

POLITECNICO DI TORINO

Master's Degree Programme in Mechanical Engineering

Master's Thesis

**Dynamic Analysis of a Rotor with a
Transverse Crack Using the Finite Element
Method**



**Politecnico
di Torino**
International
University

Supervisor

Prof. Teresa Maria Berruti

Candidate

Behzad Hosseinpour

Academic Year 2025/2026

Abstract

This thesis develops a finite-element framework for the lateral dynamics of a rotor containing a transverse breathing crack. The additional flexibility of a fully open crack in a solid circular shaft is derived from linear elastic fracture mechanics using a strain-energy-release, edge-strip representation and Castigliano’s theorem. The two retained direct bending compliances are converted into equivalent directional properties of the crack-containing shaft element. A healthy rotor model is then formulated with distributed shaft inertia, a rigid disk, flexible linear bearings, gravity, gyroscopic coupling, and synchronous unbalance. Rotation of the crack-fixed stiffness axes and modulation by the Mayes–Davies cosine breathing law lead to a linear time-periodic governing equation, solved numerically by a fixed-step fourth-order Runge–Kutta scheme.

The case study is adapted from the rotor configuration in Friswell’s book, Example 10.14.1, and compares the healthy rotor with different crack depths. The fully open model is used as a modal reference, while breathing-crack simulations provide disk time histories, orbits, and $1X$ – $2X$ – $3X$ harmonic amplitudes. Increasing crack depth produces directional modal splitting, downward frequency shifts, vertically elongated orbits, and progressively stronger second- and third-harmonic content. A three-level mesh-sensitivity study shows practical mesh independence of the modal predictions, with a maximum spread of about 0.23% for the cracked model. A control simulation at 2600 rpm further shows that the strong $2X$ response of the deepest crack is generated by periodic stiffness modulation and is additionally amplified at 2000 rpm by proximity to a modal branch. The framework therefore provides a physically interpretable basis for evaluating several crack-sensitive indicators together, while retaining explicit assumptions concerning the prescribed breathing law, linear supports, and fixed crack geometry.

Keywords: rotor dynamics; transverse breathing crack; finite element method; strain energy release rate; linear time-periodic system; Runge–Kutta method; harmonic analysis.

Nomenclature

Symbol	Definition	Unit
A	Shaft cross-sectional area	m^2
a	Physical crack depth	m
β	Prescribed unbalance phase	rad
$\mathbf{C}$	Global damping matrix	DOF-dependent
$C_{ij}^{(c)}$	Crack-induced local flexibility coefficient	variable
E	Young's modulus	Pa
$f_{\text{br}}(t)$	Prescribed crack-breathing function	—
$\mathbf{G}$	Global gyroscopic matrix	DOF-dependent
I	Second moment of area of the healthy shaft	m^4
$I_x^{(\text{eq})}, I_y^{(\text{eq})}$	Equivalent directional second moments of area	m^4
$\mathbf{K}_0$	Healthy global stiffness matrix	DOF-dependent
$\mathbf{K}_c(t)$	Time-periodic crack stiffness-reduction matrix	DOF-dependent
$\mathbf{M}$	Global mass matrix	DOF-dependent
R	Shaft radius	m
T_{rev}	Period of one rotor revolution	s
U	Unbalance parameter	kg m
$\mathbf{q}_{\text{dy}}$	Dynamic generalized-displacement vector about static equilibrium	m/rad
$\mu = a/R$	Nondimensional crack-depth ratio	—
ν	Poisson's ratio	—
Ω	Rotor angular speed	rad/s
ω_n	Natural circular frequency	rad/s

Abbreviation	Meaning
--------------	---------

DOF	Degree of freedom
FEM	Finite element method
FFT	Fast Fourier transform
LEFM	Linear elastic fracture mechanics
LTP	Linear time-periodic
RK4	Classical fourth-order Runge–Kutta method
SERR	Strain energy release rate
SIF	Stress intensity factor

Contents

Abstract	i
Nomenclature	ii
1 Introduction	1
1.1 Background and Motivation	1
1.2 Research Gap and Objectives of the Thesis	1
1.3 Thesis Structure	2
2 Literature Review and Historical Background	3
2.1 Historical Development of Rotor Dynamics	3
2.2 Rotor Faults and the Significance of Shaft Cracks	5
2.3 Fracture Mechanics Applied to Rotating Shafts	8
2.3.1 The Local-Flexibility Formulation and Its Origins	8
2.3.2 Discrete Finite-Element Versus Continuous Distributed- Parameter Representations	10
2.4 Evolution of Breathing-Crack Models	11
2.4.1 Prescribed Breathing Functions	11
2.4.2 Models Restricted to the Instantaneously Open Crack Region .	12
2.4.3 Response-Dependent (Nonlinear) Breathing Models	13
2.4.4 Justification of the Modelling Choice Adopted in This Thesis .	13
2.5 Vibration-Based Diagnostic Indicators	14
2.6 Recent Trends: Data-Driven and Hybrid Diagnosis	15
2.7 Research Gap and Positioning of the Present Thesis	16
3 Mathematical Modelling of a Rotor with a Transverse Breathing Crack	18
3.1 Fracture-Mechanics Formulation of the Transverse Open Crack	18
3.1.1 Crack Geometry and Modelling Scope	18
3.1.2 LEFM Basis and Finite-Strip Correction Functions	19
3.1.3 Circular-Section Strip Geometry	21
3.1.4 Local Flexibility and Direct Bending Compliances	22
3.1.5 Compliance Variation and Transition to the Rotor Model	23
3.1.6 Assumptions and Limits of the Local Crack Model	23
3.2 Finite-Element Model of the Healthy Rotor	23

3.2.1	Rayleigh Beam Kinematics and Degrees of Freedom	23
3.2.2	Shaft-Element Stiffness Matrix	24
3.2.3	Consistent Mass Matrix with Rotary Inertia	25
3.2.4	Disk, Bearings, and External Loads	26
3.2.5	Global Assembly and Healthy Governing Equation	27
3.3	Coupled Model of the Breathing Cracked Rotor	27
3.3.1	Equivalent Directional Stiffness of the Fully Open Crack	27
3.3.2	Rotating Crack Frame and Breathing Law	28
3.3.3	Final Governing Equation	30
3.3.4	Consolidated Assumptions and Limitations	31
4	Numerical Implementation and Solution Procedure	32
4.1	Introduction	32
4.2	Numerical Construction of the Rotor Model	33
4.2.1	Crack-Compliance Evaluation and Cracked-Element Definition	33
4.2.2	Assembly of the Healthy, Fully Open, and Breathing Cracked Rotor Models	36
4.3	Time-Domain Solution Procedure	43
4.3.1	Fourth-Order Runge–Kutta Integration Scheme	43
4.3.2	Simulation Settings, Initial Conditions, and Analysis Duration	46
4.4	Response Extraction and Post-Processing	48
4.4.1	Time Histories and Disk Orbits	48
4.4.2	FFT Analysis and Harmonic Components	50
4.4.3	Periodicity and Steady-State Response Selection	52
4.5	Chapter Summary	54
5	Numerical Case Studies and Results	56
5.1	Case-Study Configuration	56
5.1.1	Rotor Parameters and Operating Conditions	56
5.1.2	Selected Crack Depths: $\mu = 0.25$, $\mu = 0.50$, and $\mu = 0.98$	61
5.2	Modal Characteristics of the Healthy and Fully Open Cracked Rotor	62
5.2.1	Natural Frequencies and Mode Shapes	62
5.2.2	Mesh-Sensitivity Assessment	65
5.2.3	Campbell Diagram and the Effect of Crack Depth	65
5.3	Time-Domain Response of the Breathing Cracked Rotor	68
5.3.1	Time Histories, Static-Sag Dominance, and Disk Orbits	68
5.3.2	Comparison with the Healthy-Rotor Response	71
5.4	Frequency-Domain Response and Effect of Crack Depth	71
5.4.1	FFT Spectra and $1X$ – $2X$ – $3X$ Components	71

5.4.2	Comparative Interpretation of the Three Crack-Depth Cases . .	73
5.4.3	Speed-Control Assessment of the Deep-Crack Response	75
5.5	Chapter Summary	77
6	Conclusions and Future Work	79
6.1	Summary of the Work	79
6.2	Principal Conclusions	79
6.3	Limitations	81
6.4	Recommendations for Future Work	81
6.5	Closing Remark	82
	References	83
	Appendices	87
A	Derivation of the Axial, Shear, Torsional, and Coupling Local Flexi- bility Terms	87
A.1	Scope and Simplifying Nominal-Stress Assumptions	87
A.2	Direct Axial Flexibility $C_{11}^{(c)}$	88
A.3	Direct Shear Flexibilities $C_{22}^{(c)}$ and $C_{33}^{(c)}$	88
A.4	Direct Torsional Flexibility $C_{66}^{(c)}$	89
A.5	Axial–Bending Coupling: $C_{14}^{(c)}$ and $C_{15}^{(c)}$	90
A.6	Shear–Torsion Coupling: $C_{26}^{(c)}$ and $C_{36}^{(c)}$	90
A.7	Summary and the Complete Local Flexibility Matrix	91
B	Complete Derivation of the Direct Bending Flexibility Coefficients and Bending-Coupling Symmetry	92
B.1	Derivation of $C_{44}^{(c)}$	92
B.2	Derivation of $C_{55}^{(c)}$	93
B.3	Symmetry of $C_{45}^{(c)}$	96
C	Explicit Hermite Shape Functions and Full Derivation of the Healthy Shaft-Element Stiffness Matrix	99
C.1	Explicit Cubic Hermite Shape Functions	99
C.2	Integration of the One-Plane Stiffness Matrix	100
C.3	Plane-Wise-to-Nodal Permutation	101
D	FFT-Based Harmonic Analysis of the Rotor Response	103
D.1	Purpose and Input Record	103
D.2	Mean Removal	103
D.3	Discrete Fourier Transform and FFT Algorithm	104

D.4	Spectral Leakage and the Hann Window	104
D.5	One-Sided Spectrum and Coherent-Gain Correction	105
D.6	Integer-Revolution Record and Order Frequencies	106
D.7	Extraction of the Harmonic Amplitudes	106
D.8	Interpretation	106

List of Figures

2.1	William J. M. Rankine—early contributor to the analysis of rotating-shaft whirling.	3
2.2	Aftermath of a catastrophic rotor/turbine failure caused by an undetected shaft or disc crack, illustrating the consequence of failing to identify crack growth before it reaches a critical size.	6
2.3	Radial cracks at the shaft–coupling interface attributed to torsional-vibration-induced fatigue, reproduced from Bilmes et al. [23].	7
2.4	Finite-element prediction of crack opening in a steam-turbine rotor under gravity-induced bending, after Nikraves and MeidanSharaf [24].	7
2.5	Alan A. Griffith and George R. Irwin—foundational contributors to fracture mechanics.	8
2.6	Representative variation of crack-induced local compliance with relative crack depth, as originally reported in the classical strain-energy-release literature.	9
2.7	Comparison of discrete finite-element and continuous distributed-parameter representations of a cracked shaft section.	10
2.8	Researchers associated with crack-compliance and cracked-rotor modelling.	11
2.9	Comparison of prescribed breathing functions used in the cracked-rotor literature.	12
3.1	Rotor-bearing configuration and crack geometry, with $\mu = a/R$	18
3.2	Schematic illustration of Griffith’s energy criterion, showing the reduction in potential energy as the crack size increases.	19
3.3	Generalized sectional loads at the cracked shaft section.	20
3.4	Strip representation and integration domain of the cracked circular section.	21
3.5	Two-node Rayleigh shaft element and nodal degrees of freedom.	24
3.6	Transformation between the rotating crack-fixed and stationary frames.	29
3.7	Mayes–Davies breathing function; the white region in the sectional sketches denotes the open part of the crack.	29
4.1	Dimensionless direct bending-compliance functions $\hat{c}_{44}(\mu)$ and $\hat{c}_{55}(\mu)$ generated from the numerical crack-compliance table over $0 \leq \mu \leq 0.98$.	34
4.2	Identification of the crack-containing finite element within the shaft mesh.	36
4.3	Numerical workflow used to construct the healthy and cracked rotor models.	40

4.4	Numerical rotor configurations considered in the present study: healthy, fully open cracked, and breathing cracked states.	43
4.5	Fixed-step fourth-order Runge–Kutta solution procedure for the breathing cracked rotor.	45
4.6	Extraction of the final response windows used for time histories, disk orbits, and FFT analysis.	50
4.7	Representative time-history and disk-orbit post-processing layout for a selected rotor condition.	50
4.8	Frequency-domain post-processing of the disk response.	51
4.9	Representative one-sided FFT spectra of the disk response.	52
4.10	Periodicity assessment based on the difference between the final disk displacement and the disk displacement one rotor revolution earlier. . .	53
5.1	Case-study rotor and 24-element finite-element discretisation, showing the end bearings, midspan disk, and crack location within element 12. .	60
5.2	Selected crack geometries for $\mu = 0.25, 0.50$, and 0.98	62
5.3	Normalised healthy and fully open cracked mode shapes; the crack-containing span is indicated.	64
5.4	Campbell diagrams of the healthy rotor and the fully open, time-invariant cracked configurations.	67
5.5	First eight rotating modal frequencies at 2000 rpm as functions of crack depth.	67
5.6	Dynamic disk time histories over the final selected response window at 2000 rpm.	70
5.7	Dynamic disk orbits of the healthy rotor and the three breathing-crack cases.	70
5.8	One-sided FFT spectra of the disk response in the x -direction at 2000 rpm.	72
5.9	One-sided FFT spectra of the disk response in the y -direction at 2000 rpm.	73
5.10	Normalised $2X/1X$ and $3X/1X$ ratios as functions of crack depth. . . .	74
5.11	Comparative summary of the principal modal, orbit, and harmonic indicators.	75
D.1	Conceptual effect of Hann windowing on the endpoints of a finite record and on spectral leakage.	105

List of Tables

3.1	Representative dimensionless and dimensional direct bending compliances for $R = 25$ mm, $E = 200$ GPa, and $\nu = 0.30$	23
3.2	Principal assumptions of the coupled rotor–crack model.	31
4.1	Numerical settings used to generate the crack-compliance table.	34
4.2	Rotor model configurations used in the numerical analyses.	42
4.3	Time-domain simulation settings for the selected operating condition.	48
5.1	Shaft properties and finite-element discretisation used in the numerical case study.	57
5.2	Locations and properties of the concentrated rotor components and supports.	58
5.3	Time-domain operating conditions and post-processing settings.	59
5.4	Selected crack-depth cases, dimensionless compliances, and equivalent bending-property ratios.	62
5.5	First eight zero-speed natural frequencies of the healthy and fully open cracked rotor.	63
5.6	Zero-speed natural frequencies for the three mesh levels.	65
5.7	First eight rotating natural frequencies at 2000 rpm.	66
5.8	Time-response statistics at the disk node and the corresponding weight-dominance ratio.	69
5.9	Orbit-range changes relative to the healthy rotor.	71
5.10	Harmonic amplitudes extracted from the one-sided FFT spectra of the disk dynamic response.	72
5.11	Secondary harmonic amplitudes normalised by the synchronous $1X$ component.	73
5.12	Increase in the synchronous $1X$ amplitude relative to the healthy rotor.	74
5.13	Fully open modal reference for the $\mu = 0.98$ control condition at 2600 rpm.	76
5.14	Harmonic amplitudes of the $\mu = 0.98$ breathing-crack control response at 2600 rpm.	76

Chapter 1

Introduction

1.1 Background and Motivation

Rotating machinery lies at the core of most energy-conversion and process infrastructure, and the reliability of the shaft is inseparable from the reliability of the plant that depends on it. Among the faults a rotor can develop—unbalance, misalignment, bearing degradation, rub, and bow—the transverse fatigue crack is the most troubling: it begins as a microscopic defect that can propagate for most of the component’s life [39] while producing only a small, easily masked change in vibration, yet can lead to rapid failure once a critical depth is reached. The 1969 brittle fracture of a low-pressure turbine disc at the Hinkley Point ‘A’ nuclear power station [16], initiated by a stress-corrosion crack only about 1.6 mm deep, remains a widely cited illustration of this asymmetry between slow, hidden growth and fast, catastrophic failure, and comparable cases continue to occur in modern plant [23].

Because a running shaft cannot be inspected directly, the practical safeguard is indirect: continuous monitoring of the lateral vibration response and interpretation of the changes a developing crack introduces into it. Doing so reliably requires a physically based model of how a transverse crack affects rotor dynamics—in particular, the fact that the crack *breathes*, opening and closing once per revolution as it sweeps through the tension and compression sides of the rotating bending field, so that the local stiffness reduction is both direction-dependent and time-periodic rather than constant. Providing and applying such a model, in a single, self-contained, and fully worked form, is the purpose of this thesis; the literature underlying it is reviewed in Chapter 2.

1.2 Research Gap and Objectives of the Thesis

Modelling a breathing cracked rotor involves a chain of steps—deriving the local crack flexibility from fracture mechanics, embedding it in a finite-element rotor model, representing the breathing motion, and solving the resulting time-periodic system numerically to obtain diagnostic indicators—whereas published studies typically address only one link of this chain at a time. Comparatively few present the complete chain, from a first-principles derivation through to a numerical case study evaluating several

diagnostic indicators together, within one reproducible formulation. This thesis is positioned to do so, with the following objectives:

1. Derive the local bending flexibility of a fully open transverse crack in a solid circular shaft from linear elastic fracture mechanics, using a strain-energy-release, strip-based formulation and Castigliano’s theorem.
2. Develop a Rayleigh-beam finite-element model of the healthy rotor-bearing system, including the shaft, a rigid disk with gyroscopic coupling, flexible bearings, gravity, and rotating unbalance.
3. Couple the crack flexibility to this model through an equivalent stiffness reduction and the Mayes–Davies cosine breathing law, justified quantitatively via the weight-dominance ratio of the specific rotor studied.
4. Implement the resulting linear time-periodic model numerically, using a crack-compliance lookup table and a fixed-step fourth-order Runge–Kutta scheme.
5. Apply the model to a representative rotor, comparing the healthy case against three breathing-crack depths and evaluating the modal, orbit, and harmonic indicators together.

Correspondingly, the thesis is scoped to a single, fixed-location transverse crack in a solid circular shaft, retaining only its direct bending flexibility, a prescribed (rather than response-dependent) breathing law, and constant-speed operation with linear bearings; the complete set of assumptions is collected in Section 3.3.4, and several of the alternatives surveyed in Chapter 2—a response-dependent breathing law, a continuous formulation, and multiple cracks—are noted there as directions for future work rather than as gaps in the present study.

1.3 Thesis Structure

Chapter 2 reviews the literature on rotor dynamics, shaft cracks, the fracture-mechanics basis of local flexibility, breathing-crack models, vibration-based indicators, recent data-driven methods, and the resulting research gap. Chapter 3 develops the mathematical model, from local crack flexibility and the healthy rotor finite-element formulation to the final linear time-periodic governing equation. Chapter 4 presents the numerical implementation, including compliance interpolation, matrix assembly, Runge–Kutta integration, and post-processing. Chapter 5 applies the model to a case study comparing the healthy rotor with three crack depths. Chapter 6 summarises the main findings and proposes future work. Appendices A–D contain supporting derivations and numerical post-processing details omitted from the main text.

Chapter 2

Literature Review and Historical Background

2.1 Historical Development of Rotor Dynamics

The dynamics of rotating shafts has been studied for more than a century and a half, and the historical trajectory of the field is directly relevant to the modelling choices made in this thesis: the Rayleigh beam finite-element formulation adopted in Chapter 3 is itself the product of a long sequence of refinements, each addressing a limitation identified in the work that preceded it.

The earliest documented analysis is usually attributed to Rankine [1], who examined the equilibrium of a rotating shaft carrying a single mass and concluded that stable operation was possible only below the first whirling speed. Because Rankine’s analysis neglected damping and the Coriolis effect, this conclusion was incomplete; it nevertheless shaped engineering practice for several decades, during which turbine shafts were deliberately kept below their first critical speed. Dunkerley [2] later proposed an empirical rule for estimating the whirling speeds of shafts carrying several discs, still used today as a quick first estimate in undamped rotor design. The practical need for the analysis was reinforced by de Laval’s steam turbine, whose designers found—contrary to Rankine’s prediction—that stable operation above the first critical speed was achievable, prompting Föppl [3] to re-examine the problem and show that supercritical operation is indeed stable once the shaft is allowed to pass through resonance with sufficient acceleration.

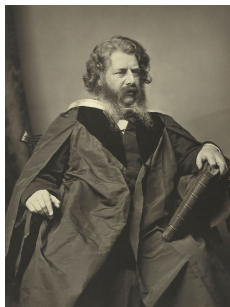


Figure 2.1: William J. M. Rankine—early contributor to the analysis of rotating-shaft whirling.

The definitive early treatment is due to Jeffcott [4], who modelled a single unbalanced disc mounted at the mid-span of a massless elastic shaft supported by rigid bearings. The resulting two-degree-of-freedom model—still referred to as the *Jeffcott rotor* or *de Laval rotor* in the current literature—correctly explained the phase relationship between the unbalance force and the whirl response, showed that the critical speed corresponds to a resonance rather than a stability limit, and demonstrated that stable supercritical operation is possible when the system is lightly damped. The Jeffcott rotor remains the standard reduced-order model for illustrating the fundamental physics of unbalance response, and, as reviewed in Section 2.4, it is also the model most frequently used in the cracked-rotor literature because it isolates the effect of the crack from the complexity of a full multi-degree-of-freedom system.

Realistic rotor-bearing systems, however, involve distributed mass and stiffness, multiple discs, and flexible anisotropic supports, none of which is captured by the two-degree-of-freedom Jeffcott model. The transition toward matrix-based, multi-degree-of-freedom analysis accelerated with the availability of digital computation in the 1960s and 1970s, leading to the finite-element formulations of rotor-bearing systems that underlie the majority of present-day rotor-dynamic software, including the model developed in this thesis. Nelson and McVaugh [5] formulated a consistent finite-element rotor model that included translational and rotary inertia, gyroscopic coupling, and axial load effects within a unified matrix framework; Nelson [6] subsequently extended the formulation to Timoshenko beam kinematics, incorporating transverse shear deformation for the analysis of short, thick shaft sections. These two formulations—and, in particular, the Rayleigh beam element of Nelson and McVaugh, which retains rotary inertia while neglecting shear deformation—form the direct methodological basis of the healthy-rotor finite-element model developed in Section 3.2. Textbooks such as that of Lalanne and Ferraris [7] subsequently consolidated these developments into a standard reference framework covering gyroscopic effects, Campbell diagrams, critical speeds, and unbalance response for general multi-disc, multi-bearing rotor-bearing systems, and are still widely used as the starting point for rotor-dynamic finite-element modelling.

It is against this established, well-validated framework for the *healthy* rotor that the crack-diagnosis literature reviewed in the remainder of this chapter has developed: the presence of a transverse crack is treated as a local modification superimposed on an otherwise conventional rotor-dynamic finite-element model, exactly as is done in Chapter 3.

2.2 Rotor Faults and the Significance of Shaft Cracks

Rotating machinery constitutes the core of most energy-conversion and process infrastructure: steam and gas turbines, generators, compressors, and pumps are all built around a rotor supported on bearings. An unplanned shutdown of such a machine carries not only the direct cost of repair, but also the cost of lost production, and in some configurations a safety risk to personnel located near the machine. This combination of high consequence and, for large power-generation units, comparatively low individual probability of failure is what has driven several decades of research into rotor condition monitoring in general, and shaft-crack detection in particular.

A rotor can develop several distinct categories of fault during its service life, and the diagnostic and modelling literature conventionally groups them as follows [15]: mass unbalance, the most common and generally the most benign fault, correctable by adding or removing trim mass; shaft misalignment between coupled machine elements; rotor-to-stator rubbing, usually a secondary consequence of another fault reaching a large enough amplitude; bearing and support faults, including oil-film instabilities in journal bearings; a residual or thermally induced shaft bow; and transverse fatigue cracks. Among these, the transverse crack occupies a distinctive position. Unlike unbalance or misalignment, whose vibration signature is present, roughly proportional to the fault magnitude, and stable from the moment the fault appears, a fatigue crack begins as a microscopic defect that can propagate for a large fraction of the component's remaining life while producing only a small and easily masked change in the overall vibration level [21]. Because the crack front is typically located away from the accessible vibration-probe planes, its presence is not always apparent even to instrumented machines under continuous monitoring [22]. Once the crack reaches a critical depth, however, failure can occur rapidly and with comparatively little additional warning, in contrast to the slow, predictable degradation associated with most other rotor faults.

Documented industrial cases illustrate both the practical significance of the problem and its persistence across machine types and decades. The consequences of a crack reaching a critical size before detection are illustrated starkly by the failure of turbine-generator No. 5 at the Hinkley Point 'A' nuclear power station (England) on 19 September 1969 [16, 17]. During a routine overspeed test at 3200 rev/min, a shrunk-on low-pressure turbine disc fractured spontaneously in a brittle manner; the initiating flaw was a stress-corrosion crack only about 1.6 mm deep, located in the stress-concentrated crown of a keyway and left undetected because the reduced fracture toughness of the disc material (a consequence of temper embrittlement) had not been anticipated. The shaft fractured completely at five locations, and fragments of three discs were thrown clear of the machine, penetrating the turbine casing; seven operators working in the immediate vicinity narrowly escaped injury. The Hinkley Point investi-

gation, together with the closely related generator-rotor failures analysed by De Forest et al. [18] and by Schabtach et al. [19] in the United States during the preceding decade, is frequently credited with motivating the systematic application of fracture mechanics—specifically the Griffith–Irwin framework underlying Section 3.1—to the design and life assessment of rotating machinery components [20].



Figure 2.2: Aftermath of a catastrophic rotor/turbine failure caused by an undetected shaft or disc crack, illustrating the consequence of failing to identify crack growth before it reaches a critical size.

More recent cases show that the problem has not been eliminated by improvements in material quality and inspection practice, only mitigated. Bilmes et al. [23] report a torsional fatigue crack found in the shaft end and coupling of a 307 MW, 3000 rpm gas–diesel generator after approximately 30,000 hours of service, detected only once the control system flagged vibration levels no longer admissible for continued operation; the shaft is shown in Figure 2.3. Nikraves and MeidanSharaf [24] analyse the growth of a circumferential crack in a steam turbine rotor under the combined action of gravity-induced bending and steady torsion, illustrated in Figure 2.4, showing how operating temperature and torque interact to control the crack-growth rate along the shaft. Cases such as these motivate the monograph of Bachschmid, Pennacchi, and Tanzi [15], which remains the most comprehensive single reference on the static and dynamic behaviour, modelling, and diagnosis of cracked rotors, and which is drawn upon repeatedly throughout the remainder of this chapter.

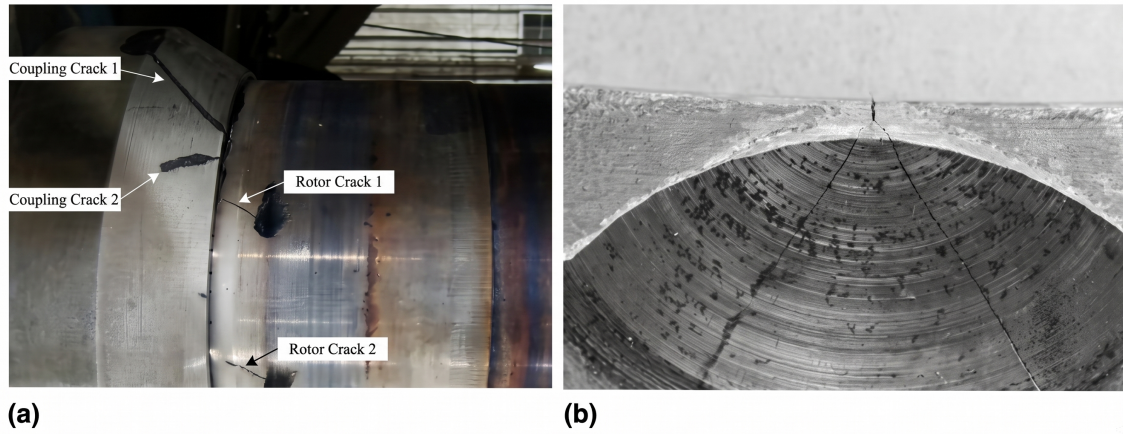


Figure 2.3: Radial cracks at the shaft–coupling interface attributed to torsional-vibration-induced fatigue, reproduced from Bilmes et al. [23].

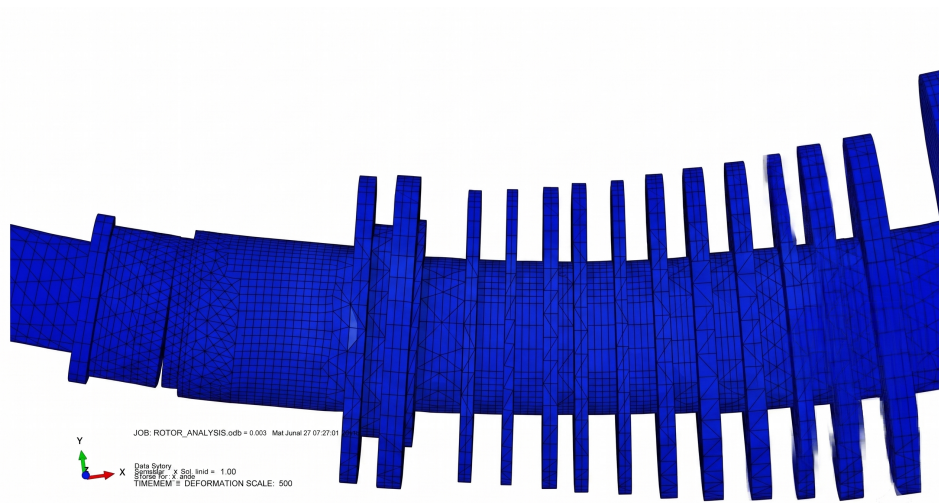


Figure 2.4: Finite-element prediction of crack opening in a steam-turbine rotor under gravity-induced bending, after Nikraves and MeidanSharaf [24].

Together, these cases—spanning half a century, several machine types, and several distinct root causes (stress-corrosion cracking, torsional fatigue, thermomechanical fatigue)—make the same point: once a transverse crack is present in a rotor, the interval between “no detectable problem” and “catastrophic failure” can be short relative to the interval between scheduled inspections, so that early, in-service detection is not merely convenient but often the only realistic safeguard.

Because direct inspection of an operating shaft is generally impossible, and scheduled offline inspections (visual examination, dye penetrant testing, ultrasonic or eddy-current testing) can only be performed during planned outages, the dominant practical approach to early crack detection is indirect: monitoring the lateral vibration response of the rotor during normal operation and identifying the characteristic changes that a breathing

crack introduces into that response. Establishing what those characteristic changes are—and how they depend on crack depth, crack location, and operating speed—requires a physically based mathematical model of the cracked rotor, which is precisely the objective of the modelling literature reviewed in Sections 2.3–2.4 and of the coupled rotor–crack model developed in Chapter 3.

2.3 Fracture Mechanics Applied to Rotating Shafts

2.3.1 The Local-Flexibility Formulation and Its Origins

The general framework of linear elastic fracture mechanics (LEFM)—the energy-balance criterion of Griffith, and the stress-intensity-factor characterisation of the crack-tip field introduced by Irwin—is presented in full in Section 3.1 and is not repeated here. What is reviewed in this section is the separate, later development that connects that general framework specifically to *rotating shafts*, since it is this connection, rather than LEFM itself, that defines the cracked-rotor literature.



Alan Arnold Griffith (1893–1963)



George R. Irwin (1907–1998)

Figure 2.5: Alan A. Griffith and George R. Irwin—foundational contributors to fracture mechanics.

The idea that a crack can be represented, for structural-dynamics purposes, as a local additional flexibility rather than as a geometric discontinuity to be meshed explicitly, predates its application to rotors: Dimarogonas and co-workers used it for stationary cracked beams and columns in the early 1970s. The representation is compact: if a crack is present at a cross-section otherwise subjected to a bending moment M , the crack introduces a discontinuity in the local slope of the elastic line,

$$\theta^+ - \theta^- = c M, \quad (2.1)$$

where c is an additional rotational compliance associated with the crack. Equation (2.1) is the same physical statement, at the level of a single generalized load, that underlies the 6×6 local flexibility matrix $\mathbf{C}^{(c)}$ derived in full in Section 3.1.4 for the general multi-axial loading case. The compliance c itself is obtained from LEFM: the energy

released by the crack under the moment M is

$$G = \frac{M^2}{2b} \frac{dC}{da}, \quad (2.2)$$

with b the cross-section width and a the crack depth, and G is related to the local stress intensity factor $K = \sigma\sqrt{\pi a} F(a/h)$ through $G = K^2/E'$, exactly as developed independently in Section 3.1.2.

The transfer of this idea to *rotating* shafts is due to Dimarogonas and Papadopoulos [8], who derived the local flexibility of a circular shaft with a transverse surface crack subjected to bending in two perpendicular planes, and showed that the resulting time-invariant local stiffness reduction, once resolved into the stationary frame as the shaft rotates, produces a shaft with direction-dependent bending stiffness—the root physical mechanism, still central to Section 3.3, that gives a cracked rotor its characteristic once- and twice-per-revolution response components. Papadopoulos and Dimarogonas subsequently extended the formulation to include the coupling terms omitted in the original bending-only analysis: coupling between bending and torsional vibration [9] and coupling between bending and longitudinal (axial) vibration [10], arriving at the complete 6×6 local flexibility matrix for a transverse surface crack under general sectional loading. Two dedicated review papers—Wauer [11] and Dimarogonas [12]—consolidated this and the parallel work of other groups into the first systematic surveys of cracked-shaft and cracked-structure dynamics, and are still the standard entry point into the older literature on the subject.

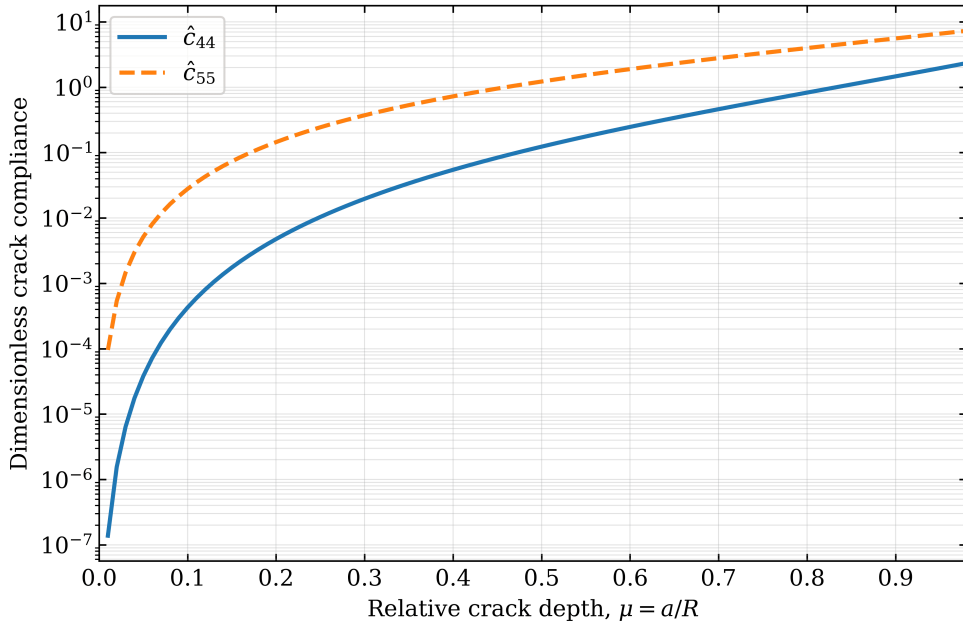


Figure 2.6: Representative variation of crack-induced local compliance with relative crack depth, as originally reported in the classical strain-energy-release literature.

2.3.2 Discrete Finite-Element Versus Continuous Distributed-Parameter Representations

Once the local flexibility of the cracked section is known, two distinct modelling philosophies have been used in the literature to embed it into the equations of motion of the complete rotor-bearing system.

The first, *discrete* route treats the shaft as an assembly of finite beam elements, as already described in Section 2.1, and modifies only the element that contains the crack—either by reducing its stiffness matrix directly using the local flexibility matrix (the approach followed in Chapter 3), or, as originally proposed by Mayes and Davies (reviewed in Section 2.4), by converting the crack flexibility into an equivalent reduction of the element’s second moment of area. This route integrates naturally with the standard finite-element assembly of disks, bearings, and gyroscopic effects established by Nelson and McVaugh [5], at the cost of requiring the crack to be assigned to a single element—typically enforced, as in Section 4.2.1, by rejecting crack positions that coincide with, or lie too close to, an existing node.

The second, *continuous* route instead retains the shaft as a distributed-parameter beam governed by its usual differential equation of motion on each side of the crack, and introduces the crack directly as a compatibility condition linking the two segments at the crack location. Christides and Barr [13] formulated this approach for a general (non-rotating) Bernoulli–Euler beam using a Hu–Washizu–Barr variational principle, obtaining—consistently with Eq. (2.1)—continuity of displacement and shear force across the crack section, but a discontinuity in slope proportional to the local bending moment through the crack compliance. Chasalevris and Papadopoulos [14] extended this continuous formulation to a rotating Timoshenko shaft-disc system with a breathing crack and resilient bearings, solving the coupled horizontal and vertical bending equations directly as a distributed-parameter system rather than through a finite-element discretisation.

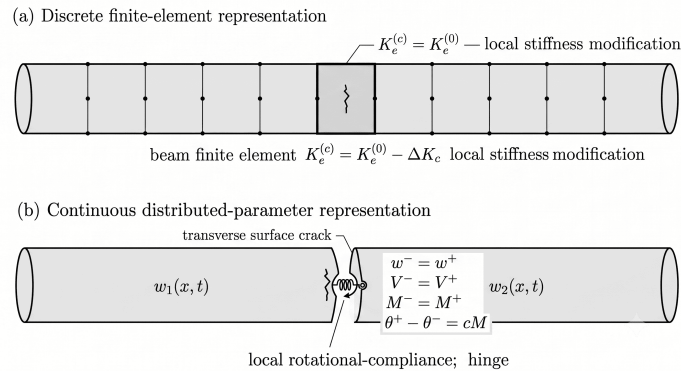


Figure 2.7: Comparison of discrete finite-element and continuous distributed-parameter representations of a cracked shaft section.

Neither route is universally preferred in the literature. The continuous formulation avoids any dependency on mesh density near the crack and represents the local discontinuity exactly rather than through a discretised element; however, it becomes considerably more cumbersome to extend to rotor-bearing systems with several discs, several bearings, and—particularly relevant to multi-crack diagnosis studies—more than one crack, because each additional discontinuity requires a new compatibility condition and a corresponding increase in the size of the analytical boundary-value problem. The discrete finite-element formulation, in contrast, extends to arbitrarily complex rotor-bearing configurations by the same standard assembly procedure regardless of how many concentrated components or cracks are present, which is the primary reason it is adopted, following Nelson and McVaugh’s original framework, for the general-purpose rotor model developed in Chapter 3.

2.4 Evolution of Breathing-Crack Models

The local flexibility formulations reviewed in Section 2.3 describe a *fully open* crack. In service, however, a transverse crack in a rotating shaft opens and closes once per revolution as the crack section sweeps between the tension and compression sides of the rotating bending field—the “breathing” behaviour that gives the cracked-rotor problem its characteristic time-periodic, rather than simply time-invariant, stiffness. How this breathing is represented mathematically is the single modelling choice that most strongly affects the predicted response, and it is the subject of a large and still-active body of literature, which this section organises into four families.



Arthur W.
Lees—cracked-rotor
dynamics



Chris A. Papadopoulos—
crack-compliance modelling



Andrew D. Dimarogonas—
cracked-structure dynamics

Figure 2.8: Researchers associated with crack-compliance and cracked-rotor modelling.

2.4.1 Prescribed Breathing Functions

The earliest and still most widely used family prescribes the crack-open fraction as an explicit function of shaft rotation angle, independent of the instantaneous dynamic response. Gasch [25] modelled the crack as switching abruptly between fully open and

fully closed states depending on the sign of the bending stress at the crack location, giving a discontinuous stepwise stiffness variation that is not representative of the gradual opening observed experimentally. Mayes and Davies [26] instead proposed the smooth cosine breathing function

$$f_{\text{br}}(t) = \frac{1 + \cos(\Omega t)}{2}, \quad (2.3)$$

which varies continuously between 0 (crack closed, full stiffness) and 1 (crack fully open) over one revolution and is the breathing law adopted in Chapter 3. Equation (2.3) is not derived from a first-principles contact analysis; it is a convenient, low-order harmonic approximation, justified by Mayes and Davies through comparison with experimental data and by its analytical tractability—it converts the cracked-rotor problem into a linear system with time-periodic coefficients, admitting the standard finite-element assembly and numerical time-integration procedure used throughout this thesis (Chapters 3 and 4).

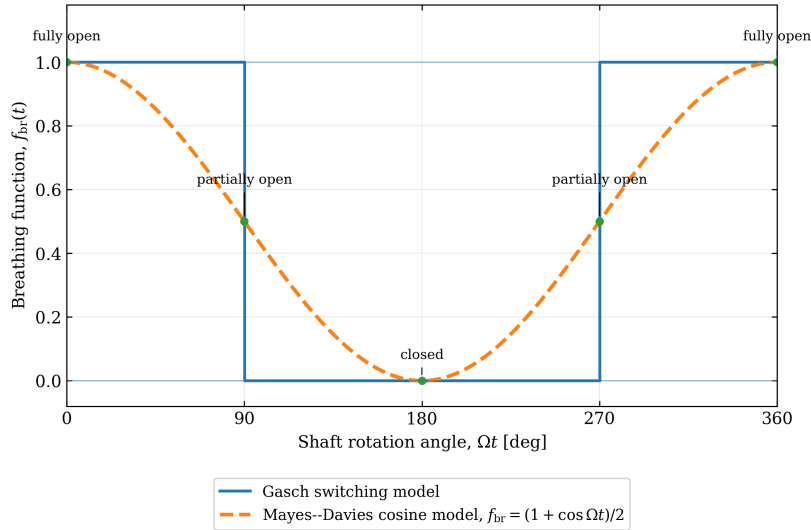


Figure 2.9: Comparison of prescribed breathing functions used in the cracked-rotor literature.

2.4.2 Models Restricted to the Instantaneously Open Crack Region

A second family avoids prescribing a closed-form breathing function altogether and instead re-evaluates the crack compliance at every time step by integrating the strain-energy-release density only over the portion of the crack front that is currently open. Jun et al. [27] formulated the local stiffness this way, determining the open/closed boundary directly from the sign of the local bending stress rather than from an assumed function of Ωt ; this removes the need for a prescribed breathing law but requires the local compliance integral of Section 3.1 to be recomputed, or interpolated from a two-dimensional table, at every integration step rather than once per crack depth as in

the present model.

2.4.3 Response-Dependent (Nonlinear) Breathing Models

Both families above determine the crack state from the rotation angle (or the associated nominal bending stress) alone, implicitly assuming—consistently with the weight-dominance approximation of Section 3.3.2—that the static gravity deflection dominates the stress field at the crack. Patel and Darpe [28] relaxed this assumption by formulating a *response-dependent* flexibility matrix, in which the open/closed boundary of the crack front is determined from the total instantaneous stress state, including the dynamic contribution of the rotor’s own vibration response, so that the crack-compliance integration limits become functions of both time and rotational speed rather than of Ωt alone; the resulting model is nonlinear, since the crack state depends on the very displacement that the equation of motion is being solved for, and is compared by the same authors against a simpler switching-crack alternative. Al-Shudeifat and Butcher [29] pursued a related but geometry-based route, deriving the time-varying area moment of inertia of the cracked cross-section directly from the evolving geometry of the open crack region as the shaft rotates, and proposed two new closed-form breathing functions as more accurate alternatives to Eq. (2.3). Darpe, Gupta, and Chawla [30] additionally showed that, at large static deflections relative to the dynamic response—again, the weight-dominance regime—the closing-crack response can be obtained analytically without a full nonlinear contact solution, providing a bridge between the response-dependent and prescribed-function families.

2.4.4 Justification of the Modelling Choice Adopted in This Thesis

Response-dependent and geometry-based models are, in principle, more physically complete than the prescribed cosine law, since they do not assume in advance that the crack state is controlled by rotation angle alone. Their cost is twofold: the local compliance must be recomputed (or looked up from a multi-dimensional table) at every time step rather than once per crack depth, and the resulting equation of motion is genuinely nonlinear rather than linear time-periodic, precluding the direct linear-time-periodic state-space formulation of Section 3.3.3 and the associated efficiency of the numerical procedure of Chapter 4. Under the weight-dominance approximation established quantitatively for the present rotor configuration in Section 5.3.1—where the static gravity sag was shown to exceed the dynamic response by between two and three orders of magnitude—the additional physical fidelity offered by a response-dependent formulation is not expected to alter the crack-opening trajectory appreciably, while the linear time-periodic formulation retains closed-form accessibility, a computationally efficient solution procedure, and direct comparability with the substantial body of prior work (Section 2.4.1) that adopts the same prescribed-function approach. The

Mayes–Davies cosine law, Eq. (2.3), is accordingly adopted for the coupled rotor–crack model developed in Chapter 3.

2.5 Vibration-Based Diagnostic Indicators

The modelling literature reviewed in Sections 2.3 and 2.4 exists, ultimately, to predict how a breathing crack changes the measurable vibration response of a rotor, so that those changes can be used as diagnostic indicators. This section reviews the indicators most commonly used in the literature and adopted, for the case study of Chapter 5, in this thesis.

Modal indicators. As reviewed in Section 2.3.1, the directional stiffness reduction introduced by a fully open crack splits each repeated bending-mode pair of an otherwise axisymmetric rotor into two distinct frequencies, and shifts the rotating natural-frequency branches of the Campbell diagram downward as crack depth increases. Sinou and Lees [31] give a comprehensive review of how these frequency shifts, together with the associated changes in mode shape, have been used for crack detection and depth estimation; the same indicators are examined for the present rotor configuration in Sections 5.2.1–5.2.3 of this thesis.

Harmonic content of the forced response. For a rotor dominated by synchronous unbalance excitation, the healthy response is concentrated almost entirely at the rotational frequency (the $1X$ component). Because the breathing crack introduces a stiffness variation with period equal to one shaft revolution, it generates additional response components at integer multiples of the rotational frequency—principally $2X$, with a smaller $3X$ contribution—superimposed on the synchronous response; Sinou [32] analyses how these super-harmonic components interact with the unbalance excitation and how their relative magnitude evolves with crack depth, while Gomez et al. [33] show that the $3X$ component specifically is also sensitive to the axial location of the crack along the shaft. The corresponding $1X$ – $2X$ – $3X$ harmonic amplitudes, extracted from the FFT of the disk response, are the primary quantitative indicator used in Section 5.4 to compare the healthy and breathing cracked rotor for the three selected crack depths.

Orbit-based indicators. Because the crack-induced stiffness reduction is direction-dependent, the shaft-centre orbit, nominally circular for an isotropic healthy rotor under synchronous unbalance, becomes progressively distorted—typically elongated along the softer bending direction—as crack depth increases; this distortion is examined quantitatively, through the orbit span ratio $\Delta y_{\text{orbit}}/\Delta x_{\text{orbit}}$, in Section 5.3.

Cross-coupled indicators. Because a crack couples bending, torsional, and longitudinal vibration (Section 2.3.1), several authors have used the appearance of torsional or longitudinal frequency content in the lateral vibration spectrum—absent in the healthy rotor, where these motions are dynamically decoupled—as an additional,

largely independent crack indicator; Muszynska, Goldman, and Bently [34] demonstrated this cross-coupled response experimentally on a laboratory rotor rig. This class of indicator is not pursued further in the present thesis, whose scope is restricted to lateral bending vibration, but is noted here as a natural extension of the coupled local flexibility matrix already derived in Section 3.1.

Interpretation caveat. None of the indicators above is, by itself, uniquely attributable to a transverse crack: $2X$ and $3X$ harmonic content can also arise from misalignment, asymmetric shaft stiffness, or rotor-to-stator rub, and modal frequency shifts can result from a change in support stiffness rather than from a crack. In practice, therefore, crack diagnosis relies on the pattern and mutual consistency of several indicators—modal, harmonic, and orbit-based—evaluated together, a point already noted in the discussion of Section 5.4.2 and consistent with the survey conclusions of Bachschmid, Pennacchi, and Tanzi [15].

2.6 Recent Trends: Data-Driven and Hybrid Diagnosis

The physically based indicators of Section 2.5 are extracted, in the classical literature, by direct inspection of modal parameters or FFT spectra. Since the early 2010s a parallel and increasingly active body of work has instead applied signal-processing and machine-learning techniques directly to the measured (or simulated) vibration signal, largely bypassing an explicit rotor–crack model. Guo et al. [35] applied empirical mode decomposition to the response of a Jeffcott rotor with a breathing crack, extracting intrinsic mode functions sensitive to crack depth without requiring an assumed harmonic decomposition; wavelet-based and time-frequency methods have been used similarly, including for the railway-axle application noted in Section 2.2 [22]. More recently, deep-learning architectures—convolutional neural networks in particular—have been trained to classify rotor faults, including cracks, directly from raw or lightly processed vibration data; Tama et al. [36] review this rapidly growing area across rotating machinery more broadly.

A recurring practical difficulty for these data-driven methods is that labelled fault data for a rare, safety-critical fault such as a shaft crack are scarce in practice, since deliberately propagating a crack to failure on an operating machine is generally not possible. Several studies have addressed this by training on *simulated* vibration responses generated from a physically based cracked-rotor model of exactly the kind reviewed in Sections 2.3 and 2.4, using the simulation to supply labelled training examples across crack depths and operating conditions that would be impractical to reproduce experimentally. Physics-based models such as the one developed in this thesis therefore retain a role even within a data-driven diagnostic pipeline—not only as a source of physically interpretable indicators in their own right, but as a controllable generator of

labelled data for such methods—although pursuing this application is outside the scope of the present work.

Recent work has extended these directions toward more realistic breathing descriptions, experimental diagnosis, and interacting cracks. Han et al. [43] coupled the strain-energy-release and neutral-axis approaches to show that increasing unbalance during acceleration can change a gravity-dominated uniformly breathing crack into a breath-hold condition. Wang et al. [44] constructed an experimental asymmetric cracked-rotor rig and applied deep learning to vibration-based crack classification. Xiong et al. [45] developed a coupled multi-crack rotor model and reported that crack interaction can markedly modify the $2X$ and $3X$ components and orbit morphology, with the theoretical trends compared against experimental observations. Together, these studies illustrate the continuing movement toward operating-condition-dependent breathing models, experimental verification, and hybrid model–data diagnostic strategies.

2.7 Research Gap and Positioning of the Present Thesis

The literature reviewed in this chapter spans four broad modelling families for the breathing crack itself (Sections 2.4.1–2.4.3), two distinct philosophies for embedding a crack into a rotor-bearing system (Section 2.3.2), and, more recently, data-driven methods that bypass an explicit crack model altogether (Section 2.6). Individually, published studies typically emphasise one part of this chain: a fracture-mechanics paper may derive a local flexibility matrix without embedding it in a complete, multi-disc, multi-bearing rotor model and carrying it through to a numerical case study; a breathing-model comparison may adopt a simplified rotor (most often the single-disc Jeffcott rotor) to isolate the effect of the breathing law; and a diagnosis-oriented paper may report harmonic or orbit indicators without deriving the underlying crack compliance from first principles.

The present thesis is deliberately positioned to traverse this entire chain within a single, self-contained, and fully reproducible formulation. Chapter 3 derives the local crack flexibility from first principles, using the strain-energy-release approach of Dimarogonas and Papadopoulos (Section 2.3.1). This local flexibility is then embedded in a general-purpose multi-disc, multi-bearing finite-element rotor model in the tradition of Nelson and McVaugh (Section 2.1). The crack is coupled to the shaft rotation through the classical Mayes–Davies prescribed breathing law (Section 2.4.1). The adoption of this law over the more physically complete response-dependent alternatives (Section 2.4.3) is justified quantitatively—rather than merely assumed—using the weight-dominance ratio computed in Section 5.3.1 for the specific rotor configuration studied. Chapters 4 and 5 then carry this formulation through to a complete numerical implementation and a systematic parametric case study across three crack depths, evaluating the modal,

orbit, and harmonic indicators reviewed in Section 2.5 together rather than in isolation, consistent with the multi-indicator diagnostic practice recommended in the literature.

Several extensions identified in this chapter are explicitly outside the scope of the present work and are noted here as directions for future development rather than as limitations discovered post hoc: a response-dependent or geometry-based breathing law (Section 2.4.3); a continuous distributed-parameter formulation (Section 2.3.2); multiple simultaneous cracks; and the data-driven diagnostic methods of Section 2.6, for which the present model could in principle serve as a labelled-data generator. The mathematical model and numerical case study developed in the remaining chapters of this thesis should accordingly be read as one complete, internally consistent instance drawn from the broader modelling landscape surveyed in this chapter, rather than as a comparison against it.

Chapter 3

Mathematical Modelling of a Rotor with a Transverse Breathing Crack

This chapter develops the rotor model in three steps. First, the flexibility caused by a fully open crack is obtained from fracture mechanics. Next, the healthy rotor is modelled using Rayleigh beam elements, a rigid disk, bearings, gravity, and unbalance. Finally, the crack effect is introduced as a rotating, time-varying stiffness reduction.

3.1 Fracture-Mechanics Formulation of the Transverse Open Crack

3.1.1 Crack Geometry and Modelling Scope

A solid circular shaft of radius R contains a single transverse surface crack of depth a . Its severity is represented by

$$\mu = \frac{a}{R}, \quad (3.1)$$

with $0 < \mu \leq 0.98$ in the numerical model. The crack depth and position are fixed, and the shaft material is assumed homogeneous, isotropic, and linearly elastic. Crack growth, plasticity, friction, and crack-face contact are neglected. Crack breathing is introduced later in the rotor model.

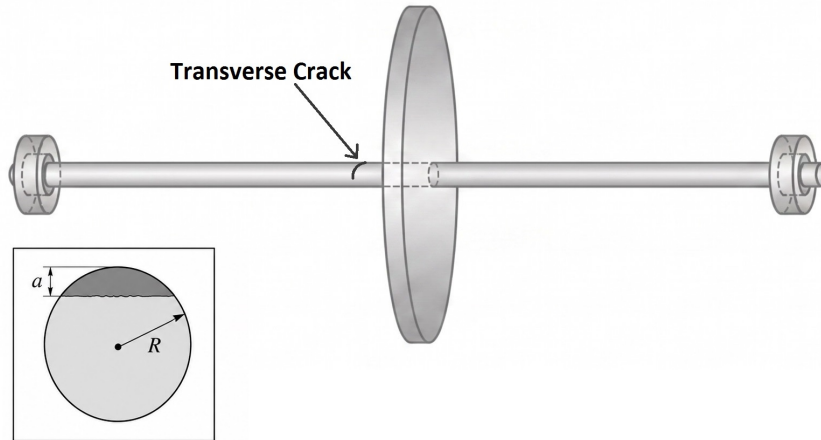


Figure 3.1: Rotor-bearing configuration and crack geometry, with $\mu = a/R$.

The crack is represented as an additional local sectional flexibility rather than as a geometric discontinuity meshed explicitly. The complete sectional formulation permits axial, shear, bending, torsional, and coupling terms; the reduced rotor model retains the two direct bending compliances. The remaining terms are collected in Appendix A.

3.1.2 LEFM Basis and Finite-Strip Correction Functions

For a cracked elastic body subjected to a uniform applied tensile stress σ , the total potential energy P_i decreases as the crack half-length a increases. For the idealised Griffith plate shown in the figure, this reduction is expressed as

$$P_i = P_{i0} - \frac{\pi\sigma^2 a^2}{E}, \quad (3.2)$$

where P_{i0} is the initial potential energy of the uncracked body and E is Young's modulus. The strain-energy-release rate is defined as the decrease in total potential energy per unit increase in crack area:

$$G = - \left(\frac{\partial P_i}{\partial A} \right)_{\sigma}, \quad (3.3)$$

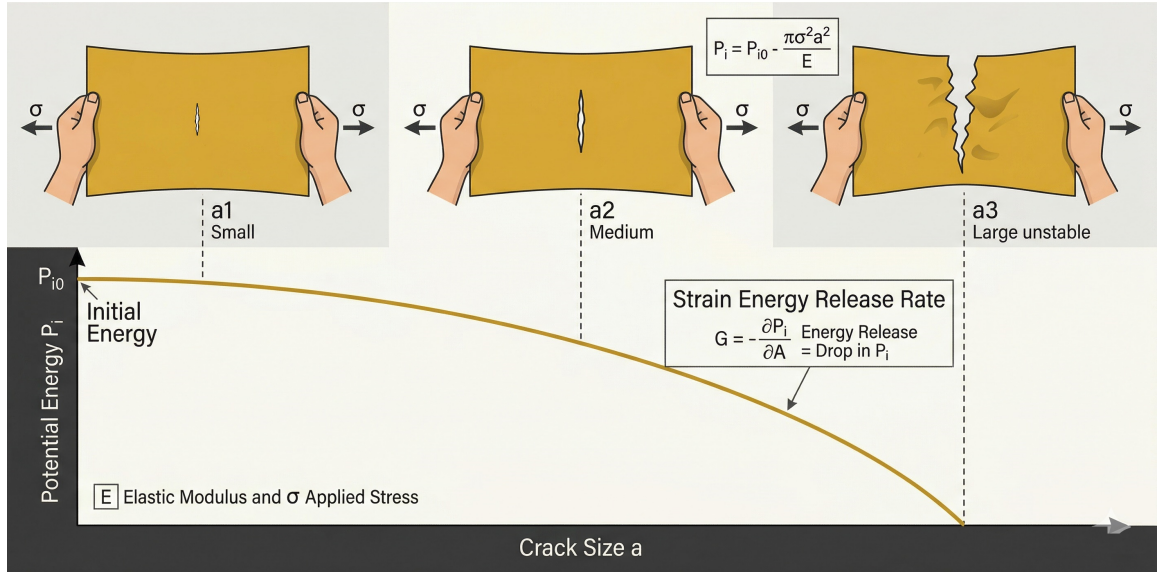


Figure 3.2: Schematic illustration of Griffith's energy criterion, showing the reduction in potential energy as the crack size increases.

where A_c is crack area and $\mathbf{P}$ denotes the applied generalized sectional loads. Under plane strain, the mixed-mode relation is

$$G = \frac{1 - \nu^2}{E} \left[K_I^2 + K_{II}^2 + \frac{1}{1 - \nu} K_{III}^2 \right], \quad (3.4)$$

where E and ν are Young's modulus and Poisson's ratio. For the homogeneous linear-

elastic material considered here, $J = G$; the algebraic LEFM relation is therefore used directly instead of evaluating a contour integral at each strip point [37, 38].

The circular cracked section is treated locally as a family of finite edge-cracked strips. A reference stress-intensity factor is corrected as

$$K = \sigma \sqrt{\pi a} F\left(\frac{a}{h}\right), \quad (3.5)$$

where h is strip height. The two opening-mode correction functions used for the retained bending terms are taken from the finite edge-crack formulations compiled by Tada, Paris, and Irwin and used in the cracked-rotor SERR literature [40, 42]

$$F_1\left(\frac{a}{h}\right) = \sqrt{\frac{2h}{\pi a} \tan\left(\frac{\pi a}{2h}\right)} \frac{0.752 + 2.02(a/h) + 0.37[1 - \sin(\pi a/2h)]^3}{\cos(\pi a/2h)}, \quad (3.6)$$

$$F_2\left(\frac{a}{h}\right) = \sqrt{\frac{2h}{\pi a} \tan\left(\frac{\pi a}{2h}\right)} \frac{0.923 + 0.199[1 - \sin(\pi a/2h)]^4}{\cos(\pi a/2h)}. \quad (3.7)$$

The corresponding F_{II} and F_{III} functions are required only for the flexibility terms listed in Appendix A.

The sectional load vector is

$$\mathbf{P} = [N \ V_x \ V_y \ M_y \ M_x \ T]^T = [P_1 \ P_2 \ P_3 \ P_4 \ P_5 \ P_6]^T. \quad (3.8)$$

Normal stresses generated by N , M_y , and M_x contribute to Mode I; shear and torsion generate Mode II and Mode III terms. Superposition within each fracture mode gives

$$J = \frac{1 - \nu^2}{E} \left[(K_{II1} + K_{II4} + K_{II5})^2 + (K_{III3} + K_{III6})^2 + \frac{(K_{III2} + K_{III6})^2}{1 - \nu} \right]. \quad (3.9)$$

The squared modal sums contain the coupling products that generate off-diagonal flexibility coefficients.

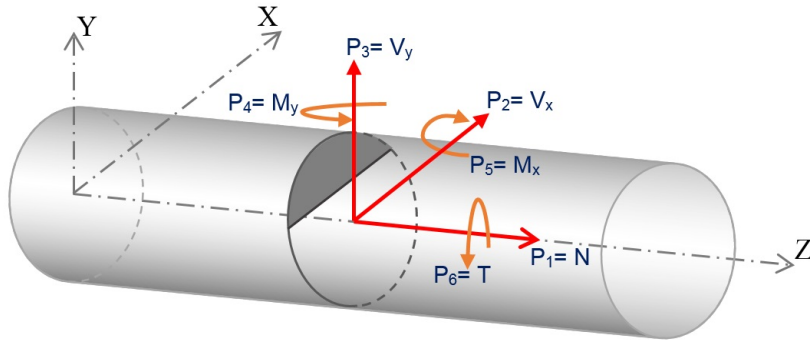


Figure 3.3: Generalized sectional loads at the cracked shaft section.

3.1.3 Circular-Section Strip Geometry

At a transverse coordinate x , the circular boundary gives the strip height

$$h_x = 2\sqrt{R^2 - x^2}. \quad (3.10)$$

For a crack of depth a , the cracked half-width and local penetration depth are

$$b = \sqrt{2Ra - a^2}, \quad a_x(x) = \sqrt{R^2 - x^2} - (R - a). \quad (3.11)$$

Hence the cracked domain is $-b \leq x \leq b$ and $0 \leq y \leq a_x(x)$.

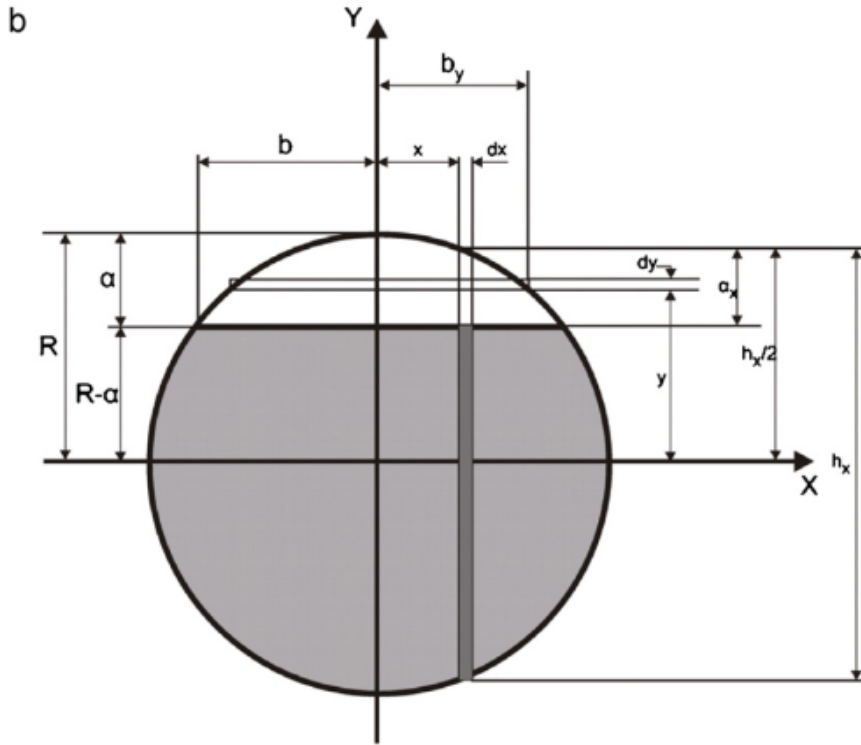


Figure 3.4: Strip representation and integration domain of the cracked circular section.

Introducing

$$\bar{x} = \frac{x}{R}, \quad \bar{y} = \frac{y}{R}, \quad \bar{b} = \sqrt{2\mu - \mu^2}, \quad (3.12)$$

leads to

$$\bar{a}_x(\bar{x}) = \sqrt{1 - \bar{x}^2} - (1 - \mu), \quad \lambda(\bar{x}, \bar{y}) = \frac{\bar{y}}{2\sqrt{1 - \bar{x}^2}}. \quad (3.13)$$

The global depth ratio μ determines the integration domain, whereas the local ratio λ is the argument of the finite-strip correction functions.

3.1.4 Local Flexibility and Direct Bending Compliances

The crack-induced strain energy is

$$U_c = \int_{-b}^b \int_0^{a_x(x)} J(x, y) dy dx. \quad (3.14)$$

Castigliano's theorem gives the additional local generalized displacements and flexibility coefficients,

$$u_i^{(c)} = \frac{\partial U_c}{\partial P_i}, \quad C_{ij}^{(c)} = \frac{\partial^2 U_c}{\partial P_i \partial P_j}. \quad (3.15)$$

The general section-level matrix is

$$\mathbf{C}_c^{(c)} = \begin{bmatrix} c_{11} & 0 & 0 & c_{14} & c_{15} & 0 \\ 0 & c_{22} & 0 & 0 & 0 & c_{26} \\ 0 & 0 & c_{33} & 0 & 0 & c_{36} \\ c_{14} & 0 & 0 & c_{44} & c_{45} & 0 \\ c_{15} & 0 & 0 & c_{45} & c_{55} & 0 \\ 0 & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix}^{(c)}, \quad (3.16)$$

which is symmetric because it derives from the scalar energy U_c .

For the lateral model, only $P_4 = M_y$ and $P_5 = M_x$ are retained. With the healthy circular-section inertia $I = \pi R^4/4$, their nominal opening stresses are proportional to x and $\sqrt{R^2 - x^2}$, respectively. The associated Mode-I terms are integrated over half the symmetric crack domain. The mixed integrand is odd in $\bar{x}$, and therefore

$$C_{45}^{(c)} = C_{54}^{(c)} = 0. \quad (3.17)$$

The final direct bending compliances are [41, 42]

$$C_{44}^{(c)} = \frac{1 - \nu^2}{ER^3} \hat{c}_{44}(\mu), \quad (3.18)$$

$$C_{55}^{(c)} = \frac{1 - \nu^2}{ER^3} \hat{c}_{55}(\mu), \quad (3.19)$$

where

$$\hat{c}_{44}(\mu) = \frac{32}{\pi} \int_0^{\sqrt{2\mu-\mu^2}} \int_0^{\sqrt{1-\bar{x}^2}-(1-\mu)} \bar{x}^2 \bar{y} F_1^2 \left(\frac{\bar{y}}{2\sqrt{1-\bar{x}^2}} \right) d\bar{y} d\bar{x}, \quad (3.20)$$

$$\hat{c}_{55}(\mu) = \frac{64}{\pi} \int_0^{\sqrt{2\mu-\mu^2}} \int_0^{\sqrt{1-\bar{x}^2}-(1-\mu)} (1 - \bar{x}^2) \bar{y} F_2^2 \left(\frac{\bar{y}}{2\sqrt{1-\bar{x}^2}} \right) d\bar{y} d\bar{x}. \quad (3.21)$$

The coefficient $32/\pi$ in Eq. (3.20) follows from the half-domain representation used after exploiting symmetry; the corresponding coefficient for $\hat{c}_{55}$ is $64/\pi$. Appendix B

gives the retained algebraic details.

3.1.5 Compliance Variation and Transition to the Rotor Model

The dimensionless functions depend only on μ ; material and dimensional effects enter through $(1 - \nu^2)/(ER^3)$. Representative values used to verify the tabulated input are listed in Table 3.1.

Table 3.1: Representative dimensionless and dimensional direct bending compliances for $R = 25$ mm, $E = 200$ GPa, and $\nu = 0.30$.

μ	$\hat{c}_{44}$	$\hat{c}_{55}$	$C_{44}^{(c)}$ [(N m) ⁻¹]	$C_{55}^{(c)}$ [(N m) ⁻¹]
0.10	4.2585×10^{-4}	2.77722×10^{-2}	1.24008×10^{-10}	8.08726×10^{-9}
0.25	1.03483×10^{-2}	2.43694×10^{-1}	3.01342×10^{-9}	7.09637×10^{-8}
0.50	1.23230×10^{-1}	1.22153	3.58846×10^{-8}	3.55711×10^{-7}
0.75	0.619680	3.33587	1.80451×10^{-7}	9.71405×10^{-7}

Both compliances increase nonlinearly with depth and $C_{55}^{(c)} > C_{44}^{(c)}$ for the cases considered. This inequality creates the directional stiffness loss used in the rotor model.

3.1.6 Assumptions and Limits of the Local Crack Model

The local formulation assumes a solid circular section, a single transverse surface crack, LEFM under plane strain, and independent edge-cracked strips. Only the direct bending terms are transferred to the rotor model. The strip approximation becomes more sensitive as the crack front approaches the section centre; consequently, the deepest case is interpreted as a severe near-limit numerical condition rather than a typical service crack. The modelling sequence is

$$\mu \longrightarrow \hat{c}_{44}, \hat{c}_{55} \longrightarrow C_{44}^{(c)}, C_{55}^{(c)} \longrightarrow I_x^{(\text{eq})}, I_y^{(\text{eq})} \longrightarrow \mathbf{K}_e^{(\text{open})}. \quad (3.22)$$

3.2 Finite-Element Model of the Healthy Rotor

3.2.1 Rayleigh Beam Kinematics and Degrees of Freedom

The theoretical shaft model is a Rayleigh beam: Euler–Bernoulli bending kinematics are used for stiffness, transverse shear deformation is neglected, and both translational and rotary inertia are retained [5, 7]. Each node has

$$\mathbf{q}_i = \begin{bmatrix} u_i & v_i & \theta_i & \psi_i \end{bmatrix}^T, \quad (3.23)$$

where u, v are transverse translations and θ, ψ are the associated section rotations. A two-node element therefore has eight degrees of freedom,

$$\mathbf{q}_e = [u_i \ v_i \ \theta_i \ \psi_i \ u_j \ v_j \ \theta_j \ \psi_j]^T. \quad (3.24)$$

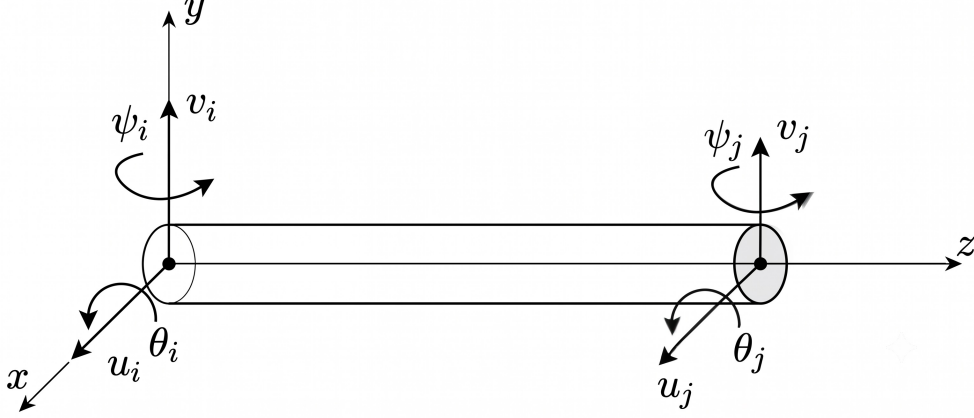


Figure 3.5: Two-node Rayleigh shaft element and nodal degrees of freedom.

For one bending plane, cubic Hermite interpolation gives $w(z, t) = \mathbf{N}(z)\mathbf{q}_b(t)$. The bending strain energy and Rayleigh kinetic energy are

$$U_b = \frac{1}{2} \int_0^{\ell_e} EI \left(\frac{\partial^2 w}{\partial z^2} \right)^2 dz, \quad (3.25)$$

$$T_b = \frac{1}{2} \int_0^{\ell_e} \rho A \dot{w}^2 dz + \frac{1}{2} \int_0^{\ell_e} \rho I \left(\frac{\partial \dot{w}}{\partial z} \right)^2 dz. \quad (3.26)$$

The first term represents translational inertia and the second rotary inertia.

3.2.2 Shaft-Element Stiffness Matrix

For one plane, the Euler–Bernoulli stiffness block is

$$\mathbf{K}_b = \frac{EI}{\ell_e^3} \begin{bmatrix} 12 & 6\ell_e & -12 & 6\ell_e \\ 6\ell_e & 4\ell_e^2 & -6\ell_e & 2\ell_e^2 \\ -12 & -6\ell_e & 12 & -6\ell_e \\ 6\ell_e & 2\ell_e^2 & -6\ell_e & 4\ell_e^2 \end{bmatrix}. \quad (3.27)$$

Using the plane-wise ordering

$$\tilde{\mathbf{q}}_e = [u_i \ \psi_i \ u_j \ \psi_j \ v_i \ \theta_i \ v_j \ \theta_j]^T, \quad (3.28)$$

the healthy circular element is

$$\widetilde{\mathbf{K}}_e^{(0)} = \begin{bmatrix} \mathbf{K}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_b \end{bmatrix}. \quad (3.29)$$

After the fixed permutation to nodal ordering,

$$\mathbf{K}_e^{(0)} = \frac{EI}{\ell_e^3} \begin{bmatrix} 12 & 0 & 0 & 6\ell_e & -12 & 0 & 0 & 6\ell_e \\ 0 & 12 & 6\ell_e & 0 & 0 & -12 & 6\ell_e & 0 \\ 0 & 6\ell_e & 4\ell_e^2 & 0 & 0 & -6\ell_e & 2\ell_e^2 & 0 \\ 6\ell_e & 0 & 0 & 4\ell_e^2 & -6\ell_e & 0 & 0 & 2\ell_e^2 \\ -12 & 0 & 0 & -6\ell_e & 12 & 0 & 0 & -6\ell_e \\ 0 & -12 & -6\ell_e & 0 & 0 & 12 & -6\ell_e & 0 \\ 0 & 6\ell_e & 2\ell_e^2 & 0 & 0 & -6\ell_e & 4\ell_e^2 & 0 \\ 6\ell_e & 0 & 0 & 2\ell_e^2 & -6\ell_e & 0 & 0 & 4\ell_e^2 \end{bmatrix}. \quad (3.30)$$

The Hermite interpolation and permutation details are given in Appendix C.

3.2.3 Consistent Mass Matrix with Rotary Inertia

The element mass matrix is the sum

$$\mathbf{M}_e^{(0)} = \mathbf{M}_e^{(\text{tr})} + \mathbf{M}_e^{(\text{rot})}, \quad (3.31)$$

with

$$\mathbf{M}_e^{(\text{tr})} = \frac{\rho A \ell_e}{840} \begin{bmatrix} 312 & 0 & 0 & 44\ell_e & 108 & 0 & 0 & -26\ell_e \\ 0 & 312 & -44\ell_e & 0 & 0 & 108 & 26\ell_e & 0 \\ 0 & -44\ell_e & 8\ell_e^2 & 0 & 0 & -26\ell_e & -6\ell_e^2 & 0 \\ 44\ell_e & 0 & 0 & 8\ell_e^2 & 26\ell_e & 0 & 0 & -6\ell_e^2 \\ 108 & 0 & 0 & 26\ell_e & 312 & 0 & 0 & -44\ell_e \\ 0 & 108 & -26\ell_e & 0 & 0 & 312 & 44\ell_e & 0 \\ 0 & 26\ell_e & -6\ell_e^2 & 0 & 0 & 44\ell_e & 8\ell_e^2 & 0 \\ -26\ell_e & 0 & 0 & -6\ell_e^2 & -44\ell_e & 0 & 0 & 8\ell_e^2 \end{bmatrix}, \quad (3.32)$$

$$\mathbf{M}_e^{(\text{rot})} = \frac{\rho I}{30\ell_e} \begin{bmatrix} 36 & 0 & 0 & 3\ell_e & -36 & 0 & 0 & 3\ell_e \\ 0 & 36 & -3\ell_e & 0 & 0 & -36 & -3\ell_e & 0 \\ 0 & -3\ell_e & 4\ell_e^2 & 0 & 0 & 3\ell_e & -\ell_e^2 & 0 \\ 3\ell_e & 0 & 0 & 4\ell_e^2 & -3\ell_e & 0 & 0 & -\ell_e^2 \\ -36 & 0 & 0 & -3\ell_e & 36 & 0 & 0 & -3\ell_e \\ 0 & -36 & 3\ell_e & 0 & 0 & 36 & 3\ell_e & 0 \\ 0 & -3\ell_e & -\ell_e^2 & 0 & 0 & 3\ell_e & 4\ell_e^2 & 0 \\ 3\ell_e & 0 & 0 & -\ell_e^2 & -3\ell_e & 0 & 0 & 4\ell_e^2 \end{bmatrix}. \quad (3.33)$$

For a rotating Rayleigh shaft element, the rotary-inertia field also produces the consistent skew-symmetric gyroscopic matrix [5]. In the nodal ordering of Eq. (3.24), it is

$$\mathbf{G}_e = \frac{\rho I}{15\ell_e} \begin{bmatrix} 0 & 36 & -3\ell_e & 0 & 0 & -36 & -3\ell_e & 0 \\ -36 & 0 & 0 & -3\ell_e & 36 & 0 & 0 & -3\ell_e \\ 3\ell_e & 0 & 0 & 4\ell_e^2 & -3\ell_e & 0 & 0 & -\ell_e^2 \\ 0 & 3\ell_e & -4\ell_e^2 & 0 & 0 & -3\ell_e & \ell_e^2 & 0 \\ 0 & -36 & 3\ell_e & 0 & 0 & 36 & 3\ell_e & 0 \\ 36 & 0 & 0 & 3\ell_e & -36 & 0 & 0 & 3\ell_e \\ 3\ell_e & 0 & 0 & -\ell_e^2 & -3\ell_e & 0 & 0 & 4\ell_e^2 \\ 0 & 3\ell_e & \ell_e^2 & 0 & 0 & -3\ell_e & -4\ell_e^2 & 0 \end{bmatrix}, \quad (3.34)$$

which satisfies $\mathbf{G}_e^T = -\mathbf{G}_e$. Its velocity-dependent contribution to the rotating element equation is $\Omega \mathbf{G}_e \dot{\mathbf{q}}_e$; the matrix introduces no dissipation because $\dot{\mathbf{q}}_e^T \mathbf{G}_e \dot{\mathbf{q}}_e = 0$.

3.2.4 Disk, Bearings, and External Loads

At a disk node d , the rigid axisymmetric disk contributes

$$\mathbf{M}_d = \text{diag}(m_d, m_d, J_d, J_d), \quad \mathbf{G}_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{pd} \\ 0 & 0 & -J_{pd} & 0 \end{bmatrix}. \quad (3.35)$$

The gyroscopic contribution is $\Omega \mathbf{G}_d \dot{\mathbf{q}}_d$ and is non-dissipative because $\mathbf{G}_d^T = -\mathbf{G}_d$.

A bearing at node b acts on the translational degrees of freedom through

$$\mathbf{f}_b^{(\text{bear})} = -\mathbf{K}_b \mathbf{r}_b - \mathbf{C}_b \dot{\mathbf{r}}_b, \quad \mathbf{r}_b = \begin{bmatrix} u_b & v_b \end{bmatrix}^T. \quad (3.36)$$

For the case study, the bearings are isotropic and uncoupled:

$$\mathbf{K}_b = K_b \mathbf{I}_2, \quad \mathbf{C}_b = C_b \mathbf{I}_2. \quad (3.37)$$

With the stationary y -axis positive upward, shaft self-weight is

$$q_g = -\rho Ag. \quad (3.38)$$

The consistent gravity vector for one element is

$$\mathbf{f}_{g,e} = \begin{bmatrix} 0 & -\rho Ag\ell_e/2 & -\rho Ag\ell_e^2/12 & 0 & 0 & -\rho Ag\ell_e/2 & \rho Ag\ell_e^2/12 & 0 \end{bmatrix}^T. \quad (3.39)$$

A disk unbalance $U = m_u e$ produces

$$\mathbf{f}_{u,d}(t) = \begin{bmatrix} U\Omega^2 \cos \Omega t & U\Omega^2 \sin \Omega t & 0 & 0 \end{bmatrix}^T. \quad (3.40)$$

3.2.5 Global Assembly and Healthy Governing Equation

Direct assembly gives the global matrices $\mathbf{M}$, $\mathbf{C}$, $\mathbf{G}$, and

$$\mathbf{K}_0 = \mathbf{K}_{\text{shaft}} + \mathbf{K}_{\text{bear}}. \quad (3.41)$$

The absolute-coordinate equation is

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{C} + \Omega\mathbf{G})\dot{\mathbf{q}} + \mathbf{K}_0\mathbf{q} = \mathbf{F}_{\text{grav}} + \mathbf{Q}_{\text{unb}}(t). \quad (3.42)$$

The healthy gravity equilibrium satisfies

$$\mathbf{K}_0\mathbf{q}_{\text{static}}^{(0)} = \mathbf{F}_{\text{grav}}. \quad (3.43)$$

With $\mathbf{q} = \mathbf{q}_{\text{static}}^{(0)} + \mathbf{q}_{\text{dy}}$, the healthy dynamic equation becomes

$$\mathbf{M}\ddot{\mathbf{q}}_{\text{dy}} + (\mathbf{C} + \Omega\mathbf{G})\dot{\mathbf{q}}_{\text{dy}} + \mathbf{K}_0\mathbf{q}_{\text{dy}} = \mathbf{Q}_{\text{unb}}(t). \quad (3.44)$$

3.3 Coupled Model of the Breathing Cracked Rotor

3.3.1 Equivalent Directional Stiffness of the Fully Open Crack

For the crack-containing element of length L_c , the healthy reference bending compliance is $c_0 = L_c/(EI)$. Adding the two crack compliances gives

$$c_x = c_0 + C_{44}^{(c)}, \quad c_y = c_0 + C_{55}^{(c)}. \quad (3.45)$$

Equivalent directional second moments are defined by $c_x = L_c/(EI_x^{(\text{eq})})$ and $c_y = L_c/(EI_y^{(\text{eq})})$, hence

$$I_x^{(\text{eq})} = \frac{L_c I}{L_c + EIC_{44}^{(c)}}, \quad (3.46)$$

$$I_y^{(\text{eq})} = \frac{L_c I}{L_c + EIC_{55}^{(c)}}. \quad (3.47)$$

The corresponding stiffness-retention ratios are

$$\eta_x = \frac{I_x^{(\text{eq})}}{I} = \frac{L_c}{L_c + EIC_{44}^{(c)}}, \quad \eta_y = \frac{I_y^{(\text{eq})}}{I} = \frac{L_c}{L_c + EIC_{55}^{(c)}}. \quad (3.48)$$

These are equivalent properties that reproduce the additional local flexibility; they are not literal geometric moments of the remaining ligament.

In the crack-fixed principal frame, the two Rayleigh-element bending-stiffness blocks are formed with $EI_x^{(\text{eq})}$ and $EI_y^{(\text{eq})}$. After permutation to nodal ordering, the fully open cracked-element stiffness is $\mathbf{K}_e^{(\text{open})}$ and the maximum local stiffness reduction is

$$\Delta \mathbf{K}_e = \mathbf{K}_e^{(0)} - \mathbf{K}_e^{(\text{open})}. \quad (3.49)$$

The element mass is unchanged. If $\mathbf{S}_c$ selects the eight global degrees of freedom of the cracked element, the fully open global reduction is

$$\mathbf{K}_c^{(\text{open})} = \mathbf{S}_c^T \Delta \mathbf{K}_e \mathbf{S}_c. \quad (3.50)$$

3.3.2 Rotating Crack Frame and Breathing Law

At constant speed, the crack-fixed frame rotates through $\phi(t) = \Omega t$. For one node,

$$\mathbf{q}_n^{(c)} = \mathbf{T}_n(\phi) \mathbf{q}_n^{(s)}, \quad (3.51)$$

with

$$\mathbf{T}_n(\phi) = \begin{bmatrix} \cos \phi & \sin \phi & 0 & 0 \\ -\sin \phi & \cos \phi & 0 & 0 \\ 0 & 0 & \cos \phi & \sin \phi \\ 0 & 0 & -\sin \phi & \cos \phi \end{bmatrix}. \quad (3.52)$$

For the two-node element,

$$\mathbf{q}_e^{(c)} = \mathbf{T}_e(\phi) \mathbf{q}_e^{(s)}, \quad \mathbf{T}_e = \text{blkdiag}(\mathbf{T}_n, \mathbf{T}_n). \quad (3.53)$$

Thus the fully open reduction in stationary coordinates is

$$\Delta \mathbf{K}_e^{(s)}(t) = \mathbf{T}_e^T(\Omega t) \Delta \mathbf{K}_e \mathbf{T}_e(\Omega t). \quad (3.54)$$

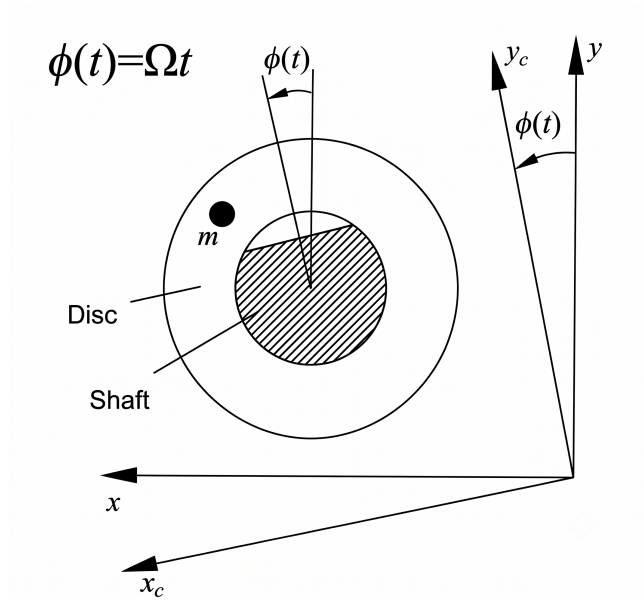


Figure 3.6: Transformation between the rotating crack-fixed and stationary frames.

The opening fraction is represented by the Mayes–Davies cosine law [26],

$$f_{\text{br}}(t) = \frac{1 + \cos(\Omega t)}{2}, \quad (3.55)$$

so $f_{\text{br}} = 1$ denotes the fully open state and $f_{\text{br}} = 0$ the closed state. At $t = 0$, the crack mouth is located on the lower side of the section, where gravity-induced bending produces the maximum tensile opening; consequently $f_{\text{br}}(0) = 1$.

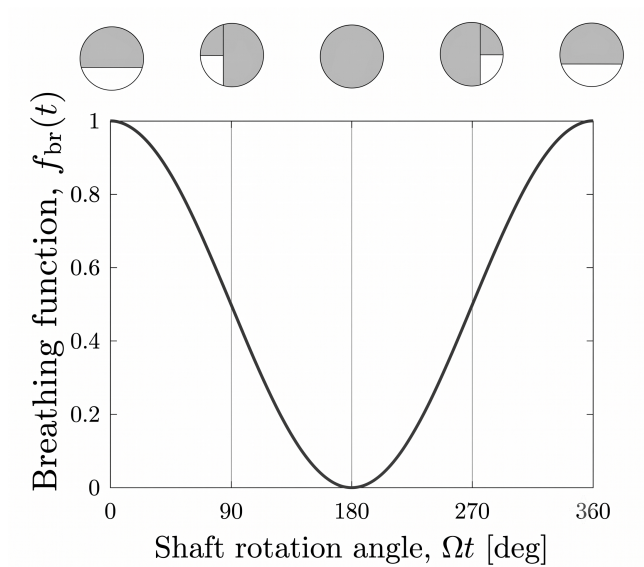


Figure 3.7: Mayes–Davies breathing function; the white region in the sectional sketches denotes the open part of the crack.

The time-dependent global stiffness reduction is

$$\mathbf{K}_c(t) = \mathbf{S}_c^T \left[f_{br}(t) \mathbf{T}_e^T(\Omega t) \Delta \mathbf{K}_e \mathbf{T}_e(\Omega t) \right] \mathbf{S}_c. \quad (3.56)$$

It is symmetric and periodic, but it contains stationary-frame cross-coupling terms because the unequal principal stiffnesses rotate with the shaft.

3.3.3 Final Governing Equation

The breathing-crack stiffness is $\mathbf{K}(t) = \mathbf{K}_0 - \mathbf{K}_c(t)$. In absolute coordinates,

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{C} + \Omega\mathbf{G})\dot{\mathbf{q}} + [\mathbf{K}_0 - \mathbf{K}_c(t)]\mathbf{q} = \mathbf{F}_{grav} + \mathbf{Q}_{unb}(t). \quad (3.57)$$

Using the healthy static equilibrium and the decomposition $\mathbf{q} = \mathbf{q}_{static}^{(0)} + \mathbf{q}_{dy}$ gives

$$\mathbf{M}\ddot{\mathbf{q}}_{dy} + (\mathbf{C} + \Omega\mathbf{G})\dot{\mathbf{q}}_{dy} + [\mathbf{K}_0 - \mathbf{K}_c(t)]\mathbf{q}_{dy} = \mathbf{Q}_{unb}(t) + \mathbf{K}_c(t)\mathbf{q}_{static}^{(0)}. \quad (3.58)$$

The final term is the periodic crack-gravity excitation. Because $\mathbf{K}_c(t)$ is prescribed by shaft angle rather than by the instantaneous response, Eq. (3.58) is a linear time-periodic (LTP) system, not a response-dependent nonlinear contact model.

The prescribed breathing law is appropriate when the gravity-induced bending field dominates the dynamic perturbation at the cracked section. Chapter 5 evaluates this condition for the case study and shows that the static sag exceeds the dynamic displacement by between two and three orders of magnitude over the analysed cases.

3.3.4 Consolidated Assumptions and Limitations

Table 3.2: Principal assumptions of the coupled rotor–crack model.

Model component	Assumption and consequence
Material and kinematics	Homogeneous, isotropic, linear elasticity; small lateral motion.
Shaft theory	Rayleigh beam theory: Euler–Bernoulli stiffness with translational and rotary inertia; transverse shear is omitted in the theoretical derivation.
Crack geometry	One fixed transverse surface crack in a solid circular section; crack growth is not simulated.
Local flexibility	Plane-strain LEFM and independent edge-cracked strips; only $C_{44}^{(c)}$ and $C_{55}^{(c)}$ are retained.
Crack faces	Contact and friction are not resolved; opening is represented by the prescribed cosine law.
Rotor components	Rigid axisymmetric disk and linear bearings; the case study uses isotropic uncoupled supports.
Operation	Constant spin speed and prescribed synchronous unbalance.
Interpretation	The model is an engineering LTP representation for modal, orbit, and harmonic trends, not a fracture-growth or nonlinear contact simulation.

Within these assumptions, the model retains the mechanisms required by the present study: directional local flexibility, rotating stiffness anisotropy, periodic opening and closure, gravity-induced crack excitation, disk gyroscopic coupling, flexible supports, and synchronous unbalance.

Chapter 4

Numerical Implementation and Solution Procedure

4.1 Introduction

This chapter describes the numerical implementation of the rotor model developed in Chapter 3 and the computational procedure used to obtain its modal and time-domain responses, converting the healthy rotor matrices, the fully open crack stiffness reduction, and the breathing-crack time-periodic correction into a practical MATLAB-based simulation framework.

The healthy rotor model is the reference configuration throughout. Shaft geometry, material properties, disk data, bearing coefficients, unbalance parameters, and operating conditions are read from a structured input file and assembled into the global mass, stiffness, gyroscopic, damping, and gravity-load matrices; this model is used directly for the uncracked rotor and also provides the baseline stiffness matrix for the cracked-rotor calculations.

For the cracked configuration, the direct bending compliances are evaluated once over a predefined range of crack depths and stored in a lookup table; a routine simulation then interpolates the compliance for the selected depth rather than repeating the double integration. The crack position is assigned to a unique shaft element, and the associated directional stiffness reduction is calculated from the equivalent second moments of area, permitting the same finite-element model to be analysed as the healthy, fully open cracked, or breathing cracked rotor. The fully open model is used for modal calculations (natural frequencies, mode shapes, speed-dependent behaviour); the breathing model, in which the local stiffness reduction is rotated into the stationary frame and modulated by the Mayes function at every Runge–Kutta stage, is used for the transient-response analysis.

The time-domain response is obtained using a fixed-step fourth-order Runge–Kutta procedure, with the time increment chosen small relative to the shortest natural period of the healthy rotor and adjusted so that each revolution is divided into an integer number of equal intervals. The computed response is post-processed at the disk location to construct time histories, shaft-centre orbits, and frequency spectra, and to assess

periodicity so that the final simulated cycles can be selected as representative late-stage response data for the comparative results of Chapter 5.

4.2 Numerical Construction of the Rotor Model

4.2.1 Crack-Compliance Evaluation and Cracked-Element Definition

The numerical crack model is implemented in two stages: the dimensionless direct bending-compliance functions are evaluated once over a prescribed range of crack depths and stored in a lookup table, and then, for a selected crack position and depth ratio, the program identifies the corresponding shaft element and constructs its fully open crack stiffness-reduction matrix, used in both the fully open modal model and the breathing-crack time-response model.

For the final case-study model, the two bearing stations coincide with the shaft end nodes at $x = 0$ and $x = 1.500$ m, the disk is located at $x_d = 0.750$ m, and the crack is fixed at $x_c = 0.710$ m inside element 12, whose interval is $0.6875 < x < 0.7500$ m. These values are used consistently in Chapters 4 and 5.

The crack is specified by two input quantities: the axial crack position x_c , measured from the left-hand shaft origin, entered in millimetres and converted internally to metres; and the nondimensional crack-depth ratio μ , defined as

$$\mu = \frac{a}{R}, \quad (4.1)$$

where a is the physical crack depth and R is the outer radius of the shaft section containing the crack. Therefore, the physical depth is evaluated only after the crack-containing element and its local radius have been identified:

$$a = \mu R. \quad (4.2)$$

The current implementation accepts crack-depth ratios within the range

$$0 < \mu \leq 0.98. \quad (4.3)$$

The lower bound excludes the trivial uncracked case. The upper bound is selected close to the upper range of the adopted compliance formulation. At $\mu = 0.98$, the crack front lies close to the centre of the circular cross-section, where the computed compliance becomes increasingly sensitive and rises rapidly.

The direct crack compliances $\hat{c}_{44}(\mu)$ and $\hat{c}_{55}(\mu)$ are evaluated in a preprocessing operation through numerical double integration over the cracked region of the circular cross-section, using the stress-intensity correction functions F_1 and F_2 and the integral definitions established in Chapter 3 (Eqs. (3.6), (3.7), (3.20), and (3.21)). The

integrations are carried out using an adaptive two-dimensional numerical integration routine, with absolute and relative tolerances of $\text{AbsTol} = 10^{-10}$ and $\text{RelTol} = 10^{-7}$, respectively.

The crack-compliance table is generated once for the discrete crack-depth sequence $\mu_k = 0, 0.01, 0.02, \dots, 0.98$ (99 values, with $\hat{c}_{44}(0) = \hat{c}_{55}(0) = 0$ assigned explicitly) and stored in the *Crack Compliance Table* worksheet of the numerical input workbook.

Table 4.1: Numerical settings used to generate the crack-compliance table.

Parameter	Symbol or setting	Value
Relative crack-depth range	μ	$0 \leq \mu \leq 0.98$
Increment in crack-depth ratio	$\Delta\mu$	0.01
Number of tabulated crack-depth values	N_μ	99
Absolute integration tolerance	AbsTol	10^{-10}
Relative integration tolerance	RelTol	10^{-7}
Interpolation method	—	Linear interpolation
Compliance at zero depth	$\hat{c}_{44}(0), \hat{c}_{55}(0)$	0

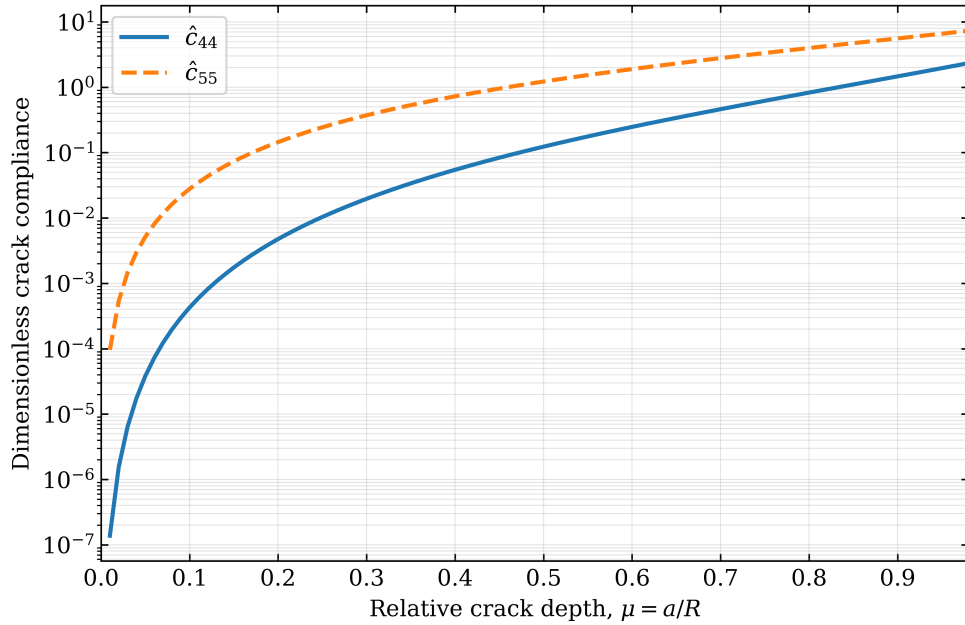


Figure 4.1: Dimensionless direct bending-compliance functions $\hat{c}_{44}(\mu)$ and $\hat{c}_{55}(\mu)$ generated from the numerical crack-compliance table over $0 \leq \mu \leq 0.98$.

During modal and time-domain simulations, the double integrations are not repeated. Instead, the selected crack-depth ratio is used to interpolate the stored compliance

values via linear interpolation, $\hat{c}_{44} = \text{interp1}(\mu_k, \hat{c}_{44,k}, \mu)$ and $\hat{c}_{55} = \text{interp1}(\mu_k, \hat{c}_{55,k}, \mu)$, reducing the computational cost of repeated analyses while preserving a continuous variation of the crack properties with crack depth.

After the crack parameters have been read, the program identifies the unique finite element containing the specified crack position. The crack must be located strictly inside the shaft and strictly between two finite-element nodes:

$$x_{e,\text{left}} < x_c < x_{e,\text{right}}, \quad (4.4)$$

where $x_{e,\text{left}}$ and $x_{e,\text{right}}$ are the axial coordinates of the left and right nodes of the candidate element. A positional tolerance $\varepsilon_x = \max(10^{-9}, 10^{-8}L_{\text{shaft}})$ prevents ambiguous crack assignment: the input crack location is rejected when it falls within ε_x of either shaft end or of any shaft node, ensuring that the crack can be assigned uniquely to one beam element.

The crack-compliance formulation is limited to solid circular shaft sections; the selected crack-containing element is therefore accepted only when its inner diameter is zero within the adopted tolerance. Once the element has been identified, its outer radius $R = D_o/2$ gives the physical crack depth from Eq. (4.2), and the interpolated dimensionless compliances are converted into dimensional local bending compliances using Eqs. (3.18) and (3.19) of Chapter 3.

Following the equivalent-inertia conversion of Chapter 3 (Eqs. (3.46), (3.47), and the stiffness-retention ratios of Eq. (3.48)), the healthy bending compliance $c_0 = L_e/(EI)$ of the crack-containing element is combined with $C_{44}^{(c)}$ and $C_{55}^{(c)}$ to give the equivalent directional second moments of area $I_x^{(\text{eq})}$ and $I_y^{(\text{eq})}$ used in the numerical model. These are, again, equivalent bending properties that reproduce the fracture-mechanics flexibility increase, not literal geometric properties of the remaining cracked section.

The healthy and fully open cracked element stiffness matrices are formed from the one-plane bending-stiffness form of the adopted Rayleigh element of Eq. (3.30), with the healthy second moment of area I replaced by $I_x^{(\text{eq})}$ and $I_y^{(\text{eq})}$ in the local x - and y -bending planes, respectively. The x -plane matrix (degrees of freedom u_i, ψ_i, u_j, ψ_j) is inserted into the element index set $\{1, 4, 5, 8\}$, and the y -plane matrix (degrees of freedom $v_i, \theta_i, v_j, \theta_j$) into $\{2, 3, 6, 7\}$, of the element vector $\mathbf{q}_e = [u_i \ v_i \ \theta_i \ \psi_i \ u_j \ v_j \ \theta_j \ \psi_j]^T$; the differing sign pattern between the two planes reflects only the rotational degree-of-freedom sign convention, not a physical difference in bending behaviour. The local fully open crack stiffness-reduction matrix is then

$$\Delta \mathbf{K}_e = \mathbf{K}_e^{(0)} - \mathbf{K}_e^{(\text{open})}, \quad (4.5)$$

symmetrised numerically as $\Delta \mathbf{K}_e \leftarrow \frac{1}{2}[\Delta \mathbf{K}_e + (\Delta \mathbf{K}_e)^T]$ to remove negligible finite-

precision asymmetry. The mass matrix of the crack-containing element is retained equal to that of the healthy element, consistent with the assumption (Chapter 3) that the mass removed by the crack is negligible compared with the crack-induced stiffness reduction.

The numerical routine stores all of the crack-model quantities defined above (position, depth, element, compliances, equivalent properties, and stiffness matrices), subsequently used to assemble the healthy, fully open, and breathing cracked rotor models.

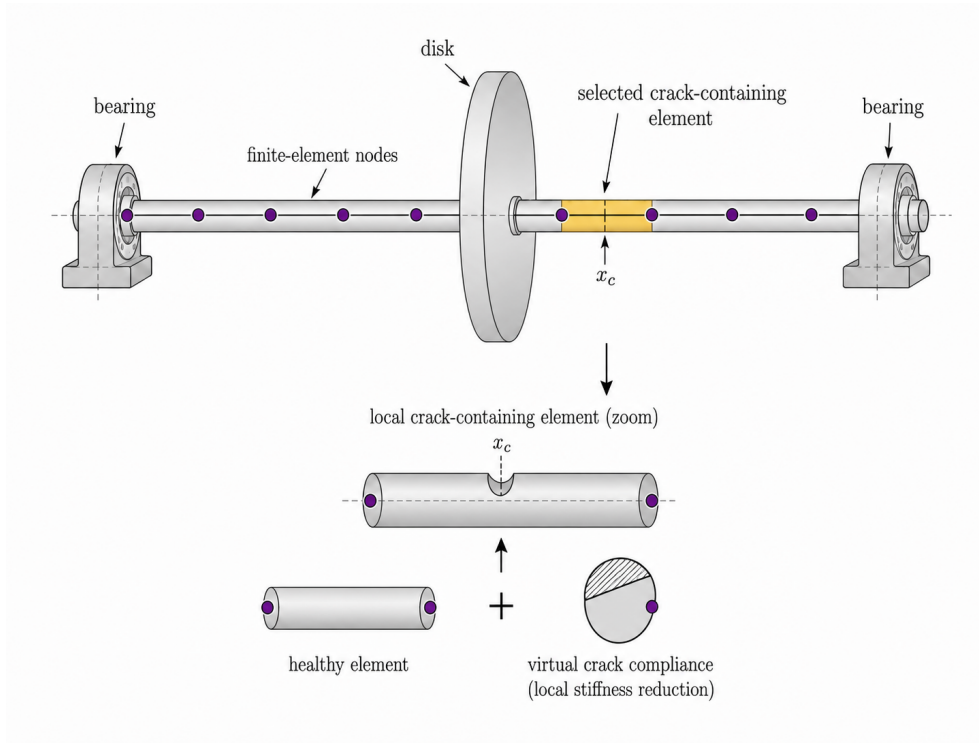


Figure 4.2: Identification of the crack-containing finite element within the shaft mesh.

4.2.2 Assembly of the Healthy, Fully Open, and Breathing Cracked Rotor Models

The numerical implementation is organised around the healthy rotor, the common reference configuration for all subsequent analyses: its global matrices are assembled first and retained throughout, and the fully open and breathing cracked configurations are obtained by applying a local crack-induced stiffness correction to this same reference model, so that the shaft geometry, mass distribution, disk properties, bearing data, and external loading remain identical across all three configurations.

The rotor geometry and material data are read from the *Shaft Geometry* worksheet of the numerical input workbook (axial length, outer and inner diameter, density, Young's modulus, Poisson's ratio, and requested element count per section) and converted internally to SI units.

For each shaft section, the cross-sectional area A and second moment of area I

are evaluated from the standard solid/hollow circular-section formulas in terms of the outer and inner diameters D_o and D_i ; in the present study every shaft section is solid ($D_i = 0$), consistent with the crack-compliance formulation of Chapter 3.

The shaft is divided into geometrical sub-sections at the original shaft-section boundaries and the axial locations of concentrated components (the rigid disk and bearing supports), and each sub-section is further divided into beam elements so that every concentrated component coincides with a finite-element node, allowing masses, inertia, bearing coefficients, gravity loads, and unbalance forces to be assigned directly to their nodal degrees of freedom.

Each shaft node possesses four lateral degrees of freedom,

$$\mathbf{q}_n = \begin{bmatrix} u_n & v_n & \theta_n & \psi_n \end{bmatrix}^T, \quad (4.6)$$

where u_n and v_n are the transverse displacements in the stationary x - and y -directions, respectively, while θ_n and ψ_n are the associated rotational degrees of freedom. Before the imposition of any ideal boundary constraints, the global number of degrees of freedom is

$$N_{\text{dof}} = 4(N_{\text{elem}} + 1), \quad (4.7)$$

where N_{elem} is the total number of beam elements in the shaft model.

For every shaft element, the numerical implementation constructs the stiffness, consistent mass, and gyroscopic matrices according to the Rayleigh beam formulation of Chapter 3, including Eq. (3.34), and then inserts them into the global system using the element degree-of-freedom indices. With $\mathbf{S}_e$ the Boolean assembly matrix of element e , the healthy shaft contributions are

$$\mathbf{K}_{\text{shaft}} = \sum_{e=1}^{N_{\text{elem}}} \mathbf{S}_e^T \mathbf{K}_e^{(0)} \mathbf{S}_e, \quad (4.8)$$

$$\mathbf{M}_{\text{shaft}} = \sum_{e=1}^{N_{\text{elem}}} \mathbf{S}_e^T \mathbf{M}_e \mathbf{S}_e, \quad (4.9)$$

and

$$\mathbf{G}_{\text{shaft}} = \sum_{e=1}^{N_{\text{elem}}} \mathbf{S}_e^T \mathbf{G}_e \mathbf{S}_e. \quad (4.10)$$

In addition to the shaft matrices, the numerical procedure assembles the concentrated mass and inertia of the rigid disk, read from the *Point Mass* worksheet (mass, diametral and polar inertia, axial location, unbalance magnitude and phase) and assigned to the shaft node at the specified axial location.

The concentrated disk mass matrix is assembled into the global system as

$$\mathbf{M}_{\text{disk}} = \mathbf{S}_d^T \mathbf{M}_d \mathbf{S}_d, \quad (4.11)$$

where $\mathbf{S}_d$ extracts the four degrees of freedom of the disk node. Similarly, the disk gyroscopic contribution is introduced through

$$\mathbf{G}_{\text{disk}} = \mathbf{S}_d^T \mathbf{G}_d \mathbf{S}_d. \quad (4.12)$$

The global mass and gyroscopic matrices of the healthy rotor are therefore written as

$$\mathbf{M} = \mathbf{M}_{\text{shaft}} + \mathbf{M}_{\text{disk}}, \quad (4.13)$$

and

$$\mathbf{G} = \mathbf{G}_{\text{shaft}} + \mathbf{G}_{\text{disk}}. \quad (4.14)$$

The translational degrees of freedom of the disk node are stored separately by the numerical routine. These degrees of freedom are used later to extract the disk displacement histories, construct the shaft-centre orbit, and perform the frequency-domain analysis.

The bearing stiffness and damping coefficients are read from the *Bearing* worksheet of the input workbook. The input file accommodates the general support matrices

$$\mathbf{K}_b = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \quad (4.15)$$

and

$$\mathbf{C}_b = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}. \quad (4.16)$$

The stiffness coefficients are entered in N/mm and the damping coefficients in Ns/mm, and are converted internally to N/m and Ns/m respectively (multiplication by 10^3). Although the numerical implementation retains the general matrix form of Eqs. (4.15) and (4.16), support anisotropy and cross-coupling are not considered in the present case studies: the bearings are assumed isotropic and uncoupled, so that

$$\mathbf{K}_b = k_b \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C}_b = c_b \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (4.17)$$

This choice eliminates externally imposed support asymmetry, so that the directional stiffness variation studied in this work originates exclusively from the transverse crack;

the general bearing-input structure remains available in the code for a future extension to unequal or cross-coupled support coefficients.

The global bearing stiffness and damping matrices are assembled as

$$\mathbf{K}_{\text{bear}} = \sum_{b=1}^{N_{\text{bear}}} \mathbf{S}_b^T \mathbf{K}_b \mathbf{S}_b, \quad (4.18)$$

and

$$\mathbf{C} = \mathbf{C}_{\text{bear}} = \sum_{b=1}^{N_{\text{bear}}} \mathbf{S}_b^T \mathbf{C}_b \mathbf{S}_b, \quad (4.19)$$

where $\mathbf{S}_b$ extracts the two translational degrees of freedom associated with bearing b . The global stiffness matrix of the healthy rotor is consequently

$$\mathbf{K}_0 = \mathbf{K}_{\text{shaft}} + \mathbf{K}_{\text{bear}}. \quad (4.20)$$

The healthy rotor is therefore completely defined by the global matrices

$$\mathbf{M}, \quad \mathbf{C}, \quad \mathbf{G}, \quad \mathbf{K}_0. \quad (4.21)$$

The gravity load vector is assembled from the distributed self-weight of the shaft and the concentrated weight of the disk. For an element with density ρ , cross-sectional area A_e , and length L_e , the uniform gravitational load per unit length is

$$q_g = -\rho A_e g, \quad (4.22)$$

where g is the gravitational acceleration. The corresponding consistent nodal gravity-load vector is assembled into the global vertical degrees of freedom, the gravitational force of the disk is added at the disk-node vertical translational degree of freedom, and the resulting total gravity-load vector is denoted $\mathbf{F}_{\text{grav}}$.

For the time-domain analysis, the healthy static gravity-sag configuration is computed once from

$$\mathbf{K}_0 \mathbf{q}_{\text{static}}^{(0)} = \mathbf{F}_{\text{grav}}. \quad (4.23)$$

The resulting displacement vector $\mathbf{q}_{\text{static}}^{(0)}$ is retained as the reference configuration for the breathing-crack response calculation.

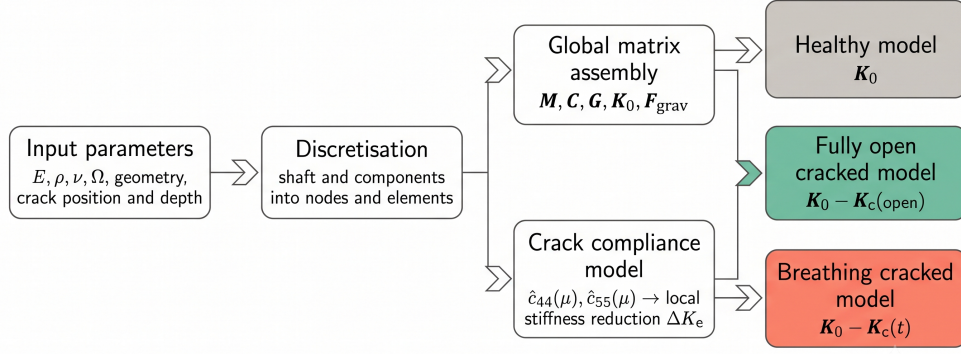


Figure 4.3: Numerical workflow used to construct the healthy and cracked rotor models.

The fully open cracked rotor is constructed from the local stiffness-reduction matrix obtained in Section 4.2.1. Let $\Delta \mathbf{K}_e$ denote the fully open local stiffness-reduction matrix of the crack-containing element, and let $\mathbf{S}_c$ be the Boolean matrix that extracts the associated eight element degrees of freedom. The corresponding global stiffness-reduction matrix is

$$\mathbf{K}_c^{(\text{open})} = \mathbf{S}_c^T \Delta \mathbf{K}_e \mathbf{S}_c. \quad (4.24)$$

All entries of $\mathbf{K}_c^{(\text{open})}$ are zero except for the 8×8 submatrix associated with the crack-containing element. The global stiffness matrix of the fully open cracked rotor is then calculated by direct subtraction from the healthy reference stiffness matrix:

$$\mathbf{K}^{(\text{open})} = \mathbf{K}_0 - \mathbf{K}_c^{(\text{open})}. \quad (4.25)$$

The crack is represented only by this local reduction in bending stiffness. Therefore, the global mass, damping, and gyroscopic matrices remain unchanged:

$$\mathbf{M}^{(\text{open})} = \mathbf{M}, \quad (4.26)$$

$$\mathbf{C}^{(\text{open})} = \mathbf{C}, \quad (4.27)$$

and

$$\mathbf{G}^{(\text{open})} = \mathbf{G}. \quad (4.28)$$

The fully open crack model is used for the modal analyses. In this configuration, the

principal crack directions are assumed to be aligned with the stationary transverse axes, and the stiffness reduction is consequently time independent. At zero speed, the natural frequencies and mode shapes are obtained from the generalized eigenvalue problem

$$\left[\mathbf{K}^{(\text{open})} - \omega_n^2 \mathbf{M} \right] \boldsymbol{\phi}_n = \mathbf{0}, \quad (4.29)$$

where ω_n and $\boldsymbol{\phi}_n$ denote the natural circular frequency and mode-shape vector of mode n , respectively.

For the Campbell-diagram calculation, the same fully open stiffness matrix is retained while the rotational-speed-dependent gyroscopic term is included. Thus, the influence of crack depth on the rotating modal characteristics can be assessed independently of the breathing mechanism.

When ideal simply supported or clamped boundary conditions are selected for a modal analysis, the constrained degrees of freedom are removed from the healthy global matrices. In the same manner, the corresponding rows and columns are removed from $\mathbf{K}_c^{(\text{open})}$ before the subtraction in Eq. (4.25) is carried out. This operation guarantees dimensional consistency between the reduced healthy stiffness matrix and the crack-reduction matrix.

The breathing cracked rotor is not constructed by permanently replacing $\mathbf{K}_0$ with one modified global stiffness matrix. Instead, the fully open local stiffness reduction $\Delta \mathbf{K}_e$ is retained as a reference matrix and is transformed at every integration instant according to the instantaneous shaft angle $\phi(t) = \Omega t$, using the crack-fixed-to-stationary transformation $\mathbf{T}_e(\phi)$ and the Mayes breathing function $f_{\text{br}}(t)$ established in Chapter 3 (Eqs. (3.51), (3.53), and (3.55)). The instantaneous local stiffness-reduction matrix in stationary coordinates is thus

$$\Delta \mathbf{K}_e^{(\text{br})}(t) = f_{\text{br}}(t) \mathbf{T}_e^T(\Omega t) \Delta \mathbf{K}_e \mathbf{T}_e(\Omega t), \quad (4.30)$$

and the equivalent global crack-reduction matrix is $\mathbf{K}_c(t) = \mathbf{S}_c^T \Delta \mathbf{K}_e^{(\text{br})}(t) \mathbf{S}_c$ (Chapter 3, Eq. (3.56)). In the numerical time-response procedure, however, this full global matrix is not assembled and stored at each integration stage. Instead, only the eight degrees of freedom of the crack-containing element are extracted from the global response vector.

The total local displacement at the cracked element is calculated as

$$\mathbf{q}_{e,\text{total}}(t) = \mathbf{q}_{e,\text{dy}}(t) + \mathbf{q}_{e,\text{static}}^{(0)}, \quad (4.31)$$

where $\mathbf{q}_{e,\text{dy}}(t)$ is the dynamic displacement vector of the cracked element and $\mathbf{q}_{e,\text{static}}^{(0)}$ is the corresponding portion of the healthy static gravity-sag vector.

The local crack-correction force is then evaluated as

$$\mathbf{Q}_{e,\text{cor}}(t) = \Delta \mathbf{K}_e^{(\text{br})}(t) \mathbf{q}_{e,\text{total}}(t). \quad (4.32)$$

This eight-component correction vector is inserted into the global force vector at the degrees of freedom of the crack-containing element, while all remaining entries are equal to zero. The global correction force is therefore equivalent to

$$\mathbf{Q}_{\text{cor}}(t) = \mathbf{K}_c(t) [\mathbf{q}_{\text{dy}}(t) + \mathbf{q}_{\text{static}}^{(0)}]. \quad (4.33)$$

The time-domain solver consequently evaluates the dynamic equation in the following form:

$$\mathbf{M} \ddot{\mathbf{q}}_{\text{dy}}(t) + [\mathbf{C} + \Omega \mathbf{G}] \dot{\mathbf{q}}_{\text{dy}}(t) + \mathbf{K}_0 \mathbf{q}_{\text{dy}}(t) = \mathbf{Q}_{\text{unb}}(t) + \mathbf{Q}_{\text{cor}}(t). \quad (4.34)$$

Equation (4.34) is algebraically equivalent to the linear time-periodic breathing-crack equation

$$\mathbf{M} \ddot{\mathbf{q}}_{\text{dy}}(t) + [\mathbf{C} + \Omega \mathbf{G}] \dot{\mathbf{q}}_{\text{dy}}(t) + [\mathbf{K}_0 - \mathbf{K}_c(t)] \mathbf{q}_{\text{dy}}(t) = \mathbf{Q}_{\text{unb}}(t) + \mathbf{K}_c(t) \mathbf{q}_{\text{static}}^{(0)}. \quad (4.35)$$

The three model configurations used in the present work are summarised in Table 4.2.

Table 4.2: Rotor model configurations used in the numerical analyses.

Rotor condition	Stiffness representation	Crack state	Primary application
Healthy rotor	$\mathbf{K}_0$	No crack	Reference modal and time-domain analyses
Fully open cracked rotor	$\mathbf{K}^{(\text{open})} = \mathbf{K}_0 - \mathbf{K}_c^{(\text{open})}$	Fixed fully open crack with principal directions aligned with the stationary axes	Natural frequencies, mode shapes, and Campbell diagram
Breathing cracked rotor	$\mathbf{K}(t) = \mathbf{K}_0 - \mathbf{K}_c(t)$	Rotating crack with periodic opening and closure	Time histories, disk orbits, FFT spectra, and harmonic analysis

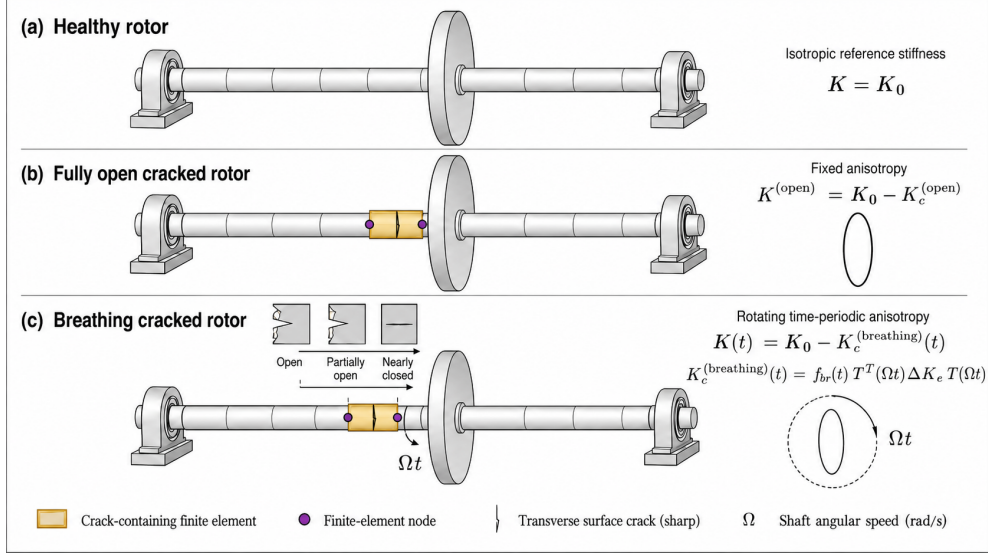


Figure 4.4: Numerical rotor configurations considered in the present study: healthy, fully open cracked, and breathing cracked states.

The use of a healthy reference model together with a local crack-correction procedure provides two important advantages. First, the healthy, fully open cracked, and breathing cracked responses can be compared directly because the global mass distribution, disk properties, bearing coefficients, and support conditions remain unchanged. Second, the breathing-crack calculation remains computationally efficient because the time-dependent matrix operations are restricted to the local degrees of freedom of the crack-containing element.

4.3 Time-Domain Solution Procedure

4.3.1 Fourth-Order Runge–Kutta Integration Scheme

The time-domain response of the rotor is obtained using a fixed-step classical fourth-order Runge–Kutta integration scheme. The procedure is applied to the dynamic displacement vector measured relative to the healthy static gravity-sag configuration. Therefore, the numerical state does not represent the absolute displacement of the rotor directly. Instead, it represents the dynamic motion about the reference equilibrium position determined from the healthy rotor under gravity loading.

For a specified rotational speed Ω , the speed-dependent damping and gyroscopic matrix is defined as

$$\mathbf{D} = \mathbf{C} + \Omega \mathbf{G}. \quad (4.36)$$

The dynamic equation solved by the integration algorithm is

$$\mathbf{M} \ddot{\mathbf{q}}_{\text{dy}}(t) + \mathbf{D} \dot{\mathbf{q}}_{\text{dy}}(t) + \mathbf{K}_0 \mathbf{q}_{\text{dy}}(t) = \mathbf{Q}_{\text{unb}}(t) + \mathbf{Q}_{\text{cor}}(t, \mathbf{q}_{\text{dy}}), \quad (4.37)$$

where $\mathbf{K}_0$ is the global stiffness matrix of the healthy rotor, $\mathbf{q}_{\text{dy}}$ is the dynamic displacement vector, and $\mathbf{Q}_{\text{cor}}$ is the crack-correction force. For the healthy rotor,

$$\mathbf{Q}_{\text{cor}} = \mathbf{0}. \quad (4.38)$$

For the breathing cracked rotor, the correction force is evaluated using

$$\mathbf{Q}_{\text{cor}}(t, \mathbf{q}_{\text{dy}}) = \mathbf{K}_c(t) [\mathbf{q}_{\text{dy}}(t) + \mathbf{q}_{\text{static}}^{(0)}]. \quad (4.39)$$

Thus, the crack stiffness reduction acts on the total displacement of the crack-containing element, including both the dynamic displacement and the healthy static gravity-sag displacement. This is important because the gravity-induced shaft deflection contributes directly to the opening and closing effect represented by the breathing-crack correction.

The second-order equation is transformed into a first-order state-space system by defining the state vector

$$\mathbf{z}(t) = \begin{bmatrix} \mathbf{q}_{\text{dy}}(t) \\ \dot{\mathbf{q}}_{\text{dy}}(t) \end{bmatrix}. \quad (4.40)$$

The state derivative is then

$$\dot{\mathbf{z}}(t) = \begin{bmatrix} \dot{\mathbf{q}}_{\text{dy}}(t) \\ \ddot{\mathbf{q}}_{\text{dy}}(t) \end{bmatrix}, \quad (4.41)$$

where the acceleration is evaluated from

$$\ddot{\mathbf{q}}_{\text{dy}}(t) = \mathbf{M}^{-1} [\mathbf{Q}_{\text{unb}}(t) + \mathbf{Q}_{\text{cor}}(t, \mathbf{q}_{\text{dy}}) - \mathbf{D}\dot{\mathbf{q}}_{\text{dy}}(t) - \mathbf{K}_0\mathbf{q}_{\text{dy}}(t)]. \quad (4.42)$$

For computational efficiency, the inverse mass operation is prepared once before entering the integration loop, and reused at every Runge–Kutta stage.

The rotating unbalance force is applied only at the translational degrees of freedom of the disk node, with magnitude $F_u = U\Omega^2$ (U the disk unbalance in kg m) and global force vector

$$\mathbf{Q}_{\text{unb}}(t) = U\Omega^2 [\cos(\Omega t + \beta)\mathbf{e}_x + \sin(\Omega t + \beta)\mathbf{e}_y], \quad (4.43)$$

where β is the prescribed unbalance phase and $\mathbf{e}_x$, $\mathbf{e}_y$ are global unit vectors with non-zero entries at the horizontal and vertical translational degrees of freedom of the disk.

Denoting the state function by $\dot{\mathbf{z}} = \mathbf{f}(t, \mathbf{z})$, the breathing cracked rotor is time periodic because the crack-correction force depends explicitly on time through the crack rotation

and the prescribed breathing function, so the classical fourth-order Runge–Kutta scheme evaluates the state derivative at four stage points within each interval Δt :

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}(t_n, \mathbf{z}_n), \\
\mathbf{k}_2 &= \mathbf{f}\left(t_n + \frac{\Delta t}{2}, \mathbf{z}_n + \frac{\Delta t}{2}\mathbf{k}_1\right), \\
\mathbf{k}_3 &= \mathbf{f}\left(t_n + \frac{\Delta t}{2}, \mathbf{z}_n + \frac{\Delta t}{2}\mathbf{k}_2\right), \\
\mathbf{k}_4 &= \mathbf{f}(t_n + \Delta t, \mathbf{z}_n + \Delta t\mathbf{k}_3), \\
\mathbf{z}_{n+1} &= \mathbf{z}_n + \frac{\Delta t}{6} [\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4].
\end{aligned} \tag{4.44}$$

For the breathing cracked rotor, the crack transformation matrix, the Mayes breathing function, the crack-correction force, and the rotating unbalance force are re-evaluated independently at all four stages (t_n , $t_n + \Delta t/2$ twice, and $t_n + \Delta t$), so that the periodic variation of the crack stiffness and the rotating excitation are resolved consistently during every integration interval.

After each state update, the numerical routine verifies that every displacement and velocity component remains finite. If a non-finite value is detected, the integration is terminated and the corresponding time step, revolution number, and integration index are reported. This check prevents unstable or invalid response histories from being passed to the orbit and frequency-domain post-processing routines.

The lateral dynamic displacement of the disk is stored after each completed integration step, converted from the internal SI units to millimetres, $x_{dy}(t) = 10^3 q_{dy,x}(t)$ and $y_{dy}(t) = 10^3 q_{dy,y}(t)$; only these dynamic components are stored, with the static gravity-sag displacement $\mathbf{q}_{static}^{(0)}$ kept separately and added only when the total physical displacement is required.

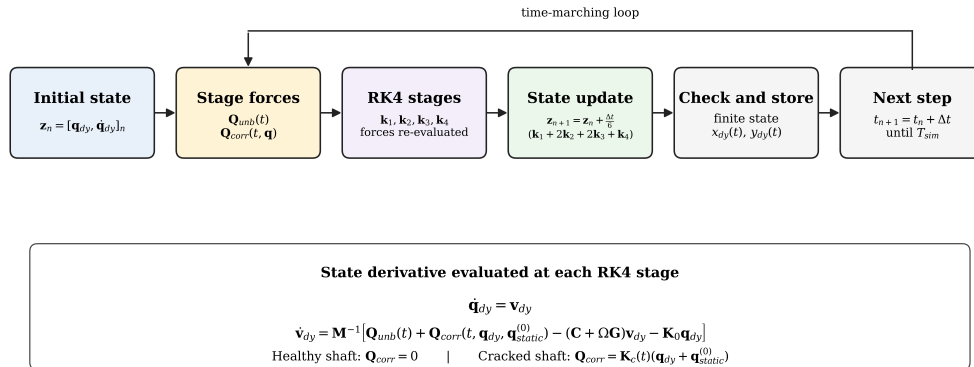


Figure 4.5: Fixed-step fourth-order Runge–Kutta solution procedure for the breathing cracked rotor.

4.3.2 Simulation Settings, Initial Conditions, and Analysis Duration

The time-domain analysis is performed at a prescribed constant rotational speed read from the *Time Response* worksheet of the numerical input workbook. The operating speed is entered in revolutions per minute and is converted internally into angular velocity according to

$$\Omega = \frac{2\pi\Omega_{\text{RPM}}}{60}. \quad (4.45)$$

The corresponding period of one complete rotor revolution is

$$T_{\text{rev}} = \frac{2\pi}{\Omega}. \quad (4.46)$$

The transient simulation is controlled by five user-defined numerical inputs, read from the *Time Response* worksheet as finite positive numbers: the operating speed and the numbers of revolutions retained for the total simulation, the time-history plot, the disk orbit, and the FFT analysis,

$$\Omega_{\text{RPM}}, \quad N_{\text{rev,total}}, \quad N_{\text{rev,time}}, \quad N_{\text{rev,orbit}}, \quad N_{\text{rev,FFT}}. \quad (4.47)$$

The revolution counts are rounded to integer values before the simulation begins, giving a total physical analysis duration $T_{\text{sim}} = N_{\text{rev,total}}T_{\text{rev}}$. For valid extraction of the final response windows, the selected numbers of retained cycles must not exceed the total,

$$N_{\text{rev,time}}, \quad N_{\text{rev,orbit}}, \quad N_{\text{rev,FFT}} \leq N_{\text{rev,total}}. \quad (4.48)$$

The integration time step Δt is fixed for a given simulation and is derived from the dynamic characteristics of the healthy rotor at the selected operating speed. The healthy state-space matrix

$$\mathbf{A}_{\text{state}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K}_0 & -\mathbf{M}^{-1}(\mathbf{C} + \Omega\mathbf{G}) \end{bmatrix} \quad (4.49)$$

gives the eigenvalues $\lambda_j = \text{eig}(\mathbf{A}_{\text{state}})$ and the characteristic eigenvalue scale

$$\lambda_{\text{max}} = \max_j |\lambda_j|, \quad (4.50)$$

from which a conservative target time increment is obtained,

$$\Delta t_{\text{target}} = \frac{2}{\lambda_{\text{max}}}. \quad (4.51)$$

Rather than using this value directly, the number of integration steps per revolution

is set to

$$N_{\text{step/rev}} = \max \left[1, \left\lceil \frac{T_{\text{rev}}}{\Delta t_{\text{target}}} \right\rceil \right], \quad (4.52)$$

so that the final time step,

$$\Delta t = \frac{T_{\text{rev}}}{N_{\text{step/rev}}}, \quad (4.53)$$

divides every revolution into an integer number of equal intervals, ensuring that the rotating unbalance force, crack orientation, and Mayes breathing function are sampled at the same angular positions during every revolution. This choice is checked against the classical fourth-order Runge–Kutta amplification polynomial,

$$P_4(\lambda \Delta t) = 1 + \lambda \Delta t + \frac{(\lambda \Delta t)^2}{2} + \frac{(\lambda \Delta t)^3}{6} + \frac{(\lambda \Delta t)^4}{24}, \quad (4.54)$$

evaluated at $\lambda = \lambda_j$: if $\max_j |P_4(\lambda_j \Delta t)|$ exceeds unity by more than a small tolerance, $N_{\text{step/rev}}$ is increased and Δt recalculated until the numerical scheme is stable at the selected time step.

The total number of Runge–Kutta integration increments is

$$N_{\text{total}} = N_{\text{rev,total}} N_{\text{step/rev}}, \quad (4.55)$$

and, since the response history also contains the initial state at $t = 0$, the number of stored time samples is $N_{\text{sample}} = N_{\text{total}} + 1$.

The dynamic initial conditions are assigned as $\mathbf{q}_{\text{dy}}(0) = \mathbf{0}$ and $\dot{\mathbf{q}}_{\text{dy}}(0) = \mathbf{0}$, so that the initial state vector is $\mathbf{z}(0) = [\mathbf{0} \ \mathbf{0}]^T$. Because the dynamic displacement is defined relative to the healthy static gravity-sag configuration, the corresponding total physical displacement at the beginning of the simulation is

$$\mathbf{q}(0) = \mathbf{q}_{\text{static}}^{(0)}. \quad (4.56)$$

Thus, the rotor initially occupies the healthy static equilibrium position under gravity. Once the time integration begins, the rotating unbalance force is applied and, for the cracked rotor, the time-dependent crack-correction force is also evaluated. The subsequent transient response therefore represents the motion from the healthy gravity reference configuration toward the periodic response associated with the selected operating condition.

Table 4.3: Time-domain simulation settings for the selected operating condition.

Parameter	Symbol	Value
Operating speed	Ω_{RPM}	2000 rpm
Angular speed	Ω	209.440 rad/s
Revolution period	T_{rev}	0.030 s
Total simulated revolutions	$N_{\text{rev,total}}$	80
Revolutions retained for time histories	$N_{\text{rev,time}}$	8
Revolutions retained for disk orbit	$N_{\text{rev,orbit}}$	3
Revolutions retained for FFT analysis	$N_{\text{rev,FFT}}$	12
Automatically selected steps per revolution	$N_{\text{step/rev}}$	4429
Time increment	Δt	6.77354×10^{-6} s
Total integration increments	N_{total}	354320
Total simulation duration	T_{sim}	2.40 s

The computed response histories are stored at every integration step. The full response is retained internally for the subsequent periodicity assessment, while only the final user-specified portions are extracted for the time-history plots, disk orbit, and spectral analysis. This final-cycle selection is based on the assumption that the initial transient response has sufficiently decayed by the end of the simulation. The verification of this assumption and the associated periodicity criterion are described in Section 4.4.3.

4.4 Response Extraction and Post-Processing

4.4.1 Time Histories and Disk Orbits

The time-domain solver stores the dynamic lateral response of the disk, $x_{\text{dy}}(t_n)$ and $y_{\text{dy}}(t_n)$ in millimetres, after every completed Runge–Kutta integration step (Section 4.3.1), giving the complete stored history $\{t_n, x_{\text{dy}}(t_n), y_{\text{dy}}(t_n)\}$, $n = 0, 1, \dots, N_{\text{total}}$. These histories contain only the dynamic displacement relative to the healthy static gravity-sag configuration $\mathbf{q}_{\text{static}}^{(0)}$, which is retained separately; the physical total disk displacement, when required for interpretation, is

$$x_{\text{total}}(t) = x_{\text{static}}^{(0)} + x_{\text{dy}}(t), \quad (4.57)$$

and

$$y_{\text{total}}(t) = y_{\text{static}}^{(0)} + y_{\text{dy}}(t). \quad (4.58)$$

In the present implementation, the displayed time histories and orbit plots use $x_{\text{dy}}(t)$ and $y_{\text{dy}}(t)$ rather than the total coordinates, so the plotted orbit describes the dynamic motion of the disk centre about the healthy gravity reference position; adding the constant static displacement would translate the orbit but not alter its shape, dimensions, or harmonic content.

The full numerical history includes both the initial transient and the later response approaching periodic steady-state conditions, so only a user-specified number of final revolutions is extracted for each post-processing task to avoid presenting the transient in the response plots. For a generic window containing the final $N_{\text{rev},w}$ revolutions ($w \in \{\text{time, orbit, FFT}\}$), the starting index is

$$i_w = N_{\text{total}} - N_{\text{rev},w} N_{\text{step/rev}} + 1, \quad (4.59)$$

and the corresponding data are extracted as $\mathbf{x}_w = \mathbf{x}_{\text{hist}}(i_w : \text{end})$, $\mathbf{y}_w = \mathbf{y}_{\text{hist}}(i_w : \text{end})$, $\mathbf{t}_w = \mathbf{t}_{\text{hist}}(i_w : \text{end})$. This single rule is applied with $N_{\text{rev,time}}$, $N_{\text{rev,orbit}}$, and $N_{\text{rev,FFT}}$ to obtain the time-history, orbit, and FFT windows, respectively, each including both the first and last sample of the selected interval.

For the time-history plots, the horizontal axis is the physical simulation time in seconds, with separate plots for the dynamic horizontal and vertical disk displacements. The disk orbit is obtained by plotting $y_{\text{dy}}(t)$ against $x_{\text{dy}}(t)$ over the orbit window,

$$\mathcal{O} = \{[x_{\text{dy}}(t), y_{\text{dy}}(t)] : t \in [t_{\text{orbit,start}}, T_{\text{sim}}]\}, \quad (4.60)$$

with equal scaling imposed on both axes to avoid visually distorting the orbit shape (e.g. a circular orbit appearing elliptical). For each orbit window, the displacement spans $\Delta x_{\text{orbit}} = \max(\mathbf{x}_{\text{orbit}}) - \min(\mathbf{x}_{\text{orbit}})$ and Δy_{orbit} (analogous) provide compact scalar measures of the lateral dynamic excursion, useful for comparing the healthy and cracked cases and assessing the effect of crack depth on the response amplitude in the two transverse directions.

The time histories and disk orbits are used together because they provide complementary information: the time histories show the temporal variation, amplitude modulation, and repeatability of the response in each stationary direction, while the orbit combines both components into a two-dimensional representation that can reveal changes in directionality, ellipticity, loop structure, and phase relationship of the disk motion.

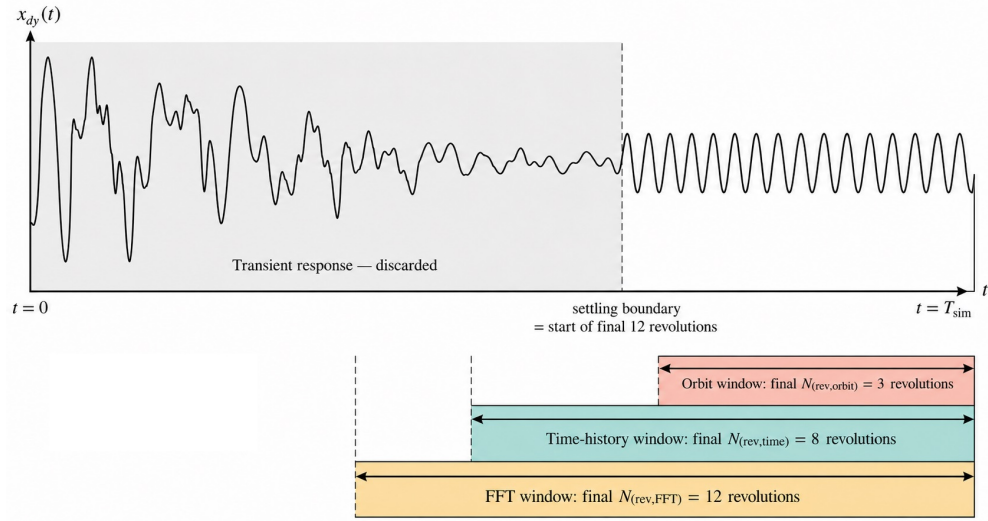


Figure 4.6: Extraction of the final response windows used for time histories, disk orbits, and FFT analysis.

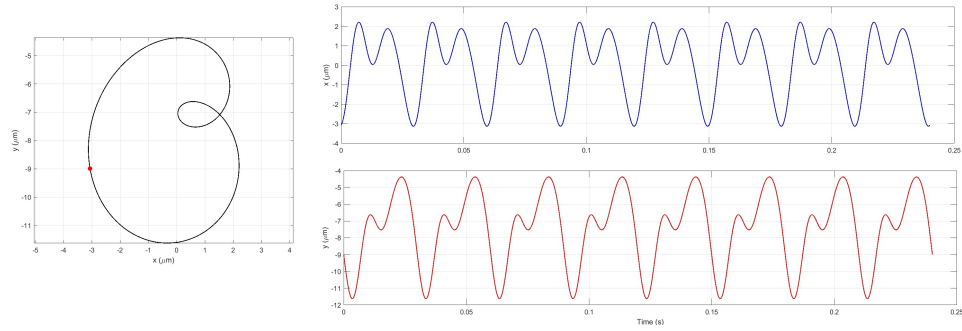


Figure 4.7: Representative time-history and disk-orbit post-processing layout for a selected rotor condition.

4.4.2 FFT Analysis and Harmonic Components

The frequency-domain analysis uses the final $N_{\text{rev,FFT}}$ complete revolutions of the disk dynamic response. For each retained direction, the mean is first removed,

$$x_0[n] = x[n] - \bar{x}, \quad y_0[n] = y[n] - \bar{y}, \quad (4.61)$$

so that the static or slowly varying offset does not appear as a dominant zero-frequency component. A Hann window is then applied to reduce spectral leakage,

$$w[n] = \frac{1}{2} \left[1 - \cos \left(\frac{2\pi n}{N-1} \right) \right], \quad n = 0, \dots, N-1. \quad (4.62)$$

The windowed discrete Fourier transforms are evaluated using the FFT algorithm.

With sampling frequency $f_s = 1/\Delta t$, the positive-frequency grid is

$$f_k = \frac{k f_s}{N}, \quad k = 0, 1, \dots, \left\lfloor \frac{N}{2} \right\rfloor. \quad (4.63)$$

The one-sided amplitudes are corrected for the coherent gain of the Hann window according to the implementation

$$A_x(f_k) = \frac{2 |X_w[k]|}{\sum_{n=0}^{N-1} w[n]}, \quad A_y(f_k) = \frac{2 |Y_w[k]|}{\sum_{n=0}^{N-1} w[n]}. \quad (4.64)$$

Because the retained record contains an integer number of rotor revolutions and the integration step divides each revolution into an integer number of samples, the principal order components are sampled consistently from cycle to cycle. The complete mathematical explanation of mean removal, the DFT/FFT, frequency resolution, spectral leakage, Hann windowing, coherent-gain correction, and one-sided amplitude scaling is provided in Appendix D.

The default plots are limited to $0 \leq f \leq 3.5 f_r$, where $f_r = \Omega/(2\pi)$, so that

$$1X = f_r, \quad 2X = 2f_r, \quad 3X = 3f_r \quad (4.65)$$

are displayed directly. For the healthy rotor, synchronous unbalance is expected to dominate the response near $1X$. Periodic stiffness modulation in the breathing cracked rotor introduces additional components at $2X$ and $3X$.

For each order $p = 1, 2, 3$, the reported amplitude is read from the frequency bin nearest to $p f_r$,

$$A_{x,pX} = A_x(f_{k_{pX}}), \quad A_{y,pX} = A_y(f_{k_{pX}}), \quad k_{pX} = \arg \min_k |f_k - p f_r|. \quad (4.66)$$

These amplitudes are used in Chapter 5 to compare the healthy and breathing cracked responses.

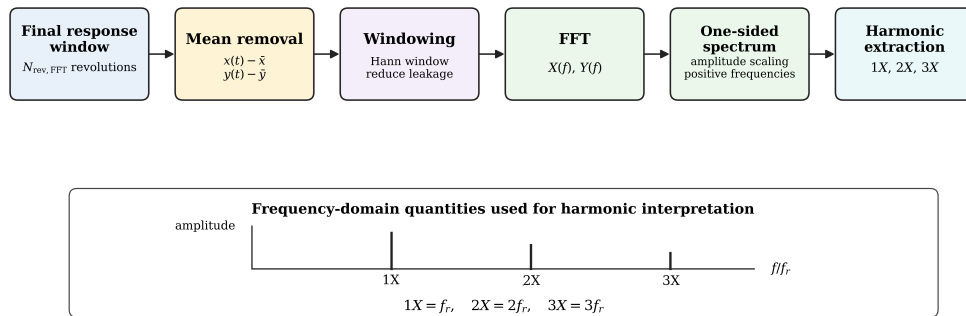


Figure 4.8: Frequency-domain post-processing of the disk response.

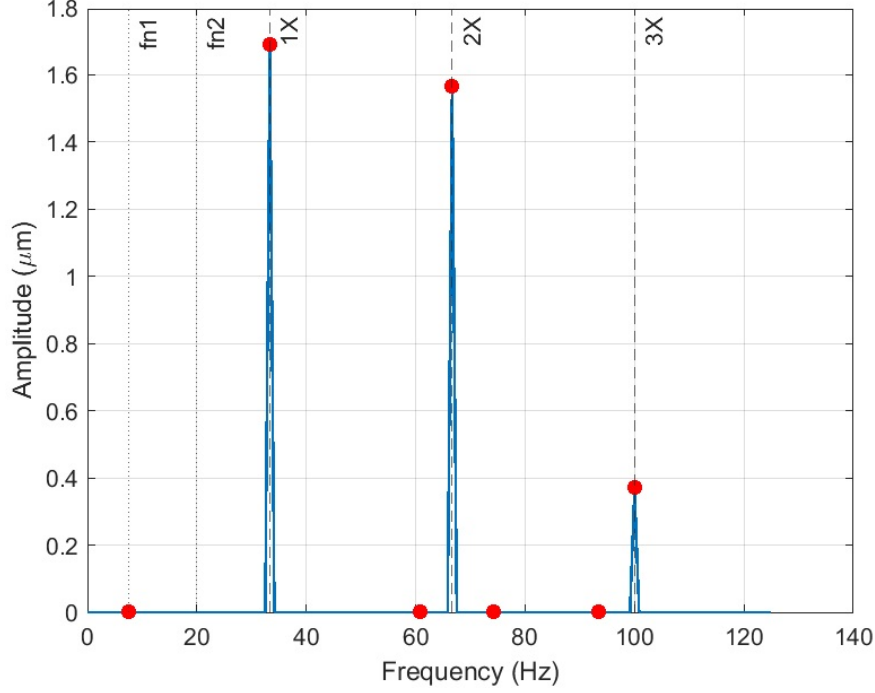


Figure 4.9: Representative one-sided FFT spectra of the disk response.

4.4.3 Periodicity and Steady-State Response Selection

The numerical simulation begins from the healthy static gravity-sag configuration with zero dynamic displacement and velocity, so the first part of the computed response contains a transient associated with this initial condition; the post-processing results reported in this study are therefore extracted from the final part of the simulation, where the response is expected to approach a repeatable periodic state.

For a rotor operating at constant angular speed Ω , the rotating unbalance force, crack orientation, and prescribed breathing function are all periodic with the revolution period

$$T_{\text{rev}} = \frac{2\pi}{\Omega}. \quad (4.67)$$

Because each revolution is divided into an integer number of equal time steps, responses one revolution apart are evaluated at the same shaft angle. After the transient has decayed, the final disk response should therefore be close to that obtained one revolution earlier. Let these two displacement vectors be

$$\mathbf{r}_{\text{end}} = \begin{bmatrix} x_{\text{hist}}(\text{end}) \\ y_{\text{hist}}(\text{end}) \end{bmatrix}, \quad \mathbf{r}_{\text{prev}} = \begin{bmatrix} x_{\text{hist}}(\text{end} - N_{\text{step/rev}}) \\ y_{\text{hist}}(\text{end} - N_{\text{step/rev}}) \end{bmatrix}, \quad (4.68)$$

where $N_{\text{step/rev}}$ is the integer number of Runge–Kutta integration steps per rotor revolution. The periodicity indicator implemented in the numerical routine is then

$$\varepsilon_{\text{per}} = \frac{\|\mathbf{r}_{\text{end}} - \mathbf{r}_{\text{prev}}\|_2}{\max[1, \|\mathbf{r}_{\text{end}}\|_2]}, \quad (4.69)$$

The lower bound of unity prevents the indicator from increasing artificially when the response amplitude is very small.

A small endpoint difference indicates an approximately periodic response. However, this does not prove complete steady state because only the final displacement is compared, not the velocity or the full waveform. The indicator is therefore interpreted together with the final time histories and disk orbits.

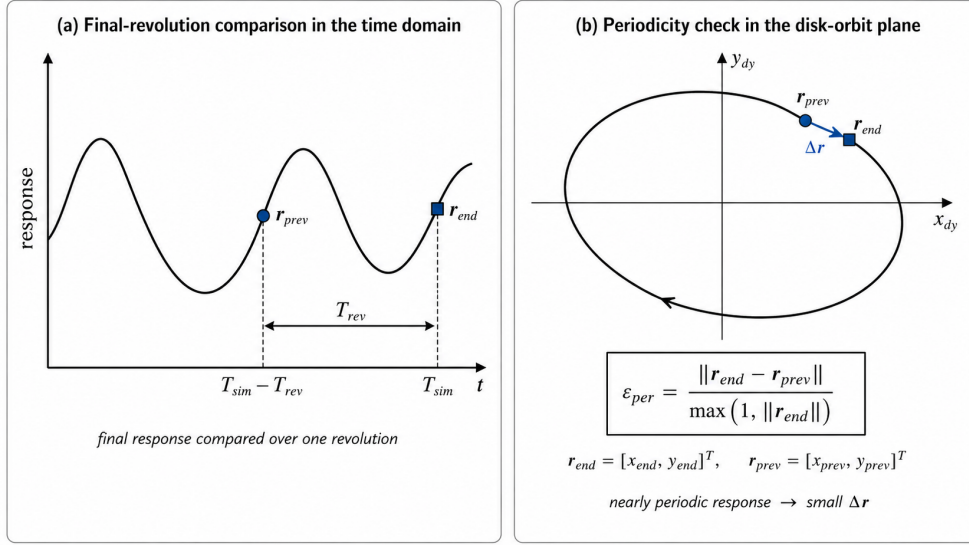


Figure 4.10: Periodicity assessment based on the difference between the final disk displacement and the disk displacement one rotor revolution earlier.

Following the periodicity check, the three final-cycle windows defined in Section 4.4.1 (Eq. (4.59)) are extracted from the complete response history for time-history plotting, disk-orbit construction, and FFT analysis, using $N_{\text{rev,time}}$, $N_{\text{rev,orbit}}$, and $N_{\text{rev,FFT}}$, respectively. The complete history is retained in the saved numerical output together with these extracted windows and the periodicity indicator, allowing the transient response, the final selected response window, and the periodicity measure to be reviewed independently after each simulation.

The response data presented in Chapter 5 are selected from these final-cycle windows. The associated periodicity indicator is reported together with the relevant numerical cases to demonstrate that the plotted time histories, disk orbits, and frequency spectra correspond to the late-stage response of the simulated rotor.

4.5 Chapter Summary

This chapter described the numerical implementation and solution procedure used to analyse the healthy, fully open cracked, and breathing cracked rotor configurations introduced in Chapter 3. The implementation was developed in a MATLAB-based environment and was organised around a healthy finite-element rotor model that provides the common reference configuration for all modal and time-domain analyses.

The crack model was implemented by first evaluating the dimensionless direct bending-compliance coefficients over a prescribed range of nondimensional crack depths. These values were stored in a crack-compliance table and subsequently interpolated for each selected crack-depth ratio. The crack position was then assigned to a unique solid circular shaft element, and the interpolated compliance values were converted into equivalent directional bending properties. The corresponding local fully open stiffness-reduction matrix was calculated from the difference between the healthy and cracked element stiffness matrices.

The healthy rotor matrices were assembled from the shaft-element, disk, bearing, and gravity-load contributions. The resulting global mass, damping, gyroscopic, and stiffness matrices were retained as the reference system. The fully open cracked rotor was obtained by subtracting the global crack stiffness-reduction matrix from the healthy stiffness matrix, while retaining the same mass, damping, and gyroscopic matrices. This time-independent configuration was used for the modal analyses.

For the time-domain analysis, the breathing crack was represented through a rotating local stiffness-reduction matrix modulated by the prescribed Mayes breathing function. Rather than reconstructing and storing a complete global time-dependent stiffness matrix at every integration stage, the crack correction was evaluated only over the degrees of freedom of the crack-containing element. The resulting local correction force was inserted into the global equations of motion together with the rotating unbalance excitation.

The transient response was obtained using a fixed-step classical fourth-order Runge-Kutta integration scheme. The time step was chosen small relative to the shortest natural period of the healthy rotor for numerical stability and adequate resolution, and was adjusted so that every shaft revolution was represented by an integer number of equal time increments. This ensured consistent sampling of the rotating unbalance force, crack orientation, and breathing function throughout the simulation. The dynamic state was initialised at zero displacement and zero velocity relative to the healthy static gravity-sag configuration.

The disk dynamic displacement histories were stored at every integration step and subsequently processed to obtain time histories, disk orbits, and one-sided FFT spectra. The final response windows were selected from the last simulated rotor

revolutions in order to exclude the initial transient response. The FFT procedure included mean removal, Hann windowing, coherent-gain-corrected construction of the one-sided amplitude spectrum, and examination of the $1X$, $2X$, and $3X$ harmonic components; the complete formulation is documented in Appendix D. A periodicity indicator based on the difference between the final disk position and the position one revolution earlier was also used to assess whether the final response window was representative of the late-stage operating condition.

The numerical procedure established in this chapter provides the computational framework for the case studies presented in Chapter 5. The following chapter applies the model to the selected crack-depth cases and compares the modal characteristics, time-domain responses, disk orbits, and harmonic content of the healthy and cracked rotor configurations.

Chapter 5

Numerical Case Studies and Results

5.1 Case-Study Configuration

5.1.1 Rotor Parameters and Operating Conditions

The numerical case study is adapted from the rotor configuration presented in Example 10.14.1 of Friswell et al. [41], with the discretisation and selected parameters defined below. It considers a single-disk flexible rotor supported by two isotropic linear bearings. The same physical configuration is retained for the healthy, fully open cracked, and breathing cracked models so that any observed variation in the response can be attributed to the crack-induced stiffness reduction rather than to changes in shaft geometry, disk properties, bearing coefficients, or external excitation.

The shaft is modelled as a uniform solid circular structural-steel beam with a total length of 1.50 m, an outer diameter of 50 mm, and no internal bore. Its cross-sectional area and second moment of area are

$$A = \frac{\pi D_o^2}{4} = 1.9635 \times 10^{-3} \text{ m}^2, \quad (5.1)$$

$$I = \frac{\pi D_o^4}{64} = 3.0680 \times 10^{-7} \text{ m}^4. \quad (5.2)$$

The shaft material has a density of 7810 kg/m³, Young's modulus of 211 GPa, and Poisson's ratio of 0.30. The resulting distributed shaft mass is approximately 23.0 kg.

For the final discretisation used in this chapter, the shaft is divided into 24 equal beam elements of length 62.5 mm, which produce 25 nodes and 100 lateral degrees of freedom. The two bearing stations coincide with the end nodes of the shaft, the disk is mounted at the midspan node, and the cracked element is the element immediately to the left of the disk. The complete shaft properties and the adopted finite-element discretisation are summarised in Table 5.1.

Table 5.1: Shaft properties and finite-element discretisation used in the numerical case study.

Shaft quantity	Symbol	Value
Total shaft length	L_{shaft}	1.500 m
Outer diameter	D_o	50 mm
Inner diameter	D_i	0 mm
Cross-sectional area	A	$1.9635 \times 10^{-3} \text{ m}^2$
Second moment of area	I	$3.0680 \times 10^{-7} \text{ m}^4$
Material	–	Structural steel
Density	ρ	7810 kg/m ³
Young’s modulus	E	200 GPa
Poisson’s ratio	ν	0.30
Distributed shaft mass	m_{shaft}	23.0 kg
Element length	L_e	62.5 mm
Total number of beam elements	N_{elem}	24
Total number of nodes	N_{node}	25
Total lateral degrees of freedom	N_{dof}	100

A rigid disk, denoted as the flange in the numerical input file, is mounted exactly at the shaft midspan:

$$x_d = 0.750 \text{ m}. \quad (5.3)$$

The disk has a mass of 52.6 kg, an outer diameter of 350 mm, and a thickness of 70 mm. Its diametral and polar mass moments of inertia are

$$I_{d,x} = I_{d,y} = 0.424197 \text{ kg m}^2, \quad (5.4)$$

$$I_{d,p} = 0.805438 \text{ kg m}^2. \quad (5.5)$$

The disk also carries a residual unbalance of

$$U = 1.0 \times 10^{-4} \text{ kg m}, \quad (5.6)$$

with an initial phase angle of

$$\beta = 0^\circ. \quad (5.7)$$

The two bearing stations are located at the shaft ends,

$$x_{b1} = 0, \quad x_{b2} = 1.500 \text{ m.} \quad (5.8)$$

Both supports are modelled as isotropic and uncoupled. Their direct stiffness and damping coefficients are equal in the two transverse directions, while the cross-coupled coefficients are set to zero:

$$\mathbf{K}_b = \begin{bmatrix} 1.0 \times 10^5 & 0 \\ 0 & 1.0 \times 10^5 \end{bmatrix} \text{ N/m,} \quad (5.9)$$

$$\mathbf{C}_b = \begin{bmatrix} 500 & 0 \\ 0 & 500 \end{bmatrix} \text{ N s/m.} \quad (5.10)$$

This support definition prevents bearing anisotropy or bearing cross-coupling from acting as an additional source of directional asymmetry. Therefore, the only intentional stiffness asymmetry in the cracked cases is introduced by the transverse crack. The locations and main properties of the concentrated rotor components and supports are collected in Table 5.2.

Table 5.2: Locations and properties of the concentrated rotor components and supports.

Concentrated component or support	Axial position [m]	Main properties
Bearing 1	0.000	$K_{11} = K_{22} = 1.0 \times 10^5 \text{ N/m}$, $C_{11} = C_{22} = 500 \text{ N s/m}$
Disk	0.750	$m_d = 52.6 \text{ kg}$, $D_d = 350 \text{ mm}$, $t_d = 70 \text{ mm}$, $U = 1.0 \times 10^{-4} \text{ kg m}$
Bearing 2	1.500	$K_{11} = K_{22} = 1.0 \times 10^5 \text{ N/m}$, $C_{11} = C_{22} = 500 \text{ N s/m}$

The modal characteristics are examined at zero speed and over a rotational-speed interval extending from

$$0 \text{ rpm} \leq \Omega_{\text{RPM}} \leq 8000 \text{ rpm.} \quad (5.11)$$

This interval is used for the Campbell-diagram calculations. The transient simulations are performed at the constant operating speed

$$\Omega_{\text{RPM}} = 2000 \text{ rpm,} \quad (5.12)$$

which corresponds to

$$\Omega = 209.440 \text{ rad/s}, \quad (5.13)$$

$$f_r = 33.333 \text{ Hz}, \quad (5.14)$$

and

$$T_{\text{rev}} = 0.030 \text{ s}. \quad (5.15)$$

At this speed, the magnitude of the rotating unbalance force is

$$F_u = U\Omega^2 = 4.386 \text{ N}. \quad (5.16)$$

Gravity is included throughout the time-domain analysis using

$$g = 9.81 \text{ m/s}^2, \quad (5.17)$$

acting in the negative stationary y -direction. The dynamic response is defined relative to the healthy static gravity-sag configuration. The complete set of time-domain operating conditions and post-processing settings is listed in Table 5.3.

Table 5.3: Time-domain operating conditions and post-processing settings.

Time-domain setting	Symbol	Value
Operating speed	Ω_{RPM}	2000 rpm
Angular speed	Ω	209.440 rad/s
Rotational frequency	f_r	33.333 Hz
Revolution period	T_{rev}	0.030 s
Total simulated revolutions	$N_{\text{rev,total}}$	80
Total simulation duration	T_{sim}	2.40 s
Automatic RK4 steps per revolution	$N_{\text{step/rev}}$	4429
Final revolutions retained for time histories	$N_{\text{rev,time}}$	8
Final revolutions retained for disk orbit	$N_{\text{rev,orbit}}$	3
Final revolutions retained for FFT analysis	$N_{\text{rev,FFT}}$	12
Campbell speed samples	N_{Ω}	200

At the beginning of each simulation, the crack orientation and the unbalance phase are prescribed as $\theta_c(0) = 0$ and $\beta = 0$, respectively, and are retained consistently in all

comparative cases.

The transverse crack is located at the fixed axial coordinate

$$x_c = 0.710 \text{ m}, \quad (5.18)$$

measured from the left-hand shaft origin. Its normalised axial location is therefore

$$\frac{x_c}{L_{\text{shaft}}} = 0.4733. \quad (5.19)$$

For the adopted mesh, this position lies inside element 12, whose boundaries are

$$0.6875 \text{ m} < x_c < 0.7500 \text{ m}. \quad (5.20)$$

Thus, the crack is placed in the element immediately adjacent to the disk on its left-hand side. The crack location is kept unchanged in all cracked-rotor simulations; only the relative crack depth $\mu = a/R$ is varied. The complete case-study configuration, including the end bearings, the midspan disk, the 24-element discretisation, and the crack location, is illustrated in Figure 5.1.

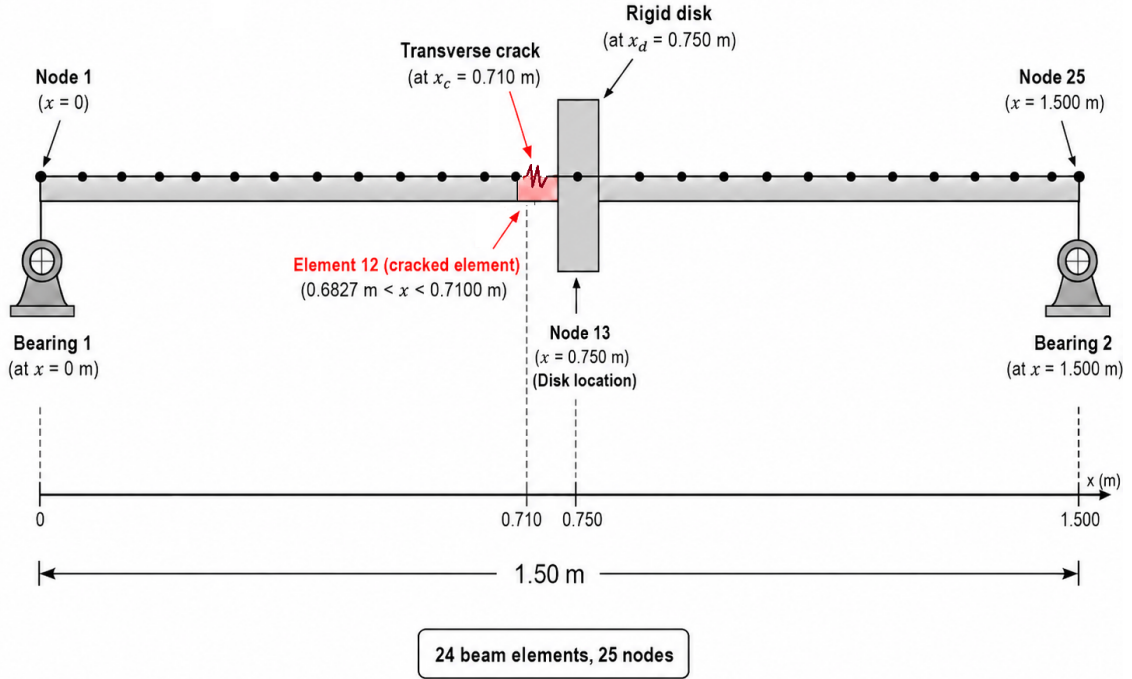


Figure 5.1: Case-study rotor and 24-element finite-element discretisation, showing the end bearings, midspan disk, and crack location within element 12.

This configuration provides a controlled baseline for studying the effect of crack

depth. The shaft, disk, support properties, gravity load, unbalance magnitude, rotational speed, crack position, and numerical post-processing windows remain unchanged in all comparative analyses presented in this chapter.

5.1.2 Selected Crack Depths: $\mu = 0.25$, $\mu = 0.50$, and $\mu = 0.98$

The effect of crack severity is investigated by considering three nondimensional crack-depth ratios,

$$\mu = \frac{a}{R} = 0.25, \quad 0.50, \quad 0.98. \quad (5.21)$$

The crack position is retained at

$$x_c = 710 \text{ mm}, \quad (5.22)$$

for all cracked-rotor cases. Since the shaft is solid and has an external radius of

$$R = 25 \text{ mm}, \quad (5.23)$$

the three nondimensional crack-depth ratios correspond to the physical crack depths

$$a = 6.25 \text{ mm}, \quad 12.50 \text{ mm}, \quad 24.50 \text{ mm}, \quad (5.24)$$

respectively.

For the three selected values of μ , the compliance coefficients are obtained directly from the crack-compliance calculations. The corresponding dimensionless values are

$$\hat{c}_{44} = 0.0103483, \quad 0.12323, \quad 2.33500, \quad (5.25)$$

and

$$\hat{c}_{55} = 0.243694, \quad 1.22153, \quad 7.28595, \quad (5.26)$$

for $\mu = 0.25$, $\mu = 0.50$, and $\mu = 0.98$, respectively.

The crack-induced flexibility is direction dependent. In particular, the calculated value of $\hat{c}_{55}$ is larger than $\hat{c}_{44}$ for all three crack depths. As a result, the equivalent bending-stiffness reduction is stronger in the direction associated with $I_y^{(\text{eq})}$ than in the direction associated with $I_x^{(\text{eq})}$. The three crack-depth cases, together with their dimensionless compliances and equivalent bending-property ratios, are summarised in Table 5.4.

Table 5.4: Selected crack-depth cases, dimensionless compliances, and equivalent bending-property ratios.

Crack case	$\mu = a/R$	a [mm]	$\hat{c}_{44}$	$\hat{c}_{55}$	$I_x^{(eq)}/I$	$I_y^{(eq)}/I$
Case 1	0.25	6.25	0.010	0.244	0.997	0.935
Case 2	0.50	12.50	0.123	1.222	0.966	0.741
Case 3	0.98	24.50	2.335	7.286	0.600	0.324

The healthy rotor is retained as the reference case, while the three crack depths are applied consistently in both parts of the numerical investigation. The fully open crack configuration is used to evaluate the influence of crack depth on the modal properties and Campbell diagram. The breathing-crack configuration is then used to investigate the resulting time-domain response, disk orbit, and harmonic content at the selected operating speed.

Finally, $\mu = 0.98$ is retained as a severe near-limit numerical case. It does not represent a typical service crack; rather, it provides a reference condition for examining the response when the adopted compliance formulation approaches the upper end of its evaluated range. The three selected crack geometries are compared in Figure 5.2.

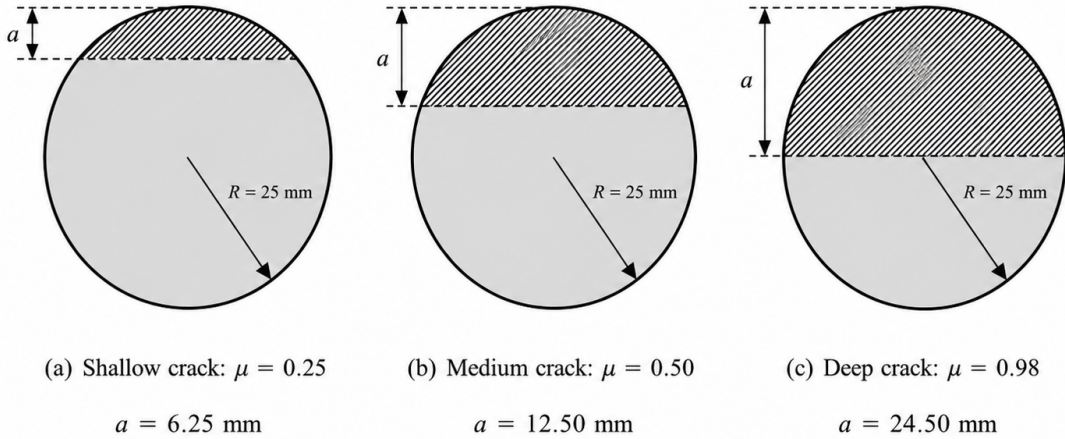


Figure 5.2: Selected crack geometries for $\mu = 0.25$, 0.50 , and 0.98 .

5.2 Modal Characteristics of the Healthy and Fully Open Cracked Rotor

5.2.1 Natural Frequencies and Mode Shapes

The zero-speed modal analysis was first performed for the healthy rotor and for the fully open cracked rotor at the three selected crack-depth ratios. Flexible isotropic bearing supports were retained in all cases. Since the healthy rotor has a circular

shaft, a centrally mounted disk, and identical support properties in the two stationary transverse directions, its zero-speed bending modes occur as repeated or nearly repeated pairs.

The first eight zero-speed natural frequencies of the healthy rotor are listed in Table 5.5 together with the corresponding frequencies of the fully open cracked configurations. In the healthy case, the first four modal pairs are repeated within numerical precision. The transverse crack removes this symmetry by introducing different equivalent bending stiffnesses in the two principal crack directions. Consequently, each repeated pair separates into two distinct frequencies.

Table 5.5: First eight zero-speed natural frequencies of the healthy and fully open cracked rotor.

Mode	Healthy [Hz]	$\mu = 0.25$ [Hz]	$\mu = 0.50$ [Hz]	$\mu = 0.98$ [Hz]
1	7.501	7.497	7.480	7.381
2	7.501	7.501	7.499	7.462
3	23.710	23.710	23.709	23.703
4	23.710	23.710	23.710	23.708
5	75.403	75.075	73.797	67.078
6	75.403	75.389	75.236	72.413
7	184.241	183.718	181.734	172.691
8	184.241	184.219	183.975	179.686

As an internal consistency check, the cracked frequencies progressively approach the corresponding healthy-rotor values as the crack depth decreases, consistent with the zero-crack-depth limit of the compliance formulation.

The zero-speed results show a systematic reduction in the natural frequencies as the crack depth increases. For the shallow crack, $\mu = 0.25$, the frequency changes are very small, particularly for the second, third, and fourth modes, which remain almost coincident with the healthy values. The crack effect becomes more visible for $\mu = 0.50$, especially in the fifth and seventh modes, which decrease from 75.403 Hz and 184.241 Hz to 73.797 Hz and 181.734 Hz, respectively. The severe crack case, $\mu = 0.98$, produces the strongest modal reduction and the clearest separation of the originally repeated healthy pairs. In this case, the fifth and sixth frequencies decrease to 67.078 Hz and 72.413 Hz, while the seventh and eighth frequencies decrease to 172.691 Hz and 179.686 Hz. The larger frequency shifts of the higher modes are consistent with a stronger interaction between these modes and the local stiffness reduction at the cracked element.

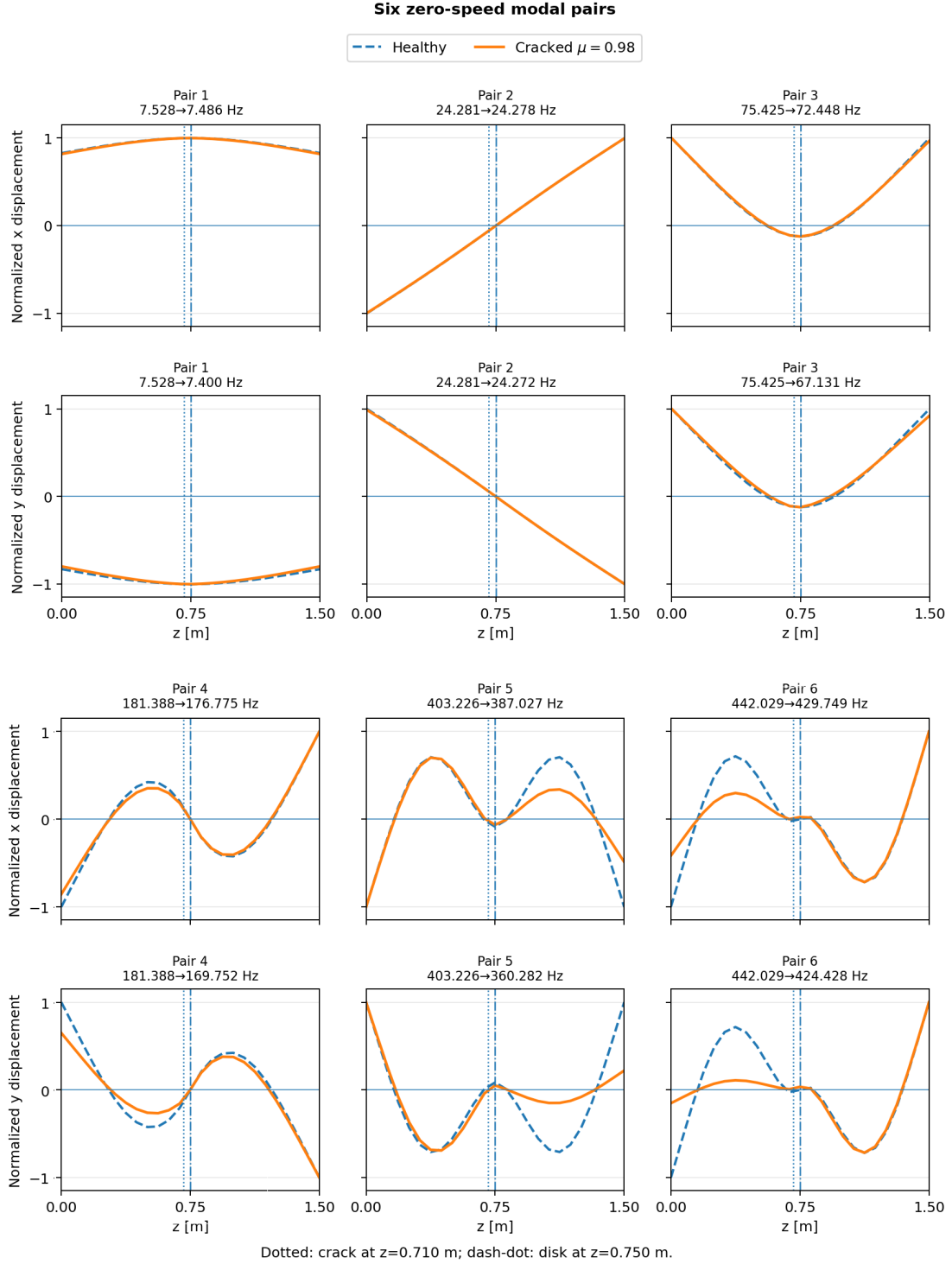


Figure 5.3: Normalised healthy and fully open cracked mode shapes; the crack-containing span is indicated.

Comparison of the matched healthy and cracked mode shapes shows that the lower modes largely preserve their global bending form, whereas the higher modes exhibit a more visible redistribution of modal amplitude and a greater deviation from the healthy configuration in the vicinity of the crack-containing span. Together with the larger frequency reductions reported in Table 5.5, this behaviour indicates a stronger

interaction between these modes and the local stiffness reduction introduced by the crack. The mode shapes are plotted with a common normalisation and the crack location is identified in Figure 5.3.

5.2.2 Mesh-Sensitivity Assessment

The sensitivity of the modal predictions to the shaft discretisation and to the length of the crack-containing element was assessed with 12, 24, and 48 uniform beam elements. The corresponding element lengths were 125, 62.5, and 31.25 mm. All physical parameters were held fixed, and the comparison was performed for the healthy rotor and for the fully open crack with $\mu = 0.50$. The resulting zero-speed natural frequencies for the three mesh levels are listed in Table 5.6.

Table 5.6: Zero-speed natural frequencies for the three mesh levels.

Mode	Healthy rotor [Hz]			Fully open crack, $\mu = 0.50$ [Hz]		
	12 elements	24 elements	48 elements	12 elements	24 elements	48 elements
1	7.501	7.501	7.501	7.482	7.480	7.481
2	7.501	7.501	7.501	7.499	7.499	7.499
3	23.710	23.710	23.710	23.709	23.709	23.709
4	23.710	23.710	23.710	23.710	23.710	23.710
5	75.404	75.403	75.403	73.968	73.797	73.885
6	75.404	75.403	75.403	75.256	75.236	75.246
7	184.250	184.241	184.240	181.734	181.734	181.721
8	184.250	184.241	184.240	183.983	183.975	183.972

For the healthy rotor, the maximum spread among the three meshes is approximately 0.0054%. For the cracked rotor, the largest spread is approximately 0.23% and occurs in the fifth mode; all other modal spreads are below 0.03%. The crack-induced reductions of the most sensitive modes also remain stable. The comparison therefore demonstrates practical mesh independence of the modal results over a fourfold change in element length and supports the 24-element discretisation used in the main study. This is a numerical sensitivity assessment rather than a claim of mathematical convergence.

5.2.3 Campbell Diagram and the Effect of Crack Depth

The rotating modal characteristics were evaluated over the speed range

$$0 \leq \Omega_{\text{RPM}} \leq 8000 \text{ rpm.} \quad (5.27)$$

For each speed, the fully open rotor stiffness matrix was retained and the gyroscopic matrix was included in the state-space eigenvalue problem. The breathing mechanism

was not included in this analysis. The cracked Campbell diagrams therefore correspond to the fully open, time-invariant crack configuration and are used as modal references for assessing the stiffness-reduction effect.

At the operating speed of 2000 rpm, the first eight rotating natural frequencies of the healthy rotor and of the three cracked cases are given in Table 5.7. The crack shifts the rotating branches downward, and the reduction becomes stronger as μ increases. The higher branches are more sensitive than the lower branches because they contain larger shaft curvature and therefore interact more strongly with the crack location.

Table 5.7: First eight rotating natural frequencies at 2000 rpm.

Branch	Healthy [Hz]	$\mu = 0.25$ [Hz]	$\mu = 0.50$ [Hz]	$\mu = 0.98$ [Hz]
1	7.518	7.514	7.498	7.397
2	7.518	7.518	7.516	7.479
3	19.814	19.814	19.812	19.795
4	25.734	25.733	25.732	25.724
5	74.100	73.818	72.552	65.864
6	74.206	74.148	73.990	71.180
7	162.821	162.501	161.181	153.451
8	201.106	200.895	200.053	194.940

The synchronous excitation line,

$$f_{1X} = \frac{\Omega_{\text{RPM}}}{60}, \quad (5.28)$$

is included in the Campbell diagrams of Figure 5.4 to identify the approximate speed locations at which the shaft rotational frequency approaches a natural-frequency branch. In the present model, the most visible gyroscopic separation occurs in the second and fourth modal families, whereas the first pair remains only weakly separated because of the dominant flexibility of the supports. The corresponding variation of the first eight rotating frequencies at 2000 rpm with crack depth is summarised in Figure 5.5.

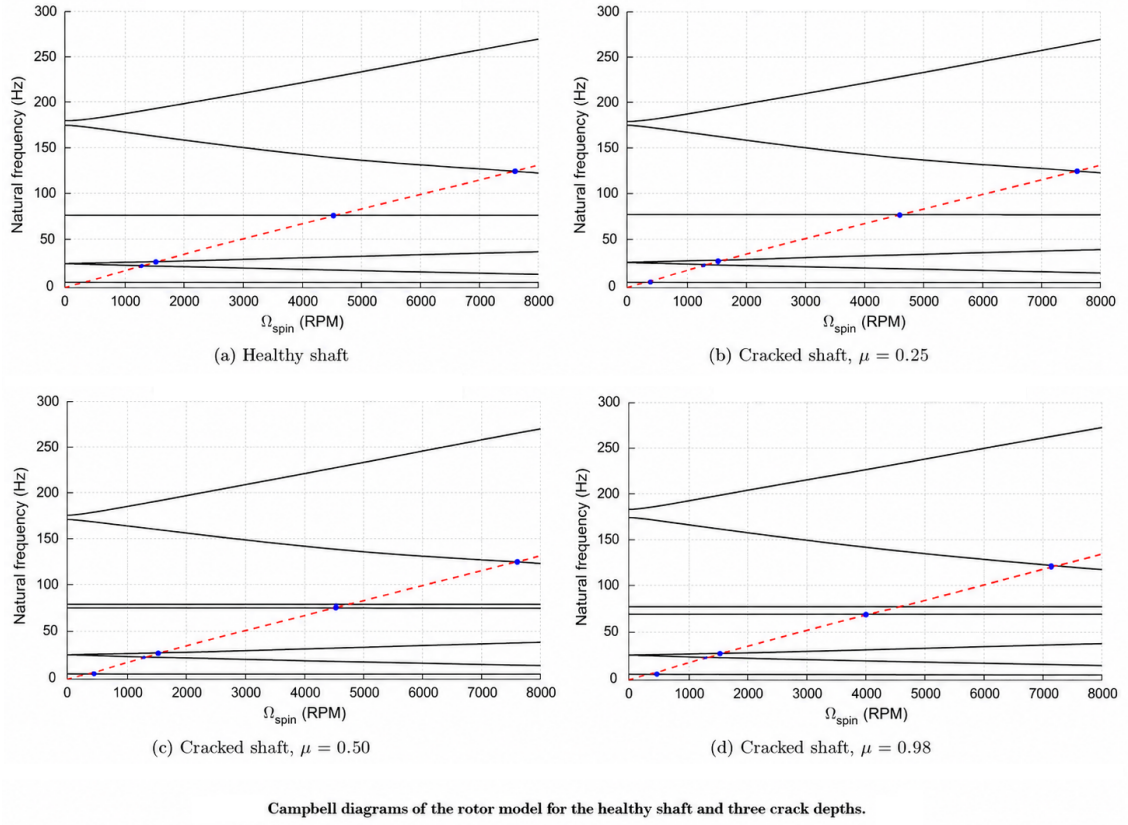


Figure 5.4: Campbell diagrams of the healthy rotor and the fully open, time-invariant cracked configurations.

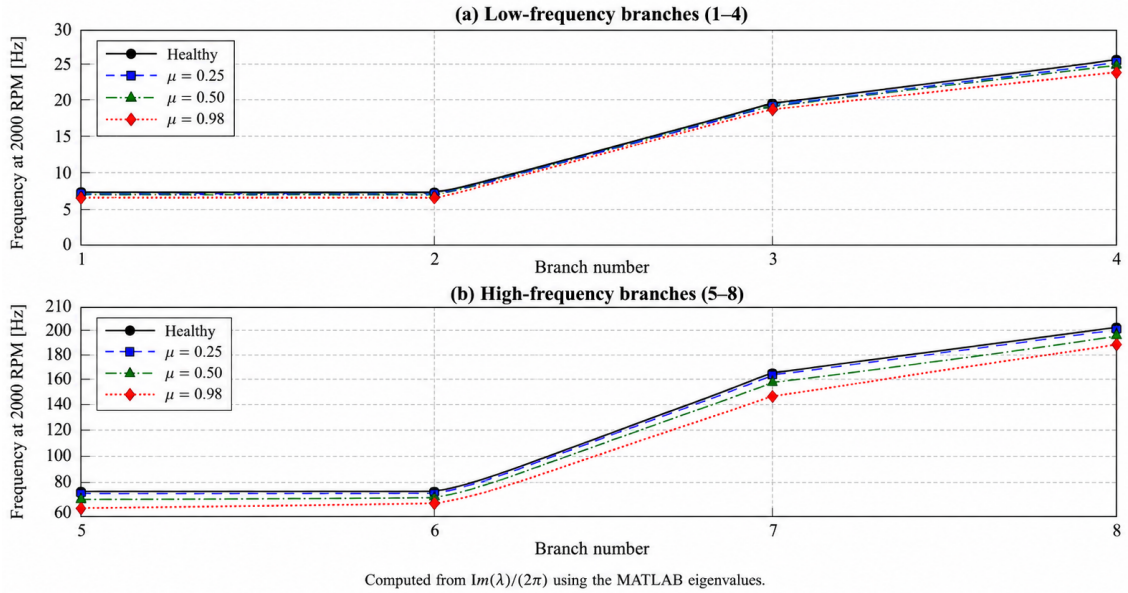


Figure 5.5: First eight rotating modal frequencies at 2000 rpm as functions of crack depth.

The Campbell results indicate that the fully open crack has two coupled effects on the rotating modal response. First, it removes the zero-speed frequency degeneracy through directional stiffness asymmetry. Second, it shifts the gyroscopic branches downward

as crack depth increases. These changes are small for the shallow crack and become substantial for the severe crack case.

5.3 Time-Domain Response of the Breathing Cracked Rotor

5.3.1 Time Histories, Static-Sag Dominance, and Disk Orbits

The time-domain response of the breathing cracked rotor was evaluated at the constant operating speed of 2000 rpm, corresponding to a rotational frequency of 33.333 Hz. For every rotor condition, the simulation covered 80 revolutions. The final eight revolutions were retained for the time-history plots, whereas the final three revolutions were used for the disk-orbit representation.

The numerical response stored by the solver is the displacement measured relative to the healthy static gravity-sag configuration. The static disk displacement of the healthy rotor is

$$x_{\text{static}}^{(0)} = 0, \quad y_{\text{static}}^{(0)} = -4.4942 \text{ mm.} \quad (5.29)$$

Therefore, the total disk coordinates are obtained from

$$x_{\text{total}}(t) = x_{\text{dy}}(t), \quad (5.30)$$

and

$$y_{\text{total}}(t) = -4.4942 + y_{\text{dy}}(t) \quad [\text{mm}]. \quad (5.31)$$

This distinction is essential for interpreting the time-domain results. The breathing crack modifies the oscillatory motion about the gravity-deflected configuration rather than moving the disk centre back toward the undeformed shaft axis.

The principal time-response statistics extracted from the final selected response window are summarised in Table 5.8. For each rotor condition, $\bar{y}$ denotes the mean vertical dynamic displacement relative to the healthy gravity-sag position. The peak-to-peak ranges are defined as

$$x_{\text{p2p}} = \max(x_{\text{dy}}) - \min(x_{\text{dy}}), \quad y_{\text{p2p}} = \max(y_{\text{dy}}) - \min(y_{\text{dy}}), \quad (5.32)$$

and represent the total dynamic excursion in the two stationary directions. The maximum vertical deviation from the mean is

$$y_{\text{max dev}} = \max |y_{\text{dy}} - \bar{y}|. \quad (5.33)$$

The weight-dominance ratio is defined as

$$\Gamma_w = \frac{|y_{\text{static}}^{(0)}|}{y_{\text{max dev}}}, \quad (5.34)$$

and compares the healthy static gravity sag with the maximum vertical dynamic deviation. The quantity ε_{per} is the one-revolution periodicity indicator defined in Eq. (4.69). The horizontal mean displacement $\bar{x}$ remains numerically negligible in all cases and is therefore not included in the table.

Table 5.8: Time-response statistics at the disk node and the corresponding weight-dominance ratio.

Case	μ	$\bar{y}$ [mm]	x_{p2p} [mm]	y_{p2p} [mm]	$y_{\text{max dev}}$ [mm]	Γ_w	ε_{per}
Healthy	–	1.869×10^{-10}	2.765×10^{-3}	2.765×10^{-3}	1.382×10^{-3}	3.251×10^3	7.157×10^{-8}
Breathing cracked	0.25	-1.439×10^{-3}	3.034×10^{-3}	3.494×10^{-3}	1.796×10^{-3}	2.503×10^3	8.332×10^{-8}
Breathing cracked	0.50	-7.421×10^{-3}	5.719×10^{-3}	7.518×10^{-3}	4.428×10^{-3}	1.015×10^3	1.376×10^{-7}
Breathing cracked	0.98	-4.630×10^{-2}	1.577×10^{-2}	3.021×10^{-2}	1.957×10^{-2}	2.296×10^2	6.893×10^{-7}

The weight-dominance ratio ranges from approximately 2.30×10^2 to 3.25×10^3 over the analysed cases. Therefore, the healthy static gravity sag remains substantially larger than the maximum vertical dynamic deviation, supporting the prescribed angle-dependent breathing law adopted in Section 3.3.2. The mean value $\bar{y}$ is reported to describe the average vertical position of the dynamic orbit relative to the healthy static-equilibrium reference .

The healthy rotor exhibits a nearly circular orbit with almost identical horizontal and vertical peak-to-peak ranges. Introducing the breathing crack increases both the response amplitude and the directional asymmetry. For $\mu = 0.25$, the orbit remains close to the healthy one, but the vertical component already becomes larger than the horizontal component. At $\mu = 0.50$, the orbit becomes clearly distorted and vertically stretched. For $\mu = 0.98$, the response is dominated by a strong downward-shifted and highly non-circular orbit, which reflects the strong periodic stiffness modulation in the cracked element.

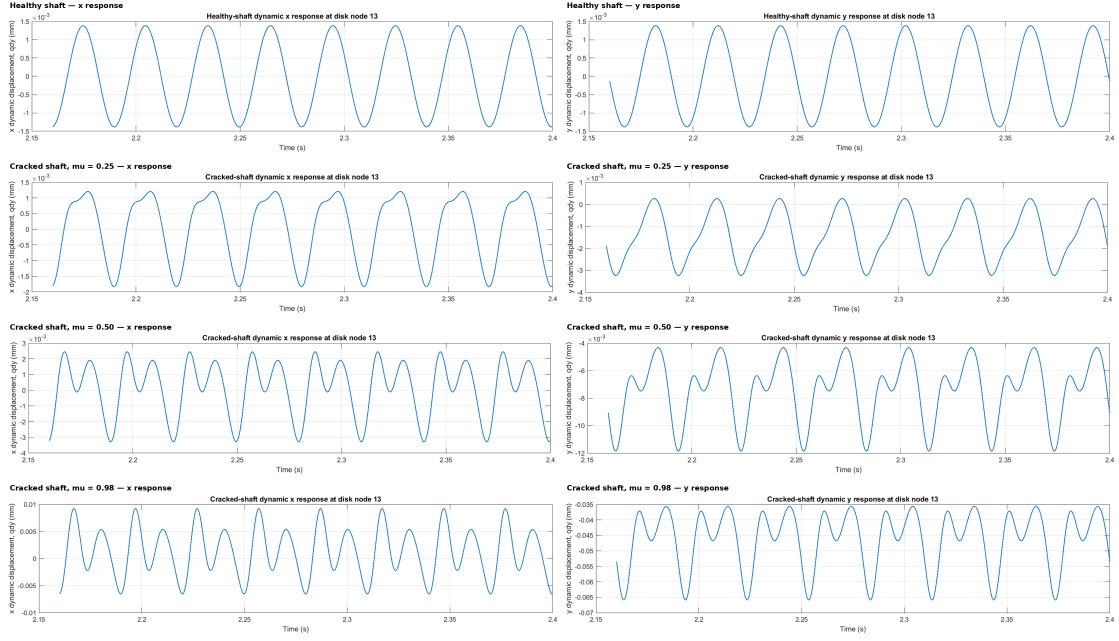


Figure 5.6: Dynamic disk time histories over the final selected response window at 2000 rpm.

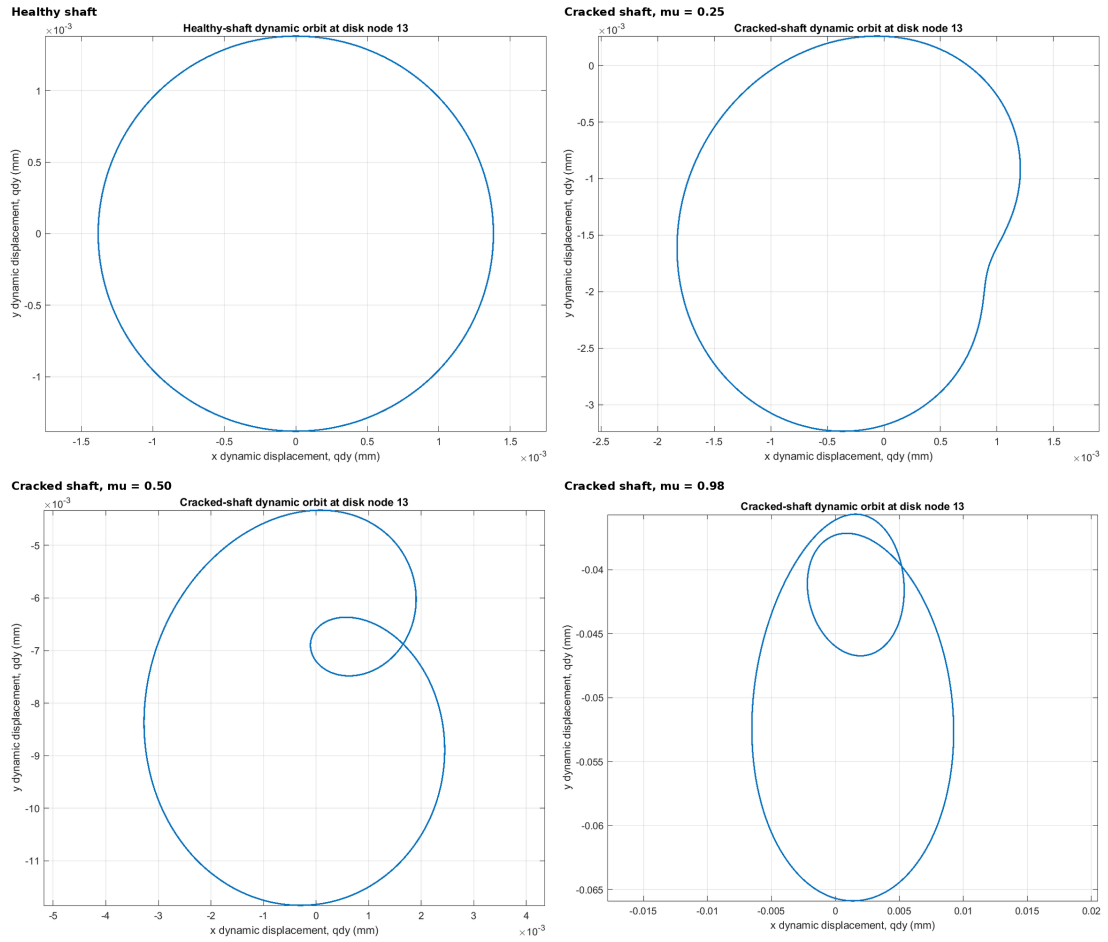


Figure 5.7: Dynamic disk orbits of the healthy rotor and the three breathing-crack cases.

5.3.2 Comparison with the Healthy-Rotor Response

The healthy rotor provides the reference response against which the breathing cracked cases are compared. Because the bearing properties, disk mass, unbalance magnitude, operating speed, gravity loading, and crack position remain unchanged, the variations reported in this section result only from the crack-induced time-periodic stiffness reduction.

Relative to the healthy rotor, the horizontal orbit range increases by approximately 9.76%, 106.87%, and 470.41% for $\mu = 0.25$, $\mu = 0.50$, and $\mu = 0.98$, respectively. Over the same crack-depth sequence, the vertical orbit range increases by approximately 26.37%, 171.91%, and 992.67%. The crack therefore affects the vertical response more strongly, in agreement with the stronger stiffness reduction associated with the crack direction linked to $I_y^{(eq)}$. These orbit-range changes are summarised in Table 5.9.

The periodicity indicators remain of the order of 10^{-7} in all cases. The small one-revolution endpoint differences indicate that the selected response windows are approximately periodic and suitable for the comparative post-processing used here.

Table 5.9: Orbit-range changes relative to the healthy rotor.

Case	μ	Increase in x_{p2p}	Increase in y_{p2p}	y_{p2p}/x_{p2p}
Healthy	–	–	–	1.000
Breathing cracked	0.25	9.76%	26.37%	1.151
Breathing cracked	0.50	106.87%	171.91%	1.314
Breathing cracked	0.98	470.41%	992.67%	1.916

5.4 Frequency-Domain Response and Effect of Crack Depth

5.4.1 FFT Spectra and 1X–2X–3X Components

The frequency-domain response was evaluated from the final 12 rotor revolutions of each transient simulation. At the operating speed of 2000 rpm,

$$1X = 33.333 \text{ Hz}, \quad 2X = 66.667 \text{ Hz}, \quad 3X = 100.000 \text{ Hz}. \quad (5.35)$$

The selected FFT record has a duration of approximately 0.36 s, corresponding to a frequency resolution of approximately

$$\Delta f \approx 2.778 \text{ Hz}. \quad (5.36)$$

The mean value of each disk-response signal was removed before constructing the one-sided amplitude spectra. The reported spectra therefore represent the oscillatory

dynamic response relative to the healthy gravity-sag reference configuration. The harmonic amplitudes extracted from these spectra are listed in Table 5.10.

Table 5.10: Harmonic amplitudes extracted from the one-sided FFT spectra of the disk dynamic response.

Case	μ	$A_{x,1X}$ [μm]	$A_{x,2X}$ [μm]	$A_{x,3X}$ [μm]	$A_{y,1X}$ [μm]	$A_{y,2X}$ [μm]	$A_{y,3X}$ [μm]
Healthy	–	1.382	3.78×10^{-7}	7.63×10^{-8}	1.382	3.60×10^{-7}	6.23×10^{-8}
Breathing cracked	0.25	1.440	0.397	0.075	1.574	0.403	0.075
Breathing cracked	0.50	1.669	1.753	0.349	2.487	1.899	0.353
Breathing cracked	0.98	2.586	5.560	1.327	9.032	9.365	1.660

The corresponding one-sided spectra in the two stationary directions are shown in Figures 5.8 and 5.9. For the healthy rotor, the frequency content is almost entirely concentrated at the synchronous $1X$ component. The breathing cracked rotor exhibits additional harmonic components at integer multiples of the rotational frequency. The crack-induced $2X$ component is already visible for $\mu = 0.25$, becomes comparable to or larger than $1X$ for $\mu = 0.50$ in the horizontal response, and dominates the horizontal spectrum for $\mu = 0.98$. In the vertical response, the synchronous component remains strong, but the $2X$ component also becomes very large for the deepest crack. The $3X$ component is always smaller than $2X$ but grows systematically with crack depth.

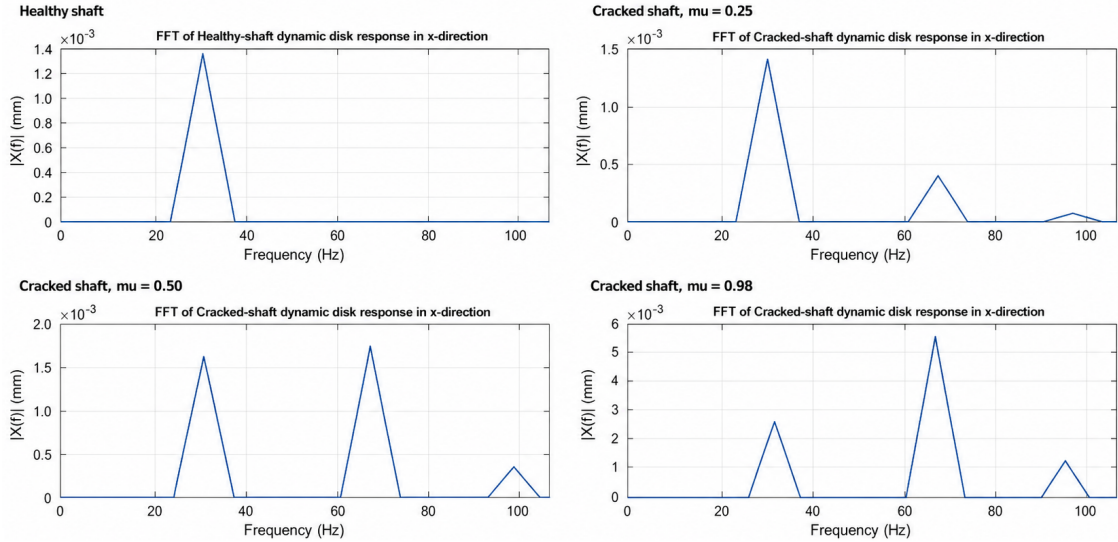


Figure 5.8: One-sided FFT spectra of the disk response in the x -direction at 2000 rpm.

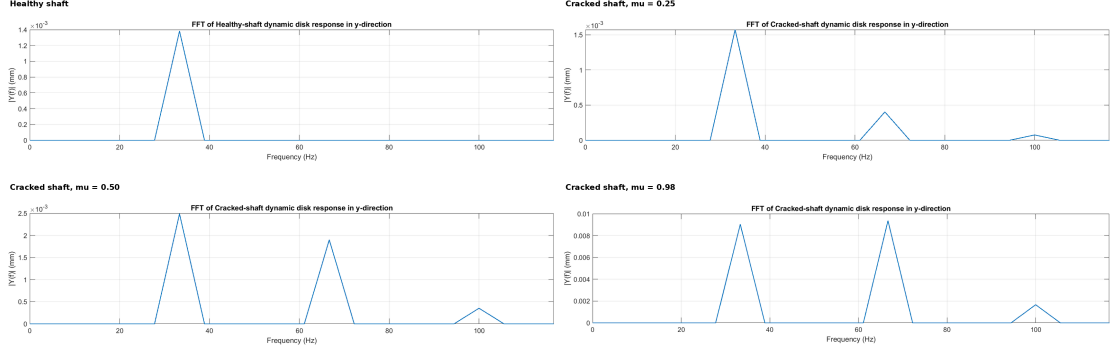


Figure 5.9: One-sided FFT spectra of the disk response in the y -direction at 2000 rpm.

5.4.2 Comparative Interpretation of the Three Crack-Depth Cases

To compare the crack-induced harmonic content independently of the absolute response level, the secondary components can be normalised with respect to the synchronous $1X$ amplitude. The resulting ratios are listed in Table 5.11 and plotted against crack depth in Figure 5.10.

Table 5.11: Secondary harmonic amplitudes normalised by the synchronous $1X$ component.

Case	μ	$A_{x,2X}/A_{x,1X}$	$A_{x,3X}/A_{x,1X}$	$A_{y,2X}/A_{y,1X}$	$A_{y,3X}/A_{y,1X}$
Breathing cracked	0.25	27.54%	5.22%	25.61%	4.78%
Breathing cracked	0.50	105.04%	20.90%	76.33%	14.18%
Breathing cracked	0.98	215.02%	51.31%	103.69%	18.38%

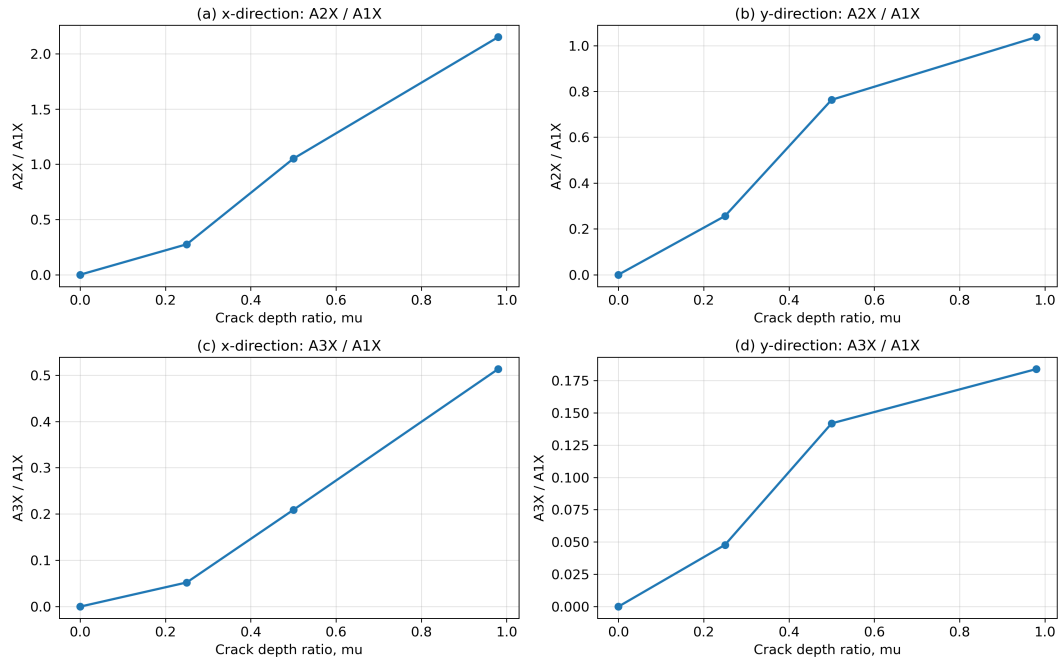
The three crack-depth cases exhibit a clear increase in harmonic distortion for the selected operating condition. For $\mu = 0.25$, the response is still primarily synchronous. For $\mu = 0.50$, the harmonic content becomes a major feature of the waveform, especially in the horizontal direction where the $2X$ component slightly exceeds $1X$. For $\mu = 0.98$, the response becomes strongly non-sinusoidal, with a very large $2X$ contribution in both stationary directions.

The variation of the synchronous component itself is also crack-depth dependent. Relative to the healthy rotor, the synchronous amplitudes increase with crack depth, and the increase is much stronger in the vertical direction because the crack compliance is larger in that direction. The corresponding increases are quantified in Table 5.12.

Table 5.12: Increase in the synchronous $1X$ amplitude relative to the healthy rotor.

Crack depth	Increase in $A_{x,1X}$	Increase in $A_{y,1X}$
$\mu = 0.25$	4.18%	13.84%
$\mu = 0.50$	20.74%	79.95%
$\mu = 0.98$	87.08%	553.28%

Within the controlled numerical model used in this thesis, the growth of $2X$ and $3X$ components is directly associated with the breathing-crack stiffness modulation. In an experimental rotating machine, however, similar harmonic components can also be influenced by other sources, such as residual support anisotropy, misalignment, rub, nonlinear bearing behaviour, or measurement imperfections. Therefore, the harmonic components should be interpreted here as model-based indicators of crack severity rather than as fault signatures that are uniquely attributable to a transverse crack under all operating conditions. Figure 5.11 collects the principal modal, orbit, and harmonic indicators established in this chapter in a single comparative view.

**Figure 5.10:** Normalised $2X/1X$ and $3X/1X$ ratios as functions of crack depth.

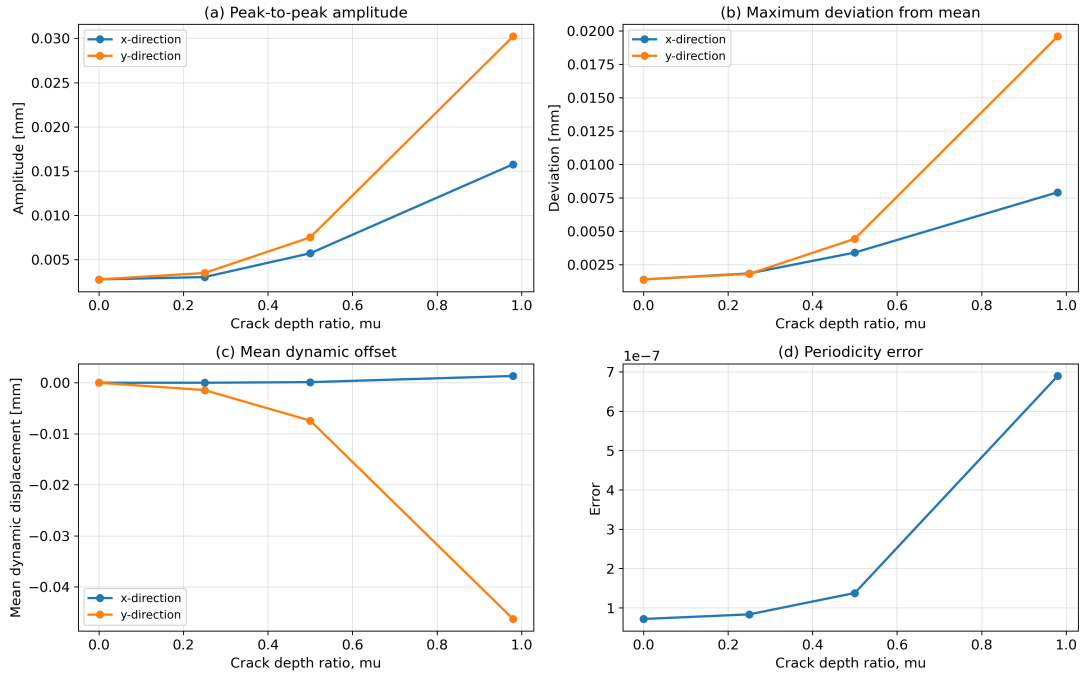


Figure 5.11: Comparative summary of the principal modal, orbit, and harmonic indicators.

5.4.3 Speed-Control Assessment of the Deep-Crack Response

A control simulation was performed for $\mu = 0.98$ at 2600 rpm to distinguish the intrinsic harmonic generation of the breathing crack from the additional amplification associated with the 2000 rpm operating point. At the control speed,

$$1X = 43.333 \text{ Hz}, \quad 2X = 86.667 \text{ Hz}, \quad 3X = 130.000 \text{ Hz}. \quad (5.37)$$

The fully open, time-invariant modal reference at the same speed gives the frequencies listed in Table 5.13, while the corresponding breathing-crack harmonic amplitudes are reported separately in Table 5.14.

Table 5.13: Fully open modal reference for the $\mu = 0.98$ control condition at 2600 rpm.

Mode	Frequency [Hz]
1	7.376
2	7.453
3	18.717
4	26.020
5	65.922
6	71.264
7	151.690
8	204.401

Table 5.14: Harmonic amplitudes of the $\mu = 0.98$ breathing-crack control response at 2600 rpm.

Direction	Harmonic	Amplitude [μm]
x	1X	2.608
x	2X	3.315
x	3X	0.404
y	1X	10.596
y	2X	5.618
y	3X	0.480

The nearest separations are 17.31 Hz between 1X and the fourth modal branch, 15.40 Hz between 2X and the sixth branch, and 21.69 Hz between 3X and the seventh branch. Thus, within the fully open Campbell reference, the control harmonics are not in direct modal proximity.

The normalised second-harmonic amplitudes at 2600 rpm are

$$\left(\frac{2X}{1X}\right)_x = 127.1\%, \quad \left(\frac{2X}{1X}\right)_y = 53.0\%. \quad (5.38)$$

The horizontal ratio is lower than the 215.020% obtained at 2000 rpm, and the vertical ratio decreases from 103.690% to 53.0%. The persistence of a pronounced 2X component away from direct modal proximity confirms that strong periodic stiffness modulation in the deep breathing crack is itself sufficient to generate the second harmonic. Its larger relative magnitude at 2000 rpm is consistent with additional near-resonant amplification caused by the proximity of 2X to the fifth modal branch.

5.5 Chapter Summary

This chapter presented the numerical case studies used to evaluate the influence of a transverse crack on the modal and dynamic behaviour of the rotor system. The healthy rotor was used as the reference configuration, while three crack-depth ratios, $\mu = 0.25$, $\mu = 0.50$, and $\mu = 0.98$, were considered at the same crack location, operating speed, shaft geometry, disk properties, bearing coefficients, gravity loading, and unbalance level.

The zero-speed modal analysis showed that the healthy rotor possesses repeated or nearly repeated bending-frequency pairs because of its symmetric shaft and support configuration. The fully open crack removed this modal degeneracy by introducing direction-dependent stiffness reduction at the crack-containing element. As the crack depth increased, the repeated healthy frequencies separated into distinct modal branches, and the frequency reduction became more pronounced in the modes with greater bending curvature near the crack position.

The fully open Campbell reference showed that the crack reduces the rotating modal branches while the gyroscopic effect continues to produce speed-dependent branch separation. The influence of the shallow crack was limited, whereas the severe crack produced a clear downward shift of the higher-frequency branches. The low-frequency modes were comparatively less sensitive because their response was more strongly influenced by the flexible bearing supports.

The breathing-crack time-domain analysis was performed at 2000 rpm. The healthy rotor exhibited an almost circular dynamic orbit with nearly equal horizontal and vertical response amplitudes. The introduction of the breathing crack progressively increased the response amplitude, particularly in the vertical direction. For $\mu = 0.25$, the modification of the orbit was limited. For $\mu = 0.50$, the orbit became visibly elongated and the time histories became clearly distorted. The severe case, $\mu = 0.98$, produced the most pronounced response asymmetry and the largest dynamic orbit span.

The time-domain results also showed that the physical disk position remained dominated by the healthy static gravity-sag displacement. The breathing crack altered the dynamic motion around this gravity-deflected reference configuration rather than producing large motion about the undeformed shaft centreline. The periodicity indicators confirmed that the final response windows used for the time histories, disk orbits, and spectral analysis were sufficiently repeatable for the selected simulation duration.

The FFT analysis showed that the healthy response is dominated by the synchronous $1X$ component, while the breathing crack introduces $2X$ and $3X$ components. For the selected 2000 rpm condition, the secondary harmonics increase strongly with crack depth. The 2600 rpm control run demonstrated that the pronounced $2X$ component of the deepest crack persists away from direct modal proximity, while its relative magnitude

decreases; the large 2000 rpm value therefore combines intrinsic stiffness modulation with additional near-resonant amplification.

Overall, the numerical results demonstrate a consistent progression from the healthy response to increasingly asymmetric modal, orbit, and spectral behaviour as crack depth increases. The shallow crack produces only limited changes, the intermediate crack produces strong harmonic distortion and clear orbit elongation, and the severe crack produces a pronounced reduction in stiffness, a large vertical response, and substantial $2X$ and $3X$ components. Within the controlled assumptions of the present model, these results establish the relationship between crack severity and the dynamic response indicators of the rotor system.

Chapter 6

Conclusions and Future Work

6.1 Summary of the Work

This thesis developed and applied a finite-element framework for the lateral dynamics of a rotor containing a transverse breathing crack. The modelling chain begins with a strain-energy-release formulation of the fully open crack, converts the resulting direct bending compliances into equivalent directional stiffnesses of the crack-containing element, and couples this local anisotropy to a rotor-bearing model through rotation of the crack-fixed frame and the Mayes–Davies breathing law. The resulting equation is linear time-periodic and was solved in the time domain using a fixed-step fourth-order Runge–Kutta procedure.

The numerical case study was adapted from the rotor configuration of Example 10.14.1 in Friswell et al. [41]. A 1.50 m solid shaft carrying a central disk and supported by two isotropic bearings was examined in the healthy condition and for crack-depth ratios $\mu = 0.25, 0.50$, and 0.98 . Modal frequencies, Campbell diagrams, disk time histories, orbits, and the $1X$ – $2X$ – $3X$ harmonic components were evaluated within a common model.

6.2 Principal Conclusions

Directional crack flexibility. The strip-based fracture-mechanics formulation produces two unequal direct bending compliances. Over the analysed range, $\hat{c}_{55} > \hat{c}_{44}$, and both quantities increase nonlinearly with crack depth. The corresponding stiffness-retention ratios decrease from $I_x^{(\text{eq})}/I = 0.997$ and $I_y^{(\text{eq})}/I = 0.935$ at $\mu = 0.25$ to 0.600 and 0.324 , respectively, at $\mu = 0.98$. This directional inequality is the local source of the modal splitting and orbit asymmetry observed at system level.

Modal behaviour. The fully open crack removes the repeated bending-frequency pairs of the symmetric healthy rotor and shifts the modal frequencies downward. The changes are small in the support-dominated low modes but become pronounced in the higher modes. At zero speed, for example, the fifth frequency decreases from 75.403 Hz in the healthy rotor to 73.797 Hz at $\mu = 0.50$ and 67.078 Hz at $\mu = 0.98$. Comparison of the mode shapes shows that the lower modes largely preserve their global bending

form, whereas the higher modes exhibit a more visible redistribution relative to the healthy configuration near the crack-containing span.

The cracked-rotor Campbell diagrams represent the fully open, time-invariant crack configuration and are therefore used as modal references. Within that interpretation, increasing crack depth shifts the rotating branches downward while preserving the overall gyroscopic branch structure.

Practical mesh independence. A mesh-sensitivity assessment was performed for 12, 24, and 48 uniform shaft elements, corresponding to element lengths of 125, 62.5, and 31.25 mm. For the healthy rotor, the maximum frequency spread among the three meshes was approximately 0.0054%. For the fully open $\mu = 0.50$ model, the largest spread was approximately 0.23%, occurring in the fifth mode. The principal crack-induced frequency reductions were also stable. These results establish practical mesh independence of the modal predictions and support the use of the 24-element discretisation in the main case study.

Time-domain and orbit response. The healthy static disk sag is -4.4942 mm. Across the analysed cases, this gravity-induced displacement exceeds the dynamic motion by between two and three orders of magnitude, supporting the prescribed angle-dependent breathing approximation. Increasing crack depth enlarges and distorts the disk orbit, with the stronger growth occurring in the vertical direction. Relative to the healthy rotor, the vertical peak-to-peak orbit range increases by approximately 26.4%, 171.9%, and 992.7% for $\mu = 0.25$, 0.50, and 0.98, respectively.

Harmonic indicators and speed dependence. At 2000 rpm, the healthy response is dominated by the synchronous $1X$ component, whereas the breathing crack introduces pronounced $2X$ and smaller $3X$ components. The horizontal $2X/1X$ ratio increases from 27.54% at $\mu = 0.25$ to 105.04% at $\mu = 0.50$ and 215.020% at $\mu = 0.98$. The vertical response follows the same general progression.

For the deepest crack, the exceptionally large $2X$ component at 2000 rpm has two contributing mechanisms. The periodic stiffness modulation generates the harmonic, while the proximity of $2X = 66.667$ Hz to the fifth fully open modal branch at 65.864 Hz provides additional near-resonant amplification. A control run at 2600 rpm placed the principal harmonics away from direct modal proximity. The $2X/1X$ ratios then decreased to 127.1% in the horizontal direction and 53.0% in the vertical direction, while $2X$ remained clearly present. The control therefore confirms that the deep breathing crack itself generates strong second-harmonic content, with the 2000 rpm operating point adding a speed-dependent amplification.

Taken together, modal splitting, orbit elongation, and the growth of $2X$ and $3X$ form a consistent multi-indicator pattern. None of these features is uniquely attributable to a crack in an actual machine, where misalignment, rub, support anisotropy, and bearing nonlinearities can produce overlapping signatures; their diagnostic value lies in their combined and physically consistent evolution.

6.3 Limitations

The conclusions apply to a single transverse crack of prescribed depth and position in a solid circular shaft. Only the two direct bending compliances are retained. Crack-face contact and friction are not resolved, and the breathing state is prescribed by shaft angle rather than computed from the instantaneous stress field. The disk is rigid, the bearings are linear, the operating speed is constant, and the numerical results are model-based rather than experimentally validated. The $\mu = 0.98$ case is a severe near-limit numerical condition close to the upper range of the adopted compliance formulation and should not be interpreted as a typical service crack.

6.4 Recommendations for Future Work

The present framework can be extended in several directions:

1. vary the axial crack position to map the dependence of modal and harmonic indicators on both location and depth;
2. replace the prescribed breathing law with a response-dependent or contact-based formulation, particularly near critical speeds;
3. perform broader speed sweeps and run-up/coast-down analyses to separate intrinsic breathing harmonics from resonance-proximity effects;
4. retain the axial, torsional, shear, and coupling flexibility terms derived in Appendix A;
5. validate the predicted orbit and harmonic trends on a laboratory rotor with a controlled transverse crack;
6. extend the model to multiple cracks, hollow shafts, and anisotropic or fluid-film bearings; and
7. use the parameterised simulator to generate labelled data for hybrid physics-based and data-driven diagnosis.

6.5 Closing Remark

Within its stated assumptions, the model links crack geometry to measurable rotor-dynamic consequences through a transparent sequence of fracture mechanics, finite-element assembly, time-periodic stiffness modulation, and response analysis. The results show that crack severity cannot be inferred from a single amplitude alone: the most defensible interpretation combines modal shifts, orbit geometry, harmonic content, and operating-speed context.

References

- [1] Rankine, W. J. M., “On the Centrifugal Force of Rotating Shafts,” *The Engineer*, London, Vol. 27, 1869, p. 249.
- [2] Dunkerley, S., “On the Whirling and Vibration of Shafts,” *Philosophical Transactions of the Royal Society A*, Vol. 185, 1894, pp. 279–360.
- [3] Föppl, A., “Das Problem der Lavalschen Turbinenwelle,” *Der Civilingenieur*, Vol. 41, 1895, pp. 335–342.
- [4] Jeffcott, H. H., “The Lateral Vibration of Loaded Shafts in the Neighbourhood of a Whirling Speed—The Effect of Want of Balance,” *Philosophical Magazine, Series 6*, Vol. 37, No. 219, 1919, pp. 304–314.
- [5] Nelson, H. D., and McVaugh, J. M., “The Dynamics of Rotor-Bearing Systems Using Finite Elements,” *ASME Journal of Engineering for Industry*, Vol. 98, No. 2, 1976, pp. 593–600.
- [6] Nelson, H. D., “A Finite Rotating Shaft Element Using Timoshenko Beam Theory,” *ASME Journal of Mechanical Design*, Vol. 102, No. 4, 1980, pp. 793–803.
- [7] Lalanne, M., and Ferraris, G., *Rotordynamics Prediction in Engineering*, 2nd ed., John Wiley & Sons, Chichester, 1998.
- [8] Dimarogonas, A. D., and Papadopoulos, C. A., “Vibration of Cracked Shafts in Bending,” *Journal of Sound and Vibration*, Vol. 91, No. 4, 1983, pp. 583–593.
- [9] Papadopoulos, C. A., and Dimarogonas, A. D., “Coupling of Bending and Torsional Vibration of a Cracked Timoshenko Shaft,” *Archive of Applied Mechanics (Ingenieur-Archiv)*, Vol. 57, No. 4, 1987, pp. 257–266.
- [10] Papadopoulos, C. A., and Dimarogonas, A. D., “Coupled Longitudinal and Bending Vibrations of a Rotating Shaft with an Open Crack,” *Journal of Sound and Vibration*, Vol. 117, No. 1, 1987, pp. 81–93.
- [11] Wauer, J., “On the Dynamics of Cracked Rotors: A Literature Survey,” *Applied Mechanics Reviews*, Vol. 43, No. 1, 1990, pp. 13–17.
- [12] Dimarogonas, A. D., “Vibration of Cracked Structures: A State of the Art Review,” *Engineering Fracture Mechanics*, Vol. 55, No. 5, 1996, pp. 831–857.

- [13] Christides, S., and Barr, A. D. S., “One-Dimensional Theory of Cracked Bernoulli–Euler Beams,” *International Journal of Mechanical Sciences*, Vol. 26, No. 11-12, 1984, pp. 639–648.
- [14] Chasalevris, A. C., and Papadopoulos, C. A., “A Continuous Model Approach for Cross-Coupled Bending Vibrations of a Rotor-Bearing System with a Transverse Breathing Crack,” *Mechanism and Machine Theory*, Vol. 44, No. 6, 2009, pp. 1176–1191.
- [15] Bachschmid, N., Pennacchi, P., and Tanzi, E., *Cracked Rotors: A Survey on Static and Dynamic Behaviour Including Modelling and Diagnosis*, Springer, Berlin, 2010.
- [16] Kalderon, D., “Steam Turbine Failure at Hinkley Point ‘A’,” *Proceedings of the Institution of Mechanical Engineers*, Vol. 186, No. 31, 1972, pp. 341–377.
- [17] Gray, J. L., “Investigation into the Consequences of the Failure of a Turbine-Generator at Hinkley Point ‘A’ Power Station,” *Proceedings of the Institution of Mechanical Engineers*, Vol. 186, No. 31, 1972, pp. 379–390.
- [18] De Forest, D. R., Grobel, L. P., Schabtach, C., and Seguin, B. R., “Investigation of the Generator Rotor Burst at the Pittsburgh Station of the Pacific Gas and Electric Company,” ASME Paper No. 57-PWR-12, 1957.
- [19] Schabtach, C., Fogleman, E. L., Rankin, A. W., and Winne, D. H., “Report of the Investigation of Two Generator Rotor Fractures,” *Transactions of the ASME*, Vol. 78, 1956, pp. 1567–1584.
- [20] Winne, D. H., and Wundt, B. M., “Application of the Griffith–Irwin Theory of Crack Propagation to the Bursting Behaviour of Discs, Including Analytical and Experimental Studies,” *Transactions of the ASME*, Vol. 80, 1958, p. 1643.
- [21] Kumar, C., and Rastogi, V., “A Brief Review on Dynamics of a Cracked Rotor,” *International Journal of Rotating Machinery*, Vol. 2009, Article ID 758108, 2009.
- [22] Gómez, M. J., Corral, E., Castejón, C., and García-Prada, J. C., “Effective Crack Detection in Railway Axles Using Vibration Signals and WPT Energy,” *Sensors*, Vol. 18, No. 5, 2018, Article 1603.
- [23] Bilmes, P., Llorente, C., Echarri, J. M., Echarri, T., Martinez, A., and Zuzulich, J., “Root Cause Analysis of Cracking in Shaft End and Coupling of a High-Power Generator,” *Journal of Failure Analysis and Prevention*, Vol. 18, No. 4, 2018, pp. 727–732.

- [24] Nikraves, M. Y., and MeidanSharaf, M., “Failure of a Steam Turbine Rotor Due to Circumferential Crack Growth Influenced by Temperature and Steady Torsion,” *Engineering Failure Analysis*, Vol. 66, 2016, pp. 296–311.
- [25] Gasch, R., “A Survey of the Dynamic Behaviour of a Simple Rotating Shaft with a Transverse Crack,” *Journal of Sound and Vibration*, Vol. 160, No. 2, 1993, pp. 313–332.
- [26] Mayes, I. W., and Davies, W. G. R., “Analysis of the Response of a Multi-Rotor-Bearing System Containing a Transverse Crack in a Rotor,” *ASME Journal of Vibration, Acoustics, Stress, and Reliability in Design*, Vol. 106, No. 1, 1984, pp. 139–145.
- [27] Jun, O. S., Eun, H. J., Earmme, Y. Y., and Lee, C.-W., “Modelling and Vibration Analysis of a Simple Rotor with a Breathing Crack,” *Journal of Sound and Vibration*, Vol. 155, No. 2, 1992, pp. 273–290.
- [28] Patel, T. H., and Darpe, A. K., “Influence of Crack Breathing Model on Nonlinear Dynamics of a Cracked Rotor,” *Journal of Sound and Vibration*, Vol. 311, No. 3-5, 2008, pp. 953–972.
- [29] Al-Shudeifat, M. A., and Butcher, E. A., “New Breathing Functions for the Transverse Breathing Crack of the Cracked Rotor System: Approach for Critical and Subcritical Harmonic Analysis,” *Journal of Sound and Vibration*, Vol. 330, No. 3, 2011, pp. 526–544.
- [30] Darpe, A. K., Gupta, K., and Chawla, A., “Transient Response and Breathing Behaviour of a Cracked Jeffcott Rotor,” *Journal of Sound and Vibration*, Vol. 272, No. 1-2, 2004, pp. 207–243.
- [31] Sinou, J.-J., and Lees, A. W., “The Influence of Cracks in Rotating Shafts,” *Journal of Sound and Vibration*, Vol. 285, No. 4-5, 2005, pp. 1015–1037.
- [32] Sinou, J.-J., “Detection of Cracks in Rotor Based on the 2X and 3X Super-Harmonic Frequency Components and the Crack–Unbalance Interactions,” *Communications in Nonlinear Science and Numerical Simulation*, Vol. 13, No. 9, 2008, pp. 2024–2040.
- [33] Gómez, M. J., Castejón, C., Corral, E., and García-Prada, J. C., “Analysis of the Influence of Crack Location for Diagnosis in Rotating Shafts Based on 3X Energy,” *Mechanism and Machine Theory*, Vol. 103, 2016, pp. 167–173.
- [34] Muszynska, A., Goldman, P., and Bently, D. E., “Torsional/Lateral Vibration Cross-Coupled Responses Due to Shaft Anisotropy: A New Tool in Shaft Crack

- Detection,” *Proceedings of the IMechE Conference on Vibrations in Rotating Machinery*, 1992, pp. 257–262.
- [35] Guo, C., Al-Shudeifat, M. A., Yan, J., Bergman, L. A., McFarland, D. M., and Butcher, E. A., “Application of Empirical Mode Decomposition to a Jeffcott Rotor with a Breathing Crack,” *Journal of Sound and Vibration*, Vol. 332, No. 16, 2013, pp. 3881–3892.
 - [36] Tama, B. A., Vania, M., Lee, S., and Lim, S., “Recent Advances in the Application of Deep Learning for Fault Diagnosis of Rotating Machinery Using Vibration Signals,” *Artificial Intelligence Review*, Vol. 56, 2023, pp. 4667–4709.
 - [37] Anderson, T. L., *Fracture Mechanics: Fundamentals and Applications*, 3rd ed., CRC Press, Boca Raton, FL, 2005.
 - [38] Janssen, M., Zuidema, J., and Wanhill, R. J. H., *Fracture Mechanics*, 2nd ed., Spon Press, London, 2004.
 - [39] Suresh, S., *Fatigue of Materials*, 2nd ed., Cambridge University Press, Cambridge, 1998.
 - [40] Tada, H., Paris, P. C., and Irwin, G. R., *The Stress Analysis of Cracks Handbook*, 3rd ed., ASME Press, New York, 2000.
 - [41] Friswell, M. I., Penny, J. E. T., Garvey, S. D., and Lees, A. W., *Dynamics of Rotating Machines*, Cambridge University Press, Cambridge, 2010.
 - [42] Papadopoulos, C. A., “The Strain Energy Release Approach for Modeling Cracks in Rotors: A State of the Art Review,” *Mechanical Systems and Signal Processing*, Vol. 22, No. 4, 2008, pp. 763–789.
 - [43] Han, B., Liu, Z., He, P., and Yan, P., “Rotor Crack Breathing under Unbalanced Disturbance,” *Journal of Sound and Vibration*, Vol. 574, 2024, Article 118236, doi:10.1016/j.jsv.2023.118236.
 - [44] Wang, C., Zheng, Z., Guo, D., Liu, T., Xie, Y., and Zhang, D., “An Experimental Setup to Detect the Crack Fault of Asymmetric Rotors Based on a Deep Learning Method,” *Applied Sciences*, Vol. 13, No. 3, 2023, Article 1327, doi:10.3390/app13031327.
 - [45] Xiong, L., Hua, C., Lv, L., Dong, D., and Ouyang, H., “A Model-Based Dynamic Characteristics Analysis of a Coupled Multi-Crack Rotor System,” *Nonlinear Dynamics*, Vol. 110, 2022, pp. 2197–2223, doi:10.1007/s11071-022-07747-2.

Appendix A

Derivation of the Axial, Shear, Torsional, and Coupling Local Flexibility Terms

Section 3.1.4 introduced the complete local crack-flexibility matrix $\mathbf{C}_c^{(c)}$ and noted that the rotor model developed in this thesis retains only the two direct bending compliances $C_{44}^{(c)}$ and $C_{55}^{(c)}$. This appendix completes the local flexibility matrix by deriving the remaining terms—the direct axial, shear, and torsional compliances, and the non-vanishing coupling terms—using the same strain-energy-release and Castigliano procedure applied to the bending terms in Section 3.1.5 and Appendix B. These terms are not used in the reduced rotor model of this thesis; they are provided here for completeness and to make the omission in Section 3.1.6 an explicit and verifiable modelling choice rather than an unstated one.

A.1 Scope and Simplifying Nominal-Stress Assumptions

The bending derivations in the main text used the exact elementary bending-stress distributions $\sigma_4(x) = 4P_4x/(\pi R^4)$ and $\sigma_5(x) = 4P_5\sqrt{R^2 - x^2}/(\pi R^4)$, evaluated at the outer surface point $(x, \sqrt{R^2 - x^2})$ of each strip. The same evaluation point is used here for the remaining sectional loads.

For the axial force $P_1 = N$ and the torque $P_6 = T$, elementary mechanics of materials gives an exact nominal stress at every point of the outer boundary, since every boundary point lies at the same radial distance R from the shaft axis. For the two transverse shear forces $P_2 = V_x$ and $P_3 = V_y$, the exact elasticity solution for the shear-stress distribution over a solid circular section is considerably more involved and is not needed at the level of rigour adopted in this thesis; consistent with common practice in strip-based crack-compliance models, the nominal shear stress is therefore approximated by its cross-sectional average, $\tau = P/A_0$. This approximation affects only the terms $C_{22}^{(c)}$, $C_{33}^{(c)}$, and their coupling with the torque terms; it does not affect $C_{11}^{(c)}$, $C_{66}^{(c)}$, or the axial-bending coupling terms, which are obtained from exact elementary formulas.

Throughout, $A_0 = \pi R^2$ is the cross-sectional area, $I_0 = \pi R^4/4$ is the second moment

of area, and $J_0 = \pi R^4/2$ is the polar second moment of area of the healthy circular section. The nondimensional variables $\bar{x}$, $\bar{y}$, $\lambda = \bar{y}/(2\sqrt{1-\bar{x}^2})$, $\bar{a}_x(\bar{x}) = \sqrt{1-\bar{x}^2} - (1-\mu)$, and $\bar{b} = \sqrt{2\mu-\mu^2}$ are used exactly as defined in Section 3.1.3.

A.2 Direct Axial Flexibility $C_{11}^{(c)}$

The axial force produces a uniform nominal stress over the section, $\sigma_1 = P_1/A_0 = P_1/(\pi R^2)$, independent of the strip position x . The associated Mode-I stress-intensity factor is

$$K_{I1}(x, y) = \frac{P_1}{\pi R^2} \sqrt{\pi y} F_1\left(\frac{y}{h_x}\right). \quad (\text{A.1})$$

Following the same steps as in Appendix B—forming $J_1 = \frac{1-\nu^2}{E} K_{I1}^2$, integrating over the cracked domain, nondimensionalising, and applying Castigliano’s theorem $C_{11}^{(c)} = \partial^2 U_1^{(c)} / \partial P_1^2$ —gives, after using the same even- $\bar{x}$ symmetry reduction to the half-domain,

$$C_{11}^{(c)} = \frac{1-\nu^2}{ER} \hat{c}_{11}(\mu), \quad \hat{c}_{11}(\mu) = \frac{4}{\pi} \int_0^{\sqrt{2\mu-\mu^2}} \int_0^{\sqrt{1-\bar{x}^2}-(1-\mu)} \bar{y} F_1^2(\lambda) d\bar{y} d\bar{x}. \quad (\text{A.2})$$

Because the nominal stress does not depend on x , the integrand no longer carries the $\bar{x}^2$ (or $1-\bar{x}^2$) weighting present in the bending compliances; the compliance depends on the crack depth only through the integration domain and the correction function F_1 . The dimensional scaling $(1-\nu^2)/(ER)$, rather than $(1-\nu^2)/(ER^3)$, reflects the fact that $C_{11}^{(c)}$ is an axial (extensional) compliance rather than a rotational one. As before, $C_{11}^{(c)} \geq 0$ and $C_{11}^{(c)} = 0$ for $\mu = 0$.

A.3 Direct Shear Flexibilities $C_{22}^{(c)}$ and $C_{33}^{(c)}$

Under the uniform-shear-stress approximation of Section A.1, $\tau_2 = P_2/A_0$ and $\tau_3 = P_3/A_0$. Recalling that $P_3 = V_y$ activates Mode II (correction function F_{II}) and $P_2 = V_x$ activates Mode III (correction function F_{III}), and that the Mode-III strain-energy-release rate carries the factor $(1+\nu)/E$ rather than $(1-\nu^2)/E$ (Eq. (3.4)), the same procedure used for $C_{11}^{(c)}$ gives

$$C_{33}^{(c)} = \frac{1-\nu^2}{ER} \hat{c}_{33}(\mu), \quad \hat{c}_{33}(\mu) = \frac{4}{\pi} \int_0^{\sqrt{2\mu-\mu^2}} \int_0^{\sqrt{1-\bar{x}^2}-(1-\mu)} \bar{y} F_{II}^2(\lambda) d\bar{y} d\bar{x}, \quad (\text{A.3})$$

and

$$C_{22}^{(c)} = \frac{1+\nu}{ER} \hat{c}_{22}(\mu), \quad \hat{c}_{22}(\mu) = \frac{4}{\pi} \int_0^{\sqrt{2\mu-\mu^2}} \int_0^{\sqrt{1-\bar{x}^2}-(1-\mu)} \bar{y} F_{\text{III}}^2(\lambda) d\bar{y} d\bar{x}. \quad (\text{A.4})$$

Both compliances are non-negative, vanish for $\mu = 0$, and increase monotonically with the crack-depth ratio, for the same reasons discussed for the bending compliances.

A.4 Direct Torsional Flexibility $C_{66}^{(c)}$

Unlike the transverse shear forces, the torque produces an exact and non-uniform elementary shear-stress field at the outer surface. At the boundary point $(x, \sqrt{R^2 - x^2})$, the torsional shear stress $\tau = Tr/J_0$ (with $r = R$ on the boundary) is directed tangentially to the circular boundary and decomposes into an anti-plane component (Mode III, parallel to the crack front, matching the role of V_x) and an in-plane component (Mode II, in the crack-extension direction, matching the role of V_y):

$$\tau_6^{(\text{II})}(x) = \frac{2P_6 x}{\pi R^4}, \quad \tau_6^{(\text{III})}(x) = \frac{2P_6 \sqrt{R^2 - x^2}}{\pi R^4}. \quad (\text{A.5})$$

This decomposition follows directly from projecting the tangential torsional shear-stress vector onto the local crack-front and crack-extension directions at the boundary point $(x, \sqrt{R^2 - x^2})$, and it reproduces the same x and $\sqrt{R^2 - x^2}$ structure already used for M_y and M_x bending in Appendix B. Substituting Eq. (A.5) into $K_{II6} = \tau_6^{(\text{II})} \sqrt{\pi y} F_{\text{II}}(y/h_x)$ and $K_{III6} = \tau_6^{(\text{III})} \sqrt{\pi y} F_{\text{III}}(y/h_x)$ and repeating the Castigliano procedure of Appendix B for each mode separately (the cross term $2K_{II6}K_{III6}$ does not contribute to $\partial^2/\partial P_6^2$, since K_{II6} and K_{III6} belong to different, additive energy terms in Eq. (3.9)) gives

$$C_{66}^{(c)} = \frac{1-\nu^2}{ER^3} \hat{c}_{66}^{(\text{II})}(\mu) + \frac{1+\nu}{ER^3} \hat{c}_{66}^{(\text{III})}(\mu), \quad (\text{A.6})$$

with

$$\hat{c}_{66}^{(\text{II})}(\mu) = \frac{16}{\pi} \int_0^{\sqrt{2\mu-\mu^2}} \int_0^{\sqrt{1-\bar{x}^2}-(1-\mu)} \bar{x}^2 \bar{y} F_{\text{II}}^2(\lambda) d\bar{y} d\bar{x}, \quad (\text{A.7})$$

$$\hat{c}_{66}^{(\text{III})}(\mu) = \frac{16}{\pi} \int_0^{\sqrt{2\mu-\mu^2}} \int_0^{\sqrt{1-\bar{x}^2}-(1-\mu)} (1-\bar{x}^2) \bar{y} F_{\text{III}}^2(\lambda) d\bar{y} d\bar{x}. \quad (\text{A.8})$$

The torsional compliance is thus the sum of a Mode-II and a Mode-III contribution, each structurally identical to $\hat{c}_{44}(\mu)$ and $\hat{c}_{55}(\mu)$, respectively, with $F_1 \rightarrow F_{\text{II}}$ and $F_2 \rightarrow F_{\text{III}}$ and the corresponding energy-release-rate prefactor.

A.5 Axial–Bending Coupling: $C_{14}^{(c)}$ and $C_{15}^{(c)}$

The coupling terms between the axial force and the two bending moments originate from the cross terms $2K_{I1}K_{I4}$ and $2K_{I1}K_{I5}$ in the expansion of $(K_I^{\text{tot}})^2$. Since $K_{I1} \propto \sigma_1$ is independent of $\bar{x}$ (an even function, trivially) while $K_{I4} \propto \sigma_4(x) \propto \bar{x}$ is an odd function of $\bar{x}$, the mixed integrand of $2K_{I1}K_{I4}$ is odd in $\bar{x}$. By the same parity argument used to show $C_{45}^{(c)} = 0$ in Appendix B, integration over the symmetric domain $-\bar{b} \leq \bar{x} \leq \bar{b}$ gives

$$C_{14}^{(c)} = C_{41}^{(c)} = 0. \quad (\text{A.9})$$

By contrast, $K_{I5} \propto \sigma_5(x) \propto \sqrt{1-\bar{x}^2}$ is an even function of $\bar{x}$, so the product $K_{I1}K_{I5}$ is even and the corresponding integral does not vanish. Carrying out the same Castigliano procedure on the cross term $2K_{I1}K_{I5}$ gives

$$C_{15}^{(c)} = C_{51}^{(c)} = \frac{1-\nu^2}{ER^2} \hat{c}_{15}(\mu), \quad (\text{A.10})$$

$$\begin{aligned} \hat{c}_{15}(\mu) = \frac{16}{\pi} \int_0^{\sqrt{2\mu-\mu^2}} \int_0^{\sqrt{1-\bar{x}^2}-(1-\mu)} \sqrt{1-\bar{x}^2} \bar{y} \\ \times F_1(\lambda) F_2(\lambda) d\bar{y} d\bar{x}. \end{aligned} \quad (\text{A.11})$$

The dimensional scaling $(1-\nu^2)/(ER^2)$, intermediate between the axial scaling $1/(ER)$ and the bending scaling $1/(ER^3)$, is consistent with $C_{15}^{(c)}$ relating an axial force to a bending rotation.

A.6 Shear–Torsion Coupling: $C_{26}^{(c)}$ and $C_{36}^{(c)}$

The same parity argument applies to the coupling between the transverse shear forces and the torque. Since τ_3 is uniform (even in $\bar{x}$) while $\tau_6^{(\text{II})} \propto \bar{x}$ is odd, the cross term $2K_{II3}K_{II6}$ is odd in $\bar{x}$ and integrates to zero:

$$C_{36}^{(c)} = C_{63}^{(c)} = 0. \quad (\text{A.12})$$

Since τ_2 is uniform (even) while $\tau_6^{(\text{III})} \propto \sqrt{1-\bar{x}^2}$ is also even, the cross term $2K_{III2}K_{III6}$ is even in $\bar{x}$ and does not vanish:

$$C_{26}^{(c)} = C_{62}^{(c)} = \frac{1+\nu}{ER^2} \hat{c}_{26}(\mu), \quad (\text{A.13})$$

$$\begin{aligned} \hat{c}_{26}(\mu) = \frac{8}{\pi} \int_0^{\sqrt{2\mu-\mu^2}} \int_0^{\sqrt{1-\bar{x}^2}-(1-\mu)} \sqrt{1-\bar{x}^2} \bar{y} \\ \times F_{\text{III}}^2(\lambda) d\bar{y} d\bar{x}. \end{aligned} \quad (\text{A.14})$$

This asymmetry— $C_{36}^{(c)} = 0$ while $C_{26}^{(c)} \neq 0$ —exactly mirrors the asymmetry found for the axial–bending coupling terms ($C_{14}^{(c)} = 0$, $C_{15}^{(c)} \neq 0$), since $\tau_6^{(\text{II})}$ and $\tau_6^{(\text{III})}$ play, with respect to the uniform shear stresses, the same odd/even roles that σ_4 and σ_5 play with respect to the uniform axial stress.

A.7 Summary and the Complete Local Flexibility Matrix

Collecting the results of this appendix with the direct and coupling bending terms from Section 3.1.5 and Appendix B, the complete local crack-flexibility matrix introduced in Eq. (3.16) is now fully populated:

$$\mathbf{C}_c^{(c)} = \begin{bmatrix} C_{11}^{(c)} & 0 & 0 & 0 & C_{15}^{(c)} & 0 \\ 0 & C_{22}^{(c)} & 0 & 0 & 0 & C_{26}^{(c)} \\ 0 & 0 & C_{33}^{(c)} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44}^{(c)} & 0 & 0 \\ C_{51}^{(c)} & 0 & 0 & 0 & C_{55}^{(c)} & 0 \\ 0 & C_{62}^{(c)} & 0 & 0 & 0 & C_{66}^{(c)} \end{bmatrix}. \quad (\text{A.15})$$

Every term in this matrix is obtained from the same strain-energy-release density and the same Castigliano differentiation used for $C_{44}^{(c)}$ and $C_{55}^{(c)}$ in the main text; only the nominal-stress expression and the associated correction function change from term to term. As stated in Section 3.1.6, the reduced rotor model developed in this thesis retains only $C_{44}^{(c)}$ and $C_{55}^{(c)}$; the terms derived in this appendix confirm that this reduction is a deliberate simplification for a bending-dominated rotor, not an assumption that these terms are identically zero.

Appendix B

Complete Derivation of the Direct Bending Flexibility Coefficients and Bending-Coupling Symmetry

The main text retains the final direct bending compliance expressions used by the rotor model. This appendix provides the complete derivations of $C_{44}^{(c)}$ and $C_{55}^{(c)}$, together with the symmetry argument giving $C_{45}^{(c)} = 0$ in the crack-fixed principal frame.

B.1 Derivation of $C_{44}^{(c)}$

The flexibility coefficient $C_{44}^{(c)}$ is associated with the generalized bending load $P_4 = M_y$. Although P_4 is a bending moment about the local y -axis, it produces bending in the local x -displacement plane; therefore, in the reduced rotor finite-element model, $C_{44}^{(c)}$ is used as the crack-induced rotational compliance associated with the local x -direction bending stiffness.

Retaining only the P_4 -dependent term of the Mode-I stress-intensity factor derived previously,

$$K_{I4}(x, y) = \frac{4P_4x}{\pi R^4} \sqrt{\pi y} F_1\left(\frac{y}{h_x}\right), \quad (\text{B.1})$$

the corresponding local strain-energy-release density is $J_{b,44} = \frac{1-\nu^2}{E} K_{I4}^2(x, y)$, and the crack-induced bending strain energy associated with P_4 is obtained by integrating this density over the cracked domain,

$$U_{b,44}^{(c)} = \int_{-b}^b \int_0^{a_x(x)} J_{b,44}(x, y) dy dx. \quad (\text{B.2})$$

Introducing the nondimensional variables $x = R\bar{x}$, $y = R\bar{y}$, and $\lambda = \bar{y}/(2\sqrt{1-\bar{x}^2})$ used throughout this chapter puts the energy in the form

$$U_{b,44}^{(c)} = \frac{16(1-\nu^2)P_4^2}{\pi ER^3} \int_{-\bar{b}}^{\bar{b}} \int_0^{\bar{a}_x(\bar{x})} \bar{x}^2 \bar{y} F_1^2(\lambda) d\bar{y} d\bar{x}. \quad (\text{B.3})$$

By Castigliano's theorem, $C_{44}^{(c)} = \partial^2 U_{b,44}^{(c)} / \partial P_4^2$. Since the integrand is an even

function of $\bar{x}$, the integral over the symmetric domain $-\bar{b} \leq \bar{x} \leq \bar{b}$ may be replaced by twice the integral over $0 \leq \bar{x} \leq \bar{b}$, where $\bar{b} = \sqrt{2\mu - \mu^2}$ is the nondimensional cracked half-width. This gives the final direct bending flexibility

$$C_{44}^{(c)} = \frac{32(1 - \nu^2)}{\pi ER^3} \int_0^{\sqrt{2\mu - \mu^2}} \int_0^{\sqrt{1 - \bar{x}^2} - (1 - \mu)} \bar{x}^2 \bar{y} \times F_1^2 \left(\frac{\bar{y}}{2\sqrt{1 - \bar{x}^2}} \right) d\bar{y} d\bar{x}. \quad (\text{B.4})$$

For practical numerical evaluation, it is convenient to define the dimensionless integral

$$\hat{c}_{44}(\mu) = \frac{32}{\pi} \int_0^{\sqrt{2\mu - \mu^2}} \int_0^{\sqrt{1 - \bar{x}^2} - (1 - \mu)} \bar{x}^2 \bar{y} \times F_1^2 \left(\frac{\bar{y}}{2\sqrt{1 - \bar{x}^2}} \right) d\bar{y} d\bar{x}, \quad (\text{B.5})$$

so that $C_{44}^{(c)} = \frac{1 - \nu^2}{ER^3} \hat{c}_{44}(\mu)$. This is the expression evaluated numerically to construct the `c44_hat` compliance table used by the rotor model; the adopted half-domain convention gives the coefficient $32/\pi$, consistent with the compliance table used by the rotor model.

Because the integrand is non-negative throughout the cracked domain, $C_{44}^{(c)} \geq 0$, and for an uncracked shaft ($\mu = 0$) the integration domain collapses so that $C_{44}^{(c)} = 0$. As the crack depth increases, the cracked domain expands and the local stress-intensity correction F_1 increases, leading to a larger crack-induced compliance and a corresponding reduction in the local x -direction bending stiffness [41, 42].

B.2 Derivation of $C_{55}^{(c)}$

The second direct bending flexibility coefficient is associated with the generalized bending moment

$$P_5 = M_x. \quad (\text{B.6})$$

Although P_5 is a moment about the local x -axis, it produces bending in the local y -displacement plane. Accordingly, in the reduced rotor finite-element model, $C_{55}^{(c)}$ is used as the crack-induced rotational compliance associated with the local y -direction bending stiffness.

For a strip located at the transverse coordinate x , the distance from the local neutral axis to the outer surface of the circular section is

$$\sqrt{R^2 - x^2}. \quad (\text{B.7})$$

The nominal normal stress generated by P_5 is therefore

$$\sigma_5(x) = \frac{P_5 \sqrt{R^2 - x^2}}{I_0}, \quad (\text{B.8})$$

where

$$I_0 = \frac{\pi R^4}{4}. \quad (\text{B.9})$$

Substitution of the healthy second moment of area gives

$$\sigma_5(x) = \frac{4P_5 \sqrt{R^2 - x^2}}{\pi R^4}. \quad (\text{B.10})$$

The corresponding Mode-I stress-intensity factor is obtained by combining this nominal stress with the local crack-depth coordinate y and the correction function F_2 :

$$K_{I5}(x, y) = \sigma_5(x) \sqrt{\pi y} F_2 \left(\frac{y}{h_x} \right). \quad (\text{B.11})$$

Hence,

$$K_{I5}(x, y) = \frac{4P_5 \sqrt{R^2 - x^2}}{\pi R^4} \sqrt{\pi y} F_2 \left(\frac{y}{h_x} \right). \quad (\text{B.12})$$

Squaring this expression gives

$$K_{I5}^2(x, y) = \frac{16P_5^2(R^2 - x^2)}{\pi R^8} y F_2^2 \left(\frac{y}{h_x} \right). \quad (\text{B.13})$$

Retaining only the energy contribution associated with P_5 , the local Mode-I strain-energy-release density becomes

$$J_{b,55}(x, y) = \frac{1 - \nu^2}{E} \frac{16P_5^2(R^2 - x^2)}{\pi R^8} y F_2^2 \left(\frac{y}{h_x} \right). \quad (\text{B.14})$$

The crack-induced bending strain energy associated with this loading case is therefore

$$U_{b,55}^{(c)} = \int_{-b}^b \int_0^{a_x(x)} J_{b,55}(x, y) dy dx. \quad (\text{B.15})$$

Using the nondimensional substitutions

$$x = R\bar{x}, \quad y = R\bar{y}, \quad dx dy = R^2 d\bar{x} d\bar{y}, \quad (\text{B.16})$$

together with

$$R^2 - x^2 = R^2(1 - \bar{x}^2), \quad (\text{B.17})$$

the energy-density contribution becomes

$$J_{b,55}(\bar{x}, \bar{y}) = \frac{16(1 - \nu^2)P_5^2}{\pi ER^5}(1 - \bar{x}^2)\bar{y} F_2^2(\lambda), \quad (\text{B.18})$$

where

$$\lambda = \frac{\bar{y}}{2\sqrt{1 - \bar{x}^2}}. \quad (\text{B.19})$$

The resulting crack-induced bending strain energy is

$$U_{b,55}^{(c)} = \frac{16(1 - \nu^2)P_5^2}{\pi ER^3} \int_{-\bar{b}}^{\bar{b}} \int_0^{\bar{a}_x(\bar{x})} (1 - \bar{x}^2)\bar{y} F_2^2(\lambda) d\bar{y} d\bar{x}. \quad (\text{B.20})$$

By Castigliano's theorem, the corresponding direct flexibility is

$$C_{55}^{(c)} = \frac{\partial^2 U_{b,55}^{(c)}}{\partial P_5^2}. \quad (\text{B.21})$$

Since

$$\frac{\partial^2 P_5^2}{\partial P_5^2} = 2, \quad (\text{B.22})$$

it follows that

$$C_{55}^{(c)} = \frac{32(1 - \nu^2)}{\pi ER^3} \int_{-\bar{b}}^{\bar{b}} \int_0^{\bar{a}_x(\bar{x})} (1 - \bar{x}^2)\bar{y} F_2^2(\lambda) d\bar{y} d\bar{x}. \quad (\text{B.23})$$

The functions

$$\bar{a}_x(\bar{x}) = \sqrt{1 - \bar{x}^2} - (1 - \mu), \quad (\text{B.24})$$

$$\lambda(\bar{x}, \bar{y}) = \frac{\bar{y}}{2\sqrt{1 - \bar{x}^2}}, \quad (\text{B.25})$$

and $F_2(\lambda)$ are even functions of $\bar{x}$. The factor $1 - \bar{x}^2$ is also even. Therefore, the full integrand is symmetric about $\bar{x} = 0$, allowing the domain to be reduced to its positive half:

$$\int_{-\bar{b}}^{\bar{b}} (\cdot) d\bar{x} = 2 \int_0^{\bar{b}} (\cdot) d\bar{x}. \quad (\text{B.26})$$

With

$$\bar{b} = \sqrt{2\mu - \mu^2}, \quad (\text{B.27})$$

the final direct bending flexibility is

$$C_{55}^{(c)} = \frac{64(1-\nu^2)}{\pi ER^3} \int_0^{\sqrt{2\mu-\mu^2}} \int_0^{\sqrt{1-\bar{x}^2}-(1-\mu)} (1-\bar{x}^2)\bar{y} \times F_2^2\left(\frac{\bar{y}}{2\sqrt{1-\bar{x}^2}}\right) d\bar{y} d\bar{x}. \quad (\text{B.28})$$

For numerical implementation, the dimensionless quantity

$$\hat{c}_{55}(\mu) = \frac{64}{\pi} \int_0^{\sqrt{2\mu-\mu^2}} \int_0^{\sqrt{1-\bar{x}^2}-(1-\mu)} (1-\bar{x}^2)\bar{y} \times F_2^2\left(\frac{\bar{y}}{2\sqrt{1-\bar{x}^2}}\right) d\bar{y} d\bar{x} \quad (\text{B.29})$$

is evaluated first. The dimensional crack-induced flexibility is then obtained from

$$C_{55}^{(c)} = \frac{1-\nu^2}{ER^3} \hat{c}_{55}(\mu). \quad (\text{B.30})$$

The integrand is non-negative over the entire cracked domain. Hence,

$$C_{55}^{(c)} \geq 0. \quad (\text{B.31})$$

For the uncracked condition, $\mu = 0$, the integration domain vanishes and therefore

$$C_{55}^{(c)} = 0. \quad (\text{B.32})$$

As the crack depth increases, both the extent of the integration domain and the local effect of the correction function F_2 increase. Consequently, the crack-induced flexibility $C_{55}^{(c)}$ increases, producing a progressive reduction in the local y -direction bending stiffness [41, 42].

B.3 Symmetry of $C_{45}^{(c)}$

The two direct flexibility coefficients derived for $C_{44}^{(c)}$ (Appendix B) and $C_{55}^{(c)}$ (Appendix B) quantify the additional rotational flexibility produced independently by the bending moments $P_4 = M_y$ and $P_5 = M_x$. The complete local flexibility matrix may also contain a bending-coupling coefficient,

$$C_{45}^{(c)} = C_{54}^{(c)}, \quad (\text{B.33})$$

which measures the additional rotation associated with one bending moment that is induced by the other bending moment.

This coefficient originates from the mixed term in the bending strain-energy-release density:

$$2K_{I4}K_{I5}. \quad (\text{B.34})$$

Using the stress-intensity-factor expressions previously derived for the two bending moments gives

$$2K_{I4}K_{I5} = \frac{32P_4P_5x\sqrt{R^2-x^2}}{\pi R^8} y F_1\left(\frac{y}{h_x}\right) F_2\left(\frac{y}{h_x}\right). \quad (\text{B.35})$$

Accordingly, the mixed contribution to the local bending strain-energy-release density is

$$J_{b,45}(x, y) = \frac{1-\nu^2}{E} \frac{32P_4P_5x\sqrt{R^2-x^2}}{\pi R^8} y F_1\left(\frac{y}{h_x}\right) F_2\left(\frac{y}{h_x}\right). \quad (\text{B.36})$$

Using the nondimensional variables

$$x = R\bar{x}, \quad y = R\bar{y}, \quad dx dy = R^2 d\bar{x} d\bar{y}, \quad (\text{B.37})$$

the corresponding nondimensional energy-density term becomes

$$J_{b,45}(\bar{x}, \bar{y}) = \frac{32(1-\nu^2)P_4P_5}{\pi ER^5} \bar{x}\sqrt{1-\bar{x}^2} \bar{y} F_1(\lambda) F_2(\lambda), \quad (\text{B.38})$$

where

$$\lambda = \frac{\bar{y}}{2\sqrt{1-\bar{x}^2}}. \quad (\text{B.39})$$

The mixed crack-induced bending energy is therefore

$$U_{b,45}^{(c)} = \frac{32(1-\nu^2)P_4P_5}{\pi ER^3} \int_{-\bar{b}}^{\bar{b}} \int_0^{\bar{a}_x(\bar{x})} \bar{x}\sqrt{1-\bar{x}^2} \bar{y} \times F_1(\lambda) F_2(\lambda) d\bar{y} d\bar{x}. \quad (\text{B.40})$$

By Castigliano's theorem, the bending-coupling flexibility is

$$C_{45}^{(c)} = \frac{\partial^2 U_{b,45}^{(c)}}{\partial P_4 \partial P_5}. \quad (\text{B.41})$$

Since

$$\frac{\partial^2 (P_4 P_5)}{\partial P_4 \partial P_5} = 1, \quad (\text{B.42})$$

it follows that

$$C_{45}^{(c)} = \frac{32(1-\nu^2)}{\pi ER^3} \int_{-\bar{b}}^{\bar{b}} \int_0^{\bar{a}_x(\bar{x})} \bar{x}\sqrt{1-\bar{x}^2} \bar{y} \times F_1(\lambda) F_2(\lambda) d\bar{y} d\bar{x}. \quad (\text{B.43})$$

The value of this integral is determined by symmetry. The factors

$$\sqrt{1 - \bar{x}^2}, \quad \bar{a}_x(\bar{x}), \quad \lambda(\bar{x}, \bar{y}), \quad F_1(\lambda), \quad F_2(\lambda) \quad (\text{B.44})$$

are all even functions of $\bar{x}$. The factor $\bar{y}$ is independent of the sign of $\bar{x}$. However, the factor $\bar{x}$ is odd. Therefore, the complete integrand is an odd function of $\bar{x}$:

$$f(-\bar{x}, \bar{y}) = -f(\bar{x}, \bar{y}). \quad (\text{B.45})$$

Because the crack domain is symmetric about $\bar{x} = 0$,

$$-\bar{b} \leq \bar{x} \leq \bar{b}, \quad (\text{B.46})$$

the contributions from the two sides of the cracked section cancel exactly. Hence,

$$C_{45}^{(c)} = 0. \quad (\text{B.47})$$

From the symmetry of the flexibility matrix,

$$C_{54}^{(c)} = C_{45}^{(c)} = 0. \quad (\text{B.48})$$

This result is not an additional approximation. It follows directly from the assumed crack geometry and the selected local principal axes. The transverse crack is symmetric about the local y -axis, while the bending moments $P_4 = M_y$ and $P_5 = M_x$ generate stress fields with opposite parity along the crack front. Their energetic interaction therefore vanishes when integrated over the complete symmetric crack domain [41, 42].

Consequently, in the crack-fixed principal coordinate system, the bending-related part of the local flexibility matrix reduces to

$$\mathbf{C}_b^{(c)} = \begin{bmatrix} C_{44}^{(c)} & 0 \\ 0 & C_{55}^{(c)} \end{bmatrix}. \quad (\text{B.49})$$

The absence of $C_{45}^{(c)}$ does not imply that all lateral coupling disappears from the complete rotating-rotor problem. Once the anisotropic local crack stiffness is transformed from the rotating crack-fixed frame to the stationary frame, time-dependent cross-coupling terms may arise through coordinate transformation. The present result states only that the local crack compliance matrix is diagonal when expressed in its principal symmetry axes.

Appendix C

Explicit Hermite Shape Functions and Full Derivation of the Healthy Shaft-Element Stiffness Matrix

Section 3.2.2 presented the final Euler–Bernoulli one-plane stiffness matrix $\mathbf{K}_b$ and the assembled 8×8 healthy shaft-element stiffness matrix $\mathbf{K}_e^{(0)}$, and noted that the explicit cubic Hermite shape functions, the integral evaluation, and the plane-wise-to-nodal permutation are given in full here for completeness and verification.

C.1 Explicit Cubic Hermite Shape Functions

Consider a straight prismatic shaft element of length ℓ_e , Young’s modulus E_e , and second moment of area I_e . For one bending plane, the transverse displacement field is interpolated using cubic Hermite shape functions. Introducing the dimensionless local coordinate

$$\xi = \frac{z}{\ell_e}, \quad 0 \leq \xi \leq 1, \quad (\text{C.1})$$

the displacement field is written as

$$w(z, t) = \mathbf{N}(\xi) \mathbf{q}_b(t), \quad (\text{C.2})$$

where

$$\mathbf{q}_b = \begin{bmatrix} w_i & \vartheta_i & w_j & \vartheta_j \end{bmatrix}^T, \quad (\text{C.3})$$

and

$$\mathbf{N}(\xi) = \begin{bmatrix} N_1(\xi) & N_2(\xi) & N_3(\xi) & N_4(\xi) \end{bmatrix}. \quad (\text{C.4})$$

The Hermite interpolation functions are

$$N_1(\xi) = 1 - 3\xi^2 + 2\xi^3, \quad (\text{C.5})$$

$$N_2(\xi) = \ell_e \left(\xi - 2\xi^2 + \xi^3 \right), \quad (\text{C.6})$$

$$N_3(\xi) = 3\xi^2 - 2\xi^3, \quad (\text{C.7})$$

and

$$N_4(\xi) = \ell_e \left(-\xi^2 + \xi^3 \right). \quad (\text{C.8})$$

These functions satisfy the displacement and slope boundary conditions at the two element nodes. Therefore, the interpolation field reproduces both the transverse displacement and the bending slope at $z = 0$ and $z = \ell_e$.

C.2 Integration of the One-Plane Stiffness Matrix

The bending curvature is obtained by differentiating the displacement field twice with respect to the local axial coordinate:

$$\kappa(z, t) = \frac{\partial^2 w}{\partial z^2} = \mathbf{B}_\kappa(z) \mathbf{q}_b(t), \quad (\text{C.9})$$

with

$$\mathbf{B}_\kappa(z) = \frac{d^2 \mathbf{N}(z)}{dz^2}. \quad (\text{C.10})$$

The elastic strain energy in one bending plane is consequently

$$U_b = \frac{1}{2} \int_0^{\ell_e} E_e I_e \kappa^2(z, t) dz. \quad (\text{C.11})$$

After substituting the finite-element interpolation, the strain energy becomes

$$U_b = \frac{1}{2} \mathbf{q}_b^T \left[\int_0^{\ell_e} \mathbf{B}_\kappa^T E_e I_e \mathbf{B}_\kappa dz \right] \mathbf{q}_b. \quad (\text{C.12})$$

Accordingly, the one-plane bending stiffness matrix is

$$\mathbf{K}_b = \int_0^{\ell_e} \mathbf{B}_\kappa^T E_e I_e \mathbf{B}_\kappa dz. \quad (\text{C.13})$$

Carrying out the differentiation of Eqs. (C.5)–(C.8) twice with respect to $z = \ell_e \xi$ and evaluating the integral term by term gives the conventional Euler–Bernoulli beam stiffness matrix:

$$\mathbf{K}_b = \frac{E_e I_e}{\ell_e^3} \begin{bmatrix} 12 & 6\ell_e & -12 & 6\ell_e \\ 6\ell_e & 4\ell_e^2 & -6\ell_e & 2\ell_e^2 \\ -12 & -6\ell_e & 12 & -6\ell_e \\ 6\ell_e & 2\ell_e^2 & -6\ell_e & 4\ell_e^2 \end{bmatrix}. \quad (\text{C.14})$$

For the healthy shaft, the same stiffness matrix applies in the two transverse bending planes because the circular cross-section has equal second moments of area:

$$I_x = I_y = I_e. \quad (\text{C.15})$$

C.3 Plane-Wise-to-Nodal Permutation

The bending degrees of freedom in the x -direction are

$$\mathbf{q}_{b,x} = [u_i \quad \psi_i \quad u_j \quad \psi_j]^T, \quad (\text{C.16})$$

whereas those in the y -direction are

$$\mathbf{q}_{b,y} = [v_i \quad \theta_i \quad v_j \quad \theta_j]^T. \quad (\text{C.17})$$

In the finite-element convention adopted in the MATLAB model, ψ and θ are the signed slope variables associated with the displacement fields $u(z, t)$ and $v(z, t)$, respectively. The plane-wise element vector $\tilde{\mathbf{q}}_e$ and the plane-wise stiffness matrix $\tilde{\mathbf{K}}_e^{(0)}$ are as given in Eqs. (3.28) and (3.29) of the main text.

To recover the nodal degree-of-freedom ordering used in the global rotor model,

$$\mathbf{q}_e = [u_i \quad v_i \quad \theta_i \quad \psi_i \quad u_j \quad v_j \quad \theta_j \quad \psi_j]^T, \quad (\text{C.18})$$

the plane-wise stiffness matrix is rearranged through the fixed permutation matrix $\mathbf{P}$, defined by

$$\tilde{\mathbf{q}}_e = \mathbf{P} \mathbf{q}_e. \quad (\text{C.19})$$

The matrix $\mathbf{P}$ is an 8×8 Boolean permutation matrix (each row and column contains exactly one entry equal to 1, all other entries zero) that maps the nodal ordering in Eq. (C.18) onto the plane-wise ordering of Eq. (3.28); explicitly, $\mathbf{P}$ places $u_i \rightarrow 1$, $\psi_i \rightarrow 2$, $u_j \rightarrow 3$, $\psi_j \rightarrow 4$, $v_i \rightarrow 5$, $\theta_i \rightarrow 6$, $v_j \rightarrow 7$, $\theta_j \rightarrow 8$. Since $\mathbf{P}$ is a permutation matrix, $\mathbf{P}^{-1} = \mathbf{P}^T$, and the congruence transformation

$$\mathbf{K}_e^{(0)} = \mathbf{P}^T \tilde{\mathbf{K}}_e^{(0)} \mathbf{P} \quad (\text{C.20})$$

simply reorders the rows and columns of $\tilde{\mathbf{K}}_e^{(0)}$ without altering its numerical entries,

and preserves symmetry and positive semi-definiteness. Carrying out this reordering explicitly reproduces the 8×8 healthy shaft-element stiffness matrix $\mathbf{K}_e^{(0)}$ given in Eq. (3.30) of the main text. The identical permutation matrix $\mathbf{P}$ is used in Section 3.2.3 to reorder the plane-wise mass matrices into the same nodal ordering.

Appendix D

FFT-Based Harmonic Analysis of the Rotor Response

D.1 Purpose and Input Record

This appendix documents the signal-processing sequence used to extract the $1X$, $2X$, and $3X$ components from the simulated disk response. The input consists of the uniformly sampled dynamic displacements $x[n]$ and $y[n]$ over the final $N_{\text{rev,FFT}}$ complete rotor revolutions. The sampling interval is the fixed Runge–Kutta step Δt , and the sampling frequency is

$$f_s = \frac{1}{\Delta t}. \quad (\text{D.1})$$

If the retained record contains N samples, its duration is approximately $T_{\text{FFT}} = N\Delta t$, and its frequency spacing is

$$\Delta f = \frac{f_s}{N} \approx \frac{1}{T_{\text{FFT}}}. \quad (\text{D.2})$$

A longer retained record therefore gives finer frequency resolution, whereas the time step controls the highest representable frequency through the Nyquist limit $f_{\text{Nyq}} = f_s/2$.

D.2 Mean Removal

The dynamic response may contain a small residual mean caused by the selected reference configuration and finite-window averaging. Before spectral analysis, the sample mean is removed:

$$\bar{x} = \frac{1}{N} \sum_{n=0}^{N-1} x[n], \quad x_0[n] = x[n] - \bar{x}, \quad (\text{D.3})$$

with the analogous definition for $y_0[n]$. This operation suppresses the zero-frequency component without changing the amplitudes of the oscillatory order components.

D.3 Discrete Fourier Transform and FFT Algorithm

For a uniformly sampled record $x_0[n]$, the discrete Fourier transform is

$$X[k] = \sum_{n=0}^{N-1} x_0[n] \exp\left(-j\frac{2\pi kn}{N}\right), \quad k = 0, 1, \dots, N-1. \quad (\text{D.4})$$

The FFT is an efficient algorithm for evaluating the same DFT; it does not define a different transform. The frequency corresponding to bin k is $f_k = kf_s/N$ for the positive-frequency part of the spectrum.

D.4 Spectral Leakage and the Hann Window

A finite record is equivalent to multiplying an indefinitely long signal by a finite observation window. If the first and last samples do not join smoothly, the resulting endpoint discontinuity spreads energy from a discrete component into neighbouring frequency bins; this effect is known as spectral leakage. The implementation reduces leakage with the Hann window

$$w[n] = \frac{1}{2} \left[1 - \cos\left(\frac{2\pi n}{N-1}\right) \right], \quad n = 0, 1, \dots, N-1. \quad (\text{D.5})$$

The windowed records are

$$x_w[n] = x_0[n]w[n], \quad y_w[n] = y_0[n]w[n]. \quad (\text{D.6})$$

The Hann window strongly reduces sidelobes at the cost of widening the main spectral lobe. It is therefore useful for distinguishing the dominant order components from smaller neighbouring content, provided that its attenuation of sinusoidal amplitudes is corrected.

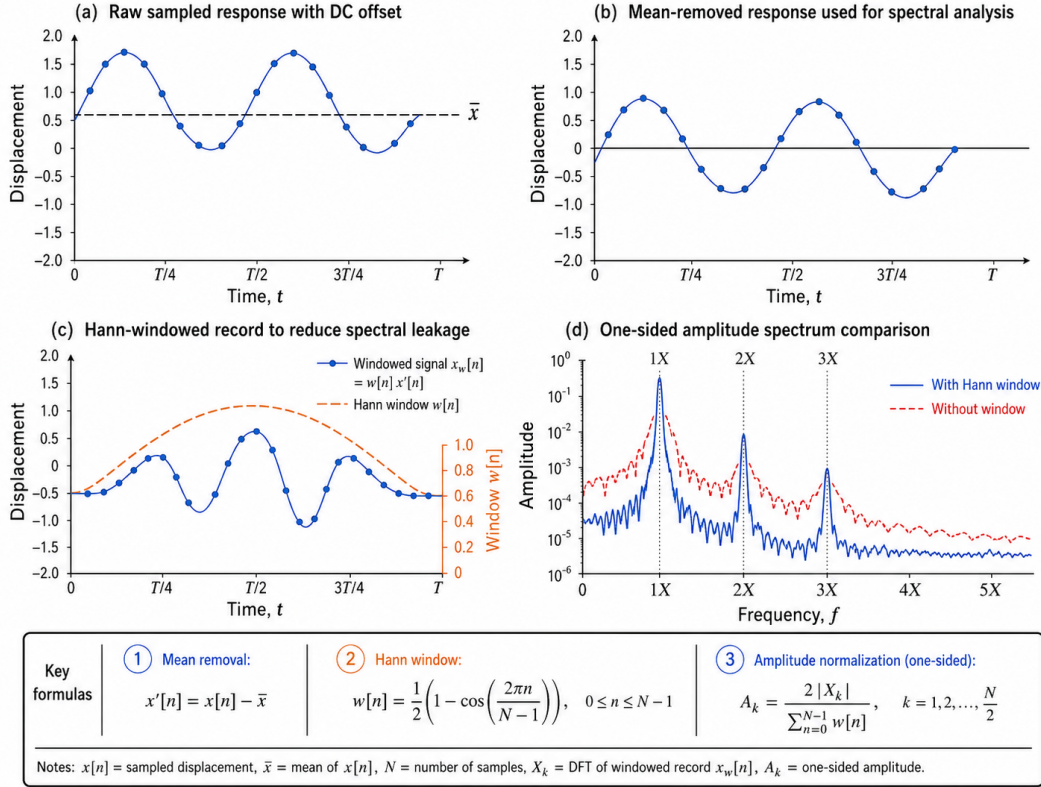


Figure D.1: Conceptual effect of Hann windowing on the endpoints of a finite record and on spectral leakage.

D.5 One-Sided Spectrum and Coherent-Gain Correction

For real-valued signals, the negative-frequency DFT coefficients are the complex conjugates of the corresponding positive-frequency coefficients. The diagnostic spectrum can therefore be represented by the non-negative-frequency half,

$$k = 0, 1, \dots, \left\lfloor \frac{N}{2} \right\rfloor. \quad (\text{D.7})$$

Windowing reduces the amplitude of a sinusoidal component by the coherent gain of the selected window. In the implemented normalisation, this attenuation is corrected using the sum of the window samples:

$$A_x[k] = \frac{2 |X_w[k]|}{\sum_{n=0}^{N-1} w[n]}, \quad A_y[k] = \frac{2 |Y_w[k]|}{\sum_{n=0}^{N-1} w[n]}. \quad (\text{D.8})$$

The factor of two transfers the energy represented by the omitted negative-frequency half to the displayed positive-frequency amplitudes. The zero-frequency component has already been suppressed by mean removal and is not used as a diagnostic order component.

D.6 Integer-Revolution Record and Order Frequencies

The selected FFT record contains an integer number of complete rotor revolutions. Since the fixed integration step also gives an integer number of samples per revolution, the same shaft angles are sampled during every retained cycle. At rotational frequency

$$f_r = \frac{\Omega}{2\pi} = \frac{\Omega_{\text{RPM}}}{60}, \quad (\text{D.9})$$

the principal diagnostic orders are

$$f_{1X} = f_r, \quad f_{2X} = 2f_r, \quad f_{3X} = 3f_r. \quad (\text{D.10})$$

Retaining complete revolutions improves the compatibility between the observation window and order-periodic content. The Hann window is nevertheless retained because the simulated response can contain transient remnants, amplitude modulation, and components that do not fall exactly at an FFT bin.

D.7 Extraction of the Harmonic Amplitudes

For each order $p = 1, 2, 3$, the code identifies the positive-frequency bin closest to pf_r ,

$$k_{pX} = \arg \min_k |f_k - pf_r|, \quad (\text{D.11})$$

and reports

$$A_{x,pX} = A_x[k_{pX}], \quad A_{y,pX} = A_y[k_{pX}]. \quad (\text{D.12})$$

The same procedure is applied to every healthy and cracked case. Consequently, the comparative $1X$, $2X$, and $3X$ values in Chapter 5 are obtained with identical sampling, windowing, amplitude correction, and bin-selection rules.

D.8 Interpretation

The FFT amplitudes quantify periodic components present in the selected late-stage response window. A strong $1X$ component is expected from synchronous unbalance, while the periodic stiffness modulation of the breathing crack generates higher-order content, particularly at $2X$ and $3X$. These components are interpreted together with modal changes and orbit morphology; by themselves, they are not unique identifiers of a transverse crack because other rotor faults can produce similar harmonic orders.