. The inequality holds also for $i = 1, ..., k-1$. Since the set Ω_η is a subsolution for the torsion energy, it is bounded and has finite perimeter, controlled only by Ω, η and n. This holds as well for the set $\widetilde{\Omega}$, with rescaling factors. $\square$

Appendix A

Sobolev spaces

We recall some fact about Sobolev spaces. We refer to [17] for proof and details.

A.1 Basic facts

Definition A.1. Let $U \subseteq \mathbb{R}^n$ be an open subset. Suppose $u, v \in L^1_{loc}(U)$, and let α be a multiindex. We say that v is the $\alpha^{th}-$*weak derivative* of u, and we write $D^\alpha u = v$, if

$$\int_U u D^\alpha \phi \, dx = (-1)^\alpha \int_U v\phi \, dx \quad \forall \phi \in C^\infty_c(U).$$

The weak derivative is unique up to sets of measure zero.

Definition A.2. Let $U \subseteq \mathbb{R}^n$ be an open subset. Let $p \in [1, +\infty]$ and let k be a positive integer. The *Sobolev space* $W^{k,p}(U)$ is the vector space of functions $u : U \longrightarrow \mathbb{R}$ in $L^p(U)$ such that, for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$. We identify functions which agree a.e.

Remark A.3. If $p = 2$ we write

$$H^k(U) = W^{k,2}(U).$$

Definition A.4. We define the norm on $W^{k,p}(U)$ as

$$||u||_{W^{k,p}(U)} := \begin{cases} (\sum_{|\alpha|\leq k} \int_U |D^\alpha u|^p \, dx)^{\frac{1}{p}} & \text{if } p \in [1, +\infty), \\ \sum_{|\alpha|\leq k} \operatorname{ess\,sup}_U |D^\alpha u| & \text{if } p = +\infty. \end{cases}$$

Definition A.5. We say that a sequence $\{u_h\} \subset W^{k,p}(U)$ is *convergent in* $W^{k,p}(U)$ to $u \in W^{k,p}(U)$ if

$$||u_h - u||_{W^{k,p}(U)} \longrightarrow 0 \quad \text{as } h \to +\infty$$

and we write $u_h \longrightarrow u$ in $W^{k,p}(U)$. We say that a sequence $\{u_h\} \subset W^{k,p}(U)$ is *convergent locally in* $W^{k,p}(U)$ to $u \in W^{k,p}(U)$ if

$$||u_h - u||_{W^{k,p}(V)} \longrightarrow 0 \quad \text{as } h \to +\infty$$

for every $V \subset\subset U$ and we write $u_h \longrightarrow u$ in $W^{k,p}_{loc}(U)$.

Definition A.6. We denote by $W^{k,p}_0(U)$ the closure of $C^\infty_c(U)$ in $W^{k,p}(U)$.

Remark A.7. We interpret this space as the space of function such that

$$"D^\alpha u = 0 \quad \text{on } \partial U" \quad \forall\ |\alpha| \leq k-1.$$

If $p = 2$ we write

$$H^k_0(U) = W^{k,2}_0(U).$$

Theorem A.8. *Let $U \subseteq \mathbb{R}^n$ be an open subset. Let $p \in [1, +\infty]$ and let k be a positive integer. The Sobolev space $W^{k,p}(U)$ is a Banach Space.*

A.2 Approximation by smooth functions

Let U be open and let $\varepsilon > 0$, then we put $U_\varepsilon := \{x \in U \,|\, \mathrm{dist}(x, \partial U) > \varepsilon\}$.

Definition A.9. We define the *standard mollifier* $\eta : \mathbb{R}^n \longrightarrow \mathbb{R}$ as

$$\eta(x) := \begin{cases} C\exp(\frac{1}{|x|^2-1}) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

where $C > 0$ is chosen in order to have $\int_{\mathbb{R}^n} \eta\, dx = 1$. For every $\varepsilon > 0$ we define

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta\Big(\frac{x}{\varepsilon}\Big),$$

that satisfies $\int_{\mathbb{R}^n} \eta_\varepsilon\, dx = 1$ and $\mathrm{supp}(\eta_\varepsilon) \subset B(0, \varepsilon)$.

Definition A.10. If $f : U \longrightarrow \mathbb{R}$ is locally integrable we define its *mollification* as

$$f^\varepsilon := \eta_\varepsilon \star f \quad \text{in } U_\varepsilon$$

that is

$$f^\varepsilon(x) = \int_U \eta_\varepsilon(x-y) f(y)\, dy = \int_{B(0,\varepsilon)} \eta_\varepsilon(y) f(x-y)\, dy \quad \forall x \in U_\varepsilon.$$

Theorem A.11 (Properties of mollifiers)**.** *Let $f : U \longrightarrow \mathbb{R}$ be locally integrable, then*

i) $f^\varepsilon \in C^\infty(U_\varepsilon)$*,*

ii) $f^\varepsilon \longrightarrow f$ *a.e. as* $\varepsilon \to 0^+$*,*

iii) *if* $f \in C^0(U)$*, then* $f^\varepsilon \longrightarrow f$ *uniformly on compact subset of* U*,*

iv) *if* $p \in [1, +\infty)$ *and* $f \in L^p_{loc}(U)$*, then* $f^\varepsilon \longrightarrow f$ *in* $L^p_{loc}(U)$*.*

Using the previous theorem one can prove the following

Theorem A.12 (Local approximation by smooth functions)**.** *Let* $u \in W^{k,p}(U)$ *for some* $p \in [1, +\infty)$ *and* $u^\varepsilon := \eta_\varepsilon \star u$ *in* U_ε*. Then*

i) $u^\varepsilon \in C^\infty(U_\varepsilon)$ *for each* $\varepsilon > 0$*,*

ii) $u^\varepsilon \longrightarrow u$ *in* $W^{k,p}_{loc}(U)$*, as* $\varepsilon \to 0^+$*.*

The previous theorem can be improved:

Theorem A.13 (Global approximation by smooth functions)**.** *Assume* U *is bounded, and suppose* $u \in W^{k,p}(U)$ *for some* $p \in [1, +\infty)$*. Then there exists a sequence* $\{u_h\} \subset W^{k,p}(U) \cap C^\infty(U)$ *such that*

$$u_h \longrightarrow u \quad \text{in } W^{k,p}(U).$$

This theorem shows that the sequence can be chosen as $\{u_h\} \subset C^\infty(\overline{U})$.

Theorem A.14 (Global approximation by smooth functions up to the boundary)**.** *Assume* U *is bounded,* ∂U *is* C^1 *and suppose* $u \in W^{k,p}(U)$ *for some* $p \in [1, +\infty)$*. Then there exists a sequence* $\{u_h\} \subset C^\infty(\overline{U})$ *such that*

$$u_h \longrightarrow u \quad \text{in } W^{k,p}(U).$$

A.3 Extension and Trace

Theorem A.15 (Extension Theorem). *Assume U is bounded and ∂U is C^1. Let V be a bounded open set such that $U \subset\subset V$. Then there exists a bounded linear operator*

$$E : W^{1,p}(U) \longrightarrow W^{1,p}(\mathbb{R}^n)$$

such that

i) *$Eu = u$ a.e. in U,*

ii) *the support of Eu is a subset of V,*

iii) *there exists a constant $C = C(p, U, V)$, depending only on p, U and V, such that*

$$||Eu||_{W^{1,p}(\mathbb{R}^n)} \leq C||u||_{W^{1,p}(U)}$$

Definition A.16. Under the hypothesis of previous theorem we call Eu *extension* of u to $\mathbb{R}^n$.

Theorem A.17 (Trace Theorem). *Let U be a bounded open subset and assume ∂U is C^1. Then there exists a bounded linear operator*

$$T : W^{1,p}(U) \longrightarrow L^p(\partial U)$$

such that

i) *$Tu = u|_{\partial U}$ if $W^{1,p}(U) \cap C^0(\overline{U})$,*

ii) *there exists a constant $C = C(p, U)$, depending only on p and U, such that*

$$||Tu||_{L^p(\partial U)} \leq C||u||_{W^{1,p}(U)} \quad \forall\, u \in W^{1,p}(U).$$

Definition A.18. Under the hypotheses of the previous theorem, we call Tu the *trace* of u on ∂U.

Theorem A.19 (Trace-zero functions in $W^{1,p}(U)$). *Let U be a bounded open subset and assume ∂U is C^1. If $u \in W^{1,p}(U)$, then*

$$u \in W_0^{1,p}(U) \;\Leftrightarrow\; Tu = 0 \;\text{ on }\; \partial U.$$

A.4 Sobolev inequalities

Definition A.20. If $p \in [1, n)$, the *Sobolev conjugate* of p is defined as

$$p^* := \frac{np}{n-p}.$$

This is equivalent to

$$\frac{1}{p^*} := \frac{1}{p} - \frac{1}{n}, \quad p^* > p.$$

Theorem A.21 (Gagliardo–Nirenberg-Sobolev inequality)**.** *If $1 < p < n$, then there exists a constant $C = C(p, n)$, such that*

$$||u||_{L^{p^*}(\mathbb{R}^n)} \leq C||Du||_{L^p(\mathbb{R}^n)}, \quad \forall\, u \in C_c^1(\mathbb{R}^n).$$

Using the Gagliardo–Nirenberg-Sobolev inequality, the extension theorem and the density results of previous section it is possible to prove the following estimate:

Theorem A.22 (Estimate for $W^{1,p}(U)$, $1 \leq p < n$)**.** *Let U be a bounded open subset of $\mathbb{R}^n$ and assume ∂U is C^1. Let $u \in W^{1,p}(U)$ with $1 \leq p < n$. Then*

$$||u||_{L^{p^*}(U)} \leq C||u||_{W^{1,p}(U)},$$

where the constant $C = C(p, n, U)$, depends only on p, n and U. In particular, $u \in L^{p^*}(U)$.

Using the estimate for $W^{1,p}(U)$ and the density results of the previous section it is possible to prove the following result:

Theorem A.23 (Estimate for $W_0^{1,p}(U)$, $1 \leq p < n$)**.** *Let U be a bounded open subset of $\mathbb{R}^n$. If $u \in W_0^{1,p}(U)$ with $1 \leq p < n$, then*

$$||u||_{L^q(U)} \leq C||Du||_{L^p(U)}, \quad \forall\, q \in [1, p^*],$$

where the constant $C = C(p, q, n, U)$, depends only on p, q, n and U.

Remark A.24. The inequality above is called *Poincaré's inequality* and implies that the norm $||Du||_{L^p(U)}$ is equivalent on $W_0^{1,p}(U)$ to the usual one.

Definition A.25. Let $\gamma \in (0,1]$. A functions $u : U \longrightarrow \mathbb{R}$ is $\gamma-$*Hölder continuous* if there exists $C > 0$ such that

$$|u(x) - u(y)| \leq C|x-y|^{\gamma} \quad \forall\, x, y \in U$$

and we define $C^{0,\gamma}(\overline{U})$ as the space of all $\gamma-$Hölder continuous functions. We define the $\gamma-$*Hölder seminorm* as

$$[u]_{C^{0,\gamma}(\overline{U})} := \sup_{x,y\in U,\, x\neq y} \left\{ \frac{|u(x)-u(y)|}{|x-y|^{\gamma}} \right\}$$

and the $\gamma-$*Hölder norm* as

$$||u||_{C^{0,\gamma}(\overline{U})} := ||u||_{C^{0}(\overline{U})} + [u]_{C^{0,\gamma}(\overline{U})}$$

where $||u||_{C^0(\overline{U})} := \sup_{x\in U} |u(x)|$.

Definition A.26. For every $\gamma \in (0,1]$ and $k \in \mathbb{N}$, we define the $k, \gamma-$*Hölder space* as the space of all functions $u \in C^k(\overline{U})$ such that

$$||u||_{C^{0,\gamma}(\overline{U})} := \sum_{|\alpha|\leq k} ||D^{\alpha}u||_{C^{0}(\overline{U})} + \sum_{|\alpha|=k} [D^{\alpha}u]_{C^{0,\gamma}(\overline{U})}$$

is finite and we denote it by $C^{k,\gamma}(U)$.

Proposition A.27. *The space $C^{k,\gamma}(U)$ is a Banach space.*

Theorem A.28 (Morrey's inequality)**.** *If $n < p \leq +\infty$, then there exists a constant $C = C(p,n)$, such that*

$$||u||_{C^{0,\gamma}(\mathbb{R}^n)} \leq C||Du||_{W^{1,p}(\mathbb{R}^n)}, \quad \forall\, u \in C^1(\mathbb{R}^n),$$

where $\gamma := 1 - \frac{n}{p}$.

Using this inequality it is possible to prove the following estimate for the case $n < p \leq +\infty$:

Theorem A.29 (Estimate for $W^{1,p}(U)$, $n < p \leq +\infty$)**.** *Let U be a bounded open subset of $\mathbb{R}^n$ and assume ∂U is C^1. If $u \in W^{1,p}(U)$ with $n < p \leq +\infty$, then there exists a representative $u^* \in C^{0,\gamma}(\overline{U})$ with $\gamma := 1 - \frac{n}{p}$ such that*

$$||u^*||_{C^{0,\gamma}(\overline{U})} \leq C||u||_{W^{1,p}(U)},$$

where the constant $C = C(p,n,U)$, depends only on p, n and U.

We finally give the general Sobolev inequalities:

Theorem A.30 (General Sobolev inequality). *let U be a bounded open subset of $\mathbb{R}^n$, with C^1 boundary. Assume $u \in W^{k,p}(U)$ with $p \in [1,+\infty)$, $k \in \mathbb{N}^+$.*

i) *If $k < \frac{n}{p}$ then*

$$W^{k,p}(U) \hookrightarrow L^q(U)$$

where $\frac{1}{q} := \frac{1}{p} - \frac{k}{n}$, more explicitly $u \in L^q(U)$ and

$$||u||_{L^q(U)} \leq C||u||_{W^{k,p}(U)},$$

where the constant C depends only on k, p, n and U.

ii) *If $k = \frac{n}{p}$ then*

$$W^{k,p}(U) \hookrightarrow L^q(U)$$

for every $q < +\infty$, more explicitly $u \in L^q(U)$ and

$$||u||_{L^q(U)} \leq C||u||_{W^{k,p}(U)},$$

where the constant C depends only on k, p, q, n and U.

iii) *If $k > \frac{n}{p}$ then*

$$W^{k,p}(U) \hookrightarrow C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\overline{U})$$

where

$$\gamma := \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p} & \text{if } \frac{n}{p} \notin \mathbb{N}^+ \\ \text{each } \alpha \in (0,1) & \text{if } \frac{n}{p} \in \mathbb{N}^+ \end{cases}$$

more explicitly, there exists $u^* \in C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\overline{U})$, $u^* = u$ a.e. and

$$||u||_{C^{k-[\frac{n}{p}]-1,\gamma}(\overline{U})} \leq C||u||_{W^{k,p}(U)},$$

where the constant C depends only on k, p, n, γ and U.

A.5 Compact embeddings

Definition A.31. Let X and Y be Banach spaces such that $X \subset Y$. The space X is *compactly embedded* in Y if

- there exists a positive constant C such that $||u||_Y \leq C||u||_X \ \forall\, u \in X$,
- the injection is compact, that is, every bounded sequence in X has a convergent subsequence in Y.

In this case we write $X \subset\subset Y$.

Theorem A.32 (Rellich–Kondrachov)**.** *Suppose that U is bounded and of class C^1. Then we have the following compact embeddings:*

i) *if $p < n$, then $W^{1,p}(U) \subset\subset L^q(U)$ for every $q \in [1, p^*)$, where $\frac{1}{p^*} := \frac{1}{p} - \frac{1}{n}$.*

ii) *If $p = n$, then $W^{1,p}(U) \subset\subset L^q(U)$ for every $q \in [p, +\infty)$.*

iii) *If $p > n$, then $W^{1,p}(U) \subset\subset C(\overline{U})$.*

Remark A.33. If U is not bounded, the injection $W^{1,p}(U) \subset L^q(U)$ is not compact in general. Moreover the injection $W^{1,p}(U) \subset\subset L^{p^*}(U)$ is never compact, even if U is bounded and smooth.

A.6 Some functional analysis results

We close this appendix recalling some useful result of functional analysis.

Theorem A.34 (Lax–Milgram)**.** *Let H be an Hilbert space and let*

$$a : H \times H \longrightarrow \mathbb{R}$$

be a bilinear form such that

- *(Coercivity) there exists a constant $\alpha > 0$ such that*

$$a(u, u) \geq \alpha||u||^2 \ \ \forall\, u \in H,$$

- *(Continuity) there exists a constant* $C > 0$ *such that*

$$|a(u,u)| \leq C||u||\,||v|| \ \ \forall\, u,\, v \in H.$$

Then, for every $f \in H^*$*, there exists a unique* $\overline{u} \in H$ *such that*

$$a(\overline{u}, v) = \,<f, v> \qquad \forall\, v \in H.$$

Moreover we have the following estimate

$$||\overline{u}|| \leq \frac{1}{\alpha}||f||_{H^*}.$$

Theorem A.35. *Let* X *be a reflexive Banach space. Then every bounded sequence in* X *has a weakly convergent subsequence.*

Theorem A.36. *Let* X *be a reflexive Banach space and let* C *be a non-empty subset of* X*. Then* C *is closed if and only if it is weakly closed.*

Theorem A.37 (Direct Method)**.** *Let* X *be a reflexive Banach space and let* C *be a non-empty closed and convex subset of* X*. Let*

$$F : C \subseteq X \longrightarrow \mathbb{R} \cup \{+\infty\}$$

be a functional lower semi-continuous with respect to the weak convergence and such that

$$\lim_{||u|| \to +\infty} F(u) = +\infty.$$

Then there exists $u \in C$ *such that*

$$F(u) = \inf_C F.$$

Remark A.38. If F is strictly convex, then the minimum is unique.

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