



**POLITECNICO
DI TORINO**

POLITECNICO DI TORINO

Master Degree course in Aerospace Engineering

Master Degree Thesis

Indirect Trajectory Optimization and Mission Profile Design for a Dual-Propulsion Active Debris Removal Spacecraft

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Abstract

The large amount of space debris currently in Earth orbit raises particular concerns regarding the current and, above all, future safety of space activities. Some future forecasts suggest that, even if we immediately stopped creating debris, this would not be enough to prevent a series of uncontrolled collisions. This situation is certainly not helped by the ever-increasing intensity of space traffic in recent years. Therefore, in addition to managing the end-of-life of active satellites and developing advanced collision avoidance systems, it is necessary to actively remove the most potentially dangerous objects and maintain the most critical Earth orbits.

This thesis addresses the problem of Active Debris Removal, studying the optimal trajectories and mission profile of a cleaner satellite that focuses on debris trapped in orbits around 800 km. In this region, natural reentry is virtually non-existent due to the almost total absence of atmospheric drag, thereby drastically increasing the danger it poses. To optimize fuel consumption without compromising mission times, a dual propulsion system has been selected and preliminary sized, comprising a Hall-effect electric thruster and a chemical thruster fueled by LMP-103S green monopropellant. The electric propulsion system will handle the approach phase to the debris target or targets (up to a total maximum of 100 kg), while the chemical counterpart will handle the more demanding perigee lowering and raising maneuvers.

The optimal trajectories for both propulsion systems were derived using an indirect method based on Pontryagin's Maximum Principle with a bang-bang control law, aimed at minimizing propellant consumption. To this end, the OCULUS tool was utilized. The optimized descent releases the debris at a perigee of 300 km, a very conservative choice that ensures that it can deorbit naturally in less than 5 years (in accordance with the latest space agency guidelines), aided by drag, while the satellite continues its cyclical cleanup mission.

The results obtained at the end of this iterative optimization process demonstrate that a 918 kg satellite with the aforementioned characteristics can remove up to 400 kg of debris from an orbit at an altitude of 800 km, with potentially higher capacities at lower altitudes. Finally, in a future perspective, a quantity of residual propellant was factored at the end of the mission to enable the spacecraft to reach an in-orbit refueling station, thus maximising its potential.

Contents

List of Figures	4
List of Tables	5
1 Introduction	8
1.1 Preface	8
1.2 The Space Debris Problem	8
1.2.1 Active Debris Removal	12
1.3 Thesis Objectives and Contributions	13
1.4 Structure of the Thesis	13
2 Dynamic Model	15
2.1 Reference Systems	15
2.1.1 Coordinate Transformation and State Vector Initialization	17
2.2 Equations of Motion in Spherical Coordinates	18
2.3 Perturbing Accelerations	19
2.3.1 Atmospheric Drag Model	20
3 Optimal Control Theory	23
3.1 Direct vs. Indirect Methods in Trajectory Optimization	23
3.2 Fundamentals of Optimal Control Theory	25
3.2.1 Optimality and Transversality Boundary Conditions	27
3.2.2 Pontryagin's Principle and TPBVP Formulation	28
3.3 Multi-Point Optimal Control Problem Formulation	29
3.4 Numerical Implementation of the Multi-Phase BVP	31
3.5 Formulation of the Spacecraft Optimal Control Problem	35
3.6 The OCULUS Computational Solver	38
4 Deorbiting Preliminary Analysis and Spacecraft Budget	40
4.1 Preliminary Deorbiting Analysis and Operational Orbit Selection	40
4.1.1 Debris Release Orbit Selection	43
4.2 Propulsion Systems Characterization	44
4.2.1 Hall-Effect Electric Propulsion	45
4.2.2 High Performance Green Monopropellant	47
4.3 Spacecraft Propulsion Systems Mass Budget	49

4.3.1	Mass Budget of the Hall-Effect Thruster System	49
4.3.2	Mass Budget of HPGP Thruster System	50
4.3.3	Total mass budget of Propulsion System	51
4.4	Spacecraft Final Mass	51
5	Mission Profile and Optimal Trajectory Solutions	53
5.1	Problem Setup and Normalization	53
5.2	Phase I: Low-Thrust Target Approach	56
5.2.1	Numerical Results and Trajectory Analysis	58
5.3	Phase II: High-Thrust Perigee Lowering	61
5.3.1	Numerical Results and Trajectory Analysis	63
5.4	Phase III: Orbit Recircularization	67
5.4.1	Numerical Results and Trajectory Analysis	69
5.5	Multi-Cycle Performance and Lifetime Analysis	71
5.5.1	Second-Cycle Analysis	72
5.5.2	Subsequent Cycles	75
6	Conclusions	79
6.1	Future Perspectives	80
A	Euler-Lagrange Equations for the Adjoint Variables	84

List of Figures

1.1	Payload Launch Traffic	9
1.2	Evolution of the number of objects across all orbits	10
1.3	Number of orbital objects by perigee altitude.	10
1.4	Number of cumulative collisions in LEO in the simulated scenarios of long-term evolution of the environment.	12
2.1	EME2000 $\{\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}}\}$, perifocal $\{\hat{\mathbf{p}}_{\text{SC}}, \hat{\mathbf{q}}_{\text{SC}}, \hat{\mathbf{w}}_{\text{SC}}\}$, and ZEN $\{\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}\}$ RSs . .	16
2.2	Density variation as a function of altitude	20
2.3	Solar cycle variations and the adopted maximum and minimum reference values.	22
4.1	Deorbit time as a function of debris diameter	41
4.2	Deorbit time as a function of debris mass	42
4.3	Orbital decay trajectory	44
4.4	Hall-effect Thruster schematic	46
4.5	Safran PPS [®] 1350 HET	46
4.6	22 N HPGP Thruster	48
5.1	Evolution of state and costate variables for the continuous low-thrust perigee-lowering trajectory	60
5.2	3D trajectory plot of the low-thrust perigee-lowering maneuver	61
5.3	Evolution of state and co-state variables for the chemical perigee-lowering trajectory	64
5.4	3D trajectory plot of the chemical perigee-lowering maneuver	65
5.5	Evolution of state and costate variables for the constrained $\lambda_{\vartheta}(t_f) = 0$ trajectory.	66
5.6	3D trajectory plot for the constrained $\lambda_{\vartheta}(t_f) = 0$ perigee-lowering maneuver	67
5.7	Evolution of the state and costate variables for the chemical recircularization trajectory	70
5.8	3D trajectory plot of the chemical recircularization maneuver	71
5.9	Optimal 2-cycle Hall-effect maneuver	72

List of Tables

4.1	Decay times expressed in years for different debris mass and diameter configurations as a function of the initial circular operational altitude. . .	43
4.2	Technical specifications of the Safran PPS®1350 Hall-effect thruster. . . .	47
4.3	Technical specifications of the High Green Propulsion (HPGP) thruster. .	48
4.4	Mass budget for the electric propulsion system.	50
4.5	Mass budget for the chemical propulsion system.	51
4.6	Overall propulsion system mass budget.	51
5.1	Non-dimensionalized final BCs for the low-thrust perigee-lowering trajectory	58
5.2	Maneuver performance parameters scaled per kilometer of perigee lowering.	59
5.3	Non-dimensionalized final BCs for the chemical perigee-lowering trajectory	63
5.4	Non-dimensionalized final BCs for the chemical circularization maneuver.	68
5.5	Comparison of the low-thrust maneuver performance between Cycle 1 and Cycle 2.	73
5.6	Performance and costate comparison for the chemical propulsion maneuvers (Maneuver 2 and Maneuver 3) between Cycle 1 and Cycle 2.	74
5.7	Comparison of the low-thrust maneuver performance across all operational cycles (Cycles 1 to 4).	75
5.8	Comparison of the second maneuver performance across all operational cycles (Cycles 1 to 4).	76
5.9	Comparison of the third maneuver performance across the completed operational cycles (Cycles 1 to 3).	77

List of Acronyms

ADN	Ammonium Dinitramide
ADR	Active Debris Removal
BC	Boundary Condition
BVP	Boundary Value Problem
CoV	Calculus of Variations
DC	Differential Corrector
ECSS	European Cooperation for Space Standardization
ESA	European Space Agency
GNC	Guidance, Navigation, and Control
GUI	Graphical User Interface
HET	Hall-Effect Thruster
HPGP	High Performance Green Propulsion
IM	Indirect Method
LEO	Low Earth Orbit
MPBVP	Multi-Point Boundary Value Problem
OCP	Optimal Control Problem
OCT	Optimal Control Theory
OTS	Off-The-Shelf
PMP	Pontryagin's Minimum Principle
PPU	Power Processing Unit
RAAN	Right Ascension of the Ascending Node
RF	Reference Frame
RS	Reference System
SC	Spacecraft

STM	State Transition Matrix
TPBVP	Two-Point Boundary Value Problem
ZEN	Zenith-East-North

Chapter 1

Introduction

1.1 Preface

"If you love space, you must love clean space"

With these words, Luisa Innocenti, ESA's Head of Clean Space, addressed one of the most critical topics in the space sector in recent years. Indeed, since the dawn of space exploration, beginning with the revolutionary launch of the first artificial satellite, Sputnik 1, and for the following decades, space was perceived as an infinite and indestructible resource. Consequently, numerous efforts and launches were carried out without evaluating their long-term consequences. Over time, and most notably in recent years due to the exponential increase in space traffic and the commercial exploitation of Earth's orbits, the consequences of this oversight have begun to manifest as critical operational risks. Tons of defunct mass, known as space debris, now crowd Earth's orbits. These objects consist of launcher upper stages, defunct or failed satellites, and fragments generated by explosions or orbital collisions. If active and prompt measures are not taken to counter this phenomenon, a cascade of collision events could occur, irreparably damaging the space environment and the population of satellites with severe repercussions on our daily quality of life. Space technologies provide services that are now essential to modern society, such as telecommunications, internet connectivity, GPS navigation, and weather forecasting, all of which have been crucial to human progress over the last 70 years. Therefore, recognizing that the preservation of our planet must not be limited to the biosphere but must extend to the surrounding space environment is the first step toward safeguarding the Earth from our own actions. This thesis aims to contribute to the field of active debris removal (ADR) to remedy past errors, ensuring a more conscious and controlled future utilization of orbital regimes.

1.2 The Space Debris Problem

Since 1957, when the first artificial satellite, Sputnik 1, revolutionized the space sector by becoming the first object to be placed into Earth's orbit, around 7270 launches have taken place, carrying about 26890 satellites [9]. The unmanaged interaction of these objects over time, combined with inadequate post-mission disposal practices, has led to a series of

inadvertent collisions. Consequently, the current orbital environment has reached a critical and non-negligible state, establishing space debris mitigation as an increasingly urgent priority. Furthermore, over the last decade, and particularly since 2015, space traffic has experienced a sharp surge. This trend is driven by the miniaturization of space systems and the deployment of massive satellite constellations, fueled by the commercialization of the space sector and the entry of important private companies. A clear idea of this recent escalation in payload launch traffic is illustrated in the following histogram [7]:

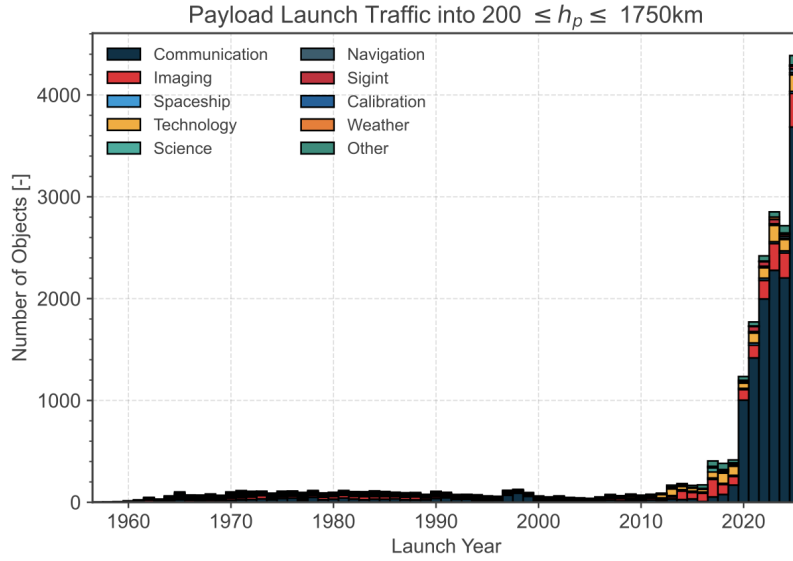


Figure 1.1. Payload Launch Traffic

As observed in Figure 1.1, the increasing trend shows a prominent peak in 2025, which is significantly higher than previous years. This demonstrates that the number of payloads in space is expanding almost exponentially, a trend facilitated by high launch frequencies and reusable technologies.

Consequently, more crowded orbits demand more rigorous monitoring, as the probability of collision inevitably rises, both between operational satellites themselves and between these objects and the fragments or debris present in orbit.

Specifically, the latest ESA report [9], updated to June 25, 2026, highlights the population of tracked and estimated objects in orbit as follows:

- 54000 space objects larger than 10 cm (which includes 9300 active payloads);
- 1.2 million space debris objects ranging from 1 cm to 10 cm;
- 140 million space debris objects ranging from 1 mm to 1 cm.

These numbers raise significant concerns, moving the perception of space, particularly the most utilized Earth orbits, from an infinite environment to a finite and congested

resource. These data are summarized in the following ESA chart, which illustrates the annual evolution of the number of objects across all space orbits, highlights the remarkable increase observed in recent years:

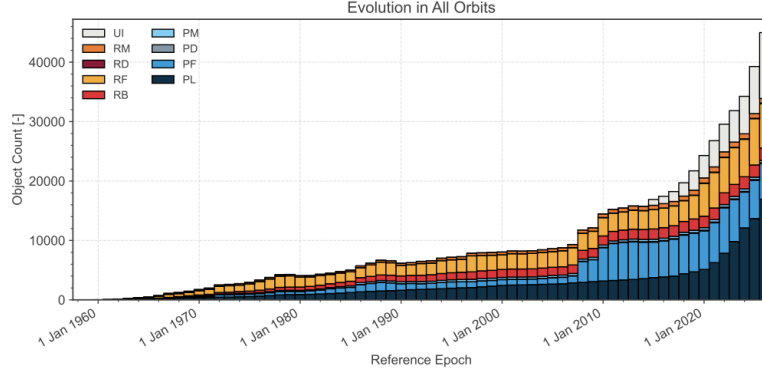


Figure 1.2. Evolution of the number of objects across all orbits

Figure 1.3, on the other hand, highlights the total number and the distribution of the various types of objects across different perigee altitudes:

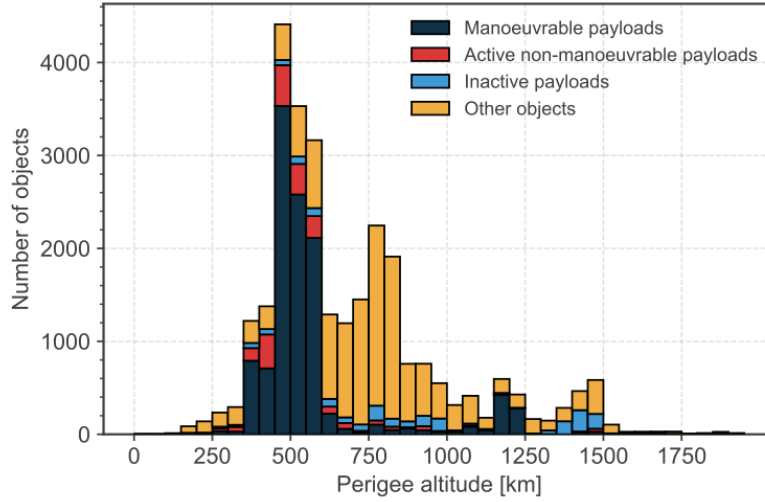


Figure 1.3. Number of orbital objects by perigee altitude.

A peak of operational satellites (indicated in blue) is located within orbits with perigee altitudes between 400 km and 600 km. In this region, the priority is to design spacecraft with maneuverability capabilities, and consequently equipped with collision avoidance systems, to prevent orbital impacts and ensure satellite safety. Conversely, it can be observed that from perigee altitudes above 600 km up to approximately 800 km,

the number of "other objects" increases significantly. These non-maneuverable objects act as uncontrolled bodies traveling at orbital velocities. Due to the low or negligible atmospheric drag at these specific altitudes, they remain in orbit for extended periods, thereby posing a continuous risk of collision.

To mitigate collision risks and slow down the growth of orbital debris, major space agencies and regulatory bodies have progressively introduced mitigation guidelines. A cornerstone of these regulations is the "25-year rule", which mandates that all satellites in Low Earth Orbit (LEO) complete deorbiting within 25 years from their end-of-life, thereby limiting the orbital lifetime of defunct spacecraft. This regulation also applies to rocket bodies that may remain trapped in orbit post-launch; increasingly, these elements are being equipped with dedicated technologies to execute controlled post-launch re-entries.

However, this guideline has proven insufficient to guarantee the long-term sustainability of the orbital environment. Consequently, in 2023, ESA deemed it necessary to implement the Zero Debris Approach. Among other measures, this policy reduces the post-mission orbital lifetime limit from 25 down to *5 years*. Currently, to be deployed into LEO, a spacecraft must demonstrate the capability to completely deorbit within 5 years of its end-of-life, while ensuring that the cumulative collision probability with orbital objects larger than 1 cm from the end-of-life until re-entry remains below 10^{-3} [7].

Despite extensive efforts and the increasingly conservative regulatory framework progressively adopted by space agencies, future projections of the current shift in orbital utilization and accelerating space traffic, combined with potential collisions and limited post-mission disposal success rates, particularly for legacy spacecraft, highlight a high probability of a cascade of collision events over the coming centuries [7].

This behavior is widely known as the Kessler syndrome, named after the space debris scientist Donald J. Kessler. He identified the phenomenon in which a single collision generates additional debris fragments, which in turn trigger subsequent collisions, resulting in a severe, self-sustaining chain reaction [10].

Remarkably, even under a hypothetical scenario where all space utilization and future launches were immediately and completely halted, the population of space debris would still continue to grow over time, as illustrated in Figure 1.4 [7]:

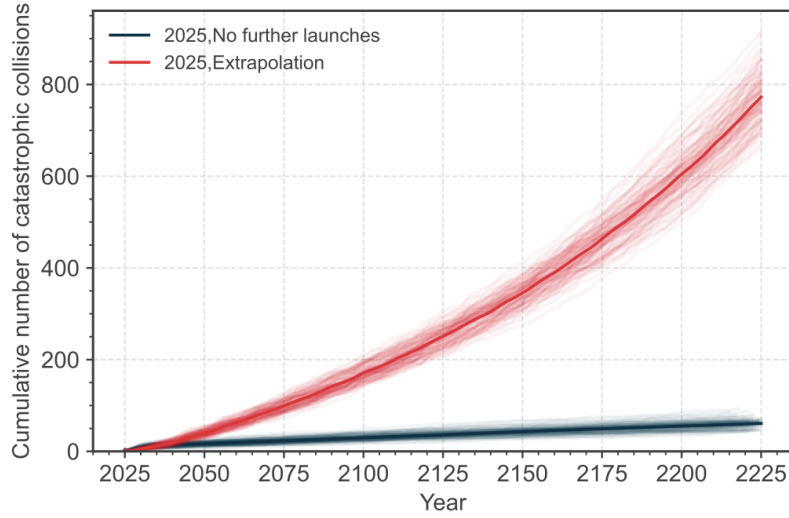


Figure 1.4. Number of cumulative collisions in LEO in the simulated scenarios of long-term evolution of the environment.

Consequently, moving toward ADR missions becomes necessary to mitigate potential cascade collision events, prevent catastrophic structural damage to active payloads, and actively intervene to reverse this critical trend.

1.2.1 Active Debris Removal

As previously discussed, given the current environment—which has seen the space debris population increase by 50% over the last five years, and considering the inherent risk of cascade collisions even under a scenario where no further launches are executed, merely cutting debris production is no longer sufficient. Active intervention is required to reduce the existing number of orbital debris. Consequently, substantial research efforts have recently been directed toward the development of specialized SC and ADR missions, aimed at mechanically capturing and deorbiting the high-risk objects currently in orbit.

Although no operational missions of this nature have been conducted to date, a demonstrative ADR mission led by ESA, named ClearSpace-1, is scheduled for 2026. The target of this mission is a Vespa (Vega Secondary Payload Adapter) upper stage, located in an orbit with a perigee of 660 km and an apogee of 800 km, with a mass of approximately 112 kg. The capture methodology developed for this mission involves the utilization of four robotic arms [6].

Within the state of the art, the Active Debris Removal by Astroscale-Japan (ADRAS-J) mission, conducted in 2024, represents another important milestone. This first mission successfully demonstrated long-range rendezvous and collision avoidance technologies, and performed the acquisition of images of the target debris. A second phase of the mission, scheduled for 2027, envisions that the chaser spacecraft will reach the previously observed debris again to execute its active removal, utilizing a series of robotic arms as well [1].

Consequently, active debris removal has emerged as a critical, urgent, and investigated field of research among the leading stakeholders in the contemporary space sector.

1.3 Thesis Objectives and Contributions

The primary objective of this thesis is to formulate and solve several indirect Optimal Control Problems (OCPs), obtaining optimal trajectories utilized within an active debris removal mission, employing two different propulsion systems.

The work begins with a preliminary analysis of the deorbiting times of objects of various sizes and masses in Low Earth Orbit (LEO) due to the effect of atmospheric drag, implemented by developing a high-fidelity algorithm to establish the target orbits of the mission. Subsequently, a realistic estimation of the mass of both propulsion systems and the spacecraft (SC) in its entirety was derived through a careful mass budget based on real-world industrial data

Having precisely defined the problem, the core of the thesis addressed the study, implementation, and resolution of trajectory optimization problems via indirect methods (IM). The OCULUS solver was utilized for the differential correction algorithms. Three different consecutive trajectories were optimized: one utilizing electric propulsion and two employing chemical propulsion. Furthermore, this cycle of three trajectories was repeated and optimized at each iteration to obtain a realistic estimation of the SC lifetime and its debris-clearing capacity.

This work wants to give a concrete contribution to a fundamental and urgent topic such as active debris removal, positioning itself within a short-term future perspective. Indeed, while current efforts focus on demonstrating the technologies required for active debris removal, this thesis already aims to optimize maneuvers to achieve the best performance and multiple, consecutive removals with the same satellite, thereby optimizing costs and more efficiently improving space sustainability.

1.4 Structure of the Thesis

A brief overview of the organization of the work is now given.

Chapter 2 deals with presenting the gravitational dynamic model used for the analysis conducted in this thesis. In particular, it first details the reference system and the coordinates chosen for this work, and then provides the fundamental equations of motion that have been implemented. In the final part, after a brief overview of orbital perturbations, it describes in detail the atmospheric drag perturbation and explains how it was developed from a computational standpoint in this study.

Chapter 3 delves into optimal control theory within the framework of indirect methods, explains its variational principles, and describes the optimality and transversality conditions, as well as the Euler-Lagrange equations, introducing Pontryagin's maximum principle and consequently the bang-bang control theory. Subsequently, the Multi-Point

Boundary Value Problem (MP-BVP) is introduced, expanding the applicability of indirect methods to more complex trajectories with interior constraints. Finally, an overview of the implementation of the boundary value problem and the OCULUS tool used to support the study is provided.

Chapter 4 shows the preliminary study at the core of the problem initialization. In the first part, an analysis of natural deorbiting times was conducted, analyzing the behavior of objects by varying their mass and exposed surface area, which led us to the definition of the target orbits of the analysis. Subsequently, the two propulsion systems used, Hall effect thruster and HPGP, were presented, highlighting their performance and performing their subsystem mass budget. From the total mass of the propulsion system, a realistic estimation of the SC mass was obtained in the final section, used as input for the subsequent analyses.

Chapter 5 shows the setting and implementation of the optimal problem under study and, consequently, the solutions obtained from its resolution. The three optimal trajectories are shown, starting from the first, which is a rendezvous maneuver performed via electric thruster, and moving through the subsequent chemical maneuvers of perigee lowering and orbit recircularization. The trajectories take into account, where necessary, the additional mass of the debris that the SC carries during its active removal mission. Finally, the last section illustrates the results of subsequent optimizations and iterations of the maneuvers, so as to visualize a complete mission profile until the depletion of the chemical propellant.

Chapter 6 conducts a brief conclusion on the work done and proposes some insights and future implementations.

Chapter 2

Dynamic Model

Chapter 2 illustrates the dynamic model implemented in this thesis work. Along with the mathematical formulation of the problem, the theoretical foundations of interest for the analysis are reviewed, describing their actual integration within the developed calculation algorithms.

The first section introduces the Reference System adopted for the study of the problem and, starting from it, defines the coordinates chosen to describe the position and the velocity vector of the spacecraft. Subsequently, the classical orbital parameters are briefly presented, highlighting the mathematical procedure used to derive from them the state vector in a form consistent with that of the problem.

In the second part of the chapter, the differential equations of motion are presented in their final formulation. Particular attention is dedicated to the treatment of the terms related to thrust and perturbative accelerations; the latter are initially discussed in a general key, to then focus the attention on the perturbation of greatest interest for the purposes of this study: atmospheric drag. Finally, the engineering choices and algorithmic strategies adopted to model this perturbative acceleration are illustrated, guaranteeing the highest level of accuracy in the most computationally efficient way.

2.1 Reference Systems

The first approach to an astrodynamics problem, even before defining the laws of motion and the physics of the problem, is the choice of a Reference System (RS), the most appropriate and convenient one for the study under consideration. There are several RSs that are uniquely defined by an origin, a fundamental plane, and an orthogonal right-handed triad that completes the Reference Frame (RF). They are mainly distinguished into inertial and non-inertial systems, whose main difference is the presence, in the second case, of additional apparent forces.

In the present thesis, the Earth Mean Equator and Equinox of Epoch J2000 (EME2000) RS has been adopted. The RF of the system under consideration has its origin coinciding with the center of the Earth, the equatorial plane as the fundamental plane and, as can be seen from Figure 2.1, its unit vectors $\{ \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}} \}$ are placed such that: $\hat{\mathbf{K}}$ is normal to the reference plane and positive towards the hemisphere containing Polaris, $\hat{\mathbf{I}}$ is aligned

in the direction of the vernal equinox, and $\hat{\mathbf{J}}$ completes the triad following the right-hand rule. Although more recent astronomical models such as the International Celestial Reference Frame (ICRF) take into account small planetary oscillation effects, EME2000 can be considered inertial with excellent approximation for the purpose of this analysis.

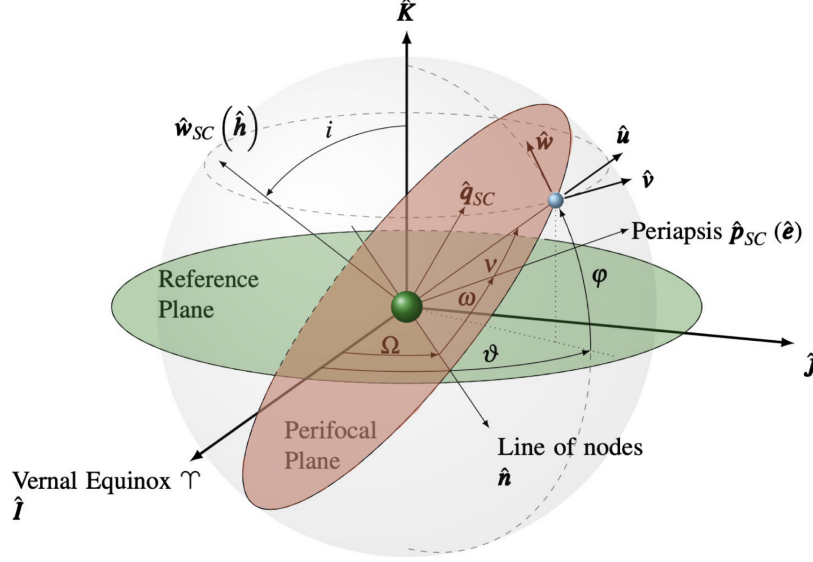


Figura 2.1. EME2000 $\{\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}}\}$, perifocal $\{\hat{\mathbf{p}}_{SC}, \hat{\mathbf{q}}_{SC}, \hat{\mathbf{w}}_{SC}\}$, and ZEN $\{\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}\}$ RSs

The position of a spacecraft(SC) within EME2000 RF can be expressed using either Cartesian or Spherical coordinates. In the present study, the choice fell on the latter, which are defined by the SC's radius r , representing the distance from the Earth's center, longitude ϑ , and latitude ϕ , all measured starting from the $\{\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}}\}$ triad.

The topocentric, rotating, and non-inertial Zenith-East-North (ZEN) reference frame has been adopted, instead, to describe the velocity. It is located at the center of mass of the SC and is identified by the set of unit vectors $\{\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}\}$, which project the velocity onto the radial, tangential, and normal directions, respectively. In detail: the radial axis develops along the extension of the body's position vector starting from the center of the Earth, while the tangential and normal directions are aligned with the local parallel and meridian of the celestial sphere, as illustrated in Figure 2.1.

The last RF to be introduced serves as an intermediate step between the transformations of interest for this work. The perifocal RF is a local coordinate system with its origin at the center of mass of the primary attracting body, the orbital plane as the fundamental plane, and is defined by the unit vectors $\hat{\mathbf{p}}, \hat{\mathbf{q}}$, and $\hat{\mathbf{w}}$. The $\hat{\mathbf{p}}$ unit vector is directed from the Earth's center toward the periapsis, coinciding with the direction of the eccentricity vector. The $\hat{\mathbf{w}}$ unit vector is normal to the orbital plane and aligned with the angular momentum vector $\mathbf{h}$. Finally, $\hat{\mathbf{q}}$ lies within the orbital plane and completes the right-handed triad.

The shape, size, and orientation in space of an orbit, together with the position of the spacecraft (SC) along the trajectory, can be uniquely defined by the six classical Keplerian elements $\{a, e, i, \Omega, \omega, \nu\}$.

Specifically, the size and energy of the orbit are indicated by the semi-major axis, a , which, for closed conics, represents the semi-distance between the periapsis and the apoapsis of the orbit:

$$a = \frac{r_p + r_a}{2} \quad (2.1)$$

The geometry of the orbit is further defined by the eccentricity, e , which determines its shape as a deviation from a circular orbit and can also be derived through r_p and r_a :

$$e = \frac{r_a - r_p}{r_a + r_p} \quad (2.2)$$

The next two parameters allow us to define the orbital plane in space. The inclination, i , represents the tilt of the orbital plane with respect to the equatorial plane and is defined as the angle between the unit vector $\hat{\mathbf{K}}$ and the angular momentum vector $\mathbf{h}$. The Right Ascension of the Ascending Node (RAAN), Ω , refers to the angle measured in the fundamental plane between the unit vector $\hat{\mathbf{I}}$ and the line of nodes, which is the point where the SC intersects the equatorial plane while moving northward.

For non-circular orbits, the orientation of the trajectory within its own plane is defined by the argument of periapsis, ω , which provides the angular distance of the periapsis from the line of nodes, measured in the direction of the body's motion. To conclude, the true anomaly, ν , identifies the instantaneous angular position of the SC with respect to the periapsis.

2.1.1 Coordinate Transformation and State Vector Initialization

The classical orbital elements have been used in this study to initialize the starting orbit of the satellite under consideration and to define the desired target trajectories at the end of the various propulsive maneuvers. Indeed, they provide a direct and geometrically intuitive visualization of the trajectories.

However, it is necessary to switch from the initial state vector, expressed by the Keplerian elements $\{a, e, i, \Omega, \omega, \nu\}$, to the integration state vector in the Spherical-ZEN RF $\{r, \theta, \phi, u, v, w\}$, adopted in this thesis for equations of motion. This transition requires the use of the perifocal frame and rotation matrices.

Therefore, starting from the state expressed in orbital elements the radius can be calculated using the conic equation, along with the semi-latus rectum p , which depends exclusively on a and e , and the magnitude of the angular momentum h :

$$p = a(1 - e^2), \quad r = \frac{p}{1 + e \cos \nu}, \quad h = \sqrt{\mu p} \quad (2.3)$$

where μ is the gravitational parameter of the primary body.

The position and velocity vectors in the perifocal frame are directly obtained as:

$$\mathbf{r}_{pqw} = \begin{Bmatrix} r \cos \nu \\ r \sin \nu \\ 0 \end{Bmatrix}, \quad \mathbf{v}_{pqw} = \begin{Bmatrix} -\frac{\mu}{h} \sin \nu \\ \frac{\mu}{h} (e + \cos \nu) \\ 0 \end{Bmatrix} \quad (2.4)$$

The rotation matrices are implemented and used to transition between two different reference frames, defined as:

$$\mathbf{R}_1(\cdot) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \cdot & \sin \cdot \\ 0 & -\sin \cdot & \cos \cdot \end{bmatrix}, \quad \mathbf{R}_2(\cdot) = \begin{bmatrix} \cos \cdot & 0 & \sin \cdot \\ 0 & 1 & 0 \\ -\sin \cdot & 0 & \cos \cdot \end{bmatrix} \quad (2.5)$$

$$\mathbf{R}_3(\cdot) = \begin{bmatrix} \cos \cdot & \sin \cdot & 0 \\ -\sin \cdot & \cos \cdot & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.6)$$

For the purposes of this study, it is interesting to evaluate the following transformations. To transition from the perifocal frame pqw to the geocentric equatorial EME2000 frame IIK , the transformation matrix $\mathbf{L}$ is applied, defined via the 3-1-3 rotation sequence:

$$\mathbf{L} = \mathbf{R}_{313}(-\Omega, -i, -w) \quad (2.7)$$

Consequently, the position and velocity vectors in the EME2000 frame are computed as:

$$\mathbf{r}_{IIK} = \mathbf{L} \mathbf{r}_{pqw}, \quad \mathbf{v}_{IIK} = \mathbf{L} \mathbf{v}_{pqw} \quad (2.8)$$

From these Cartesian components, the spherical coordinates are extracted as follows:

$$\theta = \arctan\left(\frac{r_J}{r_I}\right), \quad \phi = \arcsin\left(\frac{r_K}{r}\right) \quad (2.9)$$

Finally, the last transformation is applied to obtain the velocity vector in the ZEN frame:

$$\mathbf{v}_{ZEN} = \mathbf{R}_{23}(\phi, \theta) \mathbf{v}_{IIK} \quad (2.10)$$

2.2 Equations of Motion in Spherical Coordinates

The evolution of a SC's state over time, specifically the variation of its position and velocity vectors, and mass, is modeled by a system of Ordinary Differential Equations (ODEs) formulated within an inertial RS centered on a primary celestial body, such as EME2000. They are given by:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \quad (2.11a)$$

$$\frac{d\mathbf{v}}{dt} = \mathbf{g} + \frac{\mathbf{T}}{m} + \frac{\mathbf{D}}{m} + \frac{\mathbf{L}}{m} + \mathbf{a}_p \quad (2.11b)$$

$$\frac{dm}{dt} = -\frac{T}{c} \quad (2.11c)$$

In Eq.(2.11b), the velocity variation is driven by the gravitational acceleration, $\mathbf{g}$, which in a two-body formulation represents the attraction exerted by the Earth. Throughout this study, the aerodynamic forces, Drag $\mathbf{D}$ and Lift $\mathbf{L}$, can be safely neglected, although drag perturbations could be incorporated at a later stage of model refinement. The last

equation, Eq.(2.11c), concerns the reduction in mass due to the fuel consumption during the thrusting process.

Following the spherical coordinate definitions and the ZEN RF selection introduced in the previous section, the vector ODEs presented in (2.11) are expanded into their components. Precisely, the spacecraft position is defined by the radius r , longitude ϑ , and latitude ϕ within the EME2000 frame. Instead, the velocity vector $\mathbf{v}$ is projected onto the local ZEN RF, decomposing it into its radial (Zenith, u), tangential (East, v), and normal (North, w) components. Hence, the Equations of Motion in Spherical Coordinates–ZEN are:

$$\frac{dr}{dt} = u \quad (2.12a)$$

$$\frac{d\vartheta}{dt} = \frac{v}{r \cos \varphi} \quad (2.12b)$$

$$\frac{d\varphi}{dt} = \frac{w}{r} \quad (2.12c)$$

$$\frac{du}{dt} = -\frac{\mu}{r^2} + \frac{v^2}{r} + \frac{w^2}{r} + \frac{T_u}{m} + (a_p)_u \quad (2.12d)$$

$$\frac{dv}{dt} = -\frac{uv}{r} + \frac{vw}{r} \tan \varphi + \frac{T_v}{m} + (a_p)_v \quad (2.12e)$$

$$\frac{dw}{dt} = -\frac{uw}{r} - \frac{v^2}{r} \tan \varphi + \frac{T_w}{m} + (a_p)_w \quad (2.12f)$$

$$\frac{dm}{dt} = -\frac{T}{c}, \quad (2.12g)$$

where c is the effective exhaust velocity, which is a characteristic parameter of the propulsion system, a_p includes the perturbative accelerations and T stands for the thrust vector. When projected into the ZEN RF, it is decomposed into the following contributions:

$$T_u = T \sin \alpha_T \quad (2.13a)$$

$$T_v = T \cos \alpha_T \cos \beta_T \quad (2.13b)$$

$$T_w = T \cos \alpha_T \sin \beta_T. \quad (2.13c)$$

in which α_T and β_T represent the out-of-plane and in-plane angles, respectively.

2.3 Perturbing Accelerations

The equations of motion, Equation 2.12, in their general form, take into account perturbing accelerations of various kinds to which the spacecraft is subjected during its time in orbit. Depending on the type of mission, altitude, and time duration of the analysis, some of these effects are predominant compared to others which, conversely, can be considered negligible.

The overall cumulative vectorial perturbing acceleration can be expressed as the sum of four main contributions:

$$\mathbf{a}_p = \mathbf{a}_{\text{isp}} + \mathbf{a}_J + \mathbf{a}_{\text{SRP}} + \mathbf{a}_D \quad (2.14)$$

As the first term, we find the luni-solar perturbation, $\mathbf{a}_{\text{ls}}$, due to the gravitational effects of the Moon and the Sun. Although these accelerations have lower intensities compared to the preponderant Earth's gravity, they can cause a rotation of the orbital plane, and their magnitude depends primarily on the relative distances between the celestial bodies.

Subsequently, the harmonic perturbation, $\mathbf{a}_{\text{J}}$, refers to the disturbances generated by the Earth's non-perfect sphericity. Indeed, planetary rotation causes a non-homogeneous distribution of mass, which is more concentrated in the equatorial region. The resulting non-perfectly spherical gravitational field mainly produces the phenomena of the regression of the line of nodes and the rotation of the line of apsides.

Moreover, the acceleration $\mathbf{a}_{\text{SRP}}$ represents the perturbation due to the solar radiation pressure exerted by the photons emitted by the Sun that hit the satellite's surface. It strictly depends on the area-to-mass ratio of the SC, its reflectivity, and the distance from the solar source. This contribution is of fundamental relevance for satellites characterized by large solar arrays or for solar sails, which intentionally exploit this pressure as a propulsive source.

Finally, the acceleration due to drag, $\mathbf{a}_{\text{D}}$, is introduced, which is due to the presence of the atmosphere, although highly rarefied, at the lowest orbital altitudes. Since this specific perturbation is of fundamental relevance to the present thesis work, it will be treated in more detail in the next section.

2.3.1 Atmospheric Drag Model

Atmospheric drag, generated by the interaction of the SC with the air particles present in the highest layers of the atmosphere, is the main cause of the natural deorbiting of satellites and necessitates periodic orbital maintenance maneuvers for SC operating in LEO. Atmospheric density ρ decays exponentially with altitude, reducing to negligible values for altitudes above 500-600 km, as shown in Figure 2.2.

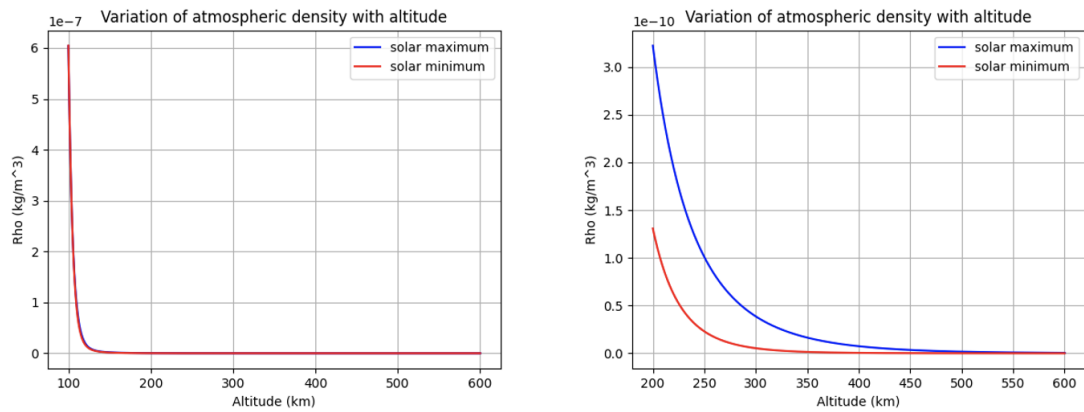


Figure 2.2. Density variation as a function of altitude

For all satellites orbiting at lower altitudes, this perturbation plays a major role. The drag force acts on the SC generating a braking acceleration, and its magnitude is expressed as:

$$D = \frac{1}{2} \rho S C_d V_r^2 \quad (2.15)$$

In which ρ represents the local and instantaneous air density, S is the frontal area of the SC projected in the direction of motion, and C_d is the drag coefficient which, as a first approximation, assumes a standard value of 2.2 [4]. Finally, V_r represents the relative velocity of the SC with respect to the air particles, thus taking into account the Earth's rotation velocity.

In the topocentric Zenith-East-North (ZEN) RF adopted in this work, the East axis is aligned with the local parallel and is aligned with the Earth's rotational motion. Therefore, the local linear velocity of the atmosphere v_{atm} , which derives from the constant angular velocity $\omega_{\oplus}$, is expressed as:

$$v_{\text{atm}} = \omega_{\oplus} r \cos \phi \quad (2.16)$$

where r is the radius and ϕ is the latitude of the SC. From this, the relative velocity vector, decomposed into ZEN components $\{u, v, w\}$, is calculated as:

$$\mathbf{v}_r = \begin{Bmatrix} u \\ v - \omega_{\oplus} r \cos \phi \\ w \end{Bmatrix}_{\text{ZEN}} \quad (2.17)$$

and its magnitude is:

$$V_r = \|\mathbf{v}_r\| = \sqrt{u^2 + (v - \omega_{\oplus} r \cos \phi)^2 + w^2} \quad (2.18)$$

Subsequently, the acceleration is calculated as $\mathbf{a}_D = -\frac{\mathbf{D}}{\mathbf{m}}$ and decomposed along the ZEN components: $\mathbf{a}_u$, $\mathbf{a}_v$, $\mathbf{a}_w$.

For the numerical implementation of this perturbation, in the present thesis, the Python library `msis` (NRLMSISE-00 model) [21] [17] was used to sample the atmospheric density values as a function of altitude, day, and time. Since the evaluation of deorbiting requires time integrations over scales of months or years, the direct use of this library would have demanded a high computational cost and long calculation times. Furthermore, these models do not predict the trend of solar fluxes in the long term, effectively preventing a predictive use of `msis` for advanced future dates.

To overcome these limitations and to improve the efficiency of the code, it was chosen to proceed by generating a discretized database of the atmosphere to map the density trend. Specifically, through `msis`, density values were extracted in an altitude range between 50 km and 805 km, with a step of 5 km. In order to consider the entire environmental variability related to solar activity cycles, the values were extracted at two dates, respectively relative to the last recorded solar maximum and minimum.[20]

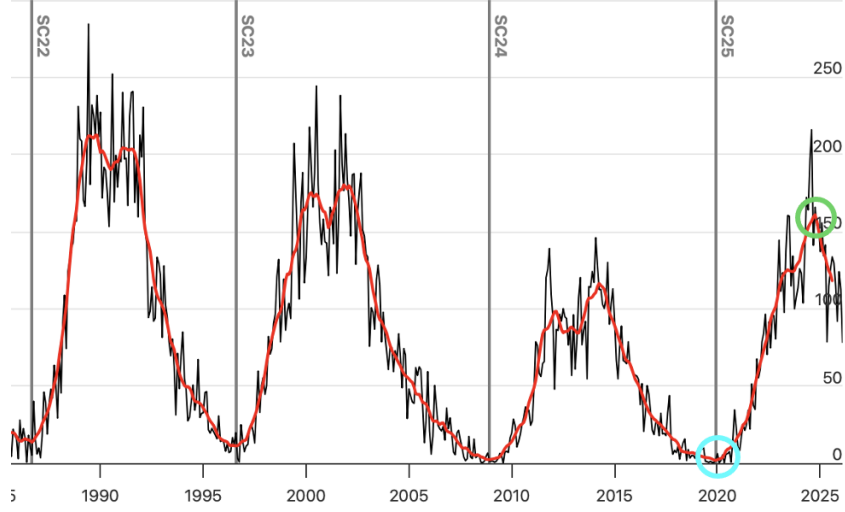


Figure 2.3. Solar cycle variations and the adopted maximum and minimum reference values.

On the basis of this database, a bilinear interpolation algorithm was implemented, which estimates the instantaneous density value by performing a linear interpolation between the two closest altitude nodes and, simultaneously, between the respective limiting cases of solar minimum and maximum for the considered altitude.

A preliminary analysis highlighted that the variations of ρ with longitude (θ) and latitude (ϕ) produce negligible oscillations for the trajectories of interest in this study, which are predominantly equatorial. Therefore, the model was simplified by considering the density as dependent only on altitude and time, which is linked to the evolution of the solar cycle.

Finally, as the last fundamental concept necessary for this analysis, we introduce the ballistic coefficient, defined as:

$$BC = \frac{m}{C_D A} \quad (2.19)$$

where BC represents the SC's ballistic coefficient (kg/m^2), m is the SC's mass (kg), C_D denotes the dimensionless drag coefficient, and A is the object's cross-sectional area (m^2). From these basic relationships, the rate of deceleration experienced by a body in the atmosphere due to aerodynamic resistance can be evaluated. Specifically, a high BC results in a lower deceleration, thereby allowing for a longer lifetime in space. Conversely, a low BC implies a faster orbital decay. This behavior can be understood intuitively from its mathematical definition: the ballistic coefficient is directly proportional to the vehicle's mass and inversely proportional to its exposed cross-sectional area. Consequently, its reciprocal, $1/BC$, appears directly within the formulation of the atmospheric drag acceleration, a_D [19].

Chapter 3

Optimal Control Theory

In the current orbital scenario, characterized by an increasingly intense space traffic, by the presence of countless space debris, and by the need to preserve space infrastructures, trajectory optimization becomes an engineering necessity. Specifically, ADR missions at the center of this thesis work, the calculation of optimal transfers is fundamental not only for the success of orbital maneuvers, but also to maximize the removal capacity of the cleaner satellite in the face of a limited propellant budget. The sought-after optimal trajectories, which specifically include the maneuvers of perigee lowering, recircularization, and approach to the debris from an initial state to a well-defined final target, are those that, among the infinite possible alternatives, minimize propellant consumption or, equivalently, maximize the final mass of the spacecraft. An Optimal Control Problem (OCP) sets exactly this as its objective, namely the search for a control law that maximizes or minimizes a certain index of merit. The essence of the method consists in controlling, along the entire trajectory, the specific parameters that influence the achievement of the final target, so as to identify the temporal evolution of the control variables capable of satisfying the imposed constraints while maximizing the performance index. This combination of optimal variables constitutes the "optimal control law", the formulation of which is derived, in this thesis, through Optimal Control Theory (OCT). It constitutes the mathematical tool employed to derive the necessary conditions that identify an optimal solution.

3.1 Direct vs. Indirect Methods in Trajectory Optimization

A numerical method aims to reduce a complex problem into a sequence of smaller, finite sub-problems and, consequently, of easier solution. Within the framework of Optimal Control Problems (OCPs) related to trajectory optimization, this approach translates into the desire to transform the continuous differential equations of the spacecraft's motion into a secondary problem characterized by a finite and limited number of variables.

To solve this class of problems, the scientific community and researchers in the sector are historically divided into two main branches: those who use direct methods and those who, conversely, opt for indirect methods. This dualism is also clearly reflected in the

architecture of the main engineering software programs developed for orbital optimization. The differences between the two approaches are substantial and outline their respective advantages and limitations.

Direct methods present as strengths an immediate implementation and a marked numerical robustness, qualities that allow them to handle highly complex scenarios. They perform a parametric optimization of the continuous-time problem, converting it into an approximated finite-dimensional model through the discretization of both the state and the optimal controls. However, the main disadvantage of this approach stems precisely from this operation: the grid (mesh) dedicated to the discretization of the domain must use an extremely high number of nodes, resulting particularly dense to guarantee an accurate solution. This entails a remarkable computational cost; nevertheless, the solution can sometimes suffer from poor precision, requiring subsequent algorithms for result refinement. Although modern computers easily tolerate such massive computational loads, the main challenge lies in the impossibility of analytically verifying whether the obtained numerical approximation is effectively optimal, since the purely discrete nature of the solution does not offer theoretical insights in this regard.

On the contrary, indirect methods guarantee an extremely high mathematical accuracy in the face of reduced computation times and a minimal computational cost compared to their direct competitors, while manifesting a greater sensitivity and criticality in numerical convergence. The most relevant aspect of indirect methods lies in their capacity to foster a deep understanding of the physics of the problem: the introduction of co-state variables provides valuable theoretical insights that guide the user in the refinement of guess profiles, steering the system toward the globally optimal solution. However, the current implementation of such methods still presents complexities, mainly linked to the need to analytically derive, via OCT, certain specific quantities of the problem and to the strong sensitivity of the solution with respect to the initial conditions of the costates, which require the intervention of an expert operator.

Of crucial importance in the evolution of indirect methods has been the introduction of internal boundaries, which allows for the mathematically rigorous management of operational constraints and dynamic discontinuities, thus shifting from a classic Two-Point Boundary Value Problem (TPBVP) to a Multi-Point Boundary Value Problem (MPBVP). If aided by adequate numerical techniques, this hybrid approach drastically improves the robustness of the indirect method, making it fully competitive in solving complex scenarios previously the exclusive domain of direct strategies.

Within this methodological panorama fits the present thesis work, which exploits this multi-phase formulation for the optimization of independent trajectories regulated by two different propulsion systems (electric and chemical). The application of specific internal boundaries allows for highlighting the adaptability, versatility, and extreme precision of the used algorithm, while simultaneously benefiting from the aforementioned theoretical insights offered by the physics of the problem to accurately guide the refinement of ADR trajectories.

3.2 Fundamentals of Optimal Control Theory

The application of OCT to space trajectories has been the subject of extensive studies in the literature. Its foundations date back to Lawden's formulation and then extended through the treatises of Kirk, Bryson, and Burghes. Born as an extension of the Calculus of Variations (CoV), dating back to a formulation by Euler in 1766, OCT is currently seeing its fundamental principles readapted into new applicative variations, since the available technology allows for the easy use of more complex numerical methods.

Within the framework of the optimization of trajectories that evolve from an initial state to a final target, as already anticipated, the OCP sets as its objective the maximization of a specific index of merit, identifying a control law that respects all the imposed constraints. Concretely, the OCP describes the temporal evolution of n state variables through a system of first-order ODEs $\dot{x}$, whose general formulation can be expressed as:

$$\dot{x}(t) = f(x(t), u(t), t) \quad (3.1)$$

in which the time t represents the independent variable, $x(t) \in \mathbb{R}^n$ is the state vector on which the ODEs depend at each instant between the initial and the final one, and, finally, $u(t) \in \mathbb{R}^m$ represents the control vector, inside of which m control variables are contained. Determining the optimal solution of the problem means identifying the optimal trajectory $x^*(t)$ subject to the optimal controls $u^*(t)$ capable of maximizing the index of merit.

The requirements of a trajectory are generally assigned at the initial time t_0 and at the final time t_f ; when these conditions at the extrema are encountered, called external boundaries, the formulation in Equation 3.1 represents a TPBVP. These conditions can involve the state variables at the external boundaries and, in some cases, affect the epochs themselves. The vector of constraint collects these Boundary Conditions (BCs) as follows:

$$\chi(x_0, x_f, t_0, t_f) = 0 \quad (3.2)$$

where $\chi[\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}, \mathbb{R}] \rightarrow \mathbb{R}^q$, with q representing the total number of constraints.

Furthermore, the control vector can also be subject to constraints that limit its components within a range of admissible values, defined by minimum and maximum thresholds. More details on the control vector u and on the admissibility criterion $u \in U$, will be addressed later on. The optimality of the OCP, and consequently of the obtained trajectory, is quantified through an index of merit $\mathcal{J}$, of which the maximum or minimum values must be determined, depending on the problem. This index is expressed as:

$$\mathcal{J} = \varphi(x_0, x_f, t_0, t_f) + \int_{t_0}^{t_f} [\Phi(x(t), u(t), t)] dt \quad (3.3)$$

This index is composed of the sum of two scalar terms. The first, represented by the function φ , depends on the values assumed by the state variables and the epochs at the external boundaries, and is therefore conditioned by the specific admissible final state reached. The second term, expressed by the integral of the function Φ , is instead linked to the values that the state variables, the controls, and the time itself assume during the time interval and, consequently, measures how the solution evolved in the transition from the initial state to the final one. The problem written in this form, in the Equations from

3.1 to 3.41 represents a classic CoV OCP, that is Bolza's problem. Furthermore, the last Equation 3.41 can be reduced, by adding opportune auxiliary variables, to the Lagrange's formulation by setting $\varphi = 0$, or to the Mayer formulation by imposing $\Phi = 0$.

For brevity, the following notation is adopted:

$$\varphi \triangleq \varphi(\mathbf{x}_0, \mathbf{x}_f, t_0, t_f) \quad (3.4)$$

$$\boldsymbol{\chi} \triangleq \boldsymbol{\chi}(\mathbf{x}_0, \mathbf{x}_f, t_0, t_f) \quad (3.5)$$

In this phase, a fundamental manipulation is implemented, deriving from the application of the principles of the IM. In order to derive the optimality conditions, an augmented index of merit $\mathcal{J}^*$ is defined, which includes a measure of how much the constraints and the state quantities are respected with respect to the ODEs of the implemented dynamic model. From a mathematical point of view, this operation translates into the introduction of the adjoint vector $\lambda(t) \in \mathbb{R}^n$ and associated with the state variables, as well as the Lagrange multipliers $\mu \in \mathbb{R}^m$ linked to the BCs. Thus, the augmented index of merit takes the form:

$$\mathcal{J}^* = \varphi + \boldsymbol{\mu}^T \boldsymbol{\chi} + \int_{t_0}^{t_f} [\Phi + \boldsymbol{\lambda}^T (\mathbf{f} - \dot{\mathbf{x}})] dt \quad (3.6)$$

From an analytical point of view, as long as the system does not reach convergence, namely $\boldsymbol{\chi} \neq 0$, Equation 3.1 is not satisfied, implying that $\dot{\mathbf{x}} \neq \mathbf{f}$. On the contrary, when the solution converges, with the consequent satisfaction of the BCs and the state equations, the constraint vector vanishes ($\boldsymbol{\chi} = 0$) and the equality $\dot{\mathbf{x}} = \mathbf{f}$ holds. In this condition, the augmented index of merit reduces exactly to the classic one: $\mathcal{J} = \mathcal{J}^*$. Therefore, once all constraints are satisfied, solving the augmented problem through the index of merit of Equation 3.6 becomes mathematically equivalent to solving that of Equation 3.41, given that both functionals remain explicit functions of the state vector $\mathbf{x}$, its time derivative $\dot{\mathbf{x}}$, and the control vector $\mathbf{u}$.

The integration by parts of the term $-\boldsymbol{\lambda}^T \dot{\mathbf{x}}$ within Equation 3.6 is therefore performed. This manipulation is necessary and allows for the exclusion from the formulation of the time derivatives of the state variables $\dot{\mathbf{x}}$, which are not known a priori since they are integrated at each step of the trajectory optimization process.

$$\int_{t_0}^{t_f} -(\boldsymbol{\lambda}^T \dot{\mathbf{x}}) dt = -(\boldsymbol{\lambda}_f^T \mathbf{x}_f) + (\boldsymbol{\lambda}_0^T \mathbf{x}_0) + \int_{t_0}^{t_f} (\dot{\boldsymbol{\lambda}}^T \mathbf{x}) dt \quad (3.7)$$

Therefore, Equation 3.6 takes the form of Equation 3.8:

$$\mathcal{J}^* = \varphi + \boldsymbol{\mu}^T \boldsymbol{\chi} + (\boldsymbol{\lambda}_0^T \mathbf{x}_0 - \boldsymbol{\lambda}_f^T \mathbf{x}_f) + \int_{t_0}^{t_f} (\Phi + \boldsymbol{\lambda}^T \mathbf{f} - \dot{\boldsymbol{\lambda}}^T \mathbf{x}) dt \quad (3.8)$$

At this point, within Equation 3.8, a quantity of fundamental importance for the discussion emerges, which takes the name of **Hamiltonian** of the system:

$$\mathcal{H} \triangleq \Phi + \boldsymbol{\lambda}^T \mathbf{f} \quad (3.9)$$

In the mathematical field, the maximization (or minimization) of a function is equivalent to requiring its stationarity at the optimum point. Consequently, in the present problem,

the optimality of the augmented index of merit is achieved by imposing that its first variation, $\delta\mathcal{J}^*$, is zero. This condition, through algebraic steps that are omitted in this discussion for brevity, takes the form:

$$\delta\mathcal{J}^* = \left(\frac{\partial\varphi}{\partial t_0} + \boldsymbol{\mu}^T \frac{\partial\boldsymbol{\chi}}{\partial t_0} - \mathcal{H}_t \right) \delta t_0 + \quad (3.10a)$$

$$+ \left(\frac{\partial\varphi}{\partial t_f} + \boldsymbol{\mu}^T \frac{\partial\boldsymbol{\chi}}{\partial t_f} + \mathcal{H}_t \right) \delta t_f + \quad (3.10b)$$

$$+ \left(\frac{\partial\varphi}{\partial \mathbf{x}_0} + \boldsymbol{\mu}^T \frac{\partial\boldsymbol{\chi}}{\partial \mathbf{x}_0} + \boldsymbol{\lambda}_0^T \right) \delta \mathbf{x}_0 + \quad (3.10c)$$

$$+ \left(\frac{\partial\varphi}{\partial \mathbf{x}_f} + \boldsymbol{\mu}^T \frac{\partial\boldsymbol{\chi}}{\partial \mathbf{x}_f} - \boldsymbol{\lambda}_f^T \right) \delta \mathbf{x}_f + \quad (3.10d)$$

$$+ \int_{t_0}^{t_f} \left[\left(\frac{\partial\mathcal{H}}{\partial \mathbf{x}} + \dot{\boldsymbol{\lambda}}^T \right) \delta \mathbf{x} + \frac{\partial\mathcal{H}}{\partial \mathbf{u}} \delta \mathbf{u} \right] dt, \quad j = 1, \dots, n_p \quad (3.10e)$$

As can be deduced from Equation 3.10, an accurate choice of the co-state variables $\boldsymbol{\lambda}$ and the Lagrange multipliers $\boldsymbol{\mu}$ allows for the cancellation of the first variation $\delta\mathcal{J}^*$, by setting to zero the coefficients of the various variational contributions, regardless of the values they assume. Furthermore, the imposition of this condition is of fundamental importance as it provides the necessary relationships to solve the problem, such as:

- **Transversality conditions:** two algebraic equations evaluated at the initial and final times, obtained by setting to zero the multiplying coefficients in rows 3.10a and 3.10b associated with δt_0 and δt_f .
- **Optimality conditions:** $2n$ algebraic equations, one for each state variable evaluated at $\mathbf{x}_0$ and $\mathbf{x}_f$, deriving from rows 3.10c and 3.10d.
- **Euler-Lagrange equations:** n ordinary differential equations for the adjoint variables, obtained from the second-to-last multiplying coefficient.
- **Equations for the optimal control $\mathbf{u}(t)$:** m algebraic equations deriving from the last multiplying coefficient.

3.2.1 Optimality and Transversality Boundary Conditions

Among the relationships obtained from setting to zero the various coefficients that composed the first variation of the augmented index of merit, the first to be analyzed in this section are those referring to the boundaries of the trajectory. The boundary conditions for optimality constitute a set of equations that govern the time behavior and the state variables of the system at the initial and final points of the trajectory, combining the 2 transversality conditions and the $2n$ boundary optimality conditions.

The $2 + 2n$ total boundary relationships, obtained by imposing that the coefficients of δt_0 , δt_f , $\delta \mathbf{x}_0$, and $\delta \mathbf{x}_f$ vanish, are expressed as follows:

$$\frac{\partial \varphi}{\partial t_0} + \boldsymbol{\mu}^T \frac{\partial \boldsymbol{\chi}}{\partial t_0} - \mathcal{H}_0 = 0 \quad (3.11a)$$

$$\frac{\partial \varphi}{\partial t_f} + \boldsymbol{\mu}^T \frac{\partial \boldsymbol{\chi}}{\partial t_f} + \mathcal{H}_f = 0 \quad (3.11b)$$

$$\frac{\partial \varphi}{\partial \mathbf{x}_0} + \boldsymbol{\mu}^T \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{x}_0} - \boldsymbol{\lambda}_0 = \mathbf{0}^T \quad (3.11c)$$

$$\frac{\partial \varphi}{\partial \mathbf{x}_f} + \boldsymbol{\mu}^T \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{x}_f} + \boldsymbol{\lambda}_f^T = \mathbf{0} \quad (3.11d)$$

Based on the two transversality equations (3.11a) and (3.11b), fundamental considerations emerge regarding the behavior of the Hamiltonian at the boundaries. In the event that time does not appear in the function φ nor is subject to constraints, the Hamiltonian is zero at the points corresponding to the initial and final times ($\mathcal{H}_0 = 0$ and $\mathcal{H}_f = 0$), and the values of the latter are determined directly by the optimization process. On the contrary, if time is constrained, that is, if conditions of the type $t_0 = a$ and/or $t_f = b$ are imposed within $\boldsymbol{\chi}$, the Hamiltonian assumes a free value at the respective boundary ($\mathcal{H}_0 \neq 0$ and/or $\mathcal{H}_f \neq 0$), which will depend on the optimization.

A similar reasoning applies to the optimality conditions (3.11c) and (3.11d) for the state variables. If a specific i -th component of the state x_i does not appear in the function ϕ nor is constrained, then its associated adjoint variable vanishes at the same point ($\lambda_{x_i} = 0$ at the initial and/or final boundary). If, on the other hand, a constraint is imposed on the variable x_i at the boundaries, the relative co-state at that boundary will be free and will result from the optimization process.

3.2.2 Pontryagin's Principle and TPBVP Formulation

Setting to zero the two multiplying coefficients present in the last row of Equation 3.10e allows for the derivation of the relations governing the time evolution of the adjoint variables and the controls. In particular, by setting the coefficient associated with $\delta \mathbf{x}$ to zero, the n Euler-Lagrange ordinary differential equations for the co-states are obtained, which take the form:

$$\frac{d\boldsymbol{\lambda}}{dt} = - \left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}} \right)^T \quad (3.12)$$

Since each state variable is uniquely associated with a corresponding co-state variable, it follows that $\dot{\boldsymbol{\lambda}} \in \mathbb{R}^n$.

On the contrary, by setting the coefficient relative to the variational term of the control $\delta \mathbf{u}$ to zero, the m algebraic equations for the optimal control are derived, expressed by:

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}} \right)^T = \mathbf{0} \quad (3.13)$$

As previously anticipated, one or more elements of the control vector $\mathbf{u}$ may be subject to admissibility limits, denoted by $\mathcal{U}$. Since, in general, $\mathbf{u} = \mathbf{u}(\mathbf{x}(t), t)$, the control action can

depend on both the state variables and time. In this context, only explicit constraints are considered; that is, the control $\mathbf{u}$ is assumed to be bounded within specific limits: $\mathbf{u}_{\min} \leq \mathbf{u} \leq \mathbf{u}_{\max}$. Therefore, the optimal control $\mathbf{u}^* \in \mathcal{U}$ for the desired trajectory is the one that, at each instant of the trajectory, extremizes the Hamiltonian defined in Equation 3.9 at that specific point. This formulation is known as Pontryagin's Maximum Principle (PMP) for a maximization problem, whereas it is known as Pontryagin's Minimum Principle (PmP) for a minimization problem.

The PMP establishes that the optimal control value coincides with the one obtained via Equation 3.13 provided that it is strictly contained within the admissible boundaries ($\mathcal{U}_{\min} < \mathbf{u} < \mathcal{U}_{\max}$). Otherwise, if the computed value falls outside this interval, the control is automatically set to the boundary values of $\mathcal{U}$.

Returning to Equation 3.13, the applicability of this condition strictly depends on the linearity of the Hamiltonian with respect to the bounded control. Indeed, two scenarios are distinguished based on the partial derivative with respect to the generic component u_i :

$$\frac{\partial \mathcal{H}}{\partial u_i} = \begin{cases} k_{u_i} & \text{if } \mathcal{H} \text{ is affine in } u_i \\ f(u_i) & \text{otherwise} \end{cases} \quad (3.14)$$

where k_{u_i} is a coefficient. If the Hamiltonian is linear with respect to the control, Equation 3.13 cannot be satisfied (except for the unique singular case where $k_{u_i} = 0$), since u_i no longer appears explicitly in the derivative. Consequently, the strategy adopted is known as *bang-bang* control.

This strategy consists in evaluating the sign of the coefficient $k_{u_i} \neq 0$ to pursue the maximization of the Hamiltonian: specifically, the control is set to its maximum possible value ($u_i = U_{i,\max}$) if $k_{u_i} > 0$, or to its minimum admissible value ($u_i = U_{i,\min}$) if $k_{u_i} < 0$. This is precisely the case implemented in the present study, where the control variable is represented by the thrust T , which is bounded between a minimum null value ($T_{\min} = 0$) and a maximum value $T_{\max}$. The complementary case where $k_{u_i} = 0$ for a finite time interval would require a different strategy involving the analysis of the so-called singular arcs, the discussion of which is omitted for brevity as it is outside the scope of the present work.

At this point, the Two-Point Boundary Value Problem (TPBVP) is fully defined. In summary, it is formulated as a system of $2 + 2\mathbf{n} + \mathbf{m}$ equations, derived from the transversality, optimality, and control conditions, which implicitly determines:

- The 2 time instants, initial (t_0) and final (t_f);
- The initial conditions for the $2n$ Ordinary Differential Equations (ODEs), which include the state vector $\mathbf{x}$ and the co-state vector $\boldsymbol{\lambda}$;
- The m additional constants associated with the Lagrange multipliers $\boldsymbol{\mu}$.

3.3 Multi-Point Optimal Control Problem Formulation

Unlike the TPBVP discussed previously, a Multi-Point Boundary Value Problem (MPBVP) features constraints imposed at internal points of the trajectory. The trajectory is

therefore subdivided into phases or arcs, namely n_p intervals allowing for the modeling of discrete events along the trajectory and enhancing the robustness of the code, thereby favoring its convergence.

In this scenario, the variables are continuous within each arc but may exhibit discontinuities at the junction points between two adjacent phases.

In this discussion, the convention is adopted whereby each j -th arc begins at time $t_{(j-1)+}$ and ends at time t_{j-} , to which the state vectors $\mathbf{x}_{(j-1)+}$ and $\mathbf{x}_{j-}$ respectively correspond. Each j -th arc extends for a duration equal to Δt_j which, being generally unknown a priori and varying from phase to phase, also constitutes a parameter subject to optimization.

Therefore, in an MPBVP, the boundary conditions, which are generally non-linear and mixed, must be imposed both at the external boundaries of the trajectory and at the internal interfaces between the various arcs. These conditions remain functions of the state variables and the independent time, assuming the following form:

$$\boldsymbol{\chi}(\mathbf{x}_{(j-1)+}, \mathbf{x}_{j-}, t_{(j-1)+}, t_{j-}) = \mathbf{0}, \quad j = 1, \dots, n_p \quad (3.15)$$

The index of merit $\mathcal{J}$ is consequently expressed as follows:

$$\mathcal{J} = \varphi(\mathbf{x}_{(j-1)+}, \mathbf{x}_{j-}, t_{(j-1)+}, t_{j-}) + \sum_{j=1}^{n_p} \int_{t_{(j-1)+}}^{t_{j-}} \Phi(\mathbf{x}(t), \mathbf{u}(t), t) dt \quad (3.16)$$

It is worth noting that the function φ becomes dependent on the values assumed by the variables and time at all nodes of the trajectory (from $j = 0$ up to $j = n_p$), thus accounting for both external conditions and internal interfaces. Similarly, the summation of the integrals of the function Φ is no longer computed over a single continuous domain: while still depending on the time evolution of states, controls, and time itself, the integration is now fragmented and summed arc by arc. Then, the augmented index of merit becomes:

$$\mathcal{J}^* = \varphi + \boldsymbol{\mu}^T \boldsymbol{\chi} + \sum_{j=1}^{n_p} \int_{t_{(j-1)+}}^{t_{j-}} \left[\Phi + \boldsymbol{\lambda}^T (\mathbf{f} - \dot{\mathbf{x}}) \right] dt \quad (3.17)$$

By applying the same integration by parts procedure carried out previously, the augmented index of merit can be rewritten as:

$$\mathcal{J}^* = \varphi + \boldsymbol{\mu}^T \boldsymbol{\chi} + \sum_{j=1}^{n_p} \left(\boldsymbol{\lambda}_{(j-1)+}^T \mathbf{x}_{(j-1)+} - \boldsymbol{\lambda}_{j-}^T \mathbf{x}_{j-} \right) + \sum_{j=1}^{n_p} \int_{t_{(j-1)+}}^{t_{j-}} \left(\Phi + \boldsymbol{\lambda}^T \mathbf{f} + \dot{\boldsymbol{\lambda}}^T \mathbf{x} \right) dt \quad (3.18)$$

By taking the derivative and setting it to zero, the variation $\delta \mathcal{J}^*$ is obtained once again,

expressed this time for multiple arcs:

$$\begin{aligned}
 \delta \mathcal{J}^* = & \left(\frac{\partial \varphi}{\partial t_{(j-1)_+}} + \boldsymbol{\mu}^T \frac{\partial \boldsymbol{\chi}}{\partial t_{(j-1)_+}} - \mathcal{H}_{(j-1)_+} \right) \delta t_{(j-1)_+} + \\
 & + \left(\frac{\partial \varphi}{\partial t_{j_-}} + \boldsymbol{\mu}^T \frac{\partial \boldsymbol{\chi}}{\partial t_{j_-}} + \mathcal{H}_{j_-} \right) \delta t_{j_-} + \\
 & + \left(\frac{\partial \varphi}{\partial \mathbf{x}_{(j-1)_+}} + \boldsymbol{\mu}^T \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{x}_{(j-1)_+}} + \boldsymbol{\lambda}_{(j-1)_+}^T \right) \delta \mathbf{x}_{(j-1)_+} + \\
 & + \left(\frac{\partial \varphi}{\partial \mathbf{x}_{j_-}} + \boldsymbol{\mu}^T \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{x}_{j_-}} - \boldsymbol{\lambda}_{j_-}^T \right) \delta \mathbf{x}_{j_-} + \\
 & + \sum_{j=1}^{n_p} \int_{t_{(j-1)_+}}^{t_{j_-}} \left[\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}} + \dot{\boldsymbol{\lambda}}^T \right) \delta \mathbf{x} + \frac{\partial \mathcal{H}}{\partial \mathbf{u}} \delta \mathbf{u} \right] dt, \quad j = 1, \dots, n_p
 \end{aligned} \tag{3.19}$$

At this stage, the MPBVP optimality and transversality conditions are more conveniently formulated directly at the j -th boundaries. Consequently, rather than considering an isolated arc, the interface itself is examined by evaluating the variables at the instants immediately preceding and following it. Therefore, these relations take the following form:

$$\frac{\partial \varphi}{\partial t_{j_+}} + \boldsymbol{\mu}^T \frac{\partial \boldsymbol{\chi}}{\partial t_{j_+}} - \mathcal{H}_{j_+} = 0, \quad j = 0, \dots, n_p - 1 \tag{3.20a}$$

$$\frac{\partial \varphi}{\partial t_{j_-}} + \boldsymbol{\mu}^T \frac{\partial \boldsymbol{\chi}}{\partial t_{j_-}} + \mathcal{H}_{j_-} = 0, \quad j = 1, \dots, n_p \tag{3.20b}$$

$$\frac{\partial \varphi}{\partial \mathbf{x}_{j_+}} + \boldsymbol{\mu}^T \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{x}_{j_+}} + \boldsymbol{\lambda}_{j_+}^T = \mathbf{0}, \quad j = 0, \dots, n_p - 1 \tag{3.20c}$$

$$\frac{\partial \varphi}{\partial \mathbf{x}_{j_-}} + \boldsymbol{\mu}^T \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{x}_{j_-}} - \boldsymbol{\lambda}_{j_-}^T = \mathbf{0}, \quad j = 1, \dots, n_p \tag{3.20d}$$

Regarding the Euler-Lagrange equations for the co-states and the control equations, the considerations previously made for the TPBVP case remain valid.

3.4 Numerical Implementation of the Multi-Phase BVP

This section details the implementation of the Boundary Value Problem (BVP) within the framework of Optimal Control Theory (OCT) for the orbital transfer under consideration. The mission profile involves an initial continuous-thrust maneuver performed by a Hall-effect thruster, followed by two impulsive chemical maneuvers dedicated to lowering the perigee and subsequently recircularizing the orbit.

When employing indirect methods, ensuring the robustness of the code is crucial due to the strong non-linearities of the dynamics and the high sensitivity of the solution with respect to the initial conditions factors that can severely hinder numerical convergence.

As previously illustrated, OCT extends the original BVP into an augmented one, where a portion of the initial states and co-states is unknown. Consequently, the objective becomes determining the optimal initial vector $\mathbf{y}_0^* = [(\mathbf{x}^*)^T, (\boldsymbol{\lambda}^*)^T]^T$ capable of satisfying the imposed boundary conditions, as well as the optimality and interface conditions between the various arcs. For the numerical solution, the single-shooting method was adopted, selected for its straightforward implementation and reduced computational time.

While the introduction of the multi-phase formulation enhances algorithmic robustness, enabling indirect methods to handle complex mission architectures, it also introduces specific difficulties related to managing the boundaries of the arcs themselves. Since the duration of these arcs (Δt) is unknown, failing to account for this non-linearity can trigger severe numerical issues, leading to a highly ill-conditioned problem. To overcome this limitation, a new independent and dimensionless time variable is introduced for the ODEs under integration:

$$t_\epsilon = j - 1 + \frac{t - t_{j-1}}{\Delta t_j}, \quad (3.21)$$

where

$$\Delta t_j \triangleq t_j - t_{j-1}, \quad j = 1, \dots, n_p$$

The fundamental utility of this approach lies in the fact that, while the durations Δt_j remain unknown parameters to be optimized, the new variable t_ϵ maps the boundaries of each arc to integer values that are entirely independent of its actual duration. Specifically, the second arc will feature fixed and independent boundaries given by $t_\epsilon = [1, 2]$, thereby rendering the numerical solution considerably more robust.

The structure of the general method is reported below; the system of ODEs in its general expression for the indirect method takes the form:

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}(t), t) \quad (3.22)$$

or, in nondimensional quantities,

$$\dot{\check{\mathbf{y}}} = \mathbf{f}(\check{\mathbf{y}}(t_\epsilon), t_\epsilon) \quad (3.23)$$

where $\check{\mathbf{y}}$ indicates that both state and adjoint variables are expressed here as functions of the dimensionless time. Since the OCP may also include constant parameters, a new augmented vector $\mathbf{z}$ is introduced to account for these quantities:

$$\mathbf{z} = [\mathbf{y}^T \quad \mathbf{c}^T]^T \quad (3.24)$$

Substituting t_ϵ for t yields:

$$\dot{\check{\mathbf{z}}} = \frac{d\check{\mathbf{z}}}{dt_\epsilon} = \mathbf{f}(\check{\mathbf{z}}(t_\epsilon), t_\epsilon) \quad (3.25)$$

where

$$\dot{\check{\mathbf{y}}} = \frac{d\check{\mathbf{y}}}{dt_\epsilon} = \frac{d\mathbf{y}(t)}{dt} \Delta t_j \quad (3.26a)$$

$$\dot{\check{\mathbf{c}}} = \frac{d\check{\mathbf{c}}}{dt_\epsilon} = \mathbf{0} \quad (3.26b)$$

Consequently, a new set BCs is defined, incorporating both the imposed and the optimal ones:

$$\chi(\check{\mathbf{s}}) = \mathbf{0} \quad (3.27)$$

Specifically, $\check{\mathbf{s}}$ is defined as the vector containing the values assumed by the variables at all interfaces, both internal and external, namely:

$$\check{\mathbf{s}} = [\check{\mathbf{y}}_0^T \quad \check{\mathbf{y}}_1^T \quad \dots \quad \check{\mathbf{y}}_{n_p-1}^T \quad \check{\mathbf{y}}_{n_p}^T \quad \check{\mathbf{c}}^T]^T \quad (3.28)$$

The primary objective now lies in determining the optimal initial values that, through integration, lead to the desired final solutions while satisfying all constraints. To this end, the single-shooting method is implemented to find the optimal initial vector $\check{\mathbf{q}}_0^*$ that satisfies the boundary conditions $\chi(\check{\mathbf{q}}^*) = \mathbf{0}$.

This search process is iterative and requires the definition of an initial guess vector $\check{\mathbf{q}}^r = \mathbf{z}_0$. In this analysis, reference will be made to the most general case, where all initial variables are unknown.

For each j -th boundary, the error associated with BCs during the r -th iteration is evaluated as $\chi(\check{\mathbf{q}}^r)$. The values assumed by the BCs at the subsequent step ($r + 1$) are expressed as functions of those recorded for the residuals and the state vector in the last performed iteration (r -th); that is, via a first-order Taylor series expansion:

$$\chi(\check{\mathbf{q}}_{r+1}) = \chi(\check{\mathbf{q}}_r) + \frac{\partial \chi(\check{\mathbf{q}}_r)}{\partial \check{\mathbf{q}}_{r+1}} (\check{\mathbf{q}}_{r+1} - \check{\mathbf{q}}_r) \quad (3.29)$$

From which the Jacobian matrix $\check{\mathbf{J}}$ is defined, which is composed of the partial derivatives of the constraint vector with respect to the free variables of the vector advanced in time. For semplicity, it was introduced the following shortenings:

$$\check{\chi}_r \triangleq \chi(\check{\mathbf{q}}_r), \quad (3.30a)$$

$$\check{\mathbf{J}}(\check{\chi}_r) \triangleq \check{\mathbf{J}}(\chi(\check{\mathbf{q}}_r), \check{\mathbf{q}}_{r+1}) \quad (3.30b)$$

If the problem admits a solution, then it will occur that $\check{\chi}^{r+1} = \mathbf{0}$, and the iterative scheme can be written as:

$$\check{\chi}_r + \check{\mathbf{J}}(\check{\chi}_r) (\check{\mathbf{q}}_{r+1} - \check{\mathbf{q}}_r) = \mathbf{0} \quad (3.31)$$

In this way, at each iteration, the state of the design vector $\check{\mathbf{q}}_r$ and the corresponding updated value of the constraint vector $\check{\chi}_r$ are obtained.

$$\check{\mathbf{q}}_{r+1} = \check{\mathbf{q}}_r - [\check{\mathbf{J}}(\check{\chi}_r)]^{-1} \check{\chi}_r \quad (3.32)$$

To determine the Jacobian matrix, one proceeds with the product between two distinct matrices:

$$\check{\mathbf{J}}(\check{\chi}_r) = \frac{\partial \check{\chi}_r}{\partial \check{\mathbf{q}}_{r+1}} = \frac{\partial \check{\chi}_r}{\partial \check{\mathbf{s}}_r} \frac{\partial \check{\mathbf{s}}_r}{\partial \check{\mathbf{q}}_{r+1}} \quad (3.33)$$

- The first is easily evaluated and is obtained by differentiating the constrained quantities with respect to the vector of the variables at the interfaces $\check{\mathbf{s}}$;

- The second matrix instead contains the partial derivatives of all the variables calculated at the boundaries of the arcs with respect to the initial conditions.

This latter matrix can be expressed more clearly through a State Transition Matrix (STM), which maps the sequence of states at step $r + 1$ with respect to the previous step r .

Considering $t_{\epsilon 0}$ as the initial dimensionless time and $t_{\epsilon} \geq t_{\epsilon 0}$ a generic subsequent instant of time, the STM can be written as:

$$\frac{\partial \check{\mathbf{z}}_r}{\partial \check{\mathbf{q}}_{r+1}} = \check{\Phi}(t_{\epsilon}, t_{\epsilon 0}) \quad (3.34)$$

Such STM is governed in turn by its own system of ODEs that describes its temporal evolution, defined as:

$$\dot{\check{\Phi}}(t_{\epsilon}, t_{\epsilon 0}) = \frac{d\check{\Phi}(t_{\epsilon}, t_{\epsilon 0})}{dt_{\epsilon}} = \frac{d}{dt_{\epsilon}} \left(\frac{\partial \check{\mathbf{z}}}{\partial \check{\mathbf{z}}_0} \right) = \frac{\partial}{\partial \check{\mathbf{z}}_0} \left(\frac{d\check{\mathbf{z}}}{dt_{\epsilon}} \right) = \frac{\partial \check{\mathbf{z}}}{\partial \check{\mathbf{z}}} \frac{\partial \check{\mathbf{z}}}{\partial \check{\mathbf{z}}_0} \quad (3.35)$$

$$\dot{\check{\Phi}}(t_{\epsilon}, t_{\epsilon 0}) = \check{\mathbf{A}}(t_{\epsilon}) \check{\Phi}(t_{\epsilon}, t_{\epsilon 0}) \quad (3.36)$$

The initial value of the STM coincides with the identity matrix, namely $\check{\Phi}(t_{\epsilon}, t_{\epsilon 0}) = \mathbf{I}$. The Jacobian matrix of the system $\check{\mathbf{A}}(t_{\epsilon})$ is in turn composed of four submatrices, structured as follows:

$$\check{\mathbf{A}}(t_{\epsilon}) = \begin{bmatrix} \check{\mathbf{A}}_{\check{\mathbf{x}}\check{\mathbf{x}}} & \check{\mathbf{A}}_{\check{\mathbf{x}}\check{\boldsymbol{\lambda}}} \\ \check{\mathbf{A}}_{\check{\boldsymbol{\lambda}}\check{\mathbf{x}}} & \check{\mathbf{A}}_{\check{\boldsymbol{\lambda}}\check{\boldsymbol{\lambda}}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \dot{\check{\mathbf{x}}}}{\partial \check{\mathbf{x}}} & \frac{\partial \dot{\check{\mathbf{x}}}}{\partial \check{\boldsymbol{\lambda}}} \\ \frac{\partial \dot{\check{\boldsymbol{\lambda}}}}{\partial \check{\mathbf{x}}} & \frac{\partial \dot{\check{\boldsymbol{\lambda}}}}{\partial \check{\boldsymbol{\lambda}}} \end{bmatrix} \quad (3.37)$$

where the vector $\check{\mathbf{x}}$ indicates the state variables, while the vector $\check{\boldsymbol{\lambda}}$ represents the co-state variables. Within this notation, the order of the subscripts identifies respectively the variable at the numerator and the one at the denominator. An advantage of this formulation lies in the capability of being able to consider potential discontinuities at the interfaces. Indeed, a generic discontinuity on the j -th boundary can be included both in the design vector $\check{\mathbf{z}}$ and in the STM, through a relationship vector $\check{\mathbf{h}}$:

$$\check{\mathbf{z}}_{j+} = \check{\mathbf{h}} \cdot \check{\mathbf{z}}_{j-} \quad (3.38a)$$

$$\check{\Phi}(t_{\epsilon-}, t_{\epsilon 0}) = \frac{\partial \check{\mathbf{h}}}{\partial \check{\mathbf{z}}_j} \check{\Phi}(\tau^-, \tau_0) \quad (3.38b)$$

This also explains the absence of the need to differentiate between the values immediately preceding and following the boundary for the vector $\check{\mathbf{s}}$ in equation 3.28. In light of the discussion carried out, the indirect OCP, formulated as an MPBVP, translates into the search for the optimal initial state $\mathbf{z}_0^*$ that allows the trajectory to evolve toward the desired final state $\check{\mathbf{z}}_f^*$, while simultaneously satisfying all boundary conditions $\boldsymbol{\chi}$.

From a computational point of view, this operation requires the simultaneous integration of the main system of ODEs and the equations associated with the STM,

introduced in equations 3.25 and 3.36, respectively. This process is performed by implementing a variable-step, variable-order implicit multistep numerical method, based on the Adams-Moulton formulations according to the scheme described by Shampine and Gordon.

The use of a linear mapping based on the STM inevitably induces errors during the iterative process of the Differential Corrector (DC), which could alter the convergence of the method. Defining the maximum error as $E_{max} = \max_i(\chi_i)$, in order to increase the robustness of the procedure, two specific control strategies have been implemented:

- Introduction of a corrective factor: a relaxation parameter is used, usually equal to $k_1 = 0.1 \div 1$, during the update phase of the variables, as follows:

$$\check{\mathbf{z}}_{r+1} = \check{\mathbf{z}}_r - k_1 \cdot \left[\check{\mathbf{J}}(\check{\chi}_r) \right]^{-1} \check{\chi}_r \quad (3.39)$$

It is noted that higher values of k_1 can be used when the solution is already in a neighborhood of the optimal solution, while lower values are indicated in an initial phase of rough estimation of the unknowns.

- Control on the subsequent error: a monitoring is performed with respect to the error on the boundary conditions of the previous step, expressed as follows:

$$E_{max,r+1} < k_2 E_{max,r} \quad (3.40)$$

The constant used in this case, k_2 (usually equal to $2 \div 3$), results suitable to support the initial phases of the iterative process, in which a temporary increase of the maximum error could occur while the algorithm orients itself toward the correct direction of optimality in the search space. If this inequality is violated, one proceeds by applying a bisection procedure on the factor k_1 up to a maximum of 5 consecutive times.

3.5 Formulation of the Spacecraft Optimal Control Problem

The dynamic model of the spacecraft used in this thesis, together with the system of ODEs that governs the evolution of the trajectory, has already been detailedly presented in Chapter 2 (see equations 2.12). The application of OCT to such a differential system aims to determine the optimal control law $\mathbf{u}^*(t)$ capable of maximizing the final mass of the vehicle at the end of the trajectory. For this purpose, the performance index of the problem coincides with the value of the spacecraft mass evaluated on the last arc ($j = n_p$):

$$J = \varphi = m_f = m_{n_p} \quad (3.41)$$

For the resolution, the Mayer formulation is adopted (described in Section 4.2), which implies $(\Phi = \mathbf{0})$. In our study, the state vector of the problem is defined as $\mathbf{x}(t) \in \mathbb{R}^n$:

$$\mathbf{x} = \begin{bmatrix} r & \vartheta & \varphi & u & v & w & m \end{bmatrix}^T \quad (3.42)$$

To which an adjoint variable is added for each state variable, obtaining the augmented state vector of the problem $\mathbf{y}(t) \in \mathbb{R}^{2n}$:

$$\mathbf{y} = \begin{bmatrix} r & \vartheta & \varphi & u & v & w & m & \lambda_\vartheta & \lambda_\phi & \lambda_u & \lambda_v & \lambda_w & \lambda_m \end{bmatrix}^T \quad (3.43)$$

We now proceed by writing the Hamiltonian function of the problem, obtained starting from the system of ODEs:

$$\mathcal{H} = \boldsymbol{\lambda}^T \mathbf{f} = \sum_{i=1}^{2n} \lambda_i f_i = \lambda_r u + \lambda_\vartheta \frac{v}{r \cos \varphi} + \lambda_\phi \frac{w}{r} + \quad (3.44)$$

$$+ \lambda_u \left(-\frac{\mu}{r^2} + \frac{v^2}{r} + \frac{w^2}{r} + \frac{Tu}{m} + (a_p)_u \right) + \quad (3.45)$$

$$+ \lambda_v \left(-\frac{uv}{r} + \frac{vw}{r} \tan \varphi + \frac{Tv}{m} + (a_p)_v \right) +$$

$$+ \lambda_w \left(-\frac{uw}{r} - \frac{v^2}{r} \tan \varphi + \frac{Tv}{m} + (a_p)_w \right) + \quad (3.46)$$

$$- \lambda_m \frac{T}{c} \quad (3.47)$$

It can also be expressed in a compact form as follows:

$$\mathcal{H} = \boldsymbol{\lambda}_r^T \mathbf{V} + \boldsymbol{\lambda}_V^T \left(\frac{\mathbf{T}}{m} - \frac{\mu}{r^3} \mathbf{r} + \mathbf{a}_p \right) - \lambda_m \frac{T}{c} \quad (3.48)$$

Isolating now the terms that multiply the thrust-to-mass ratio and engulfing them into the factor $\mathcal{S}_{\mathcal{F}}$, a further version of the Hamiltonian is obtained:

$$\mathcal{H} = \boldsymbol{\lambda}_r^T \mathbf{V} + \boldsymbol{\lambda}_V^T \left(-\frac{\mu}{r^3} \mathbf{r} + \mathbf{a}_p \right) + \frac{T}{m} \mathcal{S}_{\mathcal{F}} \quad (3.49)$$

in which, precisely, $\mathcal{S}_{\mathcal{F}}$ results to be equal to:

$$\mathcal{S}_{\mathcal{F}} = \boldsymbol{\lambda}_V^T \frac{\mathbf{T}}{T} - \lambda_m \frac{m}{c} \quad (3.50)$$

In this context, the control vector $\mathbf{u}(t)$ consists of the thrust vector $\mathbf{T}$, having the freedom to act both on its direction and on its intensity. Based on the PMP, the optimal control law $\mathbf{u}^*$ of the problem is the one that maximizes the performance index J of the trajectory, and must be sought by maximizing the Hamiltonian function.

As already discussed previously, the linearity of the Hamiltonian with respect to the thrust T , combined with the application of the PMP, leads to a discontinuous control of the bang-bang type (neglecting singular arcs). It is therefore deduced that the value of the thrust is governed directly by the sign of the switching function $\mathcal{S}_{\mathcal{F}}$ according to the following scheme:

- $T = T_{max}$ if $\mathcal{S}_{\mathcal{F}} > 0$ (maximum thrust);

- $T = 0$ if $\mathcal{S}_{\mathcal{F}} < 0$ (zero thrust).

As for the optimal direction of the thrust instead, Lawden has demonstrated that the maximum of $\mathcal{S}_{\mathcal{F}}$ (and therefore of the Hamiltonian) is obtained by orienting the thrust vector in a direction parallel to the costate vector of the velocity $\boldsymbol{\lambda}_V$, known in literature as primer vector:

$$\boldsymbol{\lambda}_V = \begin{bmatrix} \lambda_u \\ \lambda_v \\ \lambda_w \end{bmatrix}, \quad \lambda_V = \|\boldsymbol{\lambda}_V\| \quad (3.51)$$

From which the scalar form of the $\mathcal{S}_{\mathcal{F}}$ is obtained:

$$\mathcal{S}_{\mathcal{F}} = \lambda_V - \lambda_m \frac{m}{c} \quad (3.52)$$

As already illustrated in Section 2.2, the thrust vector $\mathbf{T}$ is decomposed into its scalar components with respect to the ZEN (Zenith-East-North) reference system. This geometric decomposition employs the elevation angle α_T (namely the out-of-plane angle) and the heading or azimuth angle β_T (the in-plane angle), translating into the following relationships:

$$\mathbf{T} = \begin{bmatrix} T_u \\ T_v \\ T_w \end{bmatrix} = T \begin{bmatrix} \sin \alpha_T \\ \cos \alpha_T \cos \beta_T \\ \cos \alpha_T \sin \beta_T \end{bmatrix}, \quad T = \|\mathbf{T}\| \quad (3.53)$$

To obtain the optimal values of the thrust angles, it is necessary to partially differentiate the Hamiltonian with respect to them and nullify these first derivatives, thus obtaining:

$$\frac{\partial \mathcal{H}}{\partial \alpha_T} = \lambda_u \cos \alpha_T - (\lambda_v \cos \beta_T + \lambda_w \sin \beta_T) \sin \alpha_T = 0 \quad (3.54a)$$

$$\frac{\partial \mathcal{H}}{\partial \beta_T} = -\lambda_v \sin \beta_T + \lambda_w \cos \beta_T = 0 \quad (3.54b)$$

From which, through algebraic manipulations, the optimal directions can be expressed as:

$$\sin \alpha_T = \frac{\lambda_u}{\lambda_V} \quad (3.55a)$$

$$\cos \alpha_T \cos \beta_T = \frac{\lambda_v}{\lambda_V} \quad (3.55b)$$

$$\cos \alpha_T \sin \beta_T = \frac{\lambda_w}{\lambda_V} \quad (3.55c)$$

These relationships define the directional cosines of the primer vector itself and coincide with the components of the thrust vector of equation 3.53. Consequently, these equations allow determining the optimal thrust angles, exploiting the costate variables obtained by integration starting from the Euler-Lagrange equations. For brevity of discussion, the explicit formulation of these equations for the problem under consideration is omitted in this section and is presented in Appendix A.

At this point, the problem results completely defined. A final fundamental observation concerns the automatic calculation of the thrust phases ($\mathcal{S}_{\mathcal{F}} > 0$) and coast phases ($\mathcal{S}_{\mathcal{F}} < 0$). Indeed, in strongly non-linear dynamic systems and for high-fidelity models like the one developed in this work, the switching function could present small fluctuations around zero, causing numerical criticalities. Consequently, even a minimal variation of the initial conditions could abruptly alter the behavior of the $\mathcal{S}_{\mathcal{F}}$, introducing an unnecessary coast arc or eliminating a desired thrust phase, thus compromising the convergence of the algorithm. The management of thrust discontinuities represents one of the most critical aspects in indirect optimization; to obviate this, various strategies have been implemented in literature, such as smoothing techniques, approaches based on homotopy and continuation, or the more recent methods of uniform trigonometrization and integrated control regularization.

In this work, it has been chosen to follow an alternative line: the switching structure, namely the sequence of coast and thrust phases, is set a priori and provided at the beginning of the analysis. The process takes off from an elementary arc structure, to then pass to progressively more complex configurations whenever suggested by the compliance with the PMP. A Multi-Point Boundary Value Problem (MPBVP) is thus generated, in which the duration of each single phase, Δt_j , is unknown and constitutes an object of optimization. To the problem thus formulated, the further boundary condition is added which imposes the nullification of the $\mathcal{S}_{\mathcal{F}}$ at the interface points, that is to say, at the ignition and shutdown instants of the thruster. This strategy allows a faster, more robust, and more precise implementation compared to the instantaneous evaluation of the thrust level based on the pointwise value of the switching function.

A further and significant advantage of this choice lies in the management of boundary solutions. Should a trajectory find itself at the limit between a single-ignition structure (Thrust-Coast) and a double-ignition one (Thrust-Coast-Thrust-Coast), the standard approach based on the instantaneous verification of the $\mathcal{S}_{\mathcal{F}}$ could oscillate indefinitely between the two configurations without converging. On the contrary, by forcing the thrust arcs a priori, the solver manages to converge to a solution even in the case where the PMP were momentarily violated in a portion of the trajectory. Then, if the provided structure perfectly responds to the optimal requirements, the PMP will result fully respected and the solution will be optimal; otherwise, the trend of the switching function will clearly suggest how to modify the sequence of the arcs, allowing to promptly obtain a new convergent solution [11].

3.6 The OCULUS Computational Solver

In the context of solving the OCP addressed in this thesis, OCULUS (Optimal Control with User-friendly Layout Unified Solver) was used, a software environment developed by Mascolo [12]. As anticipated in Section 3.1, IM are extremely efficient but suffer from a strong sensitivity to the initial guesses of the adjoint variables, significantly increasing the user's manual effort in initializing the problem.

OCULUS fits into this context precisely to mitigate this criticality, providing an interactive graphical user interface (GUI) updated in real time. This possibility allows the

user to deeply understand the physical structure of the trajectory, making it possible to vary the initial guesses in a targeted and logical way, based on the visual behavior of the variables, rather than proceeding by purely mechanical trials.

The software allows solving TPBVP and MPBVP for any system of ordinary differential equations expressible in Bolza form, using robust correction algorithms, implementing an accurate calculation of the Jacobian via a finite difference scheme, and an advanced integration.

The adoption of OCULUS was fundamental in this thesis because it provided a "non-black-box" solution interface. This approach made it possible to fully master the theoretical foundations of optimal control and, simultaneously, to navigate effectively through the software environment. The support of the graphical interface accelerated the delicate work of initializing and refining the OCP.

At the same time, the essential component of mathematical modeling was entirely preserved: the explicit formulation of the problem, including the derivation and writing of the ODEs, boundary conditions (BC), and the cost functional, remained under the direct and manual control of the author.

Chapter 4

Deorbiting Preliminary Analysis and Spacecraft Budget

Chapter 4 will be dedicated to the conceptual definition and setup of the problem. It will justify the adopted engineering choices, the operational orbits targeted for trajectory optimization, as well as the specifications related to the satellite type, propulsion system, and initial mass of the SC. The preliminary study of these aspects is fundamental to guarantee a rigorous adherence to the physical reality of the modeled problem. A reasonable SC mass value, for instance, is crucial because in accordance with the Newton's second law of motion, for a given thrust, a higher or lower mass determines an inversely proportional acceleration. This aspect is important in an optimization problem since it directly influences the acceleration profile and, consequently, the propellant mass consumed along the trajectory.

In particular, the first section will present the analysis related to the deorbiting of space debris, evaluating their re-entry times through the implementation of the equations of motion perturbed by the action of atmospheric drag. Subsequently, the next section will present the two chosen propulsion systems and their main characteristics. Finally, starting from these data, the third section will calculate the mass budget of the propulsion system, thus providing an overall estimate of the total mass of the satellite.

4.1 Preliminary Deorbiting Analysis and Operational Orbit Selection

As a first step in this thesis, the decay times of various debris pieces were evaluated based on their diameter (and thus their cross-sectional area) and mass. This was done to determine which objects are the most persistent in space and to select the operational altitude and the specific target debris for this ADR mission.

To achieve this, an algorithm that integrates the equations of motion with atmospheric drag perturbations was implemented, as explained in detail in Section 2.3.1. To optimize the process and reduce computational time, the code was accelerated using the `@njit` decorator from the Numba library [16]; adhering to its compilation constraints allowed multi-year orbit integrations to be completed in just a few minutes. Additionally, a

termination event function was implemented to stop the numerical integration when the debris reaches the Kármán line at an altitude of 100 km, where the atmosphere becomes dense enough that the object can be considered successfully deorbited.

Circular and equatorial orbits starting from 500 km and above were taken into account in this study. Specifically, initial operational altitudes of 500, 520, 550, 580, 600, 620, 650, and 680 km were selected to evaluate the decay timelines, with the option to expand the analysis to 700 km and beyond if required.

At this point, a circular cross-sectional area was considered as a first approximation, using the diameter as a variable parameter. Furthermore, a mass range between 10 and 100 kg was chosen for the target debris.

Firstly, considering the worst-case scenario of 100 kg, the analysis was conducted by calculating the decay time for three different diameters (0.8, 1, and 2 m), as shown in Figure 4.1:

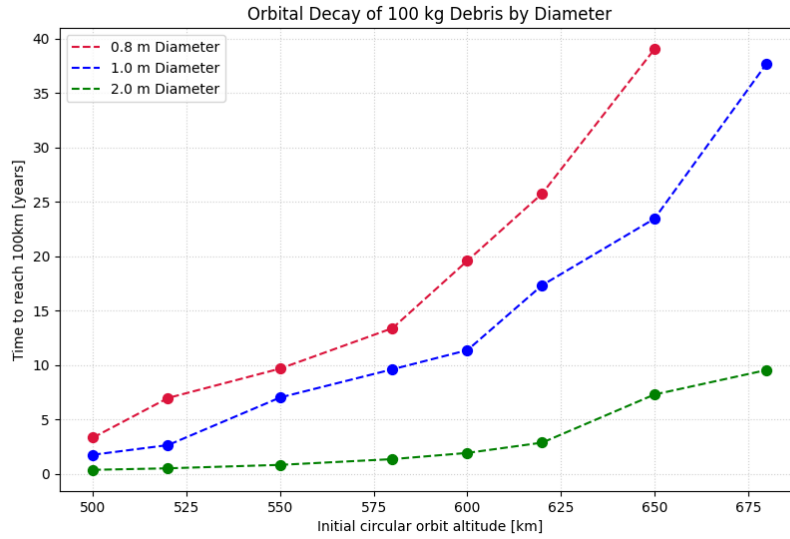


Figure 4.1. Deorbit time as a function of debris diameter

In this graph, the abscissa represents the altitude in kilometers, while the ordinate shows the decay time in years. The points correspond to the specific values computed by the algorithm at the selected altitudes. As expected, the decay time of a larger object is shorter because the atmospheric drag force acts on a larger cross-sectional area, in accordance with Equation 2.15. This implies that a 100 kg debris object with a diameter of 0.8 m can remain in space for more than 5 years starting from an altitude of 520 km, with the lifetime increasing almost exponentially with altitude. In particular, the data point at 680 km for this specific object does not appear on the plot because the integration concluded after 200 years without reaching the deorbiting condition. Consequently, it can be concluded that a 100 kg debris with a 0.8 m diameter orbiting above 680 km will remain trapped there for centuries unless actively removed. Conversely, the lifetimes for

a 1 m diameter object are slightly shorter, exceeding the 5-year threshold starting around 550 km. For a relatively large object with a 2 m diameter, the decay times are even more compressed, drastically reducing its overall orbital permanence. In terms of orbital hazard, while a shorter lifetime inherently lowers the cumulative collision probability over time, this advantage is partially counterbalanced by the object's larger physical cross-section.

Subsequently, taking into examination the worst-case section scenario (0.8 m diameter), that is the configuration that persists longest in orbit, the two limiting mass cases were analyzed, yielding the following results:

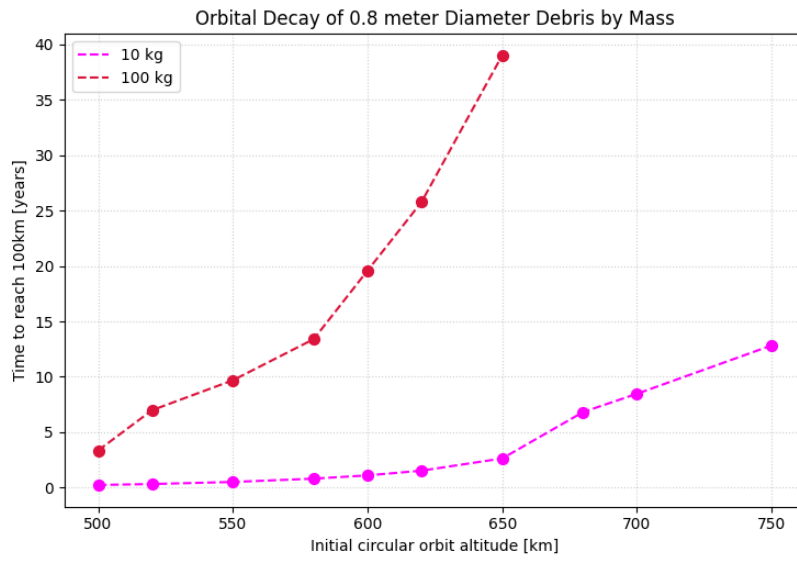


Figura 4.2. Deorbit time as a function of debris mass

In accordance with Newton's second law of motion, a smaller mass experiences a greater deceleration when subjected to the same aerodynamic force acting on an identical surface area, thus deorbiting in a shorter time. Indeed, the same 0.8-m diameter debris that previously required approximately 7 years to deorbit from 520 km now completes its decay from 680 km within an equivalent timeframe when its mass is reduced to 10 kg. It should be noted that the 10 kg case for a 0.8 m diameter represents a hypothetical limiting scenario for study purposes; in reality, debris of these dimensions, generally composed of metallic alloys, would likely possess a higher mass, placing actual targets within the envelope bounded by these two curves.

As theoretically predicted by the ballistic coefficient, Eq. 2.19, this analysis highlights that the most critical and persistent debris are those characterized by a small cross-section combined with a high mass. Furthermore, the numerical simulations conducted herein yield a concrete and precise timeline for the deorbiting processes across the different scenarios investigated:

Tabella 4.1. Decay times expressed in years for different debris mass and diameter configurations as a function of the initial circular operational altitude.

Initial Altitude [km]	Mass: 100 kg			Mass: 10 kg
	Diameter: 0.8 m [years]	Diameter: 1.0 m [years]	Diameter: 2.0 m [years]	Diameter: 0.8 m [years]
500	3.35	1.77	0.38	0.24
520	6.99	2.66	0.52	0.32
550	9.69	7.03	0.84	0.51
580	13.40	9.62	1.37	0.81
600	19.59	11.38	1.94	1.10
620	25.77	17.35	2.88	1.53
650	39.03	23.46	7.30	2.63
680	—	37.72	9.56	6.82
700	—	—	—	8.47
750	—	—	—	12.83

However, the overall hazard is also driven by the spatial density at specific altitudes and the physical size of the debris, since an increased area directly expands the collision cross-section. Therefore, by balancing these factors with the parametric results of the deorbiting timelines across the different mass-to-area scenarios, a circular operational orbit around 800 km was selected as the optimal trade-off for the mission. At this altitude small and massive debris (close to the 100 kg target limit of the chaser satellite) are trapped for centuries and thus they require removal. Targetting small, lightweight debris at this altitude is also justified, as their lifetime reasonably exceeds the 5 year threshold, making the economic investment worthwhile. A similar reasoning applies to large, massive objects, with the additional critical factor that their extended cross-sectional area poses a significantly higher geometric threat to operational constellations.

More precisely, for the parametric analysis of this study, the hypothetical debris targets are placed at a nominal altitude of 790 km. This specific choice serves purely as a representative baseline to conduct a parametric evaluation of the Hall effect thruster's propellant consumption and transfer duration during the approach phase. A perfectly analogous analysis can be effortlessly extended to targets located a few kilometers above or below the 800 km operational orbit. Ultimately, selecting this altitude represents a conservative design choice. Because operations at 800 km are more energetically demanding than those at lower altitudes, meeting the requirements for this altitude ensures a longer spacecraft lifetime when adapting to potentially critical targets in lower orbits.

4.1.1 Debris Release Orbit Selection

A similar line of reasoning was followed to choose the debris release orbit, whether for a single target or multiple objects. The primary objective of the mission is to ensure an orbital decay of less than 5 years for the captured space junk, which would otherwise remain in orbit for centuries without intervention. While executing this phase, it is crucial to select a release target where the debris does not pose a significant hazard or interference to operational constellations. Based on the deorbiting analysis conducted for

various debris sizes and masses, it was evident that a 500 km perigee would guarantee the fulfillment of the primary objective. However, since this altitude corresponds to a highly crowded orbit, releasing debris there should be avoided. For this reason, despite requiring a more energetically expensive maneuver, it was decided to release the object or objects at a 300 km perigee of an 800×300 km elliptical orbit. Furthermore, to validate this choice, the worst-case decay scenario based on the ballistic coefficient was simulated. This configuration combines the smallest cross-sectional area taken into account (0.5 m^2 , that correspond to 0.8 m diameter) with the maximum mass transportable by the spacecraft (100 kg). The implementation results demonstrate that, when released into the chosen orbit, even this highly conservative debris profile will completely deorbit within 1.10 years.

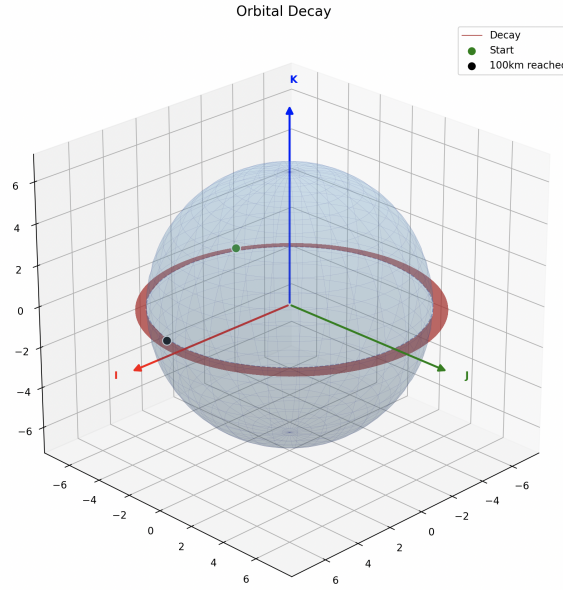


Figura 4.3. Orbital decay trajectory

Automatically, extending this scenario to all other possible cases, the decay time remains shorter than one year. These alternative configurations include, for instance, a 100 kg debris piece with a larger cross-sectional area, or single and multiple debris objects with a 0.5 m^2 area but a smaller mass. For example, a 100 kg target with a 2 m^2 surface area will decay in approximately 0.26 years.

4.2 Propulsion Systems Characterization

For the purposes of this mission it was decided to implement a dual propulsion system. The first phase, which involves approaching the target debris, is performed by an electric Hall effect thruster. Its aim is to transfer the chaser SC from its operational, circular, and equatorial 800 km orbit to the position of the target debris within a few kilometers of this orbit. Since these maneuvers involve distances on the order of a few kilometers to

a few tens of kilometers, the longer transfer times resulting from the lower thrust level are acceptable and fully justified by the significantly lower propellant consumption.

Conversely, the more demanding maneuvers of lowering the perigee to 300 km and recircularizing the orbit back at 800 km, are executed via chemical propulsion. A green monopropellant system was selected to provide a higher thrust and ensure that the SC can complete his mission within just a few days.

4.2.1 Hall-Effect Electric Propulsion

Hall-effect thrusters (HET) are one of the most diffused and reliable among the advanced electric propulsion systems. The first prototype was flown by the Soviet Union on the Meteor-1-10 spacecraft in 1971, followed by subsequent flights from European and U.S. manufacturers. In recent years, particularly since 2019, the deployment of HETs has registered a rapid increase due to constellations such as SpaceX's Starlink and OneWeb, where each spacecraft is equipped with them [14].

The points of strength of these thrusters are their simplicity, reliability, high specific impulse, high thrust-to-power ratio and good efficiency. For all these reasons, it has been chosen as one of the main thrust source of the developed SC.

The operating principle of HET is based on the ionization and electrostatic acceleration of the propellant. Traditionally the best choice to feed HET is xenon, due to its elevated molecular weight, low ionization energy and favorable handling qualities. However, especially in recent times, other gases such as krypton and argon have been adopted in these thrusters due to their lower cost. This is, for instance, the path followed by Starlink, which preferred to sacrifice some efficiency while optimizing costs. In fact, these last two propellants show a lower efficiency than xenon, along with some other storability issues [14].

From the physical point of view, the thruster is structured as shown in Figure 4.4. The propellant is injected from the base of the discharge chamber, at the annular anode. The coexistence of a strong axial electric field $\vec{E}$ and a radial magnetic field $\vec{B}$, produces an $\vec{E} \times \vec{B}$ force acting on the electrons, giving rise to an intense azimuthal electron current. This current, which is many times greater than the beam current, ionizes the propellant via collision. The resulting positive ions are then accelerated by the electrostatic force. Before being expelled, the ion flow is neutralized by electrons emitted from a cathode. The cathode is generically positioned externally to the HET body, recently configurations with a centrally mounted cathode have also been developed, such as the one illustrated in Figure 4.4 [14]. It is precisely the expulsion of this net-neutral jet that generates the thrust, in accordance with Newton's third law.

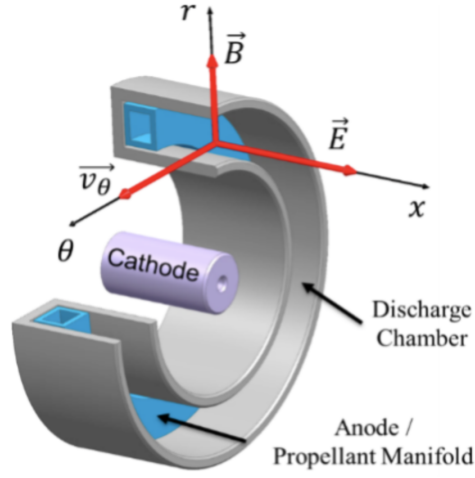


Figura 4.4. Hall-effect Thruster schematic

Among all existing and flight-proven HETs, the PPS[®]1350, shown in Figure 4.5, was chosen for this study. Developed by Safran Spacecraft Propulsion [18], this thruster has a long heritage of reliability, having been the first European HET to fly, back in 2003, onboard the SMART-1 lunar probe [14].



PPS[®]1350 [Stationary plasma thruster]
 • Designed for orbit-topping and station-keeping
 • 40% weight savings versus conventional propulsion
 • Excellent thrust-to-electrical power ratio
 • Set a record for cumulated operating time in space

Figura 4.5. Safran PPS[®]1350 HET

The specifications of this thruster adapt well to the chaser SC project requirements and are briefly summarized in Table 4.2:

Tabella 4.2. Technical specifications of the Safran PPS®1350 Hall-effect thruster.

Parameter	Value	Unit
Propellant	Xenon	—
Nominal Power	1500	W
Thrust	90	mN
Specific Impulse (I_{sp})	1660	s
Total Impulse	3.4	MN.s
Number of cycles	7300	—
Discharge current	4.28	A
Efficiency	50	%
Power supply voltage	350	V
Xenon supply system pressure	2.50 to 2.80	bar
Weight(including xenon flow control system)	4.8	kg

4.2.2 High Performance Green Monopropellant

While electric thrusters offer high efficiency for long-term maneuvers, chemical propulsion represent the most consolidated technology in the space propulsion field, producing the high thrust required for rapid orbital trajectories.

Among the chemical propulsion alternatives, especially liquid-based ones, a monopropellant feed system was chosen due to its simplicity and reduced mass. Fundamentally, a chemical liquid monopropellant thruster produces thrust by accelerating high-pressure gases, which originate from propellant decomposition, and expelling them through a convergent-divergent nozzle [14]. The system uses a catalyst to decompose the monopropellant, thereby generating hot gases.

The most common and historically widely used monopropellant in space is hydrazine. It shows good properties such as long-term storability, remaining liquid at ambient temperature, and being easy to ignite; however, efforts are being made to restrict its use due to its high toxicity, corrosiveness, and potential carcinogenicity. Thus, several "green" propellants have been developed, which are characterized by the absence of vapor hazards, making them safe to handle with appropriate precautions. For this purpose, monopropellants based on ionic liquids have been intensively developed in the recent years until becoming space-proven. Essentially, they consist of ionic salts dissolved in water and blended with a fuel. Combustion is achieved through a catalyst that interacts with the aqueous ionic salt and fuel mixture [14].

Among the two mature ionic liquid monopropellant blends, the one chosen to feed the chemical propulsion system of the SC in this study is LMP-103S, which is based on ammonium dinitramide (ADN). The advantages of using this propellant are primarily the absence of vapor hazard, alongside a higher specific impulse and higher density-specific impulse (1.3 times greater) compared to hydrazine. Additionally, it exhibits excellent long-term storability. On the other hand, this solution requires a higher preheat temperature for the catalyst and leads to a higher combustion temperature, thereby requiring

the use of more advanced, high-performance, and expensive catalysts and materials.

ECAPS offers a range of High-Performance Green Propulsion (HPGP) thrusters, including the 22 N model chosen for this study, which is shown in Figure 4.6. Specifically, it was decided to install 10 of these 22 N thrusters to achieve a total maximum thrust of 220 N.

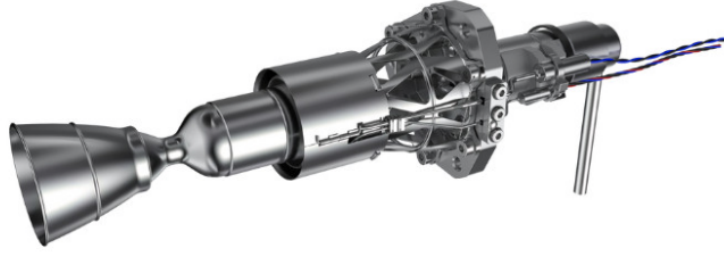


Figura 4.6. 22 N HPGP Thruster

The most relevant operating parameters are reported below in Table 4.3 [5]:

Tabella 4.3. Technical specifications of the High Green Propulsion (HPGP) thruster.

Parameter	Value	Unit
Propellant	LMP-103S	—
Propellant Density	1242	kg/m ³
Propellant Temperature	10 to 50	°C
Operating Pressure	5.5 to 24	bar
Thrust (Steady State)	22 to 5.5	N
Specific Impulse (Steady State)	>242 to 232	s
Propellant Throughput	53	kg
Total Impulse	Qualification target: 150	kg
	125	kNs
	Qualification target: >345	kNs
Pulse Length	20 to 38	ms to minutes
Thruster Mass (incl. harness)	1.1	kg

4.3 Spacecraft Propulsion Systems Mass Budget

In this section, the mass budget of the dual propulsion system will lead to a realistic estimation of the spacecraft's total mass. This approach was chosen to guarantee an accurate study based on realistic parameters, especially since the mass of the propulsion system could be significantly larger than that of a standard SC.

For precision, the ECSS margin philosophy [8] was taken into account both in the mass evaluation of single components and on the total dry mass of the subsystem. In particular, according to this regulation, a 5% margin was adopted at the equipment level for the majority of the components, as they consist of "Off-The-Shelf" (OTS) items. However, for a few specific elements, more conservative margins of 10% and 20% were applied instead. Additionally, a 20% margin was imposed on the nominal dry mass at launch; in this specific case, this margin was applied individually to the dry mass of the propulsion system and then scaled proportionally to the final mass of the spacecraft. Furthermore, a mass equivalent to 5% of the nominal total dry mass was added to account for the harness.

4.3.1 Mass Budget of the Hall-Effect Thruster System

For the mass budget of the electric propulsion system, the evaluation began with the mass of the HET, which is equal to 4.8 kg (as shown in Table 4.2) and includes the xenon flow control system. Subsequently, to complete the dry architecture, a suitable PPU Mk1 [2], a high-voltage electrical power converter with a mass of 10.9 kg, was added, along with the smallest available 7 L titanium xenon tank [13], weighing 3.5 kg. The tank capacity determines the maximum onboard xenon volume, thereby imposing a physical constraint on the maximum mass of gas propellant that can be loaded.

Once the dry mass was obtained, the propellant mass was added to calculate the total wet mass of the subsystem. To this end, although the xenon tank can be underfilled below its maximum capacity, it was chosen to load it to its full capacity. Consequently, for a 7 L tank and assuming a storage density of 1.6 kg/L [22], a propellant mass of 11.2 kg was calculated. This density accounts for xenon being in its supercritical phase due to the storage conditions inside the tank.

To summarize these evaluations and design choices, Table 4.4 is presented:

Tabella 4.4. Mass budget for the electric propulsion system.

Component	Nominal Mass [kg]	Margin [%]	Margined Mass [kg]
Thruster	4.8	5%	5.04
PPU Mk1	10.9	5%	11.44
Xenon Tank (7L)	3.5	5%	3.67
Total Dry Mass	20.15	20%	24.18
Propellant (Xenon)	—	—	11.2
Total Wet Mass	—	—	35.38

Therefore, a total dry mass with an ECSS margin of 24.18 kg and a total wet mass of 35.38 kg are obtained.

4.3.2 Mass Budget of HPGP Thruster System

Moving to the chemical propulsion system, an analogous first-approximation reasoning was conducted. In particular, ten 22 N HPGP thrusters were adopted, resulting in a total combined mass of 11.0 kg. To contain the propellant required by all these thrusters, a single tank configuration was considered, and further evaluations were carried out to select the most suitable model.

First of all, according to the thruster specifications reported in Table 4.3, each unit has been tested and flight-proven up to a propellant throughput of 53 kg. Although this limit can hypothetically be extended up to a qualification target of 150 kg, for reliability reasons it was decided to adhere to the demonstrated limit. Therefore, a maximum mass of 350 kg of propellant can be utilized by the chemical propulsion system, applying a margin factor of 1.5 with respect to the limit. This represents the propellant mass constraint for the mission, which translates into a constraint on the number of optimized trajectories that can be performed during the ADR mission.

From this data, the required volume for propellant storage was determined to be 282.2 L, considering the density indicated in Table 4.3. Along with the propellant, the tank must contain helium as the pressurant gas, operating in a blowdown mode. To determine the total tank capacity, the initial gas volume (ullage) is assumed to be 25% of the total tank volume. Starting from the required liquid propellant volume of 282.2 L, the necessary helium gas volume is calculated to be 94.1 L, leading to an estimated total internal volume of 376.3 L. Based on these requirements, a suitable tank with a mass of 23 kg was selected from the Northrop Grumman reference catalog [15].

Finally, to complete the dry architecture, an estimated mass of 2.8 kg was allocated for the Propulsion Control Unit (PCU) shared between both propulsion systems. This component, along with an estimated set of lines, valves, and filters, represents a realistic mass allocation based on preliminary design estimates for multi-thruster configurations.

As a final step, the propellant mass and the pressurant gas mass were added to the total dry mass of this propulsion system, resulting in a total wet mass of 400.5 kg. This primary HPGP thruster system mass budget can be summarized as follows:

Tabella 4.5. Mass budget for the chemical propulsion system.

Component	Nominal Mass [kg]	Margin [%]	Margined Mass [kg]
Thrusters (10x HPGP 22N)	11.00	5%	11.55
PCU (Shared System)	2.80	10%	3.08
PMD Tank (LMP-103S + Helium)	23.00	5%	24.15
Lines, Valves, and Filters	2.70	20%	3.24
Total Dry Mass	39.50	20%	42.02
Propellant (LMP-103S)	—	—	350.00
Pressurant Gas (Helium)	—	—	0.50
Total Wet Mass	—	—	400.52

4.3.3 Total mass budget of Propulsion System

By combining the values obtained separately from each propulsion architecture, the overall mass budget for the entire propulsion system is determined and summarized in Table 4.6:

Tabella 4.6. Overall propulsion system mass budget.

Mass Category	Chemical Propulsion [kg]	Electric Propulsion [kg]	Total Propulsion System [kg]
Total Dry Mass	50.00	24.18	74.18
Total Propellant Mass	350.50	11.20	361.70
Total Wet Mass	400.50	35.38	435.88

4.4 Spacecraft Final Mass

The final step required to gather all the necessary data to initiate the analysis, and consequently the trajectory optimization for the selected mission, is to define the initial SC mass. The approach followed in this study utilizes an inverse proportion. Specifically, a percentage of the total dry mass was allocated to the propulsion system, from which the overall mass of the spacecraft was subsequently derived.

A reference publication titled "Active Debris Removal mission design in Low Earth Orbit" [3] suggests a 12% mass allocation for the propulsion system relative to the total dry mass for an ADR mission. Given that the current study implements a dual propulsion

architecture, this allocation was increased to 14%. Consequently, a total dry mass of 529.86 kg (which already incorporates the 20% system-level ECSS margin) was calculated. To this value, an additional 5% was added to account for the harness, yielding a final total dry mass of **556.35 kg** [8].

From this, the total wet mass is calculated by adding the total propellant mass as:

$$M_{wet} = 556.35 \text{ kg} + 361.70 \text{ kg} = \mathbf{918.05 \text{ kg}}$$

This value represents the initial spacecraft mass variable utilized in the optimization problem described in the next chapter.

It is worth noting that, compared to standard LEO satellites designed for other operational missions, the mass budget allocation demands higher percentages for both the propulsion system and the payload. Indeed, the payload must include robotic arms (or nets) and advanced Guidance, Navigation, and Control (GNC) systems specifically dedicated to debris capture operations. Assuming approximately 15% of the total dry mass is dedicated to the robotic system, a reasonable allocation for the payload system mass would be around 22%.

Chapter 5

Mission Profile and Optimal Trajectory Solutions

This chapter presents the optimal trajectories for an ADR mission. In particular, three types of maneuvers are analyzed in detail: a first Hall-effect transfer aimed at lowering the perigee to intercept the debris, a second perigee-lowering maneuver down to 300 km to release the debris, and a final recircularization trajectory to return to the operational orbit and restart the cycle.

The first transfer is framed within a TPBVP to explore the case of continuous thrust throughout the entire maneuver. Conversely, for the other two trajectories, a MPBVP was implemented by imposing a structured sequence of switching arcs. The optimal solutions were obtained by applying the two-body gravitational model presented in Chapter 2 through the IM described in Chapter 3.

In this context, the OCULUS computational solver represented a solid and valuable tool to support the search for the initial guesses, offering both a graphical interface to visualize how the problem behaves when changing the initial co-state values, and a robust and stable differential correction algorithm.

Following these initial solutions, the analysis was extended in the final section by optimizing the same mission profile with a lower initial spacecraft mass, accounting for the propellant consumption. This approach allowed for the evaluation of the satellite's operational lifetime and its total debris removal capability, determining how many complete mission cycles could be supported by the spacecraft. Since the initial mass changed for each subsequent maneuver, a different set of initial guesses was required, thereby defining a new problem to be optimized.

5.1 Problem Setup and Normalization

This section presents the formulation of the problem required by the OCULUS tool. This software environment was developed to solve two-point and multi-point OCPs via the IM; specifically, it implements a differential correction algorithm to determine the initial guesses required to close the trajectory while satisfying the assigned BCs. However, because

this is an indirect method, it demands coherent initial guesses to achieve numerical convergence, alongside a strong user intuition regarding the physical problem. Furthermore, the complete definition of the mathematical structure of the OCP is left entirely to the user.

In particular, the equations of motion for the two-body problem in spherical ZEN coordinates, as presented in Chapter 2, were implemented and provided to the solver, along with the complete set of adjoint differential equations obtained from the optimality conditions and reported in Appendix A. Furthermore, the Hamiltonian formulation and the switching function expression were defined. In addition to these, the specific parameters characterizing the problem must be supplied in the appropriate format. These include the propulsion characteristics, namely the maximum thrust $T_{\max}$ required by the bang-bang control law and the exhaust velocity c , the switching structure defining the sequence of thrust and coast arcs into which the trajectory is divided, and the initial state vector of the SC at t_0 . Finally, the boundary conditions specifying the target state of the SC must be enforced, together with the transversality conditions and, if required, the switching and Hamiltonian boundary relations.

Additionally, a specific function was developed to calculate the boundary residuals based on the problem. The OCULUS solver aims to nullify these errors by iteratively adjusting the initial guesses in order to find the optimal initial co-states that satisfy the assigned BCs.

Focusing more specifically on the problem under consideration, the initial state vector must be provided in the following format:

$$\mathbf{x}_0 = \begin{pmatrix} r_0 \\ \vartheta_0 \\ \varphi_0 \\ u_0 \\ v_0 \\ w_0 \\ m_0 \end{pmatrix} \quad (5.1)$$

The state variables in spherical-ZEN coordinates govern the equations of motion according to the formulations implemented in Chapter 2. To enhance numerical stability, these quantities were normalized using the Earth's mean radius $r_E = 6371$ km and a reference circular velocity computed as:

$$v_{ref} = \sqrt{\frac{\mu}{r_E}} \quad (5.2)$$

where μ represents the Earth's gravitational parameter. Furthermore, the gravitational parameter itself was non-dimensionalized with respect to the terrestrial one, and the SC mass was normalized by its initial mass at the beginning of each maneuver, so that both parameters are equal to 1 at the start of the integration.

Consequently, the corresponding initial adjoint vector, directly correlated to the state

variables, is composed of the following elements:

$$\boldsymbol{\lambda}_0 = \begin{pmatrix} \lambda_{r,0} \\ \lambda_{\vartheta,0} \\ \lambda_{\varphi,0} \\ \lambda_{u,0} \\ \lambda_{v,0} \\ \lambda_{w,0} \\ \lambda_{m,0} \end{pmatrix} \quad (5.3)$$

These elements represent the initial guesses supplied to the IM, which must be properly estimated to ensure numerical convergence. Interestingly, some of these costate variables exhibit an intuitive behavior directly related to specific mission performances or physical trends. For instance, the absolute value of each velocity costate provides insight into the relevance of the corresponding velocity component at a given instant. Meanwhile, their signs dictate the optimal thrust direction, implying a thrust action oriented opposite to the reference frame if the sign is negative.

Similarly, radial costate, λ_r , measures the significance of the time variation of its associated state variable, whereas the remaining co-states exhibit a less predictable behavior.

Subsequently, the boundary conditions must be defined at the beginning of the analysis to specify the targeted final state. In addition to the final constraints on the state variables, which can restrict the final position and velocity, specific conditions on the co-states may be enforced, arising from the transversality equations. In this analysis, both the final position and velocity are fully constrained. However, the condition on the angular coordinate ϑ is relaxed across all maneuvers since it is irrelevant to the mission objectives; indeed, different values of ϑ can almost always be achieved by simply adding a coast arc before or after the obtained optimal solution. Furthermore, based on the Mayer formulation aimed at maximizing the final spacecraft mass (m_f), the boundary condition for the corresponding costate is derived as $\lambda_m(t_f) = 1$.

Furthermore, based on the optimality conditions, since the final longitude $\vartheta(t_f)$ is unconstrained and does not appear in the objective function ϕ , a homogeneous boundary condition on the corresponding costate is theoretically expected, namely $\lambda_{\vartheta}(t_f) = 0$. However, examining the differential equations for the adjoint variables derived from the Euler-Lagrange equations, it is evident that the time derivative of λ_{ϑ} is influenced exclusively by perturbative effects. Consequently, in the absence of such perturbations, as is the case in the simplified model under consideration, this costate remains strictly constant. While its constant value affects the evolution of λ_{φ} , λ_r , and λ_v , its stationarity is analytically preserved; thus, the resulting numerical solution is fully accurate and acceptable, depending on the optimal initial value determined by the solver.

More practical considerations, supported by application cases and examples, will be addressed later in this discussion. Based on these premises, the expected boundary

conditions are formulated as follows:

$$BC = \begin{pmatrix} r_f \\ \varphi_f \\ u_f \\ v_f \\ w_f \\ \lambda_m = 1 \end{pmatrix} \quad (5.4)$$

As a final theoretical consideration, the switching structure of the thrust is analyzed. In a MPBVP, the sequence of arcs is provided at the beginning of the analysis within the problem definition. Subsequently, based on the Pontryagin's Minimum Principle (PMP) results once numerical convergence is achieved, the user can decide whether to retain or modify the sequence to converge toward a globally optimal architecture.

In this study, a single continuous thrust profile, modeled as a two-point boundary value problem, was implemented for the electric propulsion system, which served to establish baseline considerations regarding the maneuver behavior and performance. Conversely, a multi-arc problem was developed for the two chemically propelled trajectories, features four and two switching arcs, respectively. The selection of these specific arc structures was driven by both physical insights and mission-specific constraints.

Finally, additional boundary and interface conditions applied to the problem are defined as follows. Specifically, for free-final-time problems, the transversality conditions dictate that the Hamiltonian must vanish at the final time:

$$\mathcal{H}(t_f) = 0 \quad (5.5)$$

Conversely, for multi-arc trajectories, the switching function must strictly nullify at each internal interface separating the consecutive thrust and coast phases to ensure consistency with the PMP, yielding:

$$\mathcal{S}_F(t_{\text{int}}) = 0 \quad (5.6)$$

where t_{int} represents the precise time instances of the switching nodes.

5.2 Phase I: Low-Thrust Target Approach

For the first phase of the mission, a preliminary low-thrust electric maneuver to close in on the debris was proposed. This initial maneuver was intended to be indicative, providing an estimate of the propellant consumption per kilometer of orbital descent or ascent, as well as the time required to traverse it, starting from an operational orbit at 800 km and assuming continuous thrust (with no coast arcs). This analysis will serve to provide a baseline estimate of the cost required to approach a single piece of debris or multiple targets in a sequential manner.

First of all, based on the technical information of the adopted Hall-effect thruster, reported in Table 4.2, the following useful data for the analysis are extrapolated:

$$\begin{aligned} T &= 90 \text{ mN} \\ I_{sp} &= 1660 \text{ s} \end{aligned}$$

From which the exhaust velocity is obtained as:

$$c = I_{sp} \cdot g_0 = 1660 \cdot 9.81 = 16\,284.6 \text{ m s}^{-1} \quad (5.7)$$

where g_0 represents the standard acceleration of gravity at sea level on Earth, expressed in m/s^2 .

Next, the initial state vector and the target boundary conditions at the end of the trajectory are defined. The initial state of the satellite is associated with a generic point on a circular equatorial orbit, characterized by the following radius and orbital velocity values:

$$\begin{aligned} r_0 &= 800 + 6371 = 7171 \text{ km} \\ v_c &= \sqrt{\frac{\mu_{\oplus}}{r_0}} = \sqrt{\frac{398600}{7171}} = 7.45 \text{ km s}^{-1} \end{aligned}$$

By expressing these quantities in the problem coordinates and applying the non-dimensionalization process, the initial state vector $\mathbf{x}_0$ takes the following form:

$$\mathbf{x}_0 = \begin{bmatrix} 1.125569 & 0.0 & 0.0 & 0.0 & 0.942570 & 0.0 & 1.0 \end{bmatrix}^T \quad (5.8)$$

It should be noted that the initial mass component appears as 1 because it is non-dimensionalized with respect to the initial mass of each specific maneuver. More precisely, in this case, the reference initial mass corresponds to the value obtained from the spacecraft mass budget estimation in Chapter 4, which is equal to 918.05 kg.

The arrival point was constrained to the perigee of an elliptic orbit, with an apogee at 800 km and a perigee at 790 km. This 10 km lowering of the perigee represents an optimal compromise for subsequent observations. Consequently, the target radius is calculated as follows:

$$r_{pf} = 790 + 6371 = 7161 \text{ km} \quad (5.9)$$

The velocity that the spacecraft must possess at the perigee is derived from the energy equation of an orbit:

$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu_{\oplus}}{r} = -\frac{\mu_{\oplus}}{2a} \quad (5.10)$$

Given that the semi-major axis a is defined by the average of the apogee and perigee radii:

$$\begin{aligned} r_{af} &= 800 + 6371 = 7171 \text{ km} \\ a &= \frac{r_{pf} + r_{af}}{2} = \frac{7161 + 7171}{2} = 7166 \text{ km} \end{aligned}$$

the required velocity at perigee v_{pf} is obtained as:

$$v_{pf} = \sqrt{\mu_{\oplus} \left(\frac{2}{r_{pf}} - \frac{1}{a} \right)} = \sqrt{398600 \cdot \left(\frac{2}{7161} - \frac{1}{7166} \right)} = 7.463\,34 \text{ km s}^{-1} \quad (5.11)$$

From this, by normalizing, the following BCs are obtained:

Tabella 5.1. Non-dimensionalized final BCs for the low-thrust perigee-lowering trajectory

Parameter	Non-Dimensional Value
r_f	1.123999
φ_f	0.0
u_f	0.0
v_f	0.943557
w_f	0.0
$\lambda_m(t_f)$	1.0

5.2.1 Numerical Results and Trajectory Analysis

Once the problem was properly set, the optimal solution for a single-arc, low-thrust, 10 km perigee lowering maneuver was obtained. The resulting optimal initial guesses are presented below:

$$\lambda_0 = \begin{bmatrix} -0.39549 \\ -0.00000052 \\ 0.0 \\ -0.21368 \\ -0.56193 \\ 0.0 \\ 0.99972 \end{bmatrix} \quad (5.12)$$

By analyzing the resulting costate values, it can be noted that, as expected, negative values are obtained for the velocity-related costates. Intuitively, lowering the perigee requires a deceleration, which corresponds to a thrust vector acting in the opposite direction to the velocity.

Regarding the longitude costate (λ_θ), even though it was not rigidly constrained to zero, it assumes a value that can be safely approximated to zero; furthermore, in accordance with the analytical derivations provided in Appendix C, its profile remains constant and null in the absence of orbital perturbations (as shown in Figure 5.1).

Moving toward the last optimized parameter derived from the problem, a non-dimensional final time of $t_f = 62.392$ was obtained. By applying the inverse non-dimensionalization process, the actual maneuver duration is found to be 50 254.58 s, which is equivalent to 13 hours, 57 minutes, and 34 seconds. Furthermore, by estimating the time required per kilometer of altitude variation, it results that the spacecraft takes approximately 1 hour, 23 minutes, and 45 seconds to achieve a 1 km lowering of the perigee.

Instead, the propellant consumption of the maneuver is now evaluated by extracting the final non-dimensional mass of the spacecraft, corresponding to the seventh component of the state vector at the final integration step. The converged value is $m_{f_{\text{adim}}} = 0.999697737$.

Consequently, the actual final mass of the spacecraft is calculated as:

$$m_f = m_{f_{\text{adim}}} \cdot m_0 = 0.999697737 \cdot 918.05 \text{ kg} = 917.7725 \text{ kg} \quad (5.13)$$

The total Xenon propellant consumed during the maneuver (m_p) is then obtained as:

$$m_p = m_0 - m_f = 918.05 - 917.7725 = 0.2775 \text{ kg} = 277.5 \text{ g} \quad (5.14)$$

Similarly, the average propellant consumption per kilometer of perigee lowering is estimated by dividing the total mass expenditure by 10:

$$m_{p/\text{km}} = \frac{m_p}{10} = \frac{277.5 \text{ g}}{10} = 27.75 \text{ g km}^{-1} \quad (5.15)$$

It is worth noting that despite the relatively long duration of the maneuver, the overall propellant cost remains extremely low. This highlights the high efficiency inherent to electric propulsion systems, even under a continuous-thrust operating profile.

In summary:

Tabella 5.2. Maneuver performance parameters scaled per kilometer of perigee lowering.

Parameter	Value (per km)
Maneuver Duration	1 h, 23 min, 45 s
Propellant Consumption (Xenon)	27.75 g

From these results, the ΔV of the mission can also be evaluated using the Tsiolkovsky rocket equation:

$$\Delta V = I_{\text{sp}} \cdot g_0 \cdot \ln \left(\frac{m_0}{m_f} \right) = 1660 \cdot 9.81 \cdot \ln \left(\frac{918.05}{917.7725} \right) = 4.929 \text{ m s}^{-1} \quad (5.16)$$

The implementation within OCULUS provides a graphical interface that shows the evolution of the state and costate variables over time during the trajectories. This tool is useful to deeply understand the physical problem and to illustrate how the adjoint variables directly influence both the trajectory evolution and the final state parameters. Furthermore, an analysis of this nature allows for a proper evaluation of the lambda variables prior to initiating the integration process, a critical step given the extreme sensitivity of the indirect method to these initial values. Figure 5.1 shows the time evolution of all the variables, along with the switching function and the Hamiltonian, corresponding to the converged optimal solution:

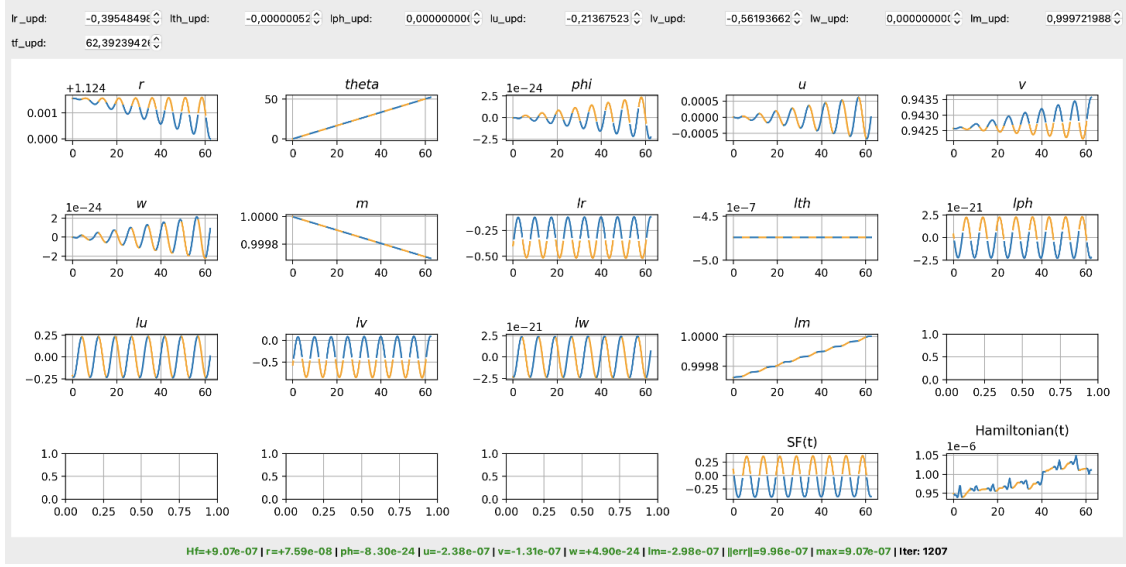


Figure 5.1. Evolution of state and costate variables for the continuous low-thrust perigee-lowering trajectory

Several important considerations can be deduced from these graphical results. The time-history plot of the radius r clearly shows that the spacecraft completes approximately eight and a half orbits; throughout these revolutions, the apogee remains almost constant while the perigee continuously decreases until the target radius is successfully reached during the ninth orbit.

Regarding the velocity profiles, the radial velocity u shows a symmetric oscillation around zero on the order of 10^{-4} , representing the natural physical behavior of an orbit that becomes increasingly eccentric over time. Conversely, the out-of-plane velocity w remains effectively null, with a minimum oscillation on the order of 10^{-24} , confirming that the maneuver is entirely planar. The tangential velocity v increases on average as the trajectory progresses, since lower perigee altitudes inherently imply higher orbital speeds, while its oscillatory amplitude widens due to the increasing eccentricity. In fact, the maximum velocity at the perigee increases while the minimum velocity at the apogee decreases. Furthermore, the spacecraft mass m decreases linearly over time, as expected for a continuous full-thrust maneuver.

Regarding the costate variables, λ_r and λ_v oscillate in sync with the orbital period while maintaining a negative mean value, whereas the mass costate λ_m increases almost linearly until reaching exactly 1.0 at the final time.

Of particular significance is the behavior of the switching function $SF(t)$. The solver successfully converged to an optimal solution under the constraint of continuous thrust. However, the resulting profile of $SF(t)$, alternating between positive (yellow arcs) and negative (blue arcs) values, indicates that this continuous-thrust strategy is energetically sub-optimal compared to an unconstrained formulation that follow the SF .

As previously seen, according to Pontryagin’s Maximum Principle (PMP) the thrust should be organized based on the sign of SF : $T_{\max}$ when positive and 0 when negative. The plot suggests that incorporating coasting arcs precisely during the blue phases (near the perigee) and concentrating the thrust around the apogee (the yellow phases) would yield a much more fuel-efficient trajectory for lowering the perigee.

The plot of the continuous thrust maneuver is shown in Figure 5.2:

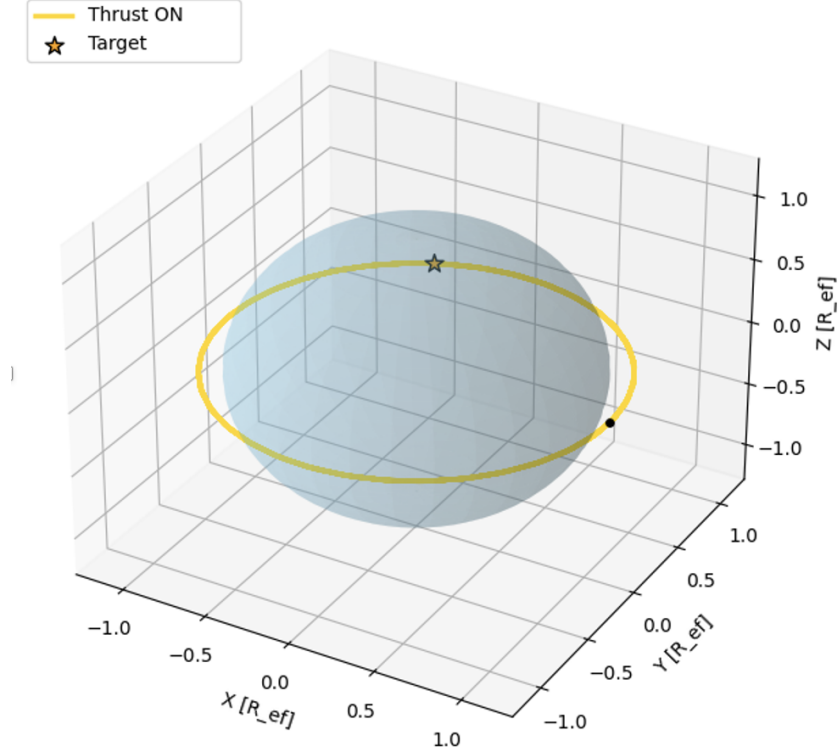


Figura 5.2. 3D trajectory plot of the low-thrust perigee-lowering maneuver

5.3 Phase II: High-Thrust Perigee Lowering

For the last two most demanding maneuvers of the mission profile, a chemical propulsion system composed of ten 22 N High Performance Green Propulsion (HPGP) thrusters was chosen. The fundamental propulsion characteristics required for the analysis are extrapolated from Table 4.3:

$$T_{\max_{\text{tot}}} = T_{\max} \cdot n_{\text{thrusters}} = 22 \text{ N} \cdot 10 = 220 \text{ N} \quad (5.17)$$

$$I_{\text{sp}} = 242 \text{ s} \quad (5.18)$$

which implies an effective exhaust velocity of:

$$c = I_{\text{sp}} \cdot g_0 = 242 \cdot 9.81 = 2374 \text{ m s}^{-1} \quad (5.19)$$

The trajectory to be optimized in this section takes place immediately after the previous phase. It is assumed that the spacecraft, currently in an orbit with a 790 km perigee and an 800 km apogee, has collected 100 kg of debris (the maximum allowable debris capacity). The spacecraft must now release this collected mass within a short time-frame into the pre-selected and previously discussed 800 km by 300 km disposal orbit. Specifically, this deployment will be executed precisely at the perigee of the target orbit.

The sequence of arcs, namely the switching structure, chosen for this analysis is defined as: coast-thrust-coast-thrust $([0,1,0,1])$. The optimal solution obtained will strictly follow this specific thrust profile. This sequence was selected to allow the spacecraft to perform the first burn at the most favorable position along the trajectory; subsequently, the final thrust arc was incorporated based on preliminary study trials, during which the behavior of the switching function (SF) clearly suggested its inclusion.

Once the problem is defined, it is necessary to set the initial conditions for the state vector along with the exact values of the BCs. As the initial state parameters, the same non-dimensional values used as BCs in the previous trajectory are applied, positioning the spacecraft at the 790 km perigee of an elliptical orbit with an 800 km apogee:

$$\mathbf{x}_0 = [1.123999 \quad 0.0 \quad 0.0 \quad 0.0 \quad 0.943557 \quad 0.0 \quad 1.0]^T \quad (5.20)$$

It is important to note that while the initial non-dimensional mass remains equal to 1.0, the reference mass used for non-dimensionalization is now the initial mass of the spacecraft before this specific maneuver. Based on the previous considerations, this value is equal to the final mass of the preceding maneuver increased by the mass of the collected debris:

$$m_{0_2} = m_{f_1} + m_{\text{debris}} = 917.7725 + 100 = 1017.7725 \text{ kg} \quad (5.21)$$

The parameters relative to the arrival point of the target orbit were now obtained in order to set the new constraints of the problem.

First, the arrival altitude is defined by a radius coinciding with the perigee of a 300 km target elliptical orbit:

$$r_{p_f} = 300 + 6371 = 6671 \text{ km} \quad (5.22)$$

The orbit remains equatorial, meaning that the final latitude φ_f is constrained to zero, while θ_f remains free based on the same considerations discussed previously.

Regarding the velocity, it is once again derived from the orbital energy conservation equation 5.10. Given that the apogee of the target transfer orbit is at 800 km ($r_{a_f} = 800 + 6371 = 7171$ km), the semi-major axis a is calculated as the mean value:

$$a = \frac{r_{p_f} + r_{a_f}}{2} = \frac{6671 + 7171}{2} = 6921 \text{ km} \quad (5.23)$$

From this, the required velocity at the perigee v_{p_f} is extracted as:

$$v_{p_f} = \sqrt{\mu \left(\frac{2}{r_{p_f}} - \frac{1}{a} \right)} \approx 7.968 \text{ km s}^{-1} \quad (5.24)$$

Finally, by normalizing these physical quantities, the following final BCs are obtained:

Tabella 5.3. Non-dimensionalized final BCs for the chemical perigee-lowering trajectory

Parameter	Non-Dimensional Value
r_f	1.047088
φ_f	0.0
u_f	0.0
v_f	0.994749
w_f	0.0
$\lambda_m(t_f)$	1.0

5.3.1 Numerical Results and Trajectory Analysis

The optimal initial guesses obtained from the optimization of this four-arc chemical maneuver are reported below:

$$\boldsymbol{\lambda}_0 = \begin{bmatrix} -0.5570 \\ -0.00098 \\ 0.0 \\ 1.4908 \\ -1.3292 \\ 0.0 \\ 0.9418 \end{bmatrix} \quad (5.25)$$

Despite this type of maneuver being fundamentally different from the previous one and requiring a completely distinct sensitivity regarding the initialization of the costate variables, it can be observed that the initial costate relative to the tangential velocity ($\lambda_{v,0}$) remains negative. This negative sign confirms the expected physical behavior for a perigee-lowering maneuver, where a deceleration along the tangential direction is required to drop the orbital altitude. Conversely, in this specific high-thrust configuration, the initial radial velocity costate ($\lambda_{u,0}$) takes a positive value, highlighting a different orientation of the radial component of the thrust compared with the last case.

Regarding the costate associated with the angular position, λ_θ , its initial value in this case is small but non-negligible. As previously seen, it acts as a constant of the motion, yet its specific value directly influences the evolution of the other costates. Further considerations on this topic are detailed at the end of this section.

Finally, the value of λ_m , which is smaller compared to the low-thrust case, suggests that this maneuver requires burns with a higher mass consumption, exactly as expected from chemical propulsion.

The optimization process yields four distinct time parameters, which define the duration of each individual thrust or coasting arc, with the final value representing the total duration of the orbital transfer. These parameters have the following non-dimensionalized values:

$$t_1 = 0.987645, \quad t_2 = 1.74849 \quad (5.26)$$

$$t_3 = 4.93334, \quad t_f = 4.94646 \quad (5.27)$$

From these results, the total duration of the orbital transfer is calculated to be 3984.18 s, which is equivalent to approximately 1 hour, 6 minutes, and 24 seconds. Regarding the active propulsion phases, the thrusters fire during two distinct intervals: first, between t_1 and t_2 for a duration of approximately 612.86 s, and finally, for a brief terminal pulse between t_3 and t_f lasting 10.57 s. This results in a total burn time of 623.39 s, which corresponds to approximately 10.39 minutes of thrust. This value widely respects the specifications of the thrusters (Table 4.3), which indicate a maximum pulse length of 38 minutes.

Shifting the focus to the maneuver's propellant consumption, the non-dimensional value of the final mass obtained during the last iteration is equal to 0.9432363. By multiplying this value by the initial mass, the dimensional final mass of the spacecraft upon arrival is determined as:

$$m_{f_2} = m_{f_{\text{adim}}} \cdot m_{0_2} = 0.9432363 \cdot 1017.7725 \text{ kg} = 959.99 \text{ kg} \quad (5.28)$$

Consequently, the total mass of propellant consumed during the chemical maneuver is calculated as:

$$m_p = m_{0_2} - m_{f_2} = 1017.7725 - 959.99 = 57.78 \text{ kg} \quad (5.29)$$

From this information, the corresponding ΔV of the maneuver is calculated using Tsiolkovsky's rocket equation:

$$\Delta V = I_{\text{sp}} \cdot g_0 \cdot \ln \left(\frac{m_{0_2}}{m_{f_2}} \right) = 242 \cdot 9.81 \cdot \ln \left(\frac{1017.7725}{959.99} \right) = 138.759 \text{ m s}^{-1} \quad (5.30)$$

The plots illustrating the time-history evolution of the state and costate variables along the trajectory are presented below in Figure 5.3.

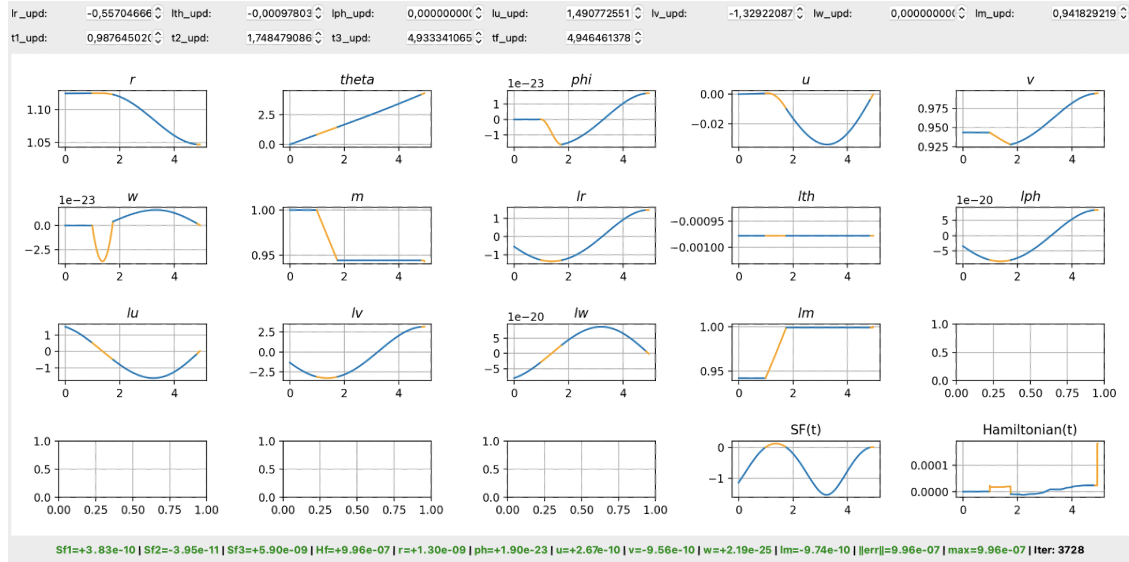


Figure 5.3. Evolution of state and co-state variables for the chemical perigee-lowering trajectory

The evolution plots of the state variables show that the radius decreases while the tangential velocity increases, both matching smoothly the final condition imposed for the problem. The angular position θ is left unconstrained and reaches a final value below 6.28 (radians 2π), proving that the optimal transfer is completed in less than one full revolution.

As expected, the SC mass decreases exclusively during the thrust phases (highlighted in yellow) due to propellant consumption, while remaining strictly constant during the coasting arcs (in blue). A perfectly complementary and specular behavior is observed in the mass costate variable λ_m .

Furthermore, the switching function (SF) perfectly matches the imposed arc structure by crossing the zero value precisely at the switching times t_1 , t_2 , and t_3 , thereby satisfying the optimality conditions of the PMP.

Finally, focusing on the Hamiltonian profile, it presents a sharp spike at the very end of the trajectory, corresponding to the final thrust phase. This peak is caused by a numerical integration error resulting from the extremely short duration of this quasi-impulsive burn.

Below, in Figure 5.4, the trajectory plot is presented, highlighting the thrust and coasting phases in yellow and blue, respectively. This visualization is consistent with the color adopted for the evolution profiles of the variables previously discussed.

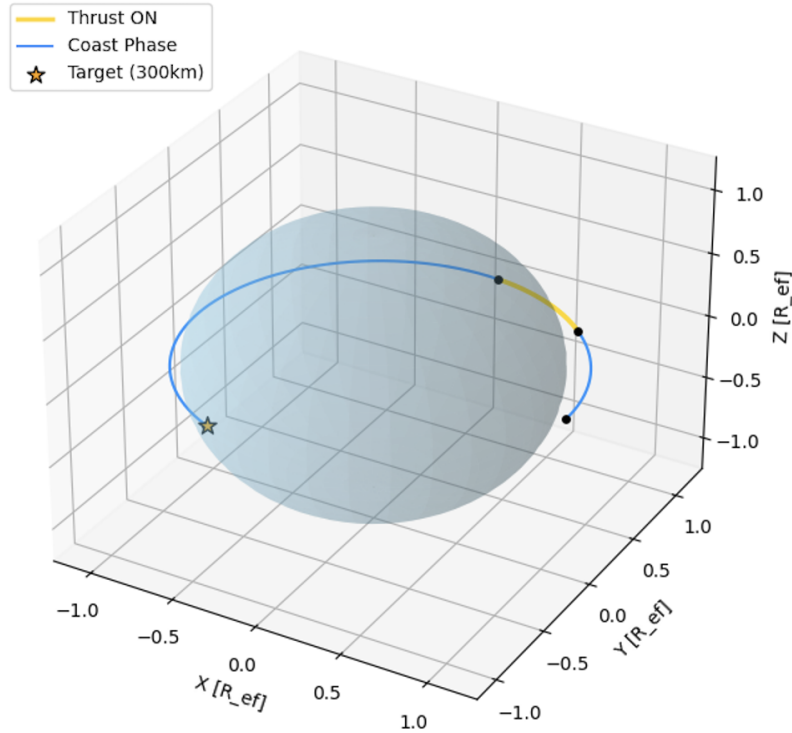


Figure 5.4. 3D trajectory plot of the chemical perigee-lowering maneuver

Effect of the Longitude Costate Constraint

For the sake of completeness, a comparative analysis is presented by explicitly enforcing the homogeneous boundary condition, namely $\lambda_\vartheta(t_f) = 0$, to assess how the optimal solution varies.

Initially, when this constraint is applied using the baseline four-arc switching structure, the solver achieves numerical convergence but introduces a physical inconsistency: the value of the first switching time t_1 becomes negative, and the evolution of the switching function $S_F(t)$ effectively eliminates the associated first coasting arc. This behavior clearly suggests that the optimization process rejects the initial coasting phase under this formulation. Consequently, the initial switching sequence was modified by providing a [1,0,1] (Thrust-Coast-Thrust) configuration. With this structural setup, the problem successfully converges to the optimal solution illustrated in Figure 5.5:

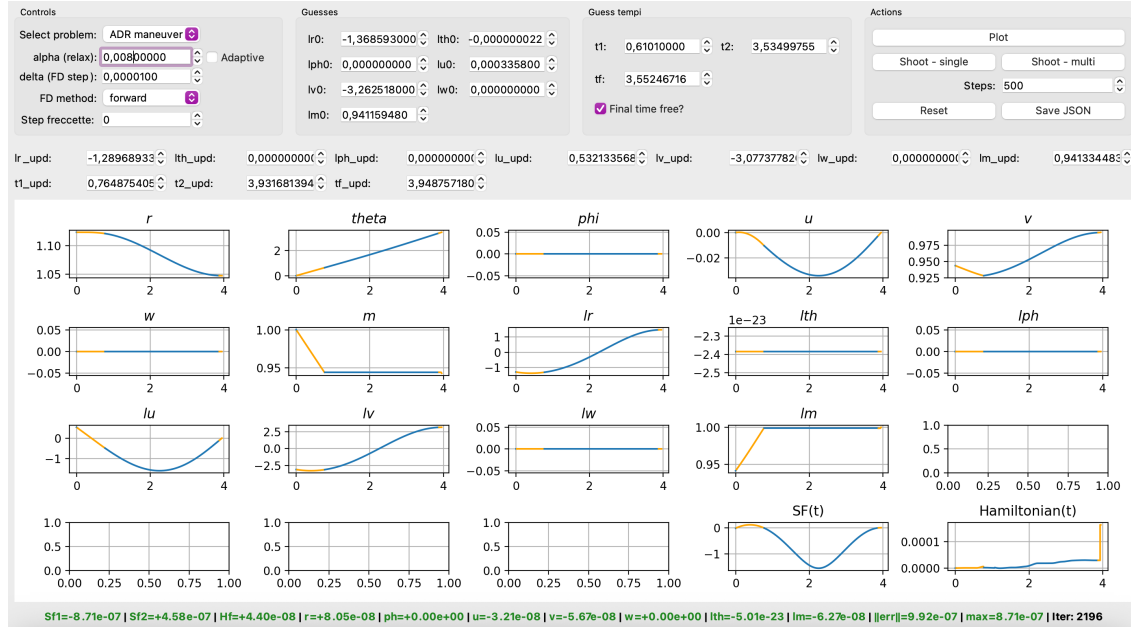


Figura 5.5. Evolution of state and costate variables for the constrained $\lambda_\vartheta(t_f) = 0$ trajectory.

The primary difference compared to the unconstrained case lies in how the solver handles the initial guess for $\lambda_{\vartheta,0}$. In the unconstrained formulation, the optimizer is free to initialize this costate, which acts as a constant of motion in an unperturbed two-body model, at a value that enables convergence while preserving the predefined coasting arc. Conversely, when the final boundary condition is strictly enforced, the solver is forced to set $\lambda_{\vartheta,0}$ identical to zero from the beginning of the integration to satisfy the final constraint. The imposition of this formal condition directly drives the physical elimination of the initial coasting phase. Although this alternative solution is mathematically rigorous and the propellant consumption remains virtually identical, the

unconstrained formulation is still preferred in order to maintain the initial arc structure unchanged. The resulting optimal trajectory is illustrated in the corresponding plot in Figure 5.6:

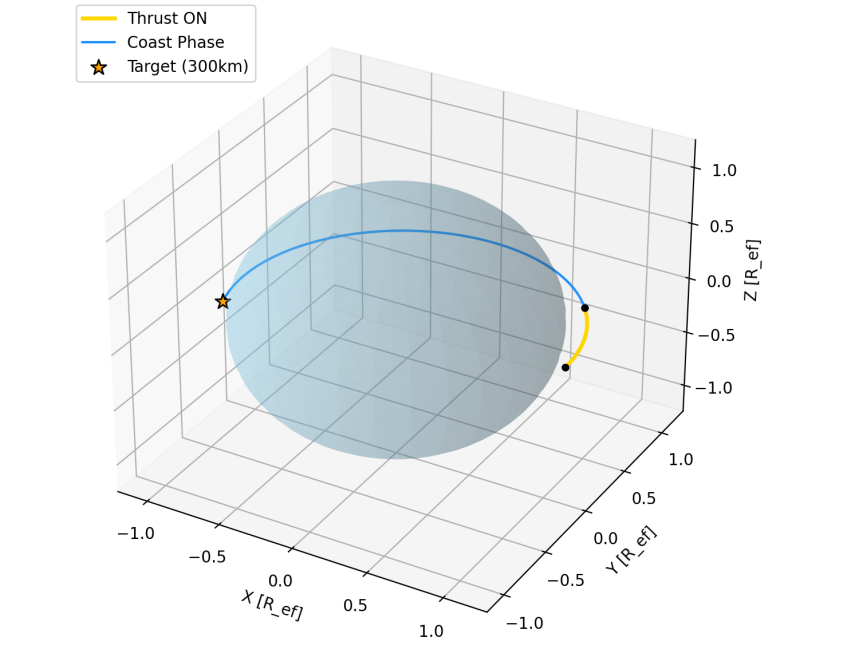


Figura 5.6. 3D trajectory plot for the constrained $\lambda_\theta(t_f) = 0$ perigee-lowering maneuver

5.4 Phase III: Orbit Recircularization

The final trajectory under analysis, which concludes a single ADR cycle, involves returning the SC to its operational orbit to restart the process. This specific maneuver, much like the second one, utilizes a HPGP monopropellant chemical system. The propulsion parameters, including maximum thrust $T_{\max_{\text{tot}}}$, specific impulse I_{sp} , and effective exhaust velocity c , were introduced and calculated in the previous section, but they are reported below for convenience:

$$T_{\max_{\text{tot}}} = 220 \text{ N} \quad (5.31)$$

$$I_{\text{sp}} = 242 \text{ s} \quad (5.32)$$

$$c = 2374 \text{ m s}^{-1} \quad (5.33)$$

It is assumed that at the end of the previous trajectory, upon reaching the 300 km perigee, the 100 kg payload of debris is instantaneously released, allowing the spacecraft to continue its mission. Therefore, the initial mass of the satellite at the beginning of this

third phase is equal to the final mass of the second phase minus the 100 kg debris mass, yielding:

$$m_{0_3} = m_{f_2} - 100 \text{ kg} = 959.99 - 100 = 859.99 \text{ kg} \quad (5.34)$$

This resulting value will subsequently be employed as the reference mass for the non-dimensionalization of this maneuver.

The switching structure adopted for this phase consists of a coasting phase followed by a thrust arc, resulting in a two-arc configuration $([0,1])$. This specific sequence was selected because the spacecraft needs to perform a circularization maneuver upon reaching the apogee of its current orbit. Therefore, a viable and effective strategy is to coast until the target altitude is reached, and then activate the propulsion system to provide the required velocity increment to match the circular velocity of the target orbit, which in this case is located at 800 km.

More specifically, the initial state is set exactly at the final position and configuration, including the angular coordinate θ , reached at the end of the second trajectory. At this stage, the spacecraft is located at the perigee of the 300×800 km elliptical orbit, expressed in non-dimensional coordinates by the following initial state vector:

$$\mathbf{x}_0 = \begin{bmatrix} 1.04709 & 4.2825 & 0.0 & 0.0 & 0.9947 & 0.0 & 1.0 \end{bmatrix}^T \quad (5.35)$$

Subsequently, the constraints of the problem at the final time are defined. As previously stated, the target is a circular equatorial orbit at an altitude of 800 km, which requires the final radius r_f and the circular tangential velocity v_c to be:

$$r_f = 800 + 6371 = 7171 \text{ km} \quad (5.36)$$

$$v_c = \sqrt{\frac{\mu_{\oplus}}{r_f}} = \sqrt{\frac{398600}{7171}} = 7.45 \text{ km s}^{-1} \quad (5.37)$$

as already calculated in the section regarding the Hall-effect thruster maneuver. Consequently, the non-dimensional final BCs are established as follows:

Tabella 5.4. Non-dimensionalized final BCs for the chemical circularization maneuver.

Parameter	Non-Dimensional Value
r_f	1.125557
φ_f	0.0
u_f	0.0
v_f	0.9425
w_f	0.0
$\lambda_m(t_f)$	1.0

5.4.1 Numerical Results and Trajectory Analysis

The converged solution for this transfer yields:

$$\lambda_0 = \begin{bmatrix} 2.9815 \\ -0.0038 \\ 0.0 \\ 1.0903 \\ 2.9699 \\ 0.0 \\ 0.9311 \end{bmatrix} \quad (5.38)$$

For this set of optimal initial costates, the same considerations discussed in the previous cases remain valid, with an important observation: unlike the other maneuvers, this phase aims to raise the perigee to circularize the orbit. Consequently, the mechanical energy and tangential velocity must be increased to match the required circular velocity. This physical behavior is directly reflected in the positive value of the velocity costate λ_v , which mathematically confirms that the thrust vector is aligned with the positive direction of the velocity vector.

The optimal solution provides the switching time t_1 , separating the initial coasting phase from the subsequent thrust phase, and the final time t_f in non-dimensional units:

$$t_1 = 3.03140 \quad (5.39)$$

$$t_f = 3.70235 \quad (5.40)$$

These values imply that the total duration of the transfer is 2982.10 s (approximately 49 min and 42 s). Concurrently, the thrust phase lasts for the time interval between t_1 and t_f , which corresponds to 540.42 s (approximately 9 min).

By extracting the non-dimensional final mass of the spacecraft, following the same procedure as in the previous maneuvers, the final mass at the conclusion of the third phase is determined:

$$m_{f_3} = m_{f_{\text{adim}}} \cdot m_{0_3} = 0.941761 \cdot 859.99 \text{ kg} = 809.91 \text{ kg} \quad (5.41)$$

Consequently, the mass of LMP-103S monopropellant consumed during the maneuver is calculated as:

$$m_p = m_{0_3} - m_{f_3} = 859.99 \text{ kg} - 809.91 \text{ kg} \approx 50.08 \text{ kg} \quad (5.23) \quad (5.42)$$

Once the final masses are obtained, the cost of the mission phase in terms of ΔV can be calculated using Tsiolkovsky's rocket equation:

$$\Delta V = I_{\text{sp}} \cdot g_0 \cdot \ln \left(\frac{m_{0_3}}{m_{f_3}} \right) = 242 \cdot 9.81 \cdot \ln \left(\frac{859.99}{809.91} \right) \approx 142.43 \text{ m s}^{-1} \quad (5.43)$$

Now, the optimal trajectory obtained is analyzed throughout the evolution of its parameters in time, thanks to the OCULUS tool visualization:

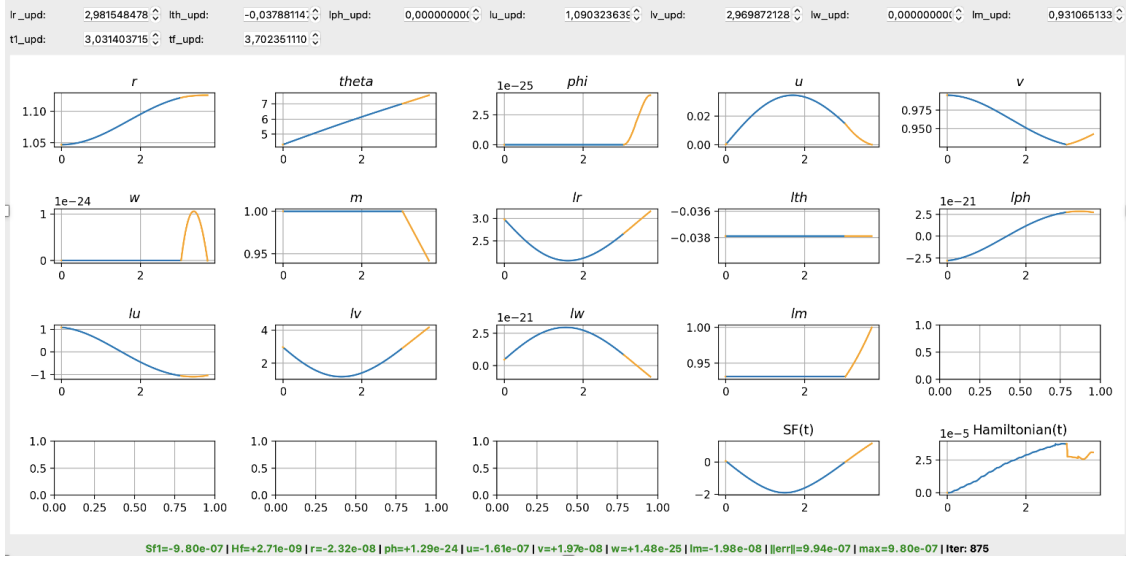


Figure 5.7. Evolution of the state and costate variables for the chemical recircularization trajectory

Based on the evolution of the state variables, it can be noted, first of all, that the radius increases up to the target value by following the profile of the initial elliptical orbit. The unconstrained angular position θ indicates that the optimal trajectory requires less than one revolution to converge. Meanwhile, the latitude φ undergoes minimal oscillations, on the order of 10^{-25} only during the thrust phase, which can be neglected, confirming the maneuver to be planar.

Shifting focus to the velocities, the radial velocity u during the transfer takes non-negligible positive values due to the elliptical nature of the trajectory. On the other hand, the tangential velocity v decreases in accordance with the gradual increase in altitude, and then increases again during the thrust phase precisely to accelerate the spacecraft to the circular velocity of the target orbit, in this case at 800 km. The out of plane velocity w is negligible because the spacecraft remains in the equatorial plane.

The mass starts from a unitary value equal to the initial reference mass; it exhibits a constant profile during the first coasting phase and then decreases linearly during the thrust phases. An opposite trend is followed by the mass costate λ_m , which starts from its initial value, remains constant at first, and then increases up to 1 during the propulsion activation phase.

Regarding the costate variables, a similar evolution profile can be observed between λ_r and λ_v , whose respective initial guesses are also the highest in absolute value, indicating a higher sensitivity of the cost function with respect to these parameters compared to the others. The added variable relating to λ_θ remains constant, and the same considerations made previously hold true. Meanwhile, λ_w is almost null, with variations on the order of 10^{-21} , which suggests that the problem is completely insensitive to this parameter.

In this case, there is an almost total correspondence between the pre-set switching structure and the SF profile obtained during the optimization, respecting the PMP according to the Bang-Bang control. However, a minimal thrust arc appears at the very beginning of the trajectory, suggesting that there could be a less demanding solution if the thrusters were turned on, even briefly, at lower altitudes. This result is not surprising since, from theory, we know it can be justified by the Oberth effect, according to which it is convenient to perform a small burn at lower altitudes to optimize consumption and minimize gravity losses. This insight confirms the accuracy and precision of the indirect method used in this project, which successfully guides the user toward the absolute optimal solution among the infinite convergent variants that exist.

Finally, the resulting trajectory is illustrated in the corresponding plot in Figure 5.8:

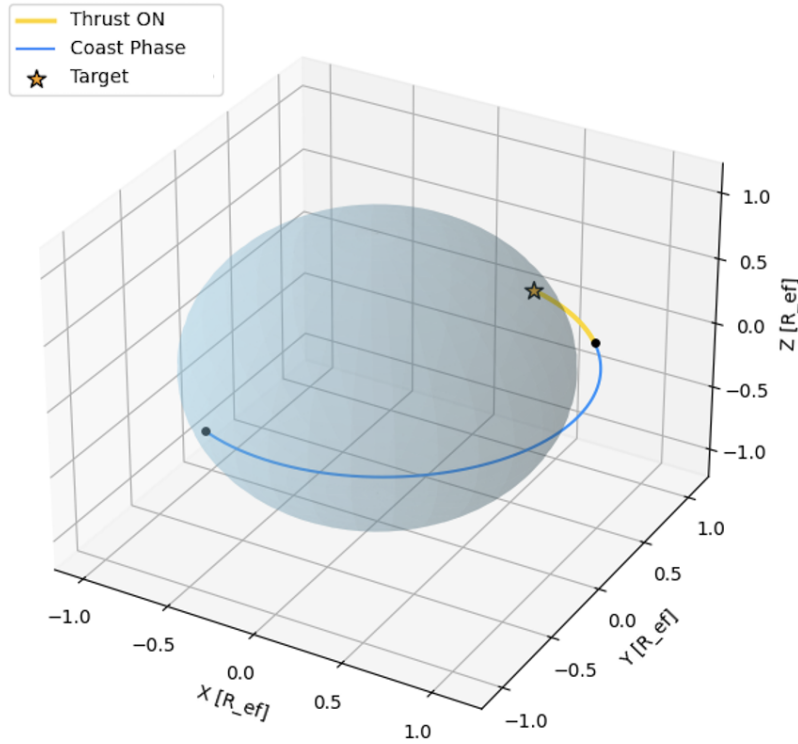


Figure 5.8. 3D trajectory plot of the chemical recircularization maneuver

5.5 Multi-Cycle Performance and Lifetime Analysis

This final section analyzes the successive optimized trajectories in order to maximize the number of operational cycles performed with the available propellant, thereby evaluating the mission lifetime and the total mass of debris successfully removed.

Consequently, the same sequence of three maneuvers is repeated iteratively for as long as propellant permits, with each phase being optimized using the indirect method.

Although each subsequent trajectory is the same of those previously studied, sharing identical boundary conditions, switching structure and starting from the same initial state vector, each iteration constitutes a distinct optimization problem. This variation requires specific and updated initial guesses, as the spacecraft's mass changes continuously after each maneuver due to propellant consumption.

5.5.1 Second-Cycle Analysis

Starting from the second cycle, it can be observed that during the first maneuver, the phase performed via Hall effect propulsion, the initial spacecraft mass is reduced by an amount equal to the total propellant consumed in the previous cycle:

$$m_{p_{\text{tot}}} = m_{p_1} + m_{p_2} + m_{p_3} = 0.2775 + 57.78 + 50.08 = 108.1375 \text{ kg} \quad (5.44)$$

Consequently, the initial mass of the spacecraft at the beginning of the second cycle is given by:

$$m_{0,2} = m_{0,1} - m_{p_{\text{tot}}} = 918.05 - 108.14 = 809.91 \text{ kg} \quad (5.45)$$

This non-negligible, substantial mass reduction, primarily driven by the large amount of chemical propellant consumed, significantly alters the spacecraft's dynamics. As a result, the low-thrust trajectory executed by the Hall thrusters requires one less full revolution to converge compared to the exact same maneuver performed during the first cycle, as demonstrated by the following convergent solution:

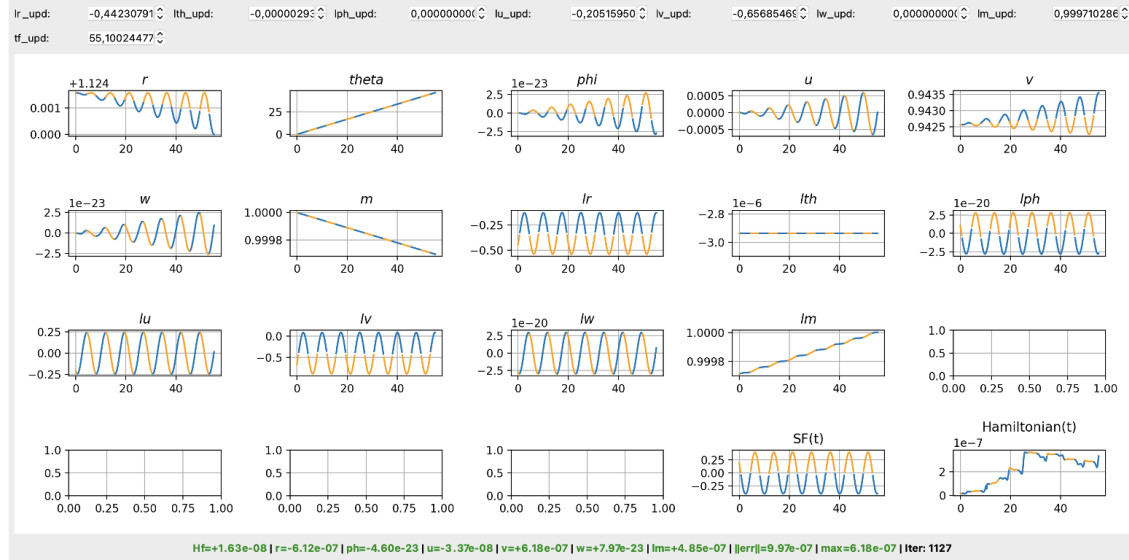


Figure 5.9. Optimal 2-cycle Hall-effect maneuver

The same considerations conducted in Section 5.2 remain valid; however, as expected, the optimal initial guesses have changed along with the final time. The new optimal costate values lie in the neighborhood of the previous ones but assume different magnitudes.

By performing the same analysis and algebraic steps described earlier, Table 5.7 summarizes the main differences in terms of initial guesses, initial mass, time of transfer, final mass, propellant consumption, and ΔV between the same maneuver executed across the first two subsequent cycles.

Tabella 5.5. Comparison of the low-thrust maneuver performance between Cycle 1 and Cycle 2.

Parameter	Cycle 1	Cycle 2
m_0 (kg)	918.05	809.91
λ_0	-0.39549	-0.44230
	0.00000	0.00000
	0.00000	0.00000
	-0.21368	-0.20500
	-0.56193	-0.65700
	0.00000	0.00000
	0.99970	0.99970
t_f (adimensional)	62.39	55.10
t_f (s)	50254.58	44381.04
t_f (h, min, s)	13h 57min 34s	12h 19min 41s
$\Delta t/\Delta h$ (per km)	1h 23min 45s	1h 13min 58s
m_f (kg)	917.7725	809.665
m_p (kg)	0.2775	0.2430
$m_p/\Delta h$ (g km ⁻¹)	27.7	24.3
ΔV (m s ⁻¹)	4.929	4.886

Requiring 7.5 orbits instead of 8.5, the final time is reduced from an adimensional value of 62.39 to 55.10. This comparison clearly demonstrates how a lighter spacecraft experiences a higher acceleration, completing the same maneuver with lower overall consumption. Indeed, the mass reduction yields a time saving of 1 hour and 38 minutes, alongside a propellant reduction of 0.0345 kg, which naturally reflects a lower total ΔV cost.

Analyzing the initial costates, the shift in the initial radial costate λ_r highlights a slightly higher sensitivity of the problem to the associated radius in this second iteration. Concurrently, the tangential velocity costate λ_v increases, while the radial velocity costate λ_u decreases, both in magnitude. Remaining negative, these values indicate that the optimal thrust is still in the opposite direction respect to positive values of the velocity, with an increased contribution along the tangential direction compared to the first cycle.

Moving on to the chemically propelled trajectories for the second cycle, in these cases as well, the initial mass is substantially different from that of the previous cycle. The converged solutions modify the optimal initial guesses only slightly, as reported in Table 5.6.

Tabella 5.6. Performance and costate comparison for the chemical propulsion maneuvers (Maneuver 2 and Maneuver 3) between Cycle 1 and Cycle 2.

Parameter	Maneuver 2		Maneuver 3	
	Cycle 1	Cycle 2	Cycle 1	Cycle 2
m_0 (kg)	1017.77	909.66	859.99	758.25
λ_0	−0.55700	−0.54450	2.98150	2.96300
	−0.00098	−0.00097	−0.00380	−0.03400
	0.00000	0.00000	0.00000	−
	1.49080	1.47500	1.09030	0.98000
	−1.32920	−1.31900	2.96990	2.99000
	0.00000	0.00000	0.00000	0.00000
	0.94180	0.94260	0.93100	0.93370
t_1 (adimensional)	0.9876	1.0587	3.0314	3.0880
t_2 (adimensional)	1.7484	1.6600	−	−
t_3 (adimensional)	4.9330	4.9287	−	−
t_f (adimensional)	4.9464	4.9380	3.7020	3.6763
t_{burn} (min)	10.39	8.23	9.00	7.88
Duration	1h 6min 24s	1h 6min 17s	49min 42s	49min 21s
m_f (kg)	959.99	858.25	809.91	714.40
m_p (kg)	57.78	51.40	50.08	43.45
Savings (%)	−	−11%	−	−13%
ΔV (m s ^{−1})	138.759	138.110	142.430	141.420

The comparison shows that in these subsequent maneuvers conducted with a lighter spacecraft, the total mission time remains nearly unchanged, whereas the engine burning times are reduced. The propellant mass consumption is lower; specifically, nearly 6 kg of propellant are saved in the second maneuver, and approximately 7 kg are saved in the third one. The switching structure has been kept unchanged, and the initial costates remain very close to those of the previous maneuvers. This behavior demonstrates that the solver successfully adjusted the costate values starting from the earlier solutions to achieve convergence in the new scenario.

The plots illustrating the time evolution of the states, costates, Switching Function (SF), and Hamiltonian are omitted for the sake of brevity. Indeed, they do not exhibit any particularly significant changes compared to the previously analyzed maneuvers, maintaining nearly identical trends.

5.5.2 Subsequent Cycles

The same optimization process was iteratively repeated for the subsequent cycles. Specifically, it was possible to successfully complete three full operational cycles, followed by a fourth cycle that included only the electric low-thrust maneuver and the subsequent descent phase.

To provide a clear overview of the mission’s evolution and to highlight the shifting parameters caused by the continuous decrease of spacecraft mass, the results are grouped by maneuver type across all completed iterations. The following tables summarize the initial costates, execution times, propellant consumption, and total ΔV for each sequential phase.

Tabella 5.7. Comparison of the low-thrust maneuver performance across all operational cycles (Cycles 1 to 4).

Parameter	Cycle 1	Cycle 2	Cycle 3	Cycle 4
m_0 (kg)	918.05	809.91	714.40	629.78
λ_0	−0.39549	−0.44230	−0.4321	−0.4347
	0.00000	0.00000	0.00000	0.00000
	0.00000	0.00000	0.00000	0.00000
	−0.21368	−0.20500	−0.09563	0.06789
	−0.56193	−0.65700	−0.69582	−0.70743
	0.00000	0.00000	0.00000	0.00000
	0.99970	0.99970	0.99976	0.99976
t_f (adimensional)	62.39	55.10	48.25	41.76
t_f (s)	50254.58	44381.04	38861.72	33635.78
t_f (h, min, s)	13h 57min 34s	12h 19min 41s	10h 47min 41s	9h 20min 35s
$\Delta t/\Delta h$ (per km)	1h 23min 45s	1h 13min 58s	1h 4min 46s	56min
m_f (kg)	917.7725	809.665	714.1855	629.59
m_p (kg)	0.2775	0.2430	0.2145	0.1889
$m_p/\Delta h$ (g km ^{−1})	27.7	24.3	21.45	18.89
ΔV (m s ^{−1})	4.929	4.886	4.890	4.914

The comparison between all four low-thrust electric trajectories demonstrates that the physical insights derived from the first two cycles can be successfully extended to the subsequent iterations.

Specifically, the initial radial costate λ_r remains relatively stable across all configurations. Regarding the velocity costates, a clear trend emerges up to the third cycle: the tangential velocity costate λ_v increases in magnitude, whereas the radial velocity costate λ_u exhibits a corresponding decrease in absolute value. Interestingly, in the fourth cycle, λ_u switches its sign from negative to positive. This indicates that a different thrust strategy was adopted, leading to a thrust vector oriented along the positive radial direction.

At the same time, the maneuver duration decreases monotonically with each subsequent cycle. This confirms the previously noted behavior, where the optimizer naturally remove approximately one full orbital revolution per iteration; indeed, the optimal trajectory for the fourth cycle converges after only 5.5 orbits compared to the initial 8.5 orbits of Cycle 1. Also the required propellant mass m_p decreases at a nearly constant rate with each iteration, while the associated kinematic ΔV remains almost stable around 4.9 m s^{-1} .

The second maneuver, which utilizes High Performance Green Propulsion (HPGP) and aims to reduce the perigee altitude to release the debris at a lower altitude, consistently exhibits an initial mass that is 100 kg higher than the spacecraft mass in that fase to account for the captured debris.

Tabella 5.8. Comparison of the second maneuver performance across all operational cycles (Cycles 1 to 4).

Parameter	Cycle 1	Cycle 2	Cycle 3	Cycle 4
m_0 (kg)	1017.77	909.66	814.19	729.59
λ_0	-0.55700	-0.54450	-0.54571	-0.54144
	-0.00098	-0.00097	-0.00097	-0.00097
	0.00000	0.00000	0.00000	0.00000
	1.49080	1.47500	1.47507	1.46981
	-1.32920	-1.31900	-1.32236	-1.31916
	0.00000	0.00000	0.00000	0.00000
	0.94180	0.94260	0.94259	0.94285
t_1 (adimensional)	0.9876	1.0587	1.0562	1.08554
t_2 (adimensional)	1.7484	1.6600	1.6631	1.62876
t_3 (adimensional)	4.9330	4.9287	4.9279	4.92633
t_f (adimensional)	4.9464	4.9380	4.9375	4.9347
t_{burn} (min)	10.39	8.23	8.27	7.40
Duration	1h 6min 24s	1h 6min 17s	1h 6min	1h 6min
m_f (kg)	959.99	858.25	768.17	688.42
m_p (kg)	57.78	51.40	46.02	41.17
ΔV (m s^{-1})	138.759	138.110	138.13	137.89

As shown in Table 5.8, the optimal initial costate guesses display a highly coherent behavior across all iterations, with only marginal variations among the cycles. This demonstrates that the solver successfully adjusts the same solution to different values of m_0 , as previously discussed.

Furthermore, while the overall transfer duration remains nearly constant across all cycles, the engine burning times (t_{burn}) significantly decrease with each subsequent iteration. This behavior proves that a lighter spacecraft requires less propulsion time to reach the exact same target state due to its higher acceleration. The other side of the same

coin is reflected in the propellant mass consumption (m_p), which decrease continuously as a direct result of these shorter thrust durations.

Regarding the ΔV , it can be noted that it decreases starting from the second cycle. This behavior is the result of a shorter thrust duration, which leads to lower gravity losses.

Finally, in Table 5.9, the last recircularization maneuver is analyzed across the different cycles.

Tabella 5.9. Comparison of the third maneuver performance across the completed operational cycles (Cycles 1 to 3).

Parameter	Cycle 1	Cycle 2	Cycle 3
m_0 (kg)	859.99	758.25	668.17
λ_0	2.98150	2.96300	2.94643
	-0.0378	-0.03400	-0.03047
	0.00000	0.00000	0.00000
	1.09030	0.98000	0.87690
	2.96990	2.99000	3.00930
	0.00000	0.00000	0.00000
	0.93100	0.93370	0.93598
t_1 (adimensional)	3.0314	3.0880	3.1414
t_f (adimensional)	3.7020	3.6763	3.6557
t_{burn} (min)	9.00	7.88	6.90
Duration	49min 42s	49min 21s	49min 4s
m_f (kg)	809.91	714.40	629.78
m_p (kg)	50.08	43.45	38.39
ΔV (m s ⁻¹)	142.430	141.420	140.476

These final optimal solutions also show initial adjoint variables that remain very close to each other, with a nearly uniform tendency to decrease, except for λ_v .

This maneuver mirrors the physical behavior previously observed in the second phase. Due to the same physical reasons, the overall transfer duration remain almost constant across iterations while the engine burning time (t_{burn}) decreases. This aspect influences both the required propellant mass, which is lower, and the ΔV , which decreases due to the lower gravity losses.

In this case, only three cycles have been optimized. In fact, at the end of the second maneuver in the fourth cycle, the remaining LMP-103S monopropellant was equal to:

$$m_p = m_{p,0} - m_{p,\text{tot}} = 350 - (57.78 + 51.40 + 46.02 + 41.17 + 50.08 + 43.45 + 38.39) = 21.71 \text{ kg} \quad (5.46)$$

This mass of propellant is kept to be used for an eventual disposal maneuver or to reach a refueling station in orbit, in order to maximize its utility.

Furthermore, a high mass of Xenon propellant remained unused, specifically:

$$m_{p,\text{Xe}} = m_{0,\text{Xe}} - m_{p,\text{Xe,tot}} = 11.2 - (0.2775 + 0.2430 + 0.2145 + 0.1889) = 10.2761 \text{ kg} \quad (5.47)$$

This residual propellant allows for numerous additional iterations in orbit if the chemical propellant could be replenished in-orbit. It would also serve as a secondary propulsion system should the remaining chemical mass be insufficient to reach the refueling station, or to accelerate the disposal maneuver if this option is unavailable.

Therefore, we can conclude that this spacecraft, as designed starting from its propulsion system and mass budget, and through the trajectory optimization, is capable of performing 4 iterations, removing a maximum debris payload of 400 kg for each chemical propellant refill. Whenever possible, it can be used continuously in its space debris clearance mission.

Chapter 6

Conclusions

The work presented in this thesis investigated an ADR mission, addressing a critical issue such as the current space congestion and the urgent need for intervention. Although other technical fields were touched, the core of the research focused on trajectory optimization implemented through indirect methods. In addition to being highly precise and computationally efficient, these methods enabled a deeper understanding of the underlying physics of the optimized maneuver, providing valuable insights for future improvements of the work.

Three distinct maneuvers were optimized by leveraging the PMP and the bang-bang formulation. The used algorithm was developed entirely from first principles, incorporating body dynamics, orbital geometry, atmospheric drag modeling, and the mathematical formulation of the optimization problem.

The optimization addressed problems of a diverse nature, ranging from a Two-Point Boundary Value Problem (TPBVP) formulated for continuous thrust to a Multi-Point Boundary Value Problem (MPBVP) defined a priori by an internal sequence of arcs alternating between coasting and thrusting phases. Furthermore, convergent solutions within the frameworks of both low-thrust electric propulsion and high-thrust chemical propulsion were analyzed and successfully obtained. This allowed for an exploration of both alternatives currently widely employed in the space sector, while identifying the suitable initial guesses for problems of an entirely different nature.

The results of the three optimal trajectories were analyzed and subsequently iterated over time, accounting for a variable initial mass due to propellant consumption, in order to define a preliminary mission profile and evaluate the lifetime and total debris removal capacity of the satellite. A SC with an initial mass of 918.05 kg, comprising 350 kg of green chemical monopropellant LMP-103S and 11.2 kg of Xenon for the electric propulsion system, is capable, under these conditions, of completing the optimized trajectory cycle 3.5 times. Consequently, it successfully transfers its maximum payload to low altitudes 4 times, deorbiting a total of 400 kg of debris.

Furthermore, the study aimed to retain a residual amount of chemical propellant at the end of the maneuvers for potential deorbiting operations and, most notably, for the future possibility of reaching an in-orbit refueling station. This capability would allow the SC to extend its debris removal operations, repeating these 3.5 mission cycles after

each refueling event.

6.1 Future Perspectives

Although the primary objective of this thesis was successfully achieved, several enhancements and future perspectives can be introduced and implemented.

First of all, it was demonstrated that the single-thrust-arc maneuver utilizing the Hall effect thruster is optimal for the imposed thrust profile, yet suboptimal with respect to the absolute global optimum. Indeed, the indirect method, through the analysis of the Switching Function (SF) profile, provided the necessary insights to identify a minimum-fuel solution, which is the primary optimization parameter of this study. Given the remarkably low propellant consumption of the electric propulsion system (amounting to 1 kg consumed over 4 cycles out of the 11 kg loaded onboard), this suboptimal solution represents an excellent trade-off for the analyzed mission profile between fuel consumption and maneuvering time. In fact, if additional thrust arcs were introduced to the trajectory, the total mission duration would increase significantly. Nevertheless, solving for a globally optimal solution remains an interesting future implementation, potentially by formulating a problem with a variable switching function that is not pre-assigned to fixed arcs, thereby allowing the thruster to fire in strict accordance with the SF profile as it evolves freely during the optimization process.

Since the other two chemical maneuvers are optimal regarding propellant consumption, the accuracy of the work could be further refined by introducing the atmospheric drag perturbation, thereby including the perturbative accelerations within the dynamic model and proceeding with the optimization exactly as conducted in this analysis.

Finally, further investigations could address the most effective disposal maneuver utilizing the remaining chemical propellant, or, better still, optimize the trajectory required to reach an in-orbit refueling station once such servicing infrastructures become operational.

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Appendix A

Euler-Lagrange Equations for the Adjoint Variables

$$\dot{\lambda} = -\frac{\partial H}{\partial \mathbf{x}} \quad (\text{A.1})$$

$$\begin{aligned} \dot{\lambda}_r = & \frac{1}{r^2} \left[\frac{\lambda_{\vartheta v}}{\cos \varphi} + \lambda_{\varphi} w + \lambda_u \left(-\frac{2\mu}{r} + v^2 + w^2 \right) + \lambda_v (-uv + vw \tan \varphi) + \right. \\ & \left. + \lambda_w (-uw - v^2 \tan \varphi) \right] - \lambda_u \frac{\partial(a_p)_u}{\partial r} - \lambda_v \frac{\partial(a_p)_v}{\partial r} - \lambda_w \frac{\partial(a_p)_w}{\partial r} - S_F \frac{\partial T}{\partial r} \end{aligned} \quad (\text{A.2a})$$

$$\dot{\lambda}_{\vartheta} = -\lambda_u \frac{\partial(a_p)_u}{\partial \vartheta} - \lambda_v \frac{\partial(a_p)_v}{\partial \vartheta} - \lambda_w \frac{\partial(a_p)_w}{\partial \vartheta} + S_F \frac{\partial T}{\partial \vartheta} \quad (\text{A.2b})$$

$$\begin{aligned} \dot{\lambda}_{\varphi} = & \frac{1}{r \cos^2 \varphi} \left(-\lambda_{\vartheta} v \sin \varphi - \lambda_v v w + \lambda_w v^2 \right) + \\ & - \lambda_u \frac{\partial(a_p)_u}{\partial \varphi} - \lambda_v \frac{\partial(a_p)_v}{\partial \varphi} - \lambda_w \frac{\partial(a_p)_w}{\partial \varphi} + S_F \frac{\partial T}{\partial \varphi} \end{aligned} \quad (\text{A.2c})$$

$$\dot{\lambda}_u = \frac{1}{r} (-\lambda_r r + \lambda_v v + \lambda_w w) \quad (\text{A.2d})$$

$$\dot{\lambda}_v = \frac{1}{r} \left[-\lambda_{\vartheta} \frac{1}{\cos \varphi} - 2\lambda_u v + \lambda_v (u - w \tan \varphi) + 2\lambda_w v \tan \varphi \right] \quad (\text{A.2e})$$

$$\dot{\lambda}_w = \frac{1}{r} (-\lambda_{\varphi} - 2\lambda_u w - \lambda_v v \tan \varphi + \lambda_w u) \quad (\text{A.2f})$$

$$\dot{\lambda}_m = \frac{T}{m^2} \lambda_V - \lambda_u \frac{\partial(a_p)_u}{\partial m} - \lambda_v \frac{\partial(a_p)_v}{\partial m} - \lambda_w \frac{\partial(a_p)_w}{\partial m} \quad (\text{A.2g})$$