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# Multiscale Analysis of Non-local Functionals

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*A mio fratello,*
*il mio compagno di viaggio da sempre,*
*il regalo più grande.*

*Ai miei genitori e a tutta la mia famiglia,*
*che ripongono fiducia in me,*
*dandomi supporto costante e incondizionato.*

*Ai miei amici,*
*che mi donano leggerezza quando la mente è appesantita,*
*e sono una parte fondamentale della persona che sono oggi.*

*A Margherita,*
*che mi guida con cura e amore giorno dopo giorno,*
*facendomi luce anche nei momenti più bui.*

**Abstract**

In recent years, non-local operators have attracted considerable attention in the calculus of variations, motivated by some works by Braides and Dal Maso. In this thesis, after illustrating the necessary preliminary tools, we review the main results concerning $\Gamma$-convergence and integral representability of such functionals. Within this setting, we then focus our study on the homogenization of a class of non-local functionals featuring a rapidly oscillating periodic weight. By means of two-scale convergence, we explicitly evaluate the $\Gamma$-limit for constant target functions and show how the interaction between periodicity and non-locality induces the formation of highly oscillatory minimizing sequences. Finally, drawing on a recent counterexample by Braides, we prove that the resulting effective functional cannot be represented through a standard double-integral formulation, highlighting the failure of integral representability in this homogenization regime.

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# Chapter 1

# Introduction

The macroscopic behavior of materials and physical systems is often linked to energy minimization. This happens because Nature is intrinsically "lazy" and always seeks the path that requires the lowest energy cost.

For instance, imagine immersing a closed wire loop into soapy water and then pulling it out: the liquid film will not assume a random shape. Indeed, the soap molecules attract each other, and organize themselves trying to expose the smallest possible number of molecules. In doing so, the macroscopic behavior of the film naturally solves a complex and non-trivial problem: finding the surface of minimum area whose border coincides exactly with the loop, thus minimizing the surface tension energy. This phenomenon is the direct physical manifestation of the Plateau problem—a cornerstone of the Calculus of Variations—which is precisely about finding the surface with the smallest possible area bounded by a specific closed contour.

Another interesting—and more complex—phenomenon is the so-called *energetic frustration*, which frequently occurs in the study of complex materials. A typical example is given by block copolymers, which consist of polymer chains formed by a monomer block A covalently bonded to a monomer block B. When dealing with billions of these chains, the system faces a conflict between two competing energies:

- Local chemical energy. Similarly to water and oil, the incompatible materials A and B tend to segregate in order to reduce unfavorable interactions between unlike monomers.

- Non-local elastic energy. Since A and B are linked to each other, they cannot physically separate by a distance greater than the length of the chain itself without requiring a huge elastic energy.

Since both requirements cannot be satisfied simultaneously, the optimal energetic solution is a compromise, a recurring feature in variational problems. Microscopically, the system avoids macroscopic phase separation by self-assembling into optimal periodic microstructures, such as thin stripes, cylinders, or spheres. However, at the macroscopic level, the system appears perfectly homogeneous. This fascinating passage from disorder to apparent macroscopic homogeneity through the formation of patterns and microstructures is a widely studied topic in the Calculus of Variations (see, for example, [7, 21, 28]), where the use of sophisticated mathematical techniques such as homogenization and Γ-convergence is essential to treat the problem in a mathematically rigorous way.

This complex phenomenology naturally leads to the study of long-range interactions. Indeed, as pointed out in the previous example, while the local chemical energy refers to the interaction between nearby points, computing the non-local elastic energy requires taking into account distant points along the chain. These long-range features are effectively captured by the mathematical framework of non-local functionals, which have recently attracted significant attention within the Calculus of Variations. Indeed, beyond long-range interactions, non-local energies—typically formulated as double integrals—naturally arise in a wide variety of models, including:

- Peridynamics, which is a continuum mechanics formulation specifically developed to model damage and fracture propagation in solids. In this framework, a material point interacts not only with its nearest neighbors, but with all the points within a certain distance, known as the *horizon*. Interesting works on this topic include [24, 36].

- Image processing, where advanced algorithms for denoising and image transformation evaluate similarities across the entire image rather than just adjacent pixels. We refer to [16, 27] for some variational studies of non-local functionals related to image processing.

## Non-local functionals

In this context, recent works by Braides and Dal Maso have investigated some fundamental properties of (sequences of) non-local functionals taking the general integral form

$$F_k(u) = \int_{\Omega\times\Omega} f_k(u(x) - u(y))\,\mathrm{d}\mu_k(x, y) + \int_\Omega g_k(x, \nabla u(x))\,\mathrm{d}x,$$

defined on the Sobolev space $W_0^{1,p}(\Omega)$ with $p > 1$, where $\Omega$ is a bounded open set in $\mathbb{R}^d$ for $d \geq 1$. Here, the non-local character of the energy is captured by the double integral, where the integrand $f_k$ depends on the difference between the values of $u$ at two distinct points in the domain. Conversely, the single integral is a purely local term, where the energy density $g_k$ depends pointwise on the position $x$ and the gradient $\nabla u(x)$.

Some interesting results regarding this class of functionals have been obtained. In [9], the authors provide suitable conditions on the measures $\mu_k$ guaranteeing the stability of the Γ-limit, meaning that the functional $F_k$ Γ-converges to a limit functional preserving the same additive structure, namely:

$$\int_{\Omega\times\Omega} f(u(x) - u(y))\,\mathrm{d}\mu(x, y) + \int_\Omega g(x, \nabla u(x))\,\mathrm{d}x.$$

As pointed out in Remark 2.4.7 below, this is a highly non-trivial property, since the Γ-limit of a sum is generally not the sum of the Γ-limits.

Similar decoupling properties of the $\Gamma$-limit are also analyzed in [8], where Braides and Dal Maso highlight how the effective local part arises from the interplay between local and non-local terms. More precisely, the function $g$ appearing in the local term of the limit functional is shown to be determined by the sequence $\{g_k\}$ and the part of the sequence $\{f_k\}$ that is concentrating on the diagonal of the domain $\Omega \times \Omega$, namely the set $\Delta := \{(x, x) \colon x \in \Omega\}$. The key idea here is to examine the part of the $\Gamma$-limit of the double integral term concentrated on a $\delta$-neighborhood of the diagonal, that is $\Delta_\delta := \{(x, y) \colon x, y \in \Omega, |x - y| < \delta\}$. It is shown that this portion admits a local integral representation, up to an error that vanishes as $\delta \to 0$.

Another particularly delicate issue, which is a central theme of this thesis, is the integral representability of the limit. As extensively reported in Chapter 3, when considering non-local potentials of the form

$$f_k(u(x) - u(y)) \sim |u(x) - u(y)|^p,$$

positive representability results can be achieved for $p = 2$, while non-representability counterexamples arise for $p \neq 2$ [10]. In particular, in [10] it emerges that a powerful tool to study the limits of such sequences is given by the theory of Dirichlet forms, which we introduce in Section 2.5.

Finally, another counterexample for the representability is provided by [6]. There, Braides considers purely non-local functionals of the form

$$F(u) = \int_{\Omega\times\Omega} f(u(x) - u(y))\,\mathrm{d}x\mathrm{d}y.$$

Since no sequence of functionals is involved in this specific setting, the counterexample is constructed for the relaxed functional.

## The choice of the potential

It is worth providing some context on the choice of the energy density $f$, which in [6] is the non-convex triple-well potential defined by

$$f(z) = \begin{cases} 0 & \text{if } z \in \{-1, 1\}, \\ 1 & \text{if } z = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.1}$$

From a physical perspective, non-convex potentials play a fundamental role, as they provide realistic models for atomic and molecular interactions. A classical example is the Lennard–Jones potential

$$V_{LJ}(r) = \frac{1}{r^{12}} - \frac{1}{r^6},$$

where $r$ is the distance between the two particles. The Lennard–Jones potential captures the essential features of atomic interactions: particles strongly repel each other at very short distances ($V_{LJ}(r) \to +\infty$ as $r \to 0$, preventing overlap), weakly attract each other at larger distances ($V_{LJ}(r) \to 0$ as $r \to +\infty$), and tend to arrange themselves at a preferred distance corresponding to the minimum of the potential.

For a system of $N$ particles located at $x_1, \dots, x_N \in \mathbb{R}^d$, the total interaction energy is given by

$$E_N^{LJ}(x_1, \dots, x_N) = \sum_{i \neq j} V_{LJ}(|x_i - x_j|).$$

The analysis of minimizers of $E_N^{LJ}$ is notoriously difficult, since every particle interacts with all the others through long-range forces. To overcome these difficulties, Heitmann and Radin [31] proposed a simplified potential, given by

$$V_{HR} = \begin{cases} +\infty & \text{if } 0 < r < 1, \\ -1 & \text{if } r = 1, \\ 0 & \text{if } r > 1, \end{cases}$$

which may be viewed as a limiting case of $V_{LJ}$ with a very steep well around the equilibrium distance. In this model, particles cannot overlap, and the minimum energy is reached when two particles are at a distance 1. The corresponding interaction energy becomes

$$E_N^{HR}(x_1, \dots, x_N) = \sum_{i \neq j} V_{HR}(|x_i - x_j|),$$

which is considerably easier to analyze. We refer to [11, 14, 22, 23] for some significant studies in this direction.

Although much simpler than realistic interaction potentials, the triple-well potential (1.1) shares the key feature of non-convexity and the presence of favored states, and it is analogous to Heitmann–Radin for three wells. Specifically, this triple-well potential is designed to model systems with preferred interacting states (at $z = 1$ and $z = -1$) and an unstable equilibrium state (at $z = 0$, where a finite energy penalty is paid).

## Homogenization

A natural further step in this investigation is to incorporate the effects of spatial heterogeneities. Indeed, the simplified models discussed so far always assume a perfectly homogeneous medium, where interactions between particles only depend on their relative distance. However, the majority of materials—such as composites, alloys, or crystalline structures with periodic impurities—features a non-constant interaction strength that oscillates across the material at a microscopic scale $\varepsilon > 0$. When dealing with materials exhibiting a heterogeneous yet periodic microstructure, homogenization provides a powerful tool to derive the effective macroscopic behavior. In [2], homogenization techniques are applied to non-local energies using Young measures, a useful yet complex tool in the Calculus of Variations. Furthermore, in [12, 13], some results on the homogenization of convolution-type energies are presented, involving the same oscillating kernels we will use below.

Motivated by these results, in Chapter 4 of this thesis, which contains our original contribution, we study the asymptotic behavior of the sequence of non-local functionals $F_\varepsilon\colon L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$F_\varepsilon(u) = \int_{\Omega\times\Omega} a\left(\frac{x-y}{\varepsilon}\right) f(u(x)-u(y))\,\mathrm{d}x\mathrm{d}y, \tag{1.2}$$

where $f$ is the potential introduced in (1.1), and the periodic weight $a \in L^\infty(\mathbb{R})$ is assumed to be strictly positive. Notice how, in our setting, the heterogeneity acts on the interaction itself; therefore, the coefficient $a$ depends on the relative position $x-y$ of the interacting points.

To tackle the homogenization problem for the energy (1.2), we apply the theory of two-scale convergence introduced by Nguetseng and Allaire [1,34]—and outlined in Section 2.6— which allows us to effectively capture the microscopic oscillations during the limit process. Moreover, we explicitly compute the Γ-limit of the sequence of functionals in (1.2) on constant functions, which is proved to correspond to the macroscopic ground state of the system. Interestingly, the effective value of the minimized energy is lower than the value one would obtain by simply plugging a constant function into the expression (1.2). Indeed, our analysis reveals the emergence of highly oscillating step-like microstructures, which lie at the core of our treatment and drastically lower the overall energy. More precisely, as stated in Remark 4.2.4, these optimal profiles synchronize their microscopic oscillations with the periodic weight $a$, strategically concentrating their mass in the regions where the interaction kernel assumes its lowest values. This crucial and complex energy-saving mechanism is entirely precluded for a simple flat function, resulting in an unexpected effective response. Once again, the apparent macroscopic homogeneity hides a non-trivial microscopic law governing the entire system, and our investigation reveals the profound and fascinating complexity of these phenomena.

# Chapter 2

# Preliminaries

In this chapter, we introduce the mathematical framework and the main theoretical tools required to analyze non-local operators and their asymptotic behavior in the context of the Calculus of Variations. Rather than providing exhaustive proofs, which can be found in classical literature, we focus on stating the results and highlighting their role while treating variational problems.

In the sequel, $\Omega \subset \mathbb{R}^d$ will always be a general open, bounded, connected set.

## 2.1 Basic results in measure theory and integration

When dealing with integral functionals, bounding the energy and passing limits inside the integrals are common operations. To this end, the following inequalities and convergence theorems are fundamental.

**Theorem 2.1.1** (Hölder's inequality)**.** *Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, where $1 \leq p, q \leq \infty$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

*Remark* 2.1.2. In the special case where $p = q = 2$, this inequality reduces to the well-known Cauchy-Schwarz inequality.

**Theorem 2.1.3** (Jensen's inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be a measurable set with finite positive measure, and let $\varphi \colon \mathbb{R} \to \mathbb{R}$ be a convex function. If $u \in L^1(\Omega)$, then*

$$\varphi\left(\frac{1}{|\Omega|}\int_\Omega u(x)\,\mathrm{d}x\right) \leq \frac{1}{|\Omega|}\int_\Omega \varphi(u(x))\,\mathrm{d}x.$$

Jensen's inequality is widely used when dealing with convexity and is essential to establish lower bounds for the limit of sequences of functionals (for example for the liminf inequality in Γ-convergence, see Definition 2.4.1). The following theorems provide two different conditions under which the limit of a sequence of functions commutes with the integral operator.

**Theorem 2.1.4** (Lebesgue, dominated convergence). *Let $\{f_n\}$ be a sequence of integrable functions $f_n\colon \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ such that the sequence converges pointwise almost everywhere to a function $f$ defined almost everywhere in $\mathbb{R}^d$. Suppose there exists an integrable function $g\colon \mathbb{R}^d \to \mathbb{R}$ such that:*

$$|f_n(x)| \leq g(x) \quad \textit{for all } n \in \mathbb{N} \textit{ and for a.e. } x \in \mathbb{R}^d.$$

*Then, the limit function $f$ is integrable and the following formulas are valid*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n \,\mathrm{d}x = \int_{\mathbb{R}^d} f \,\mathrm{d}x \qquad \textit{and} \qquad \lim_{n\to\infty} \int_{\mathbb{R}^d} |f_n - f| \,\mathrm{d}x = 0.$$

**Theorem 2.1.5** (Beppo Levi, monotone convergence). *Let $\{f_n\}$ be a sequence of integrable functions $f_n\colon \mathbb{R}^d \to [0, +\infty]$ such that the sequence is monotonically non-decreasing almost everywhere; that is:*

$$0 \leq f_1(x) \leq f_2(x) \leq \dots \leq f_n(x) \leq \dots \quad \textit{for a.e. } x \in \mathbb{R}^d.$$

*If $f(x) = \lim_{n\to\infty} f_n(x)$ almost everywhere, then $f$ is integrable and:*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n \,\mathrm{d}x = \int_{\mathbb{R}^d} f \,\mathrm{d}x.$$

The following two theorems, usually known together as the Fubini–Tonelli Theorem, are a powerful tool to compute multiple integrals as iterations of simple integrals, establishing when it is possible to reverse the order of integration.

**Theorem 2.1.6** (Fubini). *Let $f\colon \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$, $(x, y) \mapsto f(x, y)$, be an integrable function in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then*

1. *the function $x \mapsto f(x, y)$ is integrable in $\mathbb{R}^{d_1}$, for almost every $y \in \mathbb{R}^{d_2}$;*
2. *the function $y \mapsto \int_{\mathbb{R}^{d_1}} f(x, y) \,\mathrm{d}x$ is integrable in $\mathbb{R}^{d_2}$;*
3. *the following formula holds*

$$\int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} f(x, y) \,\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f(x, y) \,\mathrm{d}x \right) \mathrm{d}y.$$

**Theorem 2.1.7** (Tonelli). *Let $f\colon \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$, $(x, y) \mapsto f(x, y)$, be a (real) non-negative and measurable function in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. If the properties 1. and 2. of Theorem 2.1.6 hold, then $f$ is integrable in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.*

Finally, a corollary of the Stone–Weierstrass Theorem will be useful while dealing with test functions depending simultaneously on multiple variables. It guarantees that any sufficiently regular test function $\psi(x, y)$ can be approximated by finite linear combinations of factorized functions $\phi(x)\psi(y)$.

**Theorem 2.1.8.** *Let $K_1 \subset \mathbb{R}^{d_1}$ and $K_2 \subset \mathbb{R}^{d_2}$ be compact sets. Any continuous function $f\colon K_1 \times K_2 \to \mathbb{R}$ can be uniformly approximated by finite sums of factorized functions. That is, for every $\varepsilon > 0$, there exist an integer $N > 0$ and continuous functions $\phi_1, \dots, \phi_N \in C^0(K_1)$ and $\psi_1, \dots, \psi_N \in C^0(K_2)$ such that*

$$\max_{(x,y)\in K_1\times K_2} \left| f(x,y) - \sum_{i=1}^{N} \phi_i(x)\psi_i(y) \right| < \varepsilon.$$

## 2.2 Sobolev spaces

In the Calculus of Variations, one often seeks minimizers of energy functionals among continuously differentiable functions. However, classical spaces such as $C^1(\Omega)$ are not complete under the natural integral norms associated with these energies (see Eq. (2.1)). This lack of completeness means that minimizing sequences might not converge to a well-behaved function within the space. Sobolev spaces resolve this fundamental issue. By replacing classical derivatives with weak derivatives and completing the spaces with respect to $L^p$ norms, Sobolev spaces provide the proper framework where the existence of minimizers can be established via the Direct Method (see Section 2.3). For a complete reference on this topic, see [18, 25].

**Definition 2.2.1** (Weak derivative)**.** Given $u, v \in L^1_{\text{loc}}(\Omega)$ and a multi-index $\alpha \in \mathbb{N}^N$, we say that $v$ is the $\alpha^{th}$ weak derivative of $u$, and we write $D^\alpha u = v$, if

$$\int_\Omega u D^\alpha \varphi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v\varphi \,\mathrm{d}x,$$

for every $\varphi \in C^\infty_c(\Omega)$.

**Definition 2.2.2** (Sobolev space)**.** Let $p \in [1, +\infty]$ and $k$ be a positive integer. The Sobolev space $W^{k,p}(\Omega)$ is the set of functions $u\colon \Omega \to \mathbb{R}$ such that $u \in L^p(\Omega)$ and $D^\alpha u \in L^p(\Omega)$, where $D^\alpha u$ are the weak partial derivatives of $u$, for every multi-index $\alpha$ with $|\alpha| \le k$.

This space is endowed with the norm

$$\|u\|_{W^{k,p}} = \begin{cases} \left( \sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^p \,\mathrm{d}x \right)^{\frac{1}{p}} & \text{if } 1 \le p < +\infty, \\ \sum_{|\alpha|\le k} \text{ess-sup}_\Omega\, |D^\alpha u| & \text{if } p = +\infty. \end{cases} \tag{2.1}$$

**Definition 2.2.3.** The set $W^{k,p}_0(\Omega)$ is the closure of the space $C^\infty_c(\Omega)$ in $W^{k,p}(\Omega)$. If $\Omega$ has Lipschitz boundary, then $W^{1,p}_0(\Omega)$ coincides with the set of Sobolev functions having zero trace on $\partial\Omega$.

**Theorem 2.2.4** (Poincaré inequality)**.** *Let $1 \le p < \infty$. There exists a constant $C > 0$, depending only on $d$, $p$, and $\Omega$, such that for every function $u \in W^{1,p}_0(\Omega)$, the following inequality holds:*

$$\|u\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)}.$$

*Remark* 2.2.5. It is worth noting that the constant $C$ in the Poincaré inequality depends heavily on the geometry of the domain $\Omega$, specifically scaling with its width. If we consider a thin domain, such as a strip of height $\varepsilon$, the constant $C$ becomes proportional to $\varepsilon$ (or $\varepsilon^2$ for the squared norms). This property is useful while dealing with problems where energy concentrates on shrinking domains (see Example 3.1.4).

**Definition 2.2.6** (Hölder continuous functions)**.** Let $0 < \gamma \leq 1$. A function $u : \Omega \to \mathbb{R}$ is said to be Hölder continuous with exponent $\gamma$ if there exists a constant $C > 0$ such that

$$|u(x) - u(y)| \leq C|x - y|^\gamma \quad \text{for all } x, y \in \Omega.$$

**Definition 2.2.7** (Hölder spaces)**.** Let $0 < \gamma \leq 1$. For a continuous function $u\colon \Omega \to \mathbb{R}$, we define the Hölder norm with exponent $\gamma$ as

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} = \|u\|_{C^0(\overline{\Omega})} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^\gamma}.$$

The Hölder space $C^{0,\gamma}(\overline{\Omega})$ is defined as the set of all continuous functions $u\colon \Omega \to \mathbb{R}$ for which this norm is finite.

*Remark* 2.2.8. Equipped with the norm $\|\cdot\|_{C^{0,\gamma}(\overline{\Omega})}$, the space $C^{0,\gamma}(\overline{\Omega})$ is a Banach space.

**Theorem 2.2.9** (Morrey's inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with $\partial\Omega \in C^1$. Let $u \in W^{1,p}(\Omega)$ with $p > d$. Then $u$ has a continuous representative $u^* \in C^{0,\gamma}(\overline{\Omega})$ with $\gamma = 1 - \frac{d}{p}$. Moreover, there exists a constant $C > 0$, depending only on $\Omega, p$, and $d$, such that*

$$\|u^*\|_{C^{0,\gamma}(\overline{\Omega})} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

**Definition 2.2.10** (Weak and weak-$*$ convergence in $L^p$ and $W^{1,p}$ spaces)**.** A sequence $\{u_k\} \subset L^p(\Omega)$, with $1 \leq p < \infty$, is said to converge weakly to an element $u \in L^p(\Omega)$, denoted by $u_k \rightharpoonup u$, if

$$\lim_{k\to\infty} \int_\Omega u_k(x)v(x)\,\mathrm{d}x = \int_\Omega u(x)v(x)\,\mathrm{d}x \quad \text{for every } v \in L^q(\Omega),$$

where $q$ is the Hölder conjugate exponent of $p$.

If $p = \infty$, the space $L^\infty(\Omega)$ is not reflexive, and we rely on weak-$*$ convergence. A sequence $\{u_k\} \subset L^\infty(\Omega)$ is said to converge weakly-$*$ to $u \in L^\infty(\Omega)$, denoted by $u_k \overset{*}{\rightharpoonup} u$, if

$$\lim_{k\to\infty} \int_\Omega u_k(x)v(x)\,\mathrm{d}x = \int_\Omega u(x)v(x)\,\mathrm{d}x \quad \text{for every } v \in L^1(\Omega).$$

Similarly, a sequence $\{u_k\}$ converges weakly to $u$ in the Sobolev space $W^{1,p}(\Omega)$ for $1 \leq p < \infty$ if $u_k \rightharpoonup u$ in $L^p(\Omega)$ and $\nabla u_k \rightharpoonup \nabla u$ in $L^p(\Omega; \mathbb{R}^d)$.

Finally, a sequence $\{u_k\}$ converges weakly-$*$ to $u$ in the Sobolev space $W^{1,\infty}(\Omega)$ if $u_k \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$ and $\nabla u_k \overset{*}{\rightharpoonup} \nabla u$ in $L^\infty(\Omega; \mathbb{R}^d)$.

### 2.2.1 Radon measures and weak-$*$ convergence

The concept of weak-$*$ convergence, introduced for $L^\infty(\Omega)$, naturally extends to the mathematical framework of measures. When dealing with sequences of functionals, energy concentrations or highly oscillating coefficients might not converge to a standard function, but rather to a measure.

**Definition 2.2.11** (Radon measures and weak-$*$ convergence)**.** Let $K \subset \mathbb{R}^d$ be a compact set (such as $\overline{\Omega}$ or $\overline{\Omega} \times \overline{\Omega}$). We denote by $\mathcal{M}(K)$ the space of finite Radon measures on $K$, that is, the dual of $C^0(K)$.

A sequence of measures $\{\mu_k\} \subset \mathcal{M}(K)$ is said to weakly-$*$ converge to a limit measure $\mu \in \mathcal{M}(K)$, denoted by $\mu_k \overset{*}{\rightharpoonup} \mu$, if

$$\lim_{k\to\infty} \int_K \varphi(z)\,\mathrm{d}\mu_k(z) = \int_K \varphi(z)\,\mathrm{d}\mu(z) \quad \text{for all } \varphi \in C^0(K).$$

**Theorem 2.2.12** (Banach–Alaoglu for measures)**.** *Let $K \subset \mathbb{R}^d$ be a compact set. If $\{\mu_k\}$ is a sequence of positive Radon measures with uniformly bounded total mass, meaning there exists a constant $C > 0$ such that $\mu_k(K) \le C$ for all $k \in \mathbb{N}$, then the sequence is weakly-$*$ relatively compact.*

*Consequently, there exists a subsequence $\{\mu_{k_j}\}$ and a positive measure $\mu \in \mathcal{M}(K)$ such that $\mu_{k_j} \overset{*}{\rightharpoonup} \mu$ as $j \to \infty$.*

**Example 2.2.13.** Let $\Omega = (-1,1)$ and consider the sequence of absolutely continuous measures $\mathrm{d}\mu_k(x) = \frac{k}{2}\mathbf{1}_{(-1/k,1/k)}(x)\mathrm{d}x$. Each $\mu_k$ has total mass equal to 1. For any continuous test function $\varphi \in C^0([-1,1])$, the mean value theorem for integrals implies that

$$\lim_{k\to\infty} \int_{-1}^{1} \varphi(x)\,\mathrm{d}\mu_k(x) = \lim_{k\to\infty} \frac{k}{2} \int_{-1/k}^{1/k} \varphi(x)\,\mathrm{d}x = \varphi(0).$$

By definition, this means that $\mu_k \overset{*}{\rightharpoonup} \delta_0$, where $\delta_0$ is the Dirac measure centered at the origin.

### 2.2.2 Embedding theorems

A crucial feature of Sobolev spaces is how they relate to classical $L^p$ spaces and continuous functions. The following definition and theorem provide the essential tools to extract strongly convergent subsequences from weakly convergent ones.

**Definition 2.2.14** (Compact embedding)**.** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces such that $X \subset Y$. We say that $X$ is compactly embedded in $Y$, denoted by $X \hookrightarrow Y$, if the following two conditions hold:

1. *Continuous embedding:* There exists a constant $C > 0$ such that

$$\|x\|_Y \le C\|x\|_X \quad \text{for all } x \in X.$$

2. *Compactness:* Every bounded sequence in $X$ admits a subsequence that is strongly convergent in $Y$.

**Theorem 2.2.15** (Rellich–Kondrachov)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with a Lipschitz continuous boundary $\partial\Omega$. Let $1 \leq p \leq \infty$. Then the Sobolev space $W^{1,p}(\Omega)$ is compactly embedded in the following spaces:*

1. *If $1 \leq p < d$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $1 \leq q < p^*$, where $p^* = \frac{dp}{d-p}$ is the Sobolev conjugate exponent.*

2. *If $p = d$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $1 \leq q < \infty$.*

3. *If $p > d$, then $W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$.*

*Remark* 2.2.16. The Rellich–Kondrachov Theorem is a cornerstone in the study of $\Gamma$-convergence. Often, the equicoercivity of a sequence of functionals (see Definition 2.3.1) guarantees that any sequence of bounded energy converges weakly in $W^{1,p}(\Omega)$. The compact embedding allows us to upgrade this weak convergence into strong convergence in $L^q(\Omega)$ (for example, in $L^2(\Omega)$ when $p = 2$). This strong convergence is essential to pass to the limit in non-linear or non-local terms of the energy, as we will see when proving the truncation property (Proposition 3.1.2).

### 2.2.3 Capacity

While the Lebesgue measure is suitable for integrating $L^p$ functions, evaluating the pointwise behavior of Sobolev functions requires a finer tool to measure the size of exceptional sets, where functions exhibit singular behavior. To this end, we introduce the following notion of capacity, for which we refer in particular to [30, 32].

**Definition 2.2.17** (Capacity)**.** For any subset $A \subset \Omega$ the capacity of $A$ in $\Omega$ is defined as

$$C_{1,p}(A) := \inf \left\{ \int_\Omega |\nabla u|^p \, \mathrm{d}x : u \in W_0^{1,p}(\Omega), u \geq 1 \text{ a.e. in a neighborhood of } A \right\}.$$

**Definition 2.2.18** ($p$-capacitary potential)**.** Let $K \subset \Omega$ be a compact set. The $p$-capacitary potential of $K$ with respect to $\Omega$ is the unique minimizer $v \in W_0^{1,p}(\Omega)$ of the variational problem defining the capacity; that is, the function achieving

$$\int_\Omega |\nabla v|^p \, \mathrm{d}x = C_{1,p}(K).$$

**Definition 2.2.19** ($C_p$-quasicontinuous function)**.** A function $f \colon \Omega \to \mathbb{R}$ is said to be $C_p$-quasicontinuous if for every $\varepsilon > 0$ there exists $A = A_\varepsilon \subset \Omega$ such that $C_{1,p}(A) < \varepsilon$ and $f|_{\Omega \setminus A}$ is continuous on $\Omega \setminus A$.

**Theorem 2.2.20.** *For every $u \in W_0^{1,p}(\Omega)$, there exists a $C_p$-quasicontinuous representative, which we denote by $\widetilde{u}$, such that $\widetilde{u} \colon \Omega \to \mathbb{R}$ is a $C_p$-quasicontinuous function and $\widetilde{u} = u$ holds almost everywhere in $\Omega$.*

*Remark* 2.2.21. Functions in $L^p(\Omega)$ or $W^{1,p}(\Omega)$ are equivalence classes defined up to sets of Lebesgue measure zero. However, when dealing with non-local functionals—which evaluate differences like $|u(x) - u(y)|$ across the domain (see Chapter 3)—the pointwise behavior of $u$ becomes highly relevant. Capacity provides a finer tool than the Lebesgue measure to evaluate the size of exceptional sets. The $C_p$-quasicontinuous representative $\widetilde{u}$ ensures that we can pointwise evaluate our functions outside of sets of arbitrarily small capacity, making the integral representation of non-local limits well-defined (as seen in Theorem 3.1.6).

**Example 2.2.22.** To illustrate why capacity provides a finer tool than the Lebesgue measure, let us analyze the capacity of a single point $A = \{x_0\}$ in the domain $\Omega \subset \mathbb{R}^d$. Without loss of generality, we can assume $x_0 = 0$ and that $\Omega$ contains the ball $B_R(0)$, for a fixed $R > 0$. While the Lebesgue measure of a point is trivially $|\{0\}| = 0$ in any dimension, its $p$-capacity exhibits a very different behavior.

Recall that $C_{1,p}(\{0\})$ is defined as an infimum. Thus, to prove that $C_{1,p}(\{0\}) = 0$, it is sufficient to construct a sequence of admissible functions $u_k \in W_0^{1,p}(\Omega)$ (meaning $u_k \geq 1$ in a neighborhood of the origin) and show that their energy $\int_\Omega |\nabla u_k|^p \,\mathrm{d}x$ vanishes as $k \to \infty$. Let us consider three different cases:

- Case $1 \leq p < d$: we define a sequence of functions $u_k$ with a shrinking support of radius $r_k \to 0$ (and $r_k < R$):

$$u_k(x) := \begin{cases} 1 & \text{if } |x| \leq r_k, \\ 2 - \frac{|x|}{r_k} & \text{if } r_k < |x| \leq 2r_k, \\ 0 & \text{if } |x| > 2r_k. \end{cases}$$

  It is easy to check that $u_k \in W_0^{1,p}(\Omega)$, and obviously satisfies $u_k = 1$ in a neighborhood of the origin. Computing the energy in polar coordinates yields:

$$\begin{aligned} \int_\Omega |\nabla u_k|^p \,\mathrm{d}x &= \int_{B_{2r_k}(0) \setminus B_{r_k}(0)} \frac{1}{r_k^p} \,\mathrm{d}x \\ &= \frac{1}{r_k^p} \omega_d \int_{r_k}^{2r_k} \rho^{d-1} \,\mathrm{d}\rho = \frac{\omega_d}{d}(2^d - 1) r_k^{d-p}, \end{aligned}$$

  where $\omega_d$ is the surface area of the unit sphere in $\mathbb{R}^d$. Since $p < d$, the exponent $d - p$ is strictly positive. As $r_k \to 0$, the energy vanishes, thus $C_{1,p}(\{0\}) = 0$.

- Case $p = d$: the previous sequence yields a constant energy $\frac{\omega_d}{d}$. To show that the capacity is still zero, we use a different $u_k$: for $r_k \to 0$, we define

$$u_k(x) := \begin{cases} 1 & \text{if } |x| \leq r_k, \\ \dfrac{\ln(R/|x|)}{\ln(R/r_k)} & \text{if } r_k < |x| < R, \\ 0 & \text{if } |x| \geq R. \end{cases}$$

These functions belong to $W_0^{1,d}(\Omega)$ and are admissible. Computing the energy we obtain

$$\begin{aligned}\int_\Omega |\nabla u_k|^d \,\mathrm{d}x &= \int_{B_R(0)\setminus B_{r_k}(0)} \left(\frac{1}{\ln(R/r_k)|x|}\right)^d \mathrm{d}x \\ &= \frac{\omega_d}{(\ln(R/r_k))^d}\int_{r_k}^R \frac{1}{\rho^d}\rho^{d-1}\,\mathrm{d}\rho = \frac{\omega_d}{(\ln(R/r_k))^{d-1}}.\end{aligned}$$

As $r_k \to 0$, we have that $\ln(R/r_k) \to \infty$, thus $C_{1,d}(\{0\}) = 0$ for any $d > 1$.

- Case $p > d$: by Morrey's inequality every $u \in W_0^{1,p}(\Omega)$ admits a continuous representative and

$$\|u^*\|_{L^\infty(\Omega)} \le C\|u\|_{W_0^{1,p}(\Omega)},$$

for a certain $C > 0$. For any admissible function $u$, the condition $u \ge 1$ a.e. in a neighborhood of 0 implies, by continuity, that its continuous representative satisfies $u^*(0) \ge 1$. Hence, using Poincaré inequality 2.2.4,

$$1 \le \|u^*\|_{L^\infty(\Omega)} \le C\|u\|_{W_0^{1,p}(\Omega)} \le \overline{C}\|\nabla u\|_{L^p(\Omega)},$$

which yields $\int_\Omega |\nabla u|^p \,\mathrm{d}x \ge \overline{C}^{-p}$. Taking the infimum over all the admissible functions, we conclude that $C_{1,p}(\{0\}) > 0$.

This behavior perfectly exemplifies why two functions that coincide almost everywhere in the Lebesgue sense might still differ on a set of zero measure but strictly positive capacity.

### 2.2.4 The Riesz–Sobolev inequality

This powerful property allows us to explicitly find the optimal profiles for non-local functionals by restricting the search to the class of symmetrically decreasing functions, see [15,35]. When dealing with periodic homogenization on a reference cell (as we will do in Proposition 4.2.1), it is useful to define this operation on the flat torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ (i.e., functions defined modulo 1).

**Definition 2.2.23** (Symmetric decreasing rearrangement on $\mathbb{T}^d$)**.** Let $f\colon \mathbb{T}^d \to \mathbb{R}^+$ be a measurable function. The symmetric decreasing rearrangement of $f$, denoted by $f^*$, is the unique non-negative function on $\mathbb{T}^d$ that is radially symmetric, non-increasing with respect to the distance from the origin, and equimeasurable with $f$; that is, for every $\tau > 0$,

$$|\{x \in \mathbb{T}^d : f^*(x) > \tau\}| = |\{x \in \mathbb{T}^d : f(x) > \tau\}|.$$

**Example 2.2.24.** If $d = 1$ and $f$ is an indicator function of a set $E \subset [0,1)$ with Lebesgue measure $|E| = t$, its symmetric decreasing rearrangement is the centered indicator function $f^*(x) = \chi_{(-t/2,t/2)}(x)$, understood modulo 1.

**Theorem 2.2.25** (Riesz–Sobolev rearrangement inequality)**.** *Let $f, g, h\colon \mathbb{T}^d \to \mathbb{R}^+$ be non-negative measurable functions defined on the flat torus. Let $f^*, g^*, h^*$ be their respective symmetric decreasing rearrangements. Then, the following inequality holds:*

$$\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} f(x)g(x-y)h(y)\,\mathrm{d}x\mathrm{d}y \le \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} f^*(x)g^*(x-y)h^*(y)\,\mathrm{d}x\mathrm{d}y.$$

## 2.3 The Direct Method of the Calculus of Variations

The classical approach to finding the minimum of a functional $F$ is the so-called Direct Method of the Calculus of Variations, for which we refer to [17, 29]. Generalizing the Weierstrass Theorem to infinite-dimensional spaces, the Direct Method guarantees the existence of a minimizer for a functional $F\colon X \to \mathbb{R} \cup \{+\infty\}$ provided two conditions are satisfied:

- Compactness (usually derived from coercivity): ensuring that minimizing sequences have a convergent subsequence.
- Lower-semicontinuity: ensuring that the limit of this subsequence does not have a jump in energy.

**Definition 2.3.1** (Coercivity and Equicoercivity)**.** Let $X$ be a normed space and $F\colon X \to \mathbb{R} \cup \{+\infty\}$. $F$ is called coercive if for any $\ell \in \mathbb{R}$, there exists a constant $C_\ell \in \mathbb{R}$ such that for every $u \in X$,

$$F(u) \le \ell \quad \Longrightarrow \quad \|u\|_X \le C_\ell.$$

A family of functionals $\{F_k\}$ on a normed space $X$ is said to be equi-coercive if for any $\ell \in \mathbb{R}$, there exists a constant $C_\ell \in \mathbb{R}$ not depending on $k$ such that for every $u \in X$,

$$F_k(u) \le \ell \quad \Longrightarrow \quad \|u\|_X \le C_\ell, \quad \text{for every } k.$$

*Remark* 2.3.2. This condition means that if the energy $F(u)$ is controlled from above, then the norm $\|u\|_X$ is also bounded.

This property is crucial for the compactness step of the Direct Method, thanks to the following proposition and the properties of reflexive spaces. Recall that a sequence $\{u_k\} \subset X$ is called a minimizing sequence for $F$ if $\lim_{k\to\infty} F(u_k) = \inf_{u\in X} F(u)$.

**Proposition 2.3.3.** *If $F$ is coercive on a normed space $X$ (and not identically $+\infty$), then every minimizing sequence is bounded in norm. Furthermore, if $X$ is a reflexive Banach space (such as $W^{1,p}$ for $1 < p < \infty$), this boundedness guarantees the existence of a weakly convergent subsequence.*

**Definition 2.3.4** (Lower-semicontinuity)**.** Let $X$ be a metric space and $F\colon X \to \mathbb{R} \cup \{+\infty\}$. $F$ is said to be lower-semicontinuous at $u \in X$ if and only if

$$F(u) \le \liminf_{u_k \to u} F(u_k),$$

for every sequence $\{u_k\}$ such that $u_k \to u$ in $X$.

*Remark* 2.3.5. If $H$ is a reflexive Banach space, then the norm is weakly-lower semicontinuous; i.e., for every sequence $\{u_k\}$ such that $u_k \rightharpoonup u$ in $H$,

$$\|u\|_H \leq \liminf_{k\to\infty} \|u_k\|_H.$$

For instance, if $H = H_0^1(\Omega)$, we have

$$\int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k(x)|^2 \,\mathrm{d}x.$$

This will be useful in Example 3.1.4.

*Remark* 2.3.6. The definitions of coercivity and lower-semicontinuity can be formulated in a more general framework, for functionals defined on a topological space, as in [20]. In this broader setting:

- a functional $F\colon X \to \mathbb{R} \cup \{+\infty\}$ is said to be coercive if, for every $t \in \mathbb{R}$, there exists a compact and closed set $K$ which contains the set $\{x \in X \mid F(x) \leq t\}$. This is equivalent to requiring that the set $\overline{\{F \leq t\}}$ is compact.
- a functional $F\colon X \to \mathbb{R} \cup \{+\infty\}$ is said to be lower-semicontinuous at $x_0 \in X$ if for every $t < F(x_0)$ there exists a neighborhood $U$ of $x_0$ such that $t < F(y)$ for all $y \in U$.

When a functional is not lower-semicontinuous, it might not admit a minimum. In such cases, one replaces the original problem with a relaxed one.

**Definition 2.3.7** (Lower-semicontinuous envelope or relaxed functional)**.** Let $X$ be a metric space and $F\colon X \to \mathbb{R} \cup \{+\infty\}$. The lower-semicontinuous envelope of $F$, or the relaxed functional of $F$, is defined as

$$\overline{F}(u) := \sup\{G(u) : G \text{ is lower-semicontinuous and } G \leq F\}.$$

## 2.4 Γ-convergence

While the Direct Method deals with the minimization of a single functional, in many physical and mathematical models (such as homogenization and phase transitions) we encounter a sequence of approximate functionals $F_k$. Introduced by Ennio De Giorgi, Γ-convergence is the natural notion of variational convergence in this case. Unlike pointwise convergence, Γ-convergence guarantees that the minimizers of $F_k$ converge to the minimizer of a limit functional $F$ under equicoercivity conditions, as stated in the Theorem 2.4.3 below. We refer to [5, 19] for this part.

**Definition 2.4.1** (Γ-convergence)**.** Let $X$ be a metric space, and $F_k\colon X \to \mathbb{R} \cup \{+\infty\}$ a sequence of functionals on $X$. We say that $\{F_k\}$ Γ-converges to $F\colon X \to \mathbb{R} \cup \{+\infty\}$ if, for every $u \in X$, the following two conditions hold:

1. (liminf inequality) for every $u_k \to u$,

$$F(u) \leq \liminf_{k\to\infty} F_k(u_k);$$

2. (limsup inequality) there exists $\overline{u}_k \to u$ such that

$$F(u) \geq \limsup_{k\to\infty} F_k(\overline{u}_k).$$

*Remark* 2.4.2. Notice that if $F_k \equiv F$, it may not be true that $\Gamma - \lim_{k\to\infty} F_k = F$. Indeed, if this holds, the liminf inequality condition would imply that for every $u \in X$ and for every $u_k \to u$, $F(u) \leq \liminf_{k\to\infty} F_k(u_k) = \liminf_{k\to\infty} F(u_k)$, which is true only if $F$ is lower-semicontinuous.

As said before, the core motivation for introducing Γ-convergence lies in the following result.

**Theorem 2.4.3** (Fundamental Theorem of Γ-convergence)**.** *Let $X$ be a metric space and $\{F_k\}, F\colon X \to \mathbb{R}\cup\{+\infty\}$ such that $\{F_k\}$ Γ-converges to $F$. If $\{F_k\}$ is an equi-coercive sequence (i.e., there exists a non-empty compact subset $K \subset X$ such that $\inf_X F_k = \inf_K F_k$ for every $k$). Then,*

- $\exists \min_X F$*;*
- $\exists \lim_{k\to\infty} \big(\inf_X F_k\big) = \min_X F$*.*

**Corollary 2.4.4.** *If a sequence $\{u_k\}$ is relatively compact (i.e., it admits a convergent subsequence) and is such that $\lim_{k\to\infty} F_k(u_k) = \lim_{k\to\infty} (\inf_X F_k)$, then every limiting point of $\{u_k\}$ is a minimizing point for $F$.*

**Example 2.4.5.** To illustrate the power of this result, and consequently of the notion of Γ-convergence, let us consider the sequence of functions $F_k\colon \mathbb{R} \to \mathbb{R}$ defined by

$$F_k(x) = x^2 + \sin(kx).$$

Here, $X = \mathbb{R}$, endowed with the standard Euclidean topology. Notice that the pointwise limit of $F_k(x)$ as $k \to \infty$ does not even exist for almost every $x \in \mathbb{R}$. However, the Γ-limit exists and can be easily computed. Since $\sin(kx) \geq -1$ everywhere, we clearly have the lower bound $F_k(x) \geq x^2 - 1$ for every $k$, which obviously implies

$$\liminf_{k\to\infty} F_k(x_k) \geq x^2 - 1.$$

Moreover, for any given $x \in \mathbb{R}$, we construct the recovery sequence $\overline{x}_k = \frac{2\pi}{k}\lfloor\frac{kx}{2\pi}\rfloor - \frac{\pi}{2k}$ and evaluate its limit as $k \to \infty$. We have that

$$\begin{aligned}
\frac{kx}{2\pi} - 1 &< \left\lfloor \frac{kx}{2\pi} \right\rfloor \leq \frac{kx}{2\pi} \\
\frac{2\pi}{k}\left(\frac{kx}{2\pi} - 1\right) - \frac{\pi}{2k} &< \frac{2\pi}{k}\left\lfloor \frac{kx}{2\pi} \right\rfloor - \frac{\pi}{2k} \leq \frac{2\pi}{k}\left(\frac{kx}{2\pi}\right) - \frac{\pi}{2k} \\
x - \frac{2\pi}{k} - \frac{\pi}{2k} &< \overline{x}_k \leq x - \frac{\pi}{2k};
\end{aligned}$$

and since $\lim_{k\to\infty}\left(x-\frac{2\pi}{k}-\frac{\pi}{2k}\right)=x$ and $\lim_{k\to\infty}\left(x-\frac{\pi}{2k}\right)=x$, the Squeeze Theorem ensures that $\lim_{k\to\infty}\overline{x}_k=x$. Moreover,

$$\limsup_{k\to\infty} F_k(\overline{x}_k)=\lim_{k\to\infty}\left(\overline{x}_k^2+\sin\left(2\pi\left\lfloor\frac{kx}{2\pi}\right\rfloor-\frac{\pi}{2}\right)\right)=x^2-1.$$

Therefore, the $\Gamma$-limit is exactly the functional

$$F(x)=x^2-1.$$

To apply Theorem 2.4.3, we first note that the sequence $\{F_k\}$ is equi-coercive. Indeed, the minima of $F_k$ are achieved within a compact subset of $\mathbb{R}$ (e.g., a closed interval around the origin). Therefore, the Fundamental Theorem of $\Gamma$-convergence 2.4.3 applies, guaranteeing that

$$\lim_{k\to\infty}\left(\inf_{x\in\mathbb{R}} F_k(x)\right)=\min_{x\in\mathbb{R}} F(x)=-1,$$

which was not obvious at all at the beginning. Although the oscillatory term prevents any pointwise convergence of the sequence $\{F_k\}$, the $\Gamma$-limit is able to capture the asymptotic behavior of the minimal energy values.

Now, let $x_k$ be a minimizer of $F_k$, so that $F_k(x_k)=\inf_{\mathbb{R}} F_k$. Thanks to the equi-coercivity, the sequence of minimizers $\{x_k\}$ is bounded, hence relatively compact in $\mathbb{R}$. We can therefore apply Corollary 2.4.4 to deduce that any limiting point of $\{x_k\}$ must be a minimizer of the $\Gamma$-limit $F$. Since $x=0$ is the unique minimizer of $F(x)=x^2-1$, we can immediately conclude that the entire sequence of minimizers must converge to the origin, i.e., $x_k\to 0$. From a geometric perspective, this is intuitively clear, since the minimizers $x_k$ are forced to concentrate in the lowest valleys of the oscillating function.

We conclude with some elementary properties of $\Gamma$-limits.

**Proposition 2.4.6.** *Let $X$ be a metric space.*

- *Given a functional $F\colon X\to\mathbb{R}\cup\{+\infty\}$, if we consider the constant sequence of functionals $F_k=F$ for all $k\in\mathbb{N}$, then the $\Gamma$-limit of $\{F_k\}$ exists and coincides with the lower-semicontinuous envelope $\overline{F}$ of $F$.*

- *The $\Gamma$-convergence is stable under continuous perturbations. Precisely, given $\{F_k\}$ which $\Gamma$-converges to $F$ and a continuous functional $G\colon X\to\mathbb{R}\cup\{+\infty\}$, then $\{F_k+G\}$ $\Gamma$-converges to $F+G$.*

- *The $\Gamma$-limit of a sequence of quadratic forms is a quadratic form (see [19, Theorem 11.10]).*

*Remark* 2.4.7. Notice that in general, if $\Gamma-\lim_{k\to\infty}F_k=F$ and $\Gamma-\lim_{k\to\infty}G_k=G$, it is not true that $\Gamma-\lim_{k\to\infty}(F_k+G_k)=F+G$ (i.e., $\Gamma$-convergence is not additive in general).

A simple counterexample is given by choosing $X = \mathbb{R}$ (endowed with the Euclidean topology), $F_k(x) = \sin^2(kx)$ and $G_k(x) = \cos^2(kx)$. In order to prove that the two functions both $\Gamma$-converge to 0, we first observe that for every sequence $x_k \to x$

$$\liminf_{k\to\infty} F_k(x_k) = \liminf_{k\to\infty} \sin^2(kx_k) \geq 0,$$

which immediately proves the liminf inequality. Notice that the same reasoning can also be applied to the functions $G_k$. To prove the limsup inequality, let us construct the recovery sequence $\overline{x}_k = \frac{\pi}{k}\lfloor\frac{kx}{\pi}\rfloor$ and evaluate its limit as $k \to \infty$. Similarly to Example 2.4.5, we have that

$$\begin{gathered}
\frac{kx}{\pi} - 1 < \left\lfloor \frac{kx}{\pi} \right\rfloor \leq \frac{kx}{\pi} \\
\frac{\pi}{k}\left(\frac{kx}{\pi} - 1\right) < \frac{\pi}{k}\left\lfloor \frac{kx}{\pi} \right\rfloor \leq \frac{\pi}{k}\left(\frac{kx}{\pi}\right) \\
x - \frac{\pi}{k} < \overline{x}_k \leq x;
\end{gathered}$$

and again the Squeeze Theorem ensures that $\lim_{k\to\infty} \overline{x}_k = x$. Evaluating $F_k$ along this recovery sequence, we obtain

$$F_k(\overline{x}_k) = \sin^2\left(\pi \left\lfloor \frac{kx}{\pi} \right\rfloor\right) = 0.$$

Therefore, $\limsup_{k\to\infty} F_k(\overline{x}_k) = 0$ and $\Gamma - \lim_{k\to\infty} F_k = 0$. An analogous argument holds for $G_k$. For every $y \in \mathbb{R}$, we consider the recovery sequence $\overline{y}_k = \frac{\pi}{k}\lfloor\frac{ky}{\pi}\rfloor + \frac{\pi}{2k}$ which is such that $\overline{y}_k \to y$ and

$$G_k(\overline{y}_k) = \cos^2\left(k\left(\frac{\pi}{k}\left\lfloor \frac{ky}{\pi} \right\rfloor + \frac{\pi}{2k}\right)\right) = \cos^2\left(\pi \left\lfloor \frac{ky}{\pi} \right\rfloor + \frac{\pi}{2}\right) = 0,$$

thus proving that $\Gamma - \lim_{k\to\infty} G_k = 0$. However, the sequence of sums is exactly

$$F_k(x) + G_k(x) = \sin^2(kx) + \cos^2(kx) = 1,$$

and this constant sequence obviously does not $\Gamma$-converge to 0.

## 2.5 Dirichlet forms

The theory of Dirichlet forms, initiated by Beurling and Deny [3, 4], is a powerful tool for studying composite media and their macroscopic behavior. The only result we will see here is the well-known Beurling–Deny formula (Theorem 2.5.3), which provides a decomposition of a regular Dirichlet form into two local and one non-local terms. For this part, we mainly refer to [26, 33].

**Definition 2.5.1** (Dirichlet form)**.** Let $a\colon D(a) \times D(a) \to \mathbb{R}$, where $D(a) \subseteq L^2(\Omega)$. Then $a$ is called a Dirichlet form if the following conditions hold.

1. $a$ is bilinear, symmetric, and non-negative.

2. $D(a)$ is a dense linear subspace in $L^2(\Omega)$.

3. $a$ is closed, i.e., $D(a)$ is complete under the inner product defined by $(u,v)_a \coloneqq (u,v)_{L^2(\Omega)} + a(u,v)$.

4. $a$ is Markovian, i.e., for every $\varepsilon > 0$ there exists a function $\eta_\varepsilon \colon \mathbb{R} \to [-\varepsilon, 1+\varepsilon]$ such that $\eta_\varepsilon(t) \equiv t$ for every $t \in [0,1]$, $0 \leq \eta_\varepsilon(t') - \eta_\varepsilon(t) \leq t' - t$ for every $t < t'$ and $a(\eta_\varepsilon \circ u, \eta_\varepsilon \circ u) \leq a(u,u)$ for every $u \in D(a)$.

   Alternatively, if $a$ is closed, then $a$ is Markovian if and only if $a(Tu,Tu) \leq a(u,u)$ for every $u \in D(a)$, with $Tu = (u \vee 0) \wedge 1$.

**Definition 2.5.2** (Regular Dirichlet form)**.** A Dirichlet form $a$ is regular if it possesses a core $C$, i.e., there exists a subset $C \subseteq C_c^0(\Omega) \cap D(a)$ such that $C$ is dense in $C_c^0(\Omega)$ with the uniform norm and in $D(a)$ with the intrinsic norm $\|\cdot\|_a$.

**Theorem 2.5.3** (Beurling–Deny formula)**.** *Let $a$ be a regular Dirichlet form on $L^2(\Omega)$. Then there exist and are uniquely determined by $a$ the form $a_c$ and the measures $j(\mathrm{d}x,\mathrm{d}y)$, $k(\mathrm{d}x)$ such that*

$$a(u,v) = a_c(u,v) + \int_\Omega \widetilde{u}(x)\widetilde{v}(x)k(\mathrm{d}x) + \iint_{\Omega\times\Omega\setminus\Delta} (\widetilde{u}(x) - \widetilde{u}(y))(\widetilde{v}(x) - \widetilde{v}(y))j(\mathrm{d}x,\mathrm{d}y), \tag{2.2}$$

*for every $u,v \in D(a)$, where $\widetilde{u}$ and $\widetilde{v}$ are the quasicontinuous representatives of $u$ and $v$, respectively; and $\Delta \coloneqq \{(x,y) \in \Omega\times\Omega : x = y\}$ is the diagonal.*

*In particular:*

- *$j(\mathrm{d}x,\mathrm{d}y)$, called the jumping measure, is the unique positive Radon measure on $\Omega\times\Omega\setminus\Delta$ such that*

$$\iint_{\Omega\times\Omega\setminus\Delta} u(x)v(y)j(\mathrm{d}x,\mathrm{d}y) = -\frac{1}{2}a(u,v),$$

  *for every $u,v \in C_c^0(\Omega) \cap D(a)$ with disjoint supports.*

- *$k(\mathrm{d}x)$, called the killing measure, is the unique positive Radon measure on $\Omega$ such that*

$$\int_\Omega u(x)v(x)k(\mathrm{d}x) = a(u,v) - \iint_{\Omega\times\Omega\setminus d} (u(x) - u(y))(v(x) - v(y))j(\mathrm{d}x,\mathrm{d}y),$$

  *for every $u,v \in C_c^0(\Omega) \cap D(a)$, with $v$ constant on a neighborhood of $\operatorname{supp} u$.*

- *The diffusion part $a_c$ is the form, with domain $D(a_c) = D(a)$, uniquely defined by the formula (2.2). Moreover, $a_c$ is bilinear, non-negative, Markovian, and strongly local, i.e., $a_c(u,v) = 0$ for every $u,v \in D(a_c)$, with $v$ constant on a neighborhood of $\operatorname{supp} u$.*

## 2.6 Two-scale convergence

In the study of homogenization problems, such as composite materials or processes with highly oscillatory periodic behaviors, classical weak limits often fail to capture the microscopic effects of the rapid oscillations. To understand this limitation, we first recall a fundamental result concerning the macroscopic behavior of highly oscillating sequences.

In the sequel, we will denote $Y := [0,1]^d$ as the unit reference cell in $\mathbb{R}^d$. We denote by $C^0_\#(Y)$ the space of continuous functions on $\mathbb{R}^d$ that are $Y$-periodic, and by $L^p_\#(Y)$ the space of $Y$-periodic functions that are locally in $L^p(\mathbb{R}^d)$.

**Lemma 2.6.1** (Riemann–Lebesgue)**.** *Let $1 \leq p \leq \infty$ and let $f \in L^p_\#(Y)$. For any $\varepsilon > 0$, define the scaled sequence $f_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right)$. Then, as $\varepsilon \to 0$, $f_\varepsilon$ converges weakly in $L^p(\Omega)$ (or weakly-$*$ in $L^\infty(\Omega)$ if $p = \infty$) to the average of $f$ over the periodicity cell $Y$. That is, for every open bounded set $\Omega \subset \mathbb{R}^d$,*

$$f_\varepsilon \rightharpoonup \frac{1}{|Y|} \int_Y f(y)\,\mathrm{d}y \quad \textit{in } L^p(\Omega).$$

**Example 2.6.2.** Let $Y = (0,1)$, $t \in Y$, and

$$f(x) = \begin{cases} z_1 & 0 < x \leq t, \\ z_2 & t < x < 1. \end{cases}$$

If we consider the scaled sequence

$$f_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right) = \begin{cases} z_1 & 0 < x \leq t\varepsilon, \\ z_2 & t\varepsilon < x < \varepsilon, \end{cases}$$

extended periodically to the interval $(0,1)$, we have that

$$f_\varepsilon \rightharpoonup \int_0^1 f(x)\mathrm{d}x = tz_1 + (1-t)z_2.$$

This lemma highlights a significant limitation: while the weak limit correctly captures the macroscopic average of the sequence $f_\varepsilon$, it completely loses the microscopic information regarding the oscillation profile $f(y)$. To overcome this, Nguetseng and Allaire [1, 34] introduced the concept of two-scale convergence, which allows us to retain both macroscopic variations and microscopic periodic profiles.

**Definition 2.6.3** (Two-scale convergence)**.** A sequence of functions $\{u_\varepsilon\} \subset L^2(\Omega)$ is said to two-scale converge to a limit function $u_0 \in L^2(\Omega \times Y)$, denoted by $u_\varepsilon \xrightharpoonup{2} u_0$, if for every test function $\psi \in \mathcal{D}(\Omega; C^\infty_\#(Y))$ we have

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \int_Y u_0(x,y) \psi(x,y)\,\mathrm{d}y\mathrm{d}x. \tag{2.3}$$

**Example 2.6.4.** Let $\Omega = (0,1)$ and consider the sequence $u_\varepsilon(x) = x\sin(2\pi x/\varepsilon)$. By the Riemann–Lebesgue Lemma 2.6.1, $u_\varepsilon \rightharpoonup x \cdot 0 = 0$ in $L^2(\Omega)$, losing all information about the wave. However, let us compute its two-scale limit; i.e.,

$$\lim_{\varepsilon\to 0} I_\varepsilon = \int_0^1 x \sin\left(\frac{2\pi x}{\varepsilon}\right) \psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x,$$

where $\psi \in \mathcal{D}(\Omega; C^\infty_\#(Y))$. To simplify the notation, we define the auxiliary function $\Phi(x,y) \coloneqq x\sin(2\pi y)\psi(x,y)$. Since $\psi$ is smooth, compactly supported in $x$, and 1-periodic in $y$, $\Phi$ shares the exact same properties. Our integral becomes:

$$I_\varepsilon = \int_0^1 \Phi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x.$$

We partition the domain (0,1) into $N_\varepsilon = \lfloor 1/\varepsilon \rfloor$ subintervals of length $\varepsilon$, plus a small remainder. Let $I_k^\varepsilon = [k\varepsilon, (k+1)\varepsilon]$ for $k = 0, \ldots, N_\varepsilon - 1$, and let $R_\varepsilon = [N_\varepsilon \varepsilon, 1]$ be the remainder interval. We can split the integral as:

$$I_\varepsilon = \sum_{k=0}^{N_\varepsilon - 1} \int_{k\varepsilon}^{(k+1)\varepsilon} \Phi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x + \int_{R_\varepsilon} \Phi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x.$$

First, notice that the remainder integral vanishes as $\varepsilon \to 0$. Indeed, $\Phi$ is bounded (say, $|\Phi| \le M$) and the length of $R_\varepsilon$ is strictly less than $\varepsilon$, so the remainder is bounded by $M\varepsilon \to 0$.

For the main sum, we perform the change of variables $x = \varepsilon(k+y)$ in each interval $I_k^\varepsilon$. As $x$ ranges from $k\varepsilon$ to $(k+1)\varepsilon$, the new microscopic variable $y$ ranges exactly in the reference cell (0,1). Furthermore, $\mathrm{d}x = \varepsilon \mathrm{d}y$ and $x/\varepsilon = k + y$. By the 1-periodicity of $\Phi$ in its second argument, we have $\Phi(x, k+y) = \Phi(x,y)$. The sum becomes:

$$\sum_{k=0}^{N_\varepsilon - 1} \varepsilon \int_0^1 \Phi(\varepsilon(k+y), y)\, \mathrm{d}y.$$

Since $\Phi(x,y)$ is smooth and compactly supported, it is uniformly Lipschitz continuous with respect to its first variable. Thus, there exists a constant $L > 0$ such that

$$|\Phi(\varepsilon(k+y), y) - \Phi(\varepsilon k, y)| \le L|\varepsilon y| \le L\varepsilon.$$

We can therefore approximate $\Phi(\varepsilon(k+y), y)$ with $\Phi(\varepsilon k, y)$,introducing an error which is bounded by:

$$\sum_{k=0}^{N_\varepsilon - 1} \varepsilon \int_0^1 L\varepsilon\, \mathrm{d}y = N_\varepsilon L \varepsilon^2 \le \left(\frac{1}{\varepsilon}\right) L\varepsilon^2 = L\varepsilon,$$

which again vanishes as $\varepsilon \to 0$.

We are left with the main term:

$$\lim_{\varepsilon\to 0} I_\varepsilon = \lim_{\varepsilon\to 0} \int_0^1 \left( \sum_{k=0}^{N_\varepsilon - 1} \varepsilon \Phi(\varepsilon k, y) \right) \mathrm{d}y.$$

For any fixed $y \in (0,1)$, the sum inside the parenthesis is exactly the Riemann sum for the function $x \mapsto \Phi(x,y)$ over the interval $(0,1)$ with step size $\varepsilon$. Since $\Phi$ is uniformly continuous, this Riemann sum converges (uniformly in $y$) to the integral $\int_0^1 \Phi(x,y)\,\mathrm{d}x$.

Swapping the limit with the integral over $y$ (thanks to Dominated Convergence 2.1.4), we finally obtain:

$$\lim_{\varepsilon\to 0} I_\varepsilon = \int_0^1 \int_0^1 \Phi(x,y)\,\mathrm{d}x\mathrm{d}y,$$

which corresponds to

$$\lim_{\varepsilon\to 0} \int_0^1 u_\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right)\,\mathrm{d}x = \int_0^1 \int_0^1 \Big(x\sin(2\pi y)\Big)\psi(x,y)\,\mathrm{d}x\mathrm{d}y.$$

By the definition of two-scale convergence, this proves that $u_\varepsilon(x)$ two-scale converges to the profile $u_0(x,y) = x\sin(2\pi y)$. This explicitly shows how two-scale convergence preserves both the macroscopic amplitude $x$ and the microscopic oscillation $\sin(2\pi y)$, successfully decoupling the two scales.

One of the main properties of this definition is the following compactness theorem [1, Theorem 1.2].

**Theorem 2.6.5.** *Let $\{u_\varepsilon\}$ be a uniformly bounded sequence in $L^2(\Omega)$. Then, there exists a subsequence (still denoted by $\{u_\varepsilon\}$) and a function $u_0 \in L^2(\Omega \times Y)$ such that $u_\varepsilon$ two-scale converges to $u_0$.*

*Remark* 2.6.6. The proof of the theorem in [1] involves test functions $\psi \in L^2(\Omega; C^0_\#(Y))$. However, other choices of the space of test functions are possible. For example, if $\Omega$ is bounded, we could have chosen $C^0(\overline{\Omega}; C^0_\#(Y))$ or $L^2_\#(Y; C^0(\overline{\Omega}))$, which would have led to the same two-scale limit $u_0(x,y)$ without affecting any part of the proof.

This result is particularly useful because it allows us to extract a profile $u_0(x,y)$ that simultaneously describes the macroscopic behavior (dependence on $x$) and the microscopic periodic oscillations (dependence on $y$). Finally, we point out the link between two-scale convergence and weak convergence with the following proposition [1, Proposition 1.6].

**Proposition 2.6.7.** *If $\{u_\varepsilon\}$ is a bounded sequence in $L^2(\Omega)$ that two-scale converges to $u_0 \in L^2(\Omega \times Y)$, then $\{u_\varepsilon\}$ converges weakly in $L^2(\Omega)$ to the average of $u_0$ over the periodic cell $Y$. That is,*

$$u_\varepsilon \overset{2}{\rightharpoonup} u_0 \in L^2(\Omega \times Y) \qquad \Longrightarrow \qquad u_\varepsilon \rightharpoonup u(x) := \int_Y u_0(x,y)\,\mathrm{d}y \quad \textit{in } L^2(\Omega).$$

*Proof.* It is sufficient to take test functions $\psi(x)$, which depend only on $x$, in (2.3). $\square$

# Chapter 3

# Recent results on non-local functionals

In this chapter, we review the recent developments on non-local functionals, for which we refer to [6,10]. The general problem concerns the asymptotic behavior and the structural properties of the Γ-limits for sequences of functionals $F_k \colon W_0^{1,p}(\Omega) \to \mathbb{R}$, featuring both a local and a non-local term:

$$F_k(u) = \int_{\Omega\times\Omega} f_k(u(x) - u(y))\,\mathrm{d}\mu_k(x,y) + \int_\Omega g_k(x, \nabla u(x))\,\mathrm{d}x.$$

In this framework, non-locality is captured by the double integral, which allows us to model long-range interactions between distant points within the domain. Specifically, the analysis [10] focuses on a particular class of such functionals, defined as:

$$F_k(u) = \int_{\Omega\times\Omega} a_k(x,y)|u(x) - u(y)|^p\,\mathrm{d}x\mathrm{d}y + \int_\Omega |\nabla u(x)|^p\,\mathrm{d}x, \tag{3.1}$$

where $u \in W_0^{1,p}(\Omega)$ and $a_k \in L^1(\Omega\times\Omega)$ are non-negative functions such that

$$\|a_k\|_{L^1(\Omega\times\Omega)} \le M \text{ for all } k, \tag{3.2}$$

for some $M > 0$. It is also assumed that $F_k$ Γ-converges, with respect to the weak convergence in $W_0^{1,p}(\Omega)$, to a certain functional $F$.

## 3.1 The case $p = 2$

When $p = 2$, the sequence of functionals in (3.1) reduces to the quadratic case, defined for $u \in H_0^1(\Omega)$ as:

$$F_k(u) = \int_{\Omega\times\Omega} a_k(x,y)|u(x) - u(y)|^2\,\mathrm{d}x\mathrm{d}y + \int_\Omega |\nabla u(x)|^2\,\mathrm{d}x. \tag{3.3}$$

*Remark* 3.1.1. We can immediately notice the following properties.

1. From (3.2) we obtain the upper bound
$$F_k(u) \leq M(\mathrm{osc}_\Omega(u))^2 + \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x, \tag{3.4}$$
where $\mathrm{osc}_\Omega(u) = \operatorname{ess\,sup}_\Omega u - \operatorname{ess\,inf}_\Omega u$ is the oscillation of $u$ in $\Omega$.

2. The domain of the limit functional $F$, defined as $D(F) \coloneqq \{u \in H_0^1(\Omega) : F(u) < +\infty\}$, is a linear space containing $H_0^1(\Omega) \cap L^\infty(\Omega)$. This inclusion follows from (3.4), since if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, then
$$F(u) \leq \liminf_{k\to\infty} F_k(u) \leq (2\|u\|_{L^\infty(\Omega)})^2 M + \|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^d)}.$$

3. Finally, a basic property of $\Gamma$-convergence ensures that $F$ is a quadratic form; that is, there exists a bilinear form $B\colon D(F) \times D(F) \to \mathbb{R}$ such that $F(u) = B(u,u)$ for every $u \in D(F)$.

**Proposition 3.1.2** (Truncation property)**.** *Each $F_k$ satisfies*
$$F_k(\psi(u)) \leq F_k(u), \tag{3.5}$$
*for every 1-Lipschitz function $\psi\colon \mathbb{R} \to \mathbb{R}$ such that $\psi(0) = 0$ and for every $u \in H_0^1(\Omega)$. Moreover, if $\{F_k\}$ weakly $\Gamma$-converges to a functional $F$, then we also have $F(\psi(u)) \leq F(u)$ for every $u \in H_0^1(\Omega)$.*

*Proof.* Since $\psi$ is a 1-Lipschitz function, it satisfies
$$|\psi(s) - \psi(t)| \leq |s - t|,$$
for every $s, t \in \mathbb{R}$. This means that $|\psi'(u(x))| \leq 1$ for every $x \in \Omega$, and we obtain
$$|\nabla\psi(u)| = |\psi'(u)\nabla u| \leq |\nabla u|.$$
We can now conclude that
$$\begin{aligned} F_k(\psi(u)) &= \int_{\Omega\times\Omega} a_k(x,y)|\psi(u(x)) - \psi(u(y))|^2 \,\mathrm{d}x\mathrm{d}y + \int_\Omega |\nabla\psi(u(x))|^2 \,\mathrm{d}x \\ &\leq \int_{\Omega\times\Omega} a_k(x,y)|u(x) - u(y)|^2 \,\mathrm{d}x\mathrm{d}y + \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x = F_k(u). \end{aligned}$$
Notice that the condition $\psi(0) = 0$ is necessary since the functionals $F_k$ are defined on $H_0^1(\Omega)$. By the definition of $\Gamma$-convergence, we know that there exists a recovery sequence $u_k \rightharpoonup u$ such that
$$\limsup_{k\to\infty} F_k(u_k) \leq F(u).$$
Moreover, we claim that
$$u_k \rightharpoonup u \text{ in } H_0^1(\Omega) \quad\Longrightarrow\quad \psi(u_k) \rightharpoonup \psi(u) \text{ in } H_0^1(\Omega). \tag{3.6}$$

By the Rellich–Kondrachov Theorem 2.2.15, we know that $u_k \to u$ in $L^2(\Omega)$. We have

$$\int_\Omega |\psi(u_k) - \psi(u)|^2 \,\mathrm{d}x \le \int_\Omega |u_k - u|^2 \,\mathrm{d}x \to 0,$$

so that $\psi(u_k) \to \psi(u)$ in $L^2(\Omega)$.
Also,

$$\|\psi(u_k)\|_{H_0^1(\Omega)} = \int_\Omega |\nabla\psi(u_k)|^2 \,\mathrm{d}x \le \int_\Omega |\nabla u_k|^2 \,\mathrm{d}x = \|u_k\|_{H_0^1(\Omega)} \le C$$

since $\{u_k\}$ weakly converges in $H_0^1(\Omega)$ so its norm is bounded.
Since $\psi(u_k)$ is bounded in $H_0^1(\Omega)$, we can extract a subsequence weakly converging to a certain limit $v$. Applying the Rellich–Kondrakov Theorem 2.2.15 again, we deduce that the convergence is strong; concluding that $v = \psi(u)$ and that the whole sequence $\{\psi(u_k)\}$ weakly converges to $\psi(u)$ in $H_0^1(\Omega)$. Finally, using (3.5) and (4.3.1) we get

$$F(\psi(u)) \le \liminf_{k\to\infty} F_k(\psi(u_k)) \le \liminf_{k\to\infty} F_k(u_k) \le \limsup_{k\to\infty} F_k(u_k) \le F(u),$$

which is (3.5) and concludes the proof. □

This property will be useful for extending the integral representation to the entire space $H_0^1(\Omega)$.

**Theorem 3.1.3.** *Assume that $\{F_k\}$ weakly $\Gamma$-converges in $H_0^1(\Omega)$ to a functional $F$. Then there exist two positive Radon measures $\mu$ on $\Omega \times \Omega$ and $\nu$ on $\Omega$, such that, for $u \in C_c^\infty(\Omega)$,*

$$F(u) = \int_{\Omega\times\Omega} |u(x) - u(y)|^2 \,\mathrm{d}\mu(x,y) + \int_\Omega |u(x)|^2 \,\mathrm{d}\nu(x) + \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x. \tag{3.7}$$

Equation (3.7) is a consequence of the Beurling–Deny formula (2.2). In particular, [26, Example 1.2.1] shows that, for every $u \in C_c^\infty(\Omega)$, $F(u)$ can be uniquely decomposed into a non-local term, a local term, and a strongly local term depending only on the derivatives:

$$F(u) = F^n(u) + F^\ell(u) + F^c(u),$$

where

$$\begin{aligned} F^n(u) &= \int_{\Omega\times\Omega} |u(x) - u(y)|^2 \,\mathrm{d}\mu(x,y), \\ F^\ell(u) &= \int_\Omega |u(x)|^2 \,\mathrm{d}\nu(x), \\ F^c(u) &= \sum_{i,j=1}^d \int_\Omega \frac{\partial u(x)}{\partial x_i}\frac{\partial u(x)}{\partial x_j} \,\mathrm{d}\mu_{ij}(x). \end{aligned}$$

Here, $\mu$ and $\nu$ are positive Radon measures, and $\mu_{ij}$ is a symmetric matrix of Radon measures. It is possible to prove that in our case $\mu_{ij} = \delta_{ij}\mathcal{L}^d$, where $\mathcal{L}^d$ denotes the $d$-dimensional Lebesgue measure (see [10] for the detailed proof). This reduces the strongly local term $F^c(u)$ exactly to $\int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x$.

**Example 3.1.4.** We will see hereby that an unexpected value for the measure $\nu$ appears in the limit if we consider $d = 2$, $\Omega = (0{,}1) \times (0{,}1)$ and $a_k(x, y) = \alpha_k(x) + \alpha_k(y)$, where

$$\alpha_k(x) = \begin{cases} k & \text{if } x \in I_k = (0{,}1) \times (0, \frac{1}{k}), \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\|a_k\|_{L^1(\Omega\times\Omega)} = 2 \int_0^1 \int_0^1 \alpha_k(x)\,\mathrm{d}x = 2,$$

so that inequality (3.2) is valid for $M = 2$. We now claim that the relevant sequence of functionals $\{F_k\}$ $\Gamma$-converges to

$$F(u) = 2 \int_\Omega |u(x)|^2\,\mathrm{d}x + \int_\Omega |\nabla u(x)|^2\,\mathrm{d}x,$$

which corresponds to $\mu = 0$ and $\nu = 2\mathcal{L}^d$ in (3.7).

In order to prove the liminf inequality we choose a sequence $u_k \rightharpoonup u$ in $H^1_0(\Omega)$, yielding

$$\begin{aligned}
&\int_{\Omega\times\Omega} |u_k(x) - u_k(y)|^2 a_k(x, y)\,\mathrm{d}x\mathrm{d}y \\
&= \int_{\Omega\times\Omega} |u_k(x)|^2 a_k(x, y)\,\mathrm{d}x\mathrm{d}y - 2\int_{\Omega\times\Omega} u_k(x)u_k(y)a_k(x, y)\,\mathrm{d}x\mathrm{d}y + \int_{\Omega\times\Omega} |u_k(y)|^2 a_k(x, y)\,\mathrm{d}x\mathrm{d}y \\
&= 2\Big(\int_\Omega |u_k(x)|^2\alpha_k(x)\,\mathrm{d}x - 2\int_\Omega u_k(x)\alpha_k(x)\,\mathrm{d}x \int_\Omega u_k(y)\,\mathrm{d}y + \int_\Omega |u_k(x)|^2\,\mathrm{d}x \int_\Omega \alpha_k(y)\,\mathrm{d}y\Big) \\
&= 2k \int_{I_k} |u_k(x)|^2\,\mathrm{d}x - 4k \int_{I_k} u_k(x)\,\mathrm{d}x \int_\Omega u_k(y)\,\mathrm{d}y + 2\int_\Omega |u_k(x)|^2\,\mathrm{d}x.
\end{aligned} \tag{3.8}$$

The Poincaré inequality 2.2.4 ensures that

$$k \int_{I_k} |u_k(x)|^2\,\mathrm{d}x \le \frac{C}{k} \int_{I_k} |\nabla u_k(x)|^2\,\mathrm{d}x,$$

since the Poincaré constant scales with the square of the thickness of the domain. This leads to

$$\lim_{k\to\infty} k \int_{I_k} |u_k(x)|^2\,\mathrm{d}x = 0,$$

due to $\{u_k\}$ being a weakly-converging sequence in $H^1_0(\Omega)$, so its norm is bounded by a constant. Moreover, by Cauchy-Schwarz inequality, we also have

$$\int_{I_k} |u_k(x)|\,\mathrm{d}x \le \sqrt{\int_{I_k} |u_k(x)|^2\,\mathrm{d}x}\sqrt{\frac{1}{k}} \quad \Longrightarrow \quad k\int_{I_k} |u_k(x)|\,\mathrm{d}x \le \sqrt{k\int_{I_k} |u_k(x)|^2\,\mathrm{d}x};$$

so that

$$\lim_{k\to\infty} k \int_{I_k} |u_k(x)|\,\mathrm{d}x = 0.$$

Therefore, passing to the limit in formula (3.8) yields

$$\lim_{k\to\infty} \int_{\Omega\times\Omega} |u_k(x) - u_k(y)|^2 a_k(x,y)\,\mathrm{d}x\mathrm{d}y = 2\int_\Omega |u(x)|^2\,\mathrm{d}x.$$

Taking into account the lower semicontinuity of the gradient we obtain that

$$\liminf_{k\to\infty} F_k(u_k) \geq F(u).$$

For the limsup inequality it is sufficient to choose $u_k = u$ as the recovery sequence, which gives $\lim_{k\to\infty} F_k(u) = F(u)$.

The next proposition states that $\nu$ is the null measure if $a_k$ do not concentrate on the boundary.

**Proposition 3.1.5.** *Suppose that the hypotheses of Theorem 3.1.3 hold. Moreover, suppose that for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \Omega$ such that*

$$\int_{(\Omega\times\Omega)\setminus(K_\varepsilon\times K_\varepsilon)} a_k(x,y)\,\mathrm{d}x\mathrm{d}y \leq \varepsilon \text{ for every } k \in \mathbb{N}. \tag{3.9}$$

*Then the measure $\nu$ is the null measure.*

In the following Theorem, $\widetilde{u}$ denotes the quasicontinuous representative of $u \in H_0^1(\Omega)$, which is unique up to sets of zero capacity. We refer to Remark 2.2.21 to clarify its role.

**Theorem 3.1.6.** *Let $\mu$ and $\nu$ be two bounded positive Radon measures on $\Omega\times\Omega$ and $\Omega$, respectively, that satisfy 1., 2. and 3. of Theorem 3.1.3 and let $F\colon H_0^1(\Omega) \to [0,+\infty)$. Suppose also that $F$ is weakly lower semicontinuous in $H_0^1(\Omega)$ and that there exists $M > 0$ such that*

$$F(u) \leq M(\operatorname{osc}_\Omega u)^2 + \int_\Omega |\nabla u(x)|^2 \mathrm{d}x$$

*holds for all $u \in C_c^\infty(\Omega)$. Then*

(a) *$\mu(B\times\Omega) = \nu(B) = 0$ for every $B \subset \Omega$ Borel set with zero capacity.*

(b)

$$F(u) = \int_{\Omega\times\Omega} |\widetilde{u}(x) - \widetilde{u}(y)|^2\,\mathrm{d}\mu(x,y) + \int_\Omega |\widetilde{u}(x)|^2\,\mathrm{d}\nu(x) + \int_\Omega |\nabla u(x)|^2\,\mathrm{d}x, \tag{3.10}$$

*for every $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$.*

(c) *if, in addition, $F(\psi_m(u)) \leq F(u)$ for all $u \in H_0^1(\Omega)$ and $m \in \mathbb{N}$, where $\psi_m(t) := (m \wedge t) \vee (-m)$, then (3.10) holds for every $u \in H_0^1(\Omega)$.*

*Remark* 3.1.7. Notice how the assumption that $F$ satisfies the truncation property is crucial to extend equality (3.10) to the whole $H_0^1(\Omega)$. Indeed, if we define $G\colon H_0^1(\Omega) \to [0,+\infty]$ by the right-hand side in (3.10), we take $u \in H_0^1(\Omega)$ and set $u_m = \psi_m(u)$; we can note that $\lim_{m\to\infty} G(u_m) = G(u)$ by the Monotone Convergence 2.1.5. Since $F(u_m) = G(u_m)$, to conclude the proof it is enough to prove that $\lim_{m\to\infty} F(u_m) = F(u)$. This follows in a direct way from the hypothesis on truncations (which guarantees that $\lim_{m\to\infty} F(u_m) \le F(u)$) and the lower semicontinuity of $F$ (which guarantees that $\lim_{m\to\infty} F(u_m) \ge F(u)$).
Conversely, if the additional assumption in (c) is dropped, the latter might not hold. For instance, defining again $G$ by the right-hand side in (3.10), and if $F$ is defined as equal to $G$ except on a single $u_0 \in H_0^1(\Omega)\backslash L^\infty(\Omega)$, where $F(u_0) = 0$, then $F$ is lower semicontinuous, $F$ and $G$ are equal on $H_0^1(\Omega) \cap L^\infty(\Omega)$, but not on the whole $H_0^1(\Omega)$.

**Corollary 3.1.8.** *Let $F_k$ be given by (3.3), with $a_k$ satisfying (3.2). Suppose that $\{F_k\}$ weakly $\Gamma$-converges to $F$ in $H_0^1(\Omega)$. Then there exist two uniquely determined positive finite Radon measures $\mu$ on $\Omega\times\Omega$ and $\nu$ on $\Omega$, such that*

1. *for every $u \in H_0^1(\Omega)$*

$$F(u) = \int_{\Omega\times\Omega} |\widetilde{u}(x) - \widetilde{u}(y)|^2 \,\mathrm{d}\mu(x,y) + \int_\Omega |\widetilde{u}(x)|^2 \,\mathrm{d}\nu(x) + \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x. \quad (3.11)$$

*Moreover, if $a_k$ satisfies property (3.9), then $\nu = 0$.*

This concludes the part about quadratic forms. We have shown that, for functionals $F_k$ of the type (3.3), the $\Gamma$-limit is given by

$$F(u) = \int_{\Omega\times\Omega} |u(x) - u(y)|^2 \mathrm{d}\mu(x,y) + \int_\Omega |u(x)|^2 \mathrm{d}\nu(x) + \sum_{i,j=1}^d \int_\Omega \frac{\partial u(x)}{\partial x_i}\frac{\partial u(x)}{\partial x_j} \mathrm{d}\mu_{ij}(x),$$

for every $u \in C_c^\infty(\Omega)$. Moreover, the last term equals $\int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x$, while if the $a_k$ do not concentrate on the boundary, $\nu$ is the null measure. The extension of this integral representation to the whole $H_0^1(\Omega)$ is provided by the truncation property, leading to the integral expression (3.11).

## 3.2 The counterexample for the case $p \neq 2$

We can now focus on the case $p \neq 2$, showing by means of a counterexample that an integral representation does not hold when $p \in (1, d]$. For this part we will provide the proofs, since they are instructive and relatively short, and because some of the novel results presented in Chapter 4 rely on a counterexample too.

We set

$$a_k(x,y) = \frac{1}{|B_{\varepsilon_k}(x_0)|}\mathbf{1}_{B_{\varepsilon_k}(x_0)}(y),$$

where $x_0 \in \Omega$ and $\{\varepsilon_k\}$ is a sequence of positive numbers converging to 0. For simplicity of notation, and without loss of generality, we can assume that $x_0 = 0 \in \Omega$ and denote the ball centered at the origin simply as $B_{\varepsilon_k}$. We obtain the functionals $F_k \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$F_k(u) = \frac{1}{|B_{\varepsilon_k}|} \int_\Omega \int_{B_{\varepsilon_k}} |u(x) - u(y)|^p \, \mathrm{d}y \mathrm{d}x + \int_\Omega |\nabla u(x)|^p \, \mathrm{d}x, \tag{3.12}$$

for a generic $p \in (1, +\infty)$.

**Definition 3.2.1.** We define $m_p(u)$ as the unique minimum point of

$$t \mapsto \int_\Omega |u(x) - t|^p \, \mathrm{d}x,$$

for every $u \in L^p(\Omega)$.

**Lemma 3.2.2.** *The map $m_p \colon L^p(\Omega) \to \mathbb{R}$ is continuous.*

*Proof.* Let $\{u_k\}$ be a sequence such that $u_k \to u$ strongly in $L^p(\Omega)$. We want to prove that $m_p(u_k) \to m_p(u)$.

As a first step, we show that the numerical sequence $\{m_p(u_k)\}$ is bounded in $\mathbb{R}$. Using the triangle inequality, we have:

$$|m_p(u_k)| \leq \|u_k - m_p(u_k)\|_{L^p(\Omega)} + \|u_k\|_{L^p(\Omega)}.$$

Since $m_p(u_k)$ minimizes the function $t \mapsto \int_\Omega |u_k(x) - t|^p \, \mathrm{d}x$ by definition, we know that

$$\|u_k - m_p(u_k)\|_{L^p(\Omega)} \leq \|u_k\|_{L^p(\Omega)}.$$

Substituting this information into the previous inequality, it follows that:

$$|m_p(u_k)| \leq 2\|u_k\|_{L^p(\Omega)}.$$

Since $\{u_k\}$ is convergent in $L^p(\Omega)$, the sequence of its norms is bounded. This guarantees that the sequence $\{m_p(u_k)\}$ is also bounded, so we can assume (up to passing to a subsequence) that $m_p(u_k) \to t_0$ for some $t_0 \in \mathbb{R}$. Now, let us fix an arbitrary real number $t \in \mathbb{R}$. It holds that

$$\int_\Omega |u_k(x) - m_p(u_k)|^p \, \mathrm{d}x \leq \int_\Omega |u_k(x) - t|^p \, \mathrm{d}x,$$

which passing to the limit gives us

$$\int_\Omega |u(x) - t_0|^p \, \mathrm{d}x \leq \int_\Omega |u(x) - t|^p \, \mathrm{d}x.$$

Since this minimizer is unique, it must necessarily be $t_0 = m_p(u)$. ☐

**Theorem 3.2.3.** *If $p \in (1, d]$, then the $\Gamma$-limit of the sequence $\{F_k\}$ defined in (3.12), with respect to the weak $W_0^{1,p}$-convergence, is the functional $F$ defined by*

$$F(u) = \int_\Omega |u(x) - m_p(u)|^p \mathrm{d}x + \int_\Omega |\nabla u(x)|^p \mathrm{d}x, \tag{3.13}$$

*for every $u \in W_0^{1,p}(\Omega)$.*

*Proof. Liminf inequality:* Let $u_k \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$, which implies $u_k \to u$ strongly in $L^p(\Omega)$. Using Jensen's inequality 2.1.3, the minimality of $m_p(u_k)$, and the continuity of $m_p$ (Lemma 3.2.2), we obtain

$$\begin{aligned}
&\liminf_{k\to\infty} \frac{1}{|B_{\varepsilon_k}|} \int_\Omega \int_{B_{\varepsilon_k}} |u_k(x) - u_k(y)|^p \,\mathrm{d}y\mathrm{d}x \\
&\quad \geq \liminf_{k\to\infty} \int_\Omega \left| u_k(x) - \frac{1}{|B_{\varepsilon_k}|} \int_{B_{\varepsilon_k}} u_k(y)\,\mathrm{d}y \right|^p \mathrm{d}x \\
&\quad \geq \liminf_{k\to\infty} \int_\Omega |u_k(x) - m_p(u_k)|^p \,\mathrm{d}x \\
&\quad \geq \int_\Omega |u(x) - m_p(u)|^p \,\mathrm{d}x.
\end{aligned}$$

The weakly lower semicontinuous gradient term $\int_\Omega |\nabla u|^p \,\mathrm{d}x$ yields the desired liminf bound.

*Limsup inequality:* Assume first that $u = 0$ in a neighborhood of 0. Let $v_k$ be the $p$-capacitary potential of $B_{\varepsilon_k}$ with respect to $\Omega$, which is

$$\operatorname{argmin}\Big\{ \int_\Omega |\nabla v(x)|^p \mathrm{d}x : v \in W_0^{1,p}, v = 1 \text{ on } B_{\varepsilon_k} \Big\}.$$

Because $p \leq d$, it is known that $v_k \to 0$ strongly in $W_0^{1,p}(\Omega)$. We define the recovery sequence as $u_k = u + m_p(u)v_k$. Evaluating the functional gives

$$\lim_{k\to\infty} F_k(u_k) = \int_\Omega |u(x) - m_p(u)|^p \,\mathrm{d}x + \int_\Omega |\nabla u(x)|^p \,\mathrm{d}x = F(u).$$

Since $F$ is continuous in $W_0^{1,p}(\Omega)$ and the set of smooth functions vanishing around 0 is dense in $W_0^{1,p}(\Omega)$, the result extends to all $u \in W_0^{1,p}(\Omega)$. $\square$

Now that we know the explicit expression of the $\Gamma$-limit $F$, we want to investigate whether $F$ can be expressed in the form

$$F(u) = \int_{\Omega\times\Omega} f(u(x), u(y))\mathrm{d}\mu(x,y) + \int_\Omega g(u(x))\mathrm{d}\nu(x) + \int_\Omega |\nabla u(x)|^p \mathrm{d}x,$$

for $u \in C_c^\infty(\Omega)$, where $f \colon \mathbb{R}^2 \to \mathbb{R}$ and $g \colon \mathbb{R} \to \mathbb{R}$ are continuous functions, while $\mu$ and $\nu$ are two positive bounded Radon measures on $\Omega \times \Omega$ and $\Omega$, respectively.

Let us focus on the integrals of $f$ and $g$ first.

**Proposition 3.2.4.** *Let $\mu$ and $\nu$ be two positive bounded Radon measures on $\Omega \times \Omega$ and $\Omega$, respectively, and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions. Suppose that*

$$\int_\Omega |u(x) - m_p(u)|^p \,\mathrm{d}x = \int_{\Omega\times\Omega} f(u(x), u(y)) \,\mathrm{d}\mu(x,y) + \int_\Omega g(u(x)) \,\mathrm{d}\nu(x)$$

*holds for $u \in C_c^\infty(\Omega)$; then it is also true that*

$$\int_\Omega |\mathbf{1}_A(x) - m_p(\mathbf{1}_A)|^p \,\mathrm{d}x = \int_{\Omega\times\Omega} f(\mathbf{1}_A(x), \mathbf{1}_A(y)) \,\mathrm{d}\mu(x,y) + \int_\Omega g(\mathbf{1}_A(x)) \,\mathrm{d}\nu(x), \quad (3.14)$$

*for all $A$ open subsets of $\Omega$.*

*Proof.* As a first step, we evaluate the assumed identity for the identically zero function $u \equiv 0$. This yields

$$0 = f(0,0)\mu(\Omega \times \Omega) + g(0)\nu(\Omega).$$

Without loss of generality, we can assume $f(0,0) = 0$ and $g(0) = 0$. If this were not the case, we could simply replace $f(s,t)$ with $f(s,t) - f(0,0)$ and $g(s)$ with $g(s) - g(0)$. Let $A$ be an open set relatively compact in $\Omega$. We now consider a sequence of smooth test functions $u_k \in C_c^\infty(\Omega)$ such that they converge pointwise to $\mathbf{1}_A$ and satisfy $0 \leq u_k(x) \leq \mathbf{1}_A(x)$ for all $x$. Due to this boundedness and the Dominated Convergence Theorem 2.1.4, the sequence $\{u_k\}$ converges to $1_A$ strongly in $L^p(\Omega)$. Thanks to Lemma 3.2.2, we have that

$$\lim_{k\to\infty} \int_\Omega |u_k(x) - m_p(u_k)|^p \,\mathrm{d}x = \int_\Omega |1_A(x) - m_p(1_A)|^p \,\mathrm{d}x.$$

For the right-hand side we can use the Dominated Convergence Theorem 2.1.4 since $f$ and $g$ are continuous functions, $u_k$ is uniformly bounded, and both $\mu$ and $\nu$ are bounded Radon measures. □

*Remark* 3.2.5. Definition 3.2.1 states that for any measurable set $A$ the constant $m_p(\mathbf{1}_A)$ is obtained by minimizing

$$\int_\Omega |\mathbf{1}_A - t|^p \,\mathrm{d}x = \int_A |1 - t|^p \,\mathrm{d}x + \int_{\Omega\setminus A} |t|^p \,\mathrm{d}x = |A||1-t|^p + (|\Omega| - |A|)|t|^p.$$

The minimal $t \in [0,1]$ is determined by deriving the previous expression with respect to $t$ and setting it to 0; that is,

$$|A|(1-t)^{p-1} = (|\Omega| - |A|)t^{p-1}.$$

Thus, we obtain

$$\frac{1-t}{t} = \left(\frac{|\Omega| - |A|}{|A|}\right)^{\frac{1}{p-1}} \qquad \text{and} \qquad m_p(\mathbf{1}_A) = \frac{|A|^{1/(p-1)}}{(|\Omega| - |A|)^{1/(p-1)} + |A|^{1/(p-1)}}.$$

At this point, we can set

$$\Phi_p(|A|) = \int_\Omega |\mathbf{1}_A(x) - m_p(\mathbf{1}_A)|^p \mathrm{d}x = |A||1 - m_p(\mathbf{1}_A)|^p + (|\Omega| - |A|)|m_p(\mathbf{1}_A)|^p; \quad (3.15)$$

where, substituting the previous expression for $m_p(\mathbf{1}_A)$, we define

$$\Phi_p(s) := \frac{s(|\Omega| - s)^{p/(p-1)} + (|\Omega| - s)s^{p/(p-1)}}{((|\Omega| - s)^{1/(p-1)} + s^{1/(p-1)})^p} = \frac{s(|\Omega| - s)}{((|\Omega| - s)^{1/(p-1)} + s^{1/(p-1)})^{p-1}}.$$

**Proposition 3.2.6.** *Let $C_f := f(1,0) + f(0,1) - f(1,1)$. If the assumptions of Proposition 3.2.4 hold, we have*

$$\Phi_p(|A|) + \Phi_p(|B|) - \Phi_p(|A| + |B|) = C_f(\mu(A \times B) + \mu(B \times A)), \quad (3.16)$$

*for all $A$, $B$ open sets in $\Omega$ such that $A \cap B = \emptyset$.*

*Moreover, if there exists a bounded Radon measure $\mu$ on $\Omega \times \Omega$ such that the previous equality holds, then we must have $p = 2$.*

*Proof.* From equations (3.14) and (3.15) we have that, for any open set $A \subset \Omega$,

$$\begin{aligned}\Phi_p(|A|) &= \int_\Omega |1_A(x) - m_p(1_A)|^p \,\mathrm{d}x \\ &= f(1,1)\mu(A \times A) + f(1,0)\mu(A \times (\Omega \setminus A)) + f(0,1)\mu((\Omega \setminus A) \times A) + g(1)\nu(A).\end{aligned}$$

Analogous relations for $\Phi_p(|B|)$ and $\Phi_p(|A|) + \Phi_p(|B|) = \Phi_p(|A \cup B|)$ lead to the first part of the claim.

For the second part, let $A_1$, $A_2$, and $B$ be disjoint open subsets of $\Omega$. Let their respective Lebesgue measures be $s_1 = |A_1|$, $s_2 = |A_2|$, and $t = |B|$. Applying equation (3.16) to $B$ and the union $A_1 \cup A_2$ (whose measure is $s_1 + s_2$), we obtain:

$$\begin{aligned}\Phi_p(s_1 + s_2) &+ \Phi_p(t) - \Phi_p(s_1 + s_2 + t) \\ &= C_f(\mu((A_1 \cup A_2) \times B) + \mu(B \times (A_1 \cup A_2))) \\ &= C_f(\mu(A_1 \times B) + \mu(B \times A_1) + \mu(A_2 \times B) + \mu(B \times A_2)) \\ &= \Phi_p(s_1) + \Phi_p(t) - \Phi_p(s_1 + t) + \Phi_p(s_2) + \Phi_p(t) - \Phi_p(s_2 + t).\end{aligned}$$

The latter means that, for a fixed $t \in (0, |\Omega|)$, the function $\psi(s) := \Phi_p(s) + \Phi_p(t) - \Phi_p(s+t)$ is additive on $(0, |\Omega| - t)$; therefore, there exists a constant $c_t$ such that $\psi(s) = c_t s$. Thus, differentiating $\psi(s)$ twice with respect to $s$ yields:

$$\Phi_p''(s) - \Phi_p''(s + t) = \psi''(s) = 0,$$

for all valid $s, t$. This implies that $\Phi_p(s)$ must be a second-order polynomial, say $P(s)$. From its definition, we can rewrite $\Phi_p(s) = s(|\Omega| - s)(h_p(s))^{1-p}$, where $h_p(s) = s^{1/(p-1)} + (|\Omega| - s)^{1/(p-1)}$. Since $h_p(s) \neq 0$, the function $\Phi_p(s)$ evaluates to 0 if and only if $s = 0$

or $s = |\Omega|$. Therefore, the polynomial must also take the form $P(s) = \kappa s(|\Omega| - s)$ for some constant $\kappa \in \mathbb{R}$. Equating the two expressions for $\Phi_p(s)$, we obtain:

$$s(|\Omega| - s)(h_p(s))^{1-p} = \kappa s(|\Omega| - s) \implies h_p(s)^{1-p} = \kappa.$$

This means $h_p(s)$ itself must be a constant for all $s \in (0, |\Omega|)$, and by continuity, also at the boundaries. Let us evaluate this constant $h_p(s)$ at $s = 0$ and at the midpoint $s = |\Omega|/2$:

$$h_p(0) = |\Omega|^{1/(p-1)}$$

$$h_p\left(\frac{|\Omega|}{2}\right) = \left(\frac{|\Omega|}{2}\right)^{1/(p-1)} + \left(\frac{|\Omega|}{2}\right)^{1/(p-1)} = 2\left(\frac{|\Omega|}{2}\right)^{1/(p-1)}.$$

Equating these two values yields:

$$|\Omega|^{1/(p-1)} = 2\left(\frac{|\Omega|}{2}\right)^{1/(p-1)} = 2 \cdot 2^{-1/(p-1)}|\Omega|^{1/(p-1)},$$

from which $2^{1/(p-1)} = 2$; so that $p = 2$. □

We can now conclude that $F$ does not admit an integral representation when $1 < p \leq d$ and $p \neq 2$.

**Corollary 3.2.7.** *Let $1 < p \leq d$. Let $F$ be the $\Gamma$-limit of $\{F_k\}$, with respect to the weak convergence in $W_0^{1,p}$. If there exist two real valued functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ and two positive bounded Radon measures $\mu$ on $\Omega \times \Omega$ and $\nu$ on $\Omega$ such that*

$$F(u) = \int_{\Omega\times\Omega} f(u(x), u(y))\,\mathrm{d}\mu(x, y) + \int_\Omega g(u(x))\,\mathrm{d}\nu(x) + \int_\Omega |\nabla u(x)|^p\,\mathrm{d}x, \tag{3.17}$$

*for every $u \in C_c^\infty(\Omega)$. Then it must be $p = 2$.*

*Moreover, in this case, $f(s, t) = |s - t|^2$ for every $s, t \in \mathbb{R}$, while $\mu = \frac{1}{2|\Omega|}\mathcal{L}^{2d}$ and $\nu = 0$.*

*Proof.* According to Theorem 3.2.3, the $\Gamma$-limit $F$ of the sequence $\{F_k\}$ is given by (3.13). Since we are supposing that (3.17) holds, the assumptions of Proposition 3.2.4 are satisfied, and we can apply Proposition 3.2.6 to claim that $p = 2$.

Finally, in the specific case where $p = 2$, we can explicitly compute that

$$\int_\Omega |u(x) - m_2(u)|^2\,\mathrm{d}x = \frac{1}{2|\Omega|}\int_{\Omega\times\Omega} |u(x) - u(y)|^2\,\mathrm{d}x\mathrm{d}y;$$

which gives the explicit expressions for $f$, $\mu$ and $\nu$. □

*Remark* 3.2.8. Notice that, for $p > d$, the situation is totally different. Indeed, let us consider the same functionals $F_k\colon W_0^{1,p} \to \mathbb{R}$ as

$$F_k(u) := \int_{\Omega\times\Omega} |u(x) - u(y)|^p a_k(x, y)\,\mathrm{d}x\mathrm{d}y + \int_\Omega |\nabla u(x)|^p\,\mathrm{d}x,$$

where $a_k$ satisfies (3.2). We can now define, for each $k$, the measure $\mu_k = a_k\mathcal{L}^{2d}$, (i.e., $\mathrm{d}\mu_k(x, y) = a_k(x, y)\mathrm{d}x\mathrm{d}y$), with total mass

$$\mu_k(\Omega \times \Omega) = \int_{\Omega\times\Omega} a_k(x, y)\,\mathrm{d}x\mathrm{d}y \le C.$$

Thanks to the compactness provided by Banach-Alaoglu Theorem 2.2.12, we can assume, up to subsequences, that $\{a_k\mathcal{L}^{2d}\}$ weakly* converges to a certain measure $\mu$ in $\overline{\Omega} \times \overline{\Omega}$. Moreover, from the continuous embedding of $W_0^{1,p}$ in $C^0(\overline{\Omega})$, we obtain that every sequence $\{u_k\}$ weakly converging to $u$ in $W_0^{1,p}$ converges uniformly to $u$, so that the sequence of functions

$$(x, y) \mapsto |u_k(x) - u_k(y)|^p$$

converges uniformly to the limit function $|u(x) - u(y)|^p$. This means that, when $p > d$, the $\Gamma$-limit of $\{F_k\}$ is the integral functional

$$F(u) = \int_{\Omega\times\Omega} |u(x) - u(y)|^p\,\mathrm{d}\mu(x, y) + \int_\Omega |\nabla u(x)|^p\,\mathrm{d}x,$$

for $u \in W_0^{1,p}$.

In particular, for the case $a_k(x, y) = \frac{1}{|B_{\varepsilon_k}|}\mathbf{1}_{B_{\varepsilon_k}}(y)$, we have that

$$a_k(x, y)\mathcal{L}^{2d} \overset{*}{\rightharpoonup} \mathcal{L}^d \times \delta_0,$$

and the $\Gamma$-limit is the integral functional

$$F(u) = \int_\Omega |u(x) - u(0)|^p\,\mathrm{d}x + \int_\Omega |\nabla u(x)|^p\,\mathrm{d}x.$$

This concludes the study of the functionals defined in (3.1), which combine local and non-local terms. We have shown that, for $p = 2$, an integral representation exists and is given by (3.11). Conversely, for $p \neq 2$ (specifically, $p \le d$), this property fails. A counterexample is provided by the sequence of kernels

$$a_k(x, y) = \frac{1}{|B_{\varepsilon_k}|}\mathbf{1}_{B_{\varepsilon_k}}(y).$$

which yields the explicit $\Gamma$-limit derived in (3.2.3). As proven in Corollary 3.2.7, this $\Gamma$-limit admits the desired integral representation if and only if $p = 2$. Finally, Remark 3.2.8 clarifies that integral representability is naturally recovered when $p > d$.

## 3.3 Non-representability result from Braides

As announced in Chapter 1, the starting point of our investigation is the result established in [6]. Therein, the author considers purely non-local functionals of the form

$$F(u) = \int_{\Omega\times\Omega} f(u(x) - u(y))\,\mathrm{d}x\mathrm{d}y.$$

A counterexample to the integral representation of the relaxed functional is provided by choosing $f$ as the same triple-well potential of formula (1.1), namely

$$f(z) = \begin{cases} 0 & \text{if } z \in \{-1, 1\}, \\ 1 & \text{if } z = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

We will not provide the proof since some of its steps are similar to the ones we will see in Chapter 4. However, it is worth noting that this example can be extended by approximation to everywhere finite integrands $f$. Indeed, we can consider a sequence of finite functions $f_n$ increasingly converging to $f$ and define the corresponding functionals

$$F_n(u) := \int_{\Omega\times\Omega} f_n(u(x) - u(y))\,\mathrm{d}x\mathrm{d}y.$$

By the properties of relaxation, the sequence of lower-semicontinuous envelopes $\overline{F}_n$ converges to $\overline{F}$. Therefore, if there exist convex functions $g_n$ such that every $\overline{F}_n$ admits a double-integral representation, then the Monotone Convergence Theorem 2.1.5 would ensure that the limit functional $\overline{F}$ also admits an integral representation.

# Chapter 4

# Novel results: homogenization

This chapter contains the main original contribution of the thesis. Here, we extend the investigation introduced in [6] to the framework of periodic homogenization, studying the asymptotic behavior of the sequence of non-local functionals $F_\varepsilon \colon L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$F_\varepsilon(u) = \int_{\Omega\times\Omega} a\left(\frac{x-y}{\varepsilon}\right) f(u(x) - u(y))\, \mathrm{d}x\mathrm{d}y, \tag{4.1}$$

when $f$ is the same triple-well potential (1.1) as in [6]

$$f(z) = \begin{cases} 0 & \text{if } z \in \{-1, 1\}, \\ 1 & \text{if } z = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

and $a \in L^\infty(\mathbb{R})$ is such that

$$a\left(\frac{x-y}{\varepsilon}\right) \geq C,$$

for almost every $(x, y) \in \Omega \times \Omega$ and for a certain $C > 0$.

The introduction of the periodic kernel $a(x/\varepsilon)$ prevents a trivial reduction to the homogeneous case and requires a careful decoupling of the macroscopic profile from the microscopic oscillations. To tackle the problem, we apply the theory of two-scale convergence, while the minimization of the cell problem is achieved by relying on the Riesz-Sobolev rearrangement inequality 2.2.25. We will then establish that the $\Gamma$-limit of the sequence of functionals defined in (4.1) does not admit an integral representation.

Without loss of generality, we can choose again $\Omega = (0,1)$ for simplicity.

## 4.1 Two-scale limit and homogenization

Following the same ideas of Braides [6], we set some properties of the functions $u_\varepsilon$ and $u$.

**Lemma 4.1.1.** *Let $u \in L^1(\Omega)$ and let $\{u_\varepsilon\}$ be a sequence weakly converging to $u$ in $L^1(\Omega)$ such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. Then, there exists a bounded sequence $\{z_\varepsilon\} \subset \mathbb{R}$ such that, up to subsequences, $z_\varepsilon \to z \in \mathbb{R}$ and*

$$u_\varepsilon(x) \in \{z_\varepsilon, z_\varepsilon + 1\} \quad \textit{for almost every } x \in \Omega.$$

*Furthermore, the limit $z$ satisfies*

$$z \leq \operatorname{ess-inf} u \qquad \textit{and} \qquad \operatorname{ess-sup} u \leq z + 1. \tag{4.2}$$

*Proof.* The strict positivity of $a$ implies that the only way to achieve $F_\varepsilon(u_\varepsilon) < +\infty$ is for $f(u_\varepsilon(x) - u_\varepsilon(y))$ to take values in $\{0,1\}$ for almost every $(x, y) \in \Omega \times \Omega$. Therefore, fixing a suitable $y \in \Omega$, there exists $z_\varepsilon$ such that $u_\varepsilon(x) \in \{z_\varepsilon, z_\varepsilon - 1, z_\varepsilon + 1\}$ for almost every $x \in \Omega$. If both values $z_\varepsilon - 1$ and $z_\varepsilon + 1$ were taken on sets of positive measure, there we would have $F_\varepsilon(u_\varepsilon) = +\infty$; hence we can restrict the values to $u_\varepsilon(x) \in \{z_\varepsilon, z_\varepsilon + 1\}$. Since the sequence is weakly convergent, $\{z_\varepsilon\}$ is bounded, and we can assume $z_\varepsilon \to z$ up to subsequences thanks to the Bolzano-Weierstrass Theorem. Given that $z_\varepsilon \leq u_\varepsilon \leq z_\varepsilon + 1$, integrating over any measurable subset $A \subset \Omega$ and passing to the limit yields

$$|A| z \leq \int_A u(x)\,\mathrm{d}x \leq |A|(z + 1).$$

Since this holds for every measurable subset $A$, inequality (4.2) follows almost everywhere, which in turn implies that $u \in L^\infty(\Omega)$. $\square$

Following Lemma 4.1.1, we define the characteristic functions

$$\chi_\varepsilon(x) := \begin{cases} 1 & \text{if } u_\varepsilon(x) = z_\varepsilon + 1, \\ 0 & \text{if } u_\varepsilon(x) = z_\varepsilon, \end{cases} \tag{4.3}$$

so that we can rewrite the potential as

$$f(u_\varepsilon(x) - u_\varepsilon(y)) = \chi_\varepsilon(x)\chi_\varepsilon(y) + (1 - \chi_\varepsilon(x))(1 - \chi_\varepsilon(y)),$$

and the functional becomes

$$F_\varepsilon(u_\varepsilon) = \int_{\Omega\times\Omega} a\left(\frac{x-y}{\varepsilon}\right)\Big[\chi_\varepsilon(x)\chi_\varepsilon(y) + (1 - \chi_\varepsilon(x))(1 - \chi_\varepsilon(y))\Big]\,\mathrm{d}x\mathrm{d}y. \tag{4.4}$$

We now pass to the limit by using a two-scale convergence approach.

Let $Y := (0,1)$ be the unit cell. Note that since $\chi_\varepsilon \in \{0,1\}$ almost everywhere and $\Omega$ has finite measure, the sequence $\{\chi_\varepsilon\}$ is uniformly bounded not only in $L^\infty(\Omega)$ but also in $L^2(\Omega)$. This allows us to apply classical two-scale convergence theory. By Theorem 2.6.5, we can extract a subsequence that two-scale converges to a limit in $L^2(\Omega \times Y)$; namely, there exists a function $\phi \in L^2(\Omega \times Y)$ such that

$$\lim_{\varepsilon\to 0} \int_\Omega \chi_\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right)\mathrm{d}x = \int_\Omega \int_Y \phi(x,\sigma)\psi(x,\sigma)\,\mathrm{d}\sigma\mathrm{d}x,$$

for every test function $\psi \in \mathcal{D}(\Omega; C^\infty_\#(Y))$.

We can now prove the following lemma, which ensures that, in our case, the two-scale limit of the product is the product of the two-scale limits.

**Lemma 4.1.2.** *If $\chi_\varepsilon \xrightharpoonup{2} \phi$, then $w_\varepsilon(x,y) := \chi_\varepsilon(x)\chi_\varepsilon(y) \xrightharpoonup{2} \phi(x,\sigma)\phi(y,\tau)$.*

*Proof.* We test the bounded sequence $\{w_\varepsilon\} \subset L^2(\Omega\times\Omega)$ against functions belonging to $\mathcal{D}(\Omega\times\Omega; C^\infty_\#(Y\times Y))$, which is the space of smooth, compactly supported functions on $\Omega\times\Omega$, taking values in the space of smooth, $(Y\times Y)$-periodic functions. By invoking the Stone–Weierstrass Theorem 2.1.8, we may restrict our analysis to test functions of the form $\Psi(x,y,\sigma,\tau) = \psi_1(x,\sigma)\psi_2(y,\tau)$, with $\psi_1,\psi_2 \in \mathcal{D}(\Omega; C^\infty_\#(Y))$. We obtain

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega\times\Omega} w_\varepsilon(x,y)\Psi\left(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\mathrm{d}x\mathrm{d}y\\
&= \lim_{\varepsilon\to 0}\left(\int_\Omega \chi_\varepsilon(x)\psi_1\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}x\right)\left(\int_\Omega \chi_\varepsilon(y)\psi_2\left(y,\frac{y}{\varepsilon}\right)\mathrm{d}y\right)\\
&= \int_{\Omega\times\Omega}\int_Y\int_Y \phi(x,\sigma)\phi(y,\tau)\psi_1(x,\sigma)\psi_2(y,\tau)\,\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}x\mathrm{d}y,
\end{aligned}
$$

where we used the Fubini–Tonelli Theorem 2.1.6; the thesis follows by density. □

We define the functional $F\colon L^\infty(Y;[0,1]) \to \mathbb{R}$ by

$$
F(\varphi) := \int_Y\int_Y a(\sigma-\tau)\Big[\varphi(\sigma)\varphi(\tau) + (1-\varphi(\sigma))(1-\varphi(\tau))\Big]\,\mathrm{d}\sigma\mathrm{d}\tau. \tag{4.5}
$$

**Proposition 4.1.3.** *Let $u \in L^1(\Omega)$ and let $\{u_\varepsilon\}$ be a sequence weakly converging to $u$ in $L^1(\Omega)$ such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. Let $\chi_\varepsilon$ be the characteristic functions associated with $u_\varepsilon$ defined by (4.3), and suppose they admit a two-scale limit $\phi \in L^2(\Omega\times Y)$. Upon defining $\varphi(\cdot) := \int_\Omega \phi(\xi,\cdot)\,\mathrm{d}\xi$, we have*

$$
\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = F(\varphi). \tag{4.6}
$$

*Proof.* While Theorem 2.6.5 typically requires test functions to be in $L^2(\Omega; C^0_\#(Y))$, a standard density argument of $C^0$ in $L^1$ allows us to use $a \in L^\infty_\#(Y)$ as an admissible test function. Lemma 4.1.2 along with the Fubini–Tonelli Theorem 2.1.6 yields

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\Omega\times\Omega} a\left(\frac{x-y}{\varepsilon}\right)\chi_\varepsilon(x)\chi_\varepsilon(y)\,\mathrm{d}x\mathrm{d}y &= \int_{\Omega\times\Omega}\int_Y\int_Y a(\sigma-\tau)\phi(x,\sigma)\phi(y,\tau)\,\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}x\mathrm{d}y\\
&= \int_Y\int_Y a(\sigma-\tau)\varphi(\sigma)\varphi(\tau)\,\mathrm{d}\sigma\mathrm{d}\tau.
\end{aligned}
$$

Defining $v_\varepsilon(x,y) := (1-\chi_\varepsilon(x))(1-\chi_\varepsilon(y))$, Lemma 4.1.2 similarly ensures $v_\varepsilon(x,y) \xrightharpoonup{2}$

$(1-\phi(x,\sigma))(1-\phi(y,\tau))$, namely

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega\times\Omega} a\left(\frac{x-y}{\varepsilon}\right) v_\varepsilon(x,y)\,\mathrm{d}x\mathrm{d}y \\
&= \lim_{\varepsilon\to 0}\int_{\Omega\times\Omega} a\left(\frac{x-y}{\varepsilon}\right)(1-\chi_\varepsilon(x)-\chi_\varepsilon(y)+\chi_\varepsilon(x)\chi_\varepsilon(y))\,\mathrm{d}x\mathrm{d}y \\
&= \int_{\Omega\times\Omega}\int_Y\int_Y a(\sigma-\tau)(1-\phi(x,\sigma)-\phi(y,\tau)+\phi(x,\sigma)\phi(y,\tau))\,\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}x\mathrm{d}y \\
&= \int_{\Omega\times\Omega}\int_Y\int_Y a(\sigma-\tau)(1-\phi(x,\sigma))(1-\phi(y,\tau))\,\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}x\mathrm{d}y \\
&= \int_Y\int_Y a(\sigma-\tau)\big(1-\varphi(\sigma)\big)\big(1-\varphi(\tau)\big)\,\mathrm{d}\sigma\mathrm{d}\tau.
\end{aligned}$$

The conclusion follows by linearity. □

**Corollary 4.1.4.** *Let $u\in L^1(\Omega)$ and let $\{u_\varepsilon\}$ be a sequence weakly converging to $u$ in $L^1(\Omega)$ such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon)<+\infty$, and let $z\in\mathbb{R}$ be the limit of the associated sequence $\{z_\varepsilon\}$ given by Lemma 4.1.1. By letting $t:=\int_Y\varphi(\sigma)\,\mathrm{d}\sigma$, the limit function $u$ satisfies the relation*

$$\int_\Omega u(x)\,\mathrm{d}x = z+t. \tag{4.7}$$

*Proof.* From Proposition 2.6.7, the two-scale convergence $\chi_\varepsilon \xrightarrow{2} \phi$ implies the weak convergence $\chi_\varepsilon(\cdot)\rightharpoonup\int_Y\phi(\cdot,\sigma)\,\mathrm{d}\sigma$ in $L^2(\Omega)$, and thus in $L^1(\Omega)$. By integrating over $\Omega$, we get

$$\int_\Omega \chi_\varepsilon(x)\,\mathrm{d}x \to \int_\Omega\left(\int_Y\phi(x,\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}x = \int_Y\varphi(\sigma)\,\mathrm{d}\sigma = t.$$

By Lemma 4.1.1, we can write $u_\varepsilon(x)=z_\varepsilon+\chi_\varepsilon(x)$. Integrating and passing to the limit yields exactly the conclusion. □

## 4.2 Minimization of the cell problem

To explicitly investigate the optimal microstructures governing the homogenized functional, we specialize our analysis to a symmetric, non-trivial piece-wise constant periodic weight. For fixed parameters $\beta>\alpha>0$ and a given $\lambda\in(0,1)$, we consider a symmetric profile for the kernel $a_\lambda$, centered around $t=1/2$:

$$a_\lambda(t)=\begin{cases}\alpha & \text{if } t\in[0,\frac{\lambda}{2})\cup[1-\frac{\lambda}{2},1),\\ \beta & \text{if } t\in[\frac{\lambda}{2},1-\frac{\lambda}{2}),\end{cases} \tag{4.8}$$

extended periodically to $\mathbb{R}$. Without loss of generality, we can restrict ourselves to the case $0<\lambda\le 1/2$. The average of the weight is denoted by $\overline{a}_\lambda=\lambda\alpha+(1-\lambda)\beta$.

The sequence of functionals associated with this weight is totally analogous to the one in (4.1), and reads

$$F_\varepsilon^\lambda(u)=\int_{\Omega\times\Omega} a_\lambda\left(\frac{x-y}{\varepsilon}\right) f(u(x)-u(y))\,\mathrm{d}x\mathrm{d}y. \tag{4.9}$$

As proved in Proposition 4.1.3, we have that

$$\lim_{\varepsilon\to 0} F^\lambda_\varepsilon(u_\varepsilon) = \int_Y\int_Y a_\lambda(\sigma-\tau)\Big[\varphi(\sigma)\varphi(\tau)+(1-\varphi(\sigma))(1-\varphi(\tau))\Big]\,\mathrm{d}\sigma\mathrm{d}\tau =: F^\lambda(\varphi), \quad (4.10)$$

which we aim to minimize over all admissible microscopic profiles $\varphi$ subject to the volume fraction constraint $\int_Y \varphi(\sigma)\,\mathrm{d}\sigma =: t \in [0,1]$. Thus, we define the cell problem energy:

$$\gamma_\lambda(t) := \min\left\{F^\lambda(\varphi) : \varphi \in L^\infty(Y;[0,1]) \text{ with } \int_Y \varphi(\sigma)\,\mathrm{d}\sigma = t\right\}. \quad (4.11)$$

**Proposition 4.2.1.** *The optimal profile minimizing the cell problem (4.11) is given by the characteristic function $\varphi_t(x) = \chi_{[0,t/2)\cup(1-t/2,1)}(x)$ modulo 1, and the explicit minimum energy is*

$$\gamma_\lambda(t) = \begin{cases} 2\alpha t^2 - 2\overline{a}_\lambda t + \overline{a}_\lambda & \text{if } t \in [0,\frac{\lambda}{2}],\\ 2\beta(t^2-t) - \frac{\alpha-\beta}{2}\lambda^2 + \overline{a}_\lambda & \text{if } t\in[\frac{\lambda}{2},1-\frac{\lambda}{2}],\\ 2\alpha(1-t)^2 + 2\overline{a}_\lambda t - \overline{a}_\lambda & \text{if } t \in [1-\frac{\lambda}{2},1]. \end{cases} \quad (4.12)$$

*Proof.* Let us expand the integrand defining $F^\lambda(\varphi)$:

$$a_\lambda(\sigma-\tau)\left[\varphi(\sigma)\varphi(\tau)+(1-\varphi(\sigma))(1-\varphi(\tau))\right] = a_\lambda(\sigma-\tau)\left[2\varphi(\sigma)\varphi(\tau)-\varphi(\sigma)-\varphi(\tau)+1\right].$$

Integrating this over $Y\times Y$ and noting that $\int_Y a_\lambda(\sigma-\tau)\mathrm{d}\tau = \overline{a}_\lambda$ for any fixed $\sigma$, the functional simplifies to

$$F^\lambda(\varphi) = 2J^\lambda(\varphi) - 2\overline{a}_\lambda t + \overline{a}_\lambda,$$

where

$$J^\lambda(\varphi) := \int_Y\int_Y a_\lambda(\sigma-\tau)\varphi(\sigma)\varphi(\tau)\,\mathrm{d}\sigma\mathrm{d}\tau. \quad (4.13)$$

Since $t$ and $\overline{a}_\lambda$ are fixed, minimizing $F^\lambda$ is equivalent to minimizing $J^\lambda$. We can rewrite the weight as $a_\lambda(x) = \alpha + (\beta-\alpha)\chi_{(\lambda/2,1-\lambda/2)}(x)$. Consequently, the functional $J^\lambda$ reads

$$\begin{aligned} J^\lambda(\varphi) &= \int_Y\int_Y \left[\alpha + (\beta-\alpha)\chi_{(\lambda/2,1-\lambda/2)}(\sigma-\tau)\right]\varphi(\sigma)\varphi(\tau)\,\mathrm{d}\sigma\mathrm{d}\tau\\ &= \alpha t^2 + (\beta-\alpha)\int_Y\int_Y \chi_{(\lambda/2,1-\lambda/2)}(\sigma-\tau)\varphi(\sigma)\varphi(\tau)\,\mathrm{d}\sigma\mathrm{d}\tau\\ &= \alpha t^2 + (\beta-\alpha)\left[t^2 - \int_Y\int_Y \chi_{(0,\frac{\lambda}{2})\cup(1-\frac{\lambda}{2},1)}(\sigma-\tau)\varphi(\sigma)\varphi(\tau)\,\mathrm{d}\sigma\mathrm{d}\tau\right], \end{aligned}$$

so that minimizing it is, in turn, equivalent to maximizing (recall that $\beta-\alpha>0$) the term

$$K^\lambda(\varphi) := \int_Y\int_Y \chi_{(-\lambda/2,\lambda/2)}(\sigma-\tau)\varphi(\sigma)\varphi(\tau)\,\mathrm{d}\sigma\mathrm{d}\tau,$$

where the intervals are understood modulo 1. Since the kernel $\chi_{(-\lambda/2,\lambda/2)}$ is symmetrically decreasing for any $\lambda\in(0,1/2]$, the Riesz–Sobolev rearrangement inequality 2.2.25

implies that $K^\lambda$ is maximized when $\varphi$ is also a symmetrically decreasing indicator function. Hence, the optimal profile is

$$\varphi_t(x) \coloneqq \chi_{(-t/2,t/2)}(x) \equiv \chi_{[0,t/2)\cup(1-t/2,1)}(x) \quad \text{modulo } 1,$$

which satisfies the constraint $\int_Y \varphi(\sigma)\,\mathrm{d}\sigma = t$.

Computing the integral $J^\lambda(\varphi_t) = \int_{-t/2}^{t/2}\int_{-t/2}^{t/2} a_\lambda(\sigma-\tau)\,\mathrm{d}\sigma\mathrm{d}\tau$ for the three regimes $t < \lambda/2$, $t \in [\lambda/2, 1-\lambda/2)$, and $t \geq 1-\lambda/2$ yields the polynomial expressions in (4.12), whose continuity with respect to $t$ is immediate to verify. We refer the reader to Appendix A for the full calculations. □

*Remark* 4.2.2. It is worth noting that the assumption $\alpha < \beta$ drives the symmetric rearrangement to concentrate the mass of the optimal profile $\varphi_t$ around the boundary of the unit cell (i.e., around $t = 0$), precisely where the low-cost kernel $\alpha$ is distributed.

If, conversely, one considers $\alpha > \beta$, the energetic penalty is inverted. To minimize the intersection with the expensive $\alpha$-phase, the optimal profile concentrates its mass symmetrically within the central low-cost plateau, yielding

$$\varphi_t(x) = \chi_{(1/2-t/2,1/2+t/2)}(x).$$

However, the minimum energy $\gamma_\lambda(t)$ has exactly the same shape as in (4.12), except that $\alpha$ and $\beta$ are swapped, as well as $\lambda$ and $1-\lambda$. In the sequel, it may be useful to take this into consideration.

We define the final homogenized functional evaluated on a generic function $u \in L^1(\Omega)$ as:

$$\overline{F}^\lambda(u) \coloneqq \min\{\gamma_\lambda(t) : t \in I_u \coloneqq [\iota(u), \varsigma(u)]\}, \tag{4.14}$$

where $\iota(u) \coloneqq \int_\Omega u(x)\,\mathrm{d}x - \operatorname{ess-inf} u$ and $\varsigma(u) \coloneqq \int_\Omega u(x)\,\mathrm{d}x - \operatorname{ess-sup} u + 1$. Note that if the essential oscillation of $u$ is strictly greater than 1 (i.e., $\operatorname{ess-sup} u - \operatorname{ess-inf} u > 1$), the interval $I_u$ is empty and $\overline{F}^\lambda(u) = \min_{t\in\emptyset}\{\gamma_\lambda(t)\} = +\infty$, which is consistent with the conditions established by Lemma 4.1.1.

The following theorem identifies the minimum of the energy as the $\Gamma$-limit evaluated on constant functions, providing its explicit value.

**Theorem 4.2.3.** *Let $u \equiv c \in \mathbb{R}$ be a constant function. Then, the $\Gamma$-limit of the sequence $\{F_\varepsilon^\lambda\}$ exists and evaluates exactly to the minimum energy of the cell problem:*

$$\Big(\Gamma - \lim_{\varepsilon\to 0} F_\varepsilon^\lambda\Big)(c) = \overline{F}^\lambda(c) = \frac{(1-(1-\lambda)^2)\alpha + (1-\lambda)^2\beta}{2}. \tag{4.15}$$

*Proof. Liminf inequality.* From Proposition 4.1.3 and formulas (4.11) and (4.14), we deduce that for any $u_\varepsilon \rightharpoonup c$ in $L^1(\Omega)$ with bounded energy, it holds

$$\liminf_{\varepsilon\to 0} F_\varepsilon^\lambda(u_\varepsilon) = F^\lambda(\varphi) \geq \gamma_\lambda(t) \geq \overline{F}^\lambda(c).$$

*Limsup inequality.* For the constant function $u \equiv c$, we have that $I_c = [0,1]$. The optimal parameter $\bar{t}$ that minimizes $\gamma_\lambda(t)$ over $I_c$ imposes the optimal shift to be $\bar{z} = \int_\Omega u(x)\,\mathrm{d}x - \bar{t}$, see (4.7). Minimizing $\gamma_\lambda(t)$ in (4.12) over [0,1] yields that the minimizer is $\bar{t} = 1/2$; consequently the optimal shift is $\bar{z} = c - 1/2$.

Recalling the definition of $\varphi_t$ as the minimizer of the problem defining $\gamma_\lambda(t)$ (see (4.11) and Proposition 4.2.1), we construct the recovery sequence as

$$u^*_\varepsilon(x) \coloneqq \bar{z} + \varphi_{\bar{t}}\left(\frac{x}{\varepsilon}\right) = c - \frac{1}{2} + \varphi_{1/2}\left(\frac{x}{\varepsilon}\right), \tag{4.16}$$

which weakly converges in $L^1(\Omega)$ to $c - 1/2 + \int_0^1 \varphi_{1/2}(\sigma)\,\mathrm{d}\sigma = c$. Plugging $u^*_\varepsilon$ into $F_\varepsilon$ and passing to the limit as $\varepsilon \to 0$ yields precisely the integral defining $F^\lambda(\varphi_{1/2})$. Indeed, from the definition (4.3) of the characteristic function, we have that in this case

$$\chi^*_\varepsilon(x) \coloneqq \begin{cases} 1 & \text{if } u^*_\varepsilon(x) = c + \frac{1}{2}, \\ 0 & \text{if } u^*_\varepsilon(x) = c - \frac{1}{2}, \end{cases} \tag{4.17}$$

which means that $\chi^*_\varepsilon(x) \equiv \varphi_{1/2}\left(\frac{x}{\varepsilon}\right)$, for almost every $x \in \Omega$. Therefore, using the Riemann-Lebesgue Lemma, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{\Omega\times\Omega} a_\lambda\left(\frac{x-y}{\varepsilon}\right) \left[\chi^*_\varepsilon(x)\chi^*_\varepsilon(y) + (1-\chi^*_\varepsilon(x))(1-\chi^*_\varepsilon(y))\right] \mathrm{d}x\mathrm{d}y \\
=&\lim_{\varepsilon\to 0} \int_{\Omega\times\Omega} a_\lambda\left(\frac{x-y}{\varepsilon}\right) \left[\varphi_{1/2}\left(\frac{x}{\varepsilon}\right)\varphi_{1/2}\left(\frac{y}{\varepsilon}\right) + \left(1-\varphi_{1/2}\left(\frac{x}{\varepsilon}\right)\right)\left(1-\varphi_{1/2}\left(\frac{y}{\varepsilon}\right)\right)\right] \mathrm{d}x\mathrm{d}y \\
=&\int_Y \int_Y a_\lambda(\sigma-\tau)\left[\varphi_{1/2}(\sigma)\varphi_{1/2}(\tau) + \Big(1-\varphi_{1/2}(\sigma)\Big)\Big(1-\varphi_{1/2}(\tau)\Big)\right] \mathrm{d}\sigma\mathrm{d}\tau \\
=&F^\lambda(\varphi_{1/2}).
\end{aligned}$$

Since $\varphi_{1/2}$ is the optimal profile for $t = 1/2$, this integral evaluates exactly to the cell-problem minimum $\gamma_\lambda(1/2)$. Furthermore, since $t = 1/2$ is the global minimizer of $\gamma_\lambda(t)$ over the interval [0,1], we obtain that $F^\lambda(\varphi_{1/2}) = \overline{F}^\lambda(c)$. Thus, we have proved that

$$\lim_{\varepsilon\to 0} F^\lambda_\varepsilon(u^*_\varepsilon) = \overline{F}^\lambda(c) = F^\lambda(\varphi_{1/2}) = \gamma_\lambda(1/2) = \frac{(1-(1-\lambda)^2)\alpha + (1-\lambda)^2\beta}{2}.$$

□

*Remark* 4.2.4. It is worth noting how the macroscopic energy forces the interactions into the zero-cost wells of the potential $f$, rejecting a flat microscopic profile (which would cost the full average weight $\bar{a}_\lambda$, strictly greater than the expression in (4.15)) in favor of a highly oscillating recovery sequence to achieve the constant macroscopic state $u \equiv c$. Therefore, taking the limit to the homogeneous cases, if $\lambda \to 1$ ($a(t) \equiv \alpha$), the high oscillations of the optimal profile between $c - \frac{1}{2}$ and $c + \frac{1}{2}$ halve the macroscopic energy: $\overline{F}^1(c) = \frac{\alpha}{2}$. Similarly, $\overline{F}^0(c) = \frac{\beta}{2}$ for $\lambda \to 0$.

## 4.3 Consequences on integral representation

The explicit evaluation of the Γ-limit on optimal microstructures allows us to deduce a strong property of the homogenized macroscopic energy: the loss of the classical double-integral form. To prove this, the following lemma regarding the strong convergence of characteristic functions will be useful.

**Lemma 4.3.1.** *Let $\{A_n\}$ be a sequence of measurable subsets of $\Omega$ and let $A \subset \Omega$. If the sequence of characteristic functions $\{\chi_{A_n}\}$ weakly converges to the characteristic function $\chi_A$ in $L^1(\Omega)$ as $n \to \infty$, then the convergence is indeed strong in $L^1(\Omega)$.*

*Proof.* The weak convergence in $L^1$ implies that for any test function $v \in L^\infty(\Omega)$, we have $\int_\Omega \chi_{A_n} v \to \int_\Omega \chi_A v$. Choosing $v \equiv 1$, we get $\int_\Omega \chi_{A_n} \to \int_\Omega \chi_A$, which means $|A_n| \to |A|$. Choosing $v = \chi_A$, we get $\int_\Omega \chi_{A_n}\chi_A \to \int_\Omega \chi_A\chi_A = \int_\Omega \chi_A$, which means that $|A_n \cap A| \to |A|$. We can now conclude by expanding the $L^1$ norm of the difference:

$$\lim_{n\to\infty} \|\chi_{A_n} - \chi_A\|_{L^1(\Omega)} = \lim_{n\to\infty} (|A_n| + |A| - 2|A_n \cap A|) = |A| + |A| - 2|A| = 0. \qquad \square$$

**Theorem 4.3.2.** *There exists no function $g$ such that the Γ-limit of $\{F_\varepsilon^\lambda\}$ can be represented as an integral functional of the form*

$$\Big(\Gamma - \lim_{\varepsilon\to 0} F_\varepsilon^\lambda\Big)(u) = \int_{\Omega\times\Omega} g(u(x) - u(y))\,\mathrm{d}x\mathrm{d}y, \tag{4.18}$$

*for every $u \in L^1(\Omega)$.*

*Proof.* We argue by contradiction, similarly to [6], and suppose that such a representation exists. We already explicitly computed the exact Γ-limit for constant functions in Theorem 4.2.3. Now, let $s \in (0,1)$ and define the target step function $u_s$ as

$$u_s(x) := \begin{cases} 1 & \text{if } x \in (0, s], \\ 0 & \text{if } x \in (s,1). \end{cases}$$

We claim that for this specific profile, the Γ-limit evaluates to

$$\Big(\Gamma - \lim_{\varepsilon\to 0} F_\varepsilon^\lambda\Big)(u_s) = \overline{a}_\lambda \int_{\Omega\times\Omega} \Big[u_s(x)u_s(y) + (1-u_s(x))(1-u_s(y))\Big]\,\mathrm{d}x\mathrm{d}y =: T^\lambda(u_s). \tag{4.19}$$

The value $T^\lambda(u_s)$ can be computed explicitly, noting that the integrand is 1 when $x$ and $y$ are on the same side of the jump $s$, and 0 otherwise, yielding

$$T^\lambda(u_s) = \overline{a}_\lambda\Big[\int_0^s\int_0^s \mathrm{d}x\mathrm{d}y + \int_s^1\int_s^1 \mathrm{d}x\mathrm{d}y\Big] = \overline{a}_\lambda(s^2 + (1-s)^2).$$

*Liminf inequality.* Let $\{u_\varepsilon\}$ be an arbitrary sequence weakly converging to $u_s$ in $L^1(\Omega)$. As a consequence of Lemma 4.1.1, see (4.3), we can write $u_\varepsilon = z_\varepsilon + \chi_\varepsilon$. Since $u_s \in \{0,1\}$ is itself a characteristic function, the weak convergence forces $z_\varepsilon \to 0$ and $\chi_\varepsilon \rightharpoonup u_s$

in $L^1(\Omega)$. By Lemma 4.3.1, the convergence of $\{\chi_\varepsilon\}$ to $u_s$ is indeed strong in $L^1(\Omega)$. This allows us to pass to the limit in the term $F_\varepsilon^\lambda(u_\varepsilon)$, coupling the strong convergence of $\chi_\varepsilon(x)\chi_\varepsilon(y)$ with the weak convergence of $a_\lambda\left(\frac{x-y}{\varepsilon}\right) \rightharpoonup \overline{a}_\lambda$. We obtain

$$\begin{aligned}
\liminf_{\varepsilon\to 0} F_\varepsilon^\lambda(u_\varepsilon) &= \lim_{\varepsilon\to 0}\int_{\Omega\times\Omega} a_\lambda\left(\frac{x-y}{\varepsilon}\right)\Big[\chi_\varepsilon(x)\chi_\varepsilon(y) + (1-\chi_\varepsilon(x))(1-\chi_\varepsilon(y))\Big]\,\mathrm{d}x\mathrm{d}y \\
&= \overline{a}_\lambda \int_{\Omega\times\Omega}\Big[u_s(x)u_s(y) + (1-u_s(x))(1-u_s(y))\Big]\,\mathrm{d}x\mathrm{d}y \\
&= T^\lambda(u_s).
\end{aligned}$$

*Limsup inequality.* We choose the constant recovery sequence $u_\varepsilon^* \equiv u_s$. Applying the Riemann–Lebesgue Lemma 2.6.1 to the oscillating weight $a_\lambda\left(\frac{x-y}{\varepsilon}\right)$, we have:

$$\begin{aligned}
\lim_{\varepsilon\to 0} F_\varepsilon^\lambda(u_\varepsilon^*) &= \lim_{\varepsilon\to 0}\left[\int_s^1\int_s^1 a_\lambda\left(\frac{x-y}{\varepsilon}\right)\mathrm{d}x\mathrm{d}y + \int_0^s\int_0^s a_\lambda\left(\frac{x-y}{\varepsilon}\right)\mathrm{d}x\mathrm{d}y\right] \\
&= \overline{a}_\lambda(s^2 + (1-s)^2).
\end{aligned}$$

We now match these two exact evaluations with the hypothetical formula (4.18), yielding a counterexample analogous to [6]. For the constant function $u \equiv c$, since $u(x) - u(y) = 0$, formula (4.18) combined with Theorem 4.2.3 implies

$$g(0) = \int_{\Omega\times\Omega} g(0)\,\mathrm{d}x\mathrm{d}y = \overline{F}^\lambda(c) = \frac{(1-(1-\lambda)^2)\alpha + (1-\lambda)^2\beta}{2}.$$

Now, evaluating (4.18) on the step function $u_s$ and equating it to (4.19), we obtain

$$T^\lambda(u_s) = \int_{\Omega\times\Omega} g(u_s(x) - u_s(y))\,\mathrm{d}x\mathrm{d}y = (s^2 + (1-s)^2)g(0) + 2s(1-s)g(1).$$

By substituting the known values for $T^\lambda(u_s)$ and $g(0)$, we obtain

$$\overline{a}_\lambda(s^2 + (1-s)^2) = (s^2 + (1-s)^2)\frac{(1-(1-\lambda)^2)\alpha + (1-\lambda)^2\beta}{2} + 2s(1-s)g(1),$$

and solving for $g(1)$ gives

$$g(1) = \frac{(s^2 + (1-s)^2)}{2s(1-s)}\,\frac{\lambda^2\alpha + (1-\lambda^2)\beta}{2}.$$

Consequently, the value of $g(1)$ depends explicitly on the arbitrary jump point $s \in (0,1)$, which is a contradiction. $\square$

*Remark* 4.3.3. Our main results rely on a function $f$ that takes the value $+\infty$ almost everywhere. However, as pointed out in [6], an extension to finite functions occurs by considering, for a fixed constant $M > 0$, the everywhere finite integrand

$$f_M(z) = \begin{cases} 0 & \text{if } z \in \{-1, 1\}, \\ 1 & \text{if } z = 0, \\ M & \text{otherwise,} \end{cases}$$

which is such that $f_M \nearrow f$. By choosing a sufficiently large $M$, one can easily show that any deviation from increments in $\{-1, 0, 1\}$ becomes strictly sub-optimal. Consequently, the optimal profiles for $f_M$ rigidly coincide with those of the infinite function, yielding the same macroscopic energy on our target functions.

## 4.4 A symmetric reduction

It is worth noting how non-representability occurs for functionals defined by (4.1) for some pathological cases of the kernel $a$. Indeed, let us choose

$$a(t) = \begin{cases} \alpha & \text{if } t \in [0, \frac{1}{2}), \\ \beta & \text{if } t \in [\frac{1}{2}, 1), \end{cases}$$

extended periodically to $\mathbb{R}$. Note that $\overline{a} = (\alpha + \beta)/2$.

**Proposition 4.4.1.** *For the choice of $a$ above, the functionals $F_\varepsilon$ defined in (4.1) are independent of $\varepsilon$ for every $u \in L^1(\Omega)$ and reduce to*

$$F_1(u) := \overline{a} \int_{\Omega\times\Omega} f(u(x) - u(y))\, \mathrm{d}x\mathrm{d}y. \tag{4.20}$$

*Consequently, its lower-semicontinuous envelope does not admit an integral representation.*

*Proof.* By exchanging the variables of integration $(x, y) \mapsto (y, x)$ and using the fact that $f$ is an even function ($f(z) = f(-z)$), we can rewrite the functional as

$$F_\varepsilon(u) = \int_{\Omega\times\Omega} a\left(\frac{y-x}{\varepsilon}\right) f(u(y) - u(x))\, \mathrm{d}y\mathrm{d}x = \int_{\Omega\times\Omega} a\left(\frac{y-x}{\varepsilon}\right) f(u(x) - u(y))\, \mathrm{d}x\mathrm{d}y.$$

By adding half of the original expression and half of the rearranged one, we obtain

$$F_\varepsilon(u) = \int_{\Omega\times\Omega} \frac{a\left(\frac{x-y}{\varepsilon}\right) + a\left(\frac{y-x}{\varepsilon}\right)}{2} f(u(x) - u(y))\, \mathrm{d}x\mathrm{d}y;$$

moreover, due to the specific choice of $a$, for almost every $t \in (0,1)$ we have $a(-t) = a(1-t)$, meaning that $\{a(t), a(-t)\} = \{\alpha, \beta\}$. Thus, their average is equal to $\frac{\alpha+\beta}{2} = \overline{a}$. This implies that $F_\varepsilon(u) = F_1(u)$ for every $\varepsilon > 0$, and the non-representability of the relaxed functional follows directly from the counterexample in [6]. $\square$

*Remark* 4.4.2. It is worth noting that this symmetric reduction strongly relies on the jump point being precisely at $t = 1/2$. Indeed, one might wonder whether the same argument can be applied to the generalized weight of the type

$$a_\lambda(t) = \begin{cases} \alpha & \text{if } t \in [0, \lambda), \\ \beta & \text{if } t \in [\lambda, 1), \end{cases}$$

$\lambda \in (0,1)$, which is also a similar case to the one we carefully studied in Section 4.2. In this scenario, however, for any $\omega \in (0,1)$, the symmetrized kernel

$$A_\lambda(t) \coloneqq \omega a_\lambda(t) + (1-\omega) a_\lambda(1-t)$$

is no longer a constant function when $\lambda \neq 1/2$. Instead, $A_\lambda(t)$ becomes a symmetric step function with a central plateau and two tails. For instance, if $\lambda < 1/2$, an explicit computation shows that

$$A_\lambda(t) = \begin{cases} \omega\alpha + (1-\omega)\beta & \text{if } t \in [0,\lambda), \\ \beta & \text{if } t \in [\lambda, 1-\lambda), \\ (1-\omega)\alpha + \omega\beta & \text{if } t \in [1-\lambda, 1). \end{cases}$$

As one can easily verify, the average of $A_\lambda(t)$ over the period is exactly the averaged weight $\overline{a}_\lambda = \lambda\alpha + (1-\lambda)\beta$. However, since $A_\lambda(t)$ is not constant, the averaged weight $\overline{a}_\lambda$ does not trivially factor out, and the homogenized functional does not reduce to the original one studied in [6]. Instead, we are facing a more complex structural form, similar to the one we explicitly defined and solved before in Section 4.2.

# Chapter 5

# Conclusions and perspectives

In this thesis, we have investigated the asymptotic behavior of a class of non-local functionals. In particular, integral representability results from Braides and Dal Maso have been summarized in Chapter 3 as a starting point for the original contribution presented in Chapter 4.

Therein, we studied the sequence (4.1), starting from the structural properties induced by the three-well potential (1.1). By restricting our analysis to the more specific and concrete setting defined by (4.8), we computed the $\Gamma$-limit of the relevant sequence on constant functions. This allowed us to determine the ground state energy and to highlight the fundamental mechanisms governing the system: the recovery sequence defined in (4.16) synchronizes its spatial oscillations with those of the kernel, naturally concentrating the mass within the low-energy valleys.

The energetic frustration, in this scenario, arises from a clear competition: on the one hand, the macroscopic constraint drives the system toward a constant overall profile; on the other hand, the non-convex potential heavily penalizes the zero state (which incurs an energy cost of 1) and forces the system into one of the two zero-cost wells. The optimal energetic compromise, therefore, consists of continuously alternating between these two wells. By doing so, the system pays a small cost for each transition, but successfully approximates the desired macroscopic constant profile. This perfectly illustrates the complex interplay between microscopic pattern formation and macroscopic apparent homogeneity, which lies at the core of our investigation.

Looking ahead, the analysis introduced in this thesis opens the doors to several future developments. First, as a consequence of the counterexample provided in Section 4.3, one may seek a "non-integral" representation of the $\Gamma$-limit (which should obviously differ from the bare definition itself) for a general function $u \in L^1(\Omega)$.

Moreover, a natural extension consists of generalizing the analysis to higher-dimensional domains. In multiple dimensions, the interfaces separating the distinct phases are no longer simple zero-dimensional jumps, but rather higher-dimensional surfaces. This geometric complexity could trigger the emergence of more complex macroscopic microstructures, such as periodic stripes, checkerboards, or droplet-like configurations.

Another interesting direction concerns stochastic homogenization. While the present work deals with a deterministic periodic kernel, introducing a random field would provide a more realistic description of disordered or amorphous media. In such a framework, the construction of recovery sequences would become more complex, since the periodic and deterministic pattern would no longer be available, and one may expect the jump set to adapt to favorable realizations of the random field.

Finally, it would be interesting to investigate whether the techniques developed here can be extended to more general multi-well potentials, allowing for a larger number of competing phases and potentially more complex microstructures.

# Appendix A

# Computation and continuity of the function $\gamma_\lambda$

To find the explicit form of the function $\gamma_\lambda$, we first evaluate the functional $J^\lambda$ defined in (4.13) on the profile $\varphi_t(x) := \chi_{(-t/2,t/2)}(x)$, which we know to be optimal from Proposition 4.2.1. Then, recalling that $\gamma_\lambda(t) = F^\lambda(\varphi_t)$, we will get the final expression (4.12) using the relation

$$F^\lambda(\varphi_t) = 2J^\lambda(\varphi_t) - 2\overline{a}_\lambda t + \overline{a}_\lambda.$$

Indeed, we have:

$$\begin{aligned} J^\lambda(\varphi_t) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} a_\lambda(\sigma-\tau)\varphi_t(\sigma)\varphi_t(\tau)\mathrm{d}\sigma\mathrm{d}\tau = \int_{-\frac{t}{2}}^{\frac{t}{2}} \int_{-\frac{t}{2}}^{\frac{t}{2}} a_\lambda(\sigma-\tau)\,\mathrm{d}\sigma\mathrm{d}\tau \\ &= \int_{-\frac{t}{2}}^{\frac{t}{2}} \int_{-\frac{t}{2}-\tau}^{\frac{t}{2}-\tau} a_\lambda(\rho)\,\mathrm{d}\rho\mathrm{d}\tau = \int_0^t \int_{-\frac{t}{2}}^{\frac{t}{2}-\rho} a_\lambda(\rho)\,\mathrm{d}\tau\mathrm{d}\rho + \int_{-t}^0 \int_{-\frac{t}{2}-\rho}^{\frac{t}{2}} a_\lambda(\rho)\,\mathrm{d}\tau\mathrm{d}\rho \\ &= \int_0^t a_\lambda(\rho)(t-\rho)\,\mathrm{d}\rho + \int_{-t}^0 a_\lambda(\rho)(t+\rho)\,\mathrm{d}\rho = 2\int_0^t a_\lambda(\rho)(t-\rho)\,\mathrm{d}\rho, \end{aligned}$$

where we used the change of variables $\rho = \sigma - \tau$ in the first equality, the Fubini–Tonelli Theorem 2.1.6 in the fourth equality and the parity of the function $a_\lambda$ in the last. Since the function $a_\lambda$ defined in (4.8) takes different values depending on $t$, we have to distinguish the following cases:

- If $t \in [0, \lambda/2]$, we simply have

$$J^\lambda(\varphi_t) = 2\int_0^t \alpha(t-\rho)\,\mathrm{d}\rho = \alpha t^2,$$

  which gives $F^\lambda(\varphi_t) = 2\alpha t^2 - 2\overline{a}_\lambda t + \overline{a}_\lambda$.

- If $t \in [\lambda/2, 1-\lambda/2]$, then we obtain

$$J^\lambda(\varphi_t) = 2\int_0^{\frac{\lambda}{2}} \alpha(t-\rho)\,\mathrm{d}\rho + 2\int_{\frac{\lambda}{2}}^t \beta(t-\rho)\,\mathrm{d}\rho = \alpha\left(t\lambda - \frac{\lambda^2}{4}\right) + \beta\left(t^2 - t\lambda + \frac{\lambda^2}{4}\right),$$

from which $F^\lambda(\varphi_t) = 2\beta(t^2 - t) - \frac{\alpha-\beta}{2}\lambda^2 + \overline{a}_\lambda$.

- Finally, $t \in [1 - \lambda/2, 1]$ yields

$$\begin{aligned} J^\lambda(\varphi_t) &= 2\int_0^{\frac{\lambda}{2}} \alpha(t-\rho)\,\mathrm{d}\rho + 2\int_{\frac{\lambda}{2}}^{1-\frac{\lambda}{2}} \beta(t-\rho)\,\mathrm{d}\rho + 2\int_{1-\frac{\lambda}{2}}^{t} \alpha(t-\rho)\,\mathrm{d}\rho \\ &= \alpha\left(t\lambda - \frac{\lambda^2}{4}\right) + 2\beta\left(t - t\lambda + \frac{\lambda}{2} - \frac{1}{2}\right) + \alpha\left(t^2 - 2t + t\lambda + 1 - \lambda + \frac{\lambda^2}{4}\right) \\ &= \alpha(1-t)^2 + \alpha\lambda(2t-1) + 2\beta t(1-\lambda) - \beta(1-\lambda). \end{aligned}$$

Therefore, in this case $F^\lambda(\varphi_t) = 2\alpha(1-t)^2 + 2\overline{a}_\lambda t - \overline{a}_\lambda$.

Combining these three cases yields the explicit expression for $\gamma_\lambda(t)$ as stated in (4.12). Moreover, the continuity of the piecewise function $\gamma_\lambda$ can be easily checked by verifying that the corresponding expressions coincide at the transition points $t = \lambda/2$ and $t = 1 - \lambda/2$.

$$\begin{aligned} \gamma_\lambda\Big(\Big(\frac{\lambda}{2}\Big)^-\Big) &= 2\alpha\Big(\frac{\lambda}{2}\Big)^2 - 2\overline{a}_\lambda\Big(\frac{\lambda}{2}\Big) + \overline{a}_\lambda \\ &= \frac{\alpha\lambda^2}{2} + \overline{a}_\lambda(1-\lambda) \\ &= \frac{\alpha\lambda^2}{2} + \big(\lambda\alpha + (1-\lambda)\beta\big)(1-\lambda) \\ &= \frac{\alpha\lambda^2}{2} + \lambda\alpha - \lambda^2\alpha + \beta(1-\lambda)^2 \\ &= \alpha\Big(\lambda - \frac{\lambda^2}{2}\Big) + \beta(1-\lambda)^2, \end{aligned}$$

which coincides with the right limit from the second interval:

$$\begin{aligned} \gamma_\lambda\Big(\Big(\frac{\lambda}{2}\Big)^+\Big) &= 2\beta\Big(\Big(\frac{\lambda}{2}\Big)^2 - \frac{\lambda}{2}\Big) - \frac{\alpha-\beta}{2}\lambda^2 + \overline{a}_\lambda \\ &= \frac{\beta\lambda^2}{2} - \beta\lambda - \frac{\alpha\lambda^2}{2} + \frac{\beta\lambda^2}{2} + \lambda\alpha + (1-\lambda)\beta \\ &= \alpha\Big(\lambda - \frac{\lambda^2}{2}\Big) + \beta(\lambda^2 - 2\lambda + 1) \\ &= \alpha\Big(\lambda - \frac{\lambda^2}{2}\Big) + \beta(1-\lambda)^2. \end{aligned}$$

Similarly, for the second transition point $t = 1 - \lambda/2$, evaluating the limit from the second interval gives:

$$\begin{aligned}
\gamma_\lambda\Big(\Big(1-\frac{\lambda}{2}\Big)^-\Big) &= 2\beta\Big(\Big(1-\frac{\lambda}{2}\Big)^2 - \Big(1-\frac{\lambda}{2}\Big)\Big) - \frac{\alpha-\beta}{2}\lambda^2 + \overline{a}_\lambda \\
&= 2\beta\Big(\Big(1-\frac{\lambda}{2}\Big)\Big(-\frac{\lambda}{2}\Big)\Big) - \frac{\alpha\lambda^2}{2} + \frac{\beta\lambda^2}{2} + \lambda\alpha + \beta - \lambda\beta \\
&= -\beta\lambda + \frac{\beta\lambda^2}{2} - \frac{\alpha\lambda^2}{2} + \frac{\beta\lambda^2}{2} + \lambda\alpha + \beta - \lambda\beta \\
&= \alpha\Big(\lambda - \frac{\lambda^2}{2}\Big) + \beta(\lambda^2 - 2\lambda + 1) \\
&= \alpha\Big(\lambda - \frac{\lambda^2}{2}\Big) + \beta(1-\lambda)^2,
\end{aligned}$$

which matches the evaluation from the third interval:

$$\begin{aligned}
\gamma_\lambda\Big(\Big(1-\frac{\lambda}{2}\Big)^+\Big) &= 2\alpha\Big(1 - \Big(1-\frac{\lambda}{2}\Big)\Big)^2 + 2\overline{a}_\lambda\Big(1-\frac{\lambda}{2}\Big) - \overline{a}_\lambda \\
&= \frac{\alpha\lambda^2}{2} + (\alpha\lambda + \beta(1-\lambda))(1-\lambda) \\
&= \alpha\Big(\lambda - \frac{\lambda^2}{2}\Big) + \beta(1-\lambda)^2.
\end{aligned}$$

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