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MASTER'S THESIS  
IN GREEN BUILDING ENGINEERING

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**Quantification of model uncertainty in non-linear  
numerical  
analysis of PRC beams presenting flexural failure  
mechanism**

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# Abstract

The application of prestressed concrete (PC) beams with internal tendons, utilizing both bonded and unbonded systems, is widespread in structural engineering due to their superior deflection control and load-carrying capacity. However, accurately predicting the non-linear response of these members, particularly regarding post-cracking behavior and ultimate failure modes, remains a complex challenge requiring advanced simulation techniques. This thesis presents a comprehensive numerical investigation into the flexural failure of rectangular PRC beams using Non-Linear Finite Element Analysis (NLFEA) within the ATENA software environment. The primary objective of this research is to evaluate the effectiveness of different solution strategies in predicting the structural response of PRC beams. Validated experimental specimens, representing both bonded and unbonded tendon configurations, were modeled to serve as benchmarks. For each specimen, a rigorous parametric study was conducted, involving six distinct numerical analysis scenarios (i.e., solution strategy). These scenarios were established to critically compare two fundamental smeared crack approaches. In the fixed crack model, the crack direction is fixed upon initiation. In this model, the sensitivity of the response to the shear retention factor was specifically examined to address stress-locking phenomena. In the rotated crack model, the crack direction rotates with the principal strain axes throughout the loading history. Furthermore, to understand the governing role of concrete softening, the fracture energy  $G_f$  was evaluated at three distinct levels for both crack models: a calibrated value to capture experimental behavior, a significantly reduced value  $G_f/100$  to simulate brittle fracture, and a maximized value to simulate pseudo-plastic behavior. The comparative analysis of the numerical load-displacement curves against experimental data was useful to quantify the uncertainty associated with assumptions performed for non-linear numerical modelling. In particular, the ratio between the experimental capacity and numerical one, named as model uncertainty ratio, will be statistically assessed to give relevant information for calibration of model uncertainty safety factors for safety assessment of PRC members according to current standards and Global Resistance Format (GRF). Moreover, the information will be the base to calibrate these factors in next generation of standards.

**Keywords:** Prestressed concrete, Bonded and Unbonded Tendons, Flexural failure, NLFEA, ATENA, Fixed crack model, Rotated crack model, Fracture energy, Sensitivity analysis.

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# List of Acronyms

|                          |                                                       |      |
|--------------------------|-------------------------------------------------------|------|
| <b>EC</b>                | Eurocode . . . . .                                    | 3    |
| <b>ECoV</b>              | Estimate of Coefficient of Variation method . . . . . | 23   |
| <b><i>fib</i> MC2010</b> | <i>fib</i> Model Code 2010 . . . . .                  | 3    |
| <b>GRF</b>               | Global Resistance Format . . . . .                    | 5    |
| <b>GRFm</b>              | Global Resistance Factor method . . . . .             | 23   |
| <b>LoA</b>               | Level of Approximation . . . . .                      | 4    |
| <b>NLFEA</b>             | Non-Linear Finite Element Analysis . . . . .          | 4    |
| <b>NLNA</b>              | Non-Linear Numerical Analysis . . . . .               | 4    |
| <b>NTC</b>               | Norme Tecniche per le Costruzioni . . . . .           | 3    |
| <b>PRC</b>               | Prestressed Reinforced Concrete . . . . .             | iv   |
| <b>PSF</b>               | Partial Safety Factor . . . . .                       | 4    |
| <b>RC</b>                | Reinforced Concrete . . . . .                         | 1    |
| <b>SLS</b>               | Serviceability Limit State . . . . .                  | 2    |
| <b>ULS</b>               | Ultimate Limit State . . . . .                        | xiii |

# List of Symbols

## *Latin symbols*

|                                                    |                                                                            |
|----------------------------------------------------|----------------------------------------------------------------------------|
| $A_c$                                              | area of the (uncracked) concrete cross-section                             |
| $A_p$                                              | total area of the prestressing tendons                                     |
| $A_s, A'_s$                                        | area of the bottom / top mild longitudinal reinforcement                   |
| $b, h$                                             | width and height of the cross-section                                      |
| $c$                                                | concrete cover                                                             |
| $c_{ts}$                                           | tension-stiffening coefficient                                             |
| $d_p$                                              | effective depth of the prestressing tendon                                 |
| $E_c, E_p, E_s$                                    | Young's modulus of concrete, prestressing steel, mild steel                |
| $E_d$                                              | design value of the local internal action                                  |
| $e$                                                | tendon eccentricity                                                        |
| $F_d$                                              | design value of the total external action                                  |
| $\mathbf{f}_{\text{ext}}, \mathbf{f}_{\text{int}}$ | external / internal nodal force vectors                                    |
| $f_c, f_{cm}$                                      | cylindrical / mean cylindrical compressive strength of concrete            |
| $f_{ct}$                                           | tensile strength of concrete                                               |
| $f_{p0.1k}, f_{pk}$                                | 0.1% proof stress / characteristic ultimate strength of prestressing steel |
| $f_y, f_t (f_u)$                                   | yield / ultimate strength of mild steel                                    |
| $G_c$                                              | crushing (compressive) energy of concrete                                  |
| $G_f$                                              | specific fracture energy of concrete                                       |
| $I_c$                                              | second moment of area of the concrete section                              |
| $\mathbf{K}_T$                                     | global tangent stiffness matrix                                            |
| $L, L_{\text{eff}}$                                | total length / effective span of the beam                                  |
| $L_e, L_t, L_c$                                    | global element size; tensile / crushing characteristic (band) length       |
| $M_{\text{ext}}$                                   | external bending moment                                                    |
| $P, \delta$                                        | applied load and mid-span displacement                                     |
| $P_0, P_e$                                         | initial / effective prestressing force                                     |
| $P_{\text{dec}}, P_{cr}, P_y, P_u$                 | decompression, cracking, yielding and ultimate load                        |
| $\mathbf{r}$                                       | residual force vector                                                      |
| $R, R_d$                                           | global structural resistance / global design resistance                    |
| $R_{\text{exp}}, R_{\text{num}}$                   | experimental / numerical ultimate capacity                                 |
| $R_m, R_k$                                         | mean / characteristic global resistance                                    |
| $s_1, s_2, s_3$                                    | characteristic slips of the bond-slip law                                  |
| $\mathbf{u}$                                       | nodal displacement vector                                                  |

|                    |                                                               |
|--------------------|---------------------------------------------------------------|
| $V_R, V_\vartheta$ | coefficient of variation of resistance / of model uncertainty |
| $w, w_c$           | crack opening / critical (stress-free) crack opening          |
| $\bar{X}$          | representative values of the random variables                 |

*Greek symbols*

|                                            |                                                                                                                                                    |
|--------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------|
| $\alpha_R$                                 | FORM sensitivity factor for the resistance ( $\approx 0.8$ )                                                                                       |
| $\beta$                                    | reliability index (target $\beta = 3.8$ at Ultimate Limit State (ULS));<br>also shear retention factor $\beta \in [0, 1]$ of the fixed crack model |
| $\gamma_{Rd}$                              | partial safety factor for model (epistemic) uncertainty                                                                                            |
| $\gamma_R$                                 | global resistance partial factor (material/geometry variability)                                                                                   |
| $\delta_{\text{peak}}$                     | mid-span displacement at peak load                                                                                                                 |
| $\Delta\sigma_p$                           | stress increment in the prestressing tendon                                                                                                        |
| $\varepsilon_{c1}, \varepsilon_{cu}$       | strain at peak compressive stress / ultimate compressive strain                                                                                    |
| $\varepsilon_{cr}$                         | concrete crack strain                                                                                                                              |
| $\varepsilon_{su}, \varepsilon_{pu}$       | ultimate strain of mild steel / of prestressing steel                                                                                              |
| $\varepsilon_{p,0}$                        | initial strain imposed on the tendon                                                                                                               |
| $\vartheta$                                | model uncertainty ratio, $\vartheta = R_{\text{exp}}/R_{\text{num}}$                                                                               |
| $\lambda_\vartheta, \zeta_\vartheta$       | parameters of the lognormal distribution of $\vartheta$                                                                                            |
| $\mu_\vartheta$                            | mean value of $\vartheta$                                                                                                                          |
| $\nu_c$                                    | Poisson's ratio of concrete                                                                                                                        |
| $\sigma_c, \sigma_{ct}$                    | concrete compressive / tensile stress                                                                                                              |
| $\sigma_{p,0}$                             | initial prestress in the tendon                                                                                                                    |
| $\tau_{b,\text{max}}, \tau_{b,\text{res}}$ | maximum / residual bond stress                                                                                                                     |
| $\chi_u$                                   | ultimate curvature of the section                                                                                                                  |



# Chapter 1

## Introduction

### 1.1 Background and motivation

The history of structural engineering can be read as a continuous effort to overcome the intrinsic limitations of construction materials. The development of prestressed concrete during the twentieth century constitutes one of the most significant breakthroughs in this respect. Conventional Reinforced Concrete (RC) already represents a remarkable improvement over plain concrete, because it combines the high compressive strength of the cementitious matrix with the tensile capacity of embedded steel reinforcement. Yet the steel can only mobilise its tensile contribution once the surrounding concrete has cracked, which means that under typical service loads cracking is essentially unavoidable. Prestressing technology fundamentally alters this paradigm by introducing an internal state of compression in the concrete *before* any external action is applied. The pre-compression counteracts the tensile stresses that arise under service loading, postpones cracking, controls deflections, and unlocks longer spans and more slender geometries than RC can offer.

#### 1.1.1 The role of PRC beams in modern construction

In a prestressed member the steel acts as an *active* reinforcement: it is tensioned at the moment of construction and remains tensioned throughout the service life of the element, applying a sustained compressive force on the surrounding concrete. The full cross-section therefore participates in resisting external actions over a much wider load range than in conventional RC, which yields improved control of deflections, enhanced shear performance, better fatigue behaviour and a substantially reduced tendency to crack [1]. For these reasons, PRC has become the material of choice for long-span bridges, large-span industrial floors, offshore platforms, transfer girders and many other structural typologies in which slenderness, durability and serviceability are critical requirements.

A further reason for the modern relevance of PRC is its inherent material efficiency. High-strength concrete and high-strength prestressing steel allow the cross-sections to be smaller and lighter than those of equivalent RC elements; this reduces self-weight, lowers foundation demand and contributes to the sustainability of the construction process

by limiting the embodied carbon of the structure. The active control of cracking also improves the protection of the embedded steel from aggressive environmental agents, extending the service life of the member. Architecturally, the capability of bridging long spans without intermediate supports opens design possibilities that would otherwise be inaccessible.

### 1.1.2 Bonded versus unbonded internal tendon systems

From a structural standpoint, the way prestress is delivered into the concrete is not unique. Two main construction technologies coexist in practice, namely *pre-tensioning* and *post-tensioning*, and within post-tensioning the tendons can be either bonded to the surrounding concrete via cementitious grouting of the ducts or left unbonded inside lubricated sheathings (or external in some cases). The construction sequences of the two prestressing technologies are summarised in fig. 1.1. Each system imposes a distinct interaction between concrete and tendons and therefore a different non-linear response at the ULS:

- In *bonded* systems (pre-tensioning, or post-tensioning with grouted ducts) strain compatibility is enforced locally between the tendon and the surrounding concrete. The tendon experiences a stress increase proportional to the deformation of the section in its fibre; the failure mechanism mobilises both the conventional reinforcement and the prestressing steel up to large strain levels.
- In *unbonded* systems the tendon is free to slide longitudinally inside its duct. The stress increase in the tendon is therefore governed by the global elongation of the cable rather than by the local sectional strain, which delays tendon yielding and modifies both the cracking pattern and the ductility of the member.

The experimental campaigns selected in this thesis cover both configurations. Bonded behaviour is represented, among others, by the work of Moawad *et al.* [2], Salah *et al.* [3], Imperatore *et al.* [4], Yu *et al.* [5], and Padmarajaiah and Ramaswamy [6], whereas the unbonded case is investigated in the experimental programme of Peera and Oukaili [7]. These six studies are introduced here only to motivate the engineering relevance of the problem; their geometric and mechanical details are systematically reported in Chapter 4 (Experimental benchmark).

### 1.1.3 The challenge of predicting non-linear flexural response

Despite the maturity of the technology, the accurate prediction of the flexural response of PRC beams up to failure remains a difficult task. The transition from the Serviceability Limit State (SLS), which is generally well described by linear elastic theory, to the ULS involves a strongly non-linear interaction among several phenomena [8]: the cracking of the concrete tensile zone, the bond-slip and tension-stiffening mechanisms, the yielding of the conventional reinforcement, the additional stress demand on the prestressing tendons beyond the effective prestress level, the redistribution of internal forces, and finally the crushing of the concrete in the compression chord.

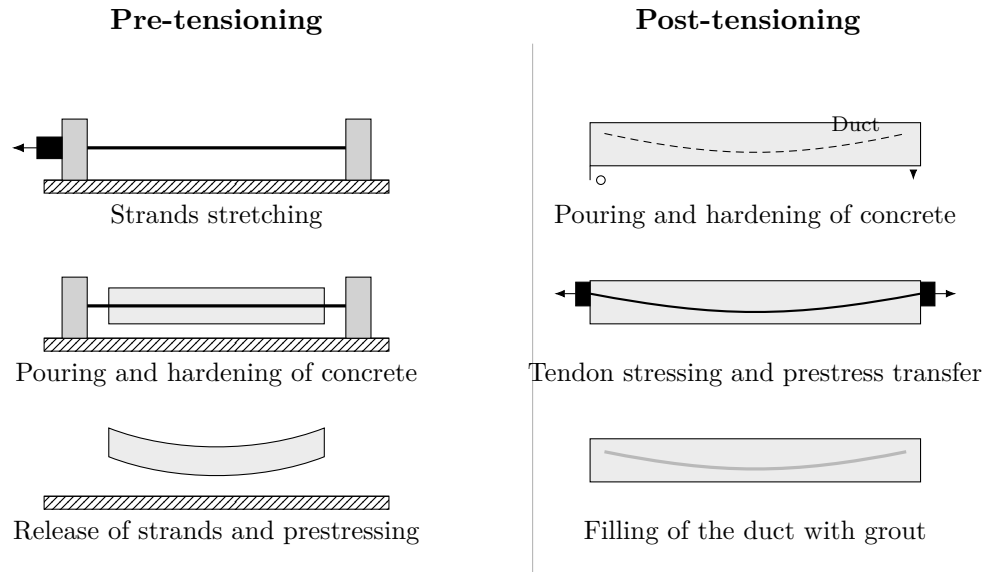


**Figure 1.1:** Construction sequence of the two prestressing technologies: pre-tensioning (left), in which the strands are tensioned against external abutments before casting, and post-tensioning (right), in which the tendons are stressed inside ducts after the concrete has hardened and, for bonded systems, the ducts are subsequently grouted.

The variability of the failure mode is, in itself, a defining feature of PRC. Depending on the prestressing ratio, the bond conditions, the residual effective prestress and the level of conventional reinforcement, the same beam type can develop a ductile failure governed by tendon yielding, a brittle failure governed by concrete crushing, or even tendon rupture. The decomposition load, the cracking load, the yielding load and the ultimate load are not equally spaced points on the load–displacement curve; their relative position changes with each design choice. A typical flexural load–displacement response, with the characteristic milestones described above, is illustrated later in fig. 2.1 of Chapter 2.

Standard analytical formulations included in design codes such as Eurocode (EC) [9], Norme Tecniche per le Costruzioni (NTC) [10] and *fib* Model Code 2010 (*fib* MC2010) [1] are calibrated to be conservative for typical configurations, but they necessarily rely on simplified stress-block assumptions and empirical coefficients that may not capture the true behaviour of complex cross-sections or beams with unusual levels of prestressing. The resulting discrepancy between the code-based prediction and the experimentally observed capacity is a quantifiable source of *epistemic* uncertainty, which this thesis addresses in a rigorous probabilistic framework.

## 1.2 The need for non-linear finite element analysis in structural safety assessment

### 1.2.1 Limits of the conventional local approach

The classical procedure adopted in structural codes consists of a *local* verification, in which a linear-elastic global analysis is performed to obtain the internal actions (bending

moment, shear force, axial load) at the relevant sections, and these internal actions are then compared with sectional resistances computed under simplified hypotheses. Symbolically, the check is expressed as

$$E_d \leq R_d, \quad (1.1)$$

where  $E_d$  is the design value of the local internal action and  $R_d$  the corresponding sectional resistance, both evaluated under the Partial Safety Factor (PSF) method.

This approach, although widely accepted, becomes inefficient and sometimes inadequate when applied to elements such as PRC beams, where: (i) the cross-sectional response is intrinsically non-linear at the ULS; (ii) the internal actions redistribute substantially as cracking and yielding progress; and (iii) the safety margin actually carried by the structure is often hidden by the conservative sectional simplifications. Figure 1.2 contrasts the local (sectional) verification, in which the design internal action  $E_d$  is compared with the sectional resistance, with the global verification, in which the total external action  $F_d$  is compared with the global resistance of the whole structure.

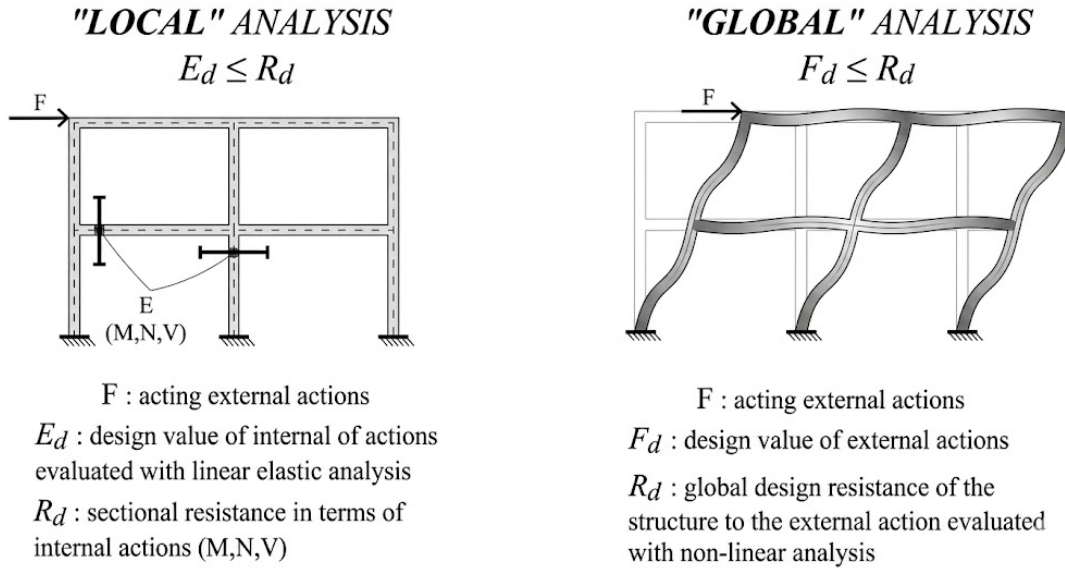


**Figure 1.2:** Comparison between the local (sectional) verification approach,  $E_d \leq R_d$ , based on a linear-elastic global analysis, and the global verification approach,  $F_d \leq R_d$ , based on non-linear numerical analysis of the whole structure.

### 1.2.2 Non-linear numerical analysis as the highest level of approximation

Non-Linear Finite Element Analysis (NLFEA), also referred to as Non-Linear Numerical Analysis (NLNA), is classified by the *fib* MC2010 [1] as the highest Level of Approximation (LoA) for the assessment of concrete structures. It allows the analyst to: (i) explicitly model the non-linear constitutive laws of concrete (cracking, crushing, softening, tension stiffening) and reinforcement (yielding, hardening, bond-slip); (ii) include geometric non-linearities such as second-order  $P$ - $\Delta$  effects whenever relevant; (iii) simulate the full load-displacement history of the member from the elastic stage to

ultimate failure; (iv) directly visualise the propagation of cracks, the spread of plasticity and the redistribution of internal forces. For PRC members this is particularly attractive because the simulation can reproduce the entire sequence from the decompression of the pre-compressed tensile chord up to the ultimate flexural collapse, including the contribution of the prestressing tendons and the interaction with the bond conditions [11].

The global verification inequality associated with NLNA is written in terms of the total external action  $F_d$  and the global resistance  $R_d$  of the structure,

$$F_d \leq R_d, \quad (1.2)$$

which constitutes the natural counterpart of eq. (1.1) when the analysis is performed on the structure as a whole rather than on isolated sections.

### 1.2.3 Inherent uncertainties of NLFEA modelling

The accuracy of an NLFEA prediction is, however, intrinsically linked to a large number of modelling choices made by the analyst: the element type and discretisation, the constitutive laws, the crack model (smeared versus discrete, fixed versus rotated), the fracture energy  $G_f$ , the shear retention factor, the bond-slip law, the prestress application procedure, the loading scheme and the convergence criteria [8, 12]. Each choice introduces an additional layer of *epistemic* uncertainty that cannot be removed by performing more analyses with the same model. The traditional PSF method is unable to handle this kind of uncertainty directly because it operates by reducing material strengths at the input stage, while in a non-linear analysis the principle of superposition does not apply and the relationship between input and output is itself unknown. A different safety framework is therefore required, one in which the uncertainty associated with the numerical model is recognised, quantified and propagated explicitly to the global resistance.

## 1.3 Model uncertainty within the Global Resistance Format

The conceptual framework adopted by the *fib* MC2010 [1] and later refined by Castaldo, Gino and Mancini [8] for the safety verification of structures analysed via NLNA is the Global Resistance Format (GRF). The basic idea is to evaluate the safety check at the *global level* of the structure rather than at the sectional level: the global design resistance  $R_d$  is obtained by reducing the structural resistance  $R$  computed with mean (or representative) material strengths through a chain of partial safety factors that account for the different sources of uncertainty,

$$R_d = \frac{R(\bar{X})}{\gamma_{Rd} \gamma_R}, \quad (1.3)$$

where  $R(\bar{X})$  is the global resistance computed with the representative values  $\bar{X}$  of the random variables,  $\gamma_R$  accounts for the geometric and other uncertainties (typically related to the variability of material properties through the safety format chosen), and

$\gamma_{Rd}$  is the *model uncertainty* safety factor, which is the central quantity investigated in this thesis.

Within the GRF several specific safety formats coexist, each characterised by a different balance between accuracy and computational effort [8, 12]:

- the *Probabilistic method*, in which a full reliability analysis is performed using probability distributions of all random variables;
- the *Global Resistance Factor method* (GRFm), which estimates the global safety from two deterministic NLNA runs (with mean and design values of material strengths);
- the *Estimate of Coefficient of Variation method* (ECoV), which infers the coefficient of variation of the global resistance from two deterministic NLNA runs and a lognormal hypothesis on  $R$ ;
- the *Strain-based method* recently proposed by Gino *et al.* [12], which provides a more accurate estimation of the global safety factors accounting for the different failure modes on their own.

All of these formats share a common feature: they rely on a coefficient  $\gamma_{Rd}$  to convert a numerical resistance, computed under specific modelling assumptions, into a value consistent with the observed physical behaviour of the structure.

### 1.3.1 Aleatory and epistemic uncertainty

Within a probabilistic framework, the sources of variability affecting the structural response are usually grouped into two classes [1, 11]. *Aleatory* uncertainty is the natural randomness of the input quantities, such as the compressive strength of concrete, the yield strength of steel, or the actual geometric dimensions of the member. It is irreducible by definition because it reflects an intrinsic property of the physical world. *Epistemic* uncertainty, on the contrary, arises from the limitations of the model used to predict the response: incomplete knowledge, mathematical idealisations and numerical approximations all contribute to it. Unlike aleatory uncertainty, epistemic uncertainty can in principle be reduced by improving the model or by gathering more experimental evidence. The model uncertainty safety factor  $\gamma_{Rd}$  is exactly the device through which the epistemic component is propagated into the global verification.

### 1.3.2 Definition of the model uncertainty random variable

Following the framework established in references [8, 11, 12], the model uncertainty is treated as a random variable, denoted by  $\vartheta$ , defined as the ratio between the experimental ultimate resistance  $R_{\text{exp}}$  of a benchmark specimen and the numerical resistance  $R_{\text{num}}$  predicted by the NLFEA model of the same specimen:

$$\vartheta = \frac{R_{\text{exp}}}{R_{\text{num}}}. \quad (1.4)$$

The interpretation of eq. (1.4) is straightforward:  $\vartheta = 1$  indicates perfect agreement between numerical and experimental capacities;  $\vartheta > 1$  corresponds to a conservative (safe-sided) prediction in which the model underestimates the resistance ( $R_{\text{num}} < R_{\text{exp}}$ );  $\vartheta < 1$  corresponds to a non-conservative (unsafe) prediction in which the model overestimates the resistance ( $R_{\text{num}} > R_{\text{exp}}$ ). By applying eq. (1.4) to a sufficiently large benchmark dataset, the statistical properties of  $\vartheta$  – its sample mean  $\mu_{\vartheta}$  and coefficient of variation  $V_{\vartheta}$ , and ultimately its probability distribution – can be characterised and used to calibrate  $\gamma_{Rd}$  for the design and safety assessment of new structures.

### 1.3.3 Current treatment of $\gamma_{Rd}$ in standards

The recommended values of  $\gamma_{Rd}$  in current standards are still derived from a relatively narrow set of benchmark data, predominantly relative to RC members and to specific failure modes [1, 8]. *fib* MC2010 [1] suggests indicative values in the range  $\gamma_{Rd} \in [1.05, 1.10]$  for well-validated numerical analyses and up to  $\gamma_{Rd} \approx 1.20$  when the modelling uncertainty is considered higher. EC [9] and NTC [10] embed the model uncertainty into the general safety verification format implicitly, without an explicit calibration tied to non-linear numerical analysis. Recent works by Castaldo *et al.* [8], Gino *et al.* [11] and Gino *et al.* [12] have demonstrated that a Bayesian updating framework can refine the distribution of  $\vartheta$  by combining prior expert knowledge with benchmark experimental evidence, and that the resulting calibration of  $\gamma_{Rd}$  depends strongly on the failure mode (flexural, shear, instability) and on the modelling choices. The present thesis adopts the same philosophy and extends it to PRC beams failing in flexure, a class of members for which a systematic calibration of  $\gamma_{Rd}$  is, to the best of the candidate’s knowledge, still missing in the open literature.

## 1.4 Research objectives and questions

In light of the considerations above, the present thesis pursues four specific objectives, summarised hereafter.

**Objective 1 – Predictive capability of NLFEA solution strategies.** To evaluate the capability of different NLFEA solution strategies, and in particular of the fixed and the rotated smeared crack models, in predicting the non-linear response, the post-cracking behaviour and the ultimate flexural capacity of PRC beams with both bonded and unbonded internal tendons.

**Objective 2 – Sensitivity to critical material parameters.** To investigate the sensitivity of the numerical predictions to two critical material parameters: the fracture energy  $G_f$ , evaluated at three distinct levels (a calibrated value, a significantly reduced value  $G_f/100$  to simulate brittle fracture, and a maximised value to simulate a pseudo-plastic post-peak response), and the shear retention factor used in the fixed crack model. The latter parameter is examined specifically to address the stress-locking phenomena that may affect the fixed crack formulation.

**Objective 3 – Statistical quantification of model uncertainty.** To statistically quantify the model uncertainty associated with each modelling choice by computing the model uncertainty ratio  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  defined in eq. (1.4) across all six numerical scenarios obtained by combining the two crack models and the three levels of fracture energy.

**Objective 4 – Bayesian calibration of  $\gamma_{Rd}$ .** To provide a rigorous probabilistic basis for the calibration of the model uncertainty safety factor  $\gamma_{Rd}$  within the GRF, using a Bayesian updating framework consistent with references [11, 12], and thereby to contribute relevant data for the next generation of structural design standards.

These objectives are translated into the following research questions, which guide the structure and the discussion of the subsequent chapters:

1. How does the choice between a fixed and a rotated smeared crack model affect the predicted flexural capacity of PRC beams with bonded and unbonded tendons?
2. What is the sensitivity of the numerical response to a variation of  $G_f$  by two orders of magnitude, and how does this sensitivity differ between the two crack models?
3. Which statistical distribution provides the most accurate description of  $\vartheta$  for PRC beams failing in flexure, and what are its parameters?
4. What value of  $\gamma_{Rd}$  should be recommended for the safety assessment of PRC flexural members analysed through NLFEA, when the experimental uncertainty is properly removed from the model uncertainty?

## 1.5 Methodology overview

The methodology adopted to answer the four research questions is structured as a sequential workflow that integrates experimental data, non-linear numerical simulations and a probabilistic post-processing stage. The principal steps are as follows:

1. **Construction of the experimental benchmark database.** An experimental benchmark of rectangular PRC beams with bonded and unbonded internal tendons is compiled from the peer-reviewed literature [2–7]. For each specimen, the geometric and mechanical properties, the prestress configuration, the loading protocol and the experimental load–displacement curve are collected.
2. **Numerical modelling in NLFEA (ATENA 3D).** Each specimen is modelled in ATENA 3D using six distinct numerical scenarios, obtained by combining the two smeared crack models (fixed and rotated) with three levels of fracture energy ( $G_f/100$ , calibrated  $G_f$ , maximised  $G_f$ ). The full load–displacement response of each model is extracted up to the ultimate state.

3. **Extraction of resistances and computation of  $\vartheta$ .** For each of the six numerical scenarios and for each specimen, the numerical ultimate capacity  $R_{\text{num}}$  is extracted from the predicted load–displacement curve and the model uncertainty ratio  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  is computed using eq. (1.4).
4. **Statistical inference on  $\vartheta$ .** The sample mean  $\mu_{\vartheta}$ , the coefficient of variation  $V_{\vartheta}$  and the candidate probability distribution of  $\vartheta$  are determined, and their compatibility with the experimental evidence is tested through goodness-of-fit inference (Chi-squared, Anderson–Darling, Lilliefors, Jarque–Bera).
5. **Bayesian updating and calibration of  $\gamma_{Rd}$ .** A Bayesian updating framework is applied to the prior distribution of  $\vartheta$  following the approach of Gino *et al.* [11, 12]. By means of a leave-one-out cross-validation scheme and by stripping away the experimental uncertainty, the *pure* posterior distribution of the model uncertainty is obtained and used to calibrate the partial safety factor  $\gamma_{Rd}$  for each of the six numerical scenarios.

A schematic representation of the entire workflow is given in fig. 1.3.

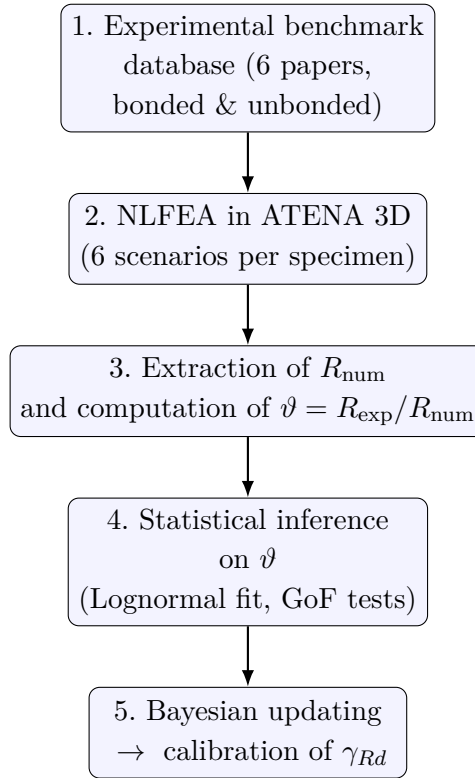


**Figure 1.3:** Schematic representation of the research methodology adopted in the thesis: from the experimental benchmark database to the Bayesian calibration of the model uncertainty safety factor  $\gamma_{Rd}$ .

### 1.5.1 Scope and limitations

The boundaries of the study are explicitly stated here in order to delimit the validity of the conclusions:

- **Structural elements.** The investigation is confined to rectangular, simply-supported PRC beams with internal tendons (bonded and unbonded). External post-tensioning, segmental construction and complex tendon profiles are not considered.
- **Loading and failure mode.** Only monotonic loading conditions leading to a flexural (bending) failure are analysed. Cyclic and dynamic loading, fatigue, pure shear failure, punching and torsion are outside the scope.
- **Numerical modelling.** The simulations are performed using NLFEA in the ATENA 3D environment, with a macro-modelling approach for concrete and a discrete (truss) representation of the reinforcement and the prestressing tendons.
- **Benchmark database.** The numerical predictions are validated against a selection of experimental tests available in the structural engineering literature; the database is restricted to rectangular, internally prestressed beams suitable for the calibration of  $\gamma_{Rd}$  for flexural failure.

## 1.6 Thesis structure

The remainder of the thesis is organised in five chapters, following the logical progression of the research workflow described in section 1.5.

**Chapter 2 – Theoretical background.** This chapter consolidates the theoretical foundations on which the work is built. It reviews the structural behaviour of PRC beams with bonded and unbonded tendons, the mechanisms of flexural failure, the constitutive behaviour of the relevant materials (concrete, conventional and prestressing steel), and the fundamentals of NLFEA, including the smeared crack approach (fixed versus rotated), the role of the fracture energy and the shear retention factor, and the solution algorithms. The chapter closes with a critical overview of the GRF, of the specific safety formats embedded in it, and of the current code treatment of  $\gamma_{Rd}$  in *fib* MC2010, EC and NTC.

**Chapter 3 – Numerical modelling framework in ATENA 3D.** Chapter 3 details the implementation of the NLFEA models in the ATENA 3D environment. It describes the 3D finite element library, the meshing rules and boundary conditions adopted, the concrete fracture–plastic constitutive model, the modelling of conventional reinforcement and prestressing tendons (bonded and unbonded), the fixed and rotated smeared crack formulations, the regularisation through the Crack Band Method, the treatment of the fracture energy at the three prescribed levels, the shear retention factor and the chosen iterative solver. The six numerical scenarios used in the parametric study are formally defined at the end of the chapter.

**Chapter 4 – Experimental benchmark.** The experimental benchmark database is presented in detail. For each of the six source references [2–7], the geometric dimensions, the reinforcement layout, the prestress configuration, the material properties and the test setup (three-point or four-point bending; load- or displacement-controlled) are summarised in a consistent format, together with the experimental load–displacement curves that will serve as ground truth for the numerical validation.

**Chapter 5 – NLFEA of PRC beams: parametric study.** This chapter applies the numerical framework of Chapter 3 to the benchmark of Chapter 4. The six numerical scenarios are executed for each specimen, the predicted load–displacement curves are compared with the experimental ones, the influence of the crack model and of the fracture energy is critically discussed, and the sensitivity of the response to the shear retention factor is examined. The chapter concludes with a synthesis of the predictive performance of the six solution strategies.

**Chapter 6 – Model uncertainty quantification and conclusions.** The final chapter is devoted to the probabilistic post-processing of the results. The model uncertainty ratio  $\vartheta$  is computed for each specimen and each scenario; its statistical properties are characterised through descriptive statistics and goodness-of-fit tests; the partial safety factor  $\gamma_{Rd}$  is calibrated through a Bayesian updating procedure consistent with [11, 12]; the influence of the experimental uncertainty is investigated; the obtained values are compared with the recommendations of EC [9], *fib* MC2010 [1] and NTC [10]; the main conclusions of the work are drawn and recommendations are formulated both for engineering practice and for the calibration of the next generation of structural design standards.

# Chapter 2

## Theoretical Background

This chapter consolidates the theoretical foundations on which the thesis is built. It opens with the mechanical behaviour of PRC beams, with an explicit distinction between bonded and unbonded internal tendon systems, and progresses to the constitutive description of the materials involved (concrete, conventional reinforcement, prestressing steel). The fundamentals of NLFEA are then introduced – sources of non-linearity, solution algorithms, smeared crack approaches and the treatment of the shear retention factor – with particular reliance on the work of Červenka and co-workers, since the numerical campaign of Chapters 3–6 is carried out in the ATENA 3D environment. The chapter closes with the safety-assessment framework adopted for non-linear numerical analysis, namely the GRF, the safety formats embedded in it, the statistical treatment of model uncertainty and the current provisions in *fib* MC2010 [1], EC [9] and NTC [10].

### 2.1 Prestressed reinforced concrete: structural behaviour

#### 2.1.1 Bonded and unbonded internal tendon systems

In PRC members the prestress is delivered to the concrete by high-strength steel tendons that are tensioned either before (*pre-tensioning*) or after (*post-tensioning*) the casting of the surrounding concrete. A crucial structural distinction arises depending on whether a continuous mechanical interaction exists between the tendons and the concrete along their entire length:

- In *bonded* systems – pre-tensioned members, or post-tensioned members in which the duct is grouted with cementitious mortar after stressing – the relative slip between tendon and surrounding concrete is restrained by adhesion, friction and mechanical interlock. Strain compatibility is therefore enforced *locally* along the member, and any increment of sectional curvature generates a corresponding stress increment in the tendon [13, 14].
- In *unbonded* systems – typically post-tensioned members with greased and sheathed mono-strands or external tendons – the tendon is free to slide longitudinally inside its duct, and a continuous slip between tendon and concrete occurs. Strain compatibility cannot be enforced locally; the stress increase in the tendon at

ULS is dictated by the *global* elongation of the cable between the two anchorages [15, 16].

The construction sequences of the two prestressing technologies (pre- and post-tensioning) have already been illustrated in fig. 1.1 of Chapter 1 and are not repeated here.

The two systems lead to substantially different non-linear responses at ULS, both in terms of ultimate flexural capacity and ductility, as discussed quantitatively in section 2.2.3.

### 2.1.2 Stress states induced by prestressing and external loading

The structural behaviour of a PRC beam under increasing external load can be conveniently described through the evolution of the stress state in the tensile chord. Immediately after transfer of the prestressing force  $P_0$ , the section is subjected to a self-equilibrating internal force system that produces a triangular compressive stress distribution (with  $\sigma_{cp, \text{btm}} > \sigma_{cp, \text{top}}$ , depending on the tendon eccentricity  $e$ ). At any subsequent stage the concrete stress at the extreme tensile fibre can be written, using the elementary beam-theory superposition [13, 14],

$$\sigma_c(y) = -\frac{P_e}{A_c} - \frac{P_e e \cdot y}{I_c} + \frac{M_{\text{ext}}(x) \cdot y}{I_c}, \quad (2.1)$$

where  $A_c$  and  $I_c$  are the area and second moment of the (uncracked) concrete section,  $P_e$  is the effective prestress after immediate and time-dependent losses,  $e$  is the tendon eccentricity,  $y$  is the distance from the centroid and  $M_{\text{ext}}$  is the bending moment produced by the external load. The first two terms are the contribution of the prestress alone, while the third term is the moment-induced bending stress. Sign convention adopts compression as negative.

A particularly significant load level is the *decompression load*, i.e. the value of  $M_{\text{ext}}$  that brings the concrete stress at the tensile fibre to zero; beyond it the section gradually transitions from the uncracked to the cracked regime. Time-dependent phenomena – creep and shrinkage of the concrete, relaxation of the prestressing steel, possible friction losses in post-tensioned ducts – progressively reduce  $P_e$  during the service life of the structure, and must be accounted for either explicitly or through code-prescribed loss coefficients [1, 9].

### 2.1.3 Phases of the flexural response

Increasing the external load monotonically up to failure, a typical PRC beam in flexure exhibits four characteristic phases on the load–displacement curve  $P$ – $\delta$  illustrated in fig. 2.1:

1. **Uncracked-elastic phase.** The full cross-section participates in resisting external actions; the response is linear and the stiffness corresponds to the gross-section stiffness  $E_c I_c$ .

2. **Cracking phase.** When the tensile stress at the extreme fibre reaches the tensile strength of the concrete  $f_{ct}$ , the first flexural crack appears. The cracking load  $P_{cr}$  is markedly higher than in an equivalent RC beam because the pre-compression delays the attainment of  $f_{ct}$ .
3. **Cracked-elastic / yielding phase.** As loading proceeds, cracks propagate towards the compression chord and the neutral axis migrates upwards; the conventional reinforcement and the prestressing tendons experience increasing strain demand. The stiffness drops to the cracked-section value  $E_c I_{cr}$  and the response remains essentially linear until either the conventional reinforcement yields or the tendons depart significantly from their effective prestress.
4. **Ultimate phase.** Plasticity spreads in the tensile chord, the compressive fibre approaches the crushing strain  $\varepsilon_{cu} \approx 3.5 \times 10^{-3}$  and the curve reaches the ultimate load  $P_u$ . Depending on the relative amount of conventional reinforcement, prestressing steel and on the bond conditions, the response may exhibit a pronounced ductile plateau (under-reinforced section) or a brittle drop (over-reinforced section) [16, 17].

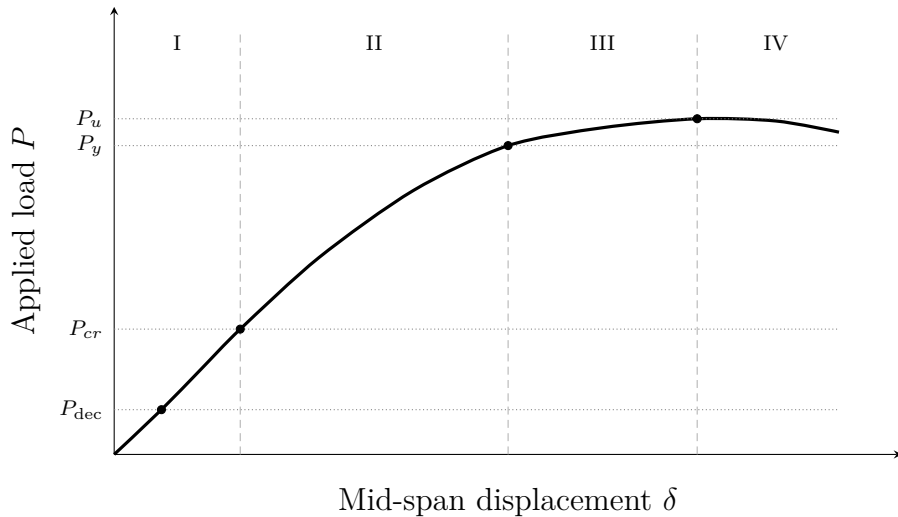


**Figure 2.1:** Idealised flexural load–displacement response of a PRC beam highlighting the four characteristic phases: uncracked elastic, cracking, cracked-elastic / yielding, and ultimate. The decompression load  $P_{dec}$ , the cracking load  $P_{cr}$ , the yielding load  $P_y$  and the ultimate load  $P_u$  are indicated. The qualitative response follows the classical description of prestressed-concrete flexure [13].

## 2.2 Flexural failure mechanisms of PRC beams

The flexural failure of a PRC beam is governed by the strain compatibility and the force equilibrium across the cracked section at ultimate. Three basic mechanisms can be distinguished, all sharing the same underlying balance between the tensile force carried by the steel (conventional and prestressing) and the compressive force carried by the concrete chord, but differing in which component reaches its limiting state first.

### 2.2.1 Concrete crushing in the compression chord

When the prestressing ratio and the conventional reinforcement ratio are high (over-reinforced section), the compression chord reaches the ultimate strain  $\varepsilon_{cu}$  before the tensile steel yields significantly. This produces a brittle crushing failure with little visible warning, a small post-peak displacement and a load-drop that is essentially governed by the compressive softening of the concrete [1, 17]. For sections of this type the available rotation capacity is small and the ratio between  $P_u$  and  $P_y$  approaches unity.

### 2.2.2 Tendon yielding and possible rupture

In under-reinforced sections, the tendons (and the conventional reinforcement) yield substantially before the compression chord crushes, leading to a ductile failure with a wide plastic plateau on the  $P$ - $\delta$  curve. If the deformation capacity of the prestressing steel is fully exhausted, the failure ultimately occurs by tendon rupture; the characteristic absence of a clear yield plateau in high-strength prestressing strands implies that the ultimate strain of the tendon is the controlling parameter [13].

### 2.2.3 Influence of the bond condition on capacity and ductility

The bond condition between tendon and concrete is one of the most influential parameters of the flexural response. In bonded members the local strain compatibility ensures that the stress increase  $\Delta\sigma_p$  in the tendon at ULS can be estimated from the curvature of the section,

$$\Delta\sigma_{p,\text{bonded}} = E_p \Delta\varepsilon_p = E_p (\chi_u d_p - \varepsilon_{cu}), \quad (2.2)$$

where  $E_p$  is the tendon modulus,  $\chi_u$  the ultimate curvature and  $d_p$  the effective depth of the tendon. In unbonded members, however, the tendon stress increment depends on the average elongation between the two anchorages [15, 16],

$$\Delta\sigma_{p,\text{unb}} = \frac{E_p}{L} \int_0^L \Delta\varepsilon_{c,p}(x) dx, \quad (2.3)$$

where  $\Delta\varepsilon_{c,p}(x)$  is the strain change at the level of the tendon and  $L$  the cable length. The strain integral in eq. (2.3) is typically much smaller than the local strain in eq. (2.2), so that unbonded tendons may not develop their full nominal strength. As a consequence, unbonded PRC beams tend to exhibit: (i) a lower flexural capacity than equivalent bonded members; (ii) fewer but wider cracks, often concentrated near the loading points; (iii) a larger displacement at ultimate but a less pronounced plastic plateau. These features are explicitly captured by the numerical models of Chapters 5 and 6.

## 2.3 Constitutive behaviour of the materials

The accuracy of any NLFEA prediction is intrinsically linked to the constitutive laws assigned to concrete, conventional reinforcement and prestressing steel. This section reviews the formulations adopted in the present thesis, all of which are implemented in

ATENA 3D through the `CC3DNonLinCementitious2` concrete model and the standard discrete-truss representation of the reinforcement [18, 19].

### 2.3.1 Concrete in compression and in tension

**Compressive behaviour.** The compressive response of concrete is modelled through a plasticity-based formulation with isotropic hardening followed by strain-softening. The failure surface in the principal-stress space is the three-parameter surface of Men  trety and Willam [20], which is smooth, convex, and able to reproduce both the meridian and the deviatoric sections of the concrete strength envelope. In ATENA, the compressive stress–strain law is described by a parabolic ascending branch up to the peak strain  $\varepsilon_{c1}$  and by a linear-descending branch in the post-peak regime, calibrated to dissipate a prescribed crushing energy  $G_c$  over a band whose width is computed from the element size, in analogy with the Crack Band Model used in tension (see below). The peak stress is reduced in the presence of large transverse tensile strains through the Vecchio–Collins compression-softening factor, which is critical for capturing the correct collapse load of members with intense diagonal cracking [18].

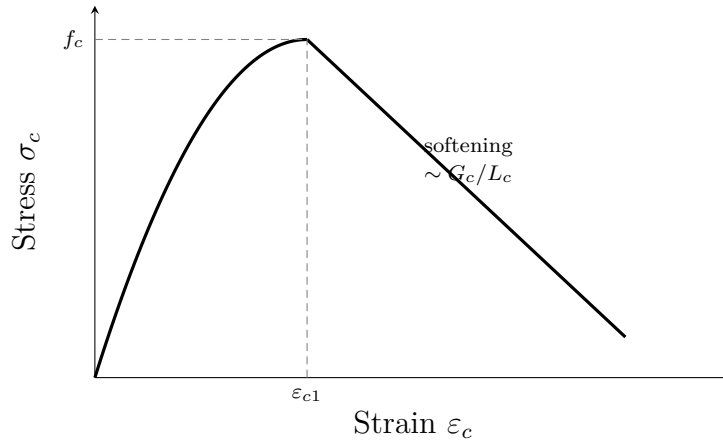


**Figure 2.2:** Uniaxial compressive stress–strain law adopted for the concrete (`CC3DNonLinCementitious2` model). Parabolic ascending branch up to the peak ( $\sigma_c = f_c$  at  $\varepsilon_{c1}$ ) and linear softening branch; the area under the post-peak curve is regularised through the crushing energy  $G_c$  and the characteristic element size  $L_c$ . Formulation after the `CC3DNonLinCementitious2` model with the Men  trety–Willam failure surface [18, 20].

**Tensile behaviour and fracture energy.** Up to the tensile strength  $f_{ct}$  the response is linear elastic; beyond this threshold, the concrete enters a softening regime governed by non-linear fracture mechanics. The total energy required to propagate a unit area of a stress-free crack, the *specific fracture energy*  $G_f$ , is the cornerstone of the formulation [21, 22]. For a localised fictitious crack,  $G_f$  is the area under the post-peak stress–crack-opening curve,

$$G_f = \int_0^{w_c} \sigma(w) dw, \quad (2.4)$$

where  $\sigma(w)$  is the cohesive stress transmitted across the crack of width  $w$  and  $w_c$  is the crack opening at which the cohesive stress vanishes. In a finite element context

this localised description must be transformed into a strain-based softening law to be compatible with the smeared crack approach (section 2.4.3). The transformation is performed via the *Crack Band Method* of Bažant and Oh [22], in which the softening law is written in terms of the equivalent crack opening  $w = \varepsilon_{cr} L_t$ , with  $L_t$  a characteristic length related to the size of the finite element along the direction of cracking. The crack-band regularisation ensures that the energy dissipated per unit area of crack remains equal to  $G_f$  independently of the mesh size, eliminating the well-known mesh dependency of strain-softening finite element analyses [23, 24]. The softening branch itself is in the present work described by the exponential law of Hordijk [25], which is widely adopted in the Červenka family of models [18].

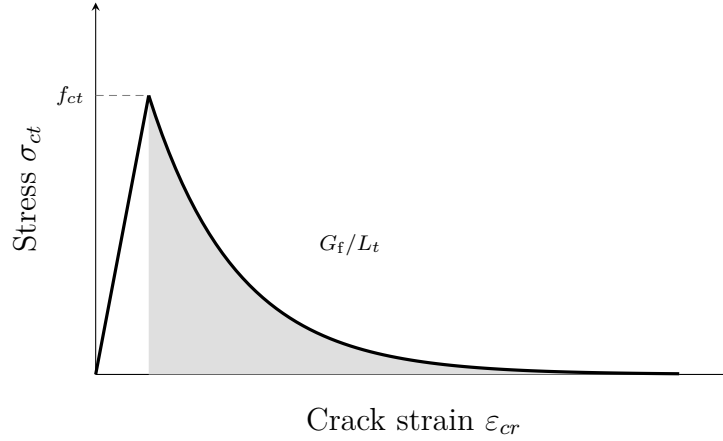


**Figure 2.3:** Uniaxial tensile response of concrete: linear elastic branch up to the tensile strength  $f_{ct}$ , followed by an exponential softening branch (Hordijk law). The shaded area below the softening curve, multiplied by the characteristic element size  $L_t$ , equals the specific fracture energy  $G_f$ . Softening law after Hordijk, regularised through the Crack Band Method [22, 25].

**Tension stiffening.** The bond between the reinforcement and the surrounding concrete allows the latter to carry residual tensile stresses between two consecutive cracks, an effect known as *tension stiffening*. In the smeared formulation adopted here, tension stiffening is included by modifying the descending branch of the concrete tensile law to leave a residual tensile stress  $\sigma_{ct}^{ts} = c_{ts} f_{ct}$  on the cracked element, with  $c_{ts}$  a scalar parameter normally taken in the range 0.4–0.6 [1, 18].

### 2.3.2 Conventional and prestressing steel

The non-prestressed (mild) reinforcement is modelled with a bilinear elastic–plastic stress–strain law with kinematic hardening, governed by the Von Mises yield criterion (fig. 2.4). The Young’s modulus  $E_s$ , the yield strength  $f_y$ , the ultimate strength  $f_t$  and the ultimate strain  $\varepsilon_{su}$  are prescribed on a specimen-by-specimen basis from the values reported in the source experimental references (Chapter 4).

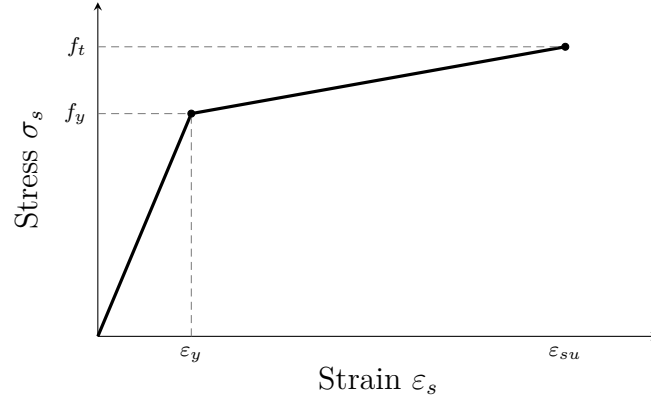


**Figure 2.4:** Bilinear stress–strain law with kinematic hardening adopted for the conventional steel reinforcement. Idealisation according to [1, 9].

Prestressing strands and wires, on the other hand, do not exhibit a clear yield plateau. A multi-linear stress–strain relationship is therefore adopted, which reproduces the gradual transition between the elastic and the strain-hardening regimes (fig. 2.5) [1, 13]. The standard reference points are the proof stress at 0.1% residual strain,  $f_{p,0.1k}$ , the characteristic ultimate strength  $f_{pk}$  and the ultimate strain  $\varepsilon_{pu}$ , all taken from the experimental tests when available or from the manufacturer’s data sheet.

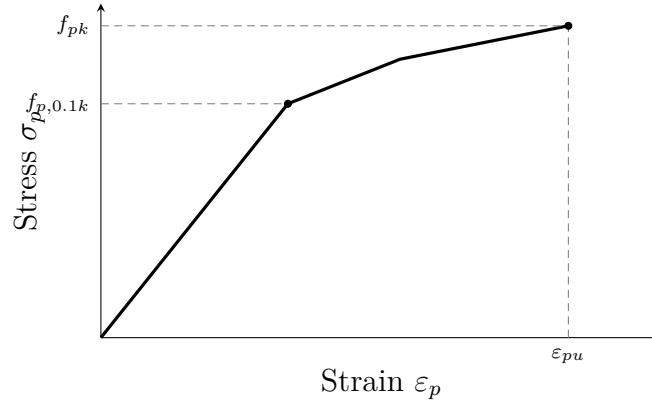


**Figure 2.5:** Multi-linear stress–strain law adopted for the prestressing strands: linear elastic up to  $f_{p,0.1k}$ , nonlinear transition zone and quasi-linear hardening up to  $f_{pk}$  at the ultimate strain  $\varepsilon_{pu}$ . Idealisation according to [1, 13].

### 2.3.3 Bond–slip and tendon–concrete interaction

In bonded systems the tendon–concrete interaction can be modelled either with a *perfect bond* assumption – the degrees of freedom of the tendon are kinematically constrained to those of the surrounding concrete elements – or with an explicit bond–slip law. In the present work the perfect bond assumption is used for pre-tensioned strands embedded in their final position, consistent with common practice in the smeared crack framework [1, 19]. For bonded post-tensioned tendons, a bilinear bond–slip law of *fib* MC2010-type is adopted, characterised by an ascending branch up to the bond strength  $\tau_{b,max}$  at the slip  $s_1$  and a descending branch towards the residual frictional bond  $\tau_{b,res}$  at the slip  $s_3$  (fig. 2.6).

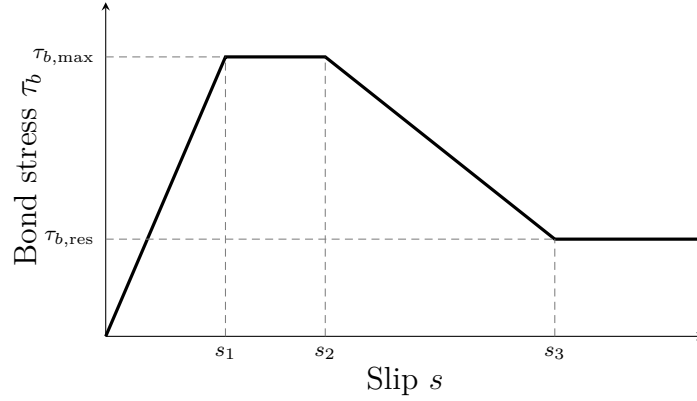


**Figure 2.6:** Schematic bilinear bond–slip law adopted for bonded post-tensioned tendons, according to the formulation of *fib* MC2010 [1].

For unbonded post-tensioned tendons the local bond is suppressed and the tendon transmits its prestress to the concrete only through the end anchorages. In the finite element model, the unbonded tendon is represented by a single truss element connecting the two anchor points, with a uniform stress along its length that depends on the average elongation of the member [15, 16].

## 2.4 Fundamentals of non-linear finite element analysis

### 2.4.1 Sources of non-linearity

Three sources of non-linearity are typically distinguished in the analysis of RC and PRC structures [26, 27]:

- *Material non-linearity*, originating from the stress–strain laws of concrete (cracking, crushing, softening, tension stiffening) and steel (yielding, hardening).
- *Geometric non-linearity*, due to large displacements and/or large rotations, including second-order  $P$ - $\Delta$  effects in slender members. For the rectangular, simply-supported, short-to-medium-span PRC beams investigated in this thesis, geometric non-linearity is small and is included only when it improves the convergence of the solver.
- *Boundary non-linearity*, associated with the modification of support conditions during the loading process (e.g. unilateral contact at the supports and at the loading plates).

The discretised governing equations of the structure can be cast in the residual form

$$\mathbf{r}(\mathbf{u}) = \mathbf{f}_{\text{ext}} - \mathbf{f}_{\text{int}}(\mathbf{u}) = \mathbf{0}, \quad (2.5)$$

in which  $\mathbf{u}$  collects the nodal displacements,  $\mathbf{f}_{\text{ext}}$  is the vector of external nodal forces and  $\mathbf{f}_{\text{int}}(\mathbf{u})$  is the vector of internal nodal forces, intrinsically non-linear in  $\mathbf{u}$ .

### 2.4.2 Solution algorithms: Newton–Raphson and continuation methods

The solution of eq. (2.5) is obtained incrementally and iteratively. In the *full Newton–Raphson* method the global tangent stiffness matrix  $\mathbf{K}_T(\mathbf{u}) = \partial \mathbf{f}_{\text{int}} / \partial \mathbf{u}$  is recomputed at every iteration  $i$  of every load step  $n$ , and the displacement increment is obtained from

$$\mathbf{K}_T(\mathbf{u}_n^{(i)}) \Delta \mathbf{u}_n^{(i+1)} = \mathbf{r}(\mathbf{u}_n^{(i)}), \quad (2.6)$$

with  $\mathbf{u}_n^{(i+1)} = \mathbf{u}_n^{(i)} + \Delta \mathbf{u}_n^{(i+1)}$ . The iteration is repeated until the residual norm  $\|\mathbf{r}\|$ , the displacement increment norm  $\|\Delta \mathbf{u}\|$  and the energy norm fall below prescribed tolerances (typically  $10^{-2}$  for force/displacement and  $10^{-4}$  for energy, in line with the ATENA recommended defaults [18]). The full Newton–Raphson scheme attains a quadratic rate of convergence in a neighbourhood of the solution, but is computationally expensive because of the repeated factorisation of  $\mathbf{K}_T$ . The *modified Newton–Raphson* variant updates the tangent stiffness less frequently and is preferred when the response is moderately non-linear; the *arc-length* continuation method of Crisfield is required when the equilibrium path exhibits snap-back or strong softening, although it is not needed for the flexural benchmarks of the present thesis.

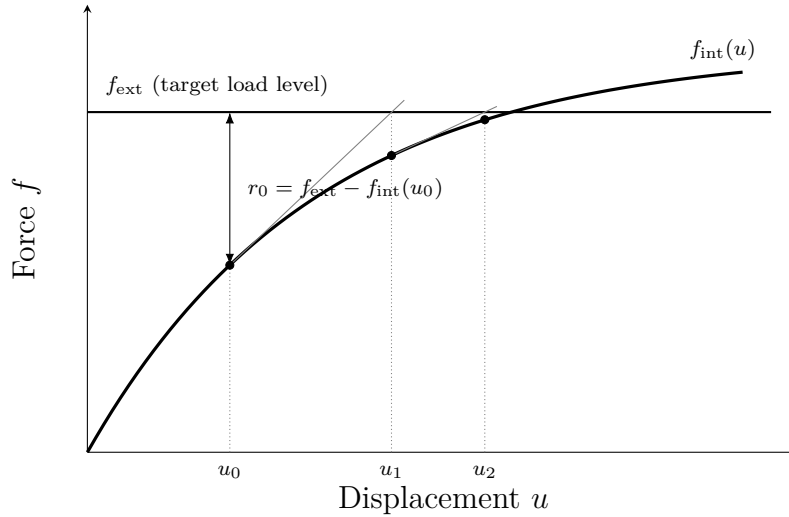


**Figure 2.7:** Schematic representation of the incremental-iterative Newton–Raphson scheme used to solve the non-linear system (2.5). Within each load step, the tangent stiffness  $\mathbf{K}_T$  is updated at every iteration and the residual  $\mathbf{r}$  is driven below the prescribed tolerance. Standard incremental-iterative formulation [18, 27].

### 2.4.3 The smeared crack approach: fixed and rotated formulations

Two main strategies exist for representing the discontinuous nature of concrete cracking inside a continuum finite element framework [23, 28, 29]:

- the *discrete crack* approach, in which cracks are modelled as actual geometric discontinuities along element boundaries – accurate for a single localised crack but cumbersome for the multi-crack patterns typical of PRC beams;

- the *smeared crack* approach, in which the effect of a crack is distributed (smeared) over the entire volume of the finite element through an anisotropic modification of the local constitutive matrix.

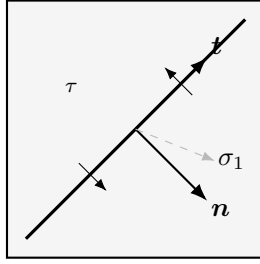
In the present work the smeared crack approach is adopted, since it is the natural choice for the diffuse cracking pattern produced by flexural loading in PRC beams and it is the formulation implemented in the Červenka family of constitutive models [18, 19]. Upon cracking, the material is treated as orthotropic with the local axes  $(n, t)$  aligned with the crack normal and tangent, respectively (fig. 2.8).

**Fixed crack model.** In the fixed crack formulation, the direction of the crack is determined at the moment of crack initiation by the orientation of the principal stress and remains *frozen* during the subsequent loading history [23, 28]. A shear stress can develop on the crack surface and is computed through the so-called *shear retention factor*  $\beta \in [0, 1]$ , which multiplies the elastic shear modulus,

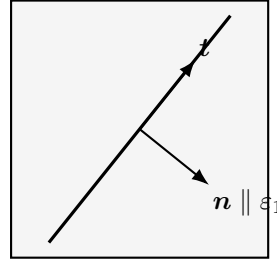
$$G_{cr} = \beta G_c. \quad (2.7)$$

A typical value  $\beta = 0.05\text{--}0.2$  is recommended for RC [18]. If  $\beta$  is overestimated, however, the crack surface transmits an unrealistically large shear stress and the principal stress can rotate away from the locked crack normal, generating a spurious increase of the structural capacity known as *stress locking* [23, 24]. The sensitivity of the predicted response to  $\beta$  is investigated explicitly in Chapter 5.

**Rotated crack model.** In the rotated crack formulation, the direction of the crack *rotates* continuously with the principal strain axes [23, 30]. The orthotropic constitutive matrix is updated at every iteration so that the crack normal coincides with the current direction of the principal tensile strain. By construction, no shear stress is transmitted across the crack plane: the rotated formulation is therefore intrinsically free from stress locking. On the other hand, it implicitly assumes that the crack heals and re-opens in a different direction, which is physically reasonable for the diffuse cracking of RC and PRC members in flexure but becomes a strong idealisation under load reversals.

**Fixed crack model**

crack fixed at initiation;  
shear  $\tau = \beta G_c \gamma$  transmitted

**Rotated crack model**

crack rotates with principal strain;  
no shear transmitted across crack

**Figure 2.8:** Schematic comparison of the fixed (left) and rotated (right) smeared crack models. In the fixed formulation the crack normal  $\mathbf{n}$  is set at the moment of initiation and does not change; a shear stress proportional to the shear retention factor  $\beta$  is transmitted across the crack. In the rotated formulation the crack normal aligns at all times with the principal tensile-strain direction and no shear stress is transmitted across the crack. After the smeared crack formulations of [18, 23].

## 2.5 Safety assessment of structures using NLFEA

The transition from a linear elastic analysis followed by sectional verification to a fully non-linear numerical analysis at the global level requires a parallel evolution of the safety assessment framework. This section reviews the basic concepts of the GRF, the safety formats embedded in it, the statistical treatment of model uncertainty and the way in which current standards address these aspects.

### 2.5.1 Local versus global approach

The traditional code-based verification format adopted, for example, in EC [9] and in NTC [10] is the *local approach*, recalled in eq. (1.1) of Chapter 1: a linear elastic global analysis is combined with a non-linear sectional verification on the critical sections, and the design values of the internal actions are compared with the sectional resistances computed under simplified hypotheses. The verification is therefore strictly sectional, and the safety is assessed by reducing the material strengths through partial factors applied at the input stage.

The *global approach*, conversely, performs a non-linear analysis of the entire structure with representative material properties and compares the global external action  $F_d$  with the global resistance  $R_d$ , as recalled in eq. (1.2). Because the principle of superposition does not hold in non-linear analysis, the partial factors cannot be applied at the input stage and must be moved to the global resistance through suitable safety formats [8, 19].

The contrast between the local and global verification approaches has already been illustrated in fig. 1.2 of Chapter 1 and is not repeated here.

### 2.5.2 The Global Resistance Format

The GRF introduced by the *fib* MC2010 [1] and later generalised by Castaldo *et al.* [8] expresses the global design resistance as

$$R_d = \frac{R(\bar{X})}{\gamma_{Rd} \gamma_R}, \quad (2.8)$$

where  $R(\bar{X})$  is the global resistance computed by NLNA with representative values  $\bar{X}$  of the random variables (typically mean strengths),  $\gamma_{Rd}$  is the partial safety factor accounting for the epistemic uncertainty of the numerical model (the central focus of this thesis) and  $\gamma_R$  collects the partial factor associated with the aleatory variability of the geometric and mechanical properties, formulated differently in each specific safety format.

### 2.5.3 Safety formats embedded in the GRF

Within the umbrella of GRF several specific safety formats coexist, each characterised by a different compromise between accuracy and computational effort [1, 8, 12]. The four most relevant for the present work are summarised hereafter.

**Probabilistic method.** The probabilistic method represents the most rigorous safety assessment. A full reliability analysis is performed by propagating the probability distributions of all the random variables (material strengths, geometric dimensions, prestress, model uncertainty) through the NLNA response. Although it provides the closest agreement with the target reliability index  $\beta$  defined by the codes, the method is computationally expensive and is typically used as a benchmark rather than as an operational design tool [1, 8].

**Global Resistance Factor method (GRFm).** In the Global Resistance Factor method (GRFm) the design resistance is obtained from two deterministic NLNA runs performed with mean and design material strengths, respectively, combined through a closed-form expression calibrated against probabilistic results [19]. The method is simple to apply but is sensitive to the choice of the representative material values and to the global coefficient of variation  $V_R$ .

**Estimate of Coefficient of Variation method (ECoV).** The Estimate of Coefficient of Variation method (ECoV) method, originally proposed by Červenka and co-workers [19], infers the coefficient of variation of the global resistance  $V_R$  from the ratio between the mean and the characteristic resistances obtained from two NLNA runs,

$$V_R = \frac{1}{1.65} \ln\left(\frac{R_m}{R_k}\right), \quad (2.9)$$

under the hypothesis that the global resistance follows a lognormal distribution. The design resistance is then obtained as

$$R_d = \frac{R_m}{\gamma_{Rd} \exp(\alpha_R \beta V_R)}, \quad (2.10)$$

with  $\alpha_R \approx 0.8$  and  $\beta = 3.8$  for the ULS verification on a 50-year reference period [1].

**Strain-based method.** The strain-based method recently proposed by Gino *et al.* [12] provides a more accurate estimation of the global safety factors by recognising that the variability of the global resistance depends on the failure mode actually mobilised by the structure (flexural, shear, instability). The method introduces a different  $\gamma_R$  for each failure mode, calibrated through extensive probabilistic analyses, and combines it with a  $\gamma_{Rd}$  derived from the model uncertainty distribution of the same failure mode.

A comparative discussion of the four methods, with explicit application to RC structures, is provided by Castaldo *et al.* [8] and Gino *et al.* [11, 12]. The present thesis extends the same conceptual framework to PRC flexural members, with particular emphasis on the calibration of  $\gamma_{Rd}$  (section 2.5.4).

#### 2.5.4 Definition and statistical treatment of model uncertainty

As anticipated in section 1.3.2 of Chapter 1, the model uncertainty is here defined as the random variable

$$\vartheta = \frac{R_{\text{exp}}}{R_{\text{num}}}, \quad (2.11)$$

i.e. the ratio between the experimentally observed ultimate resistance and the numerical resistance predicted by the NLFEA model of the same specimen. By applying eq. (2.11) to a sample of benchmark specimens, the statistical properties of  $\vartheta$  – mean  $\mu_\vartheta$ , coefficient of variation  $V_\vartheta$ , and the probability distribution itself – can be estimated.

Following the literature [1, 8, 11, 12],  $\vartheta$  is in general assumed to follow a *lognormal* distribution,

$$\ln \vartheta \sim \mathcal{N}(\lambda_\vartheta, \zeta_\vartheta^2), \quad (2.12)$$

with  $\lambda_\vartheta$  and  $\zeta_\vartheta$  the parameters of the underlying normal distribution. The hypothesis is tested through standard goodness-of-fit techniques (Chi-squared, Anderson–Darling, Lilliefors, Jarque–Bera) in Chapter 6. The partial safety factor  $\gamma_{Rd}$  is then derived from the characteristic value of  $\vartheta$  at the prescribed reliability level,

$$\gamma_{Rd} = \exp(\alpha_R \beta \zeta_\vartheta) / \mu_\vartheta, \quad (2.13)$$

in which  $\alpha_R \beta$  is the design fractile factor consistent with the chosen ULS target reliability [1].

A Bayesian updating framework, fully developed in Chapter 6, allows the posterior distribution of  $\vartheta$  to be obtained by combining prior information (typically a wide, weakly-informative prior on  $\mu_\vartheta$  and  $V_\vartheta$ ) with the experimental–numerical evidence collected on the benchmark specimens, following the approach of Gino *et al.* [11, 12].

#### 2.5.5 Current code provisions and motivation for the thesis

The treatment of  $\gamma_{Rd}$  in current standards is still based on a relatively limited set of benchmark data, predominantly on RC members. *fib* MC2010 [1] suggests indicative values in the range  $\gamma_{Rd} \in [1.05, 1.10]$  for well-validated NLFEA predictions, and up to

$\gamma_{Rd} \approx 1.20$  when the modelling uncertainty is considered higher. EC [9] does not provide an explicit value for  $\gamma_{Rd}$  calibrated against NLFEA, but embeds the model uncertainty in the more general partial-factor format applied to material strengths. NTC [10], in line with the European framework, implicitly accepts the values recommended by the *fib* MC2010 for the non-linear assessment of existing structures.

The reference values recommended by these standards have been derived mainly from RC benchmarks (slender columns, beams in flexure and shear), and the dependence of  $\gamma_{Rd}$  on the modelling assumptions (crack model, fracture energy, shear retention factor) has been explored in detail only recently [8, 11, 12]. A systematic calibration of  $\gamma_{Rd}$  for PRC beams failing in flexure, in which the additional layer of complexity introduced by the active prestress and by the bonded/unbonded interaction is explicitly taken into account, is to the best of the candidate's knowledge still missing in the open literature. This is precisely the gap that the present thesis aims to fill, by applying the framework summarised above to the benchmark database of Chapter 4 and to the six numerical scenarios defined in Chapter 3.

# Chapter 3

## Numerical Modelling Framework in ATENA 3D

The theoretical concepts introduced in Chapter 2 – constitutive description of concrete and steel, smeared crack approach, fracture energy regularisation and Newton–Raphson solution – are translated in the present chapter into the concrete implementation choices that constitute the numerical backbone of the thesis. All non-linear simulations are performed in the ATENA 3D environment developed by Červenka Consulting [18, 19], a software package specifically tailored to the analysis of reinforced and prestressed concrete structures. The chapter is conceived as a reproducible description of the modelling protocol: the type and resolution of the mesh, the constitutive laws assigned to each material, the modelling strategy adopted for bonded and unbonded tendons, the formulations of the fixed and rotated smeared crack models, the three levels of fracture energy  $G_f$ , the iterative solver settings and the convergence criteria. The chapter closes with the formal definition of the six numerical scenarios that are then applied to every specimen of the benchmark database in Chapter 5.

### 3.1 Overview of the ATENA 3D environment

ATENA (*Advanced Tool for Engineering Non-linear Analysis*) is a finite element package developed by Červenka Consulting since the late 1990s with the explicit aim of providing a reliable numerical platform for reinforced and prestressed concrete structures [18, 26]. Compared with general-purpose codes such as ABAQUS or ANSYS, ATENA provides a built-in library of fracture–plastic constitutive models and pre-coded smeared crack formulations, removing the need for user-defined subroutines and ensuring that the numerical response is consistent with the formulations validated in the literature [19]. The 3D Engineering environment used in the present work supports triaxial stress states and confinement effects that cannot be captured by 2D plane-stress idealisations – a feature that becomes essential when modelling prestressing tendons embedded in a 3D concrete volume.

### 3.1.1 Pre-processing, analysis and post-processing

The work-flow adopted to build each NLFEA model in ATENA 3D is divided into three sequential phases, schematically illustrated in fig. 3.1:

1. **Pre-processing.** The 3D geometry of the beam is defined through macro-elements (concrete volumes, loading and support plates), the reinforcement layout (longitudinal bars, stirrups, prestressing tendons) is inserted as discrete line elements, the material properties are assigned to each macro-element and the boundary conditions (supports, loading plates, monitoring points) are specified. The finite element mesh is generated automatically with the element size prescribed by the user.
2. **Analysis.** The non-linear solver assembles the global tangent stiffness matrix, applies the prestressing action and the external load through user-defined load cases, and solves the incremental-iterative non-linear system through the Newton-Raphson algorithm. At each load step, the convergence is checked on the force, displacement and energy norms.
3. **Post-processing.** Once the analysis is completed, the load-displacement response is extracted at the monitoring points; the crack pattern, the principal stresses and the strain fields are visualised in 3D and exported as required.

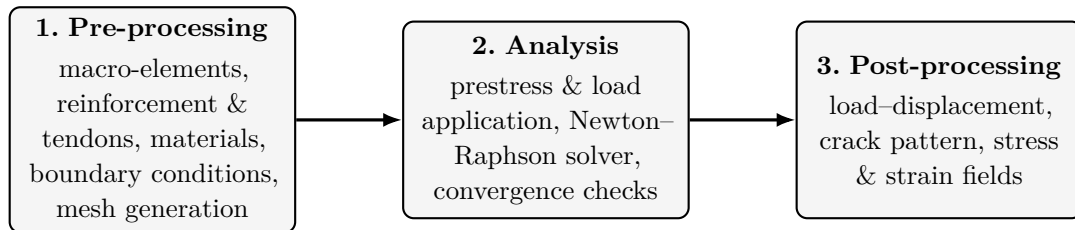


**Figure 3.1:** Schematic of the ATENA 3D work-flow adopted in the present thesis: pre-processing of the macro-elements, reinforcement and boundary conditions; non-linear incremental-iterative analysis; post-processing of load-displacement, crack pattern and stress-strain fields. Work-flow of the ATENA 3D environment [18].

### 3.1.2 Finite element library and meshing rules

The concrete macro-elements are discretised with eight-node isoparametric brick (hexahedral) elements (CCLsoBrick), which provide a good compromise between geometric flexibility and computational cost for prismatic members such as rectangular PRC beams [18]. Where the geometry of the specimen requires it, the brick mesh is locally complemented by tetrahedral elements, ensuring full compatibility at the interfaces.

**Global mesh size.** A global element size of  $L_e = 0.05$  m is adopted for the concrete volume of every specimen, on the basis of a preliminary sensitivity analysis aimed at balancing accuracy and computational cost. This resolution is fine enough to:

(i) capture the propagation of flexural cracks across the depth of the beam; (ii) resolve the stress concentrations close to the loading and support plates; (iii) remain consistent with the Crack Band Method regularisation, since the characteristic element size  $L_t = L_e = 0.05$  m enters the softening law of the concrete as discussed in section 2.3 of Chapter 2.

**Reinforcement discretisation.** Longitudinal bars, stirrups and prestressing tendons are modelled through discrete one-dimensional truss elements (**CCIsoTruss**) embedded in the concrete mesh. The truss nodes are kinematically constrained to the surrounding concrete nodes either through perfect bond (section 3.2.3) or through an explicit bond-slip law (section 3.2.3).

**Loading and support plates.** Rigid steel plates are explicitly modelled at the loading points and at the supports to spread the concentrated reactions over a finite contact area and to avoid spurious stress concentrations. The plates are connected to the concrete macro-elements through master-slave contact constraints, which prevent penetration but allow free uplift.

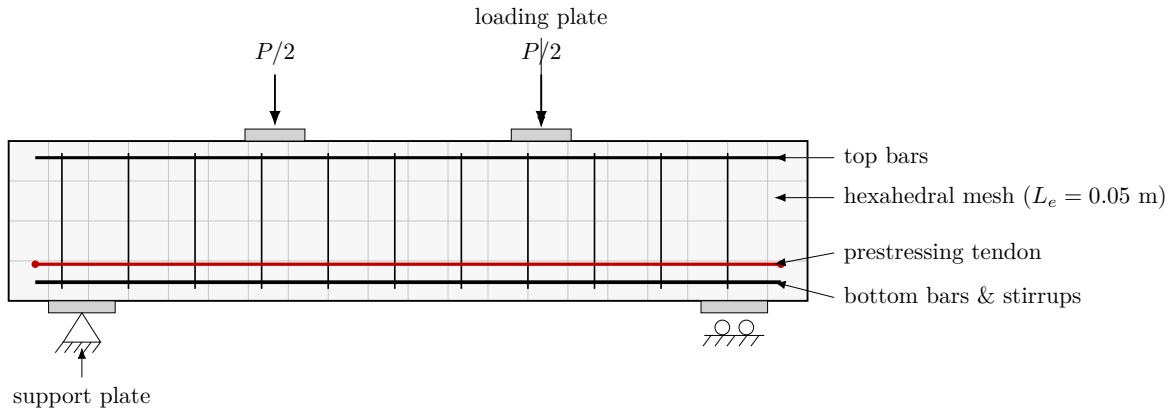


**Figure 3.2:** Schematic elevation of a typical rectangular PRC beam model adopted in ATENA 3D, showing the hexahedral mesh of the concrete volume ( $L_e = 0.05$  m), the embedded top and bottom longitudinal bars and stirrups, the prestressing tendon with its end anchorages, and the rigid loading and support plates (four-point bending example). Model built in ATENA 3D [18].

## 3.2 Constitutive models adopted

This section details the constitutive laws assigned to concrete and steel in the ATENA 3D environment. The theoretical foundations have been discussed in section 2.3 of Chapter 2; here the emphasis is on the specific ATENA implementation and the input parameters used.

### 3.2.1 Concrete: CC3DNonLinCementitious2 fracture–plastic model

The concrete material is modelled with the `CC3DNonLinCementitious2` formulation, a fracture–plastic constitutive model that combines a tensile constitutive law based on non-linear fracture mechanics with a compressive plasticity model governed by the Menétrey–Willam failure surface [18, 20]. The material is assumed to be isotropic before cracking or crushing occurs.

**Compressive behaviour.** The compressive response follows a parabolic ascending branch up to the peak stress  $f_c$  at the strain  $\varepsilon_{c1}$ , and a linear softening branch beyond the peak. The post-peak descending branch is regularised through the crushing band length  $L_c$ , in analogy with the Crack Band Method used in tension, in order to dissipate a prescribed crushing energy  $G_c$  independently of the mesh size. The Vecchio–Collins reduction factor is automatically applied to the peak compressive strength in regions experiencing large transverse tensile strains, which is critical for capturing the correct collapse load of members with significant diagonal cracking.

**Tensile behaviour.** The tensile response is linear elastic up to the tensile strength  $f_{ct}$ . Beyond this threshold, the concrete enters a softening regime described by the exponential law of Hordijk [25], which is the default option in the Červenka family of models. The softening law is regularised through the Crack Band Method [22]: the equivalent crack opening  $w = \varepsilon_{cr} L_t$  is computed from the strain at the integration point and from the characteristic element size  $L_t$  (here equal to the global mesh size  $L_e = 0.05$  m), so that the energy dissipated per unit area of crack remains equal to the specific fracture energy  $G_f$  (see section 3.4 below for the three levels of  $G_f$  adopted in the parametric study).

**Tension stiffening.** The contribution of the concrete between cracks (tension stiffening) is included by leaving a residual tensile stress  $\sigma_{ct}^{ts} = c_{ts} f_{ct}$  on the cracked element, with  $c_{ts} = 0.4$  adopted as the default value [1, 18].

**Input parameters.** For each specimen the following concrete parameters are assigned from the values reported in the source experimental references (Chapter 4): the Young’s modulus  $E_c$ , the Poisson’s ratio  $\nu_c$ , the cylindrical compressive strength  $f_c$ , the tensile strength  $f_{ct}$ , the specific fracture energy  $G_f$ , the strain at peak compressive stress  $\varepsilon_{c1}$ , and the ultimate compressive strain  $\varepsilon_{cu}$ . When one of these quantities is not directly reported in the source paper, it is computed from the others through the empirical relations recommended by *fib* MC2010 [1].

### 3.2.2 Conventional reinforcement: discrete truss embedding

The longitudinal mild steel reinforcement and the transverse stirrups are represented through discrete `CCIsoTruss` elements embedded inside the concrete mesh. This formulation is preferred over a smeared reinforcement layer because it preserves the explicit position of every bar and allows the local strain compatibility (or slip) with

the surrounding concrete to be enforced exactly. The constitutive law adopted for the reinforcement is the bilinear elastic–plastic relation with kinematic hardening described in section 2.3.2 of Chapter 2, characterised by the Young’s modulus  $E_s$ , the yield strength  $f_y$ , the ultimate strength  $f_t$  and the ultimate strain  $\varepsilon_{su}$ . A condition of *perfect bond* is assumed between the truss nodes and the surrounding concrete nodes for all conventional bars and stirrups, in line with the smeared crack framework adopted for flexural failure [18, 19].

### 3.2.3 Prestressing tendons: bonded and unbonded modelling

The modelling of the prestressing tendons differs depending on the construction technology of the specimen:

**Pre-tensioned strands – perfect bond.** For pre-tensioned beams, the prestressing strands are modelled through discrete **CCIsoTruss** elements with a multi-linear stress–strain law (section 2.3.2) and a *perfect bond* condition with the surrounding concrete. The kinematic constraint enforces strain compatibility at every node along the tendon, which is consistent with the mechanism of stress transfer of a fully bonded pre-tensioned strand. The initial prestressing force  $P_0$  is introduced through an *initial-strain* condition on the tendon elements,  $\varepsilon_{p,0} = P_0/(A_p E_p)$ , which generates a self-equilibrating system of internal forces on the concrete prior to the application of the external load.

**Bonded post-tensioned strands – explicit bond–slip.** For bonded post-tensioned beams, the interaction between the tendon and the surrounding grouted duct is described by an explicit bilinear bond–slip law of *fib* MC2010-type [1], schematically recalled in fig. 2.6 of Chapter 2. The bond strength  $\tau_{b,\max}$  is taken from the recommendations of *fib* MC2010 as a function of the concrete compressive strength  $f_c$  and of the duct/strand interface conditions, while the residual frictional bond  $\tau_{b,\text{res}}$  corresponds to the value at the slip  $s_3$ . As in the pre-tensioned case, the initial prestress is applied through an initial-strain condition on the tendon elements; the subsequent bond–slip law mobilises progressively during the loading phase.

**Unbonded post-tensioned tendons – truss between anchorages.** For unbonded post-tensioned beams, the tendon is represented by a single **CCIsoTruss** element that connects the two anchorages at the ends of the member, with no local bond to the surrounding concrete and a uniform stress along its entire length. The strain in the tendon at any stage of the analysis is computed from the relative displacement of the two anchor nodes, which reproduces the global elongation principle of eq. (2.3) (Chapter 2) [15, 16]. The initial prestress  $P_0$  is again imposed as an initial-strain on the truss element, and is transferred to the concrete exclusively through the anchor nodes; no continuous interaction is mobilised along the tendon length.

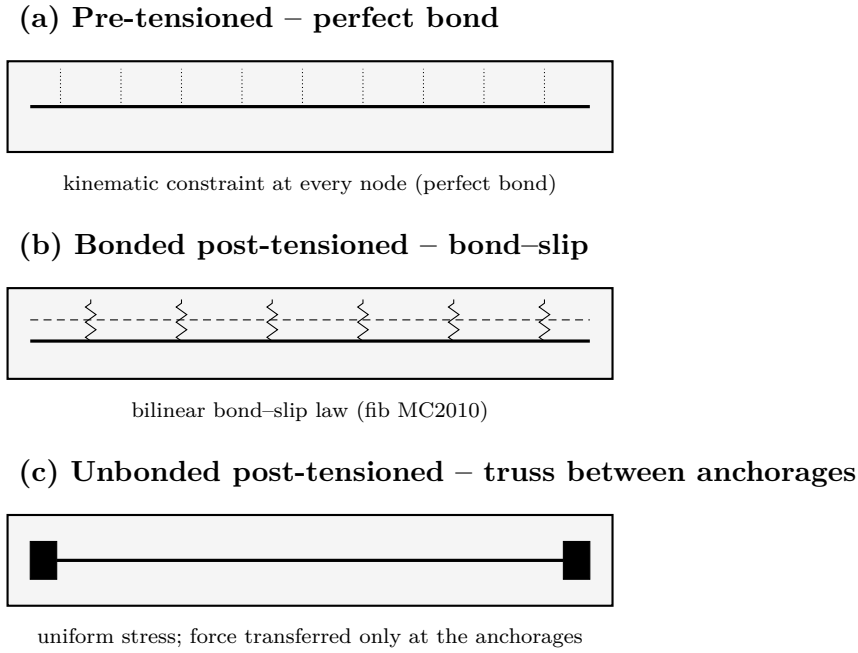


**Figure 3.3:** Schematic representation of the three modelling strategies adopted for the prestressing tendons in ATENA 3D: (a) pre-tensioned strand with perfect bond to the surrounding concrete (kinematic constraint on every truss node); (b) bonded post-tensioned strand with explicit bilinear bond-slip law of *fib* MC2010-type; (c) unbonded post-tensioned tendon, modelled as a single truss element connecting the two anchorages and a uniform stress distribution along its length. Modelling strategies after [1, 15, 16].

### 3.3 Crack modelling strategies

The smeared crack framework adopted in section 2.4.3 of Chapter 2 admits two distinct formulations, both implemented in ATENA 3D and both employed in the parametric study of the present thesis: the *fixed* crack model and the *rotated* crack model. The selection between the two is made at the material card of the concrete macro-element, and identifies the way in which the local crack normal and the corresponding constitutive matrix evolve during the loading history.

#### 3.3.1 Fixed crack formulation in ATENA 3D

In the fixed crack formulation implemented in ATENA 3D [18], the direction of the first crack at a Gauss point is determined by the orientation of the principal stress at the moment of crack initiation and remains *frozen* for the remainder of the analysis. The local constitutive matrix is rotated into the crack reference system  $(n, t)$  and the normal stress component is updated through the softening law described in section 3.2.1, while the shear component is updated through the *shear retention factor*  $\beta$  (see eq. (2.7) in Chapter 2). ATENA implements the shear retention factor as a variable that decreases from  $\beta = 1$  at crack initiation to  $\beta \approx 0.05$  at large crack openings, in accordance with the variable- $\beta$  formulation of Červenka and co-workers [18].

The fixed crack formulation is computationally robust and preserves the history

of crack orientation, which is useful for the post-processing of the crack pattern. Its known limitation is the potential onset of *stress locking* when the principal stress rotates significantly away from the initial crack direction; the sensitivity of the predicted response to the shear retention factor is therefore examined explicitly in the parametric study of Chapter 5.

### 3.3.2 Rotated crack formulation in ATENA 3D

In the rotated crack formulation, the direction of the crack *rotates* continuously with the principal strain axes during the loading history [23, 30]. The orthotropic constitutive matrix is updated at every iteration so that the crack normal coincides with the current direction of the principal tensile strain. By construction, no shear stress is transmitted across the crack plane and the formulation is therefore intrinsically free from stress locking. This makes the rotated crack model particularly suitable for the flexural response of PRC beams, in which the principal stress direction changes progressively as cracks open and the neutral axis migrates upwards.

## 3.4 Fracture energy and its role in concrete softening

The specific fracture energy  $G_f$  governs the area below the post-peak tensile branch of the concrete and is one of the most influential parameters of the smeared crack response [18, 21, 22]. Three different values of  $G_f$  are adopted in the parametric study of the present thesis, in order to span the range from a brittle behaviour to a pseudo-plastic post-peak response, while keeping all the other parameters of the concrete model unchanged.

**Calibrated value  $G_f$ .** The calibrated value is computed for each specimen from the empirical relation recommended by *fib* MC2010 [1],

$$G_f = 73 f_{cm}^{0.18} \quad [\text{N/m}], \quad (3.1)$$

where  $f_{cm}$  is the mean cylindrical compressive strength of the concrete in MPa. This value provides the reference solution that, combined with the default settings of the concrete model, captures the experimental load–displacement response of PRC beams in flexure with good accuracy.

**Reduced value  $G_f/100$  – brittle behaviour.** A significantly reduced fracture energy, equal to  $G_f/100$ , is used to simulate a brittle post-peak response of the concrete. With this very low energy dissipation, the softening branch of the tensile constitutive law drops sharply after the attainment of  $f_{ct}$  and the cracks become highly localised. The corresponding numerical response is expected to underestimate the post-cracking stiffness and to anticipate the ultimate state.

**Maximised value – pseudo-plastic behaviour.** A maximised fracture energy is adopted to simulate a pseudo-plastic post-peak response, in which the cohesive stress decays very slowly after the attainment of  $f_{ct}$  and the cracks behave almost as plastic hinges. The numerical response is expected to overestimate the post-cracking stiffness and the ultimate capacity. Together with the calibrated  $G_f$  and the reduced  $G_f/100$ , this third level brackets the physical range of  $G_f$  from below and above and is essential for the sensitivity analysis of the model uncertainty in Chapter 6.

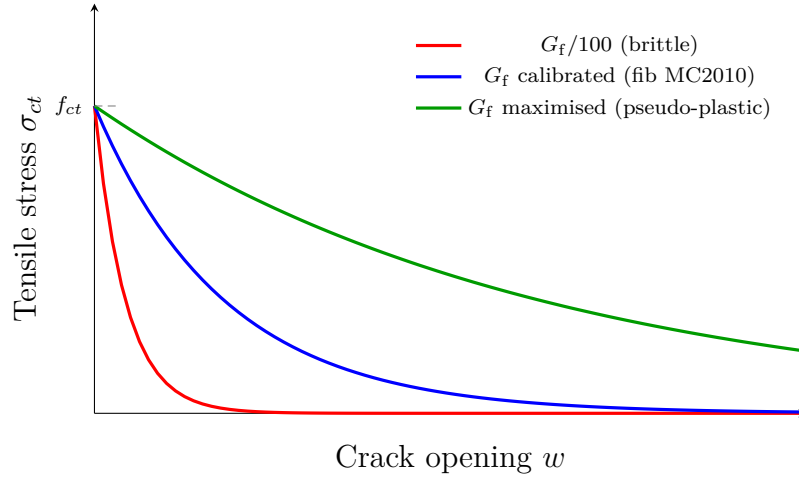


**Figure 3.4:** Concrete tensile stress–crack-opening curves corresponding to the three levels of fracture energy  $G_f$  adopted in the parametric study: a brittle response obtained with  $G_f/100$  (red), the reference calibrated response obtained with the *fib* MC2010 value (3.1) (blue), and a pseudo-plastic response obtained with a maximised fracture energy (green). The area under each curve, multiplied by the characteristic length  $L_t = L_e = 0.05$  m, equals the corresponding  $G_f$ . Softening law and  $G_f$  relations after [1, 22, 25].

### 3.5 Solution strategy and convergence criteria

The non-linear system of equilibrium equations is solved incrementally and iteratively through the Newton–Raphson method already introduced in section 2.4.2 of Chapter 2. ATENA 3D offers different variants of the algorithm; in the present work the **Newton–Raphson – Optimal** variant is selected, in which the tangent stiffness matrix is updated at every iteration but the number of iterations per load step is adapted automatically by the solver according to the convergence behaviour. The line-search option is enabled to improve convergence in regions of strong material softening (cracking, yielding).

#### 3.5.1 Loading protocol

The application of the prestress and of the external load is split into two distinct phases:

1. **Prestress phase.** The initial prestress  $P_0$  is applied through an initial-strain condition on the tendon elements as described in section 3.2.3. The corresponding self-equilibrating internal force system is mobilised over a small number of load steps until the target prestress is fully transferred to the concrete.

2. **External loading phase.** The external load is then applied through a *displacement-controlled* scheme at the loading plate, consistently with the experimental protocols of most of the benchmark specimens (Chapter 4). Displacement increments are kept sufficiently small to resolve the cracking, yielding and ultimate stages of the flexural response.

### 3.5.2 Convergence criteria

At each iteration of every load step, the convergence of the solution is assessed against three norms simultaneously: force, displacement and energy. The tolerances adopted in the present work, summarised in table 3.1, are the recommended ATENA defaults for quasi-static structural problems with cracking and yielding [18].

**Table 3.1:** Convergence tolerances adopted for the Newton–Raphson iterations in the present thesis (ATENA 3D defaults).

| Convergence norm                                                                    | Tolerance |
|-------------------------------------------------------------------------------------|-----------|
| Force residual $\ \mathbf{r}\ /\ \mathbf{f}_{\text{ext}}\ $                         | 1%        |
| Displacement increment $\ \Delta\mathbf{u}\ /\ \mathbf{u}\ $                        | 1%        |
| Energy $\Delta\mathbf{u}^\top \mathbf{r} / \mathbf{u}^\top \mathbf{f}_{\text{ext}}$ | 0.01%     |

When convergence is not achieved within the prescribed maximum number of iterations, ATENA automatically reduces the size of the current load step and restarts the iteration, ensuring that the equilibrium path is followed without spurious divergence.

## 3.6 Definition of the six numerical analysis scenarios

The cornerstone of the parametric study presented in this thesis is the systematic comparison of six numerical scenarios per specimen, obtained by combining the two smeared crack models of section 3.3 with the three levels of fracture energy of section 3.4. The six scenarios are formally defined in table 3.2 and graphically organised in fig. 3.5; this nomenclature is preserved through the rest of the thesis and through the legends of all the load–displacement plots presented in Chapter 5.

**Table 3.2:** The six numerical scenarios (i.e. solution strategies) adopted in the parametric study. Each scenario is identified by a combination of a smeared crack model and a fracture energy level.

| Scenario | Label             | Crack model | Fracture energy  |
|----------|-------------------|-------------|------------------|
| 1        | Fixed – Model 1   | Fixed       | $G_f/100$        |
| 2        | Fixed – Model 2   | Fixed       | $G_f$ calibrated |
| 3        | Fixed – Model 3   | Fixed       | $G_f$ maximised  |
| 4        | Rotated – Model 1 | Rotated     | $G_f/100$        |
| 5        | Rotated – Model 2 | Rotated     | $G_f$ calibrated |
| 6        | Rotated – Model 3 | Rotated     | $G_f$ maximised  |

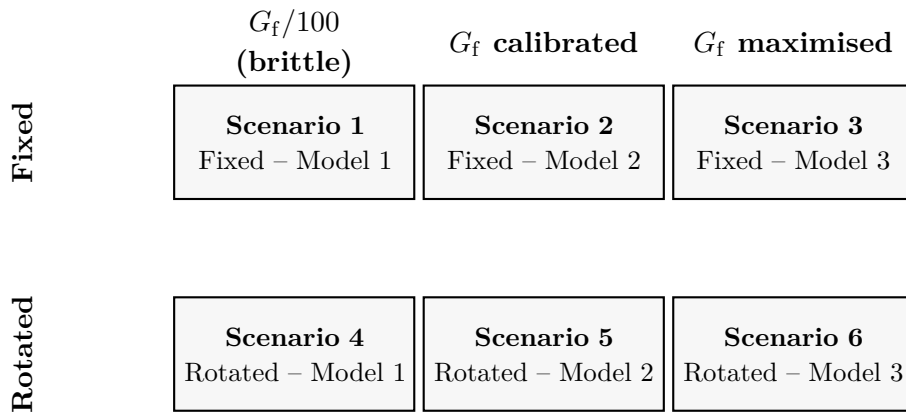


**Figure 3.5:** Graphical representation of the  $2 \times 3$  factorial matrix of the six numerical scenarios investigated in the parametric study: two smeared crack models (fixed, rotated) combined with three levels of fracture energy ( $G_f/100$ , calibrated  $G_f$ , maximised  $G_f$ ). For each specimen of the experimental benchmark database (Chapter 4) all six scenarios are executed, yielding six values of the model uncertainty ratio  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  that are statistically processed in Chapter 6.

For each scenario, the predicted load–displacement curve is extracted and the ultimate numerical capacity  $R_{\text{num}}$  is identified as the peak load reached during the displacement-controlled run (see Chapter 5 for the operational definition). The corresponding model uncertainty ratio  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  defined in eq. (1.4) of Chapter 1 will be computed for every specimen and every scenario, and the six resulting populations of  $\vartheta$  will be statistically processed to characterise the model uncertainty associated with each modelling choice.

### 3.7 General modelling hypotheses adopted for all specimens

To guarantee that the six numerical scenarios isolate the effect of the crack model and of the fracture energy, all the other modelling choices are kept identical across the six runs of a given specimen and consistent across the specimens of the database. The general hypotheses adopted are summarised hereafter:

- **Geometry.** The full geometry of each specimen (length, cross-section, reinforcement, tendon profile) is reproduced as reported in the source experimental reference (Chapter 4). Symmetry is exploited only when it does not interfere with the simulation of the 3D stress state at the loading and support plates; otherwise the full beam is modelled.
- **Mesh.** The global element size is fixed at  $L_e = 0.05$  m for the concrete volume of every specimen. The mesh is refined locally only at the contact zones with the loading and support plates.
- **Materials.** The mean values of the material strengths reported in the source paper are used as input. When a quantity is not directly reported, it is computed from the others through the empirical relations of *fib* MC2010 [1].
- **Reinforcement.** All conventional bars and stirrups are modelled as discrete truss elements embedded in the concrete mesh with a perfect bond condition.
- **Tendons.** Pre-tensioned strands are modelled with perfect bond; bonded post-tensioned strands with an explicit bilinear bond-slip law of *fib* MC2010-type; unbonded post-tensioned tendons with a single truss element between the two anchorages.
- **Boundary conditions.** Each specimen is supported on two rigid plates that allow rotation about the transverse axis and prevent vertical translation; one of the two supports also prevents longitudinal translation in order to suppress rigid-body motion.
- **Loading.** The prestress is applied first through an initial-strain condition; the external load is then applied through a displacement-controlled scheme at the loading plates.
- **Solver.** The non-linear system is solved with the Newton–Raphson Optimal scheme of section 3.5, with line-search enabled and the convergence tolerances of table 3.1.
- **Geometric non-linearity.** Given the short-to-medium span of the rectangular beams of the benchmark database and the absence of significant second-order effects, geometric non-linearity is included only when it improves the convergence of the solver, in line with the recommendations of *fib* MC2010 [1, 18].

With these hypotheses fixed, every specimen of the benchmark database of Chapter 4 is simulated under the six numerical scenarios of table 3.2, and the resulting load–displacement curves constitute the raw data of the parametric study of Chapter 5 and of the statistical analysis of Chapter 6.

# Chapter 4

## Experimental Benchmark

The numerical campaign described in Chapter 3 is benchmarked against an experimental database of ten rectangular PRC beams collected from the peer-reviewed literature. The present chapter introduces the database in detail: the selection criteria are first defined, the test typologies adopted by the original investigators are recalled, and each of the six source papers is then described in a dedicated subsection in which the geometry, the reinforcement layout, the prestress configuration, the material properties and the experimental ultimate capacity  $R_{\text{exp}}$  are reported. The chapter closes with a consolidated master summary table that gathers the relevant data for all the specimens, ready to be used in the validation of the numerical models of Chapter 5 and in the statistical analysis of Chapter 6.

### 4.1 Selection criteria for the benchmark specimens

The selection of the experimental campaigns adopted in this thesis was guided by the need to build a benchmark that is, at the same time, representative of the engineering reality of PRC beams in flexure and rigorous enough to support the statistical quantification of the model uncertainty  $\vartheta$  defined in eq. (1.4) of Chapter 1. The following criteria were systematically applied during the literature survey, which was performed on the bibliographic databases ScienceDirect, Wiley Online Library, ResearchGate and Google Scholar:

- **Failure mode.** Only specimens that failed in pure flexure under monotonic loading were retained. Tests dominated by shear, by anchorage failure, by torsion or by cyclic/dynamic actions were discarded, in line with the scope of the thesis stated in Chapter 1.
- **Cross-section and structural typology.** Only simply-supported beams with a *rectangular* cross-section were considered, so that the geometric discretisation could be standardised across the database and the comparison between specimens was meaningful.
- **Prestressing technology.** The database was constructed to cover the three main configurations of internal prestressing relevant to current practice: pre-tensioning

with bonded strands, bonded post-tensioning, and unbonded post-tensioning. Both bonded and unbonded behaviours are therefore represented, which is essential for the comparison discussed in section 2.2.3 of Chapter 2.

- **Completeness of the reported data.** The single most important selection criterion was the completeness of the experimental information. For every specimen the source paper had to report unambiguously the beam dimensions, the reinforcement layout, the prestressing details, the mean concrete and steel strengths, the prestress force or effective prestress, the test setup and the full experimental load–displacement curve up to failure, so that the experimental ultimate capacity  $R_{\text{exp}}$  could be read directly from the original data.
- **Reliability of the experimental source.** Preference was given to papers published in indexed journals or in conference proceedings of recognised scientific organisations.

The systematic application of these criteria led to a final benchmark of *ten* rectangular PRC beam specimens drawn from six source papers [2–7], all failing in flexure under monotonic loading.

## 4.2 Test typologies considered

The ten specimens of the benchmark database were tested under two standard simply-supported configurations: three-point bending (3PB) and four-point bending (4PB), schematically illustrated in fig. 4.1. Although both setups are designed to assess the flexural stiffness and the ultimate flexural strength of the specimen, they impose distinct stress states in the central region of the beam, which must be acknowledged when interpreting the experimental load–displacement curves and when comparing them with the numerical predictions.

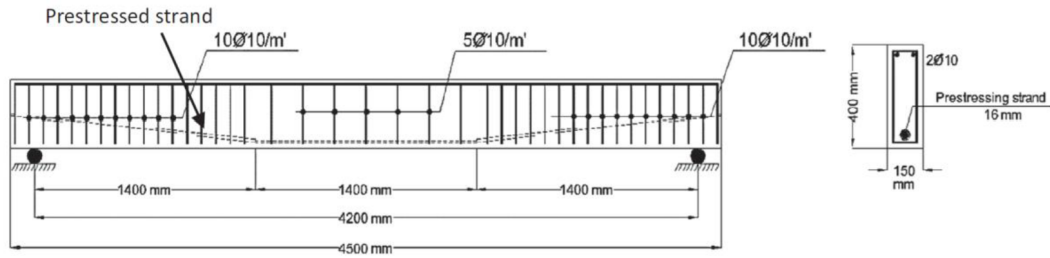
**Three-point bending (3PB).** A single concentrated load  $P$  is applied at mid-span. The bending moment diagram is triangular and peaks under the loading point, where the shear force is also non-zero. The critical section therefore experiences a combined state of bending and shear stresses, which is particularly relevant for short-to-medium-span beams. Among the specimens of the present database, the 3PB configuration was adopted in the experimental programme of Salah *et al.* [3] (specimens TS1, TS2 and TS3).

**Four-point bending (4PB).** Two concentrated loads, each equal to  $P/2$ , are applied symmetrically with respect to the mid-span, generating a central zone of *pure* bending moment with zero shear force, flanked by two shear spans in which shear and bending act simultaneously. The pure-bending zone allows the isolation of the flexural response of the section, which is the conceptual basis of any flexural benchmark. Most of the specimens of the present database were tested under 4PB [2, 4–7].



**Table 4.1:** Geometric, material and test properties of specimen B3 [2].

| Property                                    | Value                                       |
|---------------------------------------------|---------------------------------------------|
| <i>Geometry</i>                             |                                             |
| Total length $L$ / span $L_{\text{eff}}$    | 4500 / 4200 mm                              |
| Cross-section $b \times h$                  | 150 $\times$ 400 mm                         |
| Concrete cover $c$                          | 25 mm                                       |
| <i>Concrete</i>                             |                                             |
| $f_{cm} / f_{ct}$                           | 50.0 / 3.15 MPa                             |
| <i>Mild reinforcement</i>                   |                                             |
| Longitudinal (top)                          | 2 $\phi$ 10                                 |
| Stirrups                                    | $\phi$ 10 @ 100–200 mm                      |
| $f_y / f_u / \varepsilon_{su}$              | 470 / 610 MPa / 7.5%                        |
| <i>Prestressing (bonded post-tensioned)</i> |                                             |
| Strand                                      | 1 $\phi$ 15.24 ( $A_p = 139 \text{ mm}^2$ ) |
| $F_{p,0} / \sigma_{p,0}$                    | 200 kN / 980 MPa                            |
| $f_{p0.1k} / f_{pk}$                        | 1674 / 1860 MPa                             |
| <i>Test</i>                                 |                                             |
| Configuration / control                     | 4PB / displacement                          |
| $R_{\text{exp}} (\delta_{\text{peak}})$     | 123.8 kN (101 mm)                           |

**Figure 4.2:** Longitudinal elevation, cross-section and reinforcement layout of specimen B3 of the experimental campaign by Moawad *et al.* [2].

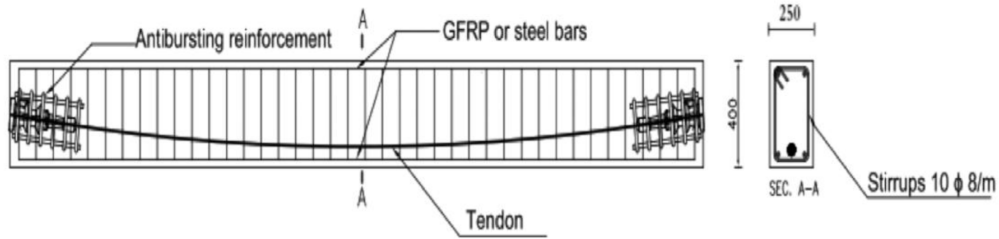
#### 4.3.2 Salah *et al.* (2024) – specimens TS1, TS2, TS3

Salah *et al.* [3] carried out an extensive experimental study on the flexural behaviour of bonded post-tensioned concrete beams reinforced with either conventional steel or GFRP bars. Three beams from their campaign – **TS1**, **TS2** and **TS3** – were retained for the present benchmark; all three use conventional steel reinforcement and bonded post-tensioning, which corresponds to the modelling framework defined in Chapter 3. The three specimens share the same geometry, the same prestressing system and the same concrete grade; they differ only in the combination of top and bottom mild steel longitudinal reinforcement, which makes them an excellent dataset for assessing the sensitivity of the model uncertainty to the mild reinforcement ratio.

The geometry, materials, prestress and capacities of the three Salah specimens are summarised in table 4.2; all three were tested in three-point bending under force control up to flexural failure.

**Table 4.2:** Geometric, material and test properties of TS1, TS2, TS3 [3]. Bar areas ( $\text{mm}^2$ ) in parentheses.

| Property                    | TS1                                                                                                          | TS2                     | TS3               |
|-----------------------------|--------------------------------------------------------------------------------------------------------------|-------------------------|-------------------|
| Cross-section $b \times h$  |                                                                                                              | 250 $\times$ 400 mm     |                   |
| $L / L_{\text{eff}}$        |                                                                                                              | 4000 / 3600 mm          |                   |
| Cover $c$                   |                                                                                                              | 25 mm                   |                   |
| $f_{cm}$                    |                                                                                                              | 40.0 MPa                |                   |
| Stirrups                    |                                                                                                              | $\phi 8$ @ 100 mm       |                   |
| Mild steel $f_y/f_u/E_s$    |                                                                                                              | 511 / 600 MPa / 200 GPa |                   |
| Prestress (bonded PT)       | 1 $\phi 15.24$ , $A_p = 140 \text{ mm}^2$ , $F_{p,0} = 180 \text{ kN}$ ( $\sigma_{p,0} = 1286 \text{ MPa}$ ) |                         |                   |
| $f_{p0.1k}/f_{pk}$          | 1670 / 1860 MPa                                                                                              |                         |                   |
| Top longitudinal            | 2 $\phi 10$ (158)                                                                                            | 2 $\phi 16$ (402)       | 2 $\phi 10$ (157) |
| Bottom longitudinal         | 2 $\phi 16$ (402)                                                                                            | 2 $\phi 16$ (402)       | 2 $\phi 12$ (226) |
| $R_{\text{exp}}$ [kN]       | 147.7                                                                                                        | 157.9                   | 127.9             |
| $\delta_{\text{peak}}$ [mm] | 44.5                                                                                                         | 65.9                    | 47.15             |

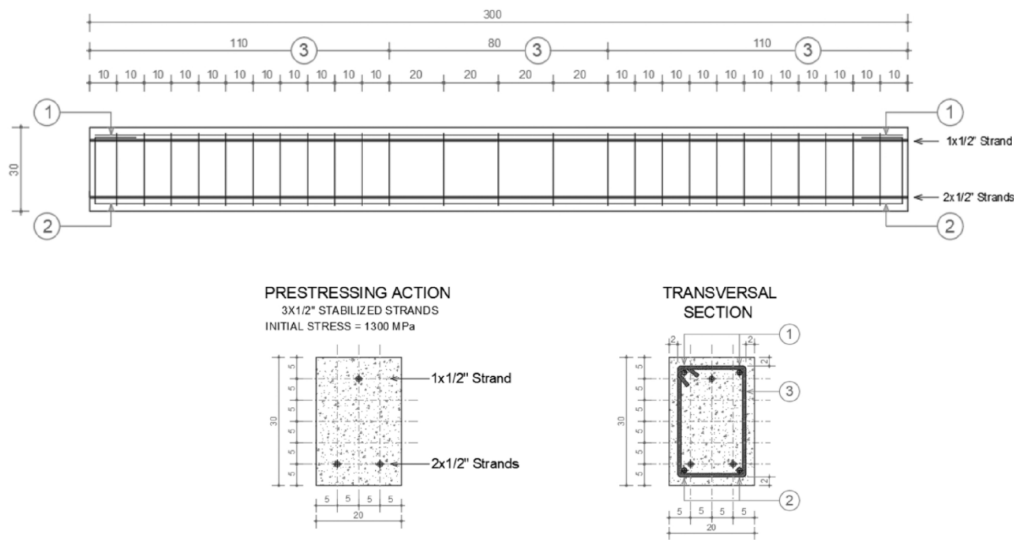
**Figure 4.3:** Longitudinal elevation, cross-section and reinforcement layout of specimens TS1, TS2 and TS3 of the experimental campaign by Salah *et al.* [3]. The three beams share the same geometry, prestressing system and concrete grade, and differ only in the longitudinal mild reinforcement.

#### 4.3.3 Imperatore *et al.* (2024) – specimen PRC-B1

Imperatore *et al.* [4] investigated the influence of corrosion on the flexural performance of pre-tensioned concrete beams. The specimen **PRC-B1** was selected from their campaign as the *uncorroded reference* member, with the most detailed and internally consistent material characterisation of all the specimens of the present database. The properties of PRC-B1 are summarised in table 4.3.

**Table 4.3:** Geometric, material and test properties of PRC-B1 [4].

| Property                                | Value                              |
|-----------------------------------------|------------------------------------|
| <i>Geometry</i>                         |                                    |
| $L / L_{\text{eff}}$                    | 3000 / 2700 mm                     |
| Cross-section $b \times h$              | 200 $\times$ 300 mm                |
| Cover $c$                               | 20 mm                              |
| <i>Concrete</i>                         |                                    |
| $f_{cm} / f_{ct}$                       | 28.21 / 1.98 MPa                   |
| <i>Mild reinforcement</i>               |                                    |
| Longitudinal (top/bottom)               | 2 $\phi$ 10 / 2 $\phi$ 10          |
| Stirrups                                | $\phi$ 8 @ 100–200 mm              |
| $f_y / f_u / \varepsilon_{su}$          | 593 / 698.18 MPa / 9.02%           |
| <i>Prestressing (pre-tensioned)</i>     |                                    |
| Strands                                 | 3 $\phi$ 12.7 (1 upper, 2 lower)   |
| $F_{p,0} / \sigma_{p,0}$                | 121 kN / 1300 MPa                  |
| $f_{p0.1k} / f_{pk} / E_p$              | 1755.36 / 1952.81 MPa / 202.98 GPa |
| $\varepsilon_{pu}$                      | 6.58%                              |
| <i>Test</i>                             |                                    |
| Configuration / control                 | 4PB / displacement                 |
| $R_{\text{exp}} (\delta_{\text{peak}})$ | 187.49 kN (23.12 mm)               |

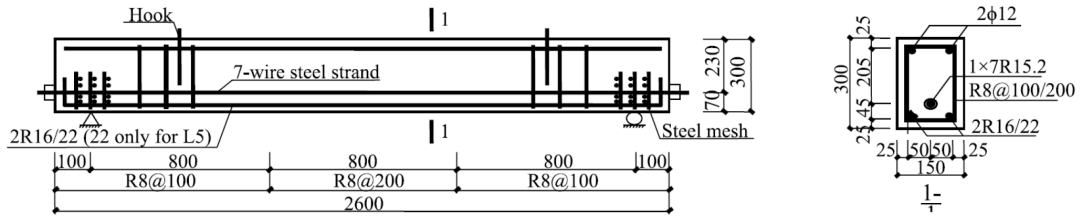
**Figure 4.4:** Longitudinal elevation, cross-section and reinforcement layout of specimen PRC-B1 of the experimental campaign by Imperatore *et al.* [4].

#### 4.3.4 Yu *et al.* (2022) – specimen L1

Yu *et al.* [5] investigated the flexural behaviour of corrosion-damaged pre-tensioned concrete beams. The specimen **L1** corresponds to the *uncorroded control* of the experimental campaign, and is therefore the appropriate benchmark for the present thesis. The properties of L1 are summarised in table 4.4.

**Table 4.4:** Geometric, material and test properties of L1 [5].

| Property                                | Value                                                 |
|-----------------------------------------|-------------------------------------------------------|
| <i>Geometry</i>                         |                                                       |
| $L / L_{\text{eff}}$                    | 2600 / 2400 mm                                        |
| Cross-section $b \times h$              | 150 $\times$ 300 mm                                   |
| <i>Concrete</i>                         |                                                       |
| $f_{cm}/f_{ct}/E_c$                     | 44.2 / 3.7 MPa / 28 GPa                               |
| <i>Mild reinforcement</i>               |                                                       |
| Longitudinal (top/bottom)               | 2 $\phi$ 12 / 2 $\phi$ 16                             |
| Stirrups                                | $\phi$ 8 @ 100–200 mm                                 |
| $f_y/f_u/E_s$                           | 365 / 570 MPa / 181 GPa                               |
| <i>Prestressing (pre-tensioned)</i>     |                                                       |
| Strand                                  | 1 $\phi$ 15.2 seven-wire ( $A_p = 140 \text{ mm}^2$ ) |
| $\sigma_{p,0}$                          | 999 MPa                                               |
| $f_{p0.1k}/f_{pk}/E_p$                  | 1751 / 1959 MPa / 203 GPa                             |
| <i>Test</i>                             |                                                       |
| Configuration / control                 | 4PB / mixed                                           |
| $R_{\text{exp}} (\delta_{\text{peak}})$ | 227.8 kN (25 mm)                                      |

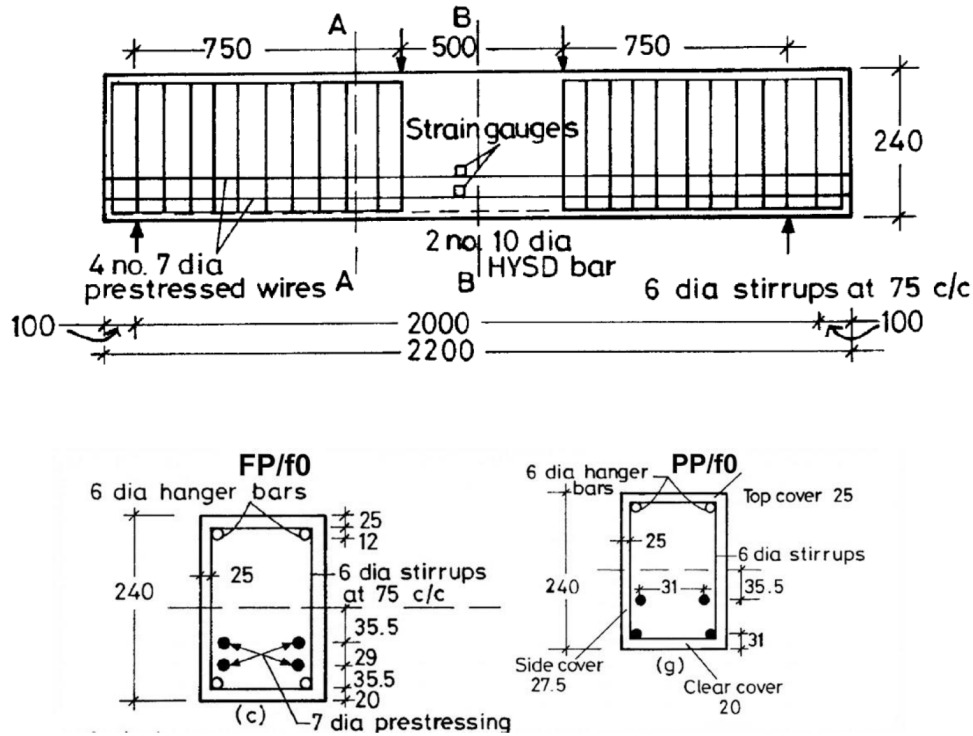
**Figure 4.5:** Longitudinal elevation, cross-section and reinforcement layout of specimen L1 of the experimental campaign by Yu *et al.* [5].

#### 4.3.5 Padmarajaiah and Ramaswamy (2002) – specimens FP/f0 and PP/f0

Padmarajaiah and Ramaswamy [6] carried out a finite-element-assisted assessment of the flexural strength of pre-tensioned concrete beams with high-strength wires. Among the specimens of the campaign, two beams without fibre reinforcement were selected for the present benchmark: **FP/f0**, *fully prestressed*, and **PP/f0**, *partially prestressed*. The properties of the two Padmarajaiah specimens are summarised in table 4.5; both were tested under a force-controlled monotonic protocol.

**Table 4.5:** Geometric, material and test properties of FP/f0 (fully prestressed) and PP/f0 (partially prestressed) [6].

| Property                                         | FP/f0                          | PP/f0            |
|--------------------------------------------------|--------------------------------|------------------|
| Cross-section $b \times h$                       | 105 $\times$ 240 mm            |                  |
| $L / L_{\text{eff}}$                             | 2200 / 2000 mm                 |                  |
| $f_{cm}/f_{ct}$                                  | $\approx 58 / \approx 5.3$ MPa |                  |
| Stirrups                                         | $\phi 6$ @ 75 mm               |                  |
| Prestress wires ( $\phi 7$ )                     | 4 wires                        | 2 wires          |
| Tendon area $A_p$ [mm <sup>2</sup> ]             | 153.9                          | 76.97            |
| $F_{p,0}$ [kN]                                   | 141.2                          | 35.3             |
| $f_{p0.1k}/f_{pk}/E_p$                           | 1542 / 1598 MPa / 202 GPa      |                  |
| Mild bottom reinf.                               | none                           | 2 $\phi 10$ HYSD |
| Mild steel $f_y/f_u$                             | —                              | 458 / 525 MPa    |
| $E_c$ [GPa]                                      | 40.58                          | 65.18            |
| $R_{\text{exp}}$ [kN] ( $\delta_{\text{peak}}$ ) | 96.99 (13.5 mm)                | 91.66 (14.3 mm)  |

**Figure 4.6:** Longitudinal elevation and cross-sections of specimens FP/f0 (fully prestressed, left) and PP/f0 (partially prestressed, right) of the experimental campaign by Padmarajaiah and Ramaswamy [6].

#### 4.3.6 Peera and Oukaili (2021) – specimens IB35-10-0.7-1 and IB60-10-0.5-2

The experimental programme of Peera and Oukaili [7] is the only one of the present database that uses *unbonded* post-tensioning, and it therefore provides the critical data points for the validation of the unbonded tendon modelling strategy described in section 3.2.3 of Chapter 3. Two specimens of the campaign were retained for the present

benchmark, denoted by the original authors as **IB35-10-0.7-1** and **IB60-10-0.5-2**. The properties of the two unbonded post-tensioned Peera specimens are summarised in table 4.6; both were tested under four-point bending with force control.

**Table 4.6:** Geometric, material and test properties of the unbonded post-tensioned specimens [7].

| Property                                         | IB35-10-0.7-1                         | IB60-10-0.5-2           |
|--------------------------------------------------|---------------------------------------|-------------------------|
| Cross-section $b \times h$                       | 250 $\times$ 350 mm                   |                         |
| $L / L_{\text{eff}}$                             | 3000 / 2700 mm                        |                         |
| Cover $c$                                        | 42.5 mm                               |                         |
| Stirrups                                         | $\phi 10$ @ 100 mm                    |                         |
| Prestress (unbonded PT)                          | 2 tendons, $A_p = 199.2 \text{ mm}^2$ |                         |
| $f_{p0.1k} / f_{pk} / E_p$                       | 1725 / 1860 MPa / 197.5 GPa           |                         |
| $f_{cm} / f_{ct} / E_c$ [GPa]                    | 36.36 / 3.2 / 27.13                   | 60.7 / 4.19 / 34.84     |
| Top/bottom longitudinal                          | $2\phi 10$ / $2\phi 25$               | $2\phi 10$ / $2\phi 16$ |
| Bottom area $A_s$ [mm <sup>2</sup> ]             | 981                                   | 398.2                   |
| $\sigma_{p,0}$ [MPa]                             | 1302                                  | 930                     |
| Mild steel $f_y / f_u$ [MPa]                     | 513 / 641.45                          | 547 / 658.59            |
| $R_{\text{exp}}$ [kN] ( $\delta_{\text{peak}}$ ) | 417.0 (22 mm)                         | 291.2 (28.5 mm)         |

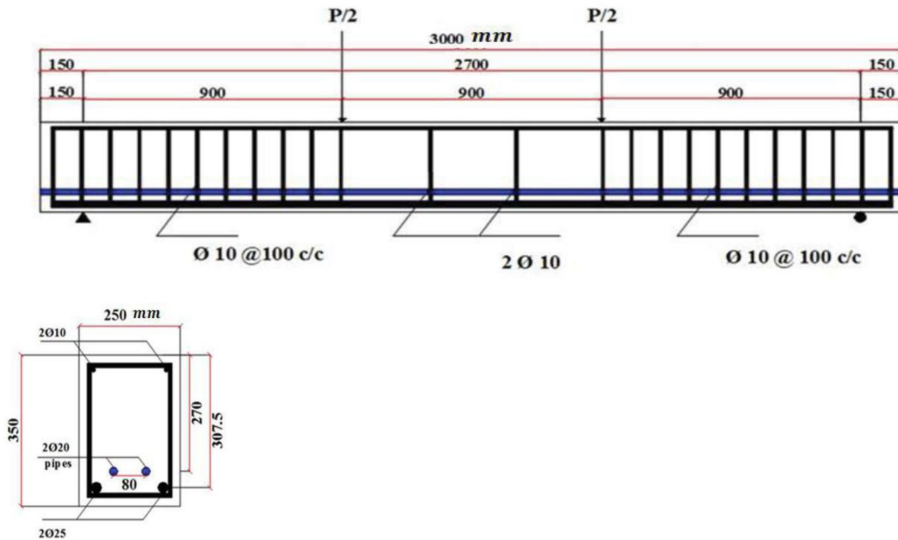


**Figure 4.7:** Longitudinal elevation, cross-section and reinforcement layout of specimens IB35-10-0.7-1 and IB60-10-0.5-2 of the experimental campaign by Peera and Oukaili [7]. Both beams share the same nominal geometry and differ in the concrete grade, in the longitudinal mild reinforcement and in the initial prestress level.

## 4.4 Master summary of geometric and mechanical properties

The ten specimens of the benchmark database are summarised in table 4.7 below, where the most relevant geometric and mechanical quantities are reported side-by-side for direct cross-comparison. The table will be referred to throughout Chapter 5 (for the

specimen-by-specimen numerical validation) and Chapter 6 (for the computation of the model uncertainty ratio  $\vartheta$ ).

**Table 4.7:** Master summary of the geometric and mechanical properties of the ten specimens of the experimental benchmark.  $L_{\text{eff}}$  is the effective span;  $b \times h$  is the cross-section;  $f_{cm}$  is the mean cylindrical concrete compressive strength;  $A_p$  is the total area of the prestressing tendons;  $F_{p,0}$  is the initial prestressing force per tendon;  $R_{\text{exp}}$  is the experimentally observed ultimate load; “Test” is the bending configuration (3PB or 4PB).

| Source           | Spec.         | System        | $L_{\text{eff}}$<br>[mm] | $b \times h$<br>[mm] | $f_{cm}$<br>[MPa] | $A_p$<br>[mm <sup>2</sup> ] | $F_{p,0}$<br>[kN] | $R_{\text{exp}}$<br>[kN] | Test |
|------------------|---------------|---------------|--------------------------|----------------------|-------------------|-----------------------------|-------------------|--------------------------|------|
| Moawad [2]       | B3            | PT bonded     | 4200                     | 150×400              | 50.0              | 139.0                       | 200.0             | 123.8                    | 4PB  |
| Salah [3]        | TS1           | PT bonded     | 3600                     | 250×400              | 40.0              | 140.0                       | 180.0             | 147.7                    | 3PB  |
| Salah [3]        | TS2           | PT bonded     | 3600                     | 250×400              | 40.0              | 140.0                       | 180.0             | 157.9                    | 3PB  |
| Salah [3]        | TS3           | PT bonded     | 3600                     | 250×400              | 40.0              | 140.0                       | 180.0             | 127.9                    | 3PB  |
| Imperatore [4]   | PRC-B1        | Pre-tensioned | 2700                     | 200×300              | 28.2              | 297.0                       | 121.0             | 187.5                    | 4PB  |
| Yu [5]           | L1            | Pre-tensioned | 2400                     | 150×300              | 44.2              | 140.0                       | 139.9             | 227.8                    | 4PB  |
| Padmarajaiah [6] | FP/f0         | Pre-tensioned | 2000                     | 105×240              | 58.0              | 153.9                       | 141.2             | 97.0                     | 4PB  |
| Padmarajaiah [6] | PP/f0         | Pre-tensioned | 2000                     | 105×240              | 58.2              | 77.0                        | 35.3              | 91.7                     | 4PB  |
| Peera [7]        | IB35-10-0.7-1 | PT unbonded   | 2700                     | 250×350              | 36.4              | 199.2                       | 259.4             | 417.0                    | 4PB  |
| Peera [7]        | IB60-10-0.5-2 | PT unbonded   | 2700                     | 250×350              | 60.7              | 199.2                       | 185.3             | 291.2                    | 4PB  |

The database covers a representative range of cross-sections (from the slender  $105 \times 240 \text{ mm}^2$  of Padmarajaiah to the  $250 \times 400 \text{ mm}^2$  of Salah), of effective spans (from 2.0 m to 4.2 m), of concrete compressive strengths (28.2 to 60.7 MPa) and of prestressing technologies (pre-tensioning with bonded strands, bonded post-tensioning and unbonded post-tensioning). Two of the ten specimens, namely IB35-10-0.7-1 and IB60-10-0.5-2, explicitly belong to the unbonded post-tensioned category that isolates the modelling assumption of section 3.2.3 of Chapter 3. The experimental ultimate capacities  $R_{\text{exp}}$  span almost an order of magnitude, from 91.7 kN (PP/f0) to 417.0 kN (IB35-10-0.7-1), ensuring that the model uncertainty ratio  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  analysed in Chapter 6 is computed over a significant range of structural sizes.

## 4.5 Experimental reference behaviour

For each of the ten specimens listed in table 4.7, the original investigators reported the full experimental load–displacement response, recorded as the relation between the applied load  $P$  and the mid-span deflection  $\delta$ . The complete experimental curves – digitised from the source papers – constitute the *reference behaviour* of the benchmark, and will be plotted against the numerical responses of the six solution strategies of section 3.6 (Chapter 3) in Chapter 5 of the thesis. The values of the experimental ultimate capacity  $R_{\text{exp}}$  tabulated in table 4.7 correspond to the peak of these experimental curves.

All ten specimens experienced a clear *flexural* failure mode under monotonic loading, consistent with the selection criteria of section 4.1, characterised by: (i) the formation of vertical flexural cracks in the central portion of the beam; (ii) the progressive opening of these cracks during the loading history; (iii) the yielding of the conventional bottom reinforcement (where present) and the substantial stress increase in the prestressing

tendons up to the attainment of the ultimate state; (iv) the eventual crushing of the concrete in the compression chord, occasionally accompanied by tendon rupture, at the peak load. No specimen developed a brittle shear failure, an anchorage failure or a torsional failure, which confirms the suitability of the database for the calibration of the model uncertainty associated with the flexural failure mechanism of PRC beams, as defined in Chapter 1.

The benchmark database described in this chapter constitutes the empirical ground on which the numerical models of Chapter 3 are validated. In Chapter 5 each of the ten specimens is modelled in ATENA 3D under the six solution strategies of section 3.6, and the predicted load–displacement curves are systematically compared with the corresponding experimental ones. In Chapter 6 the same database is processed statistically: the model uncertainty ratio  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  is computed for every specimen and every scenario, the resulting populations are characterised by their mean  $\mu_{\vartheta}$  and coefficient of variation  $V_{\vartheta}$ , and the partial safety factor  $\gamma_{Rd}$  is finally calibrated through the Bayesian updating framework described in Chapters 1 and 2.

# Chapter 5

## NLFEA Results and Parametric Study

This chapter applies the numerical framework of Chapter 3 to the experimental benchmark of Chapter 4. For each of the ten specimens of the database, the six numerical scenarios formally defined in table 3.2 of Chapter 3 are executed in ATENA 3D, and the predicted load–displacement curves are systematically compared with the experimental responses. The numerical ultimate capacity  $R_{\text{num}}$  of every run is extracted as the peak of the corresponding numerical  $P$ – $\delta$  curve, and the sixty values are collected in a master synthesis table at the end of the chapter. These values constitute the direct input of the statistical analysis of Chapter 6, where the model uncertainty ratio  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  is processed within the Bayesian updating framework adopted in the thesis.

### 5.1 General modelling protocol applied to each specimen

The modelling protocol of Chapter 3 is applied identically to all the specimens of the benchmark database, in order to guarantee that the six numerical scenarios isolate exclusively the effect of the crack model and of the fracture energy and that the resulting populations of  $R_{\text{num}}$  are mutually comparable. The recurring choices are recalled hereafter.

- **Geometry.** The full 3D geometry of each specimen is reproduced from the source experimental reference of Chapter 4, including the rectangular concrete volume, the longitudinal and transverse reinforcement, the prestressing tendons and the loading and support plates.
- **Mesh.** The concrete volume is discretised with eight-node isoparametric brick elements (CCIsoBrick) of global size  $L_e = 0.05$  m (section 3.1.2 of Chapter 3). Local refinement is applied only at the loading and support plates.
- **Materials.** The mean values of the material strengths reported in the original references are used as input. Empirical relations of *fib* MC2010 [1] are used to derive any missing parameter.

- **Crack model and shear retention.** Each specimen is analysed with both the fixed and the rotated smeared-crack formulations of section 3.3.1 of Chapter 3, which generate the six numerical scenarios. In all the fixed crack model analyses performed on the hexahedral mesh, the shear retention factor is kept constant at  $\beta = 0.2$ , within the range recommended in ATENA, in order to limit the stress-locking effect while preserving the numerical stability of the cracked elements.
- **Reinforcement and tendons.** All conventional bars and stirrups are modelled as discrete truss elements embedded in the concrete mesh with perfect bond. Pre-tensioned strands are modelled with perfect bond; bonded post-tensioned strands with the bilinear bond-slip law of section 3.2.3; unbonded post-tensioned tendons (Peera specimens) with a single truss element connecting the two anchorages (section 3.2.3).
- **Boundary conditions and loading.** The beam is simply supported on two rigid plates that allow rotation about the transverse axis and prevent vertical translation; one of the two supports also prevents longitudinal translation in order to remove the rigid-body mode. The prestressing force is applied first through an initial-strain condition on the tendon elements; the external load is then applied through a displacement-controlled scheme at the loading plate(s) up to the failure of the specimen.
- **Solver and convergence.** The non-linear system is solved through the Newton-Raphson Optimal algorithm with line search, using the convergence tolerances of table 3.1: 1% on the force and displacement residuals and 0.01% on the energy.

A representative finite element model of a benchmark beam – the hexahedral mesh of the concrete volume, the discrete reinforcement and prestressing tendons, and the loading and support plates – has already been illustrated schematically in fig. 3.2 of Chapter 3 and is not repeated here.

## 5.2 Inverse analysis and calibration of the fracture energy

### 5.2.1 Definition of the numerical ultimate capacity $R_{\text{num}}$

For each of the sixty numerical analyses (ten specimens  $\times$  six scenarios), the predicted load-displacement curve is recorded at the displacement-controlled loading point and the numerical ultimate capacity  $R_{\text{num}}$  is defined as the peak of this curve,

$$R_{\text{num}} = \max_t P(\delta(t)), \quad (5.1)$$

where  $P(\delta(t))$  is the reaction force at the support as a function of the imposed displacement  $\delta(t)$ . This operational definition is consistent with the one used in the experimental campaigns of Chapter 4 for  $R_{\text{exp}}$ , and is therefore appropriate for the direct computation of the model uncertainty ratio  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  defined in eq. (1.4) of Chapter 1.

### 5.2.2 Calibrated fracture energy from *fib* MC2010

The reference value of the fracture energy used in Models 2 and 5 of the parametric study is computed for each specimen from the empirical relation of *fib* MC2010 [1] introduced in eq. (3.1) of Chapter 3,

$$G_f = 73 f_{cm}^{0.18} \quad [\text{N/m}], \quad (5.2)$$

where  $f_{cm}$  is the mean cylindrical compressive strength of the concrete in MPa. The values obtained for the ten specimens of the benchmark are collected in table 5.1. The brittle scenario (Models 1 and 4) is obtained by dividing the calibrated value by 100, while the pseudo-plastic scenario (Models 3 and 6) is obtained by adopting a maximised value of  $G_f$  that drives the post-peak tensile response towards a quasi-plastic plateau.

**Table 5.1:** Calibrated value of the specific fracture energy  $G_f$  used in Models 2 and 5, computed from the empirical relation of *fib* MC2010 [1],  $G_f = 73 f_{cm}^{0.18}$  N/m.

| Specimen      | $f_{cm}$ [MPa] | $G_f$ [N/m] (Models 2, 5) | $G_f/100$ [N/m] (Models 1, 4) |
|---------------|----------------|---------------------------|-------------------------------|
| B3            | 50.0           | 147.6                     | 1.48                          |
| TS1, TS2, TS3 | 40.0           | 141.8                     | 1.42                          |
| PRC-B1        | 28.2           | 133.2                     | 1.33                          |
| L1            | 44.2           | 144.4                     | 1.44                          |
| FP/f0         | 58.0           | 151.6                     | 1.52                          |
| PP/f0         | 58.2           | 151.7                     | 1.52                          |
| IB35-10-0.7-1 | 36.4           | 139.4                     | 1.39                          |
| IB60-10-0.5-2 | 60.7           | 152.9                     | 1.53                          |

The maximised value of  $G_f$  used in Models 3 and 6 corresponds to a fracture energy three orders of magnitude larger than the *fib* MC2010 value, sufficient to suppress almost entirely the post-peak softening of the concrete tensile response and to drive the crack-band stress–strain law towards a quasi-plastic plateau.

### 5.2.3 Inverse-analysis logic

The six scenarios are not used independently as competing calibrations of  $G_f$ ; rather, they constitute a deliberate parametric sweep that brackets the physical range of the softening behaviour of the concrete, from a fully brittle response ( $G_f/100$ ) to a pseudo-plastic one ( $G_f$  max), with the *fib* MC2010 calibrated value at the centre. The motivation is the systematic quantification of the model uncertainty  $\vartheta$  as a function of the modelling choice – crack model and fracture energy – so that the sensitivity of the structural prediction to these two parameters can be measured directly from the experimental–numerical comparison. This procedure is consistent with the framework adopted in the recent literature on the calibration of  $\gamma_{Rd}$  for NLFEA of RC structures [8, 11, 12], and provides the empirical evidence on which the Bayesian updating of Chapter 6 is based.

### 5.3 Specimen-by-specimen results

For each of the ten specimens of the benchmark, the experimental  $P$ - $\delta$  curve and the six numerical  $P$ - $\delta$  curves are compared in a dedicated figure; the corresponding values of  $R_{\text{num}}$  and the resulting model uncertainty ratios  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  are reported and discussed. The discussion focuses on three aspects: (i) which scenario produces the value of  $R_{\text{num}}$  closest to the experimental  $R_{\text{exp}}$ ; (ii) how the prediction varies between the fixed and the rotated crack models; (iii) how it varies across the three levels of fracture energy.

The principal characteristics of the ten specimens were summarised in table 4.7 of Chapter 4 and are not repeated here.

#### 5.3.1 Moawad *et al.* (2018) – specimen B3

Specimen B3 is a bonded post-tensioned beam with a single strand and moderate concrete strength ( $f_{cm} = 50$  MPa), tested under four-point bending and displacement control (Chapter 4, section 4.3.1). The numerical and experimental load–displacement responses are compared in fig. 5.1a. The six numerical values of  $R_{\text{num}}$  and the corresponding  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  ratios are summarised in table 5.2.

**Table 5.2:** Numerical ultimate capacity  $R_{\text{num}}$  and model uncertainty ratio  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  for specimen B3.  $R_{\text{exp}} = 123.8$  kN.

| Scenario | Crack model | Fracture energy  | $R_{\text{num}}$ [kN] | $\vartheta$ [-] |
|----------|-------------|------------------|-----------------------|-----------------|
| $M_1$    | Fixed       | $G_f/100$        | 112.80                | 1.097           |
| $M_2$    | Fixed       | calibrated $G_f$ | 121.55                | 1.018           |
| $M_3$    | Fixed       | maximised $G_f$  | 120.42                | 1.028           |
| $M_4$    | Rotated     | $G_f/100$        | 123.32                | 1.004           |
| $M_5$    | Rotated     | calibrated $G_f$ | 123.93                | 0.999           |
| $M_6$    | Rotated     | maximised $G_f$  | 120.92                | 1.024           |

**Discussion.** The rotated calibrated scenario (Model 5) provides an almost perfect match with the experimental peak, with  $\vartheta = 0.999$ ; the rotated brittle scenario (Model 4) is also extremely close, with  $\vartheta = 1.004$ . The fixed brittle scenario (Model 1), in contrast, is the only one that exhibits a marked overestimation in the direction of safety, with  $\vartheta = 1.097$ , indicating a  $\sim 10\%$  underprediction of  $R_{\text{num}}$  due to the premature stress-locked softening of the fixed crack model with very low  $G_f$ . The remaining four scenarios cluster between  $\vartheta = 1.018$  and 1.028, indicating that for this specimen the choice of the crack model is more influential than the level of  $G_f$ .

#### 5.3.2 Salah *et al.* (2024) – specimens TS1, TS2 and TS3

The three Salah specimens share the same bonded post-tensioned configuration and the same concrete grade ( $f_{cm} = 40$  MPa,  $L_{\text{eff}} = 3.6$  m), and differ only in the longitudinal mild reinforcement (Chapter 4, section 4.3.2). The three sets of numerical and experi-

mental curves are compared in figs. 5.1b, 5.2a and 5.2b; the corresponding  $R_{\text{num}}$  and  $\vartheta$  values are collected in table 5.3.

**Table 5.3:** Numerical ultimate capacity  $R_{\text{num}}$  [kN] and model uncertainty ratio  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  (in parentheses) for the three Salah specimens [3].  $R_{\text{exp}}$ : TS1 = 147.7, TS2 = 157.9, TS3 = 127.9 kN.

| Scenario | Crack model | Fracture energy  | TS1                          | TS2                          | TS3                          |
|----------|-------------|------------------|------------------------------|------------------------------|------------------------------|
|          |             |                  | $R_{\text{num}} (\vartheta)$ | $R_{\text{num}} (\vartheta)$ | $R_{\text{num}} (\vartheta)$ |
| $M_1$    | Fixed       | $G_f/100$        | 170.41 (0.867)               | 170.10 (0.928)               | 132.30 (0.967)               |
| $M_2$    | Fixed       | calibrated $G_f$ | 156.40 (0.944)               | 170.10 (0.928)               | 132.30 (0.967)               |
| $M_3$    | Fixed       | maximised $G_f$  | 174.46 (0.847)               | 166.95 (0.946)               | 135.45 (0.944)               |
| $M_4$    | Rotated     | $G_f/100$        | 174.71 (0.845)               | 173.25 (0.911)               | 132.30 (0.967)               |
| $M_5$    | Rotated     | calibrated $G_f$ | 161.15 (0.917)               | 173.25 (0.911)               | 132.30 (0.967)               |
| $M_6$    | Rotated     | maximised $G_f$  | 155.64 (0.949)               | 173.25 (0.911)               | 119.70 (1.068)               |

**Discussion.** All three Salah specimens systematically experience a slight overestimation of the flexural capacity by the numerical models ( $\vartheta < 1$  for almost every scenario), reflecting the fact that the simulations capture the full nominal mechanical contribution of the bonded post-tensioning cable while the experimental response appears to be limited by secondary effects (anchorage slippage, local debonding) not explicitly modelled. The dependence on the crack model and on  $G_f$  is moderate for TS2 and TS3, while TS1 shows the largest scatter ( $\vartheta$  ranges from 0.845 to 0.949), suggesting a higher sensitivity to the softening branch when the bottom reinforcement ratio is the lowest of the three.

### 5.3.3 Imperatore *et al.* (2024) – specimen PRC-B1

The PRC-B1 specimen is a pre-tensioned beam with three  $\frac{1}{2}$ " strands and the lowest concrete strength of the database ( $f_{cm} = 28.2$  MPa), tested under four-point bending and displacement control (Chapter 4, section 4.3.3). The numerical and experimental responses are compared in fig. 5.3a; the values of  $R_{\text{num}}$  and  $\vartheta$  are reported in table 5.4.

**Table 5.4:** Numerical ultimate capacity  $R_{\text{num}}$  and model uncertainty ratio  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  for specimen PRC-B1.  $R_{\text{exp}} = 187.49$  kN.

| Scenario | Crack model | Fracture energy  | $R_{\text{num}}$ [kN] | $\vartheta$ [-] |
|----------|-------------|------------------|-----------------------|-----------------|
| $M_1$    | Fixed       | $G_f/100$        | 205.91                | 0.911           |
| $M_2$    | Fixed       | calibrated $G_f$ | 212.29                | 0.883           |
| $M_3$    | Fixed       | maximised $G_f$  | 222.29                | 0.843           |
| $M_4$    | Rotated     | $G_f/100$        | 210.62                | 0.890           |
| $M_5$    | Rotated     | calibrated $G_f$ | 211.16                | 0.888           |
| $M_6$    | Rotated     | maximised $G_f$  | 218.72                | 0.857           |

**Discussion.** The entire family of numerical predictions systematically overestimates the experimental capacity, with  $\vartheta$  ranging from 0.843 to 0.911. The closest match is obtained with the fixed brittle scenario (Model 1), the worst with the fixed pseudo-plastic

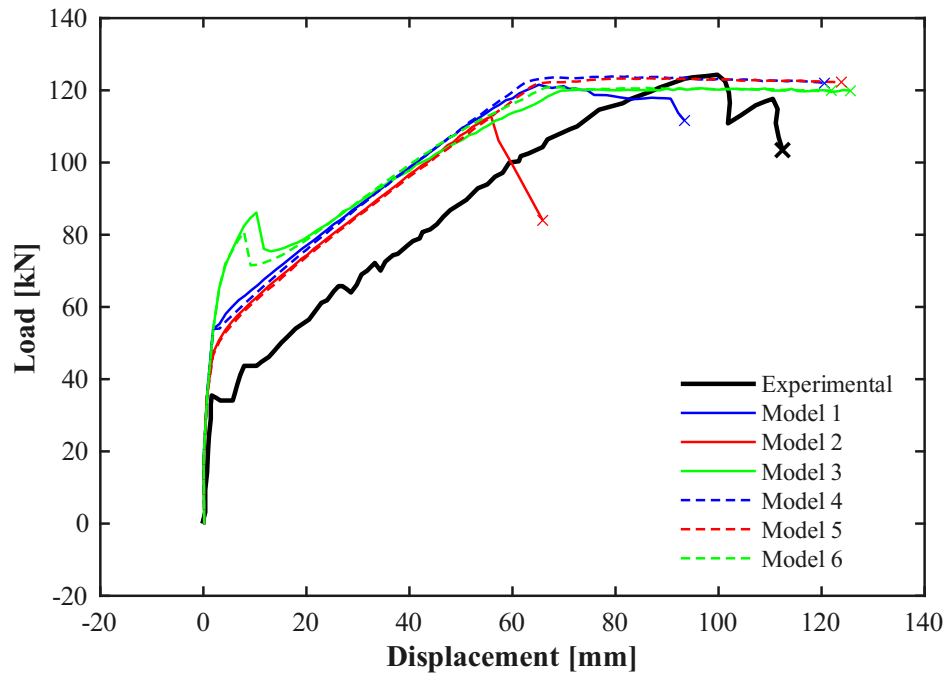
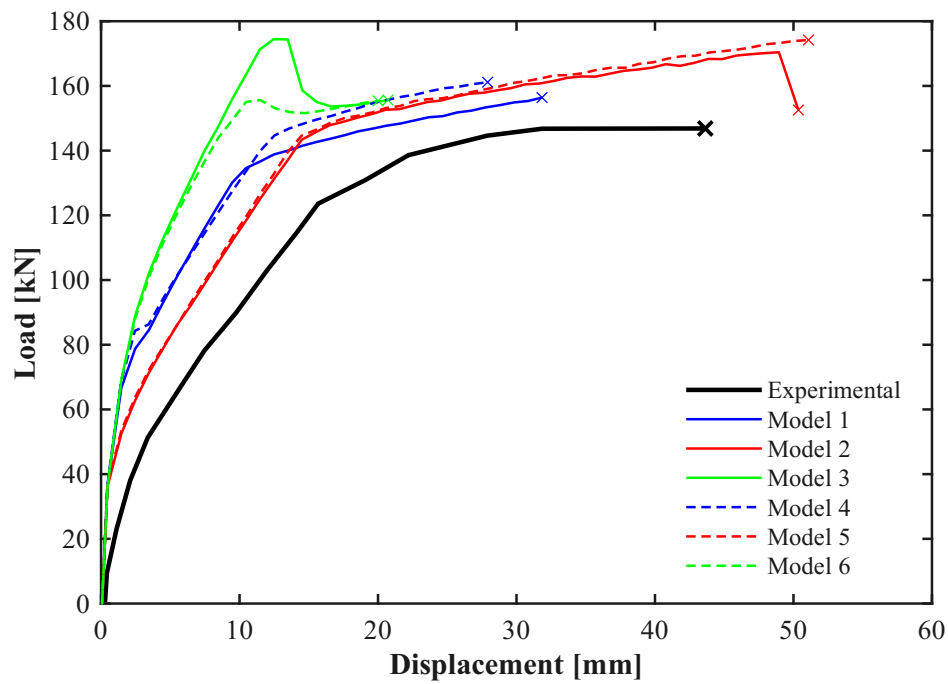
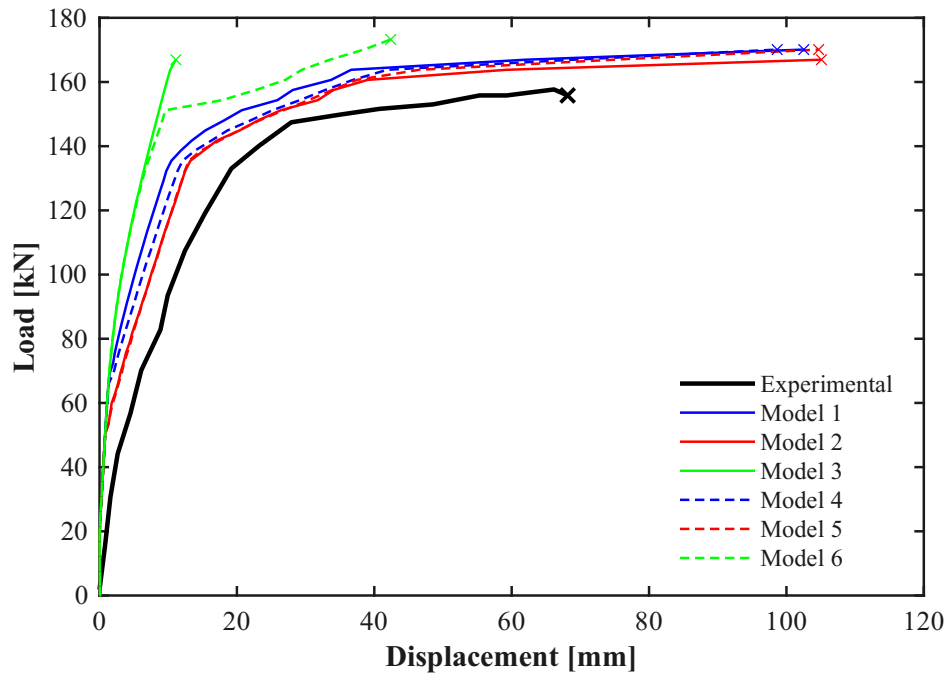
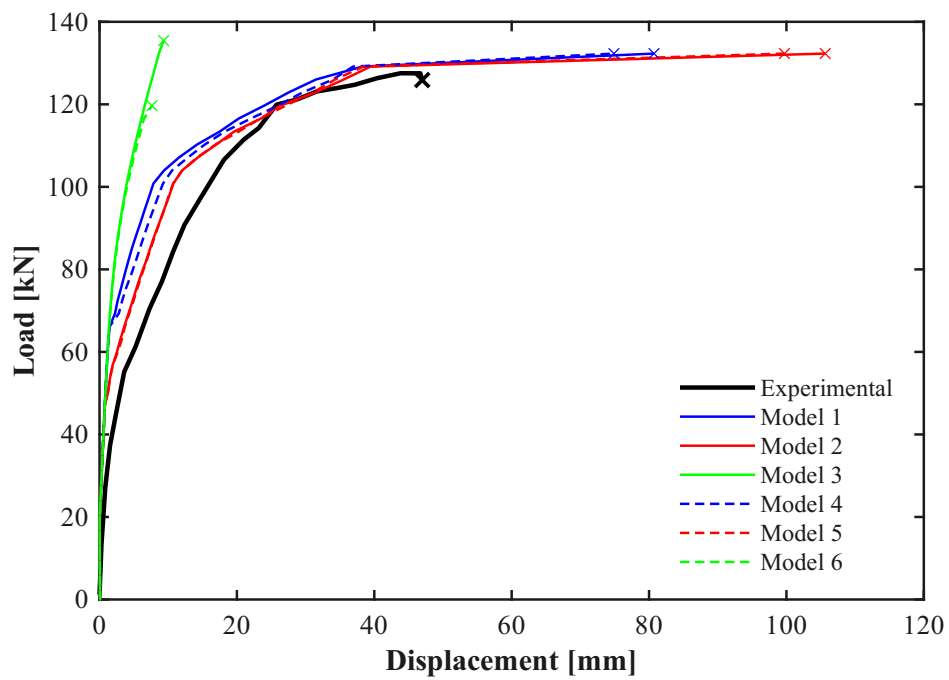
(a) B3 – Moawad *et al.* [2](b) TS1 – Salah *et al.* [3]

**Figure 5.1:** Experimental versus the six numerical load–displacement curves for the bonded post-tensioned specimens B3 and TS1.



(a) TS2 – Salah *et al.* [3]



(b) TS3 – Salah *et al.* [3]

**Figure 5.2:** Experimental versus the six numerical load–displacement curves for the bonded post-tensioned specimens TS2 and TS3.

scenario (Model 3). The dependence on  $G_f$  is monotonic: increasing the fracture energy increases  $R_{\text{num}}$  in both the fixed and the rotated crack formulations. The persistent overestimation – not observed on any other specimen of the database – suggests that the experimental peak of PRC-B1 may be governed by secondary effects (e.g. uneven prestress transfer, local bond slippage, anchorage block deformation) that are not explicitly captured by the smeared crack model used for the concrete.

#### 5.3.4 Yu *et al.* (2022) – specimen L1

The L1 specimen is a pre-tensioned beam with a single  $\phi 15.2$  mm strand and a relatively short span ( $L_{\text{eff}} = 2.4$  m), tested under four-point bending with a mixed load–displacement control protocol (Chapter 4, section 4.3.4). The comparison between the experimental and the numerical responses is shown in fig. 5.3b; the  $R_{\text{num}}$  and  $\vartheta$  values are reported in table 5.5.

**Table 5.5:** Numerical ultimate capacity  $R_{\text{num}}$  and model uncertainty ratio  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  for specimen L1.  $R_{\text{exp}} = 227.8$  kN.

| Scenario | Crack model | Fracture energy  | $R_{\text{num}}$ [kN] | $\vartheta$ [-] |
|----------|-------------|------------------|-----------------------|-----------------|
| $M_1$    | Fixed       | $G_f/100$        | 199.78                | 1.140           |
| $M_2$    | Fixed       | calibrated $G_f$ | 222.76                | 1.023           |
| $M_3$    | Fixed       | maximised $G_f$  | 214.11                | 1.064           |
| $M_4$    | Rotated     | $G_f/100$        | 214.87                | 1.060           |
| $M_5$    | Rotated     | calibrated $G_f$ | 224.67                | 1.014           |
| $M_6$    | Rotated     | maximised $G_f$  | 207.36                | 1.099           |

**Discussion.** Specimen L1 is characterised by a small but consistent underestimation of the experimental capacity by all the scenarios ( $\vartheta > 1$ ). The best agreement is reached with the calibrated fracture energy:  $\vartheta = 1.023$  for the fixed crack model (Model 2) and  $\vartheta = 1.014$  for the rotated crack model (Model 5). The brittle scenarios overpredict the safe-sided gap ( $\vartheta = 1.140$  for Model 1) because the very low  $G_f$  anticipates the post-peak softening and clips the numerical peak; the maximised scenarios partially recover, but never reach the quality of the calibrated ones. This specimen confirms the general expectation that the *fib* MC2010 calibration of  $G_f$  provides the most realistic peak.

#### 5.3.5 Padmarajaiah and Ramaswamy (2002) – specimens FP/f0 and PP/f0

Specimens FP/f0 and PP/f0 are the smallest beams of the database ( $b \times h = 105 \times 240$  mm,  $L_{\text{eff}} = 2.0$  m). FP/f0 is *fully* prestressed with four high-strength wires (no mild bottom bars), whereas PP/f0 is *partially* prestressed with two wires supplemented by mild bottom reinforcement (Chapter 4, section 4.3.5). The numerical and experimental responses are compared in figs. 5.4a and 5.4b; the  $R_{\text{num}}$  and  $\vartheta$  values are summarised in table 5.6.

**Table 5.6:** Numerical ultimate capacity  $R_{\text{num}}$  [kN] and model uncertainty ratio  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  (in parentheses) for specimens FP/f0 and PP/f0 [6].  $R_{\text{exp}}$ : FP/f0 = 96.99, PP/f0 = 91.66 kN.

| Scenario | Crack model | Fracture energy  | FP/f0                        | PP/f0                        |
|----------|-------------|------------------|------------------------------|------------------------------|
|          |             |                  | $R_{\text{num}} (\vartheta)$ | $R_{\text{num}} (\vartheta)$ |
| $M_1$    | Fixed       | $G_f/100$        | 102.17 (0.949)               | 82.08 (1.117)                |
| $M_2$    | Fixed       | calibrated $G_f$ | 109.78 (0.884)               | 87.99 (1.042)                |
| $M_3$    | Fixed       | maximised $G_f$  | 120.43 (0.805)               | 102.01 (0.899)               |
| $M_4$    | Rotated     | $G_f/100$        | 108.11 (0.897)               | 86.06 (1.065)                |
| $M_5$    | Rotated     | calibrated $G_f$ | 110.07 (0.881)               | 88.67 (1.034)                |
| $M_6$    | Rotated     | maximised $G_f$  | 116.95 (0.829)               | 93.74 (0.978)                |

**Discussion.** The two Padmarajaiah specimens display two different behaviours. The fully prestressed beam FP/f0 is systematically overestimated ( $\vartheta < 1$ ) for every scenario, with the best agreement provided by the fixed brittle Model 1 ( $\vartheta = 0.949$ ) and the worst by the fixed pseudo-plastic Model 3 ( $\vartheta = 0.805$ ). The partially prestressed beam PP/f0 is, on the contrary, underestimated by the brittle and calibrated scenarios ( $\vartheta = 1.117$  for Model 1 and 1.042 for Model 2), and correctly captured only by the pseudo-plastic Models 3 and 6. The combination of high-strength concrete and very small cross-section makes these specimens markedly sensitive to the softening parameters, consistent with the literature on slender PRC beams [8, 11].

### 5.3.6 Peera and Oukaili (2021) – specimens IB35-10-0.7-1 and IB60-10-0.5-2

The two Peera specimens are the only beams of the database with *unbonded* post-tensioning, and therefore the only ones for which the modelling strategy of section 3.2.3 of Chapter 3 is mobilised. The numerical and experimental load–displacement curves are compared in figs. 5.5a and 5.5b; the  $R_{\text{num}}$  and  $\vartheta$  values are reported in table 5.7.

**Table 5.7:** Numerical ultimate capacity  $R_{\text{num}}$  [kN] and model uncertainty ratio  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  (in parentheses) for the two unbonded post-tensioned specimens [7].  $R_{\text{exp}}$ : IB35-10-0.7-1 = 417.0, IB60-10-0.5-2 = 291.2 kN.

| Scenario | Crack model | Fracture energy  | IB35-10-0.7-1                | IB60-10-0.5-2                |
|----------|-------------|------------------|------------------------------|------------------------------|
|          |             |                  | $R_{\text{num}} (\vartheta)$ | $R_{\text{num}} (\vartheta)$ |
| $M_1$    | Fixed       | $G_f/100$        | 403.11 (1.034)               | 247.00 (1.179)               |
| $M_2$    | Fixed       | calibrated $G_f$ | 444.81 (0.937)               | 312.00 (0.933)               |
| $M_3$    | Fixed       | maximised $G_f$  | 500.41 (0.833)               | 377.01 (0.772)               |
| $M_4$    | Rotated     | $G_f/100$        | 458.71 (0.909)               | 337.99 (0.861)               |
| $M_5$    | Rotated     | calibrated $G_f$ | 458.71 (0.909)               | 350.99 (0.829)               |
| $M_6$    | Rotated     | maximised $G_f$  | 486.51 (0.857)               | 377.01 (0.772)               |

**Discussion.** The two unbonded specimens display the widest scatter of the entire database:  $\vartheta$  varies from 0.772 to 1.179 for IB60-10-0.5-2, and from 0.833 to 1.034 for IB35-10-0.7-1. For both specimens the fixed brittle scenario (Model 1) yields the value

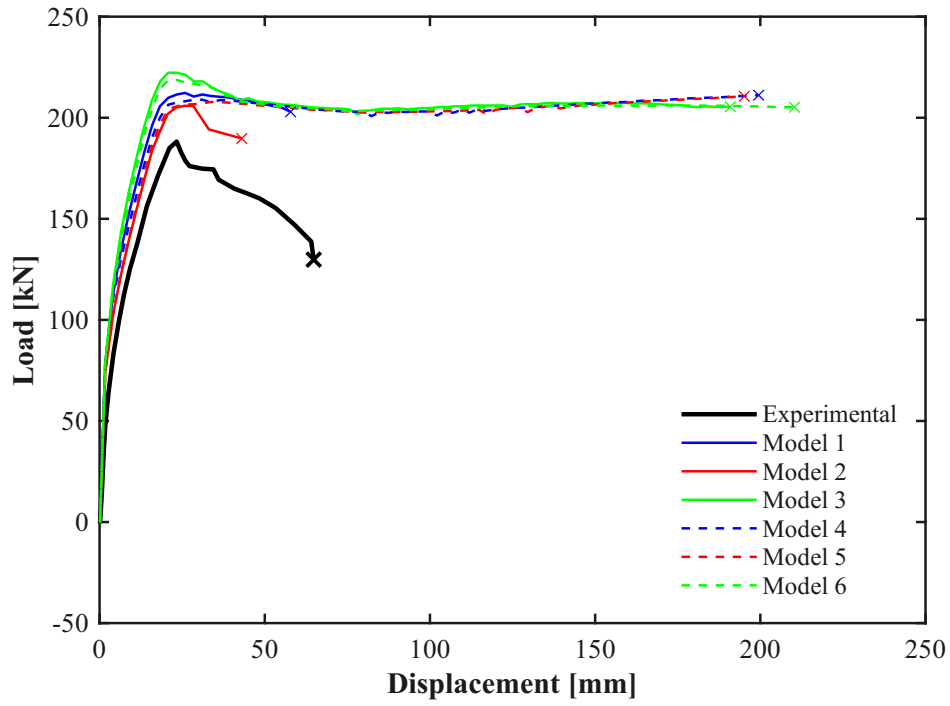
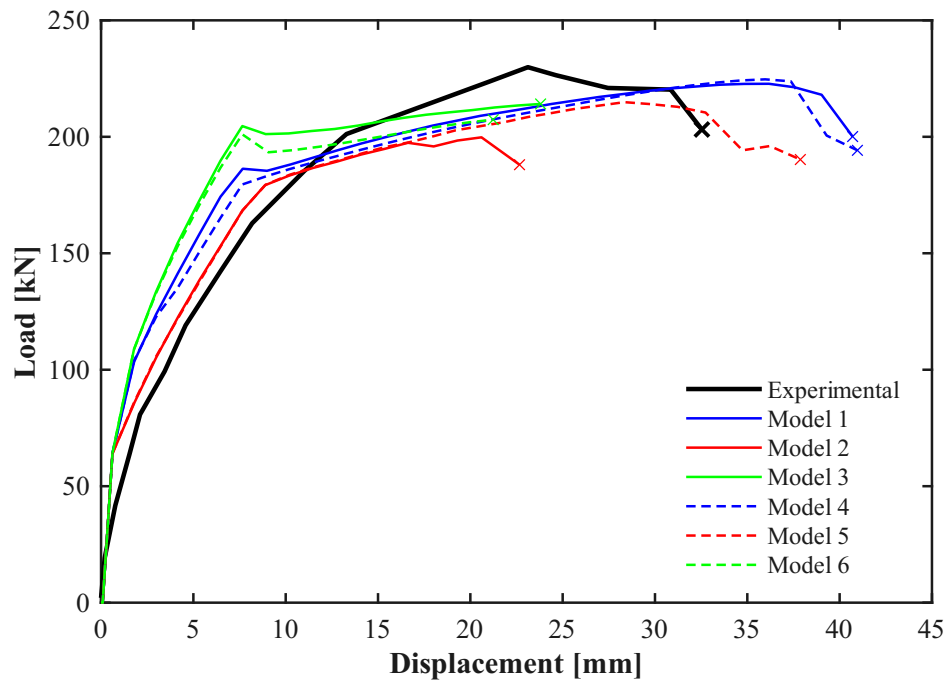
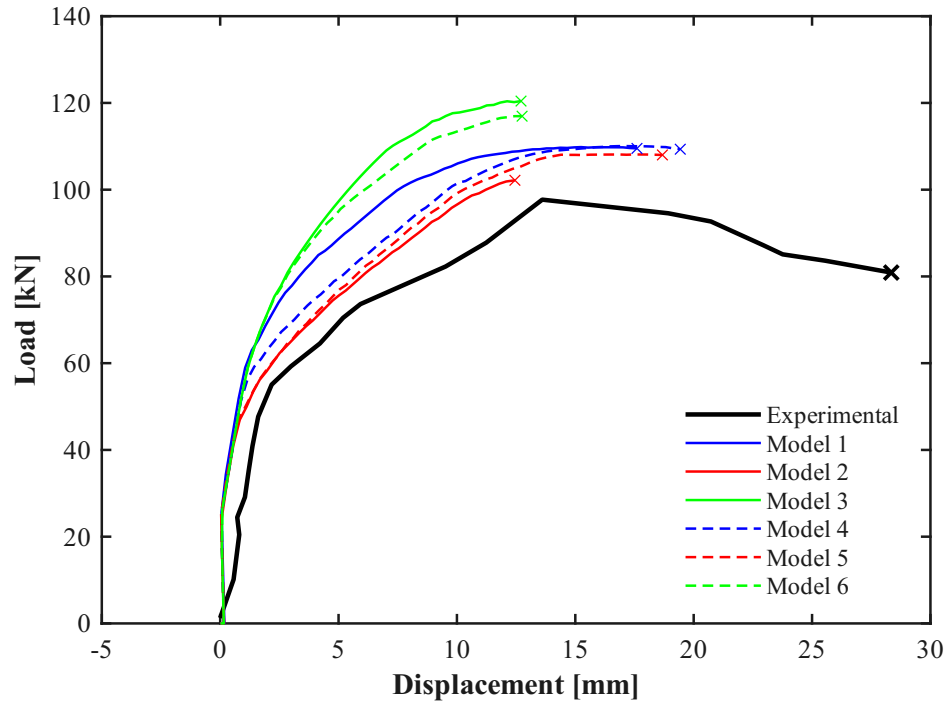
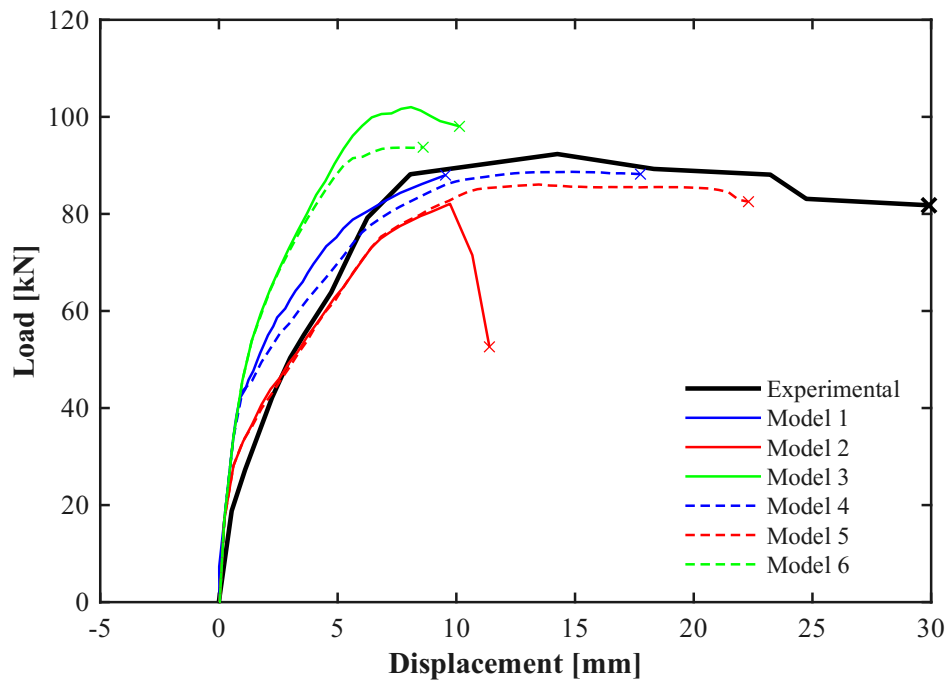
(a) PRC-B1 – Imperatore *et al.* [4](b) L1 – Yu *et al.* [5]

**Figure 5.3:** Experimental versus the six numerical load–displacement curves for the pre-tensioned specimens PRC-B1 and L1.



(a) FP/f0 – Padmarajaiah & Ramaswamy [6]



(b) PP/f0 – Padmarajaiah & Ramaswamy [6]

**Figure 5.4:** Experimental versus the six numerical load–displacement curves for the pre-tensioned specimens FP/f0 and PP/f0.

of  $\vartheta$  closest to unity (or slightly above it), while the pseudo-plastic scenarios (Models 3 and 6) systematically and significantly overestimate  $R_{\text{num}}$ . This behaviour is consistent with the fact that, in the absence of a continuous tendon–concrete interaction, the post-cracking stiffness of the beam is governed almost exclusively by the conventional reinforcement and by the residual prestress; a fracture energy larger than the *fib* MC2010 value therefore overestimates the ductility of the bottom chord and shifts the numerical peak upwards. The Peera specimens are thus the most sensitive of the database to the choice of  $G_f$ , and a clear case for which the calibrated *fib* MC2010 value provides a balanced compromise.

## 5.4 Cross-specimen synthesis

### 5.4.1 Influence of the crack model

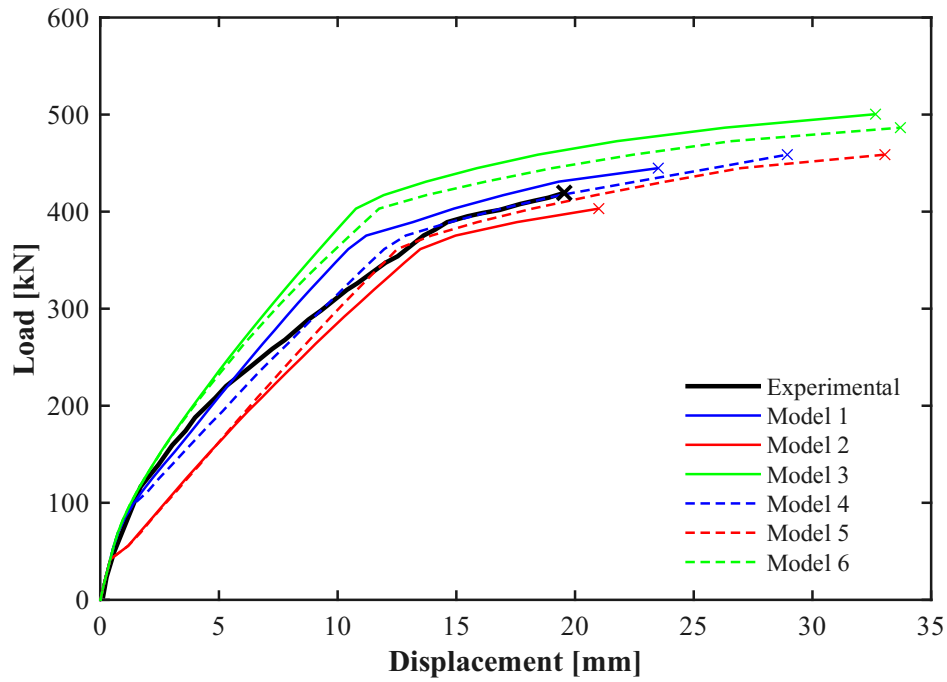
Aggregating the sixty numerical predictions, three general trends emerge regarding the crack model. First, the fixed and the rotated formulations produce, on average, similar values of  $R_{\text{num}}$  when the calibrated  $G_f$  is used (Models 2 vs 5): the mean of  $\vartheta$  across the ten specimens is  $\mu_{\vartheta,2} = 0.956$  and  $\mu_{\vartheta,5} = 0.935$  respectively. Second, the rotated formulation produces a slightly tighter distribution of  $\vartheta$  – the sample coefficient of variation is lower than for the corresponding fixed scenario – consistent with the absence of stress-locking phenomena. Third, the fixed brittle scenario (Model 1) is the only one in which the mean is essentially equal to one ( $\mu_{\vartheta,1} = 1.019$ ); this apparently favourable outcome is, however, the result of two compensating effects, namely the premature softening of the very low  $G_f$  and the stress-locking of the fixed crack, and is therefore not a reliable calibration target.

### 5.4.2 Influence of the fracture energy

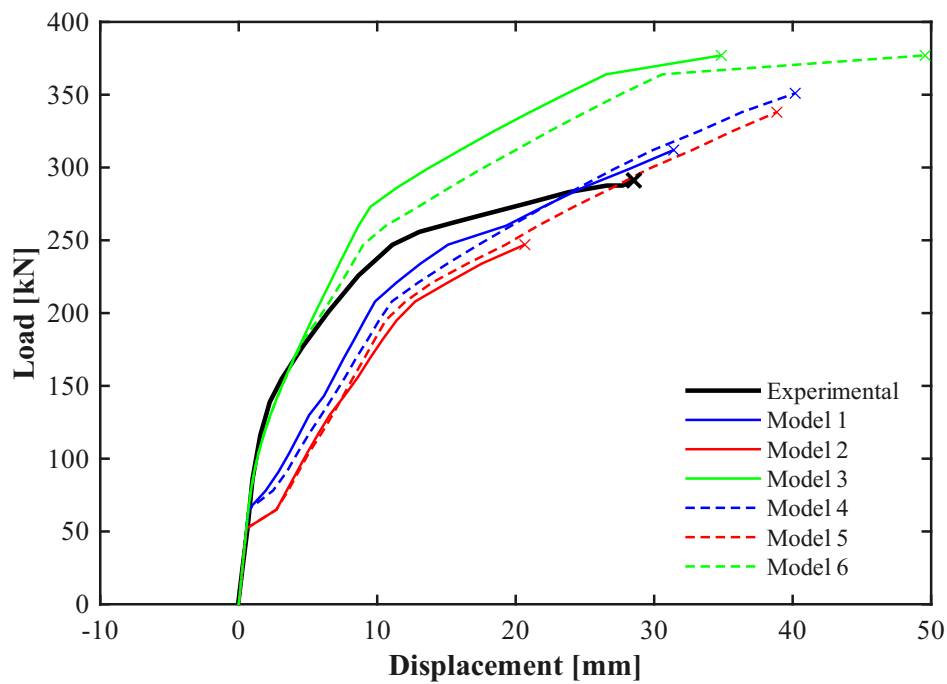
The dependence of  $\vartheta$  on the level of  $G_f$  is monotonic for every specimen: increasing  $G_f$  increases  $R_{\text{num}}$  and decreases  $\vartheta$ . The amplitude of this dependence varies markedly across the database. On the four Salah specimens and on Moawad B3, the three fracture-energy levels produce values of  $\vartheta$  within  $\pm 0.06$  of each other, indicating a relatively limited sensitivity of the global response to the softening branch. On the two Padmarajaiah specimens (small cross-section, high-strength concrete) and on the two Peera unbonded specimens (no continuous tendon–concrete interaction), the same variation of  $G_f$  produces  $\vartheta$  ranges of the order of 0.3, confirming the expectation that the brittle/pseudo-plastic bracket is the controlling parameter for these geometries.

### 5.4.3 Bonded versus unbonded behaviour

The two unbonded specimens of the database (Peera IB35 and IB60) provide the empirical evidence supporting the modelling strategy of section 3.2.3. The width of the  $\vartheta$  envelope on these two specimens is larger than on any of the bonded ones, but the average error is comparable to that of the bonded post-tensioned beams (Moawad B3, Salah TS1–TS3), indicating that the single-truss representation of the unbonded



(a) IB35-10-0.7-1 – Peera & Oukaili [7]



(b) IB60-10-0.5-2 – Peera & Oukaili [7]

**Figure 5.5:** Experimental versus the six numerical load–displacement curves for the unbonded post-tensioned specimens IB35-10-0.7-1 and IB60-10-0.5-2.

tendon captures the global elongation principle of eq. (2.3) of Chapter 2 satisfactorily, provided that the fracture energy is not pushed beyond the *fib* MC2010 value.

## 5.5 Stress-locking and softening: discussion

The results of sections 5.3 and 5.4 allow a critical discussion of the two competing mechanisms that govern the numerical response of the fixed crack model: the stress-locking phenomenon and the post-peak softening.

The fixed crack model relies on a frozen crack normal that, in combination with a non-zero shear retention factor, may transmit shear stresses inconsistent with the current principal stress direction. This stress-locking artificially stiffens the response and can produce a numerical overestimation of  $R_{\text{num}}$  when the cracks realign during the loading history – a situation particularly relevant for flexural members in which the neutral axis migrates upwards as cracking progresses. The rotated crack model, by aligning the crack normal with the principal strain direction at every iteration, is intrinsically free from stress-locking and provides a more conservative prediction of  $R_{\text{num}}$ . This is reflected in the systematically smaller scatter of  $\vartheta$  observed for the rotated calibrated scenario (Model 5) compared with the fixed calibrated scenario (Model 2) in section 5.4.

The fracture energy  $G_f$  controls the area below the post-peak tensile branch and therefore the rate at which the concrete loses its tensile capacity after cracking. A very low  $G_f$  ( $G_f/100$ , Models 1 and 4) produces a sharp, quasi-elastic–brittle drop that anticipates the post-peak softening and reduces  $R_{\text{num}}$ ; a maximised  $G_f$  (Models 3 and 6) drives the post-peak response towards a pseudo-plastic plateau that overestimates  $R_{\text{num}}$ . The *fib* MC2010 calibrated value occupies an intermediate position that, combined with the rotated crack formulation (Model 5), produces the most balanced statistical performance across the benchmark database.

## 5.6 Master synthesis table of $R_{\text{num}}$ across the six scenarios

The sixty numerical capacities obtained in this chapter are collected in table 5.8, together with the corresponding model uncertainty ratios  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  reported in parentheses. This table constitutes the direct input of the statistical analysis of Chapter 6, where the ten populations of  $\vartheta$  generated by the six scenarios will be characterised by their mean, coefficient of variation and probability distribution, and used to calibrate the partial safety factor  $\gamma_{Rd}$  through the Bayesian updating framework of Castaldo, Gino, Miceli and Mancini [8, 11, 12].

**Table 5.8:** Master synthesis of the numerical ultimate capacity  $R_{\text{num}}$  [kN] and, in parentheses, the model uncertainty ratio  $\vartheta = R_{\text{exp}}/R_{\text{num}}$ , for every specimen of the benchmark database under each of the six numerical scenarios of table 3.2.  $M_1$ – $M_3$  correspond to the fixed crack model with  $G_f/100$ , calibrated  $G_f$  and maximised  $G_f$  respectively;  $M_4$ – $M_6$  correspond to the rotated crack model with the same three fracture-energy levels.

| Specimen      | $R_{\text{exp}}$ [kN] | $M_1$          | $M_2$          | $M_3$          | $M_4$          | $M_5$          | $M_6$          |
|---------------|-----------------------|----------------|----------------|----------------|----------------|----------------|----------------|
| B3            | 123.80                | 112.80 (1.097) | 121.55 (1.018) | 120.42 (1.028) | 123.32 (1.004) | 123.93 (0.999) | 120.92 (1.024) |
| TS1           | 147.70                | 170.41 (0.867) | 156.40 (0.944) | 174.46 (0.847) | 174.71 (0.845) | 161.15 (0.917) | 155.64 (0.949) |
| TS2           | 157.90                | 170.10 (0.928) | 170.10 (0.928) | 166.95 (0.946) | 173.25 (0.911) | 173.25 (0.911) | 173.25 (0.911) |
| TS3           | 127.90                | 132.30 (0.967) | 132.30 (0.967) | 135.45 (0.944) | 132.30 (0.967) | 132.30 (0.967) | 119.70 (1.068) |
| PRC-B1        | 187.49                | 205.91 (0.911) | 212.29 (0.883) | 222.29 (0.843) | 210.62 (0.890) | 211.16 (0.888) | 218.72 (0.857) |
| L1            | 227.80                | 199.78 (1.140) | 222.76 (1.023) | 214.11 (1.064) | 214.87 (1.060) | 224.67 (1.014) | 207.36 (1.099) |
| IB35-10-0.7-1 | 417.00                | 403.11 (1.034) | 444.81 (0.937) | 500.41 (0.833) | 458.71 (0.909) | 458.71 (0.909) | 486.51 (0.857) |
| FP/f0         | 96.99                 | 102.17 (0.949) | 109.78 (0.884) | 120.43 (0.805) | 108.11 (0.897) | 110.07 (0.881) | 116.95 (0.829) |
| PP/f0         | 91.66                 | 82.08 (1.117)  | 87.99 (1.042)  | 102.01 (0.899) | 86.06 (1.065)  | 88.67 (1.034)  | 93.74 (0.978)  |
| IB60-10-0.5-2 | 291.15                | 247.00 (1.179) | 312.00 (0.933) | 377.01 (0.772) | 337.99 (0.861) | 350.99 (0.829) | 377.01 (0.772) |

The values in table 5.8 provide the empirical material on which the statistical inference, the Bayesian updating and the calibration of the partial safety factor  $\gamma_{Rd}$  are developed in Chapter 6.

## 5.7 Synthesis of the model uncertainty ratios

The complete set of model uncertainty ratios  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  generated by the parametric study is collected in table 5.9. This ratio matrix is the immediate input to the statistical inference, the goodness-of-fit testing and the Bayesian calibration of the partial safety factor  $\gamma_{Rd}$  developed in Chapter 6: each column  $M_1$ – $M_6$  constitutes a population of ten observations of  $\vartheta$  whose mean and dispersion characterise the model uncertainty associated with the corresponding modelling choice.

**Table 5.9:** Model uncertainty ratio  $\vartheta = R_{\text{exp}}/R_{\text{num}}$  for every specimen of the benchmark database under each of the six numerical scenarios ( $M_1$ – $M_6$  as defined in table 3.2). This matrix is the direct input to the statistical analysis and Bayesian calibration of Chapter 6.

| Specimen      | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ |
|---------------|-------|-------|-------|-------|-------|-------|
| B3            | 1.097 | 1.018 | 1.028 | 1.004 | 0.999 | 1.024 |
| TS1           | 0.867 | 0.944 | 0.847 | 0.845 | 0.917 | 0.949 |
| TS2           | 0.928 | 0.928 | 0.946 | 0.911 | 0.911 | 0.911 |
| TS3           | 0.967 | 0.967 | 0.944 | 0.967 | 0.967 | 1.068 |
| PRC-B1        | 0.911 | 0.883 | 0.843 | 0.890 | 0.888 | 0.857 |
| L1            | 1.140 | 1.023 | 1.064 | 1.060 | 1.014 | 1.099 |
| FP/f0         | 0.949 | 0.883 | 0.805 | 0.897 | 0.881 | 0.829 |
| PP/f0         | 1.117 | 1.042 | 0.899 | 1.065 | 1.034 | 0.978 |
| IB35-10-0.7-1 | 1.034 | 0.938 | 0.833 | 0.909 | 0.909 | 0.857 |
| IB60-10-0.5-2 | 1.179 | 0.933 | 0.772 | 0.861 | 0.830 | 0.772 |

The model uncertainty ratios of table 5.9 confirm the trends already discussed in section 5.4: the calibrated scenarios  $M_2$  and  $M_5$  are the closest to unity, while the

brittle and maximised fracture-energy scenarios bracket the response from above and below. This ratio matrix constitutes the direct input to the Bayesian calibration of the partial safety factor  $\gamma_{Rd}$  developed in Chapter 6.

# Chapter 6

## Model Uncertainty Quantification, Calibration of $\gamma_{Rd}$ and Conclusions

The empirical material accumulated in Chapter 5 – the sixty values of the numerical ultimate capacity  $R_{\text{num}}$  obtained for the ten benchmark specimens under the six solution strategies of table 3.2 – is processed in this chapter within the Bayesian probabilistic framework adopted by Castaldo, Gino, Miceli and Mancini [8, 11, 12] for the calibration of the partial safety factor for resistance model uncertainty. The chapter is structured around the five-stage Bayesian roadmap summarised in fig. 6.1: (i) construction of the prior information from the experimental–numerical benchmark; (ii) internal updating through a leave-one-out Bayesian scheme; (iii) aggregation into an average posterior distribution conditional on all the solution strategies; (iv) deprivation of the experimental uncertainty; (v) calibration of the partial safety factor  $\gamma_{Rd}$  for model uncertainty. This chapter first develops the *prior-information* stage (section 6.2, section 6.3) and the associated *statistical inference* (section 6.4); the Bayesian updating, the posterior aggregation, the deprivation of the experimental uncertainty and the calibration of  $\gamma_{Rd}$  follow in the subsequent sections.

### 6.1 Introduction and probabilistic framework recap

#### 6.1.1 The five-stage Bayesian roadmap

The quantification of the model uncertainty pursued in this thesis follows a fully probabilistic, Bayesian rationale. In the GRF of *fib* MC2010 [1], the resistance model uncertainty is represented by the random variable  $\vartheta = R_{\text{exp}}/R_{\text{num}}$ , whose statistics must be inferred from the systematic comparison between physical experiments and their numerical reproduction. Because the present benchmark is necessarily limited in size and heterogeneous in failure mode, prestressing technology and modelling choice, a direct frequentist estimate of the moments of  $\vartheta$  would be both unstable and difficult to defend on a sample of ten specimens. A Bayesian treatment, in which an initial description of the uncertainty is progressively refined by the available evidence, provides a more robust and transparent route to the calibration of the partial safety factor  $\gamma_{Rd}$ , and is the approach consistently adopted in the recent literature on the model

uncertainty of non-linear analyses [8, 11, 12].

The complete inferential procedure is summarised in the flowchart of fig. 6.1 and is organised into five sequential stages, each of which transforms a well-defined probabilistic object into the input of the following one, from the raw experimental-to-numerical ratios up to the final design coefficient.

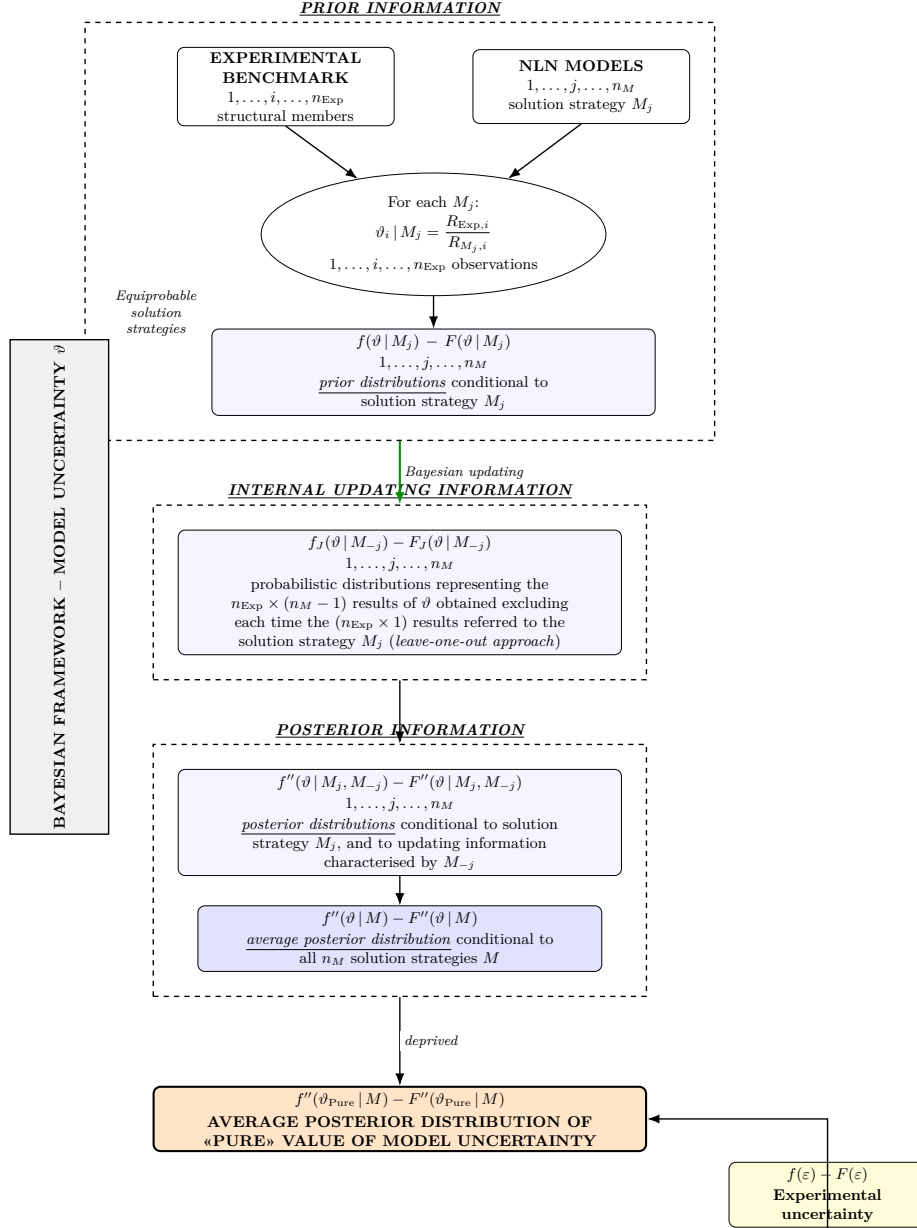


**Figure 6.1:** Bayesian framework adopted in the present thesis for the calibration of the model uncertainty  $\vartheta$ . The framework is organised in five stages: prior information, internal updating with a leave-one-out scheme, posterior information, deprivation of the experimental uncertainty  $\varepsilon$ , and the resulting average posterior of the pure model uncertainty used for the calibration of  $\gamma_{Rd}$ .

The first stage, *prior information*, establishes the raw empirical basis of the calibration. For every specimen  $i$  of the experimental benchmark and every solution strategy

$M_j$ , the ratio  $\vartheta_i | M_j = R_{\text{Exp},i}/R_{M_j,i}$  between the measured and the computed ultimate resistance is evaluated. Under the equiprobable-solution-strategy hypothesis, by which no numerical scenario is assigned an *a priori* preference over the others, these ratios are collected, separately for each strategy, into the six prior distributions  $f(\vartheta | M_j)$ . This stage therefore condenses the sixty individual comparisons of Chapter 5 into six conditional populations of the model uncertainty, one for each combination of crack model and fracture-energy level.

The second stage, *internal updating*, introduces the Bayesian learning mechanism. Rather than relying on an external, subjective prior, the framework refines the description of each strategy using the evidence carried by all the remaining strategies, by means of a leave-one-out scheme. For the  $j$ -th strategy, the new information is the pooled set  $M_{-j}$  of the other five strategies, and the prior  $f(\vartheta | M_j)$  is combined with this evidence through the conjugate normal-inverse- $\chi^2$  updating recalled in section 6.5.2. The leave-one-out construction guarantees that each posterior is informed by genuinely independent evidence and is not corroborated by the very observations it is meant to describe, which would otherwise lead to an artificial reduction of the dispersion.

The third stage, *posterior information*, collects the outcome of the updating. For each strategy a posterior distribution  $f''(\vartheta | M_j, M_{-j})$  is obtained, conditional both on the strategy itself and on the updating information  $M_{-j}$ . The six posteriors are subsequently aggregated, under the same equiprobability assumption, into a single average posterior distribution  $f''(\vartheta | M)$  that no longer depends on the particular modelling choice and that represents the model uncertainty of the NLFEA procedure as a whole, rather than that of any individual scenario.

The fourth stage, *deprivation of the experimental uncertainty*, isolates the purely numerical component of the model uncertainty. The measured resistance  $R_{\text{exp}}$  is itself affected by an experimental uncertainty  $\varepsilon$ , of unit mean and small coefficient of variation, which is unavoidably embedded in the average posterior. By removing the contribution of  $\varepsilon$  from the variance, the framework yields the average posterior of the pure value of the model uncertainty  $\vartheta_{\text{Pure}}$ , which reflects only the discrepancy attributable to the numerical model and is the quantity of interest for a model-uncertainty safety factor.

The fifth and final stage, *calibration*, maps the pure average posterior onto a single design value. Following the first-order reliability format of section 6.8, the mean and the coefficient of variation of  $\vartheta_{\text{Pure}}$  are combined with the resistance sensitivity factor  $\alpha_R$  and the target reliability index  $\beta$  to produce the partial safety factor  $\gamma_{Rd}$  for model uncertainty. The five stages thus form a closed inferential chain that begins with the raw experimental-to-numerical ratios and terminates in a single, code-compatible coefficient.

The remainder of this chapter develops each stage in turn: the prior information and its distributions in sections 6.2 and 6.3, the statistical inference and the goodness-of-fit testing of the lognormal hypothesis in section 6.4, the leave-one-out updating and the resulting posterior distributions in section 6.5, the average and pure posteriors in sections 6.6 and 6.7, and finally the calibration of  $\gamma_{Rd}$  together with its comparison against the current code provisions in sections 6.8 and 6.9.

### 6.1.2 Notation and index conventions

The following indices and quantities are used systematically throughout the chapter:

- $i = 1, \dots, n_{\text{Exp}} = 10$  – index over the benchmark specimens of Chapter 4.
- $j = 1, \dots, n_M = 6$  – index over the solution strategies (numerical scenarios) of table 3.2. The scenarios are equiprobable, i.e. no *a priori* preference is assigned to any specific combination of crack model and fracture energy.
- $R_{\text{Exp},i}$  – experimental ultimate capacity of the  $i$ -th specimen, reported in Chapter 4.
- $R_{M_j,i}$  – numerical ultimate capacity of the  $i$ -th specimen predicted by the  $j$ -th solution strategy, reported in table 5.8 of Chapter 5.
- $\vartheta_i | M_j = R_{\text{Exp},i}/R_{M_j,i}$  – *conditional* value of the model uncertainty for the  $i$ -th specimen under the  $j$ -th solution strategy.
- $f(\vartheta | M_j)$  – prior probability density function of  $\vartheta$  conditional on the  $j$ -th solution strategy.

The prior distributions  $f(\vartheta | M_j)$  obtained in the present chapter constitute the input of the leave-one-out Bayesian updating developed in section 6.5.

## 6.2 Prior information: computation of $\vartheta_i | M_j$ from the benchmark

For each of the ten specimens of the benchmark database and for each of the six numerical scenarios, the conditional value of the model uncertainty is computed as

$$\vartheta_i | M_j = \frac{R_{\text{Exp},i}}{R_{M_j,i}}, \quad i = 1, \dots, 10, \quad j = 1, \dots, 6, \quad (6.1)$$

following the definition introduced in eq. (1.4) of Chapter 1 and operationally used in section 5.6 of Chapter 5. The sixty values produced by eq. (6.1) are collected in the second half of table 5.8; they constitute the empirical sample from which the prior probability distributions are constructed in the next section. Each of the six numerical scenarios therefore provides one prior sample  $\{\vartheta_1 | M_j, \dots, \vartheta_{10} | M_j\}$  of size  $n_{\text{Exp}} = 10$ .

## 6.3 Prior distributions $f(\vartheta | M_j)$ under the equiprobable-solution-strategy hypothesis

### 6.3.1 Descriptive statistics of the six prior samples

The descriptive statistics of the six prior samples obtained from eq. (6.1) are reported in table 6.1: the sample mean  $\mu_{\vartheta | M_j}$ , the standard deviation  $\sigma_{\vartheta | M_j}$ , the coefficient of variation  $V_{\vartheta | M_j} = \sigma_{\vartheta | M_j}/\mu_{\vartheta | M_j}$ , the lognormal parameters  $\lambda_j = \mathbb{E}[\ln \vartheta_i | M_j]$  and  $\zeta_j = \text{SD}[\ln \vartheta_i | M_j]$ , the sample skewness and the sample kurtosis. The lognormal

parameters  $\lambda_j$  and  $\zeta_j$  are estimated from the log-transformed sample,  $\lambda_j \approx \ln \mu_{\vartheta|M_j} - \frac{1}{2}\zeta_j^2$  and  $\zeta_j \approx \sqrt{\ln(1 + V_{\vartheta|M_j}^2)}$ , in accordance with the standard relations of the lognormal family [1, 8, 11].

**Table 6.1:** Descriptive statistics of the six prior samples of the model uncertainty  $\vartheta|M_j$  obtained from eq. (6.1) on the ten benchmark specimens.  $\mu$  is the sample mean,  $\sigma$  the sample standard deviation,  $V = \sigma/\mu$  the coefficient of variation;  $\lambda$  and  $\zeta$  are the parameters of the fitted lognormal distribution.

| Statistic                    | $M_1$ Fixed- $G_f/100$ | $M_2$ Fixed-cal. | $M_3$ Fixed-max | $M_4$ Rot.- $G_f/100$ | $M_5$ Rot.-cal. | $M_6$ Rot.-max |
|------------------------------|------------------------|------------------|-----------------|-----------------------|-----------------|----------------|
| $\mu_{\vartheta M_j}$ [-]    | 1.0195                 | 0.9561           | 0.8986          | 0.9413                | 0.9351          | 0.9351         |
| $\sigma_{\vartheta M_j}$ [-] | 0.1094                 | 0.0557           | 0.0948          | 0.0778                | 0.0658          | 0.1086         |
| $V_{\vartheta M_j}$ [-]      | 0.1073                 | 0.0582           | 0.1055          | 0.0826                | 0.0704          | 0.1161         |
| $\lambda_j$ [-]              | 0.0136                 | -0.0466          | -0.1124         | -0.0639               | -0.0696         | -0.0738        |
| $\zeta_j$ [-]                | 0.1070                 | 0.0582           | 0.1052          | 0.0825                | 0.0703          | 0.1157         |
| Skewness                     | 0.1148                 | 0.2451           | 0.4737          | 0.5257                | 0.1186          | 0.1163         |
| Kurtosis                     | 1.5492                 | 1.8408           | 2.0418          | 1.8757                | 1.9159          | 1.8214         |

### 6.3.2 Observations on the prior statistics

The descriptive statistics of table 6.1 highlight three features that are central to the subsequent Bayesian analysis.

**Sample mean.** The sample mean  $\mu_{\vartheta|M_j}$  varies between 0.899 (Model  $M_3$ , Fixed- $G_f$  maximised) and 1.020 (Model  $M_1$ , Fixed- $G_f/100$ ). Five scenarios out of six yield a sample mean below unity, i.e. they overestimate the experimental capacity on average; only the fixed brittle scenario produces a marginally safe-sided mean. The two calibrated scenarios (Model  $M_2$  Fixed-cal., Model  $M_5$  Rot.-cal.) give means very close to each other (0.956 and 0.935), confirming that the choice of the crack model has a moderate effect when the fracture energy follows the *fib* MC2010 [1] calibration.

**Coefficient of variation.** The coefficient of variation  $V_{\vartheta|M_j}$  varies between 5.8% (Model  $M_2$  Fixed-cal.) and 11.6% (Model  $M_6$  Rot.-max). The Fixed-cal. scenario produces the tightest prior distribution of the database, with a coefficient of variation that is almost half of the one obtained with  $G_f/100$  or  $G_f$  maximised. The fixed-brittle ( $M_1$ ), fixed-max ( $M_3$ ) and rotated-max ( $M_6$ ) scenarios cluster around  $V \approx 10$ –12%, suggesting that both extremes of the fracture energy bracket inflate the spread of the prior independently of the crack model.

**Shape of the distribution.** The sample skewness is positive for every scenario and ranges from 0.11 (Model  $M_1$ , Model  $M_5$ , Model  $M_6$ ) to 0.53 (Model  $M_4$  Rot.- $G_f/100$ ), consistent with the right-skewed shape characteristic of a lognormal variable. The sample kurtosis varies between 1.55 and 2.04, i.e. slightly below the value 3 of a perfectly normal distribution; this is the expected behaviour for a small lognormal sample ( $n_{\text{Exp}} = 10$ ) in which the heavy upper tail is not yet fully resolved. Both indicators support the choice of the lognormal distribution as the parametric family for

$\vartheta | M_j$ , in line with the recommendations of *fib* MC2010 [1] and the practice established by Castaldo, Gino and Mancini for NLFEA of RC structures [8, 11, 12].

### 6.3.3 Normal probability plots of the prior samples

For each of the six scenarios, the ten  $\vartheta_i | M_j$  values are assessed through a *normal probability plot* in the log-domain: the ordered log-transformed observations are plotted against the theoretical quantiles of the fitted lognormal distribution characterised by the parameters  $\lambda_j$  and  $\zeta_j$  reported in table 6.1, so that a good lognormal fit appears as an approximately straight line. The resulting probability plots are reported in figs. 6.2a to 6.2f.

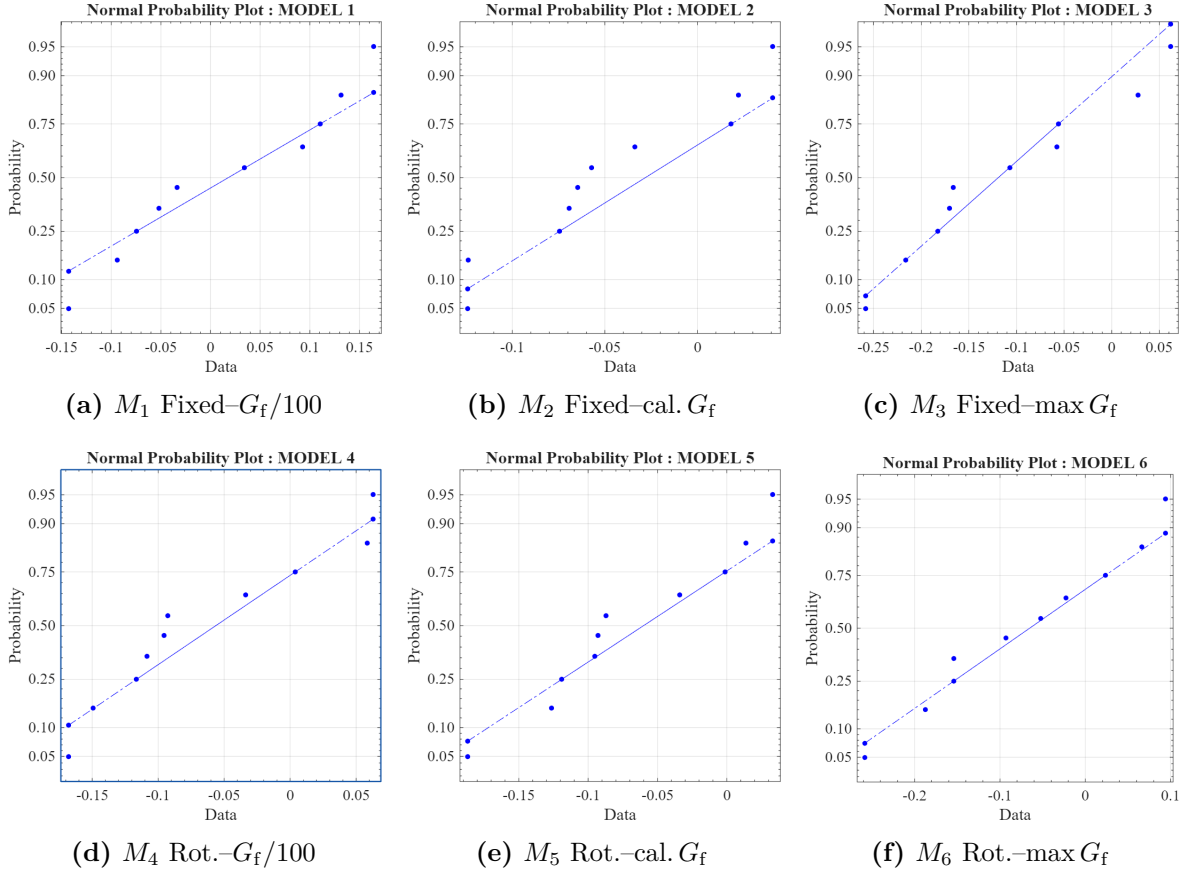


**Figure 6.2:** Normal probability plots of the ten  $\vartheta_i | M_j$  values against the fitted lognormal distribution  $\text{LN}(\lambda_j, \zeta_j)$  of table 6.1, for the six solution strategies  $M_1$ – $M_6$ . An approximately straight alignment supports the lognormal hypothesis.

## 6.4 Statistical inference: goodness-of-fit of the lognormal hypothesis

### 6.4.1 Choice of the lognormal distribution and goodness-of-fit strategy

The lognormal distribution is by now the standard parametric family used to describe model-uncertainty random variables in the framework of the GRF [1, 8, 11, 12]. Three

reasons support this choice: (i) the support  $\vartheta > 0$  is consistent with the physical nature of a ratio of two non-negative resistances; (ii) the right-skewed shape matches the empirical evidence of section 6.3.2; (iii) the lognormal family admits closed-form expressions for the fractiles used in the calibration of the partial safety factor  $\gamma_{Rd}$  (see section 2.5.4 of Chapter 2).

To formally test the lognormal hypothesis on each of the six prior samples, an *Anderson–Darling* goodness-of-fit test is performed against the null hypothesis

$$H_0 : F(\vartheta | M_j) = \Phi\left(\frac{\ln \vartheta - \lambda_j}{\zeta_j}\right), \quad j = 1, \dots, 6, \quad (6.2)$$

where  $\Phi$  is the standard normal cumulative distribution function and  $(\lambda_j, \zeta_j)$  are the parameters of the lognormal distribution fitted to the prior sample of the  $j$ -th scenario (table 6.1). The Anderson–Darling test is particularly suitable for small samples ( $n_{\text{Exp}} = 10$ ) because it weighs more heavily the discrepancies in the distribution tails, which are the regions relevant for the calibration of  $\gamma_{Rd}$ . The test is implemented in MATLAB through the function `adtest`, with a significance level  $\alpha = 5\%$ . The Anderson–Darling test is adopted in the present thesis as the sole and definitive evidence for the lognormal hypothesis, since it provides the most reliable goodness-of-fit assessment for small samples drawn from a strictly positive, right-skewed distribution.

#### 6.4.2 Results of the Anderson–Darling test

The outcome of the Anderson–Darling test on each of the six prior samples is reported in table 6.2, where  $p_{AD}$  is the  $p$ -value returned by the test and  $H_{AD}$  is the binary decision indicator ( $H_{AD} = 0$  means that the null hypothesis is *not rejected* at  $\alpha = 5\%$ ;  $H_{AD} = 1$  means that the null hypothesis is rejected).

**Table 6.2:** Anderson–Darling goodness-of-fit test of the lognormal hypothesis (6.2) on each of the six prior samples  $\vartheta | M_j$ .  $p_{AD}$  is the  $p$ -value of the test;  $H_{AD} = 0$  indicates that the null hypothesis is not rejected at the  $\alpha = 5\%$  significance level.

| Indicator                   | $M_1$ Fixed- $G_f/100$ | $M_2$ Fixed-cal. | $M_3$ Fixed-max | $M_4$ Rot.- $G_f/100$ | $M_5$ Rot.-cal. | $M_6$ Rot.-max |
|-----------------------------|------------------------|------------------|-----------------|-----------------------|-----------------|----------------|
| $p_{AD}$                    | 0.5159                 | 0.3724           | 0.6586          | 0.2941                | 0.5793          | 0.9223         |
| $H_{AD}$ ( $\alpha = 5\%$ ) | 0                      | 0                | 0               | 0                     | 0               | 0              |
| Decision                    | accept                 | accept           | accept          | accept                | accept          | accept         |

The  $p$ -values reported in table 6.2 are all substantially larger than the chosen significance level  $\alpha = 5\%$ , ranging from 0.294 (Model  $M_4$ , Rotated- $G_f/100$ ) to 0.922 (Model  $M_6$ , Rotated- $G_f$  max); the corresponding decision indicators  $H_{AD}$  are all equal to zero. The Anderson–Darling test therefore *accepts* the lognormal hypothesis (6.2) for every solution strategy, with a particularly strong acceptance for the rotated-max ( $M_6$ ) and the fixed-max ( $M_3$ ) scenarios. The four scenarios that include the calibrated or the brittle fracture energy display intermediate  $p$ -values ( $p_{AD}$  between 0.29 and 0.58), still comfortably within the acceptance region.

### 6.4.3 Implications for the Bayesian framework

The acceptance of the lognormal hypothesis for all six prior samples is a necessary condition for the validity of the closed-form Bayesian updating implemented in the subsequent part of the chapter, which is based on the lognormal-conjugate framework adopted by Castaldo, Gino and Mancini [8, 11, 12]. Concretely, it ensures that: (i) the leave-one-out posterior distributions  $f_J(\vartheta | M_{-j})$  developed in section 6.5 remain in the lognormal family; (ii) the average posterior  $f''(\vartheta | M)$  of section 6.6 admits an analytical expression in terms of the conditional means and variances of the six posteriors; and (iii) the calibration of  $\gamma_{Rd}$  developed in section 6.8 can be performed through the standard lognormal fractile of the design quantile of  $\vartheta_{\text{Pure}}$ .

The prior information stage of the Bayesian framework is thus formally completed: the six prior samples have been computed (6.1), their descriptive statistics have been extracted (table 6.1), and the lognormal hypothesis has been accepted by the Anderson–Darling test (table 6.2). Section 6.5 develops the internal updating, the leave-one-out posterior distributions and the average posterior conditional on all the solution strategies.

## 6.5 Bayesian updating: leave-one-out posterior distributions

### 6.5.1 The leave-one-out updating scheme

The second stage of the Bayesian framework summarised in fig. 6.1 – the *internal updating* stage – combines the six prior distributions  $f(\vartheta | M_j)$ ,  $j = 1, \dots, n_M$ , characterised in section 6.3, with the experimental evidence collected on the benchmark database to produce six *leave-one-out* posterior distributions  $f_J(\vartheta | M_{-j})$ .

The leave-one-out scheme adopted in the present thesis is the one used by Castaldo, Gino and Mancini [8, 11, 12]: the update of the prior conditional on the  $j$ -th solution strategy is performed using the model uncertainty data obtained from the *remaining* five solution strategies,

$$M_{-j} = \{ M_k : k = 1, \dots, n_M, k \neq j \}, \quad |M_{-j}| = n_M - 1 = 5, \quad (6.3)$$

so that the updating sample collects  $n_{\text{Exp}} \times (n_M - 1) = 10 \times 5 = 50$  values of the model uncertainty for each leave-one-out posterior. This choice is consistent with the principle of *equiprobable solution strategies* introduced in section 6.1.1: by removing one strategy at a time, the remaining strategies provide an unbiased empirical updating source for the strategy being assessed.

### 6.5.2 Lognormal-conjugate Bayesian update

The acceptance of the lognormal hypothesis on the six prior samples (table 6.2) allows the update to be performed entirely in the log-transformed space. Let  $y_{i,k} = \ln \vartheta_{i,k} = \ln(R_{\text{Exp},i}/R_{M_k,i})$  denote the log-transformed value of the model uncertainty for the  $i$ -th

specimen and the  $k$ -th solution strategy, and let

$$\hat{\lambda}_{-j} = \frac{1}{n_{\text{Exp}} (n_M - 1)} \sum_{k \neq j} \sum_{i=1}^{n_{\text{Exp}}} y_{i,k}, \quad \hat{\sigma}_{-j}^2 = \frac{1}{n_{\text{Exp}} (n_M - 1) - 1} \sum_{k \neq j} \sum_{i=1}^{n_{\text{Exp}}} (y_{i,k} - \hat{\lambda}_{-j})^2, \quad (6.4)$$

be the sample mean and the sample variance of the  $n_{\text{Exp}} \times (n_M - 1)$  log-transformed observations of the updating set  $M_{-j}$ .

Before the conjugate update is carried out, it is instructive to summarise the statistics of the new information itself, i.e. of the pooled leave-one-out evidence  $M_{-j}$  that updates each prior. The first- and second-order moments of this evidence, recovered in the original (non-logarithmic) space, are reported in table 6.3. The mean of the new information  $\mu_{\text{ni}}$  is consistently smaller than unity, ranging from 0.933 for  $M_1$  (Fixed- $G_f/100$ ) to 0.957 for  $M_3$  (Fixed- $G_f$  max), so that the pooled evidence is safe-sided by 4 to 7%. The dispersion is moderate and stable across the strategies: the coefficient of variation  $V_{\text{ni}}$  stays within the narrow band 8.8% ( $M_1$ ) to 10.3% ( $M_2$ ), the two calibrated scenarios  $M_2$  and  $M_5$  exhibiting the largest values (10.3% and 10.1%) and the brittle fixed scenario  $M_1$  the smallest. This homogeneity of the updating evidence is a direct consequence of the equiprobable-solution-strategy hypothesis and underpins the stability of the posterior parameters derived below.

**Table 6.3:** Mean  $\mu_{\text{ni}}$ , standard deviation  $\sigma_{\text{ni}}$  and coefficient of variation  $V_{\text{ni}}$  of the leave-one-out new information  $M_{-j}$ , computed on the  $n_{\text{Exp}} \times (n_M - 1) = 50$  pooled observations of  $\vartheta$  that update each solution strategy  $M_j$ .

| Statistic                | M-1 Fixed- $G_f/100$ | M-2 Fixed-cal. | M-3 Fixed-max | M-4 Rot.- $G_f/100$ | M-5 Rot.-cal. | M-6 Rot.-max |
|--------------------------|----------------------|----------------|---------------|---------------------|---------------|--------------|
| $\mu_{\text{ni}}$ [-]    | 0.933                | 0.946          | 0.957         | 0.949               | 0.950         | 0.950        |
| $\sigma_{\text{ni}}$ [-] | 0.082                | 0.097          | 0.088         | 0.095               | 0.096         | 0.088        |
| $V_{\text{ni}}$ [-]      | 0.088                | 0.103          | 0.092         | 0.100               | 0.101         | 0.093        |

Following the conjugate normal-inverse- $\chi^2$  updating scheme adopted by Castaldo, Gino and Mancini [8, 11], the prior conditional on the  $j$ -th strategy is assigned an equivalent prior sample size  $n_0$ , and is updated with the leave-one-out evidence of size  $n = n_{\text{Exp}}(n_M - 1)/(n_M - 1) = 50$ ; in the present calibration  $n_0 = n = 50$ . With  $v_0 = n_0 - 1$  and  $v_n = v_0 + n$ , the posterior log-parameters are

$$\lambda_j^{\text{post}} = \frac{n_0 \lambda_j + n \hat{\lambda}_{-j}}{n_0 + n}, \quad (6.5)$$

$$(\zeta_j^{\text{post}})^2 = \frac{1}{v_n} \left[ v_0 \zeta_j^2 + (n - 1) \hat{\sigma}_{-j}^2 + \frac{n_0 n}{n_0 + n} (\hat{\lambda}_{-j} - \lambda_j)^2 \right], \quad (6.6)$$

where  $(\lambda_j, \zeta_j)$  are the prior parameters of table 6.1. The posterior mean  $\mu_j^{\text{post}}$  and coefficient of variation  $V_j^{\text{post}}$  of  $\vartheta | M_{-j}$  are then recovered from the lognormal relations

$$\mu_j^{\text{post}} = \exp\left(\lambda_j^{\text{post}} + \frac{1}{2}(\zeta_j^{\text{post}})^2\right), \quad V_j^{\text{post}} = \sqrt{\exp(\zeta_j^{\text{post}})^2 - 1}, \quad (6.7)$$

following standard lognormal-lognormal moment conversion [1].

### 6.5.3 Leave-one-out posterior parameters

The leave-one-out Bayesian update of eqs. (6.5) to (6.7) is implemented in MATLAB for the six solution strategies of table 3.2 and applied to the prior distributions of table 6.1. The resulting posterior parameters are collected in table 6.4.

**Table 6.4:** Leave-one-out posterior parameters of the model uncertainty  $\vartheta | M_{-j}$  obtained from the lognormal-conjugate Bayesian update (6.5)–(6.7).  $\lambda_j^{\text{post}}$  and  $\zeta_j^{\text{post}}$  are the parameters of the posterior lognormal distribution;  $\mu_j^{\text{post}}$  and  $V_j^{\text{post}}$  are the corresponding mean and coefficient of variation.

| Quantity                      | $M_1$ Fixed- $G_f/100$ | $M_2$ Fixed-cal. | $M_3$ Fixed-max | $M_4$ Rot.- $G_f/100$ | $M_5$ Rot.-cal. | $M_6$ Rot.-max |
|-------------------------------|------------------------|------------------|-----------------|-----------------------|-----------------|----------------|
| $\lambda_j^{\text{post}}$ [-] | -0.0298                | -0.0539          | -0.0802         | -0.0608               | -0.0631         | -0.0648        |
| $\zeta_j^{\text{post}}$ [-]   | 0.1068                 | 0.0832           | 0.1034          | 0.0909                | 0.0868          | 0.1048         |
| $\mu_j^{\text{post}}$ [-]     | 0.9762                 | 0.9508           | 0.9278          | 0.9449                | 0.9424          | 0.9424         |
| $V_j^{\text{post}}$ [-]       | 0.1071                 | 0.0833           | 0.1036          | 0.0911                | 0.0870          | 0.1051         |

### 6.5.4 Comparison between prior and posterior distributions

The prior distribution  $f(\vartheta | M_j)$  of section 6.3.1 and the leave-one-out posterior distribution  $f_j(\vartheta | M_{-j})$  of section 6.5.3 are compared panel by panel in fig. 6.3, where each graph is identified by the scenario label  $M_{-j}$  of table 3.2 of Chapter 3. The six graphs show directly that, for every solution strategy, the posterior density is narrower than and shifted with respect to the corresponding prior: the shift is most pronounced for the four extreme fracture-energy scenarios ( $M_{-1}$  and  $M_{-3}$  of the fixed crack model,  $M_{-4}$  and  $M_{-6}$  of the rotated one) and least pronounced for the two calibrated scenarios ( $M_{-2}$  and  $M_{-5}$ ), whose prior is already close to the cross-strategy consensus.

### 6.5.5 Discussion of the posterior shift

The Bayesian update of table 6.4 produces three systematic effects that are visible by direct comparison with the prior of table 6.1.

**Shrinkage of the posterior mean towards the global average.** The posterior means  $\mu_j^{\text{post}}$  are systematically *shifted* towards the global average across the six scenarios ( $\sim 0.94$ ), pulling away from the prior means of the more extreme scenarios. For example, the prior of Model  $M_1$  (Fixed- $G_f/100$ ) has  $\mu = 1.020$ , while its leave-one-out posterior has  $\mu_1^{\text{post}} = 0.9762$ : the updating information of the remaining five strategies pulls the brittle estimate downwards by  $\approx 4.4\%$ . Conversely, the prior of Model  $M_3$  (Fixed- $G_f$  max), with  $\mu = 0.899$ , is pulled upwards to  $\mu_3^{\text{post}} = 0.9278$  by the same updating evidence. The two calibrated scenarios ( $M_2$  and  $M_5$ ), which already lie close to the global average, are shifted only marginally ( $\mu_2^{\text{post}} = 0.9508$ ,  $\mu_5^{\text{post}} = 0.9424$ ).

**Stabilisation of the posterior coefficient of variation.** The posterior coefficients of variation  $V_j^{\text{post}}$  are all comprised between 8% and 11%, with a notable reduction of the extreme values observed in the prior. The two scenarios that had the smallest prior

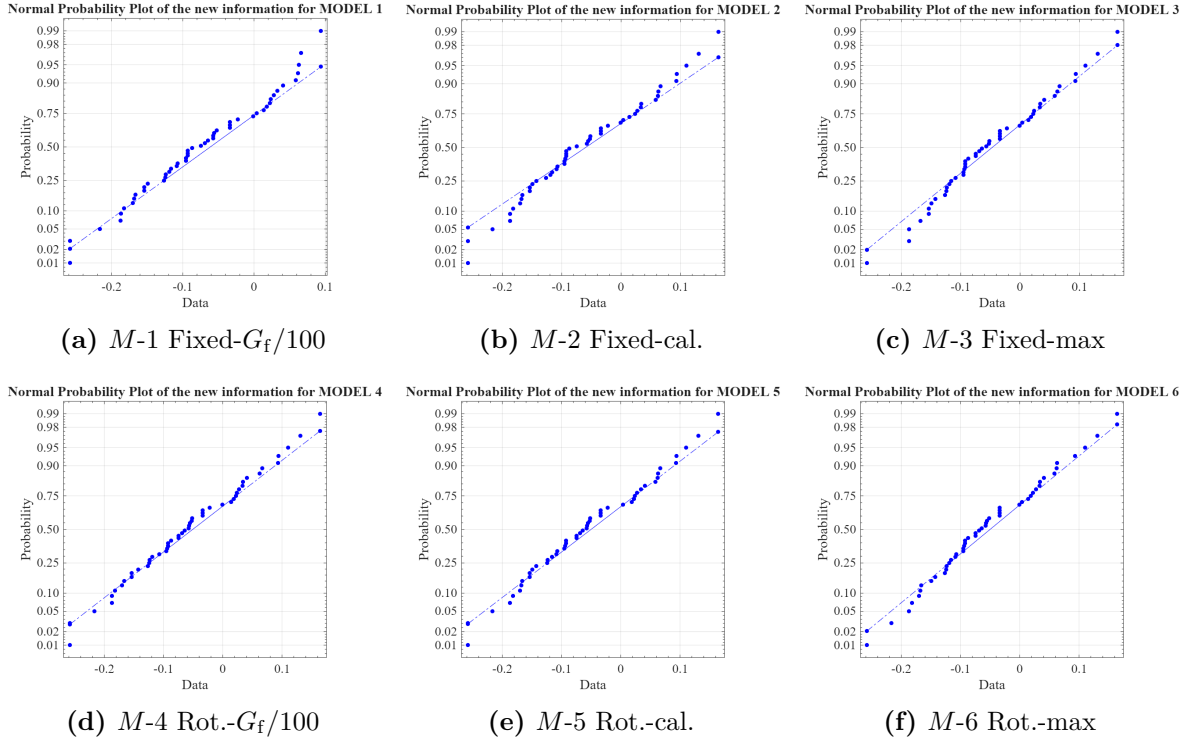


**Figure 6.3:** Comparison between the prior distribution  $f(\vartheta | M_j)$  and the leave-one-out posterior  $f_J(\vartheta | M_{-j})$  for each of the six solution strategies  $M_1$ – $M_6$ .

coefficient of variation ( $M_2$  Fixed-cal.,  $V = 5.8\%$ , and  $M_5$  Rot.-cal.,  $V = 7.0\%$ ) show a posterior coefficient of variation that *increases* (to  $8.3\%$  and  $8.7\%$  respectively), because the update incorporates the additional dispersion of the other five scenarios. Conversely, the two scenarios with the largest prior coefficient of variation ( $M_1$  Fixed- $G_f/100$  and  $M_6$  Rot.-max,  $V \approx 11\%$ ) keep an approximately constant posterior coefficient of variation, indicating that the updating information neither inflates nor compresses their already wide spread.

**Convergence of the posterior across solution strategies.** The six posterior distributions  $f_J(\vartheta | M_{-j})$  are markedly closer to each other than the six priors  $f(\vartheta | M_j)$ . In particular, the range of  $\mu_j^{\text{post}}$  across the six scenarios is reduced to  $0.928$ – $0.976$ , whereas the corresponding prior means spanned the wider interval  $0.899$ – $1.020$ . This convergence is a direct consequence of the equiprobable solution strategy hypothesis adopted in section 6.1.1: by allowing every strategy to be updated by all the others, the posterior distributions become statistically homogeneous, which is the necessary condition for the aggregation into the average posterior of section 6.6.

## 6.6 Average posterior distribution $f''(\vartheta | M)$

### 6.6.1 Aggregation across the six solution strategies

The third stage of the Bayesian framework consists in aggregating the six leave-one-out posterior distributions  $f_J(\vartheta | M_{-j})$  into a single *average posterior distribution*  $f''(\vartheta | M)$

that is no longer conditional on any specific solution strategy. Under the *equiprobable solution strategy* hypothesis adopted in section 6.1.1, each of the six posteriors carries the same statistical weight, so that the average mean and coefficient of variation of the aggregated posterior are computed as the arithmetic means of the six per-scenario values [8, 11, 12]:

$$\mu_M^{\text{post}} = \frac{1}{n_M} \sum_{j=1}^{n_M} \mu_j^{\text{post}}, \quad V_M^{\text{post}} = \frac{1}{n_M} \sum_{j=1}^{n_M} V_j^{\text{post}}, \quad n_M = 6. \quad (6.8)$$

For the calibration of the partial safety factor  $\gamma_{Rd}$  developed in section 6.8, it is convenient to approximate  $f''(\vartheta | M)$  by a single lognormal distribution with parameters  $(\lambda_M^{\text{post}}, \zeta_M^{\text{post}})$  matched on the mean and coefficient of variation of eq. (6.8),

$$\zeta_M^{\text{post}} = \sqrt{\ln(1 + (V_M^{\text{post}})^2)}, \quad \lambda_M^{\text{post}} = \ln \mu_M^{\text{post}} - \frac{1}{2} (\zeta_M^{\text{post}})^2, \quad (6.9)$$

following the standard moment-matching procedure recommended by *fib* MC2010 [1].

### 6.6.2 Parameters of the average posterior

Substitution of the six posterior values of table 6.4 into eqs. (6.8) and (6.9) produces the parameters of the average posterior reported in table 6.5.

**Table 6.5:** Parameters of the average posterior distribution  $f''(\vartheta | M)$  obtained from the arithmetic-mean aggregation (6.8) of the six leave-one-out posteriors  $f_J(\vartheta | M_{-j})$  of table 6.4, with moment-matched lognormal approximation (6.9).

| Quantity                      | Value   |
|-------------------------------|---------|
| $\lambda_M^{\text{post}}$ [-] | -0.0588 |
| $\zeta_M^{\text{post}}$ [-]   | 0.0960  |
| $\mu_M^{\text{post}}$ [-]     | 0.9473  |
| $V_M^{\text{post}}$ [-]       | 0.0962  |

### 6.6.3 Comparison and physical interpretation

The average posterior distribution  $f''(\vartheta | M)$  is the quantity that summarises, in a single probabilistic object, the information carried by the entire experimental-numerical benchmark and by all the six solution strategies considered in this thesis. A graphical overview of the relation between the six leave-one-out posteriors  $f_J(\vartheta | M_{-j})$  and their equal-weight aggregation  $f''(\vartheta | M)$  is provided in fig. 6.4.

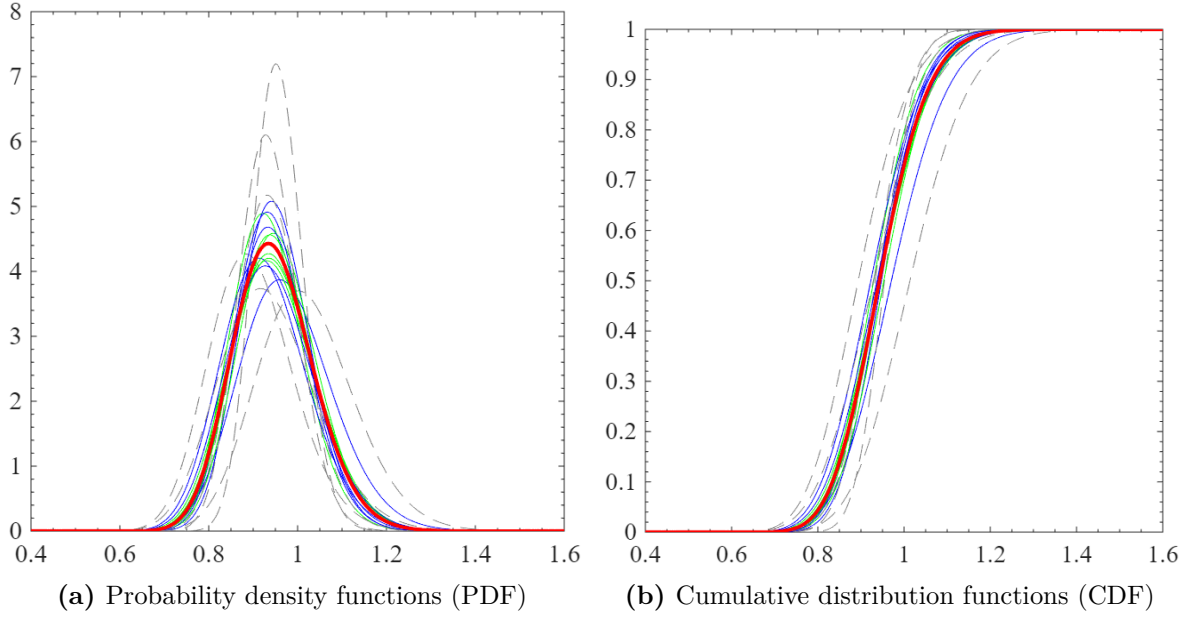


**Figure 6.4:** Probability density functions (a) and cumulative distribution functions (b) of the six leave-one-out posteriors  $f_J(\vartheta | M_{-j})$ ,  $j = 1, \dots, 6$ , compared with the average posterior distribution  $f''(\vartheta | M)$  obtained from eqs. (6.8) and (6.9).

**Mean of the average posterior.** The mean of the average posterior is  $\mu_M^{\text{post}} = 0.9473$ , which sits very close to the posterior means of the rotated calibrated scenario ( $M_5$ ,  $\mu_5^{\text{post}} = 0.9424$ ) and of the rotated-max scenario ( $M_6$ ,  $\mu_6^{\text{post}} = 0.9424$ ), and between the extremes set by  $M_3$  Fixed- $G_f$  max ( $\mu_3^{\text{post}} = 0.9278$ , the lowest) and  $M_1$  Fixed- $G_f/100$  ( $\mu_1^{\text{post}} = 0.9762$ , the highest). The aggregate model uncertainty is therefore mildly non-conservative on average ( $\mu < 1$ ), i.e. the ensemble of the six solution strategies tends to overestimate the experimental capacity of PRC beams by approximately 5.3% on average.

**Coefficient of variation of the average posterior.** The coefficient of variation of the average posterior is  $V_M^{\text{post}} = 0.0962$ , which corresponds to the arithmetic mean of the six posterior coefficients of variation of table 6.4. As such,  $V_M^{\text{post}}$  sits in the central part of the range  $[V_2^{\text{post}} = 0.0833, V_1^{\text{post}} = 0.1071]$  spanned by the six leave-one-out posteriors, and is markedly lower than the coefficient of variation of the most dispersed scenarios ( $M_1$ ,  $M_3$  and  $M_6$ ). A value  $V_M^{\text{post}} \approx 9.6\%$  is in line with the typical range reported for the model uncertainty of NLFEA of RC members in flexure ( $V \in [8\%, 15\%]$ ) [8, 11, 12], indicating that the spread observed for PRC beams is fully comparable to that of conventional RC beams.

**Outlook.** The average posterior  $f''(\vartheta | M)$  obtained in this section constitutes the input of the fourth stage of the Bayesian framework (*deprivation of the experimental uncertainty*), which is developed in section 6.7 (stage 4 of fig. 6.1). The deprivation will isolate the pure model-uncertainty distribution  $f''(\vartheta_{\text{Pure}} | M)$  from  $f''(\vartheta | M)$  by removing the contribution of the experimental uncertainty  $f(\varepsilon)$ , and the pure distribution

will finally be used to calibrate the partial safety factor  $\gamma_{Rd}$  for model uncertainty in section 6.8.

## 6.7 Pure model uncertainty: deprivation of the experimental uncertainty

### 6.7.1 Definition of the experimental uncertainty $\varepsilon$

The average posterior distribution  $f''(\vartheta | M)$  obtained in section 6.6 combines the epistemic uncertainty of the numerical model with the residual aleatory uncertainty of the experimental measurements (load-cell calibration, displacement transducer resolution, support compliance, friction at the loading plates, small size effects). To isolate the *pure* model uncertainty  $\vartheta_{\text{Pure}}$  – the quantity that has to be used for the calibration of the partial safety factor  $\gamma_{Rd}$  – the experimental component must be removed from  $f''(\vartheta | M)$ . Following the convention adopted by Castaldo, Gino, Miceli and Mancini [8, 11, 12] and consistently with the JCSS Probabilistic Model Code [31], the experimental uncertainty is modelled as a lognormal random variable  $\varepsilon \sim \text{LN}(\lambda_\varepsilon, \zeta_\varepsilon)$  with

$$\mu_\varepsilon = 1.00, \quad V_\varepsilon = 0.05, \quad (6.10)$$

i.e. a mean equal to unity (unbiased experiment) and a coefficient of variation of 5%. The corresponding lognormal parameters are

$$\zeta_\varepsilon = \sqrt{\ln(1 + V_\varepsilon^2)} \approx 0.0500, \quad \lambda_\varepsilon = \ln \mu_\varepsilon - \frac{1}{2}\zeta_\varepsilon^2 \approx -0.00125. \quad (6.11)$$

### 6.7.2 Deprivation: variance subtraction in log-space

By definition, the observed model uncertainty  $\vartheta$  collected from the experimental-numerical benchmark is the product of a pure model component  $\vartheta_{\text{Pure}}$  and the experimental component  $\varepsilon$ ,

$$\vartheta = \vartheta_{\text{Pure}} \cdot \varepsilon, \quad (6.12)$$

where  $\vartheta_{\text{Pure}}$  and  $\varepsilon$  are assumed to be statistically independent lognormal random variables [8, 11, 12]. Taking the natural logarithm of eq. (6.12) and exploiting the closure of the lognormal family under multiplication,

$$\ln \vartheta = \ln \vartheta_{\text{Pure}} + \ln \varepsilon, \quad \ln \vartheta_{\text{Pure}} \sim \mathcal{N}(\lambda_{\text{Pure}}, \zeta_{\text{Pure}}^2), \quad \ln \varepsilon \sim \mathcal{N}(\lambda_\varepsilon, \zeta_\varepsilon^2), \quad (6.13)$$

the deprivation reduces to a simple subtraction of means and variances in the log-transformed space,

$$\lambda_{\text{Pure}}^{\text{post}} = \lambda_M^{\text{post}} - \lambda_\varepsilon \approx \lambda_M^{\text{post}}, \quad (\zeta_{\text{Pure}}^{\text{post}})^2 = (\zeta_M^{\text{post}})^2 - \zeta_\varepsilon^2, \quad (6.14)$$

where  $\lambda_M^{\text{post}}$  and  $\zeta_M^{\text{post}}$  are the parameters of the average posterior reported in table 6.5. The pure mean and pure coefficient of variation are then recovered through the lognormal moment relations

$$\mu_{\text{Pure}}^{\text{post}} = \exp\left(\lambda_{\text{Pure}}^{\text{post}} + \frac{1}{2}(\zeta_{\text{Pure}}^{\text{post}})^2\right), \quad V_{\text{Pure}}^{\text{post}} = \sqrt{\exp((\zeta_{\text{Pure}}^{\text{post}})^2) - 1}. \quad (6.15)$$

### 6.7.3 Parameters of the pure model uncertainty

Substitution of the experimental-uncertainty parameters of eq. (6.11) and of the average posterior parameters of table 6.5 into eqs. (6.14) and (6.15) produces the lognormal parameters of the pure model uncertainty distribution  $f''(\vartheta_{\text{Pure}} | M)$  reported in table 6.6.

**Table 6.6:** Parameters of the average posterior distribution of the pure model uncertainty  $f''(\vartheta_{\text{Pure}} | M)$  obtained from the deprivation (6.14)–(6.15) of the average posterior  $f''(\vartheta | M)$  of table 6.5 from the experimental uncertainty  $f(\varepsilon)$  defined in eq. (6.10).

| Quantity                                  | Value   |
|-------------------------------------------|---------|
| $\lambda_{\text{Pure}}^{\text{post}} [-]$ | −0.0575 |
| $\zeta_{\text{Pure}}^{\text{post}} [-]$   | 0.0820  |
| $\mu_{\text{Pure}}^{\text{post}} [-]$     | 0.9473  |
| $V_{\text{Pure}}^{\text{post}} [-]$       | 0.0822  |

### 6.7.4 Comparison between aggregated and pure posteriors

The comparison between the average posterior  $f''(\vartheta | M)$  and the pure model uncertainty  $f''(\vartheta_{\text{Pure}} | M)$  highlights two effects of the deprivation operation, which are visible by direct juxtaposition of tables 6.5 and 6.6.

**Mean.** The mean is virtually unchanged ( $\mu_{\text{Pure}}^{\text{post}} \approx \mu_M^{\text{post}} = 0.9473$ , with  $\lambda$  shifting from −0.0588 to −0.0575, i.e. a change of the order of 0.001 in the log-domain). This is the expected behaviour, because the experimental uncertainty is centred at unity ( $\mu_\varepsilon = 1.00$ ) and therefore does not bias the mean of the multiplicative decomposition. The mild non-conservative bias of the aggregated model ( $\sim 5.3\%$  overestimation of  $R_{\text{exp}}$ ) is therefore inherent to the numerical modelling and not an artefact of the experimental measurement.

**Dispersion.** The coefficient of variation of the pure model uncertainty,  $V_{\text{Pure}}^{\text{post}} = 0.0822$  (8.22%), is markedly smaller than that of the average posterior  $V_M^{\text{post}} = 0.0962$  (9.62%). The reduction  $(V_M^{\text{post}})^2 - (V_{\text{Pure}}^{\text{post}})^2 = 0.0962^2 - 0.0822^2 \approx 0.0025 = V_\varepsilon^2$  reflects the explicit removal of the experimental variance from the aggregated posterior, in agreement with the variance subtraction of eq. (6.14). The pure model uncertainty therefore concentrates around the mean more tightly than the aggregated posterior, which is the expected basis for a sharper calibration of  $\gamma_{Rd}$ .

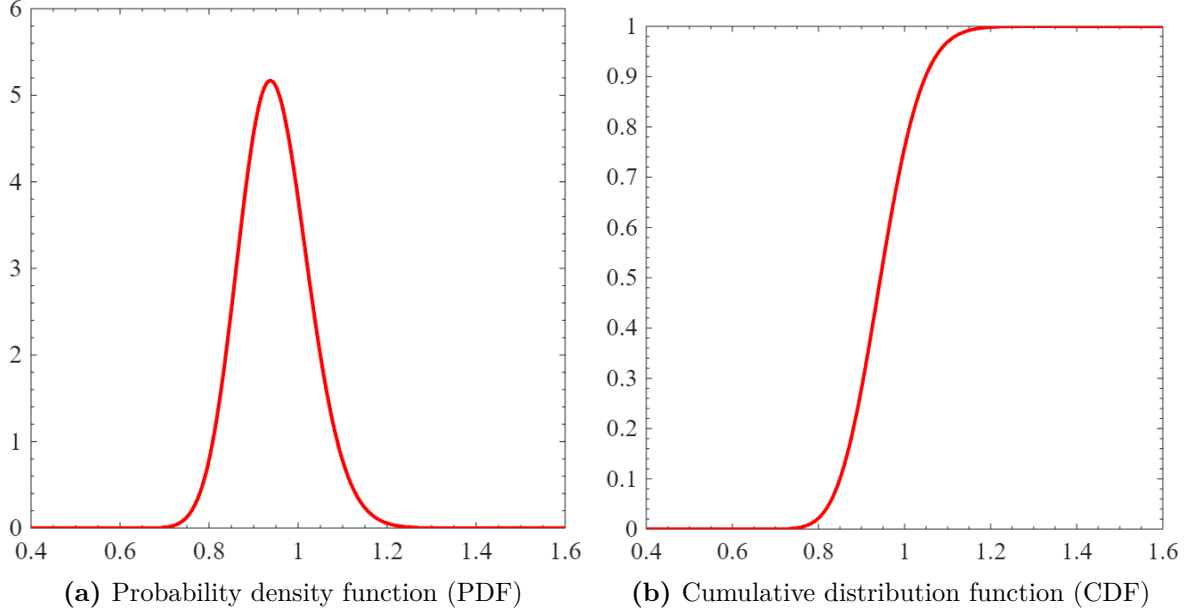


**Figure 6.5:** Probability density function (a) and cumulative distribution function (b) of the average posterior distribution of the pure model uncertainty  $f''(\vartheta_{\text{Pure}} | M)$ , characterised by  $\mu_{\text{Pure}}^{\text{post}} = 0.9473$  and  $V_{\text{Pure}}^{\text{post}} = 0.0822$  (table 6.6).

## 6.8 Calibration of the partial safety factor $\gamma_{Rd}$

### 6.8.1 Closed-form expression of $\gamma_{Rd}$

The partial safety factor for resistance model uncertainty  $\gamma_{Rd}$  is defined as the ratio between the mean (representative) and the design (lower-fractile) values of the lognormal pure model uncertainty, evaluated at the design fractile prescribed by the reliability target [1, 8, 11, 12]. For a lognormal random variable  $\vartheta_{\text{Pure}} \sim \text{LN}(\lambda_{\text{Pure}}^{\text{post}}, \zeta_{\text{Pure}}^{\text{post}})$  the closed-form expression is

$$\gamma_{Rd} = \frac{1}{\mu_{\text{Pure}}^{\text{post}}} \exp(\alpha_R \beta \zeta_{\text{Pure}}^{\text{post}}), \quad (6.16)$$

where  $\alpha_R$  is the FORM sensitivity factor for the resistance and  $\beta$  is the target reliability index of the verification.

### 6.8.2 Reliability parameters adopted

In agreement with the Castaldo–Gino framework [8, 11, 12], the reliability parameters adopted in eq. (6.16) are those recommended by *fib* MC2010 [1] for the ULS verification of an ordinary structure (reliability class RC2) over a 50-year reference period:

$$\alpha_R = 0.32, \quad \beta = 3.8, \quad \text{reference period} = 50 \text{ years}, \quad \text{Reliability class RC2}. \quad (6.17)$$

The value  $\alpha_R = 0.32$  corresponds to the *non-dominant random variable assumption*, in which the resistance model uncertainty is treated as an accompanying (non-leading) random variable of the verification; it is the reduced FORM sensitivity factor obtained

as 0.4 times the leading-resistance value. The dependence of  $\gamma_{Rd}$  on the reliability target under this assumption is reported in table 6.7.

### 6.8.3 Final calibrated value of $\gamma_{Rd}$

Substitution of the pure model uncertainty parameters of table 6.6 and of the reliability parameters (6.17) into eq. (6.16) produces the headline calibrated value of the partial safety factor for model uncertainty,

$$\gamma_{Rd} = \frac{1}{0.9473} \exp(0.32 \times 3.8 \times 0.0820) = \boxed{1.16}. \quad (6.18)$$

This is the central result of the thesis: the partial safety factor for model uncertainty associated with the NLFEA of PRC beams failing in flexure, calibrated through the Bayesian framework of fig. 6.1 on the experimental benchmark of Chapter 4 and on the six numerical scenarios of table 3.2, is  $\gamma_{Rd} \approx 1.16$ .

### 6.8.4 Sensitivity to the reliability target

To complement the headline value (6.18), the same average pure posterior was evaluated at a range of standard reliability targets under the adopted non-dominant random variable assumption ( $\alpha_R = 0.32$ ). The resulting nine values of  $\gamma_{Rd}$  are reported in table 6.7; they correspond to different target reliability indices  $\beta$  (new vs existing structures, reference period and consequence class) and bracket the headline value of eq. (6.18).

**Table 6.7:** Partial safety factor for model uncertainty  $\gamma_{Rd}$  obtained from eq. (6.16) applied to the *single* average pure posterior of table 6.6 ( $\mu_{\text{Pure}}^{\text{post}} = 0.9473$ ,  $V_{\text{Pure}}^{\text{post}} = 0.0822$ ), evaluated at nine reliability targets (new vs existing structures, reference period and consequence class, each defining a target reliability index  $\beta$ ) under the non-dominant random variable assumption ( $\alpha_R = 0.32$ ). Row 2 is the recommended ULS target adopted as the headline value of eq. (6.18).

| # | Structure | Ref. period / consequence     | $\beta$ | $\gamma_{Rd} (\alpha_R=0.32)$ |
|---|-----------|-------------------------------|---------|-------------------------------|
| 1 | New       | 50 yr – low                   | 3.1     | 1.145                         |
| 2 | New       | 50 yr – moderate <sup>†</sup> | 3.8     | <b>1.16</b>                   |
| 3 | New       | 50 yr – high                  | 4.3     | 1.182                         |
| 4 | Existing  | 50 yr                         | 3.1     | 1.145                         |
| 5 | Existing  | 50 yr                         | 3.8     | 1.16                          |
| 6 | Existing  | 15 yr                         | 3.4     | 1.154                         |
| 7 | Existing  | 15 yr                         | 4.1     | 1.176                         |
| 8 | Existing  | 1 yr                          | 4.1     | 1.176                         |
| 9 | Existing  | 1 yr                          | 4.7     | 1.195                         |

<sup>†</sup> recommended target (new, 50-year reference period, moderate consequence class); headline value  $\gamma_{Rd} = 1.16$ .

Under the adopted non-dominant random variable assumption ( $\alpha_R = 0.32$ ),  $\gamma_{Rd}$  varies between 1.145 and 1.195 across the nine reliability targets, the variation being driven by the target reliability index  $\beta$ , the reference period and the consequence class

(*not* by the modelling choice). The recommended new-structure, 50-year, moderate-consequence target ( $\beta = 3.8$ ) gives the headline value  $\gamma_{Rd} = 1.16$ . These values lie within the indicative range recommended by the *fib* MC2010, so that the calibrated factor is consistent with the current code provisions (see section 6.9).

## 6.9 Comparison with current code provisions

The calibrated value of  $\gamma_{Rd}$  produced by the present thesis is compared in this section with the recommendations of the three codes referenced in Chapter 1: *fib* MC2010, EC and NTC, summarised in table 6.8.

**Table 6.8:** Comparison between the partial safety factor for model uncertainty calibrated in the present thesis for PRC beams failing in flexure ( $\gamma_{Rd} = 1.16$ ) and the recommended values of *fib* MC2010 [1], EC [9] and NTC [10].

| Source                               | Recommended $\gamma_{Rd}$ | Remarks                                                                                              |
|--------------------------------------|---------------------------|------------------------------------------------------------------------------------------------------|
| <i>fib</i> MC2010 [1]                | 1.06                      | well-validated NLFEA                                                                                 |
| <i>fib</i> MC2010 [1]                | $\approx 1.10$            | moderate model uncertainty                                                                           |
| <i>fib</i> MC2010 [1]                | up to 1.20                | higher model uncertainty                                                                             |
| EC [9]                               | implicit                  | embedded in partial-factor format on materials                                                       |
| NTC [10]                             | implicit                  | by reference to EC                                                                                   |
| Eurocode 2, 2nd gen.                 | 1.30                      | explicit $\gamma_{Rd}$ for non-linear analysis                                                       |
| <b>Present thesis (PRC, flexure)</b> | <b>1.16</b>               | average pure posterior, non-dominant random variable assumption ( $\alpha_R = 0.32$ ), $\beta = 3.8$ |

Among the documents listed in table 6.8, the forthcoming second-generation Eurocode 2 is of particular interest, as it introduces an explicit partial factor for resistance model uncertainty of  $\gamma_{Rd} = 1.30$  for non-linear analysis. Under the non-dominant random variable assumption adopted here, the present calibration gives  $\gamma_{Rd} = 1.16$ , which lies within the indicative *fib* MC2010 range and below the more conservative second-generation Eurocode 2 value. The present result is therefore consistent with the current code provisions, while the explicit treatment of the unbonded post-tensioned specimens identifies the source of model uncertainty that the next generation of standards is beginning to address.

The calibrated value  $\gamma_{Rd} = 1.16$  obtained in this thesis lies within the upper part of the indicative range currently recommended by the *fib* MC2010 ( $\approx 1.06$ – $1.20$ ). Three concurrent factors explain why the calibrated value sits towards the upper end of this range rather than at its lower bound.

**Limited size of the benchmark database.** The Bayesian framework applied here is based on  $n_{\text{Exp}} = 10$  specimens, whereas the calibration of the *fib* MC2010 indicative values relied on substantially larger RC benchmarks aggregating data on shear, flexural and instability failures. The smaller sample inflates the spread of the prior distributions of table 6.1 and propagates into a higher  $\zeta_{\text{Pure}}^{\text{post}}$ , which through eq. (6.16) places  $\gamma_{Rd}$  towards the upper part of the indicative range. This effect is structural and is expected to attenuate when the benchmark is enlarged in future work.

**Wide bracket of fracture energy levels.** The six numerical scenarios of table 3.2 were designed to bracket the post-peak softening behaviour of the concrete from a fully

brittle response ( $G_f/100$ ) to a pseudo-plastic one ( $G_f$  maximised). This is a deliberate, methodologically clean choice that allows the sensitivity of  $\gamma_{Rd}$  to be quantified, but it also enlarges the prior dispersion compared with a single- $G_f$  calibration. The values reported by *fib* MC2010 were obtained under the implicit assumption of a single, calibrated  $G_f$  value, and are therefore not directly comparable.

**Distinct behaviour of unbonded post-tensioned tendons.** The two unbonded post-tensioned specimens of the database (IB35-10-0.7-1 and IB60-10-0.5-2) contribute the largest spread of  $\vartheta$  in the entire benchmark (section 5.3.6 of Chapter 5), with  $\vartheta$  ranging from 0.77 to 1.18 for the same specimen. The unbonded configuration mobilises the tendon stress through the global elongation between the two anchorages (eq. (2.3) of Chapter 2), which is intrinsically more difficult to capture in the smeared crack framework than the local strain compatibility of bonded specimens. The increased epistemic uncertainty of unbonded behaviour is *the dominant driver* of the upper-range position of the  $\gamma_{Rd}$  obtained in this thesis – a feature that is not reflected in the *fib* MC2010 recommendations, which were calibrated without explicit unbonded benchmarks.

## 6.10 Implications for the next generation of structural design standards

The calibration of  $\gamma_{Rd}$  developed in sections 6.8 and 6.9 suggests three specific contributions of the present thesis to the discussion that is currently shaping the next generation of structural design standards [8, 11, 12]:

- **Differentiated  $\gamma_{Rd}$  categories by member type.** The single, generic value of  $\gamma_{Rd}$  currently recommended by *fib* MC2010 for the NLFEA of concrete members is calibrated mainly on RC benchmarks. The value obtained for PRC beams in flexure, which sits in the upper part of the *fib* MC2010 range and is sensitive to the unbonded contribution, supports the introduction of differentiated  $\gamma_{Rd}$  categories that explicitly distinguish between conventional RC, bonded PRC and unbonded PRC members.
- **Explicit dependence on the modelling choices.** The model uncertainty itself depends on the modelling choices: the per-strategy prior and posterior statistics of tables 6.1 and 6.4 show that the crack model (fixed vs rotated) and the fracture-energy level shift  $\mu_\vartheta$  and  $V_\vartheta$ , and hence the calibrated  $\gamma_{Rd}$ . The next generation of standards could therefore prescribe a small number of standardised solution strategies together with the corresponding  $\gamma_{Rd}$ , instead of leaving the choice unconstrained.
- **Recognition of the unbonded tendon contribution.** The specific contribution of unbonded post-tensioning to the model uncertainty – documented quantitatively for the first time in table 5.7 of Chapter 5 and translated here into

a  $\sim 15\%$  increase of  $\gamma_{Rd}$  – should be reflected in the safety formats applicable to unbonded structures.

## 6.11 Conclusions

The present thesis has applied the Bayesian framework of Castaldo, Gino, Miceli and Mancini [8, 11, 12] to the quantification of the model uncertainty associated with the NLFEA of PRC beams failing in flexure, and has produced the first calibration of the partial safety factor  $\gamma_{Rd}$  for this class of members within the GRF adopted by current and next generation structural codes.

The main findings can be summarised as follows.

**Predictive capability of the six solution strategies.** The systematic comparison of six numerical scenarios – two smeared crack models (fixed and rotated) combined with three levels of fracture energy ( $G_f/100$ , calibrated  $G_f$ , maximised  $G_f$ ) – across the ten specimens of the benchmark database shows that the *rotated calibrated* scenario ( $M_5$ ) produces the most consistent agreement with the experimental capacities, with  $\mu_{\vartheta|M_5} = 0.9351$  and the smallest dispersion of the calibrated cases ( $V_{\vartheta|M_5} = 7.0\%$ ). The fixed calibrated scenario ( $M_2$ ) follows closely, with  $\mu_{\vartheta|M_2} = 0.9561$  and the tightest prior of the entire study ( $V_{\vartheta|M_2} = 5.8\%$ ). The brittle and pseudo-plastic scenarios brackets the response above and below the calibrated value, and are not adequate calibration targets in isolation but are essential to quantify the sensitivity of  $\gamma_{Rd}$  to the modelling assumptions.

**Lognormal hypothesis and Bayesian updating.** The Anderson–Darling goodness-of-fit test (table 6.2) unconditionally accepts the lognormal hypothesis on all the six prior samples, with  $p$ -values comprised between 0.29 and 0.92. This validates the closed-form Bayesian framework used in the rest of the thesis and provides the necessary statistical regularity for the leave-one-out updating and the deprivation of the experimental uncertainty.

**Calibrated partial safety factor for model uncertainty.** The headline contribution of the thesis is the calibrated value

$$\gamma_{Rd} = 1.16$$

obtained from the average posterior of the pure model uncertainty ( $\mu_{\text{Pure}}^{\text{post}} = 0.9473$ ,  $\zeta_{\text{Pure}}^{\text{post}} = 0.0820$ ) and from the reliability parameters  $\alpha_R = 0.32$  (non-dominant random variable assumption),  $\beta = 3.8$ , RC2, 50-year reference period. Evaluated across the standard reliability targets under the non-dominant random variable assumption ( $\alpha_R = 0.32$ ),  $\gamma_{Rd}$  spans the range  $[1.145, 1.195]$ , the spread being governed by the target reliability index  $\beta$ , the reference period and the consequence class; the recommended new-structure, 50-year, moderate-consequence target ( $\beta = 3.8$ ) gives the headline  $\gamma_{Rd} = 1.16$ .

**Comparison with current code provisions and the role of unbonded tendons.**

The calibrated value  $\gamma_{Rd} = 1.16$  lies within the upper part of the indicative range  $[1.05, 1.20]$  currently recommended by the *fib* MC2010 [1]. Its position towards the upper end of this range is driven primarily by the explicit inclusion of two unbonded post-tensioned specimens in the benchmark, whose behaviour is governed by the global elongation principle rather than by the local strain compatibility of bonded systems [15, 16]. The unbonded specimens produce the widest scatter of  $\vartheta$  in the database ( $\vartheta \in [0.77, 1.18]$  for IB60-10-0.5-2), confirming quantitatively that the epistemic uncertainty associated with the NLFEA of PRC flexural members is member-type dependent, and that a single universal  $\gamma_{Rd}$  value does not adequately capture the difference between bonded and unbonded prestressing technologies.

**Original contribution.** To the best of the candidate’s knowledge, the present work provides the first systematic calibration of  $\gamma_{Rd}$  for PRC flexural members in which both bonded and unbonded internal tendons are explicitly represented, and in which the influence of the crack model and of the fracture energy on  $\gamma_{Rd}$  is quantified. The methodology, the numerical implementation and the values of  $\gamma_{Rd}$  obtained in this thesis constitute a self-contained contribution to the calibration data that will inform the next generation of safety formats within the GRF.

## 6.12 Limitations and future research

### 6.12.1 Limitations of the present study

The conclusions summarised in section 6.11 are subject to the following limitations, explicitly acknowledged for the benefit of subsequent studies that may extend or refine the calibration of  $\gamma_{Rd}$  for PRC members.

- **Size of the experimental benchmark.** The Bayesian framework relies on a benchmark of  $n_{\text{Exp}} = 10$  specimens. Although the database is balanced across pre-tensioning, bonded post-tensioning and unbonded post-tensioning, it is significantly smaller than the benchmarks adopted in the calibration of the current *fib* MC2010 values, which aggregated data over multiple failure modes. The narrow sample size inflates the dispersion of the prior distributions and propagates into the calibrated  $\gamma_{Rd}$ .
- **Single finite-element environment (ATENA 3D).** All numerical analyses were performed in ATENA 3D [18]. The use of a single NLFEA software embeds in the calibration the specific implementation of the fracture–plastic model, of the Crack Band regularisation and of the Newton–Raphson solver of that environment. A cross-software calibration (e.g. ATENA vs DIANA vs ABAQUS) would be required to disentangle the contribution of the software-specific implementation from the intrinsic model uncertainty of the smeared crack approach.
- **Rectangular cross-sections and monotonic flexure.** The benchmark is limited to rectangular cross-sections and monotonic, statically determinate flexure.

T-sections, I-sections and box girders – characteristic of long-span PRC bridges – are not represented. Cyclic, dynamic, fatigue and combined-actions loading conditions are not included in the calibration.

- **Two unbonded specimens.** The unbonded post-tensioned configuration is represented by two specimens only (IB35-10-0.7-1 and IB60-10-0.5-2, both from the campaign of Peera and Oukaili [7]). Although these specimens dominate the spread of the model uncertainty, the small subsample limits the robustness of any conclusion specific to the unbonded case.
- **Bracketed range of  $G_f$ .** The maximised fracture energy adopted in scenarios  $M_3$  and  $M_6$  is deliberately set far above the calibrated *fib* MC2010 value to bracket the upper limit of the softening behaviour. A finer parametric study of  $G_f$  around the calibrated value (e.g.  $0.5G_f$ ,  $1.0G_f$ ,  $1.5G_f$ ,  $2.0G_f$ ) would provide a higher-resolution sensitivity of  $\gamma_{Rd}$  to the fracture energy.

### 6.12.2 Recommendations for future research

The methodology established in this thesis can be extended in several complementary directions, summarised hereafter.

- **Enlarged benchmark database.** Extending the database to  $n_{\text{Exp}} \approx 30\text{--}50$  specimens, with a balanced distribution across the three prestressing technologies and a wider spectrum of cross-sections (rectangular, T, I, box), would substantially tighten the prior distributions and provide a more robust calibration of  $\gamma_{Rd}$ . The same Bayesian framework (fig. 6.1) can be applied verbatim to the enlarged dataset.
- **Multi-software cross-calibration.** Replicating the six numerical scenarios in independent NLFEA environments (DIANA, ABAQUS with user subroutines, ATENA) would allow the software-specific component of  $\gamma_{Rd}$  to be isolated. This step is essential to convert the present thesis-specific calibration into a software-agnostic recommendation for the safety formats of the next generation of standards.
- **Refined treatment of the unbonded contribution.** The unbonded post-tensioned configuration deserves a dedicated calibration with a larger subsample of unbonded specimens and an explicit modelling of the anchorage slip, the friction in the duct and the global elongation law of Naaman and Alkhairi [15]. A separate  $\gamma_{Rd}^{\text{unb}}$  for unbonded members could be derived and compared with the present aggregated value.
- **Extension to other failure modes.** The same framework can be applied to other failure modes of PRC members, in particular shear, torsion, punching, anchorage and combined actions. A modal calibration of  $\gamma_{Rd}^{(F)}$ , with  $F$  ranging over the failure modes, would provide the level of granularity advocated by the strain-based method of Gino *et al.* [12].

- **Cyclic and dynamic loading.** The flexural model uncertainty of PRC members under cyclic actions, fatigue and dynamic excitation is currently outside the scope of this thesis. The extension of the present calibration to non-monotonic loading is an obvious future research direction.
- **Sensitivity to the shear retention factor.** A systematic sensitivity analysis of the response to the shear retention factor  $\beta$  of the fixed crack model – whose value was held at  $\beta = 0.2$  throughout the present analyses – can be extended to a full quantification of its influence on  $\gamma_{Rd}$ ; this constitutes one immediate extension of the present work.

The Bayesian framework summarised in fig. 6.1, the experimental benchmark of Chapter 4 and the calibration procedure developed in this final chapter are designed to be reusable, extensible and software-agnostic. The candidate hopes that this self-contained methodology will constitute a useful starting point for the calibration data that will inform the next generation of structural design standards for PRC members.

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