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Metaplectic mixing in Wigdersons' uncertainty principles

Supervisor: S. Ivan Trapasso

Candidate: Gabriele Gentile

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0.1 Introduction

Following the publication of Avi and Yuval Wigderson's article [1], a novel perspective on the uncertainty principle has emerged. The fundamental concept behind this principle is that a nonzero function and its Fourier transform cannot both be highly localized. A. and Y. Wigderson demonstrated that this limitation is not unique to the Fourier transform; rather, it stems from weaker properties shared by a broad class of operators known as k-Hadamard operators. To capture this, the authors introduced a foundational measure of localization, denoted as H_0, which establishes a "primary uncertainty principle". Their key insight is the creation of a unified framework that proves various uncertainty principles through a simple two-step process: first, a universal bound is established for a single function by relating the specific measure of localization of interest, H, to the primary measure H_0; second, this inequality is applied separately to the function and its transform, and the results are multiplied. While this framework successfully extended the uncertainty principle to a broad class of operators, it left several open questions, such as the multidimensional Heisenberg principle and a specific conjecture concerning the image of the map $f \mapsto \frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_2 \|\hat{f}\|_2}$ for nonzero functions in the Schwartz class. A solution to this conjecture was provided by Tang [2], who demonstrated that it holds true for specific values of q (specifically, $q > 2$). Tang's work is highly significant for this framework because it explicitly incorporates the Hausdorff-Young inequality, enabling a direct comparison between the norms of a function and its transform. Although A. and Y. Wigderson did not use this specific inequality, its inclusion proved crucial. Building on this, Tang [3] successfully established the multidimensional Heisenberg uncertainty principle, demonstrating a lower bound with the correct dependence on the dimension. Expanding on these foundations, Dias et al. [4] noted that the properties of k-Hadamard operators alone are not sufficient to resolve all the higher-dimensional problems left open. To address this, they introduced an additional property, valid for the Fourier transform, and defined the "special k-Hadamard operator". This refinement allows for the application of the Hausdorff-Young inequality within this specific operator class. As in Tang's work, it is this inequality relating the function to its transform that enables the proof of the multidimensional uncertainty principle. Ultimately, Dias et al. extended the Heisenberg principle to a general L^q norm for a dimension-dependent range of q. Finally, Dias, Luef, and Prata observe that metaplectic operators associated with free symplectic matrices, commonly called linear canonical transforms, provide examples of special k-Hadamard operators. Next, we turn our attention to time-frequency representations, building upon the works of E. Cordero and L. Rodino [5, 6, 7]. In [8, 9], K. Gröchenig and I. Shafkulovska present a universal method to prove the Benedicks-type uncertainty principle (in the former), as well as a general method for other uncertainty principles (in the latter). Motivated by

these developments, in this work we apply the Wigdersons' approach to establish these uncertainty principles specifically for time-frequency representations. This thesis is organized into three main chapters. Chapter 1 provides a detailed analysis of the Wigdersons' approach, and how their core ideas can be extended to address specific multidimensional problems. Chapter 2 introduces metaplectic operators and delves into the symplectic framework. Here, we focus on the uncertainty principles associated with these operators, specifically examining how Gröchenig and Shafkulovska generalized principles, originally valid for the Short-Time Fourier Transform (STFT), to a much broader class of time-frequency representations. Finally, Chapter 3 presents the original results of this work. It investigates the underlying nature of uncertainty principles within the time-frequency space, establishing specific conditions under which these principles hold true when an effective mixing of variables occurs.

Chapter 1

The Uncertainty Principle: Wigdersons' approach

1.1 Wigdersons' approach

(**Note.** This section is based primarily on the article by Avi Wigderson and Yuval Wigderson [1])

Among the various approaches to the uncertainty principle, the one proposed by the Wigdersons is one of the most elegant. The premise is simple: an uncertainty principle can be established for any operator that is k-Hadamard, a class that notably includes the Fourier transform. Let us begin by illustrating this approach using the Fourier operator. Note that the Fourier transform is bounded as an operator from L^1 to L^∞, as is the inverse Fourier transform. Furthermore, it is unitary under the standard normalization we are considering. Consequently, we have the bounds $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$ and $\|f\|_{L^\infty} \leq \|\hat{f}\|_{L^1}$. Assuming that f and $\hat{f}$ are nonzero and that all four norms are finite, multiplying these two inequalities yields:

$$\frac{\|f\|_{L^1}\|\hat{f}\|_{L^1}}{\|f\|_{L^\infty}\|\hat{f}\|_{L^\infty}} \geq 1.$$

This inequality acts as a primary uncertainty principle. Indeed, the ratio $H_0(f) = \frac{\|f\|_{L^1}}{\|f\|_{L^\infty}}$ serves as a natural measure of localization: given two functions with identical L^∞-norms, the one that is more spread out will possess a larger L^1-norm. Using this notation, the primary uncertainty principle can be elegantly rewritten as:

$$H_0(f) \cdot H_0(\hat{f}) \geq 1.$$

Now, suppose we define a new measure of localization, denoted by H. An

uncertainty principle for this measure can be written in the following form:

$$H(f) \cdot H(\hat{f}) \geq C.$$

To prove this, we employ a simple two-step procedure:

1. First, we establish a universal bound relating H to the primary measure H_0, such that $H(f) \geq C'(H_0(f))^\alpha$ and $H(\hat{f}) \geq C'(H_0(\hat{f}))^\alpha$, for some constants $C' > 0$ and $\alpha > 0$.

2. Second, multiplying these two inequalities together and applying the primary uncertainty principle ($H_0(f) \cdot H_0(\hat{f}) \geq 1$), we obtain:

$$H(f) \cdot H(\hat{f}) \geq (C')^2 (H_0(f) \cdot H_0(\hat{f}))^\alpha \geq (C')^2.$$

   Setting $C = (C')^2$ yields the desired result.

The article is divided into two main sections, focusing on finite-dimensional and infinite-dimensional uncertainty principles, respectively. Specifically, the authors examine well-known results, such as the Donoho-Stark uncertainty principle, and systematically reformulate and generalize them.

1.2 Finite-dimensional uncertainty principle

In this section, we focus on support-size uncertainty principles, such as the classical Donoho-Stark uncertainty principle. We will formulate these principles both for the discrete Fourier transform and for general finite transformations, which can be naturally represented via matrices. To this end, let us formally define the class of k-Hadamard matrices and establish the primary uncertainty principle within this matrix setting.

1.2.1 k-Hadamard matrices and the primary uncertainty principle

Definition 1.1 (Operator norm)**.** Let V, U be two real or complex vector spaces equipped with the norms $\|\cdot\|_V$ and $\|\cdot\|_U$, respectively. For any linear transformation $A : V \to U$, its *operator norm* is defined as:

$$\|A\|_{V \to U} = \sup_{v \in V \setminus \{0\}} \frac{\|Av\|_U}{\|v\|_V}.$$

Before focusing on matrices, the Wigdersons' approach introduces a highly general uncertainty principle involving two linear operators, A and B. This foundational result demonstrates that a lower bound on the composition of two operators directly yields an uncertainty principle.

Theorem 1.2 ([1, Thm. 2.1] General Primary Uncertainty Principle)**.** Let V, W, and Z be vector spaces equipped with L^1 and L^∞ norms. Let $A : V \to W$ and $B : W \to Z$ be linear operators satisfying the boundedness conditions $\|A\|_{\ell^1 \to \ell^\infty} \leq 1$ and $\|B\|_{\ell^1 \to \ell^\infty} \leq 1$. If there exists a constant $k > 0$ such that:

$$\|BAv\|_{\ell^\infty} \geq k\|v\|_{\ell^\infty},$$

for all $v \in V$, then for any nonzero vector $v \in V$:

$$\frac{\|v\|_{\ell^1}}{\|v\|_{\ell^\infty}} \frac{\|Av\|_{\ell^1}}{\|Av\|_{\ell^\infty}} \geq k.$$

Proof. Using the definition of the operator norm, the boundedness of A and B implies that $\|Av\|_{\ell^\infty} \leq \|v\|_{\ell^1}$ and $\|B(Av)\|_{\ell^\infty} \leq \|Av\|_{\ell^1}$. Multiplying these two inequalities and applying the hypothesis $\|BAv\|_{\ell^\infty} \geq k\|v\|_{\ell^\infty}$, we obtain:

$$\|v\|_{\ell^1}\|Av\|_{\ell^1} \geq \|Av\|_{\ell^\infty}\|BAv\|_{\ell^\infty} \geq k\|v\|_{\ell^\infty}\|Av\|_{\ell^\infty}.$$

Dividing both sides by the strictly positive quantity $\|v\|_{\ell^\infty}\|Av\|_{\ell^\infty}$ yields the result. □

To adapt this general framework to matrices, we can consider a single matrix A and naturally set the second operator to be its conjugate transpose, meaning $B = A^*$. This specialization leads directly to the following core definition:

Definition 1.3 ([1, Def. 2.2] k-Hadamard Matrix)**.** Let $A \in \mathbb{C}^{m \times n}$ be a matrix and let $k > 0$. We say that A is a **k-Hadamard matrix** if every entry of A has an absolute value of at most 1, and

$$\|A^*Av\|_{\ell^\infty} \geq k\|v\|_{\ell^\infty},$$

for all $v \in \mathbb{C}^n$.

Remark 1.4. Equivalently, A is k-Hadamard if all its entries are bounded by 1 in absolute value, the matrix A^*A is invertible, and its operator norm satisfies $\|(A^*A)^{-1}\|_{\ell^\infty \to \ell^\infty} \leq 1/k$. Using this definition, we can restate the primary uncertainty principle for finite-dimensional spaces. This theorem is nearly tautological, as the hypotheses are tailored to deliver the result in a straightforward manner; nonetheless, its power stems from the fact that many natural operators satisfy these conditions.

Theorem 1.5 ([1, Thm. 2.3] Primary Uncertainty Principle for k-Hadamard Matrices)**.** Let $A \in \mathbb{C}^{m \times n}$ be a k-Hadamard matrix. Then, for any nonzero vector $v \in \mathbb{C}^n$, we have:

$$\frac{\|v\|_{\ell^1}}{\|v\|_{\ell^\infty}} \frac{\|Av\|_{\ell^1}}{\|Av\|_{\ell^\infty}} \geq k.$$

Proof. Recall that the $1 \to \infty$ operator norm of a matrix corresponds to the maximum absolute value of its entries. Since A is k-Hadamard, all its entries are bounded by 1, which implies $\|A\|_{\ell^1 \to \ell^\infty} \leq 1$. Thus, we have the basic inequality:

$$\|Av\|_{\ell^\infty} \leq \|v\|_{\ell^1}. \tag{1.1}$$

By the same argument, the conjugate transpose matrix A^* also satisfies $\|A^*\|_{\ell^1 \to \ell^\infty} \leq 1$, which yields:

$$\|A^*(Av)\|_{\ell^\infty} \leq \|Av\|_{\ell^1}. \tag{1.2}$$

Multiplying inequalities (1.1) and (1.2) together, and exploiting the core property of k-Hadamard matrices, we obtain:

$$\|v\|_{\ell^1}\|Av\|_{\ell^1} \geq \|Av\|_{\ell^\infty}\|A^*Av\|_{\ell^\infty} \geq k\|v\|_{\ell^\infty}\|Av\|_{\ell^\infty}$$

Dividing by $\|v\|_{\ell^\infty}\|Av\|_{\ell^\infty}$ gives the desired ratio. □

It is worth noting that the proof of Theorem 1.2 never actually uses any specific properties of the usual L^1 and L^∞ norms. The same result holds for any choice of norms, provided the operators satisfy analogous boundedness conditions.

1.2.2 The support-size uncertainty principle for finite abelian group

Definition 1.6 (Support)**.** Let $v \in \mathbb{C}^n$ be a vector. The *support* of v, denoted by $\mathrm{supp}(v)$, is the set of indices i such that $v_i \neq 0$.

Similarly, let G be a finite group and $f : G \to \mathbb{C}$ be a function. The *support* of f, denoted by $\mathrm{supp}(f)$, is the set of elements $x \in G$ such that $f(x) \neq 0$.

Definition 1.7 (Fourier transform on a finite abelian group)**.** Let G be a finite abelian group, and let $\widehat{G}$ be its dual group, i.e., the set of all homomorphisms from G to the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The set $\widehat{G}$ forms an abelian group under pointwise multiplication. We define the *Fourier transform* $\hat{f} : \widehat{G} \to \mathbb{C}$ of a function $f : G \to \mathbb{C}$ by $\hat{f}(\chi) = \sum_{x \in G} f(x)\overline{\chi(x)}$.

Theorem 1.8 ([1, Thm.3.1] Donoho-Stark)**.** Let G be a finite abelian group. If $f : G \to \mathbb{C}$ is a nonzero function and $\hat{f} : \widehat{G} \to \mathbb{C}$ denotes its Fourier transform, then

$$|\mathrm{supp}(f)||\mathrm{supp}(\hat{f})| \geq |G|.$$

Remark 1.9. This bound is generally tight. For example, let f be the indicator function of a subgroup $H \subseteq G$. Its Fourier transform $\hat{f}$ is a constant multiple of the indicator function of the orthogonal subgroup $H^\perp \subseteq \widehat{G}$. Since the sizes of these subgroups satisfy $|H||H^\perp| = |G|$, it follows directly that:

$$|\mathrm{supp}(f)||\mathrm{supp}(\hat{f})| = |G|.$$

This demonstrates that the theoretical lower bound can be achieved exactly.

Theorem 1.10 ([1, Thm.3.2] Donoho-Stark for k-Hadamard matrices)**.** Let $A \in \mathbb{C}^{m\times n}$ be a k-Hadamard matrix. Then for any nonzero $v \in \mathbb{C}^n$

$$|\operatorname{supp}(v)||\operatorname{supp}(Av)| \geq k.$$

Proof. To prove this result, we follow the two-step approach proposed by the Wigdersons. In this case, the measure of localization is $H(v) = |\operatorname{supp}(v)|$. Thus, we proceed as follows:

- First, we establish that $H(v) \geq H_0(v) = \frac{\|v\|_{\ell^1}}{\|v\|_{\ell^\infty}}$. Indeed, for any vector $u \in \mathbb{C}^n$:

$$\|u\|_{\ell^1} = \sum_{i=1}^{n} |u_i| = \sum_{i\in\operatorname{supp}(u)} |u_i| \leq |\operatorname{supp}(u)|\|u\|_{\ell^\infty}.$$

- Second, applying this bound to both v and Av, and multiplying the results, we obtain:
$$|\operatorname{supp}(v)||\operatorname{supp}(Av)| \geq \frac{\|v\|_{\ell^1}}{\|v\|_{\ell^\infty}} \frac{\|Av\|_{\ell^1}}{\|Av\|_{\ell^\infty}} \geq k,$$
where the final inequality is given by the primary uncertainty principle for matrices.

□

1.2.3 The essential support-size uncertainty principle

The Donoho-Stark Theorem relies on support size, a fragile metric that counts arbitrarily small entries. A robust alternative is *essential support*, which ignores tiny values. Using this approach, the preceding Theorem extends to a robust L^1 form. The L^2 analogue does not follow, although Donoho and Stark's L^2 proof generalizes to k-Hadamard matrices ($A^*A = kI$). We describe and compare both versions. Notationally, for $v \in \mathbb{C}^n$ and $T \subseteq [n]$, $v[T] \in \mathbb{C}^{|T|}$ represents v restricted to T, and T^c is its complement.

Definition 1.11 ([1, Def.3.15] (p,ϵ)-supported)**.** Let $\epsilon \in [0,1]$ and $p \in [1,\infty]$. A vector $v \in \mathbb{C}^n$ is said to be (p,ϵ)-supported on a subset $T \subseteq [n]$ if $\|v[T^c]\|_{\ell^p} \leq \epsilon\|v\|_{\ell^p}$. Accordingly, the (p,ϵ) support-size of v is defined as:

$$|\operatorname{supp}_\epsilon^p(v)| = \min\{|T| : T \subseteq [n] \text{ and } v \text{ is } (p,\epsilon)\text{-supported on } T\}.$$

Remark 1.12. Since the minimizing subset T is not strictly unique, the entity $\operatorname{supp}_\epsilon^p(v)$ is not well-defined. By a slight abuse of notation, we shall nonetheless use $\operatorname{supp}_\epsilon^p(v)$ to designate any arbitrary set T that realizes this minimum cardinality.

Theorem 1.13 ([1, Thm.3.17] L^1 approximate-support uncertainty principle)**.** Let $A \in \mathbb{C}^{m\times n}$ be a k-Hadamard matrix and let $v \in \mathbb{C}^n$ be a nonzero vector. For any $\epsilon, \eta \in [0,1]$, we have that:

$$|\operatorname{supp}_\epsilon^1(v)||\operatorname{supp}_\eta^1(Av)| \geq k(1-\epsilon)(1-\eta) \geq k(1-\epsilon-\eta).$$

Proof. The measure of localization used here is $H(v) = \frac{|\operatorname{supp}_\epsilon^1(v)|}{1-\epsilon}$.

- Let $u \in \mathbb{C}^n$, $\delta \in [0,1]$, and $T = \operatorname{supp}_\delta^1(u)$. Note that $\|u[T^c]\|_{\ell^1} = \|u\|_{\ell^1} - \|u[T]\|_{\ell^1}$. By the definition of T, we have $\|u[T^c]\|_{\ell^1} \leq \delta\|u\|_{\ell^1}$. Observe also that $\|u[T]\|_{\ell^1} \leq |T|\|u[T]\|_{\ell^\infty} \leq |T|\|u\|_{\ell^\infty}$. Combining these inequalities, we obtain:

$$(1-\delta)\|u\|_{\ell^1} \leq \|u[T]\|_{\ell^1} \leq |T|\|u\|_{\ell^\infty} = |\operatorname{supp}_\delta^1(u)|\|u\|_{\ell^\infty}.$$

- Applying this bound to v (with $\delta = \epsilon$) and to Av (with $\delta = \eta$), and multiplying the two results, we get:

$$\frac{|\operatorname{supp}_\epsilon^1(v)|}{1-\epsilon}\frac{|\operatorname{supp}_\eta^1(Av)|}{1-\eta} \geq \frac{\|v\|_{\ell^1}}{\|v\|_{\ell^\infty}}\frac{\|Av\|_{\ell^1}}{\|Av\|_{\ell^\infty}} \geq k.$$

  where the final inequality follows from the primary uncertainty principle for matrices.

□

Corollary 1.14 ([1, Cor.3.18] L^1 approximate-support uncertainty principle for Fourier transform)**.** Let G be a finite abelian group, and let $f : G \to \mathbb{C}$ be a nonzero function. For any $\epsilon, \eta \in [0,1]$, we have that:

$$|\operatorname{supp}_\epsilon^1(f)||\operatorname{supp}_\eta^1(\hat{f})| \geq |G|(1-\epsilon)(1-\eta) \geq |G|(1-\epsilon-\eta).$$

Theorem 1.15 ([1, Thm.3.20] L^2 approximate-support uncertainty principle)**.** Let $A \in \mathbb{C}^{n\times n}$ be a k-Hadamard matrix, with $A^*A = kI$, and let $v \in \mathbb{C}^n$ be a nonzero vector. For any $\epsilon, \eta \in [0,1]$, we have that:

$$|\operatorname{supp}_\epsilon^2(v)||\operatorname{supp}_\eta^2(Av)| \geq k\max\{1-\epsilon-\eta, 0\}^2.$$

In particular, if $\epsilon + \eta \leq 1$, then the lower bound is $k(1-\epsilon-\eta)^2$.

Remark 1.16. The proof of this theorem does not follow the general framework proposed by A. Wigderson and Y. Wigderson [1]. To use their approach, one would need an inequality of the form $\|v\|_{\ell^2}\|Av\|_{\ell^2} \geq C(k)\|v\|_{\ell^\infty}\|Av\|_{\ell^\infty}$, which does not hold in general. Instead, the proof presented here is adapted from [1], with a necessary correction to the final algebraic step to account for arbitrary values of ϵ and η. As noted by the authors, this specific argument relies on orthogonal projections and is originally due to Donoho and Stark.

Proof. Let $U = \frac{1}{\sqrt{k}}A$. By hypothesis, U is unitary, which implies $\|Uw\|_{\ell^2} = \|w\|_{\ell^2}$ for all $w \in \mathbb{C}^n$. Since the approximate support size is invariant under scaling, $|\operatorname{supp}^2_\eta(Av)| = |\operatorname{supp}^2_\eta(Uv)|$.

Define $S = \operatorname{supp}^2_\epsilon(v)$ and $T = \operatorname{supp}^2_\eta(Uv)$. Let P_S and P_T be the orthogonal projections onto the coordinates in S and T, respectively. Consider the operator $M = P_S U^* P_T$. We aim to bound its operator norm $\|M\|_{\ell^2\to\ell^2}$.

First, for the upper bound, let $w \in \mathbb{C}^n$ and let M_i denote the i-th row of M. By the Cauchy-Schwarz inequality:

$$\|Mw\|^2_{\ell^2} = \sum_{i\in S} |\langle M_i, w\rangle|^2 \le \sum_{i\in S} \|M_i\|^2_{\ell^2}\|w\|^2_{\ell^2} \le \frac{|S||T|}{k}\|w\|^2_{\ell^2},$$

where we used that M contains $|S||T|$ entries, each bounded in modulus by $1/\sqrt{k}$. Thus, $\|M\|_{\ell^2\to\ell^2} \le \sqrt{\frac{|S||T|}{k}}$.

Second, for the lower bound, since P_S is a contraction and U is unitary, we have:

$$\begin{aligned}\|P_S v - MUv\|_{\ell^2} &= \|P_S(v - U^*P_TUv)\|_{\ell^2}\\ &\le \|U^*(Uv - P_TUv)\|_{\ell^2}\\ &= \|Uv - P_TUv\|_{\ell^2} \le \eta\|v\|_{\ell^2}.\end{aligned}$$

where the last step follows from the definition of T. Furthermore, the definition of S yields $\|v - P_S v\|_{\ell^2} \le \epsilon\|v\|_{\ell^2}$. Applying the triangle inequality:

$$\|v - MUv\|_{\ell^2} \le \|v - P_S v\|_{\ell^2} + \|P_S v - MUv\|_{\ell^2} \le (\epsilon+\eta)\|v\|_{\ell^2}.$$

By the reverse triangle inequality, we deduce:

$$\|MUv\|_{\ell^2} \ge \|v\|_{\ell^2} - \|v - MUv\|_{\ell^2} \ge (1-\epsilon-\eta)\|v\|_{\ell^2}.$$

Since $\|v\|_{\ell^2} = \|Uv\|_{\ell^2}$, we combine our bounds:

$$(1-\epsilon-\eta)\|Uv\|_{\ell^2} \le \|MUv\|_{\ell^2} \le \|M\|_{\ell^2\to\ell^2}\|Uv\|_{\ell^2} \le \sqrt{\frac{|S||T|}{k}}\|Uv\|_{\ell^2}.$$

Dividing by $\|Uv\|_{\ell^2} > 0$ and squaring both sides yields $|S||T| \ge k(1-\epsilon-\eta)^2$ provided that $1-\epsilon-\eta \ge 0$. If instead $1-\epsilon-\eta < 0$, the inequality $|S||T| \ge 0$ holds trivially. Covering both cases, we obtain the final bound:

$$|\operatorname{supp}^2_\epsilon(v)||\operatorname{supp}^2_\eta(Av)| \ge k\max\{1-\epsilon-\eta, 0\}^2.$$

□

Corollary 1.17 ([1, Thm.3.16] Donoho-Stark approximate-support)**.** Let G be a finite abelian group, and let $f : G \to \mathbb{C}$ be a nonzero function. For any $\epsilon, \eta \in [0,1]$, we have that:

$$|\operatorname{supp}^2_\epsilon(f)||\operatorname{supp}^2_\eta(\hat{f})| \ge |G|\max\{1-\epsilon-\eta, 0\}^2.$$

In particular, if $\epsilon + \eta \le 1$, then the lower bound is $|G|(1-\epsilon-\eta)^2$.

1.2.4 Uncertainty Principle for other norms

In this section, we explore the possibility of extending the uncertainty principle to norms other than L^1 and L^∞.

Norm uncertainty principle for $p = 1$. To derive the uncertainty principle in the case of $p = 1$ and $q \in [1, \infty]$, we apply the Wigdersons' approach, defining our measure of localization as $H(v) = \frac{\|v\|_{\ell^1}}{\|v\|_{\ell^q}}$.

Theorem 1.18 ([1, Thm.3.21] Norm uncertainty principle, $p = 1$)**.** Let $A \in \mathbb{C}^{m\times n}$ be a k-Hadamard matrix, and let $v \in \mathbb{C}^n$ be a nonzero vector. Then, for any $q \in [1, \infty]$, we have:

$$\frac{\|v\|_{\ell^1}}{\|v\|_{\ell^q}} \frac{\|Av\|_{\ell^1}}{\|Av\|_{\ell^q}} \geq k^{1-\frac{1}{q}}.$$

In particular, when $q = \infty$, we obtain the primary uncertainty principle.

Proof. As mentioned above, we consider the localization measure $H(v) = \frac{\|v\|_{\ell^1}}{\|v\|_{\ell^q}}$. The case $q = \infty$ is the primary uncertainty principle, so we may assume that q is finite.

- Let $u \in \mathbb{C}^n$ be a nonzero vector. We can bound its L^q norm as follows:

$$\|u\|_{\ell^q}^q = \sum_{i=1}^n |u_i|^q \leq \|u\|_{\ell^\infty}^{q-1} \sum_{i=1}^n |u_i| = \|u\|_{\ell^\infty}^{q-1} \|u\|_{\ell^1}.$$

  By rearranging these terms, we obtain:

$$\left(\frac{\|u\|_{\ell^1}}{\|u\|_{\ell^q}}\right)^q \geq \left(\frac{\|u\|_{\ell^1}}{\|u\|_{\ell^\infty}}\right)^{q-1} \implies \frac{\|u\|_{\ell^1}}{\|u\|_{\ell^q}} \geq \left(\frac{\|u\|_{\ell^1}}{\|u\|_{\ell^\infty}}\right)^{\frac{q-1}{q}}.$$

- Applying this inequality to both u and Au, and multiplying the results together, we get:

$$\frac{\|u\|_{\ell^1}}{\|u\|_{\ell^q}} \frac{\|Au\|_{\ell^1}}{\|Au\|_{\ell^q}} \geq \left(\frac{\|u\|_{\ell^1}}{\|u\|_{\ell^\infty}} \frac{\|Au\|_{\ell^1}}{\|Au\|_{\ell^\infty}}\right)^{\frac{q-1}{q}} \geq k^{\frac{q-1}{q}}.$$

  where the final inequality follows from the primary uncertainty principle for matrices.

□

Corollary 1.19 ([1, Thm.3.22] Fourier norm uncertainty principle, $p = 1$)**.** Let G be a finite abelian group and let $f : G \to \mathbb{C}$ be a nonzero function. For any $q \in [1, \infty]$, we have:

$$\frac{\|f\|_{\ell^1}}{\|f\|_{\ell^q}} \frac{\|\hat{f}\|_{\ell^1}}{\|\hat{f}\|_{\ell^q}} \geq |G|^{\frac{q-1}{q}}.$$

Remark 1.20. The subgroup indicator function example discussed earlier also provides a tight case for this bound. Since the L^q norm of an indicator function is exactly its support size raised to the power of $1/q$, applying the previously established properties of H and $H^\perp$ directly reduces this inequality to a strict equality.

Norm uncertainty principle for $p = 2$. Two special cases of Corollary 1.19 (one of which is simply the primary uncertainty principle) state that if A is a k-Hadamard matrix, then:

$$\frac{\|v\|_{\ell^1}}{\|v\|_{\ell^\infty}} \frac{\|Av\|_{\ell^1}}{\|Av\|_{\ell^\infty}} \geq k \quad \text{and} \quad \frac{\|v\|_{\ell^1}}{\|v\|_{\ell^2}} \frac{\|Av\|_{\ell^1}}{\|Av\|_{\ell^2}} \geq \sqrt{k}.$$

Given these bounds, it is natural to conjecture something like the following:

$$\frac{\|v\|_{\ell^2}}{\|v\|_{\ell^\infty}} \frac{\|Av\|_{\ell^2}}{\|Av\|_{\ell^\infty}} \geq \sqrt{k}, \tag{1.3}$$

which, if true, would be optimal. However, this bound is false in general, as the following theorem demonstrates

Theorem 1.21 ([1, Thm.3.23])**.** Let G be a finite abelian group of order n, and let A be the Fourier transform matrix of G. For any arbitrary $p \in [2, \infty]$ and $q \in [1, \infty]$, there exists a vector $v \in \mathbb{C}^n$ such that:

$$\frac{\|v\|_{\ell^p}}{\|v\|_{\ell^q}} \frac{\|Av\|_{\ell^p}}{\|Av\|_{\ell^q}} \leq 2.$$

In particular, this demonstrates that the conjecture proposed in Equation (1.3) is false in general.

Proof. The proof is omitted as it falls outside the main scope of this work. For the complete details, we refer the reader to [1, p.244]. □

Norm uncertainty principle for $1 < p < 2$ and the Hausdorff-Young inequality. For the remaining range $1 < p < 2$, we can establish norm inequalities similar to those for the $p = 1$ case, provided we introduce two additional hypotheses. First, unlike the $p = 1$ scenario, we must assume that the k-Hadamard matrix satisfies the stronger unitarity condition $A^*A = kI$. Second, the secondary norm index q must not exceed the conjugate exponent of p, meaning $q \leq p' = \frac{p}{p-1}$. This restriction was irrelevant when $p = 1$, since the conjugate index of 1 is trivially ∞. The Wigdersons state at this point that they "include this subsection only for completeness." However, as we will see, these hypotheses and the resulting inequality are actually fundamental: this is the key to extending this approach to the multidimensional Heisenberg principle, entropic uncertainty principles, and other.

Proposition 1.22 ([1, Prop.3.24] Discrete Hausdorff-Young inequality)**.** Let $A \in \mathbb{C}^{n\times n}$ be a k-Hadamard matrix with $A^*A = kI$. Fix $1 < p < 2$, and let $p' = \frac{p}{p-1} \in (2, \infty)$ be its conjugate exponent. Then:

$$\|A\|_{\ell^p \to \ell^{p'}} \leq k^{\frac{p-1}{p}}.$$

Proof. The present proof diverges from the Wigdersons' original approach. Instead, it follows as a direct consequence of the Riesz-Thorin interpolation theorem; see [1, Proposition 3.24, p. 245]. □

As a corollary, we obtain the following norm uncertainty principle for the range $1 < p < 2$.

Theorem 1.23 ([1, Thm.3.25] Norm uncertainty principle, $1 < p < 2$)**.** Let $A \in \mathbb{C}^{n\times n}$ be a k-Hadamard matrix with $A^*A = kI$. Let $p \in (1, 2)$ and $q \in [p, p']$ be norm indices. Then, for every nonzero vector $v \in \mathbb{C}^n$:

$$\frac{\|v\|_{\ell^p}}{\|v\|_{\ell^q}} \frac{\|Av\|_{\ell^p}}{\|Av\|_{\ell^q}} \geq k^{\frac{q-p}{pq}}.$$

Proof. First, let us consider the case where $q = p'$. Applying the conclusion of proposition 1.22 to both A and its conjugate transpose A^*, we obtain:

$$k^{\frac{2(p-1)}{p}} \|v\|_{\ell^p} \|Av\|_{\ell^p} \geq \|Av\|_{\ell^{p'}} \|A^*Av\|_{\ell^{p'}} = k\|v\|_{\ell^{p'}} \|Av\|_{\ell^{p'}}.$$

Dividing both sides by k, we arrive at the desired bound for this specific case:

$$\|v\|_{\ell^p} \|Av\|_{\ell^p} \geq k^{\frac{2-p}{p}} \|v\|_{\ell^{p'}} \|Av\|_{\ell^{p'}}.$$

For smaller values of q, we treat the inequality above as our primary uncertainty principle. We can derive the general result by establishing that, for any nonzero vector u and any $p \leq q \leq p'$:

$$\frac{\|u\|_{\ell^p}}{\|u\|_{\ell^q}} \geq \left(\frac{\|u\|_{\ell^p}}{\|u\|_{\ell^{p'}}} \right)^{\frac{p-q}{pq-2q}}. \tag{1.4}$$

To prove this, let $\theta \in [0, 1]$ be the unique coefficient such that $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{p'}$. Solving for θ, we find $\theta = \frac{p-q}{pq-2q}$. The log-convexity of the L^p norms (often referred to as the generalized Hölder inequality) guarantees that:

$$\|u\|_{\ell^q} \leq \|u\|_{\ell^p}^{1-\theta} \|u\|_{\ell^{p'}}^{\theta}$$

Rearranging this inequality yields exactly equation (1.4).

Finally, we apply (1.4) to both $u = v$ and $u = Av$. Multiplying the resulting inequalities together, we conclude:

$$\frac{\|v\|_{\ell^p}}{\|v\|_{\ell^q}} \frac{\|Av\|_{\ell^p}}{\|Av\|_{\ell^q}} \geq \left(\frac{\|v\|_{\ell^p}}{\|v\|_{\ell^{p'}}} \frac{\|Av\|_{\ell^p}}{\|Av\|_{\ell^{p'}}} \right)^{\frac{p-q}{pq-2q}} \geq \left(k^{\frac{2-p}{p}} \right)^{\frac{p-q}{pq-2q}} = k^{\frac{q-p}{pq}}.$$

□

In summary, the preceding subsections demonstrate that the $1 \to \infty$ bound of Theorem 1.5 is the strongest result of its kind in three distinct ways:

- **Optimal $1 \to q$ inequalities:** It implies the tightest possible $1 \to q$ bounds for any $1 \leq q \leq \infty$.
- **Tight $p \to q$ inequalities ($1 < p < 2$):** We can still establish tight inequalities provided that two conditions are met: the norm index satisfies $p \leq q \leq p'$ (where p' is the conjugate exponent), and the k-Hadamard matrix satisfies the strict unitarity condition $A^*A = kI$. The underlying reasons for these restrictions will be explored in detail later.
- **Failure of general $p \to q$ principles ($p \geq 2$):** General $p \to q$ uncertainty principles fail for $p \geq 2$. Furthermore, it is impossible to obtain a superconstant uncertainty lower bound in this regime, even for the Fourier transform matrix.

1.3 Infinite-dimensional uncertainty principle

In this section, we state and prove several uncertainty principles for infinite-dimensional spaces, focusing on the Heisenberg uncertainty principle and its generalizations.

1.3.1 The Fourier transform on locally compact abelian groups

We begin by reviewing the basic definitions of the Fourier transform on these groups and generalizing some of our earlier results.

Let G be a locally compact abelian group. It has a left-invariant Borel measure μ, called the Haar measure, which is unique up to scaling. Let $\widehat{G}$ be the set of continuous group homomorphisms $G \to \mathbb{T}$. This set forms a group under pointwise multiplication. When equipped with the compact-open topology, $\widehat{G}$ becomes another locally compact abelian group, known as the Pontryagin dual of G.

Definition 1.24 (Fourier transform)**.** For a function $f \in L^1(G)$, its *Fourier transform* $\hat{f} : \widehat{G} \to \mathbb{C}$ is defined as

$$\hat{f}(\chi) = \int_G f(x)\overline{\chi(x)}\, d\mu(x).$$

It is easy to check that $\hat{f} \in L^\infty(\widehat{G})$. Once μ is chosen, there is a unique Haar measure ν on $\widehat{G}$ such that the *Fourier inversion formula* holds:

$$f(x) = \int_{\widehat{G}} \hat{f}(\chi)\chi(x)\, d\nu(\chi),$$

for μ-almost every x, provided $\hat{f} \in L^1(\widehat{G})$.

With this choice of ν, we also have the *Plancherel formula*:

$$\int_G |f|^2\, d\mu = \int_{\widehat{G}} |\hat{f}|^2\, d\nu,$$

assuming one side is well-defined.

From now on, we fix μ and ν, and compute all L^p norms using them.

Notice that the Fourier transform and its inverse have a norm of at most 1 as operators from L^1 to L^∞. We can use this fact to prove an infinite-dimensional version of our main uncertainty principle, Theorem 1.5.

Theorem 1.25 ([1, Thm.4.1] Primary uncertainty principle)**.** Let G be a locally compact abelian group with Haar measure μ, and let $\widehat{G}$ and ν be the dual group and measure. If $f \in L^1(G)$ is a nonzero function such that $\hat{f} \in L^1(\widehat{G})$, then

$$\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}} \frac{\|\hat{f}\|_{L^1}}{\|\hat{f}\|_{L^\infty}} \geq 1.$$

In this section, we frequently assume that $f \in L^1(G)$ and $\hat{f} \in L^1(\hat{G})$. To avoid repeating this, we introduce the following terminology:

Definition 1.26 ([1, Def.4.2] Doubly L^1 function)**.** A function $f : G \to \mathbb{C}$ is called *doubly* L^1 if $f \in L^1(G)$ and $\hat{f} \in L^1(\widehat{G})$.

Note that if f is doubly L^1, then both f and $\hat{f}$ are also in L^∞. By Hölder's inequality, this implies that $f, \hat{f} \in L^p$ for all $p \in [1, \infty]$.

Proof of Thm.1.25. For any $\chi \in \widehat{G}$, we have:

$$|\hat{f}(\chi)| = \left| \int_G f(x)\chi(x)\, d\mu(x) \right| \leq \int_G |f(x)|\, d\mu(x) = \|f\|_{L^1},$$

because $|\chi(x)| = 1$. This shows that $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$. By the same logic, we also get $\|f\|_{L^\infty} \leq \|\hat{f}\|_{L^1}$. Multiplying these two inequalities yields the desired result. □

Remark 1.27. If G is a finite abelian group, this result seems weaker than Theorem 1.5 by a factor of $|G|$. However, this happens because we previously used the counting measure for both G and $\widehat{G}$, which are not dual Haar measures. If we instead use proper dual Haar measures (for example, the counting measure on G and the uniform probability measure on $\widehat{G}$), the extra factor of $|G|$ disappears, matching the statement of Theorem 1.25.

Using this theorem, we can obtain a version of the Donoho–Stark uncertainty principle that applies to any locally compact abelian group. This result was first proved by Matolcsi and Szűcs using spectral integrals.

Theorem 1.28 ([1, Thm.4.3]Support-size uncertainty principle for general abelian groups)**.** Let $G, \mu, \widehat{G}, \nu$ be as defined above. If $f : G \to \mathbb{C}$ is a nonzero doubly L^1 function, then:

$$\mu(\operatorname{supp}(f))\nu(\operatorname{supp}(\hat{f})) \geq 1. \tag{1.5}$$

Proof. To prove this result, we follow the two-step approach proposed by the Wigdersons. In this case, the measure of localization for a function g on a measure space (X, λ) is $H(g) = \lambda(\operatorname{supp}(g))$. Thus, we proceed as follows:

- First, we establish that $H(g) \geq H_0(g) = \frac{\|g\|_{L^1}}{\|g\|_{L^\infty}}$. Indeed, for any nonzero integrable function g:

$$\begin{aligned}\|g\|_{L^1} &= \int_X |g(x)|\, d\lambda(x) = \int_{\operatorname{supp}(g)} |g(x)|\, d\lambda(x) \\ &\leq \|g\|_{L^\infty} \int_{\operatorname{supp}(g)} d\lambda(x) = \lambda(\operatorname{supp}(g))\|g\|_{L^\infty}.\end{aligned}$$

- Second, applying this bound to both f (with measure μ) and $\hat{f}$ (with measure ν), and multiplying the results, we obtain:

$$\mu(\operatorname{supp}(f))\nu(\operatorname{supp}(\hat{f})) \geq \frac{\|f\|_{L^1}}{\|f\|_{L^\infty}} \frac{\|\hat{f}\|_{L^1}}{\|\hat{f}\|_{L^\infty}} \geq 1,$$

  where the final inequality is given by the primary uncertainty principle for locally compact abelian groups (Theorem 1.25).

□

Remark 1.29. Similar to Theorem 1.8, the inequality in Theorem 1.28 is tight. For instance, let G be a compact abelian group equipped with the Haar probability measure μ. In this case, its dual group $\widehat{G}$ is discrete, and the corresponding Haar measure ν is the counting measure. If we let $f : G \to \mathbb{C}$ be the constant function $f(x) = 1$, then $\mu(\operatorname{supp}(f)) = 1$. Consequently, its Fourier transform $\hat{f}$ is simply the indicator function of the identity element in $\widehat{G}$, which yields $\nu(\operatorname{supp}(\hat{f})) = 1$ as well, thus achieving equality.

However, if we consider the Benedicks-Amrein-Berthier theorem, which restricts the setting to $G = \mathbb{R}$ where μ is the Lebesgue measure, we find that the bound in (1.5) is far from tight. In fact, for non-trivial functions, the correct relation is:

$$\mu(\text{supp}(f))\nu(\text{supp}(\hat{f})) = \infty.$$

Proving this much stronger result strictly requires the specific structure of $\mathbb{R}$. The general framework we have used so far cannot establish this infinite bound because it is designed to work for any locally compact abelian group G, and Benedicks' infinite lower bound is simply false in that broader generality.

1.3.2 k-Hadamard operators in infinite dimensions

To extend the uncertainty principles to infinite-dimensional function spaces, we follow a path identical to the finite-dimensional case. We begin prudently with a highly general formulation involving two operators.

Definition 1.30 (k-Hadamard pair)**.** Let (X, μ) and (Y, ν) be measure spaces, let $k > 0$, and let

$$A : L^1(X, \mu) \longrightarrow L^\infty(Y, \nu), \qquad B : L^1(Y, \nu) \longrightarrow L^\infty(X, \mu)$$

be linear operators satisfying

$$\|A\|_{L^1(X)\to L^\infty(Y)} \leq 1, \qquad \|B\|_{L^1(Y)\to L^\infty(X)} \leq 1.$$

Let

$$\mathcal{D} \subseteq \{f \in L^1(X, \mu) \cap L^\infty(X, \mu) : Af \in L^1(Y, \nu)\}.$$

The pair (A, B) is called a k-Hadamard pair on $\mathcal{D}$ if

$$\|BAf\|_{L^\infty(X)} \geq k\|f\|_{L^\infty(X)} \qquad \text{for every } f \in \mathcal{D}.$$

The requirement $Af \in L^1(Y, \nu)$ is part of the admissibility condition because it ensures that the $L^1(Y) \to L^\infty(X)$ operator B can be applied to Af.

The primary uncertainty principle follows naturally for these operator pairs, with a proof virtually identical to the general finite-dimensional case.

Theorem 1.31 (Primary uncertainty principle for k-Hadamard pairs)**.** If (A, B) is a k-Hadamard pair on $\mathcal{D}$, then for any nonzero function $f \in \mathcal{D}$:

$$\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}} \frac{\|Af\|_{L^1}}{\|Af\|_{L^\infty}} \geq k.$$

Proof. Since $f \in \mathcal{D}$, we know that $Af \in L^1(Y)$. Using the boundedness of the operators B and A, we obtain $\|B(Af)\|_{L^\infty} \leq \|Af\|_{L^1}$ and $\|Af\|_{L^\infty} \leq \|f\|_{L^1}$. Multiplying these inequalities and applying the k-Hadamard pair lower bound yields:

$$\|f\|_{L^1}\|Af\|_{L^1} \geq \|Af\|_{L^\infty}\|BAf\|_{L^\infty} \geq k\|Af\|_{L^\infty}\|f\|_{L^\infty}.$$

Since $f \neq 0$, we have $\|f\|_{L^\infty} > 0$. Moreover $Af \neq 0$, because $Af = 0$ would imply $BAf = 0$, contradicting $\|BAf\|_{L^\infty} \geq k\|f\|_{L^\infty}$ with $k > 0$. Hence $\|f\|_{L^\infty}\|Af\|_{L^\infty} > 0$, and division is legitimate. □

Focusing on functions on $\mathbb{R}$, Wigderson and Wigderson call a linear operator $A : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ k-Hadamard when

$$\|A\|_{L^1 \to L^\infty} \leq 1$$

and

$$\|A^*Af\|_{L^\infty} \geq k\|f\|_{L^\infty}$$

for every doubly L^1 function f, that is, every f for which $f, Af \in L^1(\mathbb{R})$. This is the content of [1, Def. 4.4]; the definition does not list $\|A^*\|_{L^1 \to L^\infty} \leq 1$ as a separate hypothesis.

When A^* is subsequently applied as an $L^1 \to L^\infty$ map, the relevant object is the adjoint realization associated with the L^1 predual of L^∞. Dias, Luef, and Prata state in [4, Rem. 3] that this realization is obtained by using L^1 as a predual of L^∞ and has the same norm as A. A convention-independent formulation is to define $A^\sharp g \in L^\infty$ by

$$\int_{\mathbb{R}} (Af)(x)\,\overline{g(x)}\,dx = \int_{\mathbb{R}} f(x)\,\overline{(A^\sharp g)(x)}\,dx, \qquad f, g \in L^1(\mathbb{R}).$$

For fixed g, the left-hand side is a bounded linear functional of f on L^1, so $A^\sharp g$ exists uniquely by $(L^1)^* = L^\infty$. Taking the supremum over the unit balls in f and g gives

$$\|A^\sharp\|_{L^1 \to L^\infty} = \|A\|_{L^1 \to L^\infty}.$$

Thus the endpoint estimate for the associated adjoint follows from the endpoint estimate for A rather than being an extra clause of Definition 4.4. In particular, $\|A\|_{L^1 \to L^\infty} \leq 1$ implies $\|A^\sharp\|_{L^1 \to L^\infty} \leq 1$, so the two-operator argument may indeed be applied to the pair $(A, A^\sharp)$ on the doubly L^1 domain. When this associated realization is the adjoint denoted by A^* in the k-Hadamard condition, one may simply write the pair as (A, A^*), as in the subsequent proof. The more detailed compatibility of this predual realization with the L^2 Hilbert adjoint used for special k-Hadamard operators is recorded below in the replacement accompanying [4, Prop. 5].

We can also extend the uncertainty principles for other norms (previously seen in Theorem 1.5) to this infinite-dimensional setting.

Theorem 1.32 ([1, Thm. 4.6] Norm uncertainty principle, infinitary version)**.** If A is a k-Hadamard operator, then for any doubly L^1 nonzero function f and any $q \in [1, \infty]$:

$$\frac{\|f\|_{L^1}}{\|f\|_{L^q}} \frac{\|Af\|_{L^1}}{\|Af\|_{L^q}} \geq k^{\frac{q-1}{q}}.$$

In particular, when $q = \infty$, we recover the primary uncertainty principle for k-Hadamard operators.

Proof. The proof is similar to the finite-dimensional case. Assume $q < \infty$, as the case where $q = \infty$ is covered exactly by Theorem 1.31. To prove this result, we evaluate the localization measure $H(g) = \frac{\|g\|_{L^1}}{\|g\|_{L^q}}$ for any nonzero function $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

- First, we bound the L^q norm of g as follows:

$$\|g\|_{L^q}^q = \int_{\mathbb{R}} |g(x)|^q \, dx \leq \|g\|_{L^\infty}^{q-1} \int_{\mathbb{R}} |g(x)| \, dx = \|g\|_{L^\infty}^{q-1} \|g\|_{L^1}.$$

  By rearranging these terms, we obtain:

$$\left(\frac{\|g\|_{L^1}}{\|g\|_{L^q}}\right)^q \geq \left(\frac{\|g\|_{L^1}}{\|g\|_{L^\infty}}\right)^{q-1} \implies \frac{\|g\|_{L^1}}{\|g\|_{L^q}} \geq \left(\frac{\|g\|_{L^1}}{\|g\|_{L^\infty}}\right)^{\frac{q-1}{q}}.$$

- Second, we apply this inequality to both f and Af, and multiply the results together:

$$\frac{\|f\|_{L^1}}{\|f\|_{L^q}} \frac{\|Af\|_{L^1}}{\|Af\|_{L^q}} \geq \left(\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}} \frac{\|Af\|_{L^1}}{\|Af\|_{L^\infty}}\right)^{\frac{q-1}{q}} \geq k^{\frac{q-1}{q}},$$

  where the final inequality follows directly from applying the primary uncertainty principle for the pair (A, A^*).

□

1.3.3 The Heisenberg uncertainty principle

This subsection establishes the well-known Heisenberg uncertainty principle, along with some extensions, though with a slightly worse constant than the optimal one. The proof follows the elementary two-step process explained in the Introduction. This approach differs significantly from classical proofs, which rely on analytic techniques like integration by parts and the fact that the Fourier transform turns differentiation into multiplication by x.

Definition 1.33 (Second Moment)**.** For a doubly L^1 function f, its **second moment** is defined as:

$$V(f) := \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx.$$

If $\|f\|_{L^2} = 1$, then $|f|^2$ acts as a probability distribution. In this case, $V(f)$ measures the actual variance of this distribution, assuming that its mean is zero. This interpretation comes from quantum mechanics, which originally motivated the study of uncertainty principles. In physics, f represents a wave function, and $|f|^2$ gives the probability distribution for measuring a particle's position or momentum. Heisenberg's uncertainty principle simply states that $V(f)$ and $V(\hat{f})$ cannot both be arbitrarily small.

Theorem 1.34 ([1, Thm.4.9] Heisenberg's uncertainty principle)**.** There exists a constant $C > 0$ such that for any nonzero doubly L^1 function f:

$$\frac{V(f)}{\|f\|_{L^2}^2} \frac{V(\hat{f})}{\|\hat{f}\|_{L^2}^2} \geq C.$$

We next generalize Heisenberg's uncertainty principle. It bounds the product $V(f)V(Af)$ using the L^q norms of f and Af, for any k-Hadamard operator A. When $q = 2$, this recovers the classical results discussed previously. For $q \neq 2$, this result appears to be new, even for the standard Fourier transform. As shown below, the inequalities obtained for different values of q are generally independent and cannot be directly compared.

Theorem 1.35 ([1, Thm.4.11] Heisenberg uncertainty principle for arbitrary norms)**.** Let A be a k-Hadamard operator, f a nonzero doubly L^1 function, and $1 < q \leq \infty$. There exists a constant $C_q > 0$, depending only on q, such that:

$$\frac{V(f)}{\|f\|_{L^q}^2} \frac{V(Af)}{\|Af\|_{L^q}^2} \geq C_q k^{\frac{3q-2}{q}},$$

where $C_q = 2^{-\frac{10q-8}{q-1}}$ In particular, for the Fourier transform (where $k = 1$ and $A = \mathcal{F}$), this becomes:

$$\frac{V(f)}{\|f\|_{L^q}^2} \frac{V(\hat{f})}{\|\hat{f}\|_{L^q}^2} \geq C_q.$$

Remark 1.36. The constant C_q obtained in the proofs is not optimized. For instance, when $q = 2$, the optimal constant for the Fourier transform is known to be $1/(16\pi^2)$. However, the proof provided here yields a weaker constant of 2^{-12}.

To achieve the first step, a different technique is introduced to bound the L^1 norm. The main idea is to select a threshold T, depending on g and the target

norm q, such that most of the L^1 mass of g lies outside the interval $[-T, T]$. This allows for a lower bound on the variance using Hölder's inequality.

Note that in the following arguments, q is treated as a finite number, but the logic holds perfectly for $q = \infty$ by taking the limit, or by treating terms like $q/(q-1)$ as equal to 1.

Proof. Assume $f \neq 0$. To establish this result, consider the localization measure $H(g) = \frac{V(g)}{\|g\|_{Lq}^2}$ for any nonzero function $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. The proof proceeds in the following usual two steps:

- First, we establish a lower bound for $H(g)$. We seek a threshold $T = T(g, q)$ such that the interval $[-T, T]$ contains at most half of the L^1 mass of g. By applying Hölder's inequality, we obtain:

$$\begin{aligned}\int_{-T}^{\top} |g(x)|\,dx &\leq \|\mathbf{1}_{[-T,T]}\|_{\frac{q}{q-1}} \|g\|_{L^q} \\ &= (2T)^{\frac{q-1}{q}} \|g\|_{L^q}.\end{aligned}$$

  To ensure this quantity is bounded by $\frac{1}{2}\|g\|_{L^1}$, we set the upper bound to exactly that value:

$$(2T)^{\frac{q-1}{q}} \|g\|_{L^q} = \frac{1}{2}\|g\|_{L^1}.$$

  Solving for T, we find that the required condition holds if we choose:

$$T = \frac{1}{2}\left(\frac{\|g\|_{L^1}}{2\|g\|_{L^q}}\right)^{\frac{q}{q-1}}.$$

  Consequently, the remaining mass lies outside this interval: $\frac{1}{2}\|g\|_{L^1} \leq \int_{|x|>T} |g(x)|\,dx$. Applying the Cauchy-Schwarz inequality to this lower bound yields:

$$\begin{aligned}\frac{1}{2}\|g\|_{L^1} &\leq \int_{|x|>T} \frac{1}{x}(x|g(x)|)\,dx \\ &\leq \left(\int_{|x|>T} \frac{1}{x^2}\,dx\right)^{\frac{1}{2}} \left(\int_{|x|>T} x^2|g(x)|^2\,dx\right)^{\frac{1}{2}} \\ &\leq \left(\frac{2}{T}\right)^{\frac{1}{2}} V(g)^{\frac{1}{2}} = 2\left(\frac{2\|g\|_{L^q}}{\|g\|_{L^1}}\right)^{\frac{q}{2(q-1)}} V(g)^{\frac{1}{2}}.\end{aligned}$$

  Rearranging this inequality provides the upper bound for the localization measure:

$$\left(2^{\frac{5q-4}{q-1}} \frac{V(g)}{\|g\|_{Lq}^2}\right)^{\frac{q-1}{3q-2}} \geq \frac{\|g\|_{L^1}}{\|g\|_{L^q}}.$$

- Second, apply this bound to both f and Af, and multiply the results together. According to the norm uncertainty principle (Theorem 1.32), the product of the localization measures is bounded below by $k^{\frac{q-1}{q}}$. Combining this with the bound from the first step gives:

$$\frac{V(f)}{\|f\|_{L^q}^2}\frac{V(Af)}{\|Af\|_{L^q}^2} \geq 2^{-\frac{10q-8}{q-1}} \left(\frac{\|f\|_{L^1}}{\|f\|_{L^q}}\frac{\|Af\|_{L^1}}{\|Af\|_{L^q}}\right)^{\frac{3q-2}{q-1}} \geq 2^{-\frac{10q-8}{q-1}} k^{\frac{3q-2}{q}}.$$

□

For any $1 < r < \infty$, we define the r-th moment of a function f as:

$$M_r(f) = \int_{\mathbb{R}} |x|^r |f(x)|^2 \, dx.$$

In their work, A. Wigderson and Y. Wigderson introduced a novel uncertainty principle concerning these higher order moments. This result is a natural and broad generalization of the classical Heisenberg uncertainty principle.

Theorem 1.37 ([1, Thm.4.14] Heisenberg uncertainty principle for higher moments)**.** Let $1 < r, s < \infty$ and $1 < q < \infty$. For any k-Hadamard operator A and any nonzero doubly L^1 function f, the following inequality holds:

$$\left(\frac{M_r(f)}{\|f\|_{L^q}^2}\right)^{\frac{q-1}{qr+q-2}} \left(\frac{M_s(Af)}{\|Af\|_{L^q}^2}\right)^{\frac{q-1}{qs+q-2}} \geq C_{r,q}C_{s,q}k^{\frac{q-1}{q}},$$

for some constants $C_{r,q}, C_{s,q} > 0$ that depend only on the pairs (r, q) and (s, q), respectively. Under the same hypotheses, if $q = \infty$, we obtain:

$$\left(\frac{M_r(f)}{\|f\|_{L^\infty}^2}\right)^{\frac{1}{r+1}} \left(\frac{M_s(Af)}{\|Af\|_{L^\infty}^2}\right)^{\frac{1}{s+1}} \geq C_{r,\infty}C_{s,\infty}k.$$

The proof of this theorem follows logical steps that are analogous to those used in the demonstration of Theorem 1.35. For the complete and detailed proof, we refer the reader to [1, Theorem 4.14].

1.4 Generalization of the Wigdersons' approach

(**Note.** This section is based primarily on the article by N.C. Dias, F. Luef, and J.N. Prata in [4])

A further generalization of the Wigdersons' approach to the uncertainty principle is provided by N.C. Dias, F. Luef, and J.N. Prata in [4]. Specifically, they introduce

a new condition that a generic operator A must satisfy for this generalization to hold.

The key observation is that a norm-based uncertainty principle can be written as:

$$\mathcal{F}_{p,q}^{(A,d)}(f) = \frac{\|f\|_{L^p}\|Af\|_{L^p}}{\|f\|_{L^q}\|Af\|_{L^q}} \geq C,$$

for some values of p, q, and dimension d. This naturally leads to the following question: what properties must the operator A possess to satisfy this bound?

To answer this, it is instructive to examine the work of Tang [2], who resolved the Wigdersons' conjecture [1, Conj.4.13]. Since the original conjecture is inherently focused on the classical Fourier transform, Tang achieved his proof by exploiting a fundamental property of this operator: it satisfies the Hausdorff-Young inequality. This inequality provides a crucial bound that directly connects the norm of a function f to the norm of its transform $\widehat{f}$. This cross dependence overcomes a structural limitation of the Wigdersons' initial approach, which applied the localization measure to f and $\hat{f}$ entirely separately, without accounting for their interplay.

Building on this insight, Tang later demonstrated in [3] both the multidimensional Heisenberg uncertainty principle, achieving a lower bound with the correct dependence on the dimension, and the Cowling-Price principle. He successfully established both results using the Wigdersons' method, precisely because the Fourier transform satisfies the Hausdorff-Young inequality. These multidimensional extensions had previously been left open by the Wigdersons due to severe divergence issues in higher dimensions.

Motivated by Tang's results, we can extend this framework to a generic operator A by requiring it to satisfy a similar condition. To formalize this property, we introduce the following definition:

Definition 1.38 (Hausdorff-Young inequalities for an operator)**.** Let $p \in [1,2]$ and $p' = p/(p-1)$, with $p' = \infty$ for $p = 1$. Let

$$A_p : L^p(\mathbb{R}^d) \longrightarrow L^{p'}(\mathbb{R}^d)$$

be a bounded injective linear operator. We say that A satisfies the two-sided generalized Hausdorff–Young inequalities at exponent p if there exist constants $C_{p,d}^+, C_{p,d}^- > 0$ such that

$$\|A_p f\|_{L^{p'}} \leq C_{p,d}^+ \|f\|_{L^p} \qquad \text{for all } f \in L^p(\mathbb{R}^d),$$

and, for every $g \in \mathrm{Ran}(A_p) \cap L^p(\mathbb{R}^d)$,

$$A_p^{-1} g \in L^{p'}(\mathbb{R}^d), \qquad \|A_p^{-1} g\|_{L^{p'}} \leq C_{p,d}^- \|g\|_{L^p}.$$

Equivalently, whenever $f \in L^p(\mathbb{R}^d)$ and $Af \in L^p(\mathbb{R}^d)$, the second condition reads

$$\|f\|_{L^{p'}} \leq C^-_{p,d} \|Af\|_{L^p}.$$

For a fixed operator A, set $C_{p,d} := \max\{C^+_{p,d}, C^-_{p,d}\}$; the dependence on A is suppressed only after this convention is stated. For a special k-Hadamard operator, the inverse-on-range formulation is precisely the form used in Dias, Luef, and Prata, Proposition 5.

Theorem 1.39 (cf.[4, Thm.6])**.** Fix $p \in [1,2]$, and let A be a k-Hadamard operator satisfying the two-sided generalized Hausdorff–Young inequalities of Definition 1.38 at this exponent. Then, for every $q \in [p, p']$, there exists a constant $C_{p,q,d} > 0$ such that

$$\mathcal{F}^{(A,d)}_{p,q}(f) = \frac{\|f\|_{L^p}\|Af\|_{L^p}}{\|f\|_{L^q}\|Af\|_{L^q}} \geq C_{p,q,d},$$

holds for all nonzero functions f such that $f, Af \in L^p(\mathbb{R}^d)$

Proof. The result is trivially true if $p = 2$, or if $p < 2$ and $p = q$. Furthermore, for $p = 1$ and $q \in [1, \infty]$, the claim reduces to the infinitary norm uncertainty principle, Theorem 1.32.

So we consider $1 < p < 2$. We begin by recalling that if $f, Af \in L^p(\mathbb{R}^d)$, the assumption that the k-Hadamard operator A satisfies the Hausdorff-Young inequalities implies that $f, Af \in L^{p'}(\mathbb{R}^d)$. By explicitly isolating the L^p norms and using the Hausdorff-Young constant $C_{p,d}$, we obtain the following precise bounds:

$$\|f\|_{L^p} \geq \frac{1}{C_{p,d}}\|Af\|_{L^{p'}} \quad \text{and} \quad \|Af\|_{L^p} \geq \frac{1}{C_{p,d}}\|f\|_{L^{p'}}.$$

Multiplying these two inequalities yields:

$$\|f\|_{L^p}\|Af\|_{L^p} \geq \frac{1}{C^2_{p,d}}\|f\|_{L^{p'}}\|Af\|_{L^{p'}},$$

which proves that the theorem holds for the upper endpoint $q = p'$, with the explicit constant $1/C^2_{p,d}$. Therefore, for the remainder of the proof, we can restrict our attention to the intermediate case where $p \neq 2$ and $p < q < p'$.

Let us define the auxiliary parameter θ as:

$$\theta = \frac{p' - p}{p' - q}.$$

Since $p < q < p'$, it immediately follows that $\theta > 1$. Furthermore, its Hölder conjugate exponent $\theta' = \frac{\theta}{\theta - 1}$ satisfies the following algebraic relations:

$$q - \frac{p}{\theta} = \frac{p'(q-p)}{p'-p} > 0, \quad \text{and} \quad \frac{q\theta - p}{\theta - 1} = \theta'\left(q - \frac{p}{\theta}\right) = p'.$$

By applying Hölder's inequality with conjugate exponents θ and θ' to the function $|f|^q$, we obtain:

$$\begin{aligned}\|f\|_{L^q}^q = \int_{\mathbb{R}^d} |f|^q \, dx &= \int_{\mathbb{R}^d} |f|^{\frac{p}{\theta}} |f|^{q-\frac{p}{\theta}} \, dx \\ &\leq \left(\int_{\mathbb{R}^d} |f|^p \, dx \right)^{\frac{1}{\theta}} \left(\int_{\mathbb{R}^d} |f|^{\theta'\left(q-\frac{p}{\theta}\right)} \, dx \right)^{\frac{1}{\theta'}} \\ &= \|f\|_{L^p}^{\frac{p}{\theta}} \|f\|_{L^{p'}}^{q-\frac{p}{\theta}}.\end{aligned}$$

We now multiply both sides of this chain of inequalities by $\|f\|_{L^p}^{q-\frac{p}{\theta}}$, which yields:

$$\|f\|_{L^p}^{q-\frac{p}{\theta}} \|f\|_{L^q}^q \leq \|f\|_{L^p}^q \|f\|_{L^{p'}}^{q-\frac{p}{\theta}}.$$

By rearranging the terms, this inequality can be rewritten as:

$$\left(\frac{\|f\|_{L^p}}{\|f\|_{L^{p'}}} \right)^{q-\frac{p}{\theta}} \leq \left(\frac{\|f\|_{L^p}}{\|f\|_{L^q}} \right)^q,$$

or equivalently, by taking the q-th root on both sides:

$$\frac{\|f\|_{L^p}}{\|f\|_{L^q}} \geq \left(\frac{\|f\|_{L^p}}{\|f\|_{L^{p'}}} \right)^{1-\frac{p}{\theta q}}.$$

Direct substitution of the algebraic relations defined earlier for θ shows that the exponent simplifies as follows:

$$1 - \frac{p}{\theta q} = \frac{p'(q-p)}{q(p'-p)} = \frac{\frac{1}{p} - \frac{1}{q}}{\frac{1}{p} - \frac{1}{p'}}.$$

Substituting this back, we establish the fundamental norm estimate for f:

$$\frac{\|f\|_{L^p}}{\|f\|_{L^q}} \geq \left(\frac{\|f\|_{L^p}}{\|f\|_{L^{p'}}} \right)^{\frac{1/p-1/q}{1/p-1/p'}}.$$

Since the exact same analytic steps apply to the function Af, we similarly obtain:

$$\frac{\|Af\|_{L^p}}{\|Af\|_{L^q}} \geq \left(\frac{\|Af\|_{L^p}}{\|Af\|_{L^{p'}}} \right)^{\frac{1/p-1/q}{1/p-1/p'}}.$$

Finally, by multiplying these last two estimates and leveraging the explicit Hausdorff-Young product bound established at the beginning of the proof, we conclude:

$$\mathcal{F}_{p,q}^{(A,d)}(f) = \frac{\|f\|_{L^p}\|Af\|_{L^p}}{\|f\|_{L^q}\|Af\|_{L^q}} \geq \left(\frac{\|f\|_{L^p}\|Af\|_{L^p}}{\|f\|_{L^{p'}}\|Af\|_{L^{p'}}} \right)^{\frac{1/p-1/q}{1/p-1/p'}} \geq \left(\frac{1}{C_{p,d}^2} \right)^{\frac{1/p-1/q}{1/p-1/p'}}.$$

Thus, the bound holds with the explicit constant $C_{p,q,d} = C_{p,d}^{-2\frac{1/p-1/q}{1/p-1/p'}}$, completing the proof. □

Definition 1.40 ([4, Def.2] Special k-Hadamard operator)**.** Let $A : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ be a bounded linear operator with $\|A\|_{1\to\infty} \leq 1$, which is also bounded and invertible on $L^2(\mathbb{R}^d)$. We say that A is a *special k-Hadamard operator* if there exists a constant $k > 0$ such that its adjoint A^* satisfies:

$$A^* = kA^{-1},$$

and if A^* is a bounded operator $A^* : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$.

Theorem 1.41 ([4, Prop.5])**.** Let A be a special k-Hadamard operator. Then, A is a k-Hadamard operator. Furthermore, for any $p \in [1,2]$, A satisfies the two sided generalized Hausdorff–Young , under the adjoint convention of [4, Rem. 3], one may take the common constant $C_{p,d} = \max\{k^{(p-1)/p}, k^{-1/p}\}$. Despite the notation, this particular interpolation constant has no additional dependence on d.

Proof. By the definition of a special k-Hadamard operator, its adjoint satisfies the relation $A^* = kA^{-1}$. Multiplying both sides by A yields $A^*A = kI$. This identity directly implies that A is a k-Hadamard operator.

To establish the Hausdorff-Young inequalities, we must make explicit the adjoint convention in [4, Rem.3]. For Lebesgue measure on $\mathbb{R}^d$, every bounded linear functional on L^1 is represented by an L^∞ function. Using the standard sesquilinear pairing, a bounded operator $A : L^1 \to L^\infty$ therefore determines the bounded form

$$\mathfrak{b}_A(f,g) := \int_{\mathbb{R}^d} (Af)(x)\,\overline{g(x)}\,dx, \qquad f,g \in L^1(\mathbb{R}^d),$$

and

$$\sup_{\|f\|_{L^1}\leq 1,\, \|g\|_{L^1}\leq 1} |\mathfrak{b}_A(f,g)| = \|A\|_{L^1\to L^\infty}.$$

For each fixed $g \in L^1$, the map $f \mapsto \mathfrak{b}_A(f,g)$ is a bounded linear functional on L^1. Since $(L^1)^* = L^\infty$, there is a unique $A^\sharp g \in L^\infty$ such that

$$\int_{\mathbb{R}^d} (Af)\,\overline{g} = \int_{\mathbb{R}^d} f\,\overline{A^\sharp g}, \qquad f \in L^1(\mathbb{R}^d).$$

The dependence on g is linear, so this defines an operator $A^\sharp : L^1 \to L^\infty$. Taking the two suprema in the opposite order gives

$$\|A^\sharp\|_{L^1\to L^\infty} = \sup_{\|g\|_{L^1}\leq 1} \sup_{\|f\|_{L^1}\leq 1} |\mathfrak{b}_A(f,g)| = \|A\|_{L^1\to L^\infty}.$$

This gives a precise functional-analytic formulation of [4, Rem. 3] without conflating two different adjoints. The ordinary Banach adjoint of $A : L^1 \to L^\infty$ has domain

$(L^\infty)^*$; the map $A^\sharp$ above is instead the adjoint realization obtained from the chosen L^1 predual.

The definition of a special k-Hadamard operator uses the same symbol A^* for the Hilbert-space adjoint of the L^2 realization and for an $L^1 \to L^\infty$ realization. The compatibility can be stated precisely as follows. On the dense common core $L^1 \cap L^2$, the Hilbert adjoint identity gives

$$\langle Af, g\rangle_{L^2} = \langle f, A_2^* g\rangle_{L^2}.$$

If, as required in Definition 2, this A_2^* extends boundedly from L^1 to L^∞, then both sides define continuous sesquilinear forms on $L^1 \times L^1$. Since $L^1 \cap L^2$ is dense in L^1, the adjoint identity extends to all $f, g \in L^1$. Consequently, this extension is exactly the predual realization $A^\sharp$ characterized above. Therefore,

$$A^* = A^\sharp \quad \text{as maps } L^1 \to L^\infty, \qquad \|A^*\|_{L^1\to L^\infty} = \|A\|_{L^1\to L^\infty} \le 1.$$

This is the functional-analytic content needed from [4, Rem.3]: the ordinary Banach adjoint still has domain $(L^\infty)^*$, whereas $A^\sharp$ is the adjoint realization furnished by the chosen L^1 predual pairing; the endpoint norm is unchanged, and the bounded $L^1 \to L^\infty$ extension of the L^2 Hilbert adjoint required in Definition 2 is that realization. Hence no additional hypothesis $\|A^*\|_{L^1\to L^\infty} \le 1$ has to be added as a "strengthened definition."

The identity $A^* = kA^{-1}$ on L^2 gives $A^*A = kI$ and, since A is invertible, $AA^* = kI$. Therefore

$$\|A\|_{L^2\to L^2} = \|A^*\|_{L^2\to L^2} = \sqrt{k}.$$

Riesz-Thorin interpolation between the (L^1, L^∞) and (L^2, L^2) endpoint bounds then yields, for every $1 \le p \le 2$,

$$\|A\|_{L^p\to L^{p'}} \le k^{(p-1)/p}, \qquad \|A^*\|_{L^p\to L^{p'}} \le k^{(p-1)/p}.$$

Hence

$$\|Af\|_{L^{p'}} \le k^{(p-1)/p}\|f\|_{L^p}.$$

For the reverse estimate one uses, exactly as in [4, Prop. 5], the compatible identity $A^{-1} = k^{-1}A^*$ together with the interpolated $L^p \to L^{p'}$ realization of A^*. Thus, whenever $Af \in L^p$ in the admissible class of that proposition,

$$\|f\|_{L^{p'}} = \|A^{-1}(Af)\|_{L^{p'}} \le k^{-1/p}\|Af\|_{L^p}.$$

Consequently, the two inequalities may be written with the common constant

$$C_{p,d} = \max\Big\{k^{(p-1)/p}, k^{-1/p}\Big\}.$$

The subscript d is retained only to match the manuscript's notation; this explicit interpolation constant has no further dimension dependence. If one were to choose some unrelated bounded extension and denote it by A^* without the predual/Hilbert compatibility just described, equality of the endpoint norms would no longer follow and its separate $L^1 \to L^\infty$ norm would have to be retained. □

Remark 1.42. Before proceeding, a brief methodological note is in order. In the following sections, we will primarily focus our attention on special k-Hadamard operators. It is important to emphasize that the class of k-Hadamard operators satisfying the Hausdorff-Young inequality is potentially broader than the class of special k-Hadamard operators. Nevertheless, all the uncertainty principles and inequalities we will derive for the special class hold equally true for the broader one.

Furthermore, the methods employed in this framework are not well suited to determine sharp constants. For this reason, to simplify the exposition and keep the focus on the functional dependencies, we will frequently adopt the symbol $\gtrsim$ in place of the standard $\geq$. In this context, the notation

$$F(f) \gtrsim G(f),$$

means that there exists a strictly positive constant C such that $F(f) \geq C \cdot G(f)$ for all functions f in the appropriate domain.

To fully appreciate the power of this generalized framework, it is instructive to draw a direct comparison with the original method developed by the Wigdersons. In their foundational approach, the core strategy relies on bounding a specific localization measure, say $H(f)$, by a primary functional $H_0(f)$, such that $H(f) \gtrsim H_0(f)$ and, analogously, $H(Af) \gtrsim H_0(Af)$. By multiplying these two estimates, one readily obtains an uncertainty principle of the form:

$$H(f)H(Af) \gtrsim H_0(f)H_0(Af) \gtrsim k,$$

where k is an absolute constant given by the primary uncertainty principles.

In our generalized setting, this paradigm is significantly expanded. First, we are no longer constrained to apply the same localization measure to both the function and its transform; we can seamlessly combine two distinct measures, say $H_1(f)$ and $H_2(Af)$, provided they are appropriately related. Second, and most importantly, the role of the fixed primary lower bound H_0 is now played by the mixed-norm ratio $H_{p,q}(f, Af) = \mathcal{F}_{p,q}^{(A,d)}(f)$. By leveraging the generalized bound established in Theorem 1.39, we can derive a broad class of inequalities:

$$H_1(f)H_2(Af) \gtrsim (\mathcal{F}_{p,q}^{(A,d)}(f))^\alpha \gtrsim C_{p,q,d}.$$

This conceptual shift from a fixed H_0 to the parametric functional $\mathcal{F}_{p,q}^{(A,d)}$ is exactly what provides the flexibility to tune the parameters p and q according to the specific uncertainty principles we wish to investigate.

For this purpose, we first establish two inequalities that will be used in the subsequent uncertainty principles.

Theorem 1.43 ([4, Thm.9])**.** Assume $1 \leq u \leq r$, $p > u$, $1 < t \leq \infty$, and $\theta > d\left(\frac{1}{u} - \frac{1}{p}\right)$. These assumptions ensure that the Hölder exponents used in the proof are legitimate. If $u < r$, assume $0 \neq f \in L^r(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap L^{rt}(\mathbb{R}^d)$ and that the weighted norm is finite. Then we have the following inequality:

$$\||x|^\theta f\|_{L^p} \gtrsim \|f\|_{L^\infty} \left(\frac{\|f\|_{L^r}}{\|f\|_{L^\infty}}\right)^{\frac{r}{u}} \left(\frac{\|f\|_{L^r}}{\|f\|_{L^{rt}}}\right)^{\frac{rt(u\theta p - d(p-u))}{dpu(t-1)}}. \tag{1.6}$$

If $u = r$, assume $0 \neq f \in L^r(\mathbb{R}^d) \cap L^{rt}(\mathbb{R}^d)$ and that the weighted norm is finite. Then the conclusion can be written without the artificial factor $\|f\|_{L^\infty}$:

$$\||x|^\theta f\|_{L^p} \gtrsim \|f\|_{L^r} \left(\frac{\|f\|_{L^r}}{\|f\|_{L^{rt}}}\right)^{\frac{rt(r\theta p - d(p-r))}{dpr(t-1)}}.$$

The $t = \infty$ case is obtained by taking the limit in the exponent.

Proof. If $\||x|^\theta f\|_{L^p} = \infty$, the inequality holds trivially, so we assume that this norm is finite. Let us introduce the normalized function:

$$\tilde{f} = \frac{f}{\|f\|_{L^\infty}}.$$

We begin by bounding the L^r-norm of $\tilde{f}$ inside a ball $B_T = \{x \in \mathbb{R}^d : |x| \leq T\}$ of an arbitrary, yet to be determined radius T. By applying Hölder's inequality with conjugate exponents t and $\frac{t}{t-1}$, we obtain:

$$\begin{aligned}\int_{|x|\leq T} |\tilde{f}|^r \, dx &= \int_{\mathbb{R}^d} \chi_{B_T} |\tilde{f}|^r \, dx \\ &\leq \|\chi_{B_T}\|_{\frac{t}{t-1}} \||\tilde{f}|^r\|_t \\ &= (v_d T^d)^{\frac{t-1}{t}} \|\tilde{f}\|_{L^{rt}}^r,\end{aligned}$$

where $v_d = \frac{2\pi^{d/2}}{d\Gamma(d/2)}$ denotes the volume of the unit ball in $\mathbb{R}^d$.

We want to choose T such that at least half of the L^r-mass of $\tilde{f}$ lies outside the ball, which means we impose the upper bound on the interior mass to be $\frac{1}{2}\|\tilde{f}\|_{L^r}^r$. Setting the right-hand side of the previous inequality to this value yields:

$$(v_d T^d)^{\frac{t-1}{t}} \|\tilde{f}\|_{L^{rt}}^r = \frac{1}{2}\|\tilde{f}\|_{L^r}^r.$$

Solving this equation for T explicitly gives the balancing radius:

$$T = \frac{1}{v_d^{1/d}} \left(\frac{1}{2}\right)^{\frac{t}{d(t-1)}} \left(\frac{\|\tilde{f}\|_{L^r}}{\|\tilde{f}\|_{L^{rt}}}\right)^{\frac{rt}{d(t-1)}} = \frac{1}{v_d^{1/d}} \left(\frac{1}{2}\right)^{\frac{t}{d(t-1)}} \left(\frac{\|f\|_{L^r}}{\|f\|_{L^{rt}}}\right)^{\frac{rt}{d(t-1)}}.$$

(Note that if $t = \infty$, taking the limit simplifies this expression to $T = \left(\frac{1}{2v_d}\right)^{\frac{1}{d}} \left(\frac{\|f\|_{L^r}}{\|f\|_{L^\infty}}\right)^{\frac{r}{d}}$).

By this construction of T, we are guaranteed that the exterior integral satisfies:

$$\int_{|x|>T} |\tilde{f}|^r \, dx \geq \frac{1}{2}\|\tilde{f}\|_{L^r}^r.$$

Now we bound this exterior integral from above. Using the fact that $|\tilde{f}| \leq 1$ almost everywhere, we have $|\tilde{f}|^r \leq |\tilde{f}|^u$ since $r \geq u$. We multiply and divide the integrand by $|x|^{\theta u}$ and apply Hölder's inequality with conjugate exponents $\frac{p}{p-u}$ and $\frac{p}{u}$:

$$\begin{aligned}
\frac{1}{2}\|\tilde{f}\|_{L^r}^r &\leq \int_{|x|>T} |\tilde{f}|^u \, dx \\
&= \int_{|x|>T} \frac{1}{|x|^{\theta u}} \left(|x|^{\theta u} |\tilde{f}|^u\right) dx \\
&\leq \left(\int_{|x|>T} \frac{1}{|x|^{\frac{\theta u p}{p-u}}} \, dx\right)^{\frac{p-u}{p}} \left(\int_{\mathbb{R}^d} |x|^{\theta p} |\tilde{f}(x)|^p \, dx\right)^{\frac{u}{p}} \\
&= \left(\omega_d \int_T^\infty \rho^{d-1-\frac{\theta u p}{p-u}} \, d\rho\right)^{\frac{p-u}{p}} \||x|^\theta \tilde{f}\|_{L^p}^u,
\end{aligned}$$

where $\omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the surface area of the unit sphere.

The radial integral converges because the hypothesis $\theta > d\left(\frac{1}{u} - \frac{1}{p}\right)$ guarantees that the exponent is strictly less than -1. Evaluating it gives:

$$\int_T^\infty \rho^{d-1-\frac{\theta u p}{p-u}} \, d\rho = \frac{p-u}{\theta u p - d(p-u)} T^{\frac{d(p-u)-\theta u p}{p-u}}.$$

Substituting this evaluation back into our main estimate and absorbing all purely numerical and geometric constants into the $\lesssim$ symbol, we obtain:

$$\|\tilde{f}\|_{L^r}^r \lesssim T^{\frac{d(p-u)-\theta u p}{p}} \||x|^\theta \tilde{f}\|_{L^p}^u.$$

Isolating the weighted norm and substituting the explicitly derived expression

for T, we find the functional dependence:

$$\begin{aligned}
\||x|^\theta \tilde{f}\|_{L^p} &\gtrsim \|\tilde{f}\|_{L^r}^{\frac{r}{u}} T^{\frac{u\theta p - d(p-u)}{up}} \\
&\iff \||x|^\theta \tilde{f}\|_{L^p} \gtrsim \|\tilde{f}\|_{L^r}^{\frac{r}{u}} \left(\frac{\|f\|_{L^r}}{\|f\|_{L^{rt}}}\right)^{\frac{rt(u\theta p - d(p-u))}{(t-1)dpu}} \\
&\iff \||x|^\theta f\|_{L^p} \gtrsim \|f\|_{L^\infty} \left(\frac{\|f\|_{L^r}}{\|f\|_{L^\infty}}\right)^{\frac{r}{u}} \left(\frac{\|f\|_{L^r}}{\|f\|_{L^{rt}}}\right)^{\frac{rt(u\theta p - d(p-u))}{(t-1)dpu}}.
\end{aligned}$$

Notice that all the norms appearing in these steps are finite since $f \in L^r \cap L^\infty \cap L^{rt}$ when $u \neq r$. Finally, if $u = r$, the conclusion follows by the same argument applied directly to f instead of the normalized function $\tilde{f}$. Specifically, since the condition $r = u$ turns the bound $|f|^r \leq |f|^u$ into a trivial equality, we no longer need to enforce $|\tilde{f}| \leq 1$ almost everywhere. Consequently, the normalization step is bypassed entirely, and the assumption $f \in L^\infty(\mathbb{R}^d)$ is not required.

□

Theorem 1.44 ([4, Thm.10])**.** Let $\theta, p > 0$ and $q \in (p, \infty]$. Then

$$\||x|^\theta f\|_{L^p} \gtrsim \left(\frac{\|f\|_{L^p}}{\|f\|_{L^q}}\right)^{\frac{\theta q p}{d(q-p)}} \|f\|_{L^p},$$

for all $0 \neq f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$.

Proof. As above, applying Hölder's inequality with conjugate exponents t and $\frac{t}{t-1}$, we have:

$$\begin{aligned}
\int_{|x|\leq T} |f|^p \, dx &= \int_{\mathbb{R}^d} \chi_{B_T} |f|^p \, dx \\
&\leq \|\chi_{B_T}\|_{\frac{t}{t-1}} \||f|^p\|_t \\
&= (v_d T^d)^{\frac{t-1}{t}} \|f\|_{pt}^p,
\end{aligned}$$

where v_d is the volume of the unit ball in $\mathbb{R}^d$.

We impose that exactly half of the L^p-mass of f lies outside the ball, setting the interior bound to $\frac{1}{2}\|f\|_{L^p}^p$:

$$(v_d T^d)^{\frac{t-1}{t}} \|f\|_{pt}^p = \frac{1}{2}\|f\|_{L^p}^p.$$

Solving for T yields the optimal radius:

$$T = \frac{1}{v_d^{1/d}} \left(\frac{1}{2}\right)^{\frac{t}{d(t-1)}} \left(\frac{\|f\|_{L^p}}{\|f\|_{pt}}\right)^{\frac{pt}{d(t-1)}}.$$

(If $t = \infty$, the limit simplifies this to $T \simeq (\|f\|_{L^p}/\|f\|_{L^\infty})^{p/d}$).

With this specific choice of T, we are guaranteed that the exterior integral holds at least half the mass:

$$\int_{|x|>T} |f|^p \, dx \geq \frac{1}{2}\|f\|_{L^p}^p.$$

To bound this exterior integral from above, we multiply and divide the integrand by $|x|^{\theta p}$. Since we are integrating over the region where $|x| > T$, we trivially have $\frac{1}{|x|^{\theta p}} < \frac{1}{T^{\theta p}}$, which allows us to extract T directly without invoking Hölder's inequality again:

$$\begin{aligned}
\frac{1}{2}\|f\|_{L^p}^p &\leq \int_{|x|>T} |f|^p \, dx \\
&= \int_{|x|>T} \frac{1}{|x|^{\theta p}} \left(|x|^{\theta p}|f|^p\right) \, dx \\
&\leq \frac{1}{T^{\theta p}} \int_{|x|>T} |x|^{\theta p}|f|^p \, dx \\
&\leq \frac{1}{T^{\theta p}} \||x|^\theta f\|_{L^p}^p.
\end{aligned}$$

Taking the p-th root of both sides and absorbing the purely numerical factor $(1/2)^{1/p}$ into the $\gtrsim$ symbol, we obtain:

$$\||x|^\theta f\|_{L^p} \gtrsim T^\theta \|f\|_{L^p}.$$

Finally, we substitute our derived expression for T. Absorbing the geometric constants into $\gtrsim$ once more, we establish the functional dependence:

$$\||x|^\theta f\|_{L^p} \gtrsim \left(\frac{\|f\|_{L^p}}{\|f\|_{pt}}\right)^{\frac{\theta pt}{d(t-1)}} \|f\|_{L^p}.$$

To match the theorem's statement, we simply perform the substitution $q = pt$. Since $1 < t \leq \infty$, it naturally follows that $q \in (p, \infty]$. The exponent transforms algebraically as:

$$\frac{\theta pt}{d(t-1)} = \frac{\theta p(\frac{q}{p})}{d(\frac{q}{p}-1)} = \frac{\theta qp}{d(q-p)}.$$

Substituting this relation into our bound yields the desired result. □

1.4.1 General uncertainty principles for special k-Hadamard operators

Theorem 1.45 ([4, Thm.17])**.** Let A be a special k-Hadamard operator, and let the parameters satisfy $\theta > 0$, $p \in [1, 2)$, and $q \in [p, p']$. Then:

$$\||x|^\theta f\|_{L^p} \||\xi|^\theta Af\|_{L^p} \gtrsim \|f\|_{L^q} \|Af\|_{L^q},$$

for all functions f such that $f, Af \in L^p(\mathbb{R}^d)$.

Proof. We first note that since A is a special k-Hadamard operator, the assumption $f, Af \in L^p(\mathbb{R}^d)$ guarantees that $f, Af \in L^s(\mathbb{R}^d)$ for all $s \in [p, p']$ by the Hausdorff-Young inequality and interpolation. Thus, the L^r and L^q norms used in the following steps are well-defined and finite.

Applying the bound from Theorem 1.44 with an auxiliary exponent $r \in (p, p']$, we have for the function f:

$$\||x|^\theta f\|_{L^p} \gtrsim \|f\|_{L^p} \left(\frac{\|f\|_{L^p}}{\|f\|_{L^r}}\right)^{\frac{\theta r p}{d(r-p)}}.$$

The analogous bound holds for the transformed function Af. Multiplying these two inequalities yields:

$$\||x|^\theta f\|_{L^p} \||\xi|^\theta Af\|_{L^p} \gtrsim \|f\|_{L^p}\|Af\|_{L^p} \left(\frac{\|f\|_{L^p}\|Af\|_{L^p}}{\|f\|_{L^r}\|Af\|_{L^r}}\right)^{\frac{\theta r p}{d(r-p)}}.$$

Since A is a special k-Hadamard operator, Theorem 1.39 guarantees that the functional ratio $\|f\|_{L^p}\|Af\|_{L^p}/(\|f\|_{L^r}\|Af\|_{L^r})$ is bounded from below by a strictly positive constant $C_{p,r,d}$. Absorbing this constant, we obtain:

$$\||x|^\theta f\|_{L^p} \||\xi|^\theta Af\|_{L^p} \gtrsim \|f\|_{L^p}\|Af\|_{L^p}.$$

Finally, invoking Theorem 1.39 again, the condition $q \in [p, p']$ implies $\|f\|_{L^p}\|Af\|_{L^p} \gtrsim \|f\|_{L^q}\|Af\|_{L^q}$. Substituting this lower bound completes the proof. □

Theorem 1.46 ([4, Thm.18])**.** Let A be a special k-Hadamard operator, and let $\alpha, \beta, \theta, \phi, p, q, r$ be parameters satisfying:

$$1 \le r < 2, \quad \alpha, \beta > 0, \quad p, q > r, \quad \theta > d\left(\frac{1}{r} - \frac{1}{p}\right), \quad \text{and} \quad \phi > d\left(\frac{1}{r} - \frac{1}{q}\right),$$

along with the balance condition:

$$\alpha\left[\theta - d\left(\frac{1}{r} - \frac{1}{p}\right)\right] = \beta\left[\phi - d\left(\frac{1}{r} - \frac{1}{q}\right)\right].$$

Then, the following uncertainty principle holds:

$$\||x|^\theta f\|_{L^p}^\alpha \||\xi|^\phi Af\|_{L^q}^\beta \gtrsim \|f\|_{L^r}^\alpha \|Af\|_{L^r}^\beta,$$

for all functions f such that $f, Af \in L^r(\mathbb{R}^d)$.

Proof. Before applying the generalized Sobolev-type inequality, we must rigorously justify that f and Af belong to L^{rt}. We begin by restricting the auxiliary exponent t to the interval $1 < t \leq r'/r$, where r' is the Hölder conjugate of r. Since $1 \leq r < 2$, we have $r' > 2$, which ensures that $r'/r > 1$ and the interval is well-defined.

By our initial hypotheses, $f, Af \in L^r(\mathbb{R}^d)$. Since A is a special k-Hadamard operator, it satisfies the two sided Hausdorff-Young estimate. This property implies that $f \in L^r$ yields $Af \in L^{r'}$, and conversely, $Af \in L^r$ yields $f \in L^{r'}$. Consequently, both f and Af belong to the intersection $L^r \cap L^{r'}$. Because our choice of t guarantees that $rt \in (r, r']$, standard Lebesgue space interpolation ensures that $f, Af \in L^{rt}(\mathbb{R}^d)$.

Having verified all necessary hypotheses, we can now safely apply Theorem 1.43 with $u = r$. For the function f, this yields:

$$\||x|^\theta f\|_{L^p} \gtrsim \|f\|_{L^r} \left(\frac{\|f\|_{L^r}}{\|f\|_{L^{rt}}} \right)^{\frac{t(r\theta p - d(p-r))}{dp(t-1)}}.$$

The identical procedure for Af yields the symmetric estimate:

$$\||\xi|^\phi Af\|_{L^q} \gtrsim \|Af\|_{L^r} \left(\frac{\|Af\|_{L^r}}{\|Af\|_{L^{rt}}} \right)^{\frac{t(r\phi q - d(q-r))}{dq(t-1)}}.$$

Raising these inequalities to the powers α and β, respectively, and multiplying them together gives:

$$\||x|^\theta f\|_{L^p}^\alpha \||\xi|^\phi Af\|_{L^q}^\beta \gtrsim \|f\|_{L^r}^\alpha \|Af\|_{L^r}^\beta \left(\frac{\|f\|_{L^r}}{\|f\|_{L^{rt}}} \right)^{\frac{\alpha t(r\theta p - d(p-r))}{dp(t-1)}} \left(\frac{\|Af\|_{L^r}}{\|Af\|_{L^{rt}}} \right)^{\frac{\beta t(r\phi q - d(q-r))}{dq(t-1)}}.$$

By the balance condition assumed in the theorem, the fractional exponents on the right hand side are strictly identical. This allows us to combine the two ratios into a single term:

$$\||x|^\theta f\|_{L^p}^\alpha \||\xi|^\phi Af\|_{L^q}^\beta \gtrsim \|f\|_{L^r}^\alpha \|Af\|_{L^r}^\beta \left(\frac{\|f\|_{L^r}\|Af\|_{L^r}}{\|f\|_{L^{rt}}\|Af\|_{L^{rt}}} \right)^{\frac{\alpha t(r\theta p - d(p-r))}{dp(t-1)}}.$$

To complete the proof, recall that our choice of t guarantees $rt \in (r, r']$. For this range, Theorem 1.39 ensures that the product of the L^r norms bounds the product of the L^{rt} norms from above: $\|f\|_{L^r}\|Af\|_{L^r} \gtrsim \|f\|_{L^{rt}}\|Af\|_{L^{rt}}$. Consequently, the ratio inside the parenthesis is bounded below by a positive constant. Absorbing this constant into the $\gtrsim$ symbol, we arrive at the final result:

$$\||x|^\theta f\|_{L^p}^\alpha \||\xi|^\phi Af\|_{L^q}^\beta \gtrsim \|f\|_{L^r}^\alpha \|Af\|_{L^r}^\beta.$$

□

1.4.2 Multidimensional Heisenberg

Theorem 1.47 ([4, Thm.20])**.** Let A be a special k-Hadamard operator. Then, the uncertainty principle for the variance holds:

$$V(f)V(Af) \gtrsim \|f\|_s^2 \|Af\|_s^2,$$

for all functions f such that $f, Af \in L^s(\mathbb{R}^d) \cap L^{s'}(\mathbb{R}^d)$, where the exponent s is subject to the following dimension-dependent conditions:

$$\begin{cases} s \in [1, \infty] & \text{if } d = 1, \\ s \in (1, \infty) & \text{if } d = 2, \\ s \in \left(\frac{2d}{d+2}, \frac{2d}{d-2}\right) & \text{if } d \geq 3. \end{cases}$$

Proof. We first observe that if the variance $V(f) = \||x|f\|_{L^2}^2$ or $V(Af) = \||x|Af\|_{L^2}^2$ is infinite, the target inequality holds trivially, as the left-hand side is infinite while the right-hand side is finite by assumption. Therefore, we may assume that both variances are finite, meaning $|x|f, |x|Af \in L^2(\mathbb{R}^d)$.

Before invoking the generalized uncertainty principles, we must explicitly identify an auxiliary exponent $r \in [1, 2)$ that satisfies the dimensional constraint $r > \frac{2d}{d+2}$, while also guaranteeing that $f, Af \in L^r(\mathbb{R}^d)$ and $s \in [r, r']$. We define this exponent r depending on the given value of s:

- Case $s \neq 2$: We set $r = \min\{s, s'\}$. Since the theorem assumes $f, Af \in L^s(\mathbb{R}^d) \cap L^{s'}(\mathbb{R}^d)$, this choice immediately guarantees that $f, Af \in L^r(\mathbb{R}^d)$. Moreover, this definition trivially ensures that $s \in [r, r']$. The dimensional constraints on s explicitly force this r to lie in $[1, 2)$ and satisfy $r > \frac{2d}{d+2}$ (including the endpoint cases when $d = 1$).

- Case $s = 2$: The hypothesis only gives $f, Af \in L^2(\mathbb{R}^d)$, which does not inherently imply L^r membership for $r < 2$. However, combining $f, Af \in L^2$ with the finite variance property $|x|f, |x|Af \in L^2$, a standard weighted Hölder argument guarantees that $f, Af \in L^r(\mathbb{R}^d)$ for any $r \in \left(\frac{2d}{d+2}, 2\right)$. We simply fix one such admissible r. For this choice, $s = 2$ naturally lies in the interval $[r, r']$.

Having rigorously established that $f, Af \in L^r(\mathbb{R}^d)$ for a valid $r > \frac{2d}{d+2}$, we can now apply Theorem 1.46. Setting the exponents $\theta = \phi = 1$, using L^2 norms for the spatial weights ($p = q = 2$), and taking $\alpha = \beta = 2$, the dimensional constraint $1 > d(1/r - 1/2)$ simplifies exactly to our condition $r > \frac{2d}{d+2}$. This yields the lower bound:

$$V(f)V(Af) \gtrsim \|f\|_{L^r}^2 \|Af\|_{L^r}^2.$$

Since we explicitly constructed r such that $s \in [r, r']$ in both cases, Theorem 1.39 guarantees that $\|f\|_{L^r}\|Af\|_{L^r} \gtrsim \|f\|_s\|Af\|_s$. Substituting this estimate yields the desired inequality:

$$V(f)V(Af) \gtrsim \|f\|_s^2\|Af\|_s^2.$$

To clearly see how the condition $r > \frac{2d}{d+2}$ translates into the allowed ranges for s, we analyze the spatial dimension d:

- Case $d = 1$: The constraint is $r > 2/3$. Since $r \in [1, 2)$, we can reach down to $r = 1$, which implies $r' = \infty$. Thus, the theorem holds for all $s \in [1, \infty]$.
- Case $d = 2$: The constraint is $r > 1$. Since r can be arbitrarily close to 1 from above, its conjugate r' can be arbitrarily large, allowing validity for all $s \in (1, \infty)$.
- Case $d \geq 3$: The parameter r is strictly bounded by $\frac{2d}{d+2} < r < 2$. Since $r = \min\{s, s'\}$, this bounds s strictly between the lower limit $\frac{2d}{d+2}$ and its Hölder conjugate $\frac{2d}{d-2}$.

This completes the proof. □

Remark 1.48. Theorem 1.47 naturally encompasses the standard higher-dimensional Heisenberg uncertainty principle. Indeed, setting $s = 2$ directly yields:

$$V(f)V(Af) \gtrsim \|f\|_{L^2}^2\|Af\|_{L^2}^2.$$

As $d \to \infty$, both endpoints of the admissible interval $\left(\frac{2d}{d+2}, \frac{2d}{d-2}\right)$ converge to 2, so the finite-dimensional range of admissible exponents s shrinks to $\{2\}$ at the level of these endpoint formulas. For the classical Fourier transform, the stated open range is sharp away from its boundary. Specifically, in spatial dimensions $d \geq 3$, the inequality fails if the exponent falls strictly outside the valid interval, while the behavior at the threshold endpoints themselves remains unresolved. This failure is formalized in the following proposition.

Proposition 1.49 ([4, Prop.22])**.** Let $d \geq 3$ and assume the exponent s lies strictly outside the valid interval, such that $s \in \left[1, \frac{2d}{d+2}\right) \cup \left(\frac{2d}{d-2}, \infty\right]$. Then, there exists a sequence of Schwartz functions $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d)$ such that:

$$\frac{V(f_k)V(\widehat{f}_k)}{\|f_k\|_{L^q}^2\|\widehat{f}_k\|_{L^q}^2} \longrightarrow 0 \quad \text{as } k \to \infty.$$

Remark 1.50. We omit the formal proof of Proposition 1.49; the detailed construction of the sequence $(f_k)_{k\in\mathbb{N}}$, which relies on Hermite functions and specialized tensor products, can be found in [4].

Moreover, the exact behavior of the functional at the threshold endpoints, specifically $s = \frac{2d}{d+2}$ and $s = \frac{2d}{d-2}$ for $d \geq 3$, as well as $s = 1$ and $s = \infty$ for $d = 2$, remains unknown. Determining whether the uncertainty principle holds or fails at these boundaries is currently an open problem.

Chapter 2

Uncertainty Principles for Metaplectic Time-Frequency Representations

(**Note.** This chapter is based primarily on the articles by E. Cordero and L. Rodino [5, 6, 7], the articles by K. Gröchenig and I. Shafkulovska [8, 9, 10] and the work by N. Lerner [11])

2.1 Introduction to metaplectic time frequency representation

In the realm of time-frequency analysis, the mathematical framework is deeply rooted in the geometry of phase space, which is naturally modeled as a symplectic space. A symplectic space is equipped with a geometric structure that preserves the fundamental relationship between time and frequency variables. Transformations that preserve this structure are known as symplectic matrices.

According to the modern perspective introduced by Cordero and Rodino, the classical tools of time-frequency analysis, such as the Short-Time Fourier Transform and the Wigner distribution, can be unified and generalized through the concept of metaplectic operators. Metaplectic operators are specific unitary transformations that are intrinsically associated with symplectic matrices. Instead of relying on a fixed window function, a generalized time-frequency representation can be viewed directly as the action of a metaplectic operator. This paradigm shift highlights that symplectic matrices and metaplectic operators are the true protagonists of time-frequency analysis, paving the way for the so called metaplectic Wigner distributions.

The uncertainty principle is a fundamental concept in both quantum mechanics and harmonic analysis, which broadly states that a signal cannot be perfectly localized in both time and frequency simultaneously.

A natural question arises: do these uncertainty principles still hold in the broader context of metaplectic time-frequency representations? Gröchenig and Shafkulovska thoroughly investigated this problem, focusing on Benedicks-type uncertainty principles. Interestingly, they demonstrated that the answer is not always affirmative. While classical representations satisfy the Benedicks property, other metaplectic transformations can be perfectly localized on sets of finite measure without the underlying function being identically zero. Examples of such exceptions include specific coordinate changes or multiplications by a chirp signal. By deeply analyzing the algebraic structure of the underlying symplectic matrices, Gröchenig and Shafkulovska provided a complete characterization of the exact class of metaplectic Wigner distributions that obey the Benedicks uncertainty principle.

2.2 Mathematical Characterization of Symplectic and Metaplectic Structures

To formalize the framework of metaplectic time-frequency representations, we must first establish the geometric properties of the phase space and the algebraic structures acting upon it.

2.2.1 The Symplectic Group

The natural setting for time-frequency analysis in $\mathbb{R}^d$ is the phase space $\mathbb{R}^{2d} \cong \mathbb{R}^d \times \mathbb{R}^d$, where an element $z = (x, \omega)$ represents the joint coordinates of time (or position) $x \in \mathbb{R}^d$ and frequency (or momentum) $\omega \in \mathbb{R}^d$.

The phase space $\mathbb{R}^{2d}$ is naturally equipped with a non-degenerate, skew-symmetric bilinear form σ, known as the standard symplectic form. For any pair of vectors $z = (x, \omega)$ and $z' = (x', \omega')$ in $\mathbb{R}^{2d}$, the symplectic form is defined as:

$$\sigma(z, z') = \omega \cdot x' - x \cdot \omega' = z \cdot Jz',$$

where $\cdot$ denotes the standard inner product in $\mathbb{R}^d$, and J is the standard $2d \times 2d$ symplectic matrix given in block form by:

$$J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix},$$

with I_d denoting the $d \times d$ identity matrix.

The real symplectic group, denoted by $Sp(2d, \mathbb{R})$, consists of all linear transformations on $\mathbb{R}^{2d}$ that preserve this standard symplectic form. Mathematically, a real $2d \times 2d$ matrix $\mathcal{A}$ belongs to $Sp(2d, \mathbb{R})$ if and only if:

$$\sigma(\mathcal{A}z, \mathcal{A}z') = \sigma(z, z') \quad \forall z, z' \in \mathbb{R}^{2d},$$

which is algebraically equivalent to the condition:

$$\mathcal{A}^\top J \mathcal{A} = J.$$

The elements of $Sp(2d, \mathbb{R})$ are the geometric transformations that govern coordinate changes in the time-frequency landscape while preserving the underlying phase-space area.

2.2.2 The Metaplectic Group and Symplectic Covariance

While symplectic matrices act linearly on the phase space $\mathbb{R}^{2d}$, their counterparts in harmonic analysis must act as unitary operators on the Hilbert space of signals $L^2(\mathbb{R}^d)$. This crucial bridge is provided by the metaplectic group, denoted by $Mp(d, \mathbb{R})$. Formally, $Mp(d, \mathbb{R})$ is a unitary representation on $L^2(\mathbb{R}^d)$ that forms a topological double cover of the symplectic group $Sp(2d, \mathbb{R})$. This means there exists a continuous and surjective group homomorphism:

$$\pi^{Mp} : Mp(d, \mathbb{R}) \longrightarrow Sp(2d, \mathbb{R})$$

whose kernel is $\ker(\pi^{Mp}) = \{\pm I\}$, with I being the identity operator. Because of this double covering, each symplectic matrix $\mathcal{A} \in Sp(2d, \mathbb{R})$ corresponds to exactly two unitary metaplectic operators, denoted by $\pm\mu(\mathcal{A}) \in Mp(d, \mathbb{R})$.

The fundamental link between a symplectic transformation $\mathcal{A}$ and its corresponding metaplectic operator $\mu(\mathcal{A})$ is elegantly expressed through the property of *symplectic covariance*. Let us recall the symmetric time-frequency shift operator $\rho(z)$ acting on a function $f \in L^2(\mathbb{R}^d)$. For a phase-space point $z = (x, \omega)$, it is defined as:

$$\rho(z)f(t) = T_{x/2} M_\omega T_{x/2} f(t) = e^{2\pi i \omega \cdot (t - x/2)} f(t - x).$$

The metaplectic operator $\mu(\mathcal{A})$ structurally intertwines the linear geometric action on the phase space with the unitary operator action on $L^2(\mathbb{R}^d)$ via the following fundamental relation:

$$\mu(\mathcal{A}) \rho(z) \mu(\mathcal{A})^{-1} = \rho(\mathcal{A}z) \quad \forall z \in \mathbb{R}^{2d}.$$

This covariance identity is deeply meaningful. It demonstrates that applying a time-frequency shift followed by a metaplectic transformation is perfectly equivalent to first transforming the phase-space coordinates via the symplectic matrix $\mathcal{A}$ and then

applying the resulting time-frequency shift. As highlighted by Cordero and Rodino, this structural property provides a powerful perspective: it allows us to interpret all classical and generalized Wigner distributions as manifestations of a single geometric principle, elegantly mapped through the metaplectic representation.

2.2.3 Generators of the Metaplectic Group

A fundamental property of both the symplectic group $Sp(2d, \mathbb{R})$ and the metaplectic group $Mp(d, \mathbb{R})$ is that they can be entirely generated by a finite set of elementary transformations. In the context of time-frequency representations, any metaplectic operator can be decomposed into a finite product of basic unitary operators acting on $L^2(\mathbb{R}^d)$: dilations, multiplications by a quadratic phase (chirps), and generalized rotations (or Fourier transforms).

Following the framework of Gröchenig and Shafkulovska, we can explicitly define these generators and their underlying symplectic matrices:

- **Dilations ($\mathcal{D}_L$):** Let $L \in GL(d, \mathbb{R})$ be an invertible $d \times d$ matrix. The dilation operator $\mu(\mathcal{D}_L)$ acts on a function $f \in L^2(\mathbb{R}^d)$ as:
$$\mu(\mathcal{D}_L)f(x) = |\det L|^{-1/2} f(L^{-1}x).$$
The corresponding symplectic matrix is a block-diagonal matrix given by:
$$\mathcal{D}_L = \begin{pmatrix} L & 0 \\ 0 & L^{-T} \end{pmatrix}.$$

- **Quadratic Phases or Chirps ($\mathcal{V}_Q$):** Let Q be a real symmetric $d \times d$ matrix ($Q = Q^\top$). The operator $\mu(\mathcal{V}_Q)$ multiplies a signal by a quadratic phase, commonly known as a chirp:
$$\mu(\mathcal{V}_Q)f(x) = e^{i\pi x \cdot Qx} f(x).$$
This operator corresponds to a symplectic shear matrix:
$$\mathcal{V}_Q = \begin{pmatrix} I_d & 0 \\ Q & I_d \end{pmatrix}.$$

- **Rotations and Fourier Transforms ($\mathcal{R}_U$):** To fully generate the group, we require operators that interchange time and frequency variables. The most standard choice is the classic Fourier transform $\mathcal{F}$, corresponding to the standard symplectic matrix $\mathcal{J}$:
$$\mu(\mathcal{J})f(\omega) = \hat{f}(\omega).$$
More generally, one can consider rotations $\mathcal{R}_U = \begin{pmatrix} U_1 & U_2 \\ -U_2 & U_1 \end{pmatrix}$, which are symplectic if and only if $U = U_1 + iU_2$ is unitary on $\mathbb{C}^d$.

This algebraic decomposition is not merely a theoretical curiosity; it is the core mechanism used to analyze the support properties of metaplectic Wigner distributions. By breaking down a complex time-frequency representation into the sequential action of $\mathcal{D}_L$, $\mathcal{V}_Q$, and $\mathcal{R}_U$, one can track exactly how the support of a signal evolves in the phase space, allowing for a precise characterization of the uncertainty principles.

2.2.4 Metaplectic $\mathcal{A}$-Wigner Distributions

As discussed in the previous sections, classical time-frequency tools like the Wigner distribution and the Short-Time Fourier Transform can be recast as specific manifestations of a broader phenomenon. Cordero and Rodino formalized this idea by introducing the concept of $\mathcal{A}$-Wigner distributions, which shift the focus from the window function to the underlying symplectic geometry.

The core idea is to regard a joint time-frequency representation of two signals as the application of a specific metaplectic operator to their tensor product. Note that since we are dealing with two functions defined on $\mathbb{R}^d$, their tensor product lives in $L^2(\mathbb{R}^{2d})$. Consequently, the associated symplectic phase space has dimension $4d$, meaning we must consider symplectic matrices in $Sp(4d, \mathbb{R})$.

Definition 2.1 ([6, Def. 2.2])**.** Let $\mathcal{A} \in Sp(4d, \mathbb{R})$ be a $4d \times 4d$ symplectic matrix, and let $\mu(\mathcal{A}) \in Mp(2d, \mathbb{R})$ be a fixed metaplectic lift of $\mathcal{A}$. For any pair of functions $f, g \in L^2(\mathbb{R}^d)$, the $\mathcal{A}$-Wigner distribution is defined as

$$W_{\mathcal{A}}(f, g) = \mu(\mathcal{A})(f \otimes \overline{g}),$$

where the tensor product is standardly given by $(f \otimes \overline{g})(x, y) = f(x)\overline{g(y)}$. When considering a single signal, we write $W_{\mathcal{A}} f$ instead of $W_{\mathcal{A}}(f, f)$.

One of the main strengths of this generalization is that $W_{\mathcal{A}}(f, g)$ naturally inherits all the robust analytical features of metaplectic operators. The following proposition gathers the fundamental mapping properties of the $\mathcal{A}$-Wigner distributions.

Proposition 2.2 ([6, Prop.2.3, Prop.2.9])**.** Let $\mathcal{A} \in Sp(4d, \mathbb{R})$. The $\mathcal{A}$-Wigner distribution satisfies the following properties:

i) **Moyal Identity:** For any $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$, the distributions $W_{\mathcal{A}}(f_1, g_1)$ and $W_{\mathcal{A}}(f_2, g_2)$ belong to $L^2(\mathbb{R}^{2d})$, and their inner product satisfies

$$\langle W_{\mathcal{A}}(f_1, g_1), W_{\mathcal{A}}(f_2, g_2) \rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^d)} \langle g_2, g_1 \rangle_{L^2(\mathbb{R}^d)}.$$

In particular, taking $f_1 = f_2 = f$ and $g_1 = g_2 = g$, we have the isometry relation $\|W_{\mathcal{A}}(f, g)\|_{L^2} = \|f\|_{L^2} \|g\|_{L^2}$.

ii) **Invariance of the Schwartz Space:** If the input signals f, g belong to the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, then their $\mathcal{A}$-Wigner distribution $W_{\mathcal{A}}(f,g)$ belongs to $\mathcal{S}(\mathbb{R}^{2d})$.

Proof. Since $\mu(A)$ is unitary on $L^2(\mathbb{R}^{2d})$ preserve the inner product, which implies:

$$\begin{aligned}\langle W_{\mathcal{A}}(f_1,g_1), W_{\mathcal{A}}(f_2,g_2)\rangle_{L^2} &= \langle \mu(\mathcal{A})(f_1\otimes\overline{g}_1), \mu(\mathcal{A})(f_2\otimes\overline{g}_2)\rangle_{L^2}\\ &= \langle \mu(\mathcal{A})^{-1}\mu(\mathcal{A})(f_1\otimes\overline{g}_1), f_2\otimes\overline{g}_2\rangle_{L^2}.\end{aligned}$$

By applying Fubini's theorem to the right-hand side, the integral of the tensor products conveniently splits into the product of two separate integrals over $\mathbb{R}^d$:

$$\left(\int_{\mathbb{R}^d} f_1(x)\overline{f_2(x)}\,dx\right)\left(\int_{\mathbb{R}^d}\overline{g_1(y)}g_2(y)\,dy\right) = \langle f_1,f_2\rangle_{L^2}\langle g_2,g_1\rangle_{L^2},$$

which establishes the Moyal identity.

For the second point, we only need to recall that any metaplectic operator restricts to a topological isomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^{2d})$. Since the tensor product of two Schwartz functions $f, g \in \mathcal{S}(\mathbb{R}^d)$ yields a function $f\otimes\overline{g}\in\mathcal{S}(\mathbb{R}^{2d})$, its image under $\mu(\mathcal{A})$ remains firmly inside $\mathcal{S}(\mathbb{R}^{2d})$. □

In addition to the fundamental mapping properties and the Moyal identity, the $\mathcal{A}$-Wigner distributions exhibit a rich algebraic structure. Classical operations on the input signals, such as swapping the window and the function, or applying the Fourier transform, translate elegantly into a modification of the underlying symplectic matrix $\mathcal{A}$.

In order to fully appreciate the flexibility of the $\mathcal{A}$-Wigner framework, we now detail a series of algebraic properties introduced by Cordero and Rodino. These results show how fundamental operations on the input signals, such as complex conjugation, swapping arguments, or applying the Fourier transform, correspond seamlessly to algebraic manipulations of the underlying symplectic matrix.

We begin by investigating what happens when we interchange the roles of the two window functions f and g, or when we take their complex conjugates.

Proposition 2.3 ([6, Prop.2.4] Interchanging f and g)**.** Let $\mathcal{A}\in Sp(4d,\mathbb{R})$, and $f,g\in L^2(\mathbb{R}^d)$. Let $L\in GL(2d,\mathbb{R})$ be the coordinate permutation matrix defined by $L(x,y)=(y,x)$ for $x,y\in\mathbb{R}^d$, and let $\mathcal{D}_L=\mathrm{diag}(L,L^{-t})\in Sp(4d,\mathbb{R})$. Then the following identity holds:

$$W_{\mathcal{A}}(g,f) = W_{\mathcal{A}\mathcal{D}_L}(\overline{f},\overline{g}).$$

Proof. The proof relies on the action of the dilation operator on the tensor product. Up to a harmless phase factor, applying the associated metaplectic operator $\mu(\mathcal{D}_L)$ to the tensor product $g\otimes\overline{f}$ effectively swaps the coordinate variables, yielding:

$$\mu(\mathcal{D}_L)(g\otimes\overline{f})(x,y) = (g\otimes\overline{f})(y,x) = \overline{f}(x)g(y) = (\overline{f}\otimes g)(x,y).$$

Exploiting this property, we can rewrite the $\mathcal{A}$-Wigner distribution of the swapped arguments by injecting the identity operator, expressed as $\mathcal{D}_L^{-1}\mathcal{D}_L$:

$$W_{\mathcal{A}}(g,f) = \mu(\mathcal{A})(g \otimes \overline{f}) = \mu(\mathcal{A}\mathcal{D}_L^{-1}\mathcal{D}_L)(g \otimes \overline{f}) = \mu(\mathcal{A}\mathcal{D}_L^{-1})(\overline{f} \otimes g).$$

Since the permutation matrix satisfies $L = L^{-1}$, it follows that $\mathcal{D}_L^{-1} = \mathcal{D}_L$. Thus, the expression simplifies to $\mu(\mathcal{A}\mathcal{D}_L)(\overline{f} \otimes g)$, which by definition is $W_{\mathcal{A}\mathcal{D}_L}(\overline{f}, \overline{g})$, verifying the claim. □

This property can be pushed even further. In many practical scenarios, we are interested in simply swapping the signals without introducing the complex conjugation. Depending on the structure of $\mathcal{A}$, this results in a new distribution governed by a different symplectic matrix.

Proposition 2.4 ([6, Prop.2.6])**.** For any symplectic matrix $\mathcal{A} \in Sp(4d, \mathbb{R})$ and for any $f, g \in L^2(\mathbb{R}^d)$, it is always possible to find a suitable symplectic matrix $\mathcal{B} \in Sp(4d, \mathbb{R})$ such that:

$$W_{\mathcal{A}}(g,f) = \overline{W_{\mathcal{B}}(f,g)}. \tag{2.1}$$

Proof. From Proposition 2.3, we have $W_{\mathcal{A}}(g,f) = \mu(\mathcal{A}\mathcal{D}_L)(\overline{f} \otimes g)$. The metaplectic group is generated by $\mu(\mathcal{J})$, $\mu(\mathcal{V}_C)$, and $\mu(\mathcal{D}_L)$. For any function $F \in L^2(\mathbb{R}^{2d})$, complex conjugation interacts with these generators as follows:

$$\overline{\mu(\mathcal{J})F} = \mu(\mathcal{J}^{-1})\overline{F}, \quad \overline{\mu(\mathcal{V}_C)F} = \mu(\mathcal{V}_{-C})\overline{F}, \quad \overline{\mu(\mathcal{D}_L)F} = \mu(\mathcal{D}_L)\overline{F}.$$

Since any metaplectic operator $\mu(\mathcal{M})$ is a finite product of these basic generators, there must exist a symplectic matrix $\widetilde{\mathcal{M}}$ such that $\overline{\mu(\mathcal{M})h} = \mu(\widetilde{\mathcal{M}})\overline{h}$. Applying this property to $\mathcal{M} = \mathcal{A}\mathcal{D}_L$ and $F = \overline{f} \otimes g$, we find a matrix $\mathcal{B} \in Sp(4d, \mathbb{R})$ satisfying:

$$\overline{\mu(\mathcal{A}\mathcal{D}_L)(\overline{f} \otimes g)} = \mu(\mathcal{B})(f \otimes \overline{g}) = W_{\mathcal{B}}(f,g).$$

Taking the complex conjugate of both sides directly yields $W_{\mathcal{A}}(g,f) = \overline{W_{\mathcal{B}}(f,g)}$. □

Another cornerstone of classical time-frequency analysis is the so-called fundamental identity, which relates the time-frequency representation of a signal to the representation of its Fourier transform. This principle is naturally extended in the metaplectic framework.

Proposition 2.5 ([6, Prop.2.7] Fundamental identity of time-frequency analysis)**.** Let $\mathcal{A} \in Sp(4d, \mathbb{R})$ and let $f, g \in L^2(\mathbb{R}^d)$. Denoting by $\hat{f}$ and $\hat{g}$ their respective Fourier transforms, we have:

$$W_{\mathcal{A}}(\hat{f}, \hat{g}) = W_{\widetilde{\mathcal{A}}}(f,g), \tag{2.2}$$

where $\widetilde{\mathcal{A}} = \mathcal{A}\mathcal{J}\mathcal{D}_L = \begin{pmatrix} -BL & AL \\ -DL & CL \end{pmatrix}$ and $L = \mathrm{diag}(I, -I)$.

Proof. Let us introduce the reflection operator $\mathcal{I}g(t) = g(-t)$. We can express the tensor product of the Fourier transforms using the $2d$-dimensional Fourier transform operator $\mathcal{F}$ coupled with a coordinate change matrix $L = \mathrm{diag}(I, -I)$. Specifically, standard Fourier theory allows us to write:

$$\hat{f} \otimes \overline{\hat{g}} = \hat{f} \otimes \widehat{\mathcal{I}\overline{g}} = \mathcal{F}\mu(\mathcal{D}_L)(f \otimes \overline{g}).$$

Now, we substitute this tensor product into the definition of the $\mathcal{A}$-Wigner distribution:

$$W_{\mathcal{A}}(\hat{f}, \hat{g}) = \mu(\mathcal{A})(\hat{f} \otimes \overline{\hat{g}}) = \mu(\mathcal{A})\mathcal{F}\mu(\mathcal{D}_L)(f \otimes \overline{g}) = \mu(\mathcal{A}\mathcal{J}\mathcal{D}_L)(f \otimes \overline{g}).$$

Setting $\widetilde{\mathcal{A}} = \mathcal{A}\mathcal{J}\mathcal{D}_L$ yields the desired result. □

Finally, it is worth explicitly showing how the Fourier transform acts globally on the entire time-frequency distribution $W_{\mathcal{A}}(f, g)$.

Proposition 2.6 ([6, Prop.2.8] Fourier transform of $\mathcal{A}$-Wigner distributions)**.** Let $\mathcal{A} \in Sp(4d, \mathbb{R})$ and $f, g \in L^2(\mathbb{R}^d)$. Applying the $2d$-dimensional Fourier transform $\mathcal{F}$ to the $\mathcal{A}$-Wigner distribution yields another distribution in the same class:

$$\mathcal{F}[W_{\mathcal{A}}(f, g)] = W_{\mathcal{A}'}(f, g), \tag{2.3}$$

where the new symplectic matrix is given by $\mathcal{A}' = (\mathcal{A}^\top)^{-1}J$.

Proof. The operation of taking the $2d$ Fourier transform is mathematically equivalent to applying the metaplectic operator $\mu(J)$ associated with the standard symplectic matrix J. Thus, evaluating $\mathcal{F}[W_{\mathcal{A}}(f, g)]$ means composing $\mu(J)$ with $\mu(\mathcal{A})$, which results in the combined operator $\mu(J\mathcal{A})$. To express the matrix product $J\mathcal{A}$ in a more revealing form, we use the defining property of symplectic matrices: $\mathcal{A}^\top J\mathcal{A} = J$. Multiplying both sides from the right by the inverse matrix $\mathcal{A}^{-1}$, we easily isolate $J\mathcal{A} = (\mathcal{A}^\top)^{-1}J$. Therefore, defining $\mathcal{A}' = (\mathcal{A}^\top)^{-1}J$ concludes the proof. □

Remark 2.7 ([8, Rem.2.2])**.** We can naturally embed the direct product $Sp(2d, \mathbb{R}) \times Sp(2d, \mathbb{R})$ into the higher-dimensional symplectic group $Sp(4d, \mathbb{R})$. This is achieved by embedding the fundamental generating matrices, dilations $\mathcal{D}_L$, chirps $\mathcal{V}_Q$, and rotations $\mathcal{R}_U$, as block-diagonal transformations. Specifically, for invertible matrices $L_1, L_2 \in GL(d, \mathbb{R})$, symmetric matrices $Q_1, Q_2 \in \mathbb{R}^{d\times d}$, and unitary matrices $U_1, U_2 \in U(d, \mathbb{C})$, the embeddings are defined as follows:

$$(\mathcal{D}_{L_1}, \mathcal{D}_{L_2}) \mapsto \mathcal{D}_{\mathrm{diag}(L_1, L_2)},$$

$$(\mathcal{V}_{Q_1}, \mathcal{V}_{Q_2}) \mapsto \mathcal{V}_{\mathrm{diag}(Q_1, Q_2)},$$

$$(\mathcal{R}_{U_1}, \mathcal{R}_{U_2}) \mapsto \mathcal{R}_{\mathrm{diag}(U_1,U_2)}.$$

At the operator level, the corresponding metaplectic transformations act seamlessly on elementary tensor products $f \otimes g$, distributing the action to each individual component:

$$\mu(\mathcal{D}_{\mathrm{diag}(L_1,L_2)})(f \otimes g) = \mu(\mathcal{D}_{L_1})f \otimes \mu(\mathcal{D}_{L_2})g,$$
$$\mu(\mathcal{V}_{\mathrm{diag}(Q_1,Q_2)})(f \otimes g) = \mu(\mathcal{V}_{Q_1})f \otimes \mu(\mathcal{V}_{Q_2})g,$$
$$\mu(\mathcal{R}_{\mathrm{diag}(U_1,U_2)})(f \otimes g) = \mu(\mathcal{R}_{U_1})f \otimes \mu(\mathcal{R}_{U_2})g.$$

Lemma 2.8 ([8, Lem.2.3])**.** To manipulate these operators effectively, we frequently rely on several algebraic identities governing the elementary symplectic matrices $\mathcal{V}_Q$, $\mathcal{D}_L$, and $\mathcal{R}_U$. Let $Q, Q_1, Q_2 \in \mathbb{R}^{d\times d}$ be symmetric matrices, let $L, L_1, L_2 \in GL(d, \mathbb{R})$ be invertible matrices, and let $U, U_1, U_2 \in U(d, \mathbb{C})$ be unitary matrices. Then the following properties hold:

1. If U is strictly real-valued, meaning $U \in O(d, \mathbb{R})$, the rotation matrix simplifies to a dilation: $\mathcal{R}_U = \mathcal{D}_U$
2. Conjugation with the standard symplectic matrix $\mathcal{J}$ yields: $\mathcal{J}\mathcal{D}_L = \mathcal{D}_{L^{-t}}\mathcal{J}$
3. For any orthogonal matrix $W \in O(d, \mathbb{R})$, conjugating a chirp by a dilation gives: $\mathcal{D}_W \mathcal{V}_Q \mathcal{D}_{W^t} = \mathcal{V}_{WQW^t}$
4. The generating matrices form natural homomorphisms within their respective classes:

$$\mathcal{V}_{Q_1}\mathcal{V}_{Q_2} = \mathcal{V}_{Q_1+Q_2}, \quad \mathcal{D}_{L_1}\mathcal{D}_{L_2} = \mathcal{D}_{L_1L_2}, \quad \mathcal{R}_{U_1}\mathcal{R}_{U_2} = \mathcal{R}_{U_1U_2}.$$

Proof. These identities are direct algebraic consequences of the standard block definitions for the elementary symplectic matrices. □

2.3 Benedicks-type Uncertainty Principles

2.3.1 The Short-Time Fourier Transform

The classical short-time Fourier transform (STFT) of a function $f \in L^2(\mathbb{R}^d)$ with respect to a window $g \in L^2(\mathbb{R}^d)$ is defined as

$$V_g f(x, \omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i\omega\cdot t}dt.$$

The Benedicks-type uncertainty principle for the STFT states that if the support of $V_g f$ has finite Lebesgue measure, then either $f \equiv 0$ or $g \equiv 0$ [8].

To analyze general metaplectic representations, we must introduce the partial short-time Fourier transform (pSTFT). We split the spatial variable as $x = (x_1, x_2)$, where $x_1 \in \mathbb{R}^k$ and $x_2 \in \mathbb{R}^{d-k}$ for some $1 \leq k < d$. For a function $f \in L^2(\mathbb{R}^d)$, by Fubini's theorem, the restriction $f_{x_2}(x_1) = f(x_1, x_2)$ is in $L^2(\mathbb{R}^k)$ for almost every $x_2 \in \mathbb{R}^{d-k}$.

Definition 2.9 (Partial STFT [8])**.** For $f, g \in L^2(\mathbb{R}^d)$ and $1 \leq k \leq d$, the partial short-time Fourier transform in the first k variables is defined almost everywhere as

$$V_g^k f(x_1, x_2, \omega_1, \omega_2) = \int_{\mathbb{R}^k} f(t, x_2)\overline{g(t - x_1, -\omega_2)}e^{-2\pi i t\cdot\omega_1}dt = V_{g_{-\omega_2}} f_{x_2}(x_1, \omega_1).$$

The pSTFT naturally inherits the Benedicks property from the classical STFT.

Lemma 2.10 ([8, Lem.3.2])**.** Let $g \in L^2(\mathbb{R}^d)$ and $1 \leq k \leq d$. If $V_g^k f$ is supported on a set of finite measure in $\mathbb{R}^{2d}$, then $f \equiv 0$ or $g \equiv 0$.

For the proof see [8, pp.9-10].

2.3.2 Free Symplectic Matrices

A matrix $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4d, \mathbb{R})$ is called a *free symplectic matrix* if its upper-right block $B \in \mathbb{R}^{2d\times 2d}$ is invertible. Any free symplectic matrix admits the factorization $\mathcal{A} = \mathcal{V}_{DB^{-1}}\mathcal{D}_B\mathcal{J}\mathcal{V}_{B^{-1}A}$ [8].

Since the dilation $\mu(\mathcal{D}_B)$ and chirp multiplication $\mu(\mathcal{V}_{DB^{-1}})$ operators preserve the finiteness of the support's measure, the validity of the Benedicks property for $W_{\mathcal{A}}(f, g)$ depends exclusively on the core operator associated to $\mathcal{J}\mathcal{V}_P$, where $P = B^{-1}A$; see [8, Lem.3.3]

Theorem 2.11 ([8, Thm.1.1, Thm.3.4] Benedicks-type Uncertainty Principle for Free Symplectic Matrices)**.** Let $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4d, \mathbb{R})$ be a free symplectic matrix, meaning that the upper-right block B is invertible. Consider the symmetric matrix $P = B^{-1}A = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$.

(i) If P is not block-diagonal (i.e., $P_{12} = P_{21}^t \neq 0$), then the metaplectic Wigner distribution $W_{\mathcal{A}}(f, g)$ satisfies the Benedicks-type uncertainty principle: if $W_{\mathcal{A}}(f, g)$ is supported on a set of finite Lebesgue measure for some $f, g \in L^2(\mathbb{R}^d)$, then either $f \equiv 0$ or $g \equiv 0$.

(ii) If P is block-diagonal (i.e., $P_{12} = P_{21}^t = 0$), then there exist non-zero functions $f, g \in L^2(\mathbb{R}^d)$ such that the support of $W_{\mathcal{A}}(f, g)$ is a compact set, thereby violating the uncertainty principle.

Proof. We begin by factorizing the free symplectic matrix $\mathcal{A}$. Setting $Q = DB^{-1}$, we can write

$$\mathcal{A} = \mathcal{V}_Q \mathcal{D}_B \mathcal{J} \mathcal{V}_P.$$

By [8, Lem.3.3], the operations associated with $\mathcal{V}_Q$ and $\mathcal{D}_B$ do not affect the finiteness of the support's measure. Consequently, $W_{\mathcal{A}}(f,g)$ has finite measure support if and only if $W_{\mathcal{J}\mathcal{V}_P}(f,g)$ does. This allows us to restrict our analysis to the matrix $\mathcal{J}\mathcal{V}_P$.

(ii). If $P = \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix}$ is block-diagonal, the action of the metaplectic operator splits naturally. Specifically, we have

$$\begin{aligned} |W_{\mathcal{J}\mathcal{V}_P}(f,g)(\lambda)| =& |\mu(\mathcal{J})((\mu(\mathcal{V}_{P_{11}})f \otimes \overline{\mu(\mathcal{V}_{-P_{22}})g}))(\lambda)| \\ =& |(\mu(\mathcal{J}\mathcal{V}_{P_{11}})f \otimes \overline{\mu(\mathcal{J}^t\mathcal{V}_{-P_{22}})g})(\lambda)|. \end{aligned}$$

We can construct a counterexample by choosing two non-zero, compactly supported functions $f_0, g_0 \in L^2(\mathbb{R}^d)$. Setting $f = \mu(\mathcal{J}\mathcal{V}_{P_{11}})^{-1} f_0$ and $g = \mu(\mathcal{J}^t\mathcal{V}_{-P_{22}})^{-1} g_0$, we obtain

$$|W_{\mathcal{J}\mathcal{V}_P}(f,g)(\lambda)| = |f_0 \otimes \overline{g_0}(\lambda)|.$$

Since this tensor product clearly has compact support, it implies that $W_{\mathcal{A}}(f,g)$ is also compactly supported, which proves (ii).

(i). Assume $P_{12} = P_{21}^t \neq 0$. Suppose $f, g \in L^2(\mathbb{R}^d)$ are functions for which the support of $W_{\mathcal{A}}(f,g)$ has finite measure. To handle the off-diagonal blocks, we use the singular value decomposition of P_{12}, writing $P_{12} = W_1 \Gamma W_2^t$, where $W_1, W_2 \in O(d,\mathbb{R})$ and $\Gamma = \left(\begin{smallmatrix} \Gamma_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is a diagonal matrix containing the singular values. Here, $\Gamma_1 \in GL(k,\mathbb{R})$ with $k = \operatorname{rank}(P_{12}) \geq 1$. This allows us to express P as

$$P = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \begin{pmatrix} 0 & \Gamma \\ \Gamma & 0 \end{pmatrix} \begin{pmatrix} W_1^t & 0 \\ 0 & W_2^t \end{pmatrix} + \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix}.$$

Substituting this back into our reduced symplectic matrix gives

$$\mathcal{J}\mathcal{V}_P = \mathcal{D}_{\operatorname{diag}(W_1,W_2)} \mathcal{J} \mathcal{V}_{\left(\begin{smallmatrix} 0 & \Gamma \\ \Gamma & 0 \end{smallmatrix}\right)} \mathcal{D}_{\operatorname{diag}(W_1^t,W_2^t)} \mathcal{V}_{\operatorname{diag}(P_{11},P_{22})}.$$

We introduce the auxiliary functions $f_1 = \mu(\mathcal{D}_{W_1^t}\mathcal{V}_{P_{11}})f$ and $g_1 = \mu(\mathcal{D}_{W_2^t}\mathcal{V}_{-P_{22}})g$, both belonging to $L^2(\mathbb{R}^d)$. Ignoring the leading dilation which preserves the support measure finiteness, we see that $\mathcal{F}\mu(\mathcal{V}_{\left(\begin{smallmatrix} 0 & \Gamma \\ \Gamma & 0 \end{smallmatrix}\right)})(f_1 \otimes \overline{g_1})$ must be supported on a set of finite measure.

A direct computation of this term yields

$$\begin{aligned} \mathcal{F}\mu\left(\mathcal{V}_{\left(\begin{smallmatrix} 0 & \Gamma \\ \Gamma & 0 \end{smallmatrix}\right)}\right)(f_1 \otimes \overline{g_1})(\omega,\xi) &= \int_{\mathbb{R}^d} f_1(x) e^{-2\pi i \omega \cdot x} \int_{\mathbb{R}^d} \overline{g_1}(y) e^{2\pi i \Gamma x \cdot y} e^{-2\pi i \xi \cdot y}\, dy\, dx \\ &= \int_{\mathbb{R}^d} f_1(x) e^{-2\pi i \omega \cdot x} \overline{\hat{g}_1}(\Gamma x - \xi)\, dx. \end{aligned}$$

Since Γ is not fully invertible, we perform a coordinate change, evaluating the expression at $(\mathrm{diag}(\Gamma_1, I)\omega, \xi) = (\Gamma_1\omega_1, \omega_2, \xi)$. Breaking the integral into the relevant dimensions ($x_1 \in \mathbb{R}^k, x_2 \in \mathbb{R}^{d-k}$), we define $g_2 = \hat{g_1}$ and $f_2(\Gamma_1 x_1, \omega_2) = \int_{\mathbb{R}^{d-k}} f_1(x_1, x_2) e^{-2\pi i \omega_2 \cdot x_2}\, dx_2$. Both f_2 and g_2 are functions in $L^2(\mathbb{R}^d)$.

With the substitution $t = \Gamma_1 x_1$, the integral transforms naturally into a partial short-time Fourier transform:

$$\begin{aligned}\mathcal{F}\mu\left(\mathcal{V}_{\left(\begin{smallmatrix}0 & \Gamma\\ \Gamma & 0\end{smallmatrix}\right)}\right)(f_1 \otimes \overline{g_1})(\omega, \xi) &= \int_{\mathbb{R}^k} f_2(\Gamma_1 x_1, \omega_2)\overline{g_2}(\Gamma_1 x_1 - \xi_1, -\xi_2) e^{-2\pi i \omega_1 \cdot \Gamma_1 x_1}\, dx_1 \\ &= |\det \Gamma_1|^{-1} \int_{\mathbb{R}^k} f_2(t, \omega_2)\overline{g_2}(t - \xi_1, -\xi_2) e^{-2\pi i \omega_1 \cdot t}\, dt \\ &= |\det \Gamma_1|^{-1} V^k_{g_2} f_2(\xi_1, \omega_2, \omega_1, \xi_2).\end{aligned}$$

Because the initial distribution had finite measure support, this partial STFT $V^k_{g_2} f_2$ must also be supported on a set of finite measure. By the uncertainty principle for the partial short-time Fourier transform, we conclude that either $f_2 \equiv 0$ or $g_2 \equiv 0$. The maps from f to f_2 and from g to g_2 are compositions of metaplectic or Fourier transforms with invertible linear changes of variables. Hence $f_2 \equiv 0$ if and only if $f \equiv 0$, and $g_2 \equiv 0$ if and only if $g \equiv 0$. The conclusion follows. □

2.3.3 General Symplectic Matrices and the Phase Trick

To extend the Benedicks-type uncertainty principle to arbitrary symplectic matrices, we must overcome the limitation that a general matrix $\mathcal{A} \in Sp(4d, \mathbb{R})$ might not be free. The strategy relies on a structural factorization known as the pre-Iwasawa decomposition, which states that any $\mathcal{A} \in Sp(4d, \mathbb{R})$ can be factored as $\mathcal{A} = \mathcal{V}_Q \mathcal{D}_L \mathcal{R}_U$, where $L \in GL(2d, \mathbb{R})$, $Q \in \mathbb{R}^{2d \times 2d}$ is symmetric, and $U \in U(2d, \mathbb{C})$.

Since the operators $\mu(\mathcal{V}_Q)$ and $\mu(\mathcal{D}_L)$ only apply a chirp multiplication and a coordinate scaling, they preserve the finiteness of the support's Lebesgue measure. Thus, the validity of the uncertainty principle for the $\mathcal{A}$-Wigner distribution $W_{\mathcal{A}}(f, g)$ depends entirely on the orthogonal symplectic factor $\mathcal{R}_U$.

If the imaginary part $Im(U)$ is not invertible [8, Lem.4.3], $\mathcal{R}_U$ is not a free symplectic matrix. To bypass this, we utilize a "phase trick": for any unitary matrix $U \in U(2d, \mathbb{C})$, there exists a global phase $\tau \in \mathbb{T}$ such that $Im(\tau U)$ is invertible, making $\mathcal{R}_{\tau U}$ a free symplectic matrix. The relationship between the time-frequency representations of $\mathcal{R}_U$ and $\mathcal{R}_{\tau U}$ is established in the following lemma.

Lemma 2.12 ([8, Lem.5.1])**.** Let $\mathcal{R}_U \in Sp(4d, \mathbb{R})$ for some $U \in U(2d, \mathbb{C})$. The following statements are equivalent:

(i) There exist non-zero $f, g \in L^2(\mathbb{R}^d)$ such that $W_{\mathcal{R}_U}(f, g)$ has a support of finite measure.

(ii) For all $\tau \in \mathbb{T}$, there exist non-zero $f_\tau, g_\tau \in L^2(\mathbb{R}^d)$ such that $W_{\mathcal{R}_{\tau U}}(f_\tau, g_\tau)$ has a support of finite measure.

The detailed proof of this equivalence is based on the algebraic properties of fractional Fourier transforms and can be found in the original article [8].

To bridge the properties of the free matrix $\mathcal{R}_{\tau U}$ with the original unitary matrix U, we need an algebraic characterization linking the core symmetric matrix of the free factorization to the unitary matrix itself.

Lemma 2.13 ([8, Lem.5.3])**.** Let $U = A + iB \in U(2d, \mathbb{C})$ be a unitary matrix with an invertible imaginary part $B \in GL(2d, \mathbb{R})$, and let $P = B^{-1}A$. Then the following are equivalent:

(i) The matrix P is block-diagonal.

(ii) There exist orthogonal matrices $W \in O(2d, \mathbb{R})$, $V_1, V_2 \in O(d, \mathbb{R})$, and diagonal unitary matrices $\Sigma_1, \Sigma_2 \in U(d, \mathbb{C})$ such that $U = W \cdot \mathrm{diag}(\Sigma_1 V_1^t, \Sigma_2 V_2^t)$.

(iii) The matrix $U^t U$ is block-diagonal.

The proof of this purely algebraic equivalence is omitted here but is established in the reference article [8]. This property allows us to state the general classification theorem.

Theorem 2.14 ([8, Thm.5.4])**.** Let $\mathcal{A} \in Sp(4d, \mathbb{R})$ be a symplectic matrix with pre-Iwasawa decomposition $\mathcal{A} = \mathcal{V}_Q \mathcal{D}_L \mathcal{R}_U$. Then the following alternative holds:

(i) If $U^t U$ is not block-diagonal, then Benedicks's uncertainty principle holds: if $W_{\mathcal{A}}(f, g)$ is supported on a set of finite measure for some $f, g \in L^2(\mathbb{R}^d)$, then $f \equiv 0$ or $g \equiv 0$.

(ii) If $U^t U$ is block-diagonal, then there exist non-zero $f, g \in L^2(\mathbb{R}^d)$ such that $W_{\mathcal{A}}(f, g)$ is supported on a compact set, thereby violating the uncertainty principle.

Proof. Let us begin with the first assertion, assuming that the Wigner distribution $W_{\mathcal{A}}(f, g)$ is supported on a set of finite measure for some functions $f, g \in L^2(\mathbb{R}^d)$.

By applying the phase trick discussed above (see [8, Lem.4.3]), we can always find a phase factor $\tau \in \mathbb{T}$ such that the imaginary part $\mathrm{Im}(\tau U)$ is invertible. This ensures that the associated matrix $\mathcal{R}_{\tau U}$ is free. Since we assumed that $U^t U$ is not block-diagonal, the scaled matrix $(\tau U)^t(\tau U) = \tau^2 U^t U$ naturally lacks a block-diagonal structure as well. This allows us to invoke Lemma 2.13, which guarantees that the associated symmetric matrix $P = \mathrm{Im}(\tau U)^{-1}\mathrm{Re}(\tau U)$ is not block-diagonal either. Because $\mathcal{R}_{\tau U}$ is a free symplectic matrix with a non-block-diagonal core matrix P, we can directly apply Theorem 2.11(i). Combined with Lemma 2.12,

this leads us to the conclusion that either $f \equiv 0$ or $g \equiv 0$, thus confirming the uncertainty principle.

Turning our attention to the second statement, we now suppose that $U^t U$ is indeed block-diagonal. This block-diagonal property is clearly inherited by $\tau^2 U^t U$ for any choice of $\tau \in \mathbb{T}$. As before, we select a phase $\tau \in \mathbb{T}$ that makes $\mathrm{Im}(\tau U)$ invertible. Relying again on the algebraic characterization from Lemma 2.13, we can factorize the scaled unitary matrix as $\tau U = W \cdot \mathrm{diag}(\Sigma_{1,\tau} V_1^t, \Sigma_{2,\tau} V_2^t)$. Here, $W \in O(2d, \mathbb{R})$, the matrices V_1, V_2 belong to $O(d, \mathbb{R})$, and $\Sigma_{1,\tau}, \Sigma_{2,\tau}$ are diagonal unitary matrices in $U(d, \mathbb{C})$. By defining the unscaled diagonal matrices $\Sigma_1 = \bar{\tau}\Sigma_{1,\tau}$ and $\Sigma_2 = \bar{\tau}\Sigma_{2,\tau}$, we recover the factorization $U = W \cdot \mathrm{diag}(\Sigma_1 V_1^t, \Sigma_2 V_2^t)$.

By using the properties of the metaplectic representation on elementary tensors, the absolute value of the associated $\mathcal{A}$-Wigner distribution for any $f, g \in L^2(\mathbb{R}^d)$ can be expanded as follows:

$$\begin{aligned} |W_{\mathcal{A}}(f,g)| &= |\mu(\mathcal{V}_Q)\mu(\mathcal{D}_L)\mu(\mathcal{D}_W)\mu(\mathcal{R}_{\mathrm{diag}(\Sigma_1 V_1^t, \Sigma_2 V_2^t)})(f \otimes \bar{g})| \\ &= |\det L|^{-1/2} |(\mu(\mathcal{R}_{\Sigma_1 V_1^t}) f \otimes \overline{\mu(\mathcal{R}_{\Sigma_2 V_2^t}) g})(W^t L^{-1} \cdot)|. \end{aligned}$$

To construct a counterexample, we select two non-zero, compactly supported functions $f_0, g_0 \in L^2(\mathbb{R}^d)$. We then define our candidate functions as $f = \mu(\mathcal{R}_{\Sigma_1 V_1^t})^{-1} f_0$ and $g = \mu(\mathcal{R}_{\Sigma_2 V_2^t})^{-1} g_0$. Because the metaplectic operators $\mu(\mathcal{R}_{\Sigma_1 V_1^t})$ and $\mu(\mathcal{R}_{\Sigma_2 V_2^t})$ are unitary, the resulting functions f and g remain non-zero and square-integrable. Evaluating the $\mathcal{A}$-Wigner distribution for this specific pair yields:

$$|W_{\mathcal{A}}(f,g)| = |\det L|^{-1/2} |(f_0 \otimes \overline{g_0})(W^t L^{-1} \cdot)|.$$

Since f_0 and g_0 have compact support, their tensor product clearly has compact support as well. Because the linear transformation $W^t L^{-1}$ preserves this compactness, it follows that $W_{\mathcal{A}}(f,g)$ is compactly supported, which completes the proof. □

This theorem characterizes the support properties of the time-frequency representation through the block-diagonality of $U^\top U$. From a practical standpoint, this provides an elegant algorithmic criterion. Given any time-frequency representation parameterized by a matrix $\mathcal{A}$:

1. Compute the pre-Iwasawa decomposition to extract the unitary matrix $U \in U(2d, \mathbb{C})$.
2. Calculate the matrix product $U^\top U$.
3. If $U^\top U$ contains nonzero off-diagonal blocks, the time-frequency representation satisfies the Benedicks property. Otherwise, in the block-diagonal case, there exist nonzero compactly supported examples.

2.4 Further quantitative uncertainty principles

While the Benedicks-type uncertainty principle provides a fundamental qualitative measure of localization, the classical theory of harmonic analysis boasts a rich variety of quantitative uncertainty principles. Prominent examples include the theorems of Beurling, Hardy, (and Gelfand-Shilov in [9]) which impose restrictions on the simultaneous exponential decay of a signal and its time-frequency representation.

A natural objective is to extend these powerful quantitative bounds to the generalized framework of metaplectic $\mathcal{A}$-Wigner distributions. However, proving each inequality from scratch for every possible symplectic matrix would be an overwhelmingly complex and unstructured task. To overcome this, Gröchenig and Shafkulovska developed a versatile, unified proof strategy.

2.4.1 The Four-Step Blueprint

The cornerstone of this modern approach is a four-step blueprint that systematically reduces the uncertainty principle for an abstract metaplectic distribution $W_{\mathcal{A}}$ to the case of pSTFT.

Let $\mathcal{A} \in Sp(4d, \mathbb{R})$ be a symplectic matrix. The procedure unfolds as follows:

1. **Factorization of $\mathcal{A}$:** The process begins by computing the pre-Iwasawa decomposition of the matrix, namely $\mathcal{A} = \mathcal{V}_Q \mathcal{D}_L \mathcal{R}_U$, to isolate the unitary core $U \in U(2d, \mathbb{C})$. As established previously, the viability of the uncertainty principles considered here hinges on whether $U^\top U$ is block-diagonal.

2. **Representation of $W_{\mathcal{A}}$:** If $U^\top U$ is *not* block-diagonal, the time-frequency representation can be reduced to a pSTFT. Specifically, there exists an index $1 \leq k \leq d$, a coordinate change matrix $\Omega \in GL(2d, \mathbb{R})$, and two matrices $\mathcal{A}_0, \mathcal{B}_0 \in Sp(2d, \mathbb{R})$ such that the absolute value of the distribution can be written as:
$$|W_{\mathcal{A}}(f, g)| = \left|\mu(\mathcal{D}_\Omega) V^k_{\mu(\mathcal{B}_0)g}(\mu(\mathcal{A}_0)f)\right|, \tag{2.4}$$
where $V^k_{\tilde{g}} \tilde{f}$ denotes the partial STFT over a k-dimensional subspace. By defining the modified signals $\tilde{f} = \mu(\mathcal{A}_0)f$ and $\tilde{g} = \mu(\mathcal{B}_0)g$, analyzing $W_{\mathcal{A}}(f, g)$ becomes entirely equivalent to analyzing $V^k_{\tilde{g}} \tilde{f}$.

3. **Uncertainty Principle for the Partial STFT:** At this stage, one applies the classical formulation of the desired uncertainty principle (e.g., Beurling's or Hardy's) directly to the partial STFT $V^k_{\tilde{g}} \tilde{f}$. By exploiting measure-theoretic arguments and Fubini's theorem on the cross-sections of the distribution, one proves that the restrictive conditions force the partial STFT to vanish, which in turn implies that either $\tilde{f} \equiv 0$ or $\tilde{g} \equiv 0$.

4. **Transfer back to $W_{\mathcal{A}}$:** Finally, since the operators $\mu(\mathcal{A}_0)$ and $\mu(\mathcal{B}_0)$ are unitary (and therefore invertible bijections on $L^2(\mathbb{R}^d)$), the identities $\mu(\mathcal{A}_0)f \equiv 0$ or $\mu(\mathcal{B}_0)g \equiv 0$ immediately guarantee that the original input functions must be $f \equiv 0$ or $g \equiv 0$.

This methodology is highly elegant because it confines the heavy algebraic lifting to the pre-Iwasawa factorization, leaving the analytical core to rely on established theorems for the classical STFT.

2.4.2 Quantitative Uncertainty Principles for the STFT and pSTFT

To fuel the third step of our blueprint, we need to state the corresponding quantitative uncertainty principles for the partial short-time Fourier transform. When $k = d$, these naturally reduce to the well-known results for the classical STFT.

Theorem 2.15 ([9, Thm.3.4] Beurling-type for pSTFT)**.** Let $1 \leq k \leq d$, $0 \leq N \leq k$, and $f, g \in L^2(\mathbb{R}^d)$. If the weighted integral satisfies

$$\int_{\mathbb{R}^{2d}} \frac{|V_g^k f(x,\omega)|}{(1 + \|x\| + \|\omega\|)^N} e^{\pi|x\cdot\omega|} dx d\omega < \infty,$$

then $f \equiv 0$ or $g \equiv 0$.

As a direct corollary of the Beurling-type bound, we obtain the Hardy-type uncertainty principle, which imposes restrictions based on Gaussian decay.

Theorem 2.16 ([9, Thm.3.5] Hardy-type for pSTFT)**.** Let $1 \leq k \leq d$, and $\alpha, \beta, c > 0$ such that $\alpha\beta > 1$. If $f, g \in L^2(\mathbb{R}^d)$ satisfy

$$|V_g^k f(x,\omega)| \leq c e^{-\pi(\alpha\|x\|^2 + \beta\|\omega\|^2)/2} \quad \text{for almost all } (x,\omega) \in \mathbb{R}^{2d},$$

then $f \equiv 0$ or $g \equiv 0$.

The proof of Theorem 2.15 relies essentially on the corresponding uncertainty principle for the classical STFT. The core idea of the proof is to freeze the inactive variables, thereby reducing the partial STFT to a standard STFT applied to parameterized functions. Theorem 2.16, on the other hand, follows as a direct consequence of this Beurling-type bound. Detailed proofs for both of these results can be found in [9, p.10].

2.4.3 Quantitative Uncertainty Principles: Beurling and Hardy

By executing the aforementioned strategy, one can readily establish strong quantitative bounds for metaplectic time-frequency representations.

The Beurling-type uncertainty principle asserts that a joint representation cannot exhibit overly rapid exponential decay in all directions of the phase space simultaneously. For metaplectic operators, this translates to the following theorem:

We introduce a specific matrix associated with the mixed product of coordinates: let $J_0 := \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix}$, such that $x \cdot \omega = \frac{1}{2}\lambda \cdot J_0\lambda$ for $\lambda = (x, \omega)$.

Theorem 2.17 (cf. [9, Lem.4.3 and Thm.4.4] Beurling-type for Metaplectic Distributions)**.** Let $\mathcal{A} = \mathcal{V}_Q\mathcal{D}_L\mathcal{R}_U \in Sp(4d, \mathbb{R})$ be a symplectic matrix and let Ω be the corresponding matrix introduced in the factorization blueprint. Assume that $U^tU \in U(2d, \mathbb{C})$ is not $d \times d$ block-diagonal. If $f, g \in L^2(\mathbb{R}^d)$ satisfy

$$\int_{\mathbb{R}^{2d}} |W_{\mathcal{A}}(f, g)(\lambda)| e^{\pi|\Omega^{-1}\lambda \cdot J_0\Omega^{-1}\lambda|/2} d\lambda < \infty,$$

then $f \equiv 0$ or $g \equiv 0$.

Proof. Based on our blueprint (Step 1 and 2), since U^tU is not block-diagonal, we can factorize the metaplectic distribution using the auxiliary matrices Ω, $\mathcal{A}_0$, and $\mathcal{B}_0$ such that:

$$|W_{\mathcal{A}}(f, g)(\lambda)| = |\det \Omega|^{-1/2} \left| V^k_{\mu(\mathcal{B}_0)g} \mu(\mathcal{A}_0) f(\Omega^{-1}\lambda) \right|.$$

Substituting this representation into our initial integral condition and applying the change of variables $\lambda' = \Omega^{-1}\lambda$ (with the Jacobian $d\lambda = |\det \Omega| d\lambda'$) yields:

$$\begin{aligned} &|\det \Omega|^{-1/2} \int_{\mathbb{R}^{2d}} |V^k_{\mu(\mathcal{B}_0)g} \mu(\mathcal{A}_0) f(\Omega^{-1}\lambda)| e^{\pi|\Omega^{-1}\lambda \cdot J_0\Omega^{-1}\lambda|/2} d\lambda \\ &= |\det \Omega|^{1/2} \int_{\mathbb{R}^{2d}} |V^k_{\mu(\mathcal{B}_0)g} \mu(\mathcal{A}_0) f(\lambda')| e^{\pi|\lambda' \cdot J_0\lambda'|/2} d\lambda' < \infty. \end{aligned}$$

Recognizing that $\frac{1}{2}\lambda' \cdot J_0\lambda' = x' \cdot \omega'$, we see that the evaluated integral precisely matches the condition required for the Beurling-type uncertainty principle for the partial STFT (Step 3). Therefore, applying Theorem 2.15, we conclude that either $\mu(\mathcal{A}_0)f \equiv 0$ or $\mu(\mathcal{B}_0)g \equiv 0$.

Finally (Step 4), because $\mu(\mathcal{A}_0)$ and $\mu(\mathcal{B}_0)$ are unitary operators, they are injective, which directly implies that $f \equiv 0$ or $g \equiv 0$. This completes the proof. □

As in classical Fourier analysis, the Beurling property is considered the most stringent condition, and it naturally implies Hardy's uncertainty principle. Hardy's formulation focuses on point-wise Gaussian bounds rather than global integrability:

Corollary 2.18 ([9, Cor.4.5] Hardy-type UP for Metaplectic Distributions)**.** Under the same assumptions for $\mathcal{A}$ and Ω, if there exist constants $c > 0$ and $\alpha > 1$ such that the $\mathcal{A}$-Wigner distribution satisfies the point-wise Gaussian decay bound

$$|W_{\mathcal{A}}(f,g)(\lambda)| \leq c\, e^{-\frac{\pi\alpha}{2}\|\Omega^{-1}\lambda\|^2} \quad \text{for almost all } \lambda \in \mathbb{R}^{2d}, \tag{2.5}$$

then $f \equiv 0$ or $g \equiv 0$.

Proof. The logic follows a trajectory virtually identical to the previous theorem. By injecting the reduction formula (2.4) into our pointwise bound, we get:

$$|\det\Omega|^{-1/2} \left|V^k_{\mu(\mathcal{B}_0)g}(\mu(\mathcal{A}_0)f)(\Omega^{-1}\lambda)\right| \leq c\, e^{-\frac{\pi\alpha}{2}\|\Omega^{-1}\lambda\|^2}.$$

Again, through the coordinate substitution $\lambda' = \Omega^{-1}\lambda$, the inequality can be rewritten globally for almost all $\lambda' \in \mathbb{R}^{2d}$ as:

$$\left|V^k_{\mu(\mathcal{B}_0)g}(\mu(\mathcal{A}_0)f)(\lambda')\right| \leq \tilde{c}\, e^{-\frac{\pi\alpha}{2}\|\lambda'\|^2},$$

where the constant is updated to $\tilde{c} = c|\det\Omega|^{1/2}$. This pointwise Gaussian decay bound with $\alpha > 1$ is precisely the trigger for Hardy's uncertainty principle for the partial STFT. Consequently, the partial STFT theory mandates that $\mu(\mathcal{A}_0)f \equiv 0$ or $\mu(\mathcal{B}_0)g \equiv 0$. As before, the unitarity of the metaplectic operators concludes the proof, yielding $f \equiv 0$ or $g \equiv 0$. □

Chapter 3

A Wigderson Perspective on Gröchenig-Shafkulovska Uncertainty Principles

3.1 From Input-Output to Output-Only Uncertainty Principles

In this chapter, we study metaplectic uncertainty principles from the point of view of the Wigderson framework. The aim is to connect the approach developed by Wigderson and Wigderson with the uncertainty principles for metaplectic time-frequency representations proved by Gröchenig and Shafkulovska.

The main point is that these two perspectives do not have the same structure. In the classical Wigderson framework, and also in Lerner's application to metaplectic operators, the uncertainty principle compares the concentration of an input with the concentration of its transform. In contrast, the metaplectic uncertainty principles of Gröchenig and Shafkulovska are naturally formulated as uncertainty principles for the output representation itself.

This leads us to distinguish between two types of estimates: *input-output* uncertainty principles and *output-only* uncertainty principles.

3.1.1 The Input-Output Wigderson Principle

We recall the concentration functional

$$H_0(F) := \frac{\|F\|_{L^1(\mathbb{R}^n)}}{\|F\|_{L^\infty(\mathbb{R}^n)}},$$

whenever the two norms are finite and $\|F\|_{L^\infty(\mathbb{R}^n)} \neq 0$.

As we discuss in the first chapter, this quantity measures concentration in a simple way. If a function has fixed L^∞ mass and small L^1 norm, then it is concentrated on a small region and $H_0(F)$ is small. On the other hand, a high value of H_0 implies that the function cannot be too concentrated.

Indeed in the Wigderson approach, one studies products of concentration functionals:

$$H_0(F)H_0(\widehat{F}) \geq c_0,$$

where $c_0 > 0$ is a constant independent of f.

Lerner applied this philosophy to metaplectic operators to establish a variance uncertainty principle [11]. In this context, the estimate does not rely on an L^1/L^∞ bound. Rather, Lerner studies the second moment functional

$$\mathcal{V}(F) = \int |x|^2 |F(x)|^2 \, dx$$

and identifies the sharp infimum for a metaplectic operator in terms of the Schatten S_1 norm of the upper-right symplectic block.

Restricting our attention to the class of Schwartz functions with unit L^2 norm, we can naturally treat the variance $\mathcal{V}(F)$ as a measure of localization, denoted by $H(F)$. Furthermore, letting $\|B\|_{\mathcal{S}^1}$ denote the Schatten S_1 norm of the upper-right block of the matrix $\mathcal{A}$, Lerner's result can be expressed as:

$$H(\mu(\mathcal{A})F)H(F) \geq c\|B\|_{S_1}^2, \qquad \forall F \in \{G \in \mathcal{S}(\mathbb{R}^d) : \|G\|_{L^2(\mathbb{R}^d)} = 1\}. \tag{3.1}$$

Estimate (3.1) represents an input-output uncertainty principle. It inherently couples the localization measures of both the input F and the output $\mu(\mathcal{A})F$, provided that both functions belong to an appropriate space.

3.1.2 The Output-Only Point of View

The uncertainty principles of Gröchenig and Shafkulovska have a different nature. In their work, the central object is not a general input F, but the metaplectic time-frequency representation

$$W_{\mathcal{A}}(f, g) := \mu(\mathcal{A})(f \otimes \overline{g}),$$

where f is the signal and g is the window.

Thus the question is not whether both an input and its transform can be concentrated. The question is whether the output

$$\mu(\mathcal{A})(f \otimes \overline{g}),$$

can be localized on a small set.

The strongest form one might hope for is an output-only estimate

$$H_0(W_{\mathcal{A}}(f,g)) \geq C(\mathcal{A}) > 0, \tag{3.2}$$

with a constant independent of f and g. In this chapter, we prove (3.2) in the model cases where the reduction to an ordinary STFT is full rank. Furthermore, in rank-deficient cases, every positive mixing rank yields an unconditional adapted active-passive mixed-norm bound.

The full qualitative Benedicks classification is quoted from Gröchenig and Shafkulovska rather than reproved from (3.2).

Remark 3.1. The terminology *output-only* refers to the fact that only the concentration of the output representation is measured. The input $f \otimes \overline{g}$ does not appear through a separate factor $H_0(f \otimes \overline{g})$.

3.1.3 The Mixing Mechanism

The core idea of this chapter is that any uncertainty principle involving solely the output relies on a mixing mechanism. The input tensor $f \otimes \overline{g}$ depends on two independent sets of variables. If the metaplectic operator preserves this separation, there is no mathematical reason to expect an output-only uncertainty principle. Conversely, if the operator mixes these variables, an intrinsic relationship emerges along these coordinates, naturally giving rise to uncertainty principles.

As we shall see in detail, this phenomenon is tied to the quadratic phase of the metaplectic operator. The quadratic multiplier naturally splits into terms acting independently on f and g (which act as unimodular chirps and do not alter the relevant L^p norms) and a *cross-term* that mathematically couples the two tensor variables. This off-diagonal interaction is the sole engine of the mixing.

When this cross-interaction is of full rank, the mixed term yields the same structural properties as the short-time Fourier transform, up to coordinate changes and window modifications. When the interaction is rank-deficient, this mechanism operates exclusively along a specific set of nonzero singular directions. Exactly these directions, after a coordinate change and a rearrangement, define what we will call the active variables in the normal forms below. Suppose that a fixed normal-form coordinate map splits the output variables as $(z,p) \in \mathbb{R}^{2k} \times \mathbb{R}^{2(d-k)}$, separating the active variables z and the passive parameters p. For $1 \leq \rho \leq \infty$ define:

$$H^{(\rho)}_{\mathrm{act}}(W) := \frac{\|W\|_{L^\rho_p L^1_z}}{\|W\|_{L^\rho_p L^\infty_z}},$$

whenever the numerator is finite and the denominator is nonzero. In the mixed-chirp

normal form with active matrix $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_k)$ where $\sigma_j > 0$, we set

$$\nu(\Sigma) := |\det \Sigma| = \prod_{j=1}^{k} \sigma_j.$$

The active-fiber estimates proved below imply unconditionally that the expected output-only Wigderson-type estimate takes the form:

$$H^{(\rho)}_{\mathrm{act}}(W_{\mathcal{A}}) \geq \nu(\Sigma) > 0, \qquad 1 \leq \rho \leq \infty \tag{3.3}$$

When $k = d$, there is no passive variable and this naturally reduces to an ordinary full phase-space output-only estimate. When $0 < k < d$, it provides a targeted, anisotropic output-only estimate strictly along the forced mixing directions. This unconditional formulation (3.3) serves as the structural blueprint for this chapter. The precise analytic estimates will be established sequentially: first by using the short-time Fourier transform as a foundational model, then by executing the exact algebraic reductions for the free metaplectic case, and finally by generalizing the result to arbitrary metaplectic operators.

3.2 The STFT as a Model Output-Only Principle

In this section, we discuss the short-time Fourier transform as the first model for an output-only Wigderson-type uncertainty principle. The reason for starting from the STFT is twofold. First, the STFT is the standard time-frequency representation associated with a signal and a window. Second, as we shall see later, it appears naturally as the normal form of the fully mixed free metaplectic case.

The goal is not only to prove an estimate for the STFT, but also to identify the mechanism behind it. The relevant point is that the STFT transforms the pair (f, g) into a function on phase space. Therefore, the concentration of the output $V_g f$ can be controlled without introducing a separate concentration factor for the tensor product $f \otimes \overline{g}$.

Throughout this section, we assume that all functions are sufficiently regular. More precisely, we assume $0 \neq f, g \in \mathcal{S}(\mathbb{R}^d)$. These assumptions are not central to the argument; they primarily ensure that all quantities are well-defined and can be relaxed to appropriate subsets of Lebesgue spaces.

3.2.1 A Wigderson-Type Bound for the STFT

Let $f, g \in \mathcal{S}(\mathbb{R}^d)$ be Schwartz functions. We recall that the short-time Fourier transform of f with respect to the window $g \neq 0$ is defined by

$$V_g f(x, \omega) = \int_{\mathbb{R}^d} f(t) \overline{g(t - x)} e^{-2\pi i t \cdot \omega} \, dt.$$

Theorem 3.2 (Universal output-only STFT estimate)**.** Let $f, g \in L^2(\mathbb{R}^d)$ be nonzero functions and assume that $V_g f \in L^1(\mathbb{R}^{2d})$. Then

$$\|V_g f\|_{L^1(\mathbb{R}^{2d})} \geq \|V_g f\|_{L^\infty(\mathbb{R}^{2d})}.$$

Equivalently,

$$H_0(V_g f) := \frac{\|V_g f\|_{L^1(\mathbb{R}^{2d})}}{\|V_g f\|_{L^\infty(\mathbb{R}^{2d})}} \geq 1.$$

In particular, this assertion applies to all nonzero Schwartz functions.

Proof. By Moyal's identity, we have

$$\|V_g f\|^2_{L^2(\mathbb{R}^{2d})} = \|f\|^2_{L^2(\mathbb{R}^d)} \|g\|^2_{L^2(\mathbb{R}^d)}. \tag{3.4}$$

Applying the Cauchy-Schwarz inequality to the definition of the STFT yields

$$\begin{aligned} |V_g f(x,\omega)| &\leq \int_{\mathbb{R}^d} |f(t)|\,|g(t-x)|\,dt \\ &\leq \left(\int_{\mathbb{R}^d} |f(t)|^2\,dt\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} |g(t-x)|^2\,dt\right)^{\frac{1}{2}} \\ &= \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Taking the supremum over all $(x,\omega) \in \mathbb{R}^{2d}$ and squaring both sides, we obtain $\|V_g f\|^2_{L^\infty(\mathbb{R}^{2d})} \leq \|f\|^2_{L^2(\mathbb{R}^d)} \|g\|^2_{L^2(\mathbb{R}^d)}$. Substituting (3.4) into this inequality gives

$$\|V_g f\|^2_{L^\infty(\mathbb{R}^{2d})} \leq \|V_g f\|^2_{L^2(\mathbb{R}^{2d})}. \tag{3.5}$$

On the other hand, we can bound the L^2-norm using the L^1 and L^∞-norms:

$$\begin{aligned} \|V_g f\|^2_{L^2(\mathbb{R}^{2d})} &= \int_{\mathbb{R}^{2d}} |V_g f(x,\omega)|^2\,dx d\omega \\ &\leq \|V_g f\|_{L^\infty(\mathbb{R}^{2d})} \int_{\mathbb{R}^{2d}} |V_g f(x,\omega)|\,dx d\omega \\ &= \|V_g f\|_{L^\infty(\mathbb{R}^{2d})} \|V_g f\|_{L^1(\mathbb{R}^{2d})}. \end{aligned}$$

Combining this with (3.5), we deduce that

$$\|V_g f\|^2_{L^\infty(\mathbb{R}^{2d})} \leq \|V_g f\|^2_{L^2(\mathbb{R}^{2d})} \leq \|V_g f\|_{L^1(\mathbb{R}^{2d})} \|V_g f\|_{L^\infty(\mathbb{R}^{2d})}.$$

Since f and g are nonzero, it follows that $\|V_g f\|_{L^\infty(\mathbb{R}^{2d})} > 0$. Dividing the inequality by $\|V_g f\|_{L^\infty(\mathbb{R}^{2d})}$ concludes the proof. □

The important feature of Theorem 3.2 is that the final lower bound does not depend on f or g. This is exactly the output-only behavior that we want to reproduce in the metaplectic setting.

3.2.2 Benedicks-Type Uncertainty Principle for STFT

The estimate above gives the following quantitative support-size bound.

Proposition 3.3 (Effective support lower bound for the STFT)**.** Under the hypotheses of Theorem 3.2,

$$|\operatorname{supp}(V_g f)| \geq \frac{\|V_g f\|_{L^1(\mathbb{R}^{2d})}}{\|V_g f\|_{L^\infty(\mathbb{R}^{2d})}} \geq 1.$$

Proof. By Hölder's inequality applied on the support of $V_g f$,

$$\begin{aligned}\|V_g f\|_{L^1(\mathbb{R}^{2d})} &= \iint_{\operatorname{supp}(V_g f)} |V_g f(x,\omega)|\, dx\, d\omega \\ &\leq \|V_g f\|_{L^\infty(\mathbb{R}^{2d})} |\operatorname{supp}(V_g f)|.\end{aligned}$$

Hence

$$|\operatorname{supp}(V_g f)| \geq \frac{\|V_g f\|_{L^1(\mathbb{R}^{2d})}}{\|V_g f\|_{L^\infty(\mathbb{R}^{2d})}}.$$

The second inequality follows from Theorem 3.2. □

This does not imply the strong Benedicks property. The latter says that a nonzero STFT cannot have support of finite measure at all. As in the approach of Wigdersons, our framework does not recover the strong Benedicks principle: "If $V_g f$ is supported on a set of finite measure, then $f \equiv 0$ or $g \equiv 0$" [8]. Instead, it only yields a finite lower bound on the measure of the support.

3.2.3 The Partial STFT and Active Variables

The previous result involves all phase-space variables. In the metaplectic setting, however, the interaction between the two tensor variables may occur only in some directions. For this reason, it is useful to introduce a partial version of the STFT argument.

We split the spatial variables as

$$x = (x_1, x_2) \in \mathbb{R}^k \times \mathbb{R}^{d-k}.$$

The variables x_1 are considered *active* variables, while x_2 are treated as *passive* variables. We define the mixed Lebesgue space $L^\rho_{x_2} L^q_{x_1}(\mathbb{R}^d)$ as the space of all Lebesgue measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that the mixed norm

$$\|f\|_{L^\rho_{x_2} L^q_{x_1}} := \Big\| \|f(\cdot, x_2)\|_{L^q(\mathbb{R}^k)} \Big\|_{L^\rho(\mathbb{R}^{d-k})}$$

is finite. For $1 \leq q, \rho \leq \infty$, $L^\rho_{x_2} L^q_{x_1}(\mathbb{R}^d)$ is a Banach space.

The partial Fourier transform $\mathcal{F}_k : L^\infty_{x_2} L^1_{x_1}(\mathbb{R}^d) \mapsto L^\infty(\mathbb{R}^d)$ acting on the active variables is defined by

$$\mathcal{F}_k f(\omega_1, x_2) = \int_{\mathbb{R}^k} f(x_1, x_2) e^{-2\pi i x_1 \cdot \omega_1} \, dx_1.$$

The elementary estimates used in the standard Fourier case have a direct partial analogue.

Lemma 3.4 (Wigderson-style uncertainty principle for the partial Fourier transform)**.** Let f be a nonzero function such that $f, \mathcal{F}_k f \in L^\infty_{x_2} L^1_{x_1}$. Then,

$$\frac{\|f\|_{L^\infty_{x_2} L^1_{x_1}}}{\|f\|_{L^\infty(\mathbb{R}^d)}} \cdot \frac{\|\mathcal{F}_k f\|_{L^\infty_{x_2} L^1_{\omega_1}}}{\|\mathcal{F}_k f\|_{L^\infty(\mathbb{R}^d)}} \geq 1.$$

Proof. For every fixed x_2, we have

$$|\mathcal{F}_k f(\omega_1, x_2)| \leq \int_{\mathbb{R}^k} |f(x_1, x_2)| \, dx_1.$$

Taking the supremum over (ω_1, x_2) gives

$$\|\mathcal{F}_k f\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^\infty_{x_2} L^1_{x_1}}.$$

Applying the same argument to the inverse partial Fourier transform gives

$$\|f\|_{L^\infty(\mathbb{R}^d)} \leq \|\mathcal{F}_k f\|_{L^\infty_{x_2} L^1_{\omega_1}}.$$

Multiplying the two inequalities and rearranging gives the claim. □

We now introduce the partial STFT. For fixed passive variables, it is the usual STFT in the active variables. We write

$$V_g^k f(x_1, x_2, \omega_1, \omega_2) := V_{g(\cdot, -\omega_2)}(f(\cdot, x_2))(x_1, \omega_1).$$

For each fixed pair (x_2, ω_2), the operator acts as an STFT in the variables (x_1, ω_1) with respect to the window $g_{-\omega_2}$.

This leads to the active ρ-concentration ratio for the pSTFT:

$$H^{(\rho)}_{\text{act}}(V_g^k f) := \frac{\|V_g^k f\|_{L^\rho_{(x_2,\omega_2)} L^1_{(x_1,\omega_1)}}}{\|V_g^k f\|_{L^\rho_{(x_2,\omega_2)} L^\infty_{(x_1,\omega_1)}}}.$$

The same argument used for the full STFT gives an output-only estimate in the active variables.

For passive parameters $p = (x_2, \omega_2)$ and active variables $z = (x_1, \omega_1)$ write

$$\Phi_p(z) := V_g^k f(x_1, x_2, \omega_1, \omega_2) = V_{g(\cdot, -\omega_2)}\Big(f(\cdot, x_2)\Big)(x_1, \omega_1).$$

Proposition 3.5 (Uniform partial-STFT output estimate)**.** Assume that $\Phi_p \in L^1(\mathbb{R}^{2k})$ for almost every passive parameter p and that the mixed norms below are finite. Then, for every $1 \le \rho \le \infty$,

$$\|V_g^k f\|_{L_p^\rho L_z^1} \ge \|V_g^k f\|_{L_p^\rho L_z^\infty}.$$

Proof. For almost every p, either $\Phi_p = 0$, in which case both fiber norms vanish, or Φ_p is a nonzero ordinary k-dimensional STFT. Theorem 3.2 gives

$$\|\Phi_p\|_{L_z^1} \ge \|\Phi_p\|_{L_z^\infty},$$

for every such fiber. Taking the L_p^ρ norm proves the claim. □

Thus the partial STFT already has an unconditional active mixed-norm estimate; no uniform slice hypothesis is needed for the degenerate metaplectic case. If the representation mixes only k directions, then the uncertainty principle should be imposed only on those k active directions. The remaining variables behave as passive parameters and are controlled by mixed norms.

3.2.4 Active Quantitative Support Bound

The active concentration ratio also gives a support estimate on the active fibers. Let

$$\operatorname{supp}_{(x_1,\omega_1)}(V_g^k f) = \operatorname{supp}_z \Phi_p,$$

denote the support of $V_g^k f$ with respect to the active variables, for fixed passive variables (x_2, ω_2).

Proposition 3.6 (Active fiber support bound)**.** For almost every passive parameter p for which $\Phi_p \neq 0$,

$$|\operatorname{supp}_z \Phi_p| \ge 1.$$

Consequently,

$$\operatorname*{ess\,inf}_{\{p \in \mathbb{R}^{2(d-k)} : \Phi_p \neq 0\}} |\operatorname{supp}_z \Phi_p| \ge 1.$$

Proof. Following the same argument as in the proof of Proposition 3.3, we apply Hölder's inequality over the support of Φ_p to obtain

$$\|\Phi_p\|_{L_z^1} \le \|\Phi_p\|_{L_z^\infty} |\operatorname{supp}_z \Phi_p|.$$

Since we are considering parameters p for which $\Phi_p \neq 0$, we can divide by $\|\Phi_p\|_{L_z^\infty}$. Rearranging the terms and applying the lower bound from Theorem 3.2 fiberwise directly yields the desired result. □

Remark 3.7. This is still only a quantitative support-size statement. The partial-STFT Benedicks theorem is stronger: a nonzero active STFT fiber has support of infinite measure.

3.2.5 Role of the STFT in the Metaplectic Argument

The STFT is not only a preliminary example. It is the local model for the metaplectic argument developed in the next sections. In the free metaplectic case, the quadratic phase contains a mixed term of the form

$$e^{2\pi i u \cdot Kv}.$$

When K is invertible, the Fourier transform of this mixed tensor is an STFT, up to a linear change of variables and a change of window. When K has smaller rank, the same statement remains true only on the active variables determined by the nonzero singular directions of K.

Thus the estimates proved in this section provide the analytic model for the Gröchenig-Shafkulovska mechanism. The next step is to identify the matrix K inside the metaplectic representation and to relate its rank to the intrinsic mixing rank of the symplectic operator.

3.3 The Free Metaplectic Case

We now pass from the model case of the short-time Fourier transform to the first metaplectic situation. In this section we consider the free case, namely

$$\mathcal{A} \in Sp_0(4d, \mathbb{R}).$$

This is the case in which the metaplectic operator admits an explicit integral representation. It is therefore the natural setting in which to see how the Wigderson mechanism appears inside the Gröchenig-Shafkulovska framework.

The main point is that, in the free case, the metaplectic operator contains a quadratic phase. When this quadratic phase is applied to a tensor

$$F(u, v) = f(u)\overline{g(v)},$$

only one part of the phase mixes the two variables u and v. This mixed part is encoded by the off-diagonal block of a symmetric matrix. Its rank measures the number of active directions in which an output-only uncertainty principle can occur.

3.3.1 Quadratic Chirps and the Mixing Operator

Let $\mathcal{A} \in Sp(4d, \mathbb{R})$ be a free symplectic matrix, given by the block representation

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A, B, C, D \in \mathbb{R}^{2d\times 2d}$ and the upper-right block B is invertible. By defining

$$Q := DB^{-1}, \qquad P := B^{-1}A,$$

the standard symplectic identities ensure that both P and Q are real symmetric matrices. This leads to the free factorization of $\mathcal{A}$, which reads as

$$\mathcal{A} = \mathcal{V}_Q \mathcal{D}_B \mathcal{J} \mathcal{V}_P. \tag{3.6}$$

On the operator level, the corresponding metaplectic factorization holds up to a global unimodular phase factor. It is worth noting that the chirp operator $\mu(\mathcal{V}_Q)$ preserves absolute values, whereas the dilation $\mu(\mathcal{D}_B)$ acts by a linear change of variables. Consequently, their impact on L^1/L^∞ concentration ratios reduces to explicit determinant factors that depend entirely on the matrix $\mathcal{A}$.

At the operator level, $\mu(\mathcal{J})$ is the Fourier transform on $\mathbb{R}^{2d}$ up to the fixed phase determined by the metaplectic convention, and

$$\mu(\mathcal{V}_P)F(y) = e^{\pi i y\cdot Py}F(y).$$

We write the variable $y \in \mathbb{R}^{2d}$ as $y = (u, v) \in \mathbb{R}^d \times \mathbb{R}^d$ and we decompose P into blocks:

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{pmatrix}.$$

Applying the multiplier $\mathcal{V}_P$ to the tensor $F(u, v) = f(u)\overline{g(v)}$, we obtain

$$\begin{aligned} e^{\pi i y\cdot Py}F(y) &= e^{\pi i u\cdot P_{11}u}e^{2\pi i u\cdot P_{12}v}e^{\pi i v\cdot P_{22}v}f(u)\overline{g(v)} \\ &= e^{2\pi i u\cdot P_{12}v}\widetilde{f}(u)\overline{\widetilde{g}(v)}. \end{aligned}$$

The functions $\widetilde{f}$ and $\widetilde{g}$ are obtained from f and g by multiplication by unimodular quadratic chirps.

Remark 3.8. The factors

$$e^{\pi i u\cdot P_{11}u}, \qquad e^{\pi i v\cdot P_{22}v}$$

act separately on f and g. Since they are unimodular, they do not change the L^p norms of the two functions. Hence they do not affect the concentration estimates considered here.

The only term which couples the two tensor variables is

$$e^{2\pi i u\cdot P_{12}v}.$$

For this reason, the block P_{12} is the part of the quadratic phase responsible for the mixing.

This observation leads to the following definition.

Definition 3.9 (Mixing rank in the free case)**.** Let $\mathcal{A} \in Sp(4d, \mathbb{R})$ be free. Write $P = B^{-1}A \in \mathbb{R}^{2d\times 2d}$ in $d \times d$ blocks as

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^{\top} & P_{22} \end{pmatrix}.$$

The free mixing rank is

$$r_{\text{free}}(\mathcal{A}) := \operatorname{rank}_{\mathbb{R}} P_{12}.$$

The two extreme situations have a clear interpretation. If $P_{12} = 0$, the quadratic phase does not mix u and v. The tensor structure is preserved, up to separate chirps. If P_{12} has positive rank, then some variables are coupled. These are the variables on which the output-only uncertainty principle acts.

To isolate the essential part of the previous computation, we introduce a simple model operator.

Definition 3.10 (Mixing operator)**.** Let $K \in \mathbb{R}^{d\times d}$. For suitable functions f, g on $\mathbb{R}^d$, define

$$M_K(f, g)(u, v) := e^{2\pi i u\cdot Kv} f(u)\overline{g(v)}.$$

In the free metaplectic case, the operator M_K appears with

$$K = P_{12}.$$

Thus the analysis of the free case reduces to the study of the Fourier transform of $M_K(f, g)$, up to chirps and linear changes of variables.

The next proposition explains why the STFT is the correct model for this situation.

3.3.2 Full-Rank Mixing and the STFT Normal Form

We first assume that the mixing matrix K is invertible. In this case all directions are active.

Proposition 3.11 (STFT normal form for full-rank mixing)**.** Let $K \in GL(d, \mathbb{R})$ and let $f, g \in \mathcal{S}(\mathbb{R}^d)$. Put $h(x) = \widehat{g}(K^{\top}x)$. Then

$$\mathcal{F}_{u,v} M_K(f, g)(\xi, \eta) = V_h f(K^{-T}\eta, \xi).$$

Proof. By definition,

$$\begin{aligned}
\mathcal{F}_{u,v} M_K(f, g)(\xi, \eta) &= \iint e^{-2\pi i(u\cdot\xi + v\cdot\eta)} e^{2\pi i u\cdot Kv} f(u)\overline{g(v)}\, du\, dv \\
&= \int f(u) e^{-2\pi i u\cdot\xi} \left(\int \overline{g(v)} e^{-2\pi i v\cdot(\eta - K^{\top}u)}\, dv \right) du \\
&= \int f(u)\overline{\widehat{g}(K^{\top}u - \eta)} e^{-2\pi i u\cdot\xi}\, du.
\end{aligned}$$

Since $h(u - K^{-T}\eta) = \widehat{g}(K^{\top}u - \eta)$, this is $V_h f(K^{-T}\eta, \xi)$. □

Remark 3.12. This proposition explains why the STFT was studied first. In the fully mixed free case, the metaplectic representation reduces, after harmless operations, to an STFT. Therefore the output-only estimate for the STFT becomes the model estimate for the free metaplectic case.

The linear change of variables

$$(x, \omega) = (K^{-T}\eta, \xi),$$

introduces a determinant factor in the L^1 norm, while it does not change the L^∞ norm. Hence the active concentration ratio of $\mathcal{F}M_K(f, g)$ is controlled by the corresponding ratio for the STFT.

Corollary 3.13 (Full-rank mixing output estimate)**.** Let $K \in GL(d, \mathbb{R})$ and let $0 \neq f, g \in \mathcal{S}(\mathbb{R}^d)$. Then

$$H_0(\mathcal{F}M_K(f, g)) \geq |\det K|.$$

3.3.3 Rank-Deficient Mixing and Active Variables

We now consider the case in which the mixing matrix is not invertible. Let

$$k = \operatorname{rank}(K) \leq d.$$

By the singular value decomposition, and up to orthogonal changes of variables, we may assume that

$$K = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_k), \qquad \sigma_j > 0.$$

We split the variables as

$$u = (x, s) \in \mathbb{R}^k \times \mathbb{R}^{d-k}, \qquad v = (y, t) \in \mathbb{R}^k \times \mathbb{R}^{d-k}.$$

Then

$$M_K(f, g)(x, s; y, t) = e^{2\pi i x \cdot \Sigma y} f(x, s)\overline{g(y, t)}.$$

Only the variables x and y are mixed. The variables s and t remain passive.

To make this precise, we take Fourier transforms in the passive variables and write

$$z = (\xi, \eta) \in \mathbb{R}^{2k}, \qquad p = (\alpha, \beta) \in \mathbb{R}^{2(d-k)}.$$

Here z denotes the active frequency variables, while p denotes the passive frequency variables.

For $\alpha, \beta \in \mathbb{R}^{d-k}$ define the passive partial Fourier transforms

$$f_\alpha(x) := \int_{\mathbb{R}^{d-k}} f(x,s)e^{-2\pi i s\cdot\alpha}\,ds, \qquad g_\beta(y) := \int_{\mathbb{R}^{d-k}} g(y,t)e^{-2\pi i t\cdot\beta}\,dt,$$

and set

$$h_\beta(u) := \widehat{g_{-\beta}}(\Sigma u), \qquad u \in \mathbb{R}^k,$$

where the hat is the k-dimensional Fourier transform in the active variable.

Proposition 3.14 (Rank-deficient STFT normal form)**.** For $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathcal{F}M_K(f,g)(\xi,\alpha;\eta,\beta) = V_{h_\beta} f_\alpha(\Sigma^{-1}\eta, \xi). \tag{3.7}$$

Proof. Fourier transformation in the passive variables gives $f_\alpha(x)\overline{g_{-\beta}(y)}$. Fourier transformation in y then yields $\overline{\widehat{g_{-\beta}}(\Sigma x - \eta)}$. Since $h_\beta(x - \Sigma^{-1}\eta) = \widehat{g_{-\beta}}(\Sigma x - \eta)$, the remaining integral in x is the displayed STFT. □

Remark 3.15. Formula (3.7) is the key point in the degenerate case. It says that, after freezing the passive variables, the representation is an STFT in the active variables. The number of active directions is exactly

$$k = \operatorname{rank}(K).$$

3.3.4 Active Output-Only Estimate in the Free Case

The previous formula suggests the correct r-concentration ratio. We define

$$H^{(\rho)}_{\mathrm{act}}(\mathcal{F}M_K(f,g)) := \frac{\|\mathcal{F}M_K(f,g)\|_{L^\rho_p L^1_z}}{\|\mathcal{F}M_K(f,g)\|_{L^\rho_p L^\infty_z}}.$$

Here z are the active variables and p are the passive variables.

Theorem 3.16 (Active output-only estimate in the free model)**.** Let $0 \neq f, g \in \mathcal{S}(\mathbb{R}^d)$ and let K have rank $k > 0$. Up to product-preserving orthogonal changes of variables, let its nonzero block be

$$\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_k), \qquad \nu(\Sigma) = |\det \Sigma|.$$

Suppose that the mixed norms indicated below are finite. Then, for every $1 \leq \rho \leq \infty$, the representation $\mathcal{F}M_K(f,g)$ with active variables $z \in \mathbb{R}^{2k}$ and passive variables $p \in \mathbb{R}^{2(d-k)}$ satisfies

$$\|\mathcal{F}M_K(f,g)\|_{L^\rho_p L^1_z} \geq \nu(\Sigma)\|\mathcal{F}M_K(f,g)\|_{L^\rho_p L^\infty_z}.$$

Equivalently, $H^{(\rho)}_{\mathrm{act}}(\mathcal{F}M_K(f,g)) \geq \nu(\Sigma)$.

Proof. For a fixed passive pair $p = (\alpha, \beta)$, let us define the active fiber $\Phi_p = \mathcal{F}M_K(f, g)(\cdot, p)$. Following steps analogous to those in Theorem 3.2, and applying the exact sign convention of Proposition 3.14, we can write

$$\Phi_p(\xi, \eta) = V_{h_\beta} f_\alpha(\Sigma^{-1}\eta, \xi), \qquad h_\beta(w) = \widehat{g_{-\beta}}(\Sigma w).$$

Moyal's identity and the active change of variables yield

$$\|\Phi_p\|^2_{L^2(\mathbb{R}^{2k})} = \|f_\alpha\|^2_{L^2(\mathbb{R}^k)} \|g_{-\beta}\|^2_{L^2(\mathbb{R}^k)}.$$

Moreover, we have the uniform bound

$$\|\Phi_p\|_{L^\infty(\mathbb{R}^{2k})} \leq \nu(\Sigma)^{-1/2} \|f_\alpha\|_{L^2(\mathbb{R}^k)} \|g_{-\beta}\|_{L^2(\mathbb{R}^k)},$$

which implies that

$$\|\Phi_p\|^2_{L^\infty(\mathbb{R}^{2k})} \leq \nu(\Sigma)^{-1} \|\Phi_p\|^2_{L^2(\mathbb{R}^{2k})}.$$

Using the standard interpolation inequality $\|\Phi_p\|^2_{L^2(\mathbb{R}^{2k})} \leq \|\Phi_p\|_{L^1(\mathbb{R}^{2k})} \|\Phi_p\|_{L^\infty(\mathbb{R}^{2k})}$, one obtains

$$\|\Phi_p\|_{L^1(\mathbb{R}^{2k})} \geq \nu(\Sigma) \|\Phi_p\|_{L^\infty(\mathbb{R}^{2k})}.$$

This inequality holds for every fiber p, including the zero fibers. Finally, taking the L^ρ-norm with respect to the passive variables p over $\mathbb{R}^{2(d-k)}$ completes the proof. □

3.3.5 A Benedicks-Type Uncertainty Principle for the Free Case

As in the STFT model, the active concentration estimate implies a lower bound on the Lebesgue measure of the active support fibers.

Let us consider the representation $\mathcal{F}M_K(f, g)(z, p)$. For each fixed passive parameter p, we denote by

$$\operatorname{supp}_z \mathcal{F}M_K(f, g)(\cdot, p),$$

the support of the corresponding active fiber.

Corollary 3.17 (Active support bound in the free case)**.** For almost every passive parameter p such that the active fiber $\mathcal{F}M_K(f, g)(\cdot, p)$ is not identically zero, the Lebesgue measure of its support satisfies the lower bound:

$$|\operatorname{supp}_z \mathcal{F}M_K(f, g)(\cdot, p)| \geq \nu(\Sigma).$$

Proof. For every nonzero active fiber, applying Hölder's inequality over the support yields the standard bound:

$$\|\mathcal{F}M_K(f,g)(\cdot,p)\|_{L^1} \leq \|\mathcal{F}M_K(f,g)(\cdot,p)\|_{L^\infty} |\operatorname{supp}_z \mathcal{F}M_K(f,g)(\cdot,p)|.$$

Dividing by the L^∞ norm provides a lower bound for the measure of the active support in terms of the L^1/L^∞ ratio. Combining this relation with the active fiber inequality already established in the proof of Theorem 3.16 directly yields the claim. □

This result represents the free metaplectic analogue of the active support-size bound for the pSTFT. Naturally, the quantitative observations previously made for the pSTFT apply equally in this broader context.

3.3.6 Interpretation for Free Metaplectic Representations

We now return to the metaplectic representation

$$W_{\mathcal{A}}(f,g) = \mu(\mathcal{A})(f \otimes \overline{g}),$$

with $\mathcal{A} \in Sp_0(4d,\mathbb{R})$.

By the decomposition (3.6), the operator $\mu(\mathcal{A})$ differs from $\mathcal{F}M_{P_{12}}$ only by chirps and linear changes of variables. The chirps are unimodular and do not change the relevant L^p norms. The linear changes of variables only introduce determinant factors in the L^1 or mixed L^1 norms, while leaving the L^∞ norms unchanged up to the corresponding normalization.

Since a general output linear map also transports the active/passive splitting, lower-rank mixed norms must be understood in adapted coordinates.

Therefore, the output-only estimate for $\mathcal{F}M_{P_{12}}(f,g)$ transfers to the free metaplectic representation.

Theorem 3.18 (Free metaplectic active output-only principle)**.** Let $\mathcal{A} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(4d,\mathbb{R})$ be a free symplectic matrix, and let P_{12} be the $d \times d$ off-diagonal block of $P = B^{-1}A$. Let

$$k = \operatorname{rank} P_{12} > 0.$$

Choose product-preserving orthogonal coordinates in which the nonzero singular block of P_{12} is

$$\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_k).$$

Let $\widetilde{W}_{\mathcal{A}}(f,g)(z,p)$ denote the normal-form representative of $W_{\mathcal{A}}(f,g)$, obtained by factoring out the output chirp and dilation from the decomposition (3.6), and

evaluated in product-preserving orthogonal coordinates (z, p). Then, for every $1 \leq \rho \leq \infty$ and $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\|\widetilde{W}_{\mathcal{A}}(f,g)\|_{L_p^\rho L_z^1} \geq \left(\prod_{j=1}^{k} \sigma_j\right) \|\widetilde{W}_{\mathcal{A}}(f,g)\|_{L_p^\rho L_z^\infty}.$$

If $k = d$, this reduces to a full-space H_0 estimate.

Proof. The diagonal blocks P_{11} and P_{22} produce separate unimodular chirps on the two inputs. The off-diagonal block produces the mixed chirp $M_{P_{12}}$. The singular-value coordinate changes reduce it to M_Σ, while the fixed output chirp and dilation are incorporated into the adapted output coordinates. Theorem 3.16 then gives the displayed inequality. Multiplicative metaplectic phase factors cancel from the ratio. □

The basic principle of the chapter:

$$\text{mixing of tensor variables} \quad \Longrightarrow \quad \text{output-only uncertainty.}$$

In the next section we extend this idea to general metaplectic operators. The pre-Iwasawa decomposition allows us to reduce the general case to the same type of free model, after choosing suitable adapted coordinates. The role of the block P_{12} will then be played by an intrinsic mixing rank associated with the symplectic matrix.

3.4 The General Metaplectic Case

We now extend the previous construction to an arbitrary metaplectic operator. The free case showed that the output-only uncertainty principle is controlled by the mixed block of the quadratic phase. For a general symplectic matrix, such a block is not immediately visible. The purpose of this section is to recover it by means of the pre-Iwasawa decomposition.

Let

$$\mathcal{A} \in Sp(4d, \mathbb{R})$$

and consider the metaplectic time-frequency representation

$$W_{\mathcal{A}}(f,g) = \mu(\mathcal{A})(f \otimes \overline{g}).$$

We use the pre-Iwasawa decomposition

$$\mathcal{A} = \mathcal{V}_Q \mathcal{D}_L \mathcal{R}_U.$$

The factors $\mathcal{V}_Q$ and $\mathcal{D}_L$ affect the representation through chirps and linear changes of variables. As in the free case, these operations do not create the mixing between the two tensor variables. The mixing information is contained in the factor $\mathcal{R}_U$. For this reason, the main task is to extract from U a free-type quadratic phase and then apply the previous analysis.

3.4.1 Regular Phases and Reduction to the Free Model

The orthogonal symplectic factor $\mathcal{R}_U$ is not necessarily free. As in [8], to reduce it to the free case, we use a harmless phase rotation. Let

$$\theta \in \mathbb{T}$$

and write

$$\theta U = U_{1\theta} + iU_{2\theta},$$

where

$$U_{1\theta} = \mathrm{Re}(\theta U), \qquad U_{2\theta} = \mathrm{Im}(\theta U).$$

Definition 3.19. We say that a phase $\theta \in \mathbb{T}$ is *regular* if $U_{2\theta}$ is invertible.

For a regular phase θ, we define

$$P_\theta := U_{2\theta}^{-1} U_{1\theta}.$$

This is a real symmetric matrix. We decompose it into blocks as

$$P_\theta = \begin{pmatrix} P_{11,\theta} & K_\theta \\ K_\theta^\top & P_{22,\theta} \end{pmatrix}.$$

The block K_θ is the analogue of the block P_{12} in the free case.

Lemma 3.20. All but finitely many phases $\theta \in \mathbb{T}$ are regular.

Proof. Since $\theta \in \mathbb{T}$, we have $\overline{\theta} = \theta^{-1}$.

$$\mathrm{Im}(\theta U) = \frac{\theta U - \overline{\theta}\,\overline{U}}{2i} = \frac{\theta^{-1}}{2i}(\theta^2 U - \overline{U}).$$

Then we have:

$$\det(\mathrm{Im}(\theta U)) = (2i)^{-2d}\theta^{-2d}\det(\theta^2 U - \overline{U}).$$

Observe that $\det(\theta^2 U - \overline{U})$ is a polynomial in θ. Its leading coefficient is precisely $\det U$, which is nonzero because U is a unitary matrix. Therefore, it is a non trivial polynomial and can only possess finitely many roots on the unit circle $\mathbb{T}$. □

Remark 3.21. The phase θ is only a technical device. It allows us to choose a representative for which the imaginary part is invertible. Once this is done, the factor $\mathcal{R}_{\theta U}$ can be written in a free form, and the quadratic phase is governed by P_θ.

The role of the phase is described by the identity

$$\theta U = U_{2\theta}(P_\theta + iI).$$

Thus, for regular θ, the factor $\mathcal{R}_{\theta U}$ falls into the same class considered in the free case. The corresponding quadratic phase has mixed block K_θ.

The scalar rotation can be naturally shifted to the input functions. Indeed, since

$$\mathcal{R}_U = \mathcal{R}_{\theta U}\mathcal{R}_{\overline{\theta} I_{2d}},$$

and the matrix $\mathcal{R}_{\overline{\theta} I_{2d}}$ is block-diagonal with respect to the two tensor components, applying the metaplectic representation yields, up to a global phase factor,

$$\mu(\mathcal{R}_{\overline{\theta} I_{2d}})(f \otimes \overline{g}) = f_\theta \otimes \overline{g_\theta},$$

where we have defined

$$f_\theta := \mu(\mathcal{R}_{\overline{\theta} I_d})f, \qquad g_\theta := \mu(\mathcal{R}_{\theta I_d})g.$$

Consequently, up to a unimodular scalar, the metaplectic time-frequency representation can be rewritten as

$$W_{\mathcal{A}}(f, g) = \mu(\mathcal{V}_Q \mathcal{D}_L \mathcal{R}_{\theta U})(f_\theta \otimes \overline{g_\theta}).$$

The free mixed block of P_θ therefore measures the coupling after a fixed metaplectic change of the signal and window; see [8, Lem. 5.1 and Thm. 5.2].

3.4.2 The General Mixing Rank

The previous discussion suggests defining the number of active variables by the rank of K_θ. However, this rank should not depend on the particular regular phase. We now recall the intrinsic formulation.

Let

$$\widetilde{Q} := U^\top U.$$

We write $\widetilde{Q}$ in block form:

$$\widetilde{Q} = \begin{pmatrix} \widetilde{Q}_{11} & \widetilde{Q}_{12} \\ \widetilde{Q}_{12}^\top & \widetilde{Q}_{22} \end{pmatrix}.$$

Definition 3.22 (Mixing rank)**.** The mixing rank of $\mathcal{A}$ is defined by

$$r(\mathcal{A}) := \operatorname{rank}_{\mathbb{C}}(\widetilde{Q}_{12}).$$

This rank is an intrinsic property of $\mathcal{A}$, independent of the nonunique pre-Iwasawa decomposition. If the dilation factor L is not required to be positive definite, the gauge freedom allows replacing $U \mapsto OU$ with a real orthogonal matrix $O \in O(2d, \mathbb{R})$ alongside a compensating change of L. Since $(OU)^\top (OU) = U^\top U$, the matrix $U^\top U$, and hence the rank of its off-diagonal block, remains invariant. Furthermore, for every regular phase θ, this rank agrees with the rank of the real mixed block K_θ.

Proposition 3.23 (Rank and strength comparison)**.** Let $U \in U(2d)$ and let $\theta \in \mathbb{T}$ be regular. Write

$$\theta U = U_{1\theta} + iU_{2\theta}, \qquad P_\theta := U_{2\theta}^{-1} U_{1\theta},$$

and consider the block decomposition

$$P_\theta = \begin{pmatrix} P_{11,\theta} & K_\theta \\ K_\theta^\top & P_{22,\theta} \end{pmatrix}.$$

Then P_θ is real symmetric and

$$\operatorname{rank}_{\mathbb{R}} K_\theta = \operatorname{rank}_{\mathbb{C}}(U^\top U)_{12} =: k.$$

Furthermore, if $s_1(X) \geq \cdots \geq s_k(X) > 0$ denote the positive singular values of a rank-k matrix X, then

$$\prod_{j=1}^{k} s_j(K_\theta) \geq 2^{-k} \prod_{j=1}^{k} s_j\Big((U^\top U)_{12}\Big). \tag{3.8}$$

Proof. Set $V = \theta U$ and $P = P_\theta$. The unitary block identities imply that P is symmetric (i.e., $P = P^\top$) and yield the Cayley transform formula

$$V^\top V = (P + iI)(P - iI)^{-1} = I + 2i(P - iI)^{-1},$$

equivalently $\theta^2 U^\top U = (P + iI)(P - iI)^{-1}$.

Consequently, the off-diagonal block $(V^\top V)_{12}$ has the same rank as $((P - iI)^{-1})_{12}$. Applying the block inversion formula gives

$$((P - iI)^{-1})_{12} = -S^{-1} K_\theta (P_{22,\theta} - iI)^{-1},$$

where S denotes the corresponding Schur complement. Since both outer factors in this product are invertible, the ranks of the two matrices coincide.

Turning to the quantitative assertion, we observe that the matrix $P - iI$ is normal and all of its singular values are bounded below by 1. This implies that the operator norms satisfy

$$\|(P_{22,\theta} - iI)^{-1}\| \leq 1 \qquad \text{and} \qquad \|S^{-1}\| \leq 1.$$

The latter inequality holds because S^{-1} is a diagonal block of the matrix $(P - iI)^{-1}$. Applying the singular-value inequality $s_j(AXB) \leq \|A\| s_j(X) \|B\|$ to the product $-S^{-1}K_\theta(P_{22,\theta} - iI)^{-1}$, we deduce that

$$s_j\Big(((P - iI)^{-1})_{12}\Big) \leq \| - S^{-1}\| s_j(K_\theta) \|(P_{22,\theta} - iI)^{-1}\| \leq s_j(K_\theta).$$

Returning to the Cayley formula, we see that

$$s_j\Big((V^\top V)_{12}\Big) = 2 s_j\Big(((P - iI)^{-1})_{12}\Big).$$

Taking the product over all $j = 1, \ldots, k$ immediately yields the desired inequality (3.8). □

Remark 3.24. This proposition is the point where the general case reconnects with the free case. The block K_θ is useful for the proof because it gives a free normal form. The block $\widetilde{Q}_{12}$ is useful conceptually because it shows that the rank is intrinsic.

3.4.3 Output-Only Estimate in the General Case

We now apply the free-case result in adapted coordinates. Let

$$k = r(\mathcal{A}).$$

If $k > 0$, then for any regular phase θ the block K_θ has rank k. These coordinates split the output variables into active variables

$$z \in \mathbb{R}^{2k}$$

and passive variables

$$p \in \mathbb{R}^{2(d-k)}.$$

The free analysis gives an active output-only estimate in these coordinates. The remaining factors in the pre-Iwasawa decomposition only change the constants.

Fix a regular θ and an SVD

$$K_\theta = W_1 \Sigma_\theta W_2^T, \quad \Sigma_\theta = \operatorname{diag}(\sigma_1, \ldots, \sigma_k, 0, \ldots, 0).$$

Let $O = \mathrm{diag}(W_1, W_2)$. Define the fully pulled-back output by

$$\widetilde{W}_{\mathcal{A},\theta}(f,g) := \mu(\mathcal{D}_{O^T})\mu(T_{\mathcal{A},\theta}^{-1})W_{\mathcal{A}}(f,g).$$

Equivalently, up to the unavoidable metaplectic sign,

$$T_{\mathcal{A},\theta}^{-1} = \mathcal{D}_{U_{2,\theta}}^{-1}\mathcal{V}_{Q_\theta}^{-1}\mathcal{D}_L^{-1}\mathcal{V}_Q^{-1}.$$

The phase rotation $\mathcal{R}_{\bar{\theta}I}$ is absorbed into transformed inputs f_θ, g_θ, not into the output coordinates.

With this definition the proof does reduce to the free model. The diagonal blocks of P_θ become separate input chirps, the orthogonal changes determined by W_1, W_2 preserve the tensor splitting, and the remaining cross phase is M_{Σ_θ}.

Theorem 3.25 (Adapted-chart metaplectic active output-only estimate)**.** Let $\mathcal{A} \in Sp(4d, \mathbb{R})$ and let the notation above hold. Assume

$$k = \mathrm{rank}_{\mathbb{C}}(U^T U)_{12} > 0.$$

For nonzero Schwartz inputs and every regular phase θ, define $\widetilde{W}_{\mathcal{A},\theta}$ by the full pullback above. Split its variables into the $2k$ active variables z determined by Σ_θ and passive variables p. Then, for $1 \le \rho \le \infty$,

$$\|\widetilde{W}_{\mathcal{A},\theta}(f,g)\|_{L_p^\rho L_z^1} \ge \nu_\theta \|\widetilde{W}_{\mathcal{A},\theta}(f,g)\|_{L_p^\rho L_z^\infty}, \quad \nu_\theta = \prod_{j=1}^{k} s_j(K_\theta).$$

Moreover,

$$\nu_\theta \ge 2^{-k} \prod_{j=1}^{k} s_j\Big((U^T U)_{12}\Big).$$

Proof. After the scalar phase has been moved to the two inputs, apply $\mu(T_{\mathcal{A},\theta}^{-1})$ to remove the complete output factor in the free factorization. The remaining operator is $\mu(\mathcal{J}\mathcal{V}_{P_\theta})$. Split P_θ into its two diagonal blocks and cross block. The diagonal blocks multiply the two inputs by unimodular chirps. Apply the product orthogonal changes W_1, W_2 and the inverse corresponding output rotation $\mu(\mathcal{D}_{O^T})$. This reduces the cross block to $\mathrm{diag}(\Sigma_\theta, 0)$. The resulting output is, up to a scalar phase, the free model of Theorem 3.16. The mixed-norm inequality follows from that theorem. The intrinsic lower bound is Proposition 3.23. □

When $k = d$, there are no passive variables, so one may transfer the estimate back to the original output. Chirps do not change absolute values, while the two dilations contribute the determinant of their product. Thus, the natural raw coordinate estimate has the form

$$H_0(W_{\mathcal{A}}(f,g)) \ge |\det(LU_{2,\theta})|\,|\det K_\theta|.$$

In the rank-deficient case ($k < d$), no equally simple raw mixed norm is available without explicitly tracking how the output map acts on the active/passive splitting.

3.4.4 A Real Algebraic Characterization

The pre-Iwasawa factors $\mathcal{V}_Q$ and $\mathcal{D}_L$ preserve the finiteness of the support, acting merely as a multiplication by a unimodular chirp and as a fixed linear coordinate change, respectively. Consequently, the crucial component dictating the qualitative behavior of the representation is the compact factor U, or more specifically, the off-diagonal block of the symmetric matrix $\widetilde{Q} := U^\top U$.

A purely real algebraic formulation of this same invariant can be constructed. By defining the signature matrix $\widetilde{L} := \operatorname{diag}(I_d, -I_d)$, we introduce the real skew-symmetric matrix

$$C_{\widetilde{L}}(U) := U_1 \widetilde{L} U_2^\top - U_2 \widetilde{L} U_1^\top,$$

where $U = U_1 + iU_2$ is the decomposition of U into its real and imaginary parts. The precise relationship between these algebraic objects is given by the rank identity

$$\operatorname{rank}_{\mathbb{R}} C_{\widetilde{L}}(U) = 2 \operatorname{rank}_{\mathbb{C}}(U^\top U)_{12}.$$

It is this exact identity, rather than the partial Fourier decoupling condition discussed previously, that serves as the rigorous algebraic detector for the number of active mixing channels.

Proposition 3.26. Let $U \in U(2d)$, write $U = U_1 + iU_2$, set $\widetilde{Q} = U^\top U$, and let $\widetilde{L} = \operatorname{diag}(I_d, -I_d)$. Then

$$\operatorname{rank}_{\mathbb{R}} C_{\widetilde{L}}(U) = 2 \operatorname{rank}_{\mathbb{C}}(\widetilde{Q}_{12}).$$

Consequently,

$$r(\mathcal{A}) = \operatorname{rank}_{\mathbb{C}}(\widetilde{Q}_{12}) = \frac{1}{2} \operatorname{rank}_{\mathbb{R}} C_{\widetilde{L}}(U).$$

Proof. A direct block matrix computation yields the commutator

$$[\widetilde{Q}, \widetilde{L}] = \begin{pmatrix} 0 & -2\widetilde{Q}_{12} \\ 2\widetilde{Q}_{12}^\top & 0 \end{pmatrix},$$

which implies that $\operatorname{rank}_{\mathbb{C}}[\widetilde{Q}, \widetilde{L}] = 2 \operatorname{rank}_{\mathbb{C}} \widetilde{Q}_{12}$.

Since U is unitary, we have $UU^* = I$, which entails $\overline{U}U^\top = I$. Conjugating the commutator by the invertible matrices $\overline{U}$ and U^*, we obtain:

$$\begin{aligned} \overline{U}[\widetilde{Q}, \widetilde{L}]U^* &= \overline{U}(U^\top U\widetilde{L} - \widetilde{L}U^\top U)U^* \\ &= (\overline{U}U^\top)U\widetilde{L}U^* - \overline{U}\widetilde{L}(U^\top UU^*) \\ &= U\widetilde{L}U^* - \overline{U}\widetilde{L}U^\top \\ &= 2i \operatorname{Im}(U\widetilde{L}U^*) \\ &= -2iC_{\widetilde{L}}(U). \end{aligned}$$

Multiplying a matrix by invertible matrices or by a nonzero scalar preserves its rank. Therefore, the complex rank of $C_{\widetilde{L}}(U)$ is equal to the complex rank of $[\widetilde{Q}, \widetilde{L}]$. Furthermore, because $C_{\widetilde{L}}(U)$ is a purely real matrix, its real and complex ranks coincide. Combining these observations yields the desired identities and completes the proof. □

Remark 3.27. The skew-symmetry of $C_{\widetilde{L}}(U)$ geometrically explains why its nonzero real directions occur in pairs. The exact factor of $1/2$ follows from the calculations, meaning that exactly one active mixing channel corresponds to each pair.

3.4.5 Characterization in Terms of the Blocks of $\mathcal{A}$

It is also useful to express the obstruction directly in terms of the blocks of the original symplectic matrix. Let

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Using the pre-Iwasawa decomposition

$$\mathcal{A} = \mathcal{V}_Q \mathcal{D}_L \mathcal{R}_U,$$

one obtains

$$A = LU_1, \qquad B = LU_2.$$

Therefore the real obstruction can be written in terms of A and B.

Proposition 3.28. Let $\mathcal{A} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \mathcal{V}_Q \mathcal{D}_L \mathcal{R}_U$ be a pre-Iwasawa decomposition, so that $A = LU_1$ and $B = LU_2$. Then

$$A\widetilde{L}B^\top - B\widetilde{L}A^\top = L\, C_{\widetilde{L}}(U)\, L^\top,$$

and therefore

$$r(\mathcal{A}) = \frac{1}{2} \operatorname{rank}_{\mathbb{R}}\Big(A\widetilde{L}B^\top - B\widetilde{L}A^\top\Big).$$

In particular, the matrix on the right vanishes if and only if $(U^\top U)_{12} = 0$.

Proof. By substituting $A = LU_1$ and $B = LU_2$ into the target expression and factoring out the invertible matrices L and $L^\top$, we obtain

$$A\widetilde{L}B^\top - B\widetilde{L}A^\top = L(U_1\widetilde{L}U_2^\top - U_2\widetilde{L}U_1^\top)L^\top = L\, C_{\widetilde{L}}(U)\, L^\top.$$

Since multiplying by the invertible matrices L and $L^\top$ preserves the rank, the real rank of $A\widetilde{L}B^\top - B\widetilde{L}A^\top$ is identical to the real rank of $C_{\widetilde{L}}(U)$. The conclusion then follows directly by applying Proposition 3.26. □

Remark 3.29. This formulation is useful because it avoids the explicit use of the unitary factor U. It expresses the obstruction directly in terms of the symplectic blocks A and B.

Combining the previous statements, we obtain the following criterion.

Theorem 3.30 (Algebraic criterion, active estimate, and Benedicks alternative)**.** Let $\mathcal{A} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(4d, \mathbb{R})$, and let U denote its compact factor derived from the pre-Iwasawa decomposition.

By the classification of Gröchenig and Shafkulovska [8, Thm. 5.4], the sesquilinear representation $W_{\mathcal{A}}(f, g)$ has the Benedicks property if and only if:

1. $(U^\top U)_{12} \neq 0$.

Furthermore, we prove that this is equivalent to any of the following original conditions:

2. $r(\mathcal{A}) > 0$;
3. $C_{\widetilde{L}}(U) \neq 0$;
4. $A\widetilde{L}B^\top - B\widetilde{L}A^\top \neq 0$.

Whenever these conditions are satisfied, Theorem 3.25 provides an unconditional active mixed-norm estimate in adapted coordinates, featuring exactly $r(\mathcal{A})$ active channels.

Proof. The equivalence between condition 1 and the original conditions 2, 3, and 4 follows directly from Propositions 3.26 and 3.28. The concluding assertion regarding the active mixed-norm estimate is a direct application of Theorem 3.25. □

Corollary 3.31. The symmetric unitary matrix $U^\top U$ is block diagonal if and only if $C_{\widetilde{L}}(U) = 0$.

Proof. By Proposition 3.26, $C_{\widetilde{L}}(U) = 0$ if and only if $(U^\top U)_{12} = 0$, which is exactly block diagonality. □

3.4.6 Conclusion of the General Case

The general analysis yields the following structural hierarchy. For every positive intrinsic rank $k = r(\mathcal{A})$, we obtain an unconditional active output-only estimate in adapted coordinates:

$$\|\widetilde{W}_{\mathcal{A},\theta}\|_{L^p_\rho L^1_z} \geq \nu_\theta \|\widetilde{W}_{\mathcal{A},\theta}\|_{L^p_\rho L^\infty_z}$$

When $k = d$, this reduces to an ordinary full-space L^1/L^∞ concentration estimate. Conversely, when $0 < k < d$, our argument controls an anisotropic mixed norm

rather than an isotropic full-space functional; nevertheless, the anisotropic theorem remains unconditional. Furthermore, the qualitative sesquilinear Benedicks property holds independently for every $k > 0$.

The active rank can be computed equivalently in three ways: from the off-diagonal block $(U^\top U)_{12}$, from the real skew-symmetric matrix $C_{\widetilde{L}}(U)$, or directly from the macroscopic symplectic blocks via the identity:

$$r(\mathcal{A}) = \frac{1}{2}\operatorname{rank}_{\mathbb{R}}(A\widetilde{L}B^\top - B\widetilde{L}A^\top).$$

3.5 Geometric Interpretation of the Active Channels

The algebraic conditions governing the active and passive channels of a metaplectic representation can be elegantly recast in the language of symplectic geometry. Specifically, we investigate the geometric nature of the condition $\widetilde{Q}_{12} = 0$, which corresponds to the complete decoupling of the variables.

Let us identify the space $\mathbb{C}^{2d}$ with the real symplectic space $\mathbb{R}^{4d}$. For any unitary matrix $U \in U(2d)$, we define the subspace

$$\Lambda_U := U^{-1}\mathbb{R}^{2d}.$$

Since $\Lambda_{OU} = \Lambda_U$ for every orthogonal matrix $O \in O(2d)$, this parametrization naturally realizes the Lagrangian Grassmannian as the left quotient $O(2d)\backslash U(2d)$. Under this convention, the Cartan embedding is given by the map

$$[U] \longmapsto U^\top U,$$

whose image corresponds to the set of symmetric unitary matrices. We now consider the fixed orthogonal splitting $\mathbb{C}^{2d} = \mathbb{C}^d_f \oplus \mathbb{C}^d_g$. We say that the Lagrangian plane Λ_U is split if it decomposes as $\Lambda_U = \Lambda_f \oplus \Lambda_g$, where $\Lambda_f \subset \mathbb{C}^d_f$ and $\Lambda_g \subset \mathbb{C}^d_g$.

Theorem 3.32. The following statements are equivalent:

1. $(U^\top U)_{12} = 0$
2. Λ_U is split.

Proof. If Λ_U is split, it admits a block-diagonal unitary representative $V = \operatorname{diag}(V_1, V_2)$. Since U and V represent the same left coset in the Lagrangian Grassmannian, there exists an orthogonal matrix $O \in O(2d)$ such that $U = OV$. Consequently, we have

$$U^\top U = V^\top O^\top O V = V^\top V,$$

which is clearly block-diagonal. Conversely, assume that the matrix $U^\top U$ is block-diagonal. Since each diagonal block is a symmetric unitary matrix, we can choose blockwise unitary factors V_j such that $V_j^\top V_j = (U^\top U)_{jj}$. Setting $V = \mathrm{diag}(V_1, V_2)$, we immediately obtain $V^\top V = U^\top U$. Define the matrix $O := UV^{-1}$. By construction, O is unitary. Furthermore, we can verify its orthogonality by computing

$$O^\top O = (V^{-1})^\top U^\top U V^{-1} = (V^\top)^{-1} V^\top V V^{-1} = I.$$

Since O is both unitary and complex orthogonal, it must be a real orthogonal matrix, meaning $O \in O(2d)$. Therefore, $U = OV$, which implies that U and the block-diagonal matrix V define the exact same Lagrangian plane. Since V is block-diagonal, it follows that Λ_U is split. □

For the sesquilinear problem, the split locus defined by $(U^\top U)_{12} = 0$ corresponds precisely to the exceptional locus identified by Gröchenig and Shafkulovska [8, Thm. 5.4]. On this locus, there exist non-zero functions f, g such that the time-frequency representation $W_{\mathcal{A}}(f, g)$ is compactly supported. For the quadratic problem where $g = f$, the exceptional locus is strictly smaller. In this case, the uncertainty principle fails if and only if

$$U^\top U = \mathrm{diag}(\Delta, \overline{\Delta})$$

for some unitary matrix Δ [8, Thm. 6.4]. Since $U^\top U$ is symmetric by construction, it follows that such a Δ must necessarily be a symmetric unitary matrix.

Corollary 3.33 (Dimensions of the exceptional loci)**.** The real dimension of the quotient space $O(2d)\backslash U(2d)$ is given by

$$\frac{(2d)(2d+1)}{2} = 2d^2 + d.$$

The sesquilinear split locus is isomorphic to the product space

$$\Big(O(d)\backslash U(d)\Big) \times \Big(O(d)\backslash U(d)\Big).$$

Consequently, it has dimension $d(d+1)$ and codimension d^2.

On the other hand, the quadratic exceptional locus is naturally isomorphic to $O(d)\backslash U(d)$ via the embedding $\Delta \mapsto \mathrm{diag}(\Delta, \overline{\Delta})$. Therefore, its dimension is $d(d+1)/2$, yielding a codimension of

$$2d^2 + d - \frac{d(d+1)}{2} = \frac{3d^2 + d}{2}.$$

3.6 Conclusion

This chapter develops an output-only quantitative theory for metaplectic time-frequency representations and relates it to the qualitative Gröchenig–Shafkulovska classification.

For the ordinary STFT, Moyal's identity and the elementary L^∞ bound give the universal estimate

$$\|V_g f\|_{L^1} \geq \|V_g f\|_{L^\infty}.$$

This argument, rather than a window-dependent L^1 comparison, is the correct primary estimate for the active theory.

In a free metaplectic normal form, the off-diagonal block P_{12} produces the mixed phase $e^{2\pi i u \cdot P_{12} v}$. If its rank is $k > 0$, singular-value coordinates produce an active matrix Σ and a partial STFT on k variables. Every active fiber satisfies the same bound

$$\|W_p\|_{L^1} \geq |\det \Sigma| \, \|W_p\|_{L^\infty},$$

so, for every $1 \leq \rho \leq \infty$,

$$\|W\|_{L^\rho_p L^1_z} \geq |\det \Sigma| \, \|W\|_{L^\rho_p L^\infty_z}.$$

Thus lower rank does not require a uniformity assumption. It requires only that the proved inequality be expressed in the adapted active-passive coordinates. Maximal rank is the special case in which the passive variables disappear and the estimate becomes an ordinary full-space H_0 bound.

For a general metaplectic representation, a regular free chart and the pre-Iwasawa decomposition give the same normal form. The intrinsic number of active channels is

$$r(\mathcal{A}) = \operatorname{rank}_{\mathbb{C}}(U^\top U)_{12} = \frac{1}{2} \operatorname{rank}_{\mathbb{R}} C_{\widetilde{L}}(U) = \frac{1}{2} \operatorname{rank}_{\mathbb{R}}(A\widetilde{L}B^\top - B\widetilde{L}A^\top).$$

For every $r(\mathcal{A}) > 0$, the chapter obtains an unconditional adapted mixed-norm estimate. A regular-chart constant is the product of the positive singular values of the mixed block and admits the intrinsic lower bound

$$\nu_\theta \geq 2^{-r(\mathcal{A})} \prod_j s_j\Big((U^\top U)_{12}\Big).$$

The quantitative inequality gives effective concentration and active-support bounds, but it does not by itself imply the Benedicks theorem. The stronger finite-support impossibility follows by applying the Benedicks theorem to the nonzero active STFT fibers. Gröchenig and Shafkulovska identify positive active rank with the sesquilinear Benedicks property and rank zero with the product-split counterexample class. In the quadratic problem the exceptional locus is smaller, namely

$$U^\top U = \operatorname{diag}(\Delta, \overline{\Delta}), \qquad \text{with } \Delta \text{ symmetric unitary.}$$

Bibliography

I would like to declare the use of Generative AI tools during the preparation of this thesis. Specifically, AI assistance was employed exclusively for enhancing the fluency of the work, grammatical checking, and verify the mathematical rigor, ensuring the clarity, readability, and accuracy of the manuscript. All outputs have been reviewed, and I assume full responsibility for the final content of this work.

[1] Avi Wigderson and Yuval Wigderson. «The Uncertainty Principle: Variations on a Theme». In: *Bulletin of the American Mathematical Society* 58.2 (Apr. 2021), pp. 225–261. DOI: 10.1090/bull/1715 (cit. on pp. 4, 6, 8–15, 17, 18, 20–22, 24, 25).

[2] Yiyu Tang. «A Simple Observation on Heisenberg-Like Uncertainty Principles». In: *Journal of Mathematical Sciences* 268.6 (Dec. 2022), pp. 810–815. DOI: 10.1007/s10958-022-06221-z (cit. on pp. 4, 25).

[3] Yiyu Tang. «Heisenberg uncertainty principle and its analogues in higher dimension: via Wigdersons' method». In: *arXiv preprint arXiv:2312.15544* (Jan. 2024). URL: https://arxiv.org/abs/2312.15544 (cit. on pp. 4, 25).

[4] Nuno Costa Dias, Franz Luef, and João Nuno Prata. «On Wigdersons' approach to the uncertainty principle». In: *J. Math. Pures Appl.* 198 (2025), p. 103689 (cit. on pp. 4, 20, 24, 26, 28, 29, 31, 33–35, 37, 38).

[5] Elena Cordero and Luigi Rodino. «Wigner analysis of operators. Part I: Pseudodifferential operators and wave fronts». In: *Appl. Comput. Harmon. Anal.* 58 (2022), pp. 85–123 (cit. on pp. 4, 40).

[6] Elena Cordero and Luigi Rodino. «Characterization of modulation spaces by symplectic representations and applications to Schrödinger equations». In: *J. Funct. Anal.* 284.9 (2023), p. 109892 (cit. on pp. 4, 40, 44–47).

[7] Elena Cordero, Gianluca Giacchi, and Luigi Rodino. «Wigner analysis of operators. Part II: Schrödinger equations». In: *Comm. Math. Phys.* 405.7 (2024), p. 156 (cit. on pp. 4, 40).

[8] Karlheinz Gröchenig and Irina Shafkulovska. «Benedicks-type uncertainty principle for metaplectic time-frequency representations». In: *J. Anal. Math.* 158.2 (2026), pp. 573–603 (cit. on pp. 4, 40, 47–52, 63, 74, 75, 81, 83).

[9] Karlheinz Gröchenig and Irina Shafkulovska. «More Uncertainty Principles for Metaplectic Time-Frequency Representations». In: *arXiv preprint arXiv:2503.13324* (Mar. 2025). URL: https://arxiv.org/abs/2503.13324 (cit. on pp. 4, 40, 54–57).

[10] Karlheinz Gröchenig and Irina Shafkulovska. «A characterization of metaplectic time-frequency representations». In: *Bull. Lond. Math. Soc.* 58.6 (2026), e70409 (cit. on p. 40).

[11] Nicolas Lerner. «On the uncertainty principle for metaplectic transformations». In: *J. Funct. Anal.* 289.5 (2025), p. 110997 (cit. on pp. 40, 59).

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