

Politecnico di Torino
Department of Mechanical and Aerospace Engineering

Master's degree in Mechanical Engineering

MSc. Thesis

Investigation of Transient Resonance Response in Bladed Disks



**Politecnico
di Torino**

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April 2026

Acknowledgements

This work represents, I hope, the start of the life I dreamt of since childhood and thus, I would like to express my gratitude to those who made it possible.

My academic supervisors, Prof. Christian Maria Firrone and Prof. Daniele Botto and the company that supported this project, GE Avio. In particular, I would like to thank Ing. Antonio Giuseppe D'Ettola, who first trusted my ability to take on this challenge, and Ing. Gianmaria Sartor, who guided me in my first experience into the 'real world'.

I would like to thank my parents, Dino and Sonia, who have always given me everything they have, and lastly Lucrezia, for constantly pushing me out of my steady state into a new transient.

Learn from yesterday, live for today, look to tomorrow, rest this afternoon ~ Snoopy

Abstract

Aero-engines certification process requires well-defined constraints on the maximum allowable stresses for all the components, based on the expected use of the turbomachinery, this is why aeromechanical analysis are needed to predict and avoid vibration issues caused by the interaction between the engine parts and fluid flow.

One of the two principal aeroelastic phenomena are forced response, vibrations caused by external forces with a defined amplitude. For the case of bladed disks, of which the stages of the axial compressor and turbine are composed in a modern aeronautical engine, forced response stems primarily from the spatial un-steady pressure field that encounters the blade.

A blade rotating through this field experiences a force characterized by a frequency that is a multiple-integer of the angular speed of the engine, called *Engine Order* (EO). If the forcing frequency matches the natural frequency of one of the system's modes, and there is a coupling between the travelling wave excitation and the mode shape, a resonance condition occurs (Synchronous Vibration). This crossing can be visualized thanks to the Campbell/Waterfall diagram.

Usually, these crossings are crossed quickly to avoid operating the engine under critical conditions.

These frequent and quick resonance crossings introduce a transient characteristic in the forced response, accompanied by a series of effects that influence not only the operational safety of the engine, but also the effectiveness of ground tests. Although the differences between a steady-state forced response and a transient resonance response (Non-Synchronous Vibration) are well known and documented in literature, little attention has been devoted to understanding this phenomenon and the underlying relationships between system parameters and the transient response itself. A recent impetus has been provided by the need to optimize the design of future aero-engines for efficiency and safety, thereby compelling designers to address this issue.

This work aims to investigate these relationships and their effects on the dynamic behavior of the system from analytical, numerical, and experimental perspectives.

The analytical study involves the implementation of a *single-degree-of-freedom* (SDOF) model to conduct a sensitivity analysis of the various parameters involved like passage speed and damping levels, followed by the implementation of a procedure to extract the steady-state response from un-steady data.

This model is subsequently compared with the results of finite element method numerical analysis and experimental tests, both performed on a dummy blisk, to verify the validity of the lumped-parameters approach and the presented procedure for modal identification.

Contents

Page

Chapter 1: Aeromechanics in turbomachinery 1

1.1 Principles of Jet Propulsion	1
1.2 The Jet Engine	2
1.2.1 Brayton-Joule cycle	2
1.2.2 Compressor	3
1.2.3 Combustion chamber	4
1.2.4 Turbine	5
1.2.5 Exhaust system	6
1.3 Bladed disks dynamics.....	7
1.3.1 Modal analysis	8
1.3.2 Complex modes	12
1.3.3 Inter Blade Phase Angle and Harmonic Indexes	13
1.3.4 Aeromechanical analysis	16
1.4 Friction Damping.....	20

Chapter 2: Sensitivity Analysis of Transient Resonance Passage in Bladed Disks 23

2.1 Transient effects in Time and Frequency domain	23
2.2 SDOF System Transient Response Sensitivity Analysis	26
2.2.1 Linear Time Invariant System	26
2.2.2 Determination of Stationary Response from Measured Transient Response..	37
2.2.3 Linear Time Variant System.....	42
2.3 Influence of Friction Damping on Transient Response.....	46
2.3.1 Jenkins Element definition	46
2.3.2 Alternate State Space Algorithm for the solution of a dry friction system.....	48
2.3.3 Sensitivity Analysis of Friction Damping on Transient Response	51

Chapter 3: Finite Element Method Analysis of Transient Resonance Response 59

3.1 Test Article	59
-------------------------------	-----------

3.2	Finite Element Method	60
3.2.1	Domain Discretization	61
3.2.2	Finite Element Method equations	61
3.2.3	3D Solid Element selection.....	64
3.3	FreND Diagram construction	64
3.3.1	Modal Analysis in Ansys Mechanical APDL	65
3.3.2	Base Sector	67
3.3.3	Tuning of the Finite Element model	68
3.3.4	FreND Diagram	70
3.4	Transient Dynamic Analysis	72
3.4.1	Travelling Wave excitation definition	72
3.4.2	Time Integration Scheme for Linear Systems	75
3.4.3	Full Transient Dynamic Analysis	77
3.4.4	Mode-Superposition Transient Dynamic Analysis	77
3.4.5	Comparison Full Vs Mode-Superposition Transient Analysis	78

Chapter 4: Experimental Evaluation of Transient Resonance Response 83

4.1	Fundamentals of the BTT technique	83
4.1.1	Working Principle	84
4.2	Strain Gauges	90
4.2.1	Working Principle	90
4.2.2	Wheatstone Bridge	91
4.3	Spinning Test Rig	93
4.3.1	Excitation System	94
4.3.2	BTT Sensors	95
4.3.3	Strain measurement system.....	95
4.4	Test Campaign	96

Chapter 5: Experimental Validation of FEM Transient Analyses and Determination of Stationary Response 99

5.1	Experimental Validation	99
5.1.1	Test description and Modal Parameters identification	99
5.1.2	FEM Pre-Stressed Modal Analysis	103
5.1.3	Test results.....	104

5.1.4	Concluding remarks	109
5.2	Determination of Stationary Response from Transient Measurements	110
5.2.1	Concluding remarks	115
5.3	Future works	116

Appendix A: Static and Dynamic Calibration of Strain Gauges

A.1	Static Calibration of Strain Gauges	119
A.2	Dynamic Calibration of Strain Gauges	122

Introduction

Motivation and Objective

The motivations behind this thesis stem directly from the needs of the aviation industry, which is currently facing increasingly complex challenges in designing more efficient, safe, and environmentally friendly aero engines. To succeed, it is essential to investigate the underlying physics of design problems more deeply, thereby enabling the identification of specific, innovative solutions.

Part of this design process involves the execution of a series of dynamic characterization and stress certification tests of the entire engine, or single components of it, like bladed disks, which are subjected to an acceleration ramp that crosses the entire engine speed range.

It is not rare that, after these tests, unexpected phenomena are observed which are not predicted by standard dynamic analysis conducted previously.

This work, developed in collaboration with GE Avio, aims to understand if these unconventional responses are caused by the introduction of transient effects due to an excessive ramp is applied, and if it is the sole factor responsible for their onset.

Consequently, the capability of commonly adopted commercial analysis tools to accurately predict these effects before a test campaign was evaluated.

Thesis Outline

This work is divided into five chapters that follow the logical process developed to realize a cohesive work of research that investigates the dynamic response of bladed disks in transient conditions under every aspect: Analytical, Numerical and Experimental.

Below is a summary of the chapters, with a brief description of their contents:

- **Chapter 1:** presents a brief introduction to the topic of Aeromechanics and its application in aeronautical engines
- **Chapter 2:** illustrates the sensitivity analysis conducted on a lumped parameter model to understand which are the factors that influence the transient resonance response
- **Chapter 3:** describes how the Finite Element Method has been used to conduct a Transient Dynamic Analysis
- **Chapter 4:** is a description of a series of tests conducted, as part of this work, to evaluate the transient response in a real test case
- **Chapter 5:** reports the results of the experimental validation of the presented numerical model

Chapter 1:

Aeromechanics in turbomachinery

This chapter introduces the topic of aeromechanics in turbomachinery, specifically in aeronautical gas turbine engines or, more commonly, jet engines.

After a brief introduction about the history, working principles and construction schemes of the more common solutions, the spotlight is shifted to vibration phenomena that characterize these systems and their underlying causes.

1.1 Principles of Jet Propulsion

Thrust is the mechanical force that moves an aircraft through the air, it is generated via a practical application of Sir Isaac Newton's third law of motion which states that: 'for every force acting on a body there is an opposite and equal reaction'.

In case of aerospace propulsion, the 'body' is a mass of gas, while the force acting on it is the one applied to change its momentum and thus accelerating it. This is achieved by some kind of propulsion system, and it's described by the General Thrust Equation [1]:

$$F = \dot{m}_e V_e - \dot{m}_o V_o + (p_e - p_o) A_e \quad (1.1)$$

- $\dot{m}$ is the mass flow rate
- V is the velocity
- p is the pressure
- A is the area

Where the subscript e refers to the exit of the device, and o to the free stream of fluid.

By looking at the equation, is it possible to notice that there are two possible ways to produce high thrust. The first one consists in making the mass flow rate as high as possible, as long as the exit velocity is greater than the free stream (entrance) velocity. This is the theory behind propeller aircraft, where a large amount of air is processed each moment, but the speed per se does not change much.

The second way to produce thrust is to accelerate a moderate amount of fluid flow to an exit velocity much greater than the entry one.

The first manned heavier-than-air powered and controlled aircraft, *The Wright Flyer*, relied for its propulsion on a four-cylinder reciprocating petrol engine that, with its 12 hp powered two hand-made propellers.

In the decades after, internal combustion engines evolved rapidly, becoming way more powerful, efficient and reliable, but the standard configuration remained the same that allowed the 1903's milestone.

The possibility of generating thrust through a high velocity airflow, the ‘jet’, had interested aircraft designers since the 1910s, but the unsuitability of piston engines to produce the jet, and the low speeds of early aircraft, rendered the task difficult.

Everything changed when in 1930 Frank Whittle patented his idea for a Turbojet engine, which relied on a gas turbine to produce a propulsive jet, eleven years before it took flight.

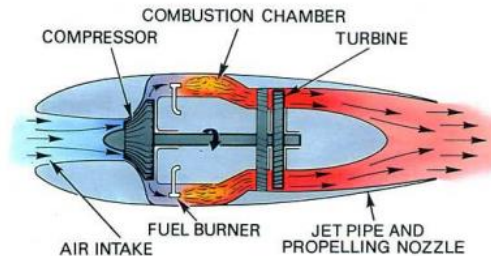


Figure 1: Whittle-Type turbo-jet engine [2]

1.2 The Jet Engine

In this work the focus is on the gas turbine engine, commonly known as simply jet engine, and all its declinations: Turbojets, Turbofans and Turboprops, the most widely used propulsion systems in the modern aerospace industry.

While each engine has its own scope and application, they all share the same base scheme and working cycle.

1.2.1 Brayton-Joule cycle

The working cycle upon which the gas turbine engine functions is the so-called Brayton-Joule cycle that, in its simplest form, can be described by four phases:

- 1) Isentropic compression (A-B)
- 2) Isobaric combustion (B-C)
- 3) Isentropic expansion (C-D)
- 4) Isobaric cooling (D-A)

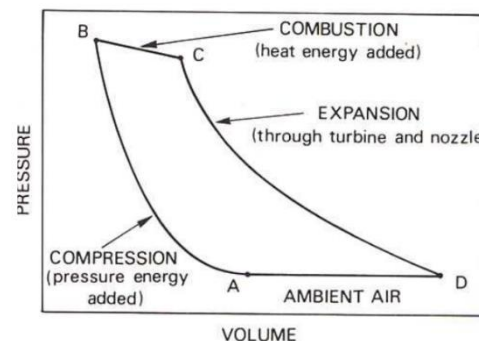


Figure 2: Brayton-Joule cycle [2]

This is a description of an ideal cycle that does not consider the effects of inevitable irreversibilities, which reduce the overall thermodynamic efficiency of the system.

Each one of the presented phases take place in a different part of the engine that, in practice, follows an open cycle.

The working fluid, contrary to the ideal cycle, is not re-introduced into the engine; exhaust gases in fact are ejected into the atmosphere, meanwhile fresh air enters through the intake.

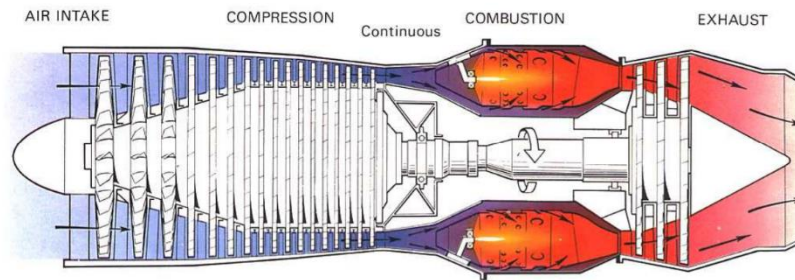


Figure 3: Gas turbine engine typical scheme [2]

1.2.2 Compressor

The compressor is the first component of the engine to enter in contact with the working fluid, in this case the atmospheric air, and where the compression phase takes place.

Two main types of compressors are adopted: centrifugal flow and axial flow compressors, each one with advantages and disadvantages.

Centrifugal flow compressors were commonly used in the first models of turbojets, while recently are progressively phased out in favor of the more efficient axial flow ones.

They usually consist of a single or two stage unit employing an impeller to accelerate the air and a diffuser to produce the required pressure rise. With respect to their counterpart, centrifugal flow compressors are usually more robust and easier to develop and manufacture, but the attainable pressure ratios and mass flow rate are low.

Axial flow compressors are multi-stage units consisting of a series of rotor assemblies that carry blades of airfoil section, called bladed disks, alternated by stator vanes; each couple of rotor and stator composes a stage.

During operation the rotor is turned at high speed by the turbine so that air is continuously induced into the compressor, which is then accelerated by the rotating blades and swept rearwards onto the adjacent row of stator vanes. The pressure rise results from the energy imparted to the air in the rotor which increases the air velocity. The air is then decelerated (diffused) in the following stator passage and the kinetic energy translated into pressure.

The more the pressure ratio of a compressor increases, the more difficult it becomes to avoid the phenomenon called compressor stall, a local disruption of the airflow on the blades.

That is the reason behind the multi-stage construction, where the total pressure rise is distributed in small progressive steps.

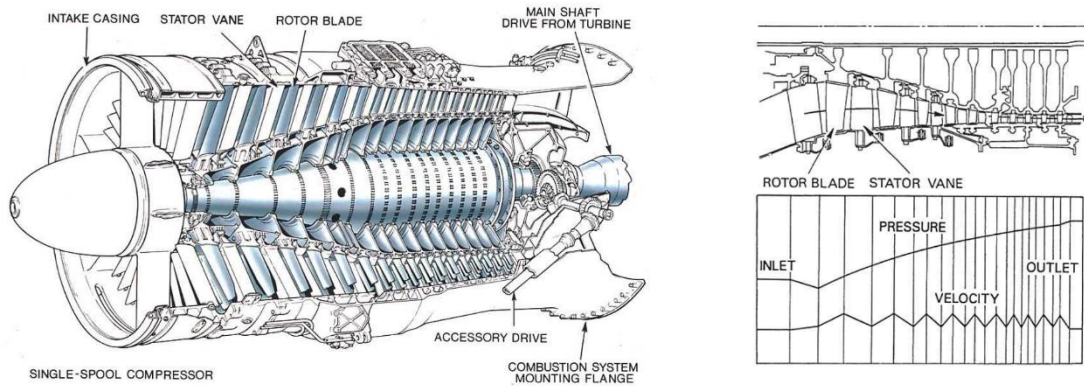


Figure 4: Typical axial flow compressor and relative pressure and velocity changes through stages [2]

1.2.3 Combustion chamber

As the name implies, the combustion chamber is where the combustion phase takes place.

Here the pressurized air from the compressor enters the chamber with a fair too high speed to enable efficient combustion, also the overall air to fuel ratio would be much greater than Kerosine stoichiometric ratio (around 15:1) thus, the first task of the combustion chamber, is to diffuse and meter the incoming air. This is achieved by a series of swirl vanes and a perforated flare that, not only reduce the airflow to around 20%, but also generate a turbulent flow that facilitates combustion.

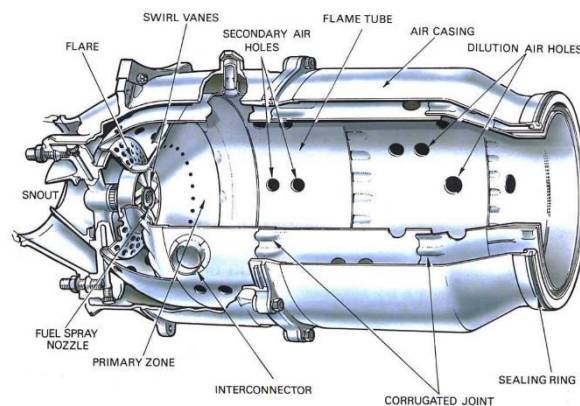


Figure 5: Combustion chamber structure [2]

The temperature of the released gases can reach values of up to 2000 degrees Celsius therefore, before reaching the turbine blades, they have to be cooled; this is done by diluting the gases with the unused air in the dilution zone, which also serves the purpose of cooling down the flame tube.

Different configurations are possible based on the disposition, shapes and numbers of combustion chambers: Multiple combustion chamber, Turbo-annular combustion chamber and the Annular combustion chamber are the main ones.

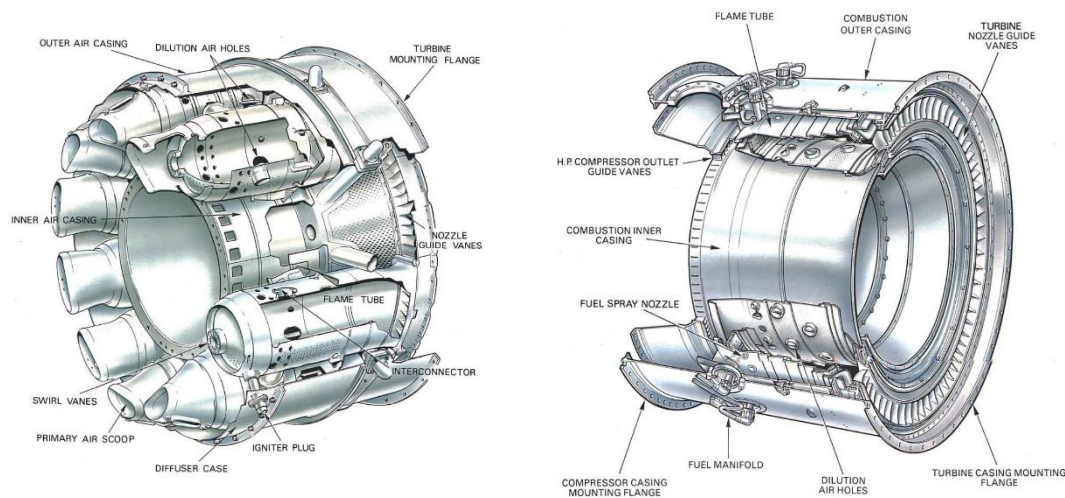


Figure 6: Turbo-Annular (left) and Annular (right) configurations [2]

1.2.4 Turbine

The turbine, composed by one or several stages, each employing one row of stationary nozzle guide vanes and one row of moving blades, drives the compressor and all the accessories; this is achieved by harvesting energy from the gases released by the combustion.

There are two main principles at the base of this energy transfer: impulse and reaction.

In the first case the total pressure drop occurs in the fixed nozzle guide vanes, while the turbine is driven only by the impulse force caused by the impact of the accelerated gas. The rotor blades do not have a change in the flow area; therefore, they do not change the pressure of the fluid, but only its direction.

The moving blades of reaction turbines, on the other hand, do have a change in flow area due to their converging blade passages designed to accelerate the fluid; the pressure drop is thus distributed on the entire stage. The result is a reaction force derived from the expansion and acceleration of the gas.

In reality, gas turbine engines do not use a pure impulse or reaction turbine, but a combination of both.

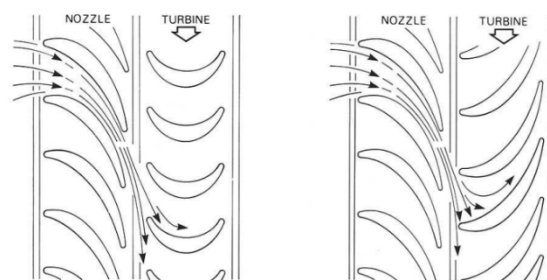


Figure 7: Impulse turbine (left) and Reaction turbine (right) [2]

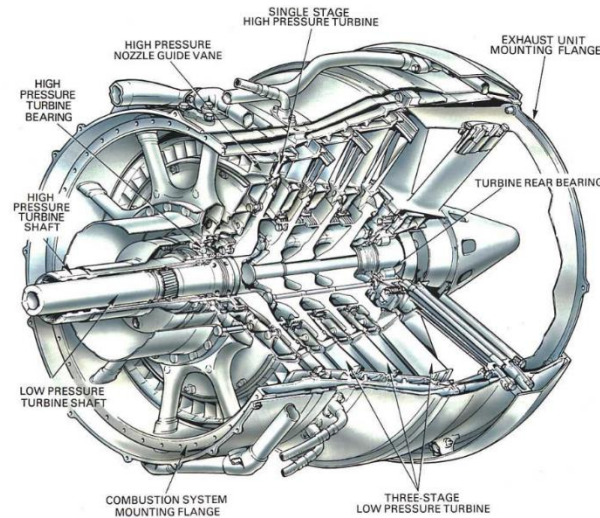


Figure 8: Twin turbine configuration [2]

1.2.5 Exhaust system

The turbine discharge gases are finally passed to the atmosphere through an exhaust system concluding the open Brayton-Joule cycle.

The design of the exhaust exerts a considerable influence on the performance of the engine as it determines the pressure, the velocity and the direction of the gases and thus the resulting thrust.

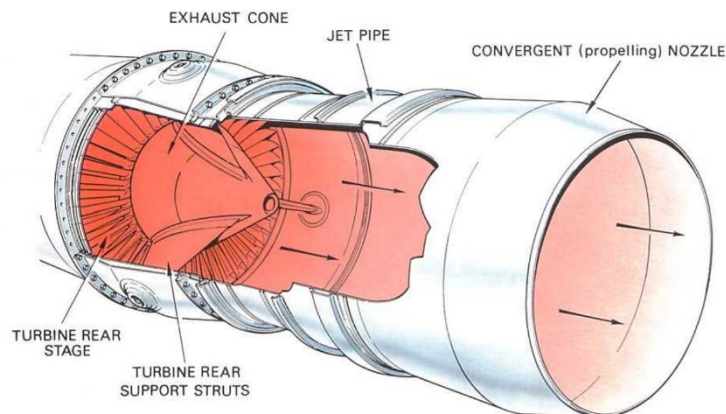


Figure 9: Basic exhaust system [2]

1.3 Bladed disks dynamics

As shown in the previous sections two of the main components of a gas turbine engine, the compressor and the turbine, rely on the same structure: one or multiple stages, each one constituted by a rotor and a stator.

In this case the rotors, object of this study, are called *Bladed Disks*, being composed of a disk with removable blades attached.

The blades are divided into:

- **Shroud:** located at the tip, has various roles in the dynamics of the system. It couples the blades changing the mode shapes of the whole row, adds stiffness to the blade and lastly damps the vibration amplitude through friction. It's used mostly, if not exclusively, in turbine blades. Compressors use so called cantilever blades.
- **Airfoil:** the aerodynamically optimized shape that encounters the airflow and exchanges work with it.
- **Root:** secures the blade to the disk, acts as a friction damper and influences the maximum rotation speed. Different types of fixings were developed through the years, mainly with the purpose of better distributing the loading on all the serrations.



Figure 10: Turbine blade sections

The disk has to rotate at high speed in a relatively cool environment and is subjected to large rotational stresses. The limiting factor which affects the useful disk life is its resistance to fatigue cracking.

Since mid-80s, *Blisks* (or *Integrally Bladed Rotors*), a component comprising both rotor disk and blades as a single part, have been used in some applications.

Bladed disks, due to their geometry, present cyclic symmetry, a type of symmetry in which a fundamental (or basic) sector, if patterned circularly about an axis of symmetry would result in the full component. As shown later this feature can be used to simplify the dynamic analysis, only if the appropriate boundary conditions are applied.

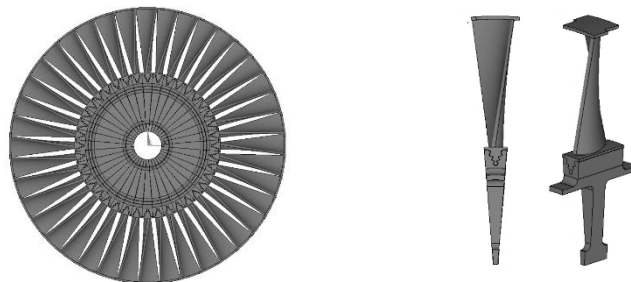


Figure 11: Full 360° Model and Fundamental sector [3]

1.3.1 Modal analysis

Modal analysis is an analytical, numerical or experimental study of the dynamic properties of an object or structure.

The objective is the identification of the modal parameters:

- **Natural frequency:** is the rate at which a body vibrates when disturbed without being subject to a driving force.
- **Mode shape:** represents the pattern of the vibrating motion at the corresponding natural frequency.
- **Damping:** energy dissipation property of the system.

These depend on the structure's mass, stiffness and damping distribution and determine how it responds when exposed to operational loads.

Natural frequencies and mode shapes are obtained solving an *eigenvalue problem* derived from the equations of motion of the system; for this reason, the natural frequencies are also called *eigen-frequencies*, since they are the system's eigenvalues, while the mode shapes are the associated eigenvector and thus, they are also known as *eigen-modes*.

A specific mode is identified by an eigenvalue and the relative eigenvector.

$$([K] - \omega^2[M])\{\phi\} = 0 \quad (1.2)$$

Where:

- $[K]$ is the Stiffness matrix
- $[M]$ is the Mass matrix
- ω is the Mode's eigenvalue
- ϕ is the Mode's eigenvector

The number of eigenvalues/eigenvectors is that of the degrees of freedom (DOFs) of the system.

When performing a modal analysis on a bladed disk, if a cylindrical coordinate system is considered, with z being the rotation axis, the same model can be used for every sector. The full model is built by replicating N times the fundamental segment, where N in case of aeronautical engines usually represents the number of blades.

A simplified 2 DOFs model with cantilever blades will be adopted:

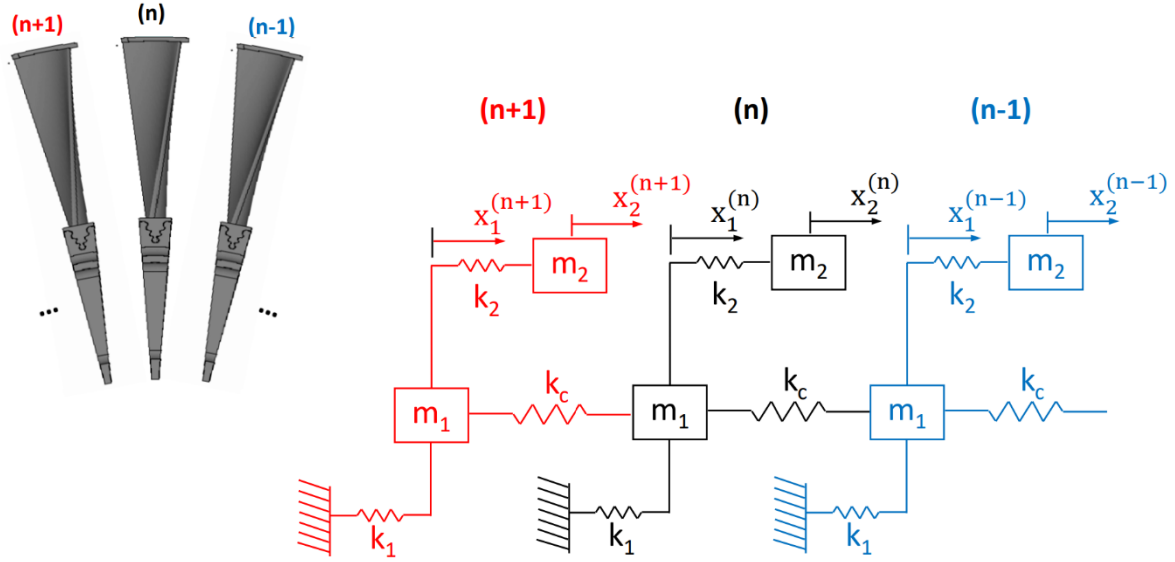


Figure 12: 2 DOFs bladed disk model with cantilever blades [3]

Where, for the single sector:

- m_1 and m_2 are respectively: the disk and the blade mass
- k_1 and k_2 are respectively: the disk and the blade stiffness
- k_c : the coupling stiffness between sectors

So, analyzing only the adjacent sectors to the base one, the equations of motions can be written as:

- **Sector n:**

$$[m^{(n)}]\{\ddot{x}^{(n)}\} + [k^{(n)}]\{x^{(n)}\} + [k_c^{(n)}]\{x^{(n+1)}\} + [k_c^{(n)}]\{x^{(n-1)}\} = 0 \quad (1.3)$$

- **Sectors 1 and N:**

$$[m^{(1)}]\{\ddot{x}^{(1)}\} + [k^{(1)}]\{x^{(1)}\} + [k_c^{(1)}]\{x^{(2)}\} + [k_c^{(1)}]\{x^{(N)}\} = 0 \quad (1.4)$$

$$[m^{(N)}]\{\ddot{x}^{(N)}\} + [k^{(N)}]\{x^{(N)}\} + [k_c^{(N)}]\{x^{(1)}\} + [k_c^{(N)}]\{x^{(N-1)}\} = 0 \quad (1.5)$$

Looking at these equations, a circulating structure can be identified, writing the full structure's equations of motion this becomes clearer:

$$[M]\{\ddot{X}\} + [K]\{X\} = 0 \quad (1.6)$$

Where:

Mass matrix is a block-diagonal matrix:

$$[M] = \begin{bmatrix} m^{(1)} & 0 & \dots & 0 \\ 0 & m^{(2)} & & \\ & \vdots & \ddots & \\ 0 & & 0 & m^{(n)} & 0 & \vdots \\ & & & 0 & \ddots & \\ & 0 & \dots & m^{(N-1)} & 0 & \\ & & & 0 & m^{(N)} \end{bmatrix} \Rightarrow \begin{bmatrix} [m] & 0 & \dots & 0 \\ 0 & [m] & & \\ & \vdots & \ddots & \\ 0 & & 0 & [m] & 0 & \vdots \\ & & & 0 & \ddots & \\ & 0 & \dots & [m] & 0 & \\ & & & 0 & [m] \end{bmatrix}$$

Stiffness matrix is a block-circulant matrix:

$$[K] = \begin{bmatrix} k^{(1)} & k_c^{(1)} & 0 & \dots & 0 & 0 & k_c^{(1)} \\ k_c^{(2)} & k^{(2)} & k_c^{(2)} & \dots & 0 & 0 & 0 \\ 0 & k_c^{(3)} & k^{(3)} & & & & \\ & \vdots & \ddots & k^{(n)} & k_c^{(n)} & & \vdots \\ & & & k_c^{(n)} & \ddots & & \\ & 0 & 0 & & k^{(N-2)} & k_c^{(N-2)} & 0 \\ & 0 & 0 & & k_c^{(N-1)} & k^{(N-1)} & k_c^{(N-1)} \\ k_c^{(n)} & 0 & & \dots & 0 & k_c^{(N)} & k^{(N)} \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} [k] & [k_c] & 0 & \dots & 0 & 0 & [k_c] \\ [k_c] & [k] & [k_c] & \dots & 0 & 0 & 0 \\ 0 & [k_c] & [k] & & & & \\ & \vdots & \ddots & [k] & [k_c] & & \vdots \\ & & & [k_c] & [k] & [k_c] & \\ & 0 & 0 & & [k] & [k_c] & 0 \\ & 0 & 0 & \dots & [k_c] & [k] & [k_c] \\ [k_c] & 0 & & & 0 & [k_c] & [k] \end{bmatrix}$$

Both matrices have an $NJ \times NJ$ dimension, with J the number of DOFs of the fundamental sector.

Generalized coordinates vector:

$$\{X\} = \begin{Bmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \\ \vdots \\ \mathbf{x}^{(n)} \\ \vdots \\ \mathbf{x}^{(N-1)} \\ \mathbf{x}^{(N)} \end{Bmatrix} \quad \text{with a dimension of } NJ \times 1$$

As already explained, an equation like (1.2) is solved for the structure to attain the eigenvectors, which in case of cyclic symmetry, must satisfy a precise requirement: an eigen-mode has to remain such even after a rigid rotation by an angle equal to $\frac{2\pi}{N_{tot}}$, the sector's amplitude.

Considering the generic eigen-mode $\{\Phi\}$:

$$\{\Phi\} = \begin{Bmatrix} \Phi^{(1)} \\ \Phi^{(2)} \\ \vdots \\ \Phi^{(n)} \\ \vdots \\ \Phi^{(N-1)} \\ \Phi^{(N)} \end{Bmatrix} \begin{matrix} J \times 1 \\ J \times 1 \\ \vdots \\ J \times 1 \\ \vdots \\ J \times 1 \\ J \times 1 \end{matrix}$$

This is possible with:

- a. Identical mode shapes in each sector: $\Phi^{(n)} = \Phi^{(n+1)}$
- b. Opposite sign mode shapes in each sector: $\Phi^{(n)} = \Phi^{(n+1)}$
- c. Linear combination of two modes associated to the same eigenvalue

Focusing on the c-type modes, if $\{\Phi\}$ is a system's eigenvector, due to the cyclic symmetry, a vector $\{\Phi'\}$ obtained through a rigid rotation of an angle equal to the sector's amplitude it's an eigenvector itself, with the same natural frequency.

It is also possible to demonstrate that $\{\Phi\}$ and $\{\Phi'\}$ are not necessarily orthogonal, however, for it to be an eigenvector, it must be:

$$\{\Phi'\} = c\{\Phi\} + s\{\hat{\Phi}\} \quad (1.7)$$

Where $\{\hat{\Phi}\}$ is an eigenvector orthogonal to $\{\Phi\}$, associated to the same eigenvalue.

This is valid for every possible mode shape resulting from a rigid rotation, hence c-type modes have multiplicity two.

Similarly, looking at the first J-blocks of equations, the ones that refer to the DOFs of the first sector:

$$\begin{Bmatrix} \Phi^{(N)} \\ \hat{\Phi}^{(N)} \end{Bmatrix} = \begin{bmatrix} I_J \cos(\varphi) & I_J \sin(\varphi) \\ -I_J \sin(\varphi) & I_J \cos(\varphi) \end{bmatrix} \begin{Bmatrix} \Phi^{(1)} \\ \hat{\Phi}^{(1)} \end{Bmatrix} \quad (1.8)$$

And for a **generic n-sector**:

$$\begin{Bmatrix} \Phi^{(n-1)} \\ \hat{\Phi}^{(n-1)} \end{Bmatrix} = \begin{bmatrix} I_J \cos(\varphi) & I_J \sin(\varphi) \\ -I_J \sin(\varphi) & I_J \cos(\varphi) \end{bmatrix} \begin{Bmatrix} \Phi^{(n)} \\ \hat{\Phi}^{(n)} \end{Bmatrix} = [R_\varphi] \begin{Bmatrix} \Phi^{(n)} \\ \hat{\Phi}^{(n)} \end{Bmatrix} \quad (1.9)$$

It emerges that, knowing the rotation matrix $[R_\varphi]$, it's possible to retrieve the displacements of every single sector of the bladed disk starting from the base sector.

It's clear that not all angles are acceptable as, after N_{tot} rotations, the displacements must be those of the starting sector.

This is a crucial property of cyclic symmetric structures that allows to reduce drastically the computational time required to conduct a modal analysis.

1.3.1 Complex modes

The coefficients shown in eq. (1.7) are generally assumed as real, but nothing prohibits them from being complex numbers. In the first case, any linear combination of the two eigenvectors yields a **real mode** in which all degrees of freedom vibrate in phase with each other, attaining their maximum or minimum displacement simultaneously (a and b types modes).

By contrast, if c and s are complex numbers, a couple of complex eigenvectors can be introduced:

$$\{\bar{\Theta}_+\} = \{\Phi\} + i\{\hat{\Phi}\} \quad \text{and} \quad \{\bar{\Theta}_-\} = \{\Phi\} - i\{\hat{\Phi}\} \quad (1.10)$$

They are also a solution to the eigenvalue problem previously introduced and, like $\{\Phi\}$ and $\{\hat{\Phi}\}$, are mutually orthogonal.

Evaluating the relationships between the modal displacements of subsequent sectors that these modes establish:

$$\bar{\Theta}_+^{(n-1)} = \bar{\Theta}_+^{(n)} e^{-i\varphi} \quad \text{and} \quad \bar{\Theta}_-^{(n-1)} = \bar{\Theta}_-^{(n)} e^{i\varphi} \quad (1.11)$$

The physical meaning of a **complex mode** is evident: homologous points vibration on identical sectors is characterized by a phase shift of an angle φ for adjacent segments; it can be viewed as a continuous spatial variation of the displacement that evolves with increasing time, i.e., it is a propagating waveform or *travelling wave*.

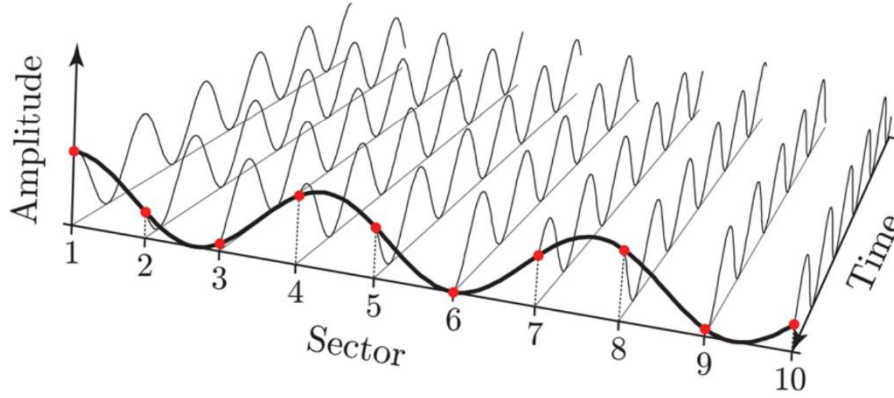


Figure 13: Temporal and Spatial harmonic variation of a traveling wave example in a cyclic symmetric structure [4]

Thus:

$$\{\bar{\Theta}_+\} \text{ and } \{\bar{\Theta}_-\}$$

Are counter-rotating waves; each mode can be therefore characterized as a pair of standing waves (SWs) with different spatial phases, or a pair of traveling waves, labeled as a forward traveling wave (FTW) and backward traveling wave (BTW) when following the terminology used in applications to rotating machinery.

1.3.2 Inter Blade Phase Angle and Harmonic Indexes

The admissible phase angles are limited since, after N_{tot} rotations, it's necessary to obtain again the base sector's eigenvector.

This translates into:

$$N_{tot}\varphi = 2\pi h \Rightarrow \varphi = h \frac{2\pi}{N_{tot}} \quad (1.12)$$

Where:

- φ is the *Inter Blade Phase Angle* or IBPA
- h is the *Harmonic Index* or HI, an integer

Remembering eq. (1.11):

- $\varphi = 0$ ($h = 0$) is linked to modes of the **a**-type (real modes), with identical mode shapes in every sector.
- $\varphi = \pi$ ($h = \frac{N_{tot}}{2}$) is linked to modes of the **b**-type (real modes), with opposite sign mode shapes in each sector. They can exist only if N_{tot} is even.
- All the other possible values of φ are associated to modes of the type **c**.

N even:

$$\mathbf{h}: \frac{-N_{tot}}{2} + 1, \dots, 0, \dots, \frac{N_{tot}}{2}$$

N odd:

$$\mathbf{h}: \frac{-(N_{tot}-1)}{2}, \dots, 0, \dots, \frac{(N_{tot}-1)}{2}$$

Due to the periodicity of the function e^{ix} , an HI greater than $\frac{N_{tot}}{2}$, can be reconducted to a value comprised between $-\frac{N_{tot}}{2}$ and $\frac{N_{tot}}{2}$.

Harmonic Indexes have also a physical meaning in the real modes linked to the complex eigenvectors through (1.10), they modulate the amplitude of the mode in the circumferential direction, for this reason for an HI of value h , there will be h *Nodal Diameters*, characterized by a null displacement.

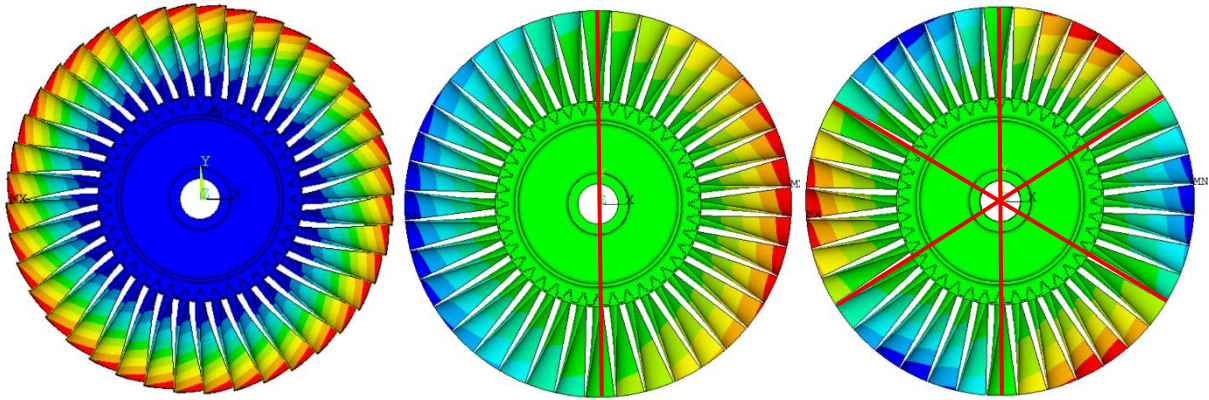


Figure 14: ND 0 (Left), ND 1 (Centre), ND 3 (Right) for the considered bladed disk [3]

The results of a modal analysis conducted on bladed disk are visualized with a *FreND* diagram (**F**requency vs **N**odal **D**iameter):

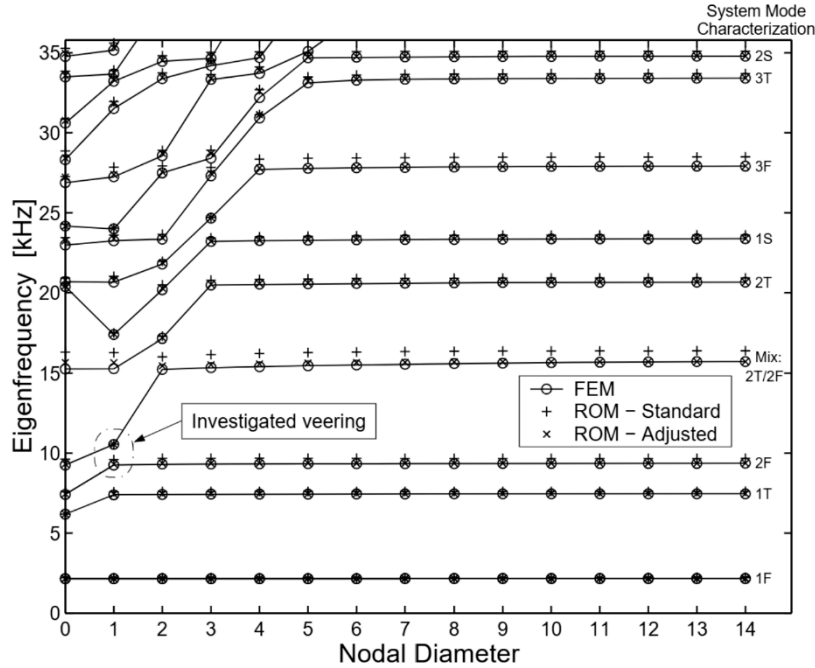


Figure 15: Example of a FreND diagram with modal families indication [5]

Modes that share similar mode shapes are classified in modal families, this means that, for a specific modal family, the dominant pattern of the vibration motion remains the same along the nodal diameters.

Typically, higher modal families show greater mode complexity, and, in addition, disk-dominant modes present a stiffening effect due to an increase in the number of nodal diameters.

Lastly, a notable feature is the veering of eigenfrequencies, like the one highlighted in Fig. 16, in which two different modal families veer away from each other.

This is a sign of a possible coupling between two different modes or, in some cases, a crossing between two families.

1.3.3 Aeromechanical analysis

Turbomachinery certification requires well-defined limits on maximum levels of allowable stresses for all the components.

For this purpose, modal analysis is insufficient: it characterizes the system's dynamic properties but not its behavior under operational conditions; therefore, another form of analysis is required.

Aeromechanics studies the interaction of elastic bodies with fluid flow in order to evaluate the occurrence of vibrations that can lead to component failures caused, mainly, by High Cycle Fatigue (HCF).

Turbomachinery, and consequently gas turbine engines, are subjected to aeromechanical issues like:

- **Forced response:** vibrations with a defined amplitude
- **Flutter:** dynamic instability with increasing vibration amplitude

Regarding aeronautical engines, the forced response, on which this work focuses, is activated by a series of phenomena: flow distortions at the stage inlet, presence of a pressure potential field or wakes from a previous row of blades.

Ideally, the steady axial gas pressure coming from the combustion chamber, or entering the compressor, might vary with radius but is otherwise uniform in the circumferential direction, thus resulting in an identical force field on each blade of a particular rotor or stator assembly, as seen in section 1.2.2 or 1.2.4. In practice, however, the fluid flow (either atmospheric air or exhaust gases) meets obstructions, such as stator vanes or rotors' blades causing blade wakes from upstream rows or potential pressure fields from downstream ones.

Therefore, even in steady operation the axial gas pressure varies circumferentially.

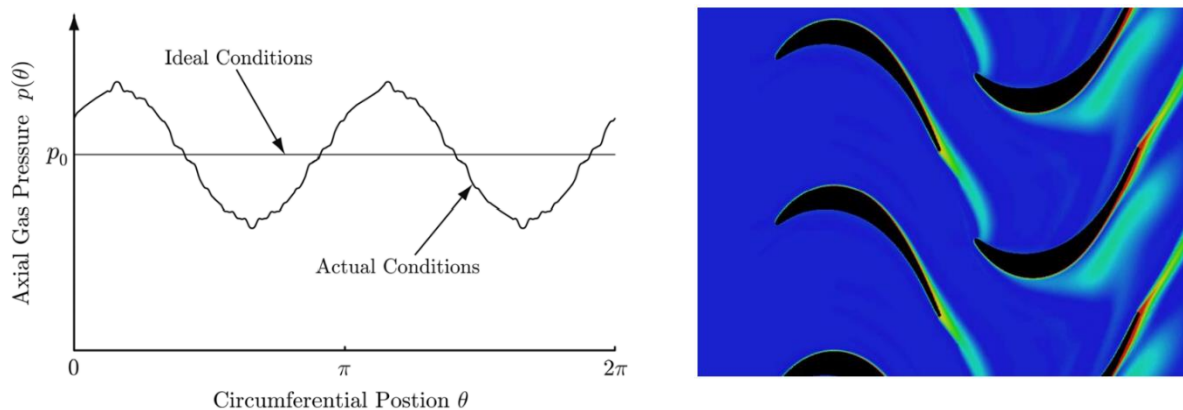


Figure 16: Axial gas pressure Ideal and Actual conditions (Left) [4], upstream blades' wakes [6]

A blade rotating through this field experiences a force with a frequency that is a multiple integer of the angular speed of the engine.

The resulting aerodynamic force is rotationally periodic and can therefore be expanded in a Fourier series:

$$F(\theta) = F_0 + \sum_{EO} F_c^{(EO)} \cos(EO \cdot \theta) + F_s^{(EO)} \sin(EO \cdot \theta) = F_0 + \sum_{EO} F^{(EO)} \cos(EO \cdot \theta + \varphi) \quad (1.13)$$

Where:

- EO is the *Engine Order*, the multiple-integer of the engine angular speed
- θ is the circumferential position of the blade

The harmonic content of the signal depends on the number of jets, and thus obstructions, in the row.

Adjacent sectors are going to experience the same force, but with a constant delay defined by the EO and the angle between them, in a similar way as defined in eq. (1.12); the only difference is that, in this case, the harmonic index is referring to the excitation, and as such it corresponds to the engine order:

$$\{F_{EO}\} = \begin{Bmatrix} I_j \\ I_j e^{-i\Psi} \\ \vdots \\ I_j e^{-i(n-1)\Psi} \\ \vdots \\ I_j e^{-i(N-2)\Psi} \\ I_j e^{-i(N-1)\Psi} \end{Bmatrix} \{F^{(EO)}\} \quad \Psi = EO \frac{2\pi}{N_{tot}}$$

As seen before in 1.3.2, these are travelling waves, which represent the primary source of forced response in bladed disk assemblies. Specifically, these are backwards traveling waves relative to the rotating hub, as they move towards descending blade numbers.

Nevertheless, depending on the value of EO relative to N_{tot} , this gives rise to discrete *standing*, *forward* or *backward* travelling waves. These additional possibilities arise due to aliasing of the sampled loads by the discrete sectors.

In general, for a system resonance to occur, the excitation frequency must match the resonance frequency, but also:

$$EO - kN_{tot} = h \quad (1.14)$$

Where $k = 0, 1, 2, 3 \dots$

For $EO \leq N_{tot}$:

$$1 \leq EO \leq \frac{N_{tot}}{2} \Rightarrow c - \text{type modes with } h > 0$$

$$EO = \frac{N_{tot}}{2} \Rightarrow b - \text{type mode (if it exists)}$$

$$\frac{N_{tot}}{2} < EO \leq N_{tot} - 1 \Rightarrow c - \text{type modes with } h < 0$$

$$EO = N \Rightarrow a - \text{type mode with } h = 0$$

For $EO > N_{tot}$, thanks to the periodicity of the exponential, it can be demonstrated that:

$$EO = kN_{tot} + EO' \quad (1.15)$$

Where $0 \leq EO' \leq N$.

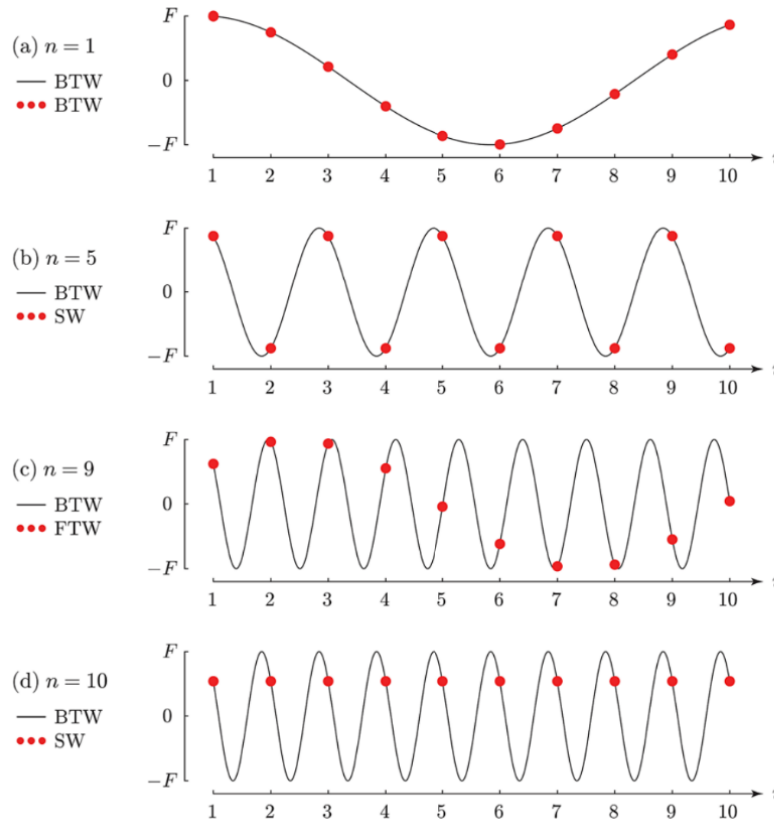


Figure 17: Discrete sampling and corresponding TW based on EO (n) and number of sectors (N) [4]

Critical crossings are graphically represented using the *Campbell's diagram* which presents, as the x-axis the angular speed of the rotor, and as the y-axis the frequency.

Engine order excitations are usually represented as straight lines passing through the origin, with a slope that depends on the EO number, whereas natural frequencies of the system, usually grouped into modal families, are horizontal lines.

In case the rotor is affected by spin softening or centrifugal stiffening for higher angular speeds, these lines could be curved instead.

Only critical crossings are highlighted in the diagram, following eq. (1.14).

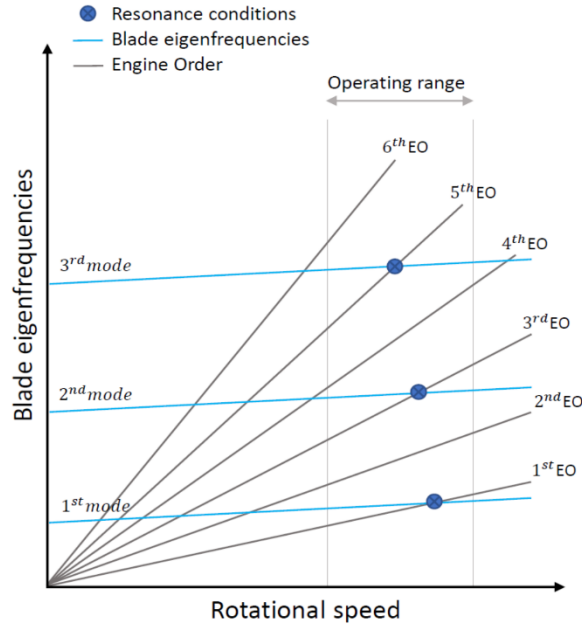


Figure 18: Typical Campbell's diagram [6]

Lastly the so-called *Zig-Zag diagram* is used to graphically represent the relationship (1.14):

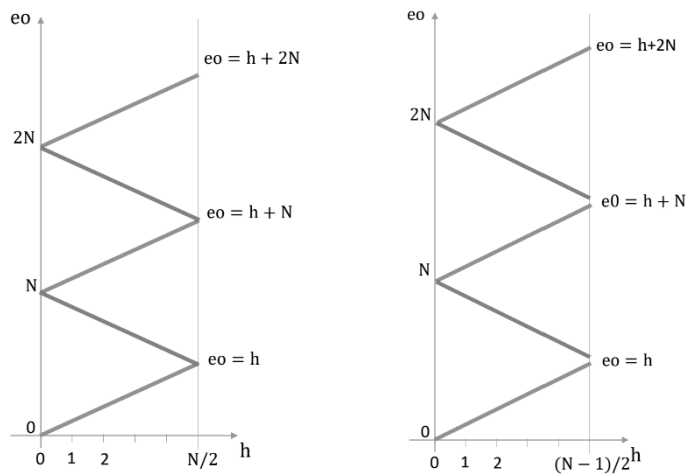


Figure 19: Zig-Zag diagram for an even (left) and odd (right) number of blades [3]

The EO excitations, previously described, originate a system response with a frequency equal to the forcing one, which corresponds to an integer-multiple of the rotating speed of the engine (the Engine Order itself), this type of oscillation is called a Synchronous Vibration.

On the other hand, other vibration mechanisms of aeroelastic nature like flutter, vortex shedding, rotating instabilities and, as it will be discussed later, a rapid crossing of system resonances, can cause vibrations with a frequency that is not an integer-multiple of the engine speed.

These kinds of oscillations are defined Non-Synchronous Vibrations or NSV.

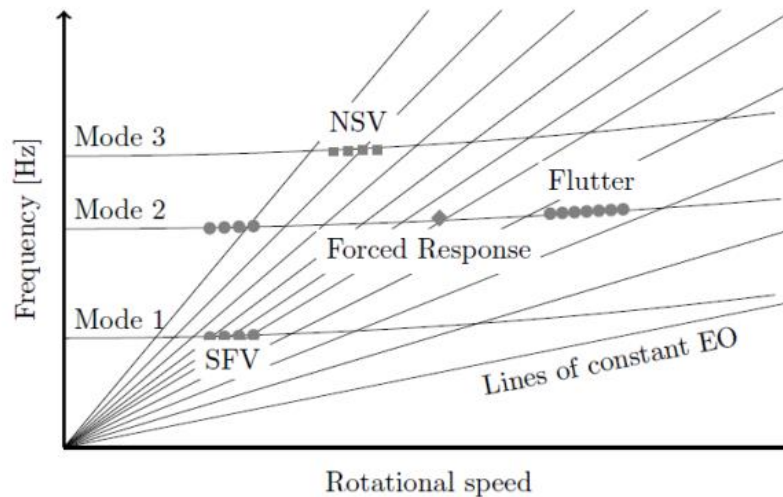


Figure 20: Campbell diagram with Synchronous and Non-Synchronous Vibrations [7]

1.4 Friction Damping

In order to reduce the stress caused by forced vibrations, especially in resonant conditions, a form of passive vibration control is needed to guarantee the bladed disk integrity in the entire operating envelope.

The solutions employed in this sense, able to withstand the harsh operating environment of an aeronautical engine, are: blade dampers, microslip damping at the root of the blades and the already mentioned shrouds.

Inertial friction dampers, also called *underplatform dampers* (UPDs), are the most employed; they consist of a properly designed piece of metal located in a cavity underneath two adjacent blade platforms.

During service the damper is pressed in a stationary position by centrifugal force against the underside of the platform, however, if the Coulomb's limit, or the static friction force, is surpassed due to the increase in displacement amplitude when a resonance is crossed, a relative displacement between the two adjacent blade platforms sets in producing a slip on the damper surfaces that dissipates the vibration energy through heat.

The oscillating nature of the dynamic response, and thus the displacement amplitude, makes this relative motion discontinuous, the system enters a stick-slip type of motion introducing a non-linearity in the dynamics.

The main challenge, as of today, is the development of friction contact models that allow to predict the damper performance through the solution of the non-linear dynamic response of the system.

Usually, the factors defined to model the damper are preload, coefficient of friction and stiffness, even though geometry is important as well.

In this work, as it will be discussed later, a standard friction damping model is used to simulate the stick-slip phenomenon, the *Jenkins Element*.

Chapter 2:

Sensitivity Analysis of Transient Resonance Response in Bladed Disks

This chapter illustrates the non-stationary effects on bladed disks' vibratory response following a frequency sweep through resonance. A sensitivity analysis will be conducted, with the help of a Single Degree of Freedom system (SDOF), to understand the dependencies on various factors like sweep rate, damping levels and varying system parameters.

2.1 Transient effects in Time and Frequency domain

In various occasions gas turbine engines are subjected to operating conditions that deviate from steady-state.

Power plants, with the growing diffusion of renewable energy resources, are required to feed the power grid on demand; this translates into more frequent start-ups and shut-downs.

In the aeronautical sector instead, engines are subjected to acceleration ramps with various rates (or sweep velocity) not only during their service time but also in case of certification tests, where slow accelerations to maximum speed and decelerations back to idle are performed and the stress responses recorded.

These frequent and quick resonance passages, coupled with the time-varying excitation frequencies, lead to a transient characteristic in the forced response accompanied by a series of effects that influence not only bladed disks dynamics, but also overall operational safety:

- Decreasing of the maximum amplitude
- Shifting of the maximum peak in the direction of the sweep
- Presence of a beating effect after the maximum peak

The investigation of the influence of these transient effects on the dynamic behavior of bladed disks is crucial for a correct estimation of fatigue cycles, and thus High Cycle Fatigue (HCF) life, but also for the parameters identification phase and model validation/update.

Observing the transient response in the time domain in fact, it becomes clear that, if these effects are ignored, the HCF life estimation, based on the number of cycles that exceed a certain mean stress threshold, becomes inadequate due to the prolonged presence of beats in the oscillation, affecting the correct design process.

Moreover, the legacy approaches for the identification of modal parameters, like the Half Power Band Method (HPB) for damping levels, present a scarce accuracy leading to a large systematic overestimation, as will be shown, due to the listed problems.

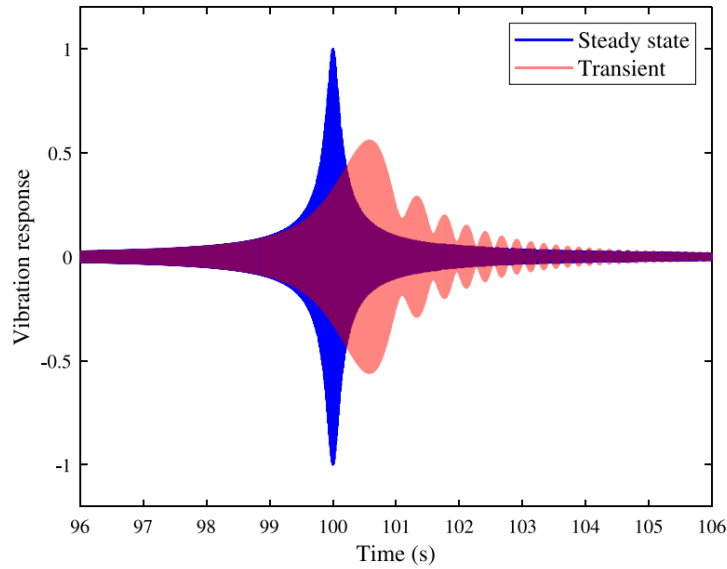


Figure 21: Time history of a vibration response in stationary and transient condition [9]

Examining the frequency content of the response through a Fast Fourier Transform (FFT) in the vicinity of the resonance, an engine order response is found as expected from the stationary forced response; however, what distinguishes the non-stationary response from it, is the presence of a second component, a strong homogeneous response is in fact triggered along the natural frequency of the system.

Through a Campbell's diagram, which highlights the energy density of the vibration, it is possible to represent this aspect of the unsteady response and give an explanation about the origin of the transient effects introduced.

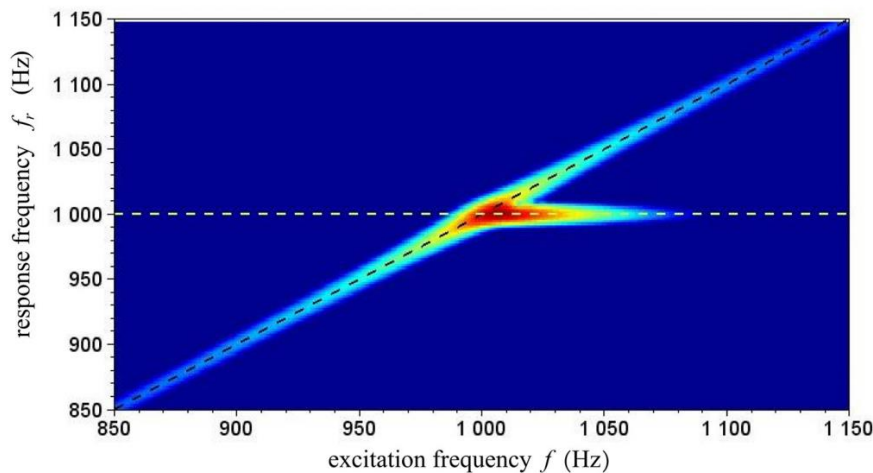


Figure 22: Campbell's diagram of a 1DOF - system transient resonance passage [8]

For a specific mode of vibration, the homogeneous response represents a transitory element that decays in accordance with the exponential decay function, dependent on the modal damping ratio.

For a steady-state or quasi steady-state operation, the sweep velocity is slow enough that this component is dissipated and does not interact with the engine order response and as such no transient effects are activated.

Above a critical value of acceleration rate, on the contrary, the excitation frequency change is comparable to the decay time of the natural frequency response, thus a superposition of two signals with close frequencies occurs, generating the beating phenomenon; also the existence of a second trace, that of the free-decaying vibration, renders the transient forced response a Non-Synchronous Vibration, differentiating it from the stationary condition discussed in 1.3.4.

The other two effects can be explained considering that, due to a high sweep velocity the time available to store energy into the system near the resonance is reduced, lowering the maximum amplitude of the response in a non-stationary condition, while the frequency shift is caused by a delay with respect to the resonance condition.

Therefore, the dependencies on damping levels and resonance passage speed of these effects have been noticed and, thanks to a simple test case, will be thoroughly investigated as well as new proposed identification methods.

Finally, the effectiveness of standard vibration control techniques will be evaluated in this context.

2.2 SDOF System Transient Response Sensitivity Analysis

A Linear Single Degree of Freedom System, of growing complexity, will be adopted for estimating the transient response dependency on the sweep velocity, damping levels, and resonance frequency; also, it will constitute the base for the extraction of the predicted stationary response in cases where the measured data are flawed by the presence of significant transient effects.

Due to the assumption of a Linear SDOF system, the approach analyzed has limitations regarding mistuned cyclic bladed disks or friction damping.

For this reason, in this work, the effect of TAMS or *Transient Amplitude Amplification of Mistuned Structures*, won't be considered, whereas a Non-Linear Two Degree of Freedom System will be introduced to evaluate the influence of friction damping, a standard vibration control technique in bladed disks, on transient sweeps.

2.2.1 Linear Time Invariant System

A tuned bladed disk without friction dampers, can be modeled as a Linear Multi Degree of Freedom system (MDOF).

The equations of motion (EoM) of an MDOF system, with appropriate assumptions on damping, can be decoupled by transformation from generalized coordinates to modal coordinates.

Recalling eq. (1.2) it is possible to define the **Modal Matrix** formed by the eigenvectors solution to the eigenvalue problem:

$$[\Phi] = [\{\phi\}_1, \{\phi\}_2, \dots, \{\phi\}_N]^T$$

Substituting the following coordinate transformation in the EoM:

$$\{x(t)\} = [\Phi]\{\eta(t)\} \quad (2.1)$$

And pre-multiplying by $[\Phi]^T$, yields:

$$[\tilde{M}]\{\ddot{\eta}(t)\} + [\tilde{C}]\{\dot{\eta}(t)\} + [\tilde{K}]\{\eta(t)\} = \{\tilde{f}_e(t)\} \quad (2.2)$$

Where:

- $[\tilde{M}] = [\Phi]^T[M][\Phi]$ is the diagonal **Modal Mass Matrix**
- $[\tilde{K}] = [\Phi]^T[K][\Phi]$ is the diagonal **Modal Stiffness Matrix**
- $[\tilde{C}] = \text{diag}(2\zeta_i\omega_i)$ is the diagonal **Modal Damping Matrix**
- $\{\tilde{f}_e(t)\} = [\Phi]^T f_e$ is the **Modal Loads Vector**

Each i -th mode decoupled equation can thus be analyzed as a separate SDOF system:

$$m_i \ddot{\eta}_i + c_i \dot{\eta}_i + k_i \eta_i = \widetilde{f}_{ie}(t) \quad (2.3)$$

Characterized by:

- Modal mass: m_i
- Modal damping ratio: ζ_i
- Modal stiffness: k_i

In this case the coefficients of this linear differential equation, represented by the listed system parameters, are considered constant, so changes linked to centrifugal force, thermal effects, bearings' characteristics etc. are neglected.

The resulting system is called Linear Time Invariant or LTI, which satisfies the superposition principle.

Given this property, the forced response of a multi-degree-of-freedom system can be expressed as a modal superposition, obtained by summing the contributions of the individual modes.

However, if the eigenfrequencies are sufficiently spaced apart, the response in proximity of a resonance condition is dominated by the response of the corresponding eigenmode and its equation of motion only.

This condition is usually verified in a system with small or null non-linearities and in case of absence of mistuning.

To obtain the transient vibration time-history, the equation of motion is solved via direct numerical integration using an 8th-order continuous extension *Runge-Kutta method*.

The model is implemented in MathWorks MATLAB R2024b.

Since this solver requires converting the SDOF's second-order ODE into a system of two first-order ODEs, a state-space formulation is adopted.

To this end, starting from Eq. (2.3), an auxiliary variable is introduced:

$$q_2 = \dot{\eta}_i \quad (2.4)$$

Substituting into eq. (2.3):

$$m_i \dot{q}_2 + c_i q_2 + k_i q_1 = \widetilde{f}_{ie}(t) \quad (2.5)$$

Writing it in monic form:

$$\begin{cases} \dot{q}_1 = q_2 \\ \dot{q}_2 = m_i^{-1} [\widetilde{f}_{ie}(t) - c_i q_2 - k_i q_1] \end{cases} \quad (2.6)$$

To compute the run-up and run-down processes, a quasi-harmonic excitation force with linearly changing frequency is assumed:

$$\widetilde{f}_{ie}(t) = \widehat{f}_{ie} \cos(\theta_e(t)) \quad (2.7)$$

(Note that from now on the i subscript is dropped from the notation since everything refers to the single mode of interest)

Where the excitation angular frequency time law is:

$$\Omega(t) = \Omega_0 + \dot{\Omega}t \quad (2.8)$$

And thus, the phase, representing the transition process, changes with time as:

$$\theta_e(t) = \int \Omega(t)dt = \frac{1}{2}\dot{\Omega}t^2 + \Omega_0t + \theta_0 \quad (2.9)$$

This is a simplifying assumption for the sensitivity analysis, as in real-world operating conditions, the sweep law may be non-linear.

As already pointed out in section 2.1, in sweeping through a resonance, transient effects appear if the frequency change reaches a critical value, the question is how to determine this limit.

In this, damping plays an important role as for increased damping, the homogenous response decays faster, and therefore the stationary response will be attained sooner, with reduced interactions between the natural frequency response and the engine order response.

So, in the definition of this critical sweep velocity, the characteristic decay time, and thus the damping level, should be related to the time it takes to cross a resonance.

In literature [8] this relationship is defined through a non-dimensional number that governs the transient nature of the response.

It is well known that the characteristic decay time is:

$$\tau = \frac{1}{2\pi\zeta f_n} \quad (2.10)$$

Where:

- τ is the characteristic decay time
- ζ is the damping ratio of the mode considered
- f_n is the natural frequency of the mode considered

While the time to cross the half-power points of the steady-state response is:

$$\Delta t_h = \frac{\Delta f_h}{\dot{f}} \quad (2.11)$$

Where:

- Δf_h is the frequency delta between the half-power points
- $\dot{f}$ is the frequency rate of change in Hz/s

The Half-Power Method is here employed to find reference points for the resonance passage, but also to relate eq. (2.10) with eq. (2.11) via the damping ratio estimation:

$$\zeta = \frac{1}{2} \frac{\Delta f_h}{f_n} \quad (2.12)$$

Combining eq. (2.11) with eq. (2.12):

$$\Delta t_h = \frac{2\zeta f_n}{\dot{f}} \quad (2.13)$$

Finally, the dimensionless group is introduced as the ratio between the decay time eq. (2.10) and the crossing time defined in eq. (2.13):

$$K = 4\pi \frac{\tau}{\Delta t_h} = \frac{\dot{f}}{\zeta^2 f_n^2} \quad (2.14)$$

K is interpreted as a *Reduced Sweep Velocity* (also referred to as *RSV* from now on).

It is worth noting that the original definition is given in terms of frequency, which is linked to the angular frequency of eq. (2.8) by:

$$f = \frac{\Omega}{2\pi} \text{ and } \dot{f} = \frac{\dot{\Omega}}{2\pi}$$

Also, regarding turbomachinery, object of study of this work, the excitation frequency can be related to the rotor speed and its acceleration:

$$f = EO \cdot \frac{N_r}{60} \quad (2.15)$$

$$\dot{f} = EO \cdot \frac{\dot{N}_r}{60} \quad (2.16)$$

With:

- N_r : Rotor Speed in RPM
- $\dot{N}_r$: Rotor Acceleration in RPM/s

It has to be noticed that the system defined is not a fully realistic model for rotor blades due to a number of different reasons.

First, as already pointed out, the natural frequency and damping, hereby considered constant, in a real bladed disk may change during the resonance crossing due to the variation of the centrifugal force.

Secondly, the unsteady pressure field that excites the disk usually is composed of a summation of several harmonic components, as described in section 1.3.4, thus the external force should be constituted a sum of chirps having frequencies multiples of the rotation speed, also blades have multimodal response and as such different resonances may be excited simultaneously by different EOs.

Furthermore, due to the coupling between adjacent sectors, present for example in shrouded turbine blades, the disk tends to have closely spaced global modes that cannot be singularly analyzed as a single degree of freedom. Finally, friction damping, employed to manage the vibratory response, introduces significant non-linearities requiring a different system for the evaluation of the damper performance.

LTI System Transient Response Sensitivity Analysis

To conduct a sensitivity analysis on the transient response of the Linear Time Invariant system shown, a large number of different parameter combinations must be considered; to simplify the task it can be demonstrated, introducing a dimensionless time variable, that variations in the natural frequency can be skipped without loss of information.

Starting from eq. (2.3), and normalizing it with respect to the modal mass:

$$m_i \ddot{\eta}_i + c_i \dot{\eta}_i + k_i \eta_i = \widetilde{f}_{ie}(t) \Rightarrow \frac{1}{m_i} \Rightarrow \ddot{\eta}_i + \frac{c_i}{m_i} \dot{\eta}_i + \frac{k_i(t)}{m_i} \eta_i = \frac{\widetilde{f}_{ie}(t)}{m_i}$$

Where:

$$\frac{k_i(t)}{m_i} = \omega_i(t)^2 \quad (2.17)$$

With ω_i the mode's natural frequency in rad/s.

For the sake of generality, a time-varying natural frequency is being considered, taking into account the effects of spin softening and centrifugal stiffening; these findings are thus relevant for the subject of this section, the simpler LTI system.

Expanding the natural and excitation frequencies:

$$\ddot{\eta}_i + \frac{c_i}{m_i} \dot{\eta}_i + (\omega_{i0} + \dot{\omega}_i t)^2 \eta_i = \frac{\widehat{f}_{ie} \cos\left(\frac{1}{2} \dot{\Omega}_i t^2 + \Omega_{i0} t + \Theta_{i0}\right)}{m_i}$$

(From now the i subscript will be dropped for clarity)

Normalizing the time in non-dimensional form with reference to the resonance frequency ω_r (intended as the intersection of the excitation and natural frequencies):

$$t' = \omega_r t \quad (2.18)$$

$$\Rightarrow \ddot{\eta} + \frac{c}{m} \dot{\eta} + \left(\omega_0 + \frac{\dot{\omega}}{\omega_r} t'\right)^2 \eta = \frac{\hat{f}}{m} \cos\left(\frac{1}{2} \frac{\dot{\Omega}}{\omega_r^2} t'^2 + \frac{\Omega_0}{\omega_r} t' + \Theta_0\right)$$

And the displacement and excitation with reference to a characteristic amplitude:

$$\begin{aligned} \frac{1}{\omega_r^2} \left(\ddot{\eta} + \frac{c}{m} \dot{\eta} + \left(\omega_0 + \frac{\dot{\omega}}{\omega_r} t'\right)^2 \eta \right) &= \frac{1}{\omega_r^2} \left(\frac{\hat{f}}{m} \cos\left(\frac{1}{2} \frac{\dot{\Omega}}{\omega_r^2} t'^2 + \frac{\Omega_0}{\omega_r} t' + \Theta_0\right) \right) \Rightarrow \\ \Rightarrow \frac{\ddot{\eta}}{\omega_r^2} + \frac{c}{m\omega_r^2} \dot{\eta} + \left(\frac{\omega_0}{\omega_r} + \frac{\dot{\omega}}{\omega_r^2} t'\right)^2 \eta &= \frac{\hat{f}}{m\omega_r^2} \cos\left(\frac{1}{2} \frac{\dot{\Omega}}{\omega_r^2} t'^2 + \frac{\Omega_0}{\omega_r} t' + \Theta_0\right) \Rightarrow \\ \Rightarrow \frac{1}{\hat{\eta}} \left(\frac{\ddot{\eta}}{\omega_r^2} + \frac{c}{m\omega_r^2} \dot{\eta} + \left(\frac{\omega_0}{\omega_r} + \frac{\dot{\omega}}{\omega_r^2} t'\right)^2 \eta \right) &= \frac{1}{\hat{\eta}} \left(\frac{\hat{f}}{m\omega_r^2} \cos\left(\frac{1}{2} \frac{\dot{\Omega}}{\omega_r^2} t'^2 + \frac{\Omega_0}{\omega_r} t' + \Theta_0\right) \right) \end{aligned}$$

The characteristic amplitude is defined in literature [11] as:

$$\hat{\eta} = \frac{\hat{f}}{m\omega_r^2} \quad (2.19)$$

Using the chain rule the derivation variable is changed from t to the dimensionless time t' :

$$\frac{d\eta}{dt} = \frac{d\eta}{dt'} \cdot \frac{dt'}{dt} \Rightarrow \dot{\eta} = \omega_r \eta', \ddot{\eta} = \omega_r^2 \eta''$$

Substituting into the main equation gives:

$$\frac{\eta''}{\hat{\eta}} + \frac{c}{m\omega_r} \frac{\eta'}{\hat{\eta}} + \left(\frac{\omega_0}{\omega_r} + \frac{\dot{\omega}}{\omega_r^2} t' \right)^2 \frac{\eta}{\hat{\eta}} = \frac{\hat{f}}{m\omega_r^2 \hat{\eta}} \cos \left(\frac{1}{2} \frac{\dot{\Omega}}{\omega_r^2} t'^2 + \frac{\Omega_0}{\omega_r} t' + \theta_0 \right)$$

A series of non-dimensional coefficients can now be defined:

$$\alpha = \frac{\dot{\Omega}}{\omega_r^2} \quad \beta = \frac{\dot{\omega}}{\omega_r^2} \quad \xi_0 = \frac{c}{2m\omega_r} \quad \rho_0 = \frac{\omega_0}{\omega_r} \quad \eta_0 = \frac{\Omega_0}{\omega_r} \quad \hat{H}_e = \frac{\hat{f}}{m\omega_r^2 \hat{\eta}} \quad \hat{\eta} = \frac{\hat{f}}{m\omega_r^2}$$

The time-dependent excitation and natural frequency become:

$$\eta(t') = \alpha t' + \eta_0; \quad \rho(t') = \rho_0 + \beta t'$$

And the final normalized equation:

$$H'' + 2\xi_0 H' + \rho(t')^2 H = \hat{H}_e \cos \left(\frac{1}{2} \alpha t'^2 + \eta_0 t' + \theta_0 \right)$$

So, the terms that govern the response of the system are:

- α the dimensionless sweep rate
- β the dimensionless rate of change of the natural frequency
- ξ_0 the dimensionless damping ratio

Together with the system's constants indicated by subscript o .

2.2 SDOF System Transient Response Sensitivity Analysis

Returning to the LTI system analyzed the inherent relationship between the response and these terms is further simplified. Due to the constant nature of the natural frequency in fact, β is null, and does not play a role in the system dynamics, also the resonance frequency ω_r corresponds to ω_0 .

The damping, instead, is usually determined by various structural factors like the material and the geometry and thus there is not much room for changes.

In conclusion, as long as the value of α stays constant the transient response is independent of the natural frequency.

Expressing α in more practical terms, the ratio is:

$$\alpha = \frac{\dot{\Omega}}{\omega_r^2} = \frac{\dot{f}}{2\pi f_n^2} \quad (2.20)$$

Now, looking again at the definition of the RSV from eq. (2.14), a similarity with eq. (2.20) can be found; α represents a part of the complete formulation of K , this implies that, for a given value of K the differentiating factor in the transient response will be the damping ratio ζ , while vice versa, for a given value of ζ what distinguishes the responses is the value of K .

This greatly reduces the number of combinations of parameters that are needed for a sensitivity study, which results are now presented.

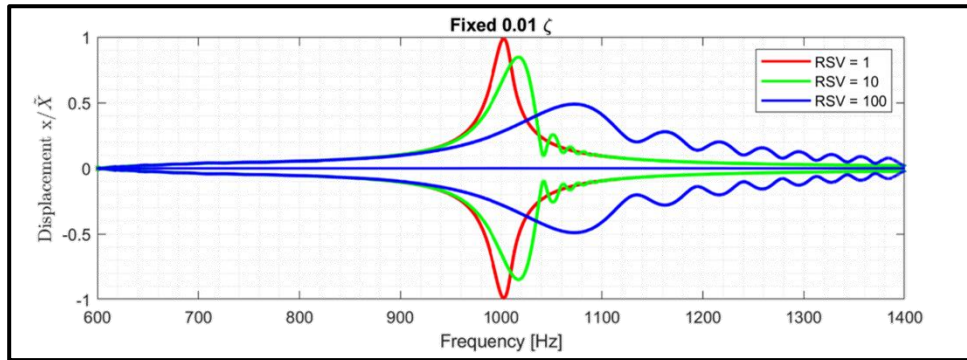


Figure 23: Response envelopes of transient sweep through a resonance for different values of Reduced Sweep Velocity K ($f_n = 1000$ Hz, $\zeta = 0.01$)

Figure 23 shows the effect of different levels of Reduced Sweep Velocity on the response envelope.

Parameters

Modal mass	[kg]	0.1
Natural frequency	[Hz]	1000
Modal damping ratio		0.01

The amplitude has been normalized with respect to the stationary response maximum amplitude:

$$\tilde{X} = \frac{\hat{f}}{2\zeta k} \quad (2.21)$$

It can be observed that, as expected from the preliminary discussion of the phenomenon in section 2.1, the transient effects are more prominent for higher values of K , in detail:

- The beating phenomenon lasts longer and is more intense, with secondary peaks closer in amplitude to the main one; due to this aspect of the response, the amplitude magnification effect of the resonance crossing interests a higher range of frequencies, and thus speeds, increasing the number of fatigue cycles to which the component is subjected.

Another problem arises the moment a test is conducted to identify the modal parameters of the disk, in fact, applying a legacy method like the HPB method or logarithmic decrement, the estimated damping would be overestimated for high values of K , as can be expected looking at the envelopes from Figure 23.

- The main peak's location shifts in frequency in the direction of the sweep with an intensity linked to the crossing speed. As for the beating, this effect interferes with the identification process since an erroneous f_n would be estimated. This would then reflect on the working life of the component which is usually intended to operate in a safe range, far from critical crossings that should be therefore correctly identified.
- The main peak's width increases for a higher value of RSV, affecting again the HCF life due to a higher number of dangerous cycles introduced.
- Finally, the maximum amplitude decreases with respect to the steady-state response, exhibiting higher differences for more rapid crossings.

Additional insight is gained observing resonance passages with different levels of damping ratio and constant RSV :

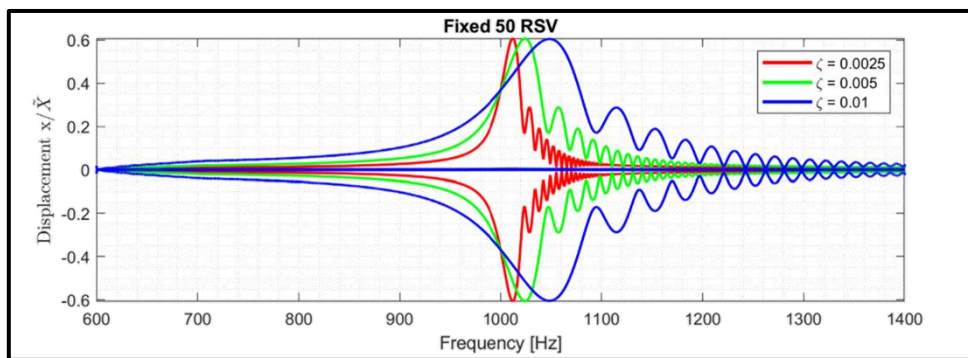


Figure 24: Resonance envelopes of transient sweep through a resonance for different values of damping ζ at equal reduced sweep velocity K ($f_n = 1000$ Hz, $K = 50$)

The parameters considered for the evaluated system are the same as the precedent one, with the exception of a varying damping ratio.

Looking at Figure 24, it is possible to notice that, even though the envelopes are similar, an interesting property, that will be recalled later, is showcased. The normalized amplitude, according to eq. (2.21), is constant for different levels of damping for the main peak, but also for the subsequent maxima, if beats are present.

These results confirm what previously was briefly mentioned, that is the existence of a limit value that acts as a discriminant for the occurrence of transient effects.

In literature [8] this limit is fixed at $K \geq 3.6$, while for $K \leq 1$ the response is virtually identical to the stationary case. The value proposed of 3.6 should not be interpreted as a physical barrier or a discrete kind of on/off switch, but rather as a conventionally accepted value based on experience, empirical trials and real-world applications.

These phenomena are, in fact, continuous in nature; beats are present for values of K lower than the indicated threshold, but their significance is considered negligible in most cases due to the low impact on the dynamic behavior of the bladed disk compared to the steady-state response.

In the envelopes analyzed only a reduced number of values were used for both the Reduced Sweep Velocity and the damping ratio.

Now, to further understand the trends highlighted and to investigate other useful properties of the transient response, results from a broader range of values are used to generate a dataset that will represent the base for the construction of a series of curves through cubic spline interpolation for various damping levels.

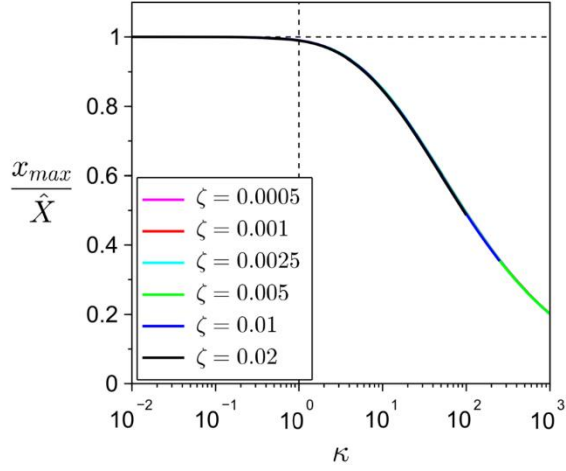


Figure 25: Ratio of maximum amplitude to the stationary response amplitude versus RSV [8]

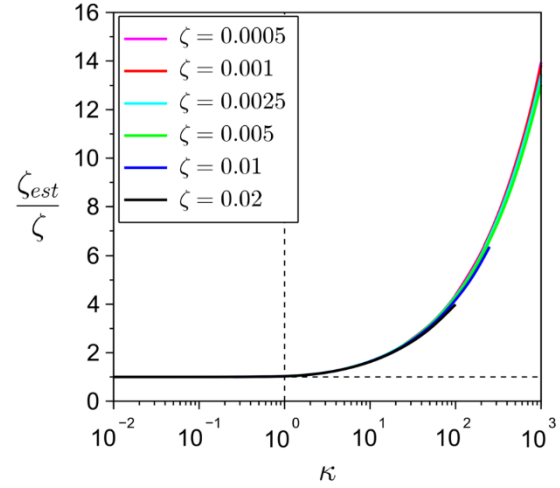


Figure 26: Ratio of estimated damping ratio to its exact value versus RSV [8]

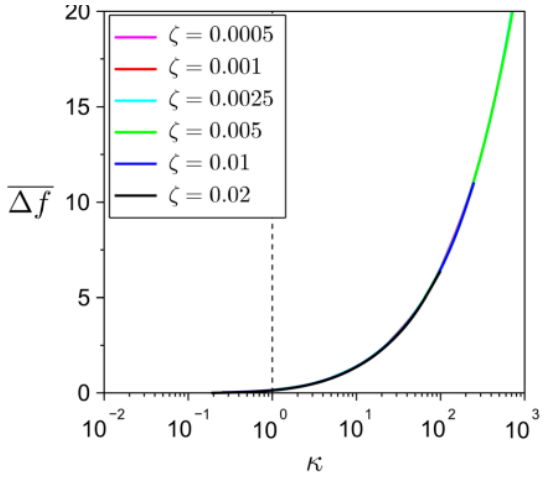


Figure 27: Frequency shift parameter versus RSV [8]

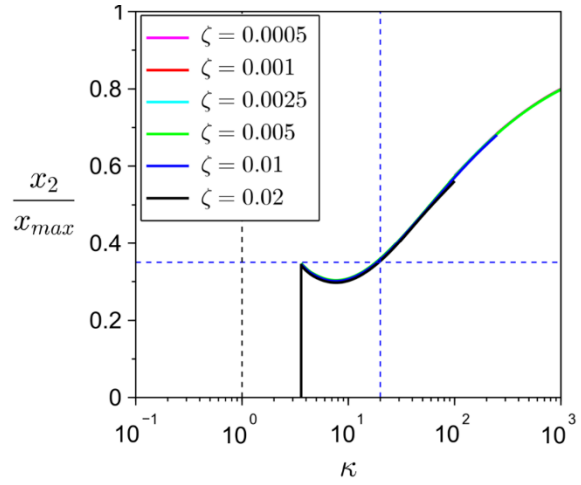


Figure 28: Ratio of amplitude of second maximum to maximum amplitude as function of RSV [8]

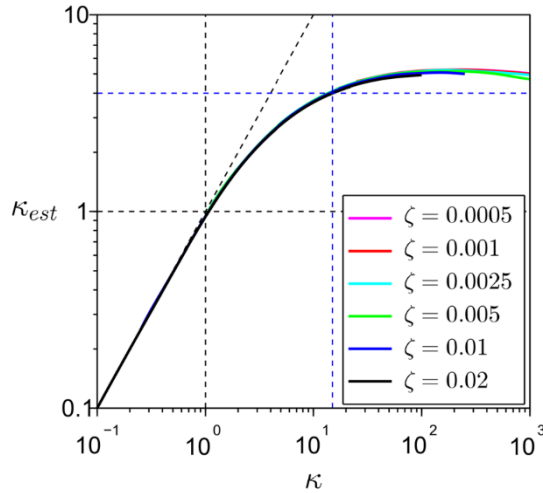


Figure 29: Estimate for RSV versus exact value [8]

For all the different cases, the curves generated for a range of values of ζ fall almost exactly on each other indicating a dependency of the y-axis parameter exclusively on the RSV. Figure 26 shows, for example, that the ratio of the estimated damping, adopting the half-power method, to its exact value depends only on K ; pay attention to the significant overestimation resulting from failing to take into account the resonance crossing speed in an identification test.

On the other hand, as previously displayed, the frequency shift is a function also of the damping ratio, for this reason a dimensionless variable is introduced to capture this dependency, as seen in Figure 27:

$$\overline{\Delta f} = \frac{|f_{n,est} - f_n|}{\zeta f_n} \quad (2.22)$$

Recalling eq. (2.10), and rewriting eq. (2.22):

$$|f_{n,est} - f_n| = \zeta f_n \overline{\Delta f}(K) = 2\pi \frac{\overline{\Delta f}(K)}{\tau} \quad (2.23)$$

It's clear that the frequency shift is more prominent the larger the damping ratio is.

2.2.2 Determination of Stationary Response from Measured Transient Response

The curves presented and the properties discussed, particularly the one relative to Figure 24, can be exploited to derive the stationary response. During an identification test, in fact, the acceleration ramp for the rotor is arbitrarily selected, and as such $\dot{N}_r$ is known; moreover, it's reasonable to assume that the analyst has a clear understanding of the eigenvalues of critical modes. However, the damping ratio ζ , which contains the structural, mechanical and aerodynamic damping, is not usually known upfront.

This means that, at priori, it is not possible to evaluate the reduced sweep velocity, and besides, once the measurements results are available, the value of K can only be estimated using values of damping ratio and natural frequency obtained through the half-power method or the logarithmic decrement, ζ_{est} and $f_{n,est}$:

$$K_{est} = \frac{\dot{f}}{\zeta_{est}^2 f_{n,est}^2} \quad (2.24)$$

Figure 29 indicates that this estimate is equal to the real reduced sweep velocity for $K_{est} \leq 1$, beyond this value the error starts to grow significantly.

K_{est} reaches an asymptotic limit at around 5.15, but the relationship displayed between the two in Figure 29 turns out to be practical for the determination of K for a maximum value of $K_{est} = 4$. Beyond this value, every small error in the determination of ζ_{est} leads to an even larger deviation in the value of K_{est} , which amplifies significantly in K due to the flattening of the curve.

For this reason, to establish the real value of RSV for $K_{est} > 4$ or $K > 15$, the properties demonstrated in Figure 24 are used.

The full methodology to determine the stationary response from the measurements is as follows [8]:

- First the HPB method is used to determine ζ_{est} , $f_{n,est}$ and K_{est} .
- If $K_{est} \leq 4$ and $\frac{x_2}{x_{max}} \leq 0.35$ (if beats are present), K is obtained through the curve of Figure 29, that links K with K_{est} .
- Otherwise, determine K from the curve of Figure 28, linking $\frac{x_2}{x_{max}}$ with K . This curve is monotonic after 0.35.
- Determine $\frac{x_{max}}{\bar{x}}$, $\frac{\zeta}{\zeta_{est}}$ and $\overline{\Delta f}$.
- Compute the exact damping ratio ζ and natural frequency f_n .

This sequence of actions has been implemented, in MATLAB R2024b, into a comprehensive algorithm that simulates the desired transient resonance passage, in the same way it would happen in an identification test, to generate a fictional transient response. This step is necessary until real measurements are available to test the methodology illustrated. The response is then analyzed to retrieve the ideally unknown, modal parameters.

Two different cases are presented to clarify the different approaches to the modal identification based on the intensity of the transient effects.

Parameters		
Modal mass	[kg]	0.1
Natural frequency	[Hz]	1000
Modal damping ratio		0.01
Modal load	[N]	100

In the time-history plot the half power points, together with the relevant peaks and the steady state resonance location are indicated.

1st Test Parameters

Sweep rate [Hz/s]	500
RSV	5

Due to the minor value of imposed RSV, the transient effects appear to have almost a negligible influence on the response of the system, with the beating phenomenon barely visible.

Now the identification results will be evaluated for both the legacy method and the proposed methodology:

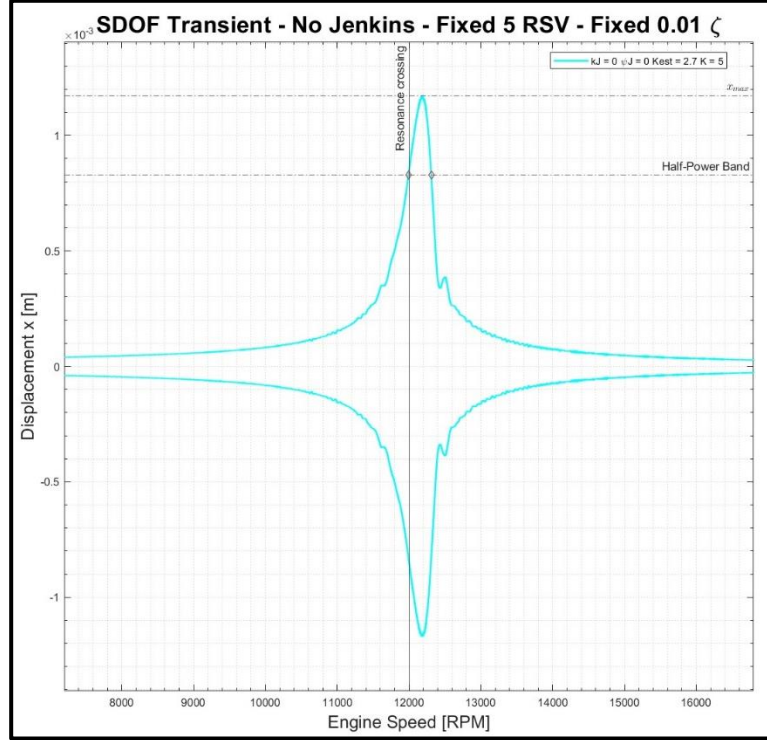


Figure 30: Time-history plot of transient resonance passage with RSV = 5

Table 1: Identification results with RSV = 5

Parameters	Values	Error %
K_{est} (HPB method)	2.71	-45.8
K_{id} identified	4.62	-7.6
K real	5	-
ζ_{est} (HPB method)	0.0133	33.0
ζ_{id} identified	0.0102	2.0
ζ real	0.0100	-
$f_{n,est}$ Hz (HPB method)	1023	2.3
$f_{n,id}$ Hz identified	1015	1.5
f_n Hz real	1000	-

The legacy method, the Half-Power Band method, as can be retrieved from Table 1, overestimates the value of the damping ratio, and slightly the natural frequency. This is a direct consequence of one of the major effects of the transient resonance crossing, the widening of the resonance peak which, in this example, is not particularly relevant but still has a considerable impact on the identification.

This discrepancy reflects on the computed value of K_{est} , that turns out to be almost half the real imposed one. Therefore, it's clear that, even for a low value of K , this method fails to accurately reconstruct the steady-state response.

The illustrated procedure instead behaves way better, especially in the reconstruction of the damping ratio and the applied reduced sweep velocity. In this instance, since the estimated value of K , K_{est} , was below the recommended limit of 4, it was possible to exploit the direct relationship between that, and the actual value depicted in Figure 29.

2nd Test Parameters

Sweep rate [Hz/s]	5000
RSV	50

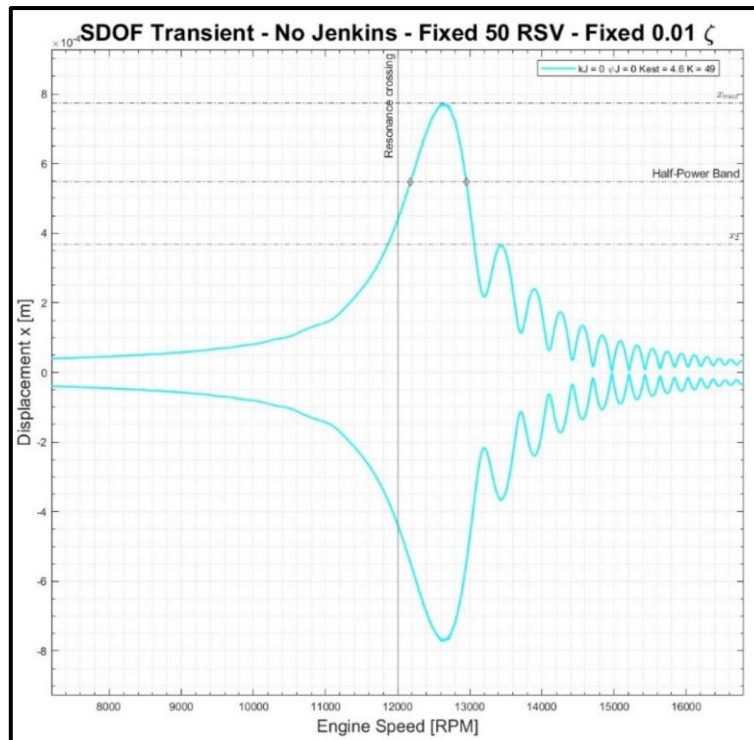


Figure 31: Time-history plot of transient resonance passage with RSV = 50

Now transient effects are clearly present, heavily affecting the response and complicating the task of modal identification. It can be noticed, in addition to dominant beating phenomenon after the crossing, an important frequency shift and enlargement of the main peak width with respect to the previous case. It's also interesting the lower maximum amplitude due to a higher RSV.

Table 2: Identification results with RSV = 50

Parameters		Values	Error %
K_{est}	(HPB method)	4.58	-90.8
K_{id}	identified	49.1	-1.8
K	real	50	-
ζ_{est}	(HPB method)	0.0312	212.0
ζ_{id}	identified	0.0100	0.0
ζ	real	0.0100	-
$f_{n,est}$	Hz (HPB method)	1059	5.9
$f_{n,id}$	Hz identified	1016	1.6
f_n	Hz real	1000	-

Again, the legacy method overestimates the modal parameters, but now due to the high increase in the crossing speed, the error committed is substantial.

The estimated damping ratio is almost triple the value of the real one, while the natural frequency is affected by a lower, but still relevant divergence. This results into an estimated reduced sweep velocity that is a tenth of what was applied to the system. In conclusion, these results are completely erroneous and show the inadequacy of the HPB method under transient conditions.

The new method, instead, proves stable even in presence of intense crossings, with an error almost null in the identification of the damping ratio ζ_{id} which is identical to the one defined, the same can be said for the natural frequency with an error well under 2%.

Lastly the identified RSV matches the one to which the system was subjected.

In this occasion the value of K_{est} is higher than the allowed limit, for this reason the identification of K is based on the ratio $\frac{x_2}{x_{max}}$, with x_2 visible in Figure 31.

If, in place of the curve of Figure 28, the relationship between K and K_{est} is used, the value of RSV identified is $K_{id} = 26.3$, a wrong result that demonstrates, as already discussed, why that correlation isn't applicable after $K_{est} = 4$.

These considerations are precious to the analyst, which not only can still retrieve the correct data even from a measurement affected by unwanted phenomena, but is now able to recalibrate the test bench to impose the correct, desired, RSV.

2.2.3 Linear Time Variant System

As previously briefly mentioned, in turbomachinery, flexible rotor components like for the case of turbine and compressor blades, the geometric stiffness induced by the centrifugal force can be comparable with the elastic stiffness, making the natural frequencies change with the rotor speed and thus, in case of a run-up or run-down test, with time.

To investigate the effects of a slowly varying natural frequency, neglected until now, on the transient resonance response, a Linear Time Variant (LTV) system is introduced.

A linear relationship with time is considered:

$$\omega(t) = \omega_0 + \dot{\omega}(t) \quad (2.25)$$

Where the rate of change of the natural frequency $\dot{\omega}(t)$ can acquire both positive and negative values, linked with two different counteracting phenomena:

- Centrifugal Stiffening: $\dot{\omega}(t) > 0$
- Spin Softening: $\dot{\omega}(t) < 0$

The SDOF equation of motion thus becomes, from eq. (2.3):

$$m_i \ddot{\eta}_i + c_i \dot{\eta}_i + k_i(t) \eta_i = \widetilde{f}_{ie}(t) \quad (2.26)$$

As the system's stiffness depends on the natural frequency.

While this kind of system can be still solved by means of direct numerical integration through the already implemented algorithm, a simplified approach is available in literature [13].

It has been demonstrated, as a matter of fact, that the LTV problem can be treated as an equivalent LTI system, but subjected to a modified frequency sweep.

As discussed in section 2.1, the origin of transient effects is the superposition between the homogeneous component of the response and the engine order response, while their intensity depends on the duration of this interaction.

In a time-invariant case, the crossing time is function only of the rate of change of the excitation frequency, hence the reduced sweep velocity, which corresponds to the slope among the constant free-response frequency and the engine order response.

In a time-variant system instead, the apparent reduced sweep velocity experienced by the system is influenced by the rate of change of the natural frequency and is different from the nominal one deriving from the rotor acceleration; this is a direct consequence of the effective slope between the two components of the response which, in case of centrifugal stiffening is smaller compared to the LTI frequency case, thereby prolonging the interval in which the system remains in the proximity of the resonance, meanwhile in case of spin softening the opposite happens, with a crossing velocity higher than the expected one for the constant frequency case.

2.2 SDOF System Transient Response Sensitivity Analysis

This can be seen by observing the Campbell's diagram for the three different possibilities:
(Note that on this occasion it's a waterfall diagram, since the x-axis is represented by time)

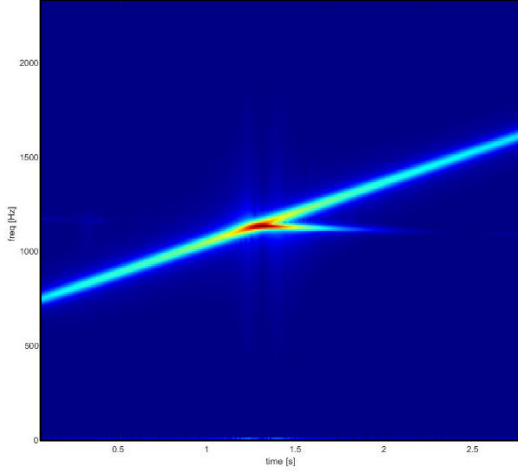


Figure 32: Waterfall diagram of transient resonance crossing, Softening case: $\dot{\omega} = -0.01\dot{\Omega}$

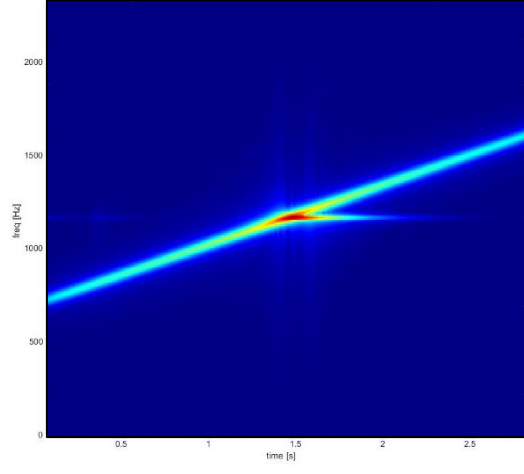


Figure 33: Waterfall diagram of transient resonance crossing, Time Invariant system: $\dot{\omega} = 0$

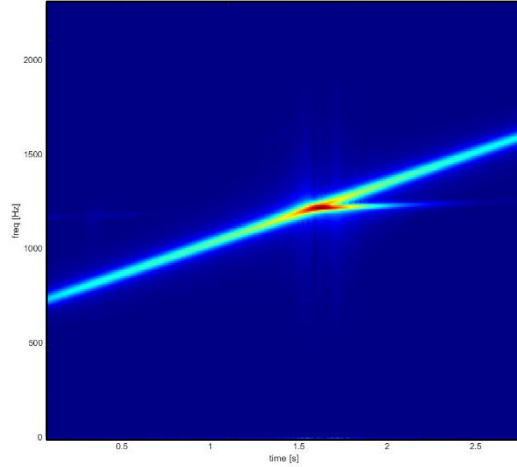


Figure 34: Waterfall diagram of transient resonance crossing, Stiffening case: $\dot{\omega} = 0.01\dot{\Omega}$

Figures 32 and 33 show the convergence or divergence of the homogeneous and EO response, caused by the non-constant nature of the mode's eigenvalue; it can so be better understood the implication on the length of the passage.

For both the softening and stiffening systems, the rate of change of the natural frequency has been expressed as a function of the time derivative of the forcing frequency, for is their ratio that matters concerning the transient response in an LTV problem. It is therefore convenient to introduce a dimensionless variable defined as the sensitivity of the natural frequency with respect to the rotor speed [13]:

$$\sigma = \frac{\dot{\omega}}{\dot{\Omega}} \quad (2.27)$$

Where the value $\sigma = 0$ characterizes the LTI system previously analyzed, a special case of LTV system.

The effective slope between the eigenfrequency and the excitation is thus $\dot{\Omega} - \dot{\omega}$, so considering an equivalent LTI system, with constant eigenfrequency, the equivalent reduced sweep velocity $\hat{K}$ is:

$$\hat{K} = \frac{\dot{\Omega} - \dot{\omega}}{\zeta^2 \omega_0^2}$$

That, recalling eq. (2.14), expressed now in terms of angular frequency:

$$\hat{K} = K \left(1 - \frac{\dot{\omega}}{\dot{\Omega}} \right) = K(1 - \sigma) \quad (2.28)$$

This, as expected from previous considerations, translates into a smaller RSV for the equivalent LTI system in presence of centrifugal stiffening, and vice versa a larger one in case of spin softening.

The equivalent LTI system parameters are:

- $m_{LTI} = m_{LTV}$
- $\omega_{0,LTI} = \omega_{r,LTV}$ where ω_r refers to the resonance frequency, not the eigenfrequency
- $\zeta_{LTI} = \zeta_{LTV}$
- $\hat{K} = K(1 - \sigma)$

The equivalence is clarified by looking at the waterfall diagram of the two systems:

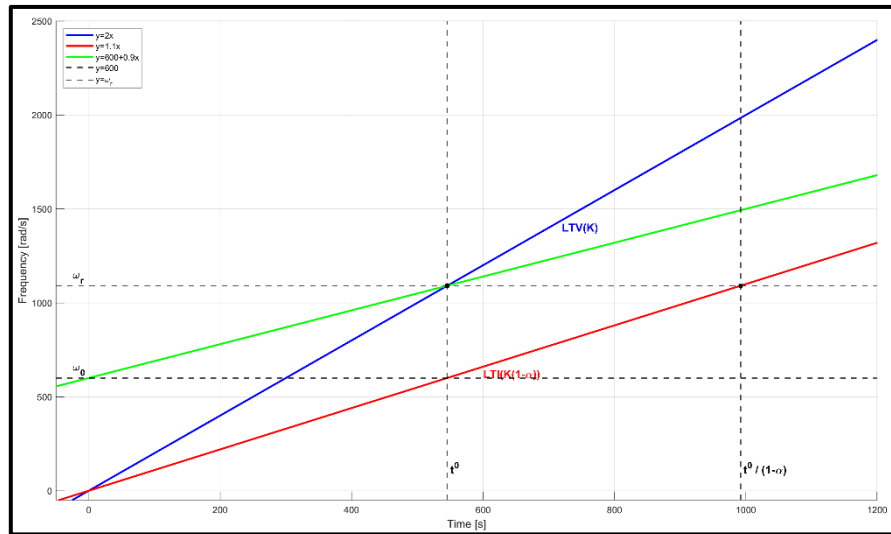


Figure 35: LTV-LTI equivalence principle

2.2 SDOF System Transient Response Sensitivity Analysis

As can be noticed in Figure 35, the time of the intersection is naturally larger in the LTI system, even though the resonance frequency ω_r is the same, this is due to the modified RSV.

The proposed equivalence has been exploited to evaluate the effect on the time-history of the transient response of the SDOF system introduced in section 2.2.1:

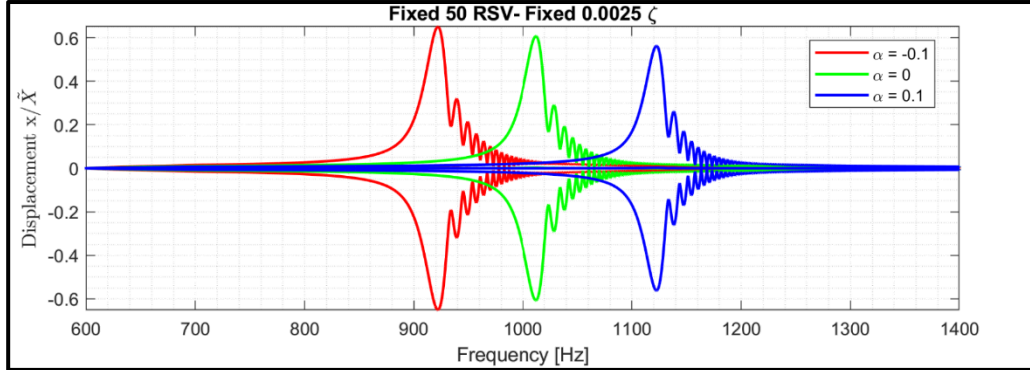


Figure 36: Time-history of transient resonance response passage for Softening, Constant and Stiffening eigenfrequency

Parameters	
Nominal RSV	50
Modal damping ratio	0.0025
σ	-0.1, 0, 0.1

Figure 36 depicts the outcome of the introduction of this new variable in the un-steady response; it can be observed that, even though the nominal RSV and the damping ratio are constant for the three cases, the responses show significant differences, especially in the frequency shift and the amplitude.

Two aspects combine in these results, the variation of stiffness and the consequently modified sweep. For the softening example, the amplitude of the displacement is higher due to a reduction in the structure stiffness but is mitigated by the higher resulting K ; the opposite verifies for the stiffening one which exhibits a smaller displacement amplitude at a higher frequency, as expected.

Although these results significantly influence the dynamic behavior of the bladed disk, they should be interpreted with caution. They are the outcome of several simplified assumptions that deviate from the conditions found in actual aeroengines, where effects related to system parameter variations are typically negligible or, in any case, reduced in scale. Nevertheless, these considerations remain valid and further clarify the implications of a transient resonance crossing.

2.3 Influence of Friction Damping on Transient Response

Friction damping, as already mentioned in section 1.4, is the most established damping technology in bladed disks, nevertheless, its effects on blades dynamics during a transient resonance passage have been rarely investigated in literature, let alone an optimized design of dampers tailored for un-steady operating conditions.

2.3.1 Jenkins Element definition

In this work, a simplified approach to the problem has been used, with the addition, in the SDOF previously discussed, of an element able to replicate the stick-slip effect typical of contact mechanics thanks to the introduction of a contact force f_c in the EoM of the now 2-DOF system.

The element in question is the Jenkins element, also known as Masing model, defined as a Coulomb slider with a threshold friction force f_J , in series with a spring of stiffness k_J .

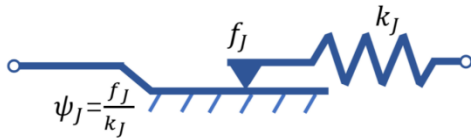


Figure 37: Jenkins element [16]

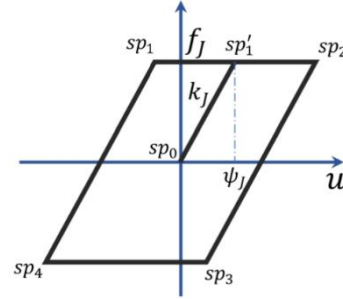


Figure 38: Hysteresis loop [16]

It models the discontinuous stick-slip phenomenon through a piecewise linear function between the friction force and the displacement u of the structure's degree of freedom to which the slider is attached. The result is a hysteric loop characterized by four transition points (sp_0 to sp_4) between stick and slip phases.

- **Slip** ($sp_{1/3} \rightarrow sp_{2/4}$): the slip phase features a constant contact force f_c equal to the threshold force of the slider; during this stage the damping effect is maximum, the slider is in fact dissipating a large quantity of the vibration energy into heat.
- **Stick** ($sp_{2/4} \rightarrow sp_{3/1}$): during the stick phase instead, the force f_c exerted by the element is lower than f_J , and is linked to u by the spring stiffness k_J ; at this point of the loop the slider is stationary acting a simple added stiffness to the overall system.

Once the value of u exceeds or falls below a critical value, the yield displacement ψ_J , the transition verifies activating or disabling the motion of the slider, and thus its damping effect.

$$\psi_J = \frac{f_J}{k_J} \quad (2.29)$$

This boundary, as seen from eq. (2.29), depends on f_J , which subsequently depends on other operational parameters:

$$f_J = \mu_s \cdot N \quad (2.30)$$

So:

$$\text{Stick} \Rightarrow f_c < \mu_s \cdot N$$

$$\text{Slip} \Rightarrow f_c = \mu_s \cdot N$$

With:

- μ_s the static dry friction coefficient
- N the slider's preload

The product between the two is also commonly known as the *Coulomb limit* (Cl). For the case of bladed disks employed in aeronautical engines, the preload is usually adjusted working on the coupling interference between the interacting bodies, such as two shrouds of adjacent blades or the blade root and the UPD.

So, combining eq. (2.29) and (2.30):

$$\psi_J = \frac{\mu_s \cdot N}{k_J} = \frac{Cl}{k_J} \quad (2.31)$$

To simplify the force-displacement description of the hysteresis loop of Figure 38, the relative motion y_J between the slider and the structure to which it is attached is used, as well as the yield displacement in place of the friction force.

It has to be noticed that y_J corresponds to the displacement of the inner spring within the Jenkins element.

The resulting piecewise linear function that describes the hysteretic loop, and thus the transition between stick and slip phases is:

$$\begin{cases} f_c(u, \dot{u}) = y_J k_J \\ \dot{y}_J = \dot{u} [H(y_J + \psi_J) - H(y_J - \psi_J) + H(y_J - \psi_J)H(-\dot{u}) + H(-y_J - \psi_J)H(\dot{u})] \end{cases} \quad (2.32)$$

Where H is the *Heaviside* step function which has value zero for negative arguments and value one for positive arguments.

From eq. (2.32) it follows that:

- For the stick phase, when the contact force f_c is under the threshold value f_j , and thus the slider is static, it holds that $|k_j y_j| < f_j$, and thus

$$y_j - \psi_j < 0, -y_j - \psi_j < 0, y_j + \psi_j > 0$$

therefore, allowing to simplify the second relation in eq. (2.32) to $\dot{y}_j = \dot{u}$.

- Instead, for the slip phase, the friction force has reached the Coulomb limit, and the slider is in motion, hence

$$y_j = \psi_j + \varepsilon, \dot{u} > 0 \text{ or } y_j = -(\psi_j + \varepsilon), \dot{u} < 0$$

it follows that $\dot{y}_j = 0$.

Note that $\varepsilon \rightarrow 0^+$ has been introduced because $|y_j| = \psi_j$ hardly occurs in time integration, where a numerical error must be expected due to the finite precision of the machine.

In conclusion, it has been possible to define a simple and complete model of dry friction with a series of linear functions that enables, not only the usage of an alternate stick-slip strategy, but also a state-space formulation of the problem, avoiding a direct numerical integration of a non-differentiable system, that would be computationally prohibitive.

2.3.2 Alternate State Space Algorithm for the solution of a dry friction system

As mentioned before, the Jenkins element is introduced connecting it to the single degree of freedom system's structure, this is achieved through a connection matrix $\mathbf{T}$ that links the slider internal displacement y_j with the SDOF's displacement, and thus the respective derivatives:

$$\dot{Y} = A T \dot{X} \quad (2.33)$$

Where $\mathbf{A}$ in eq. (2.33) is a diagonal indicator matrix that indicates whether the slider is in a stick or slip phase of the hysteretic loop. The diagonal elements a_{ii} of the indicator matrix can only assume values of 0 or 1.

Note that in this specific case, with a single slider and an SDOF system, the matrices $\mathbf{T}$ and $\mathbf{A}$ are scalar values, but this notation is valid for a more general MDOF system.

The state-space formulation, for a general MDOF is, as a result:

$$\dot{Z}(t) = HZ(t) + F_z(t), \quad Z = [X, \dot{X}, Y] \quad (2.34)$$

With:

$$H = \begin{bmatrix} 0 & I & 0 \\ -M^{-1}K & -M^{-1}C & -M^{-1}T^TK_J \\ 0 & AT & 0 \end{bmatrix} \quad (2.35)$$

And:

$$F_z(t) = \begin{bmatrix} 0 \\ M^{-1}F(t) \\ 0 \end{bmatrix} \quad (2.36)$$

As can be seen from eq. (2.35), the dynamic matrix $\mathbf{H}$ changes with the value assumed by $\mathbf{A}$, depending on the condition of the damper, consequently, it is possible to treat the described system as a conventional linear problem characterized by two different configurations; these are solved separately across the hysteric cycle. The analytical solution can be obtained through various direct integration methods, for example Runge-Kutta, as already implemented to solve the previous LTI system.

The key challenge lies in identifying the exact transition points between the two phases, from stick to slip and vice versa, during the system's discontinuous response; this is achieved adopting an iterative procedure that exploits the phase transition conditions.

Initially it is necessary to check if in a generic time step $\Delta t = t_k - t_{k-1}$, a transition occurred, to do so y_J and $\dot{u}$ are evaluated at the boundaries of the step and transition conditions verified.

- Stick to Slip:

According to the hysteresis loop in Figure 38, the points sp_1 , sp_1' and sp_3 , mark the transition from stick to slip, i.e., $|y_J(t_{k-1})| < \psi_J$, $\dot{y}_J(t_{k-1}) = \dot{u}(t_{k-1})$ shifts to slip if, at t_k , $|y_J(t_k)| \geq \psi_J$. This leads to:

$$\begin{aligned} sp_1 \text{ or } sp_1' &\Rightarrow H(y_J(t_k) - \psi_J)H(\dot{u}(t_k)) > 0 \\ sp_3 &\Rightarrow H(-y_J(t_k) - \psi_J)H(-\dot{u}(t_k)) > 0 \end{aligned}$$

- Slip to Stick:

The points sp_2 and sp_4 indicate the transition from slip to stick, characterized by the change of the sign of $\dot{u}$, thus the transition condition from $|y_J(t_{k-1})| = \psi_J$ and $\dot{y}_J(t_{k-1}) = 0$, to a stick in t_k is:

$$sp_2 \text{ or } sp_4 \Rightarrow \text{sign}(\dot{u}(t_k)) \neq \text{sign}(\dot{u}(t_{k-1}))$$

If then a transition is detected, the subsequent step involves the computation of an indicator μ , used to determine when, inside of the considered time step, the transition occurred and is defined as:

- Stick to Slip:

$$sp_1 \text{ or } sp_1' \Rightarrow \mu = \left| \frac{y_J(t_{k-1}) - \psi_J}{y_J(t_{k-1}) - y_J(t_k)} \right| \quad (2.37)$$

$$sp_3 \Rightarrow \mu = \left| \frac{y_J(t_{k-1}) + \psi_J}{y_J(t_{k-1}) - y_J(t_k)} \right| \quad (2.38)$$

Since the transition points have to satisfy $|y_J| = \psi_J$.

- Slip to Stick:

$$sp_2 \text{ or } sp_4 \Rightarrow \mu = \left| \frac{\dot{u}(t_{k-1}) - 0}{\dot{u}(t_{k-1}) - \dot{u}(t_k)} \right| \quad (2.39)$$

Since the transition points have to satisfy $\dot{u} = 0$.

Once the indicator μ has been calculated it is possible to update the boundaries of the considered time step from $[t_{k-1}, t_k]$ to $[t_{k-1}, t_e]$, where t_e is the time of the identified transition:

$$t_e = t_{k-1} + \mu \Delta t \quad (2.40)$$

Finally, once the time step is updated, the indicator matrix is transitioned to the next phase of the cycle. Numerical time integration then resumes from with the system in this new configuration, continuing until a subsequent transition is identified.

This algorithm not only allows to simplify the resolution of non-linear dry friction systems, but also to extrapolate simultaneously the hysteresis loop, in addition to the time-history of the dynamic response.

2.3.3 Sensitivity Analysis of Friction Damping on Transient Response

The response of the 2-DOFs system presented is confronted qualitatively with experimental measurements and numerical simulations from recent literature, [15] and [17], to verify the consistency of the results regarding the influence of friction damping on the transient response.

Parameters

Modal mass	[kg]	0.1
Natural frequency	[Hz]	200
Modal damping ratio		0.0045
Initial engine speed	[RPM]	0
Target engine speed	[RPM]	1600
Engine acceleration	[RPM/s]	400
Engine Order		14
Coulomb limit	[N]	125
Jenkins internal stiffness	[N/m]	9.50E4

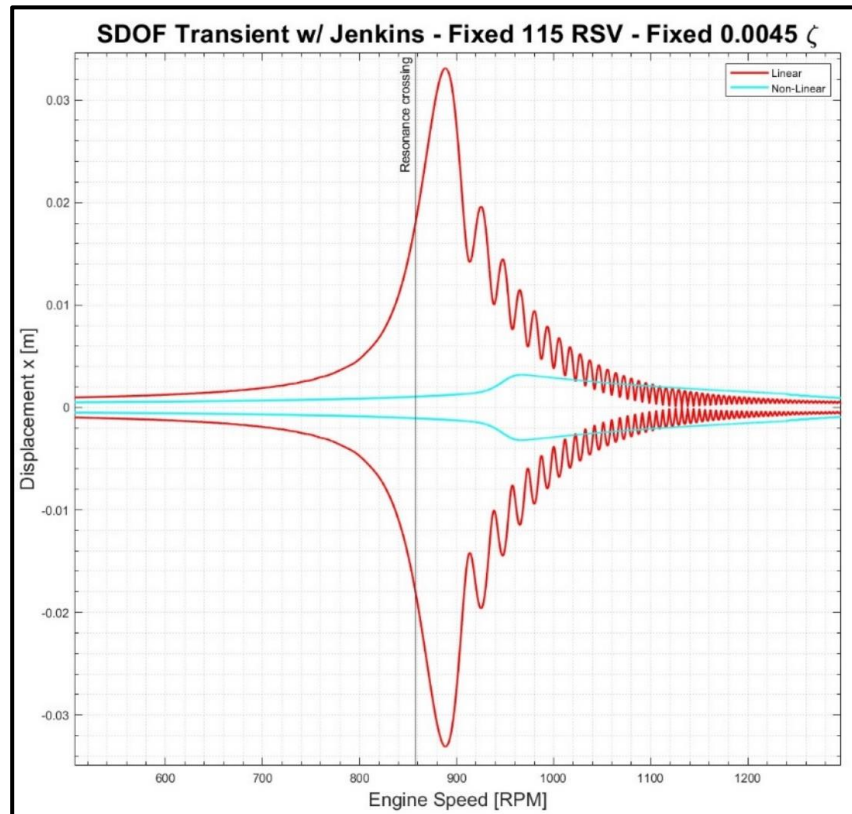


Figure 39: Effects of friction damping on transient resonance passage.
Linear (No Jenkins) vs Non-Linear (With Jenkins) systems.

Figure 39 depicts the comparison between the time-histories of the transient resonance crossing for the linear and non-linear systems. The inclusion of a single Jenkins element in the non-linear model serves to illustrate the effect of friction damping on the system response. Looking at the envelopes (red for the linear system and blue for the damped one) it can be noticed indeed that, thanks to the added damping effect of the slider, the characteristic beating phenomenon is no longer present; this is due to the faster decay of the homogeneous component of the response. Other clear differences are the reduced amplitude of the vibration and an increased frequency shift; the latter is attributable to the added stiffness provided by the Jenkins internal spring.

Equivalent findings emerge from the solution of a multi-mass oscillator with a cyclic segment structure from [15], and the finite element model of a sector composed by two simplified shrouded blades with a patch where the friction contacts interact [17].

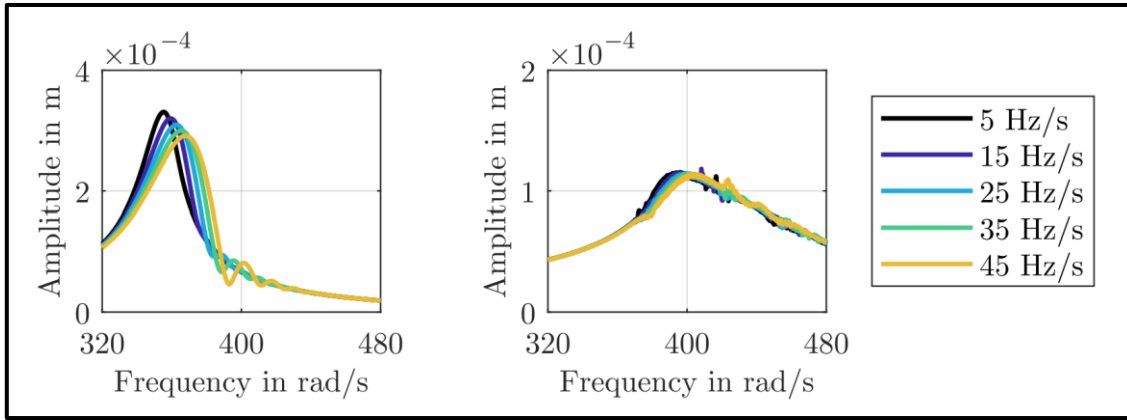


Figure 40: Run-ups result of 12-DOF cyclic multi-mass oscillator: (Left) No frictional contacts, (Right) With frictional contacts [15]

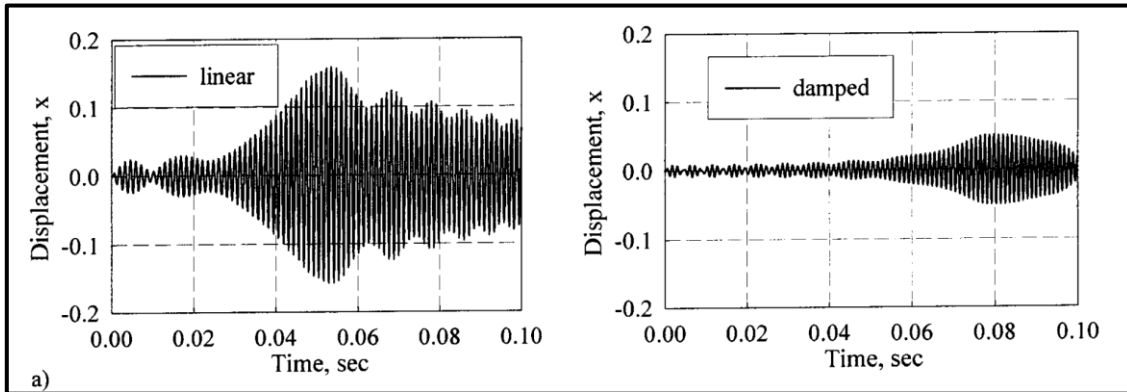


Figure 41: FEM analysis of a two shrouded blades sector with and without friction damping [17]

To further understand the relationships between the parameters associated with the friction damper and the resulting system dynamics, a sensitivity analysis is conducted on the 2-DOFs model.

Variation of Coulomb limit

First, a fixed Reduced Sweep Velocity (higher than the discussed limit of 3.6) is applied to the non-linear system in which every parameter is kept constant except for the Coulomb limit (CI), varied across a range of values, from 12.5 N to 1250 N, to simulate a change in the preload or the friction coefficient. In addition, two cases have been added: the response of the *free* system, and the slider in the *full-stick* configuration, which only contribution is an increase in the stiffness of the overall structure.

Parameters

Engine acceleration [RPM/s]	200
Reduced Sweep Velocity	58

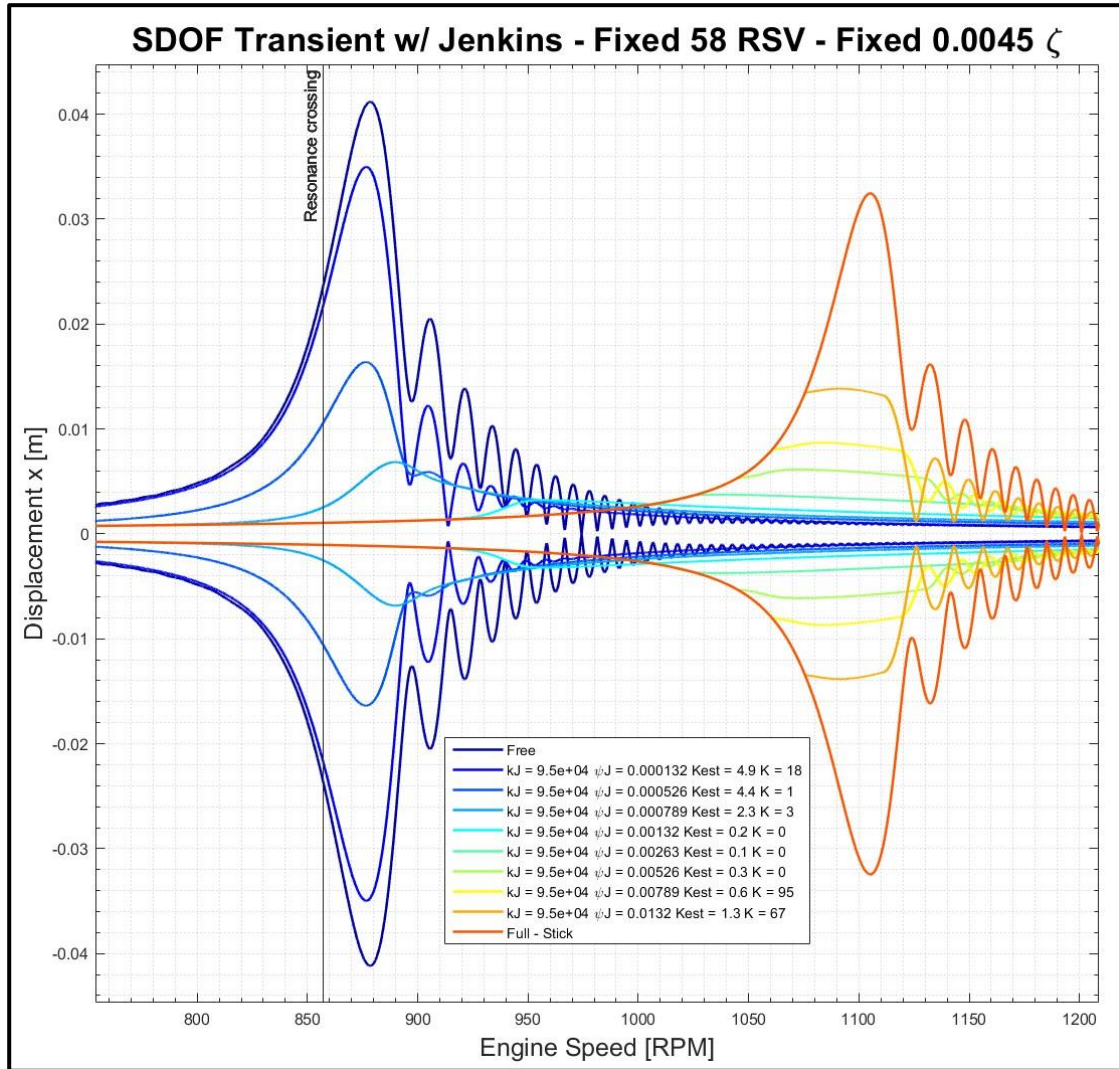


Figure 42: Effects of the variation of the Coulomb limit on the transient resonance crossing in a 2-DOFs non-linear system

The results of this first analysis, depicted in Figure 42, show a behavior similar to what verifies in a non-linear steady forced response in the frequency domain.

The reduction in the displacement amplitude, along with the shift of the resonance frequency, exhibits a non-linear trend across the Coulomb limit range.

Closer to the first extreme linear calculation, the free condition, the Jenkins element enters the slip phase due to the low threshold friction force compared to the applied load, nevertheless the added damping is scarce as well as the shift in the peak position. As the value of Cl increases, a minimum at the resonance response is reached for a particular one ($Cl = 125$ N).

Lastly, when the product of friction coefficient and preload reaches a large value with respect to the external load, the slider is close to the full-stick extreme and only briefly enters the slip condition; this results in a minimal damping effect but a large increase to the system's stiffness, followed by a significant shift in the peak frequency.

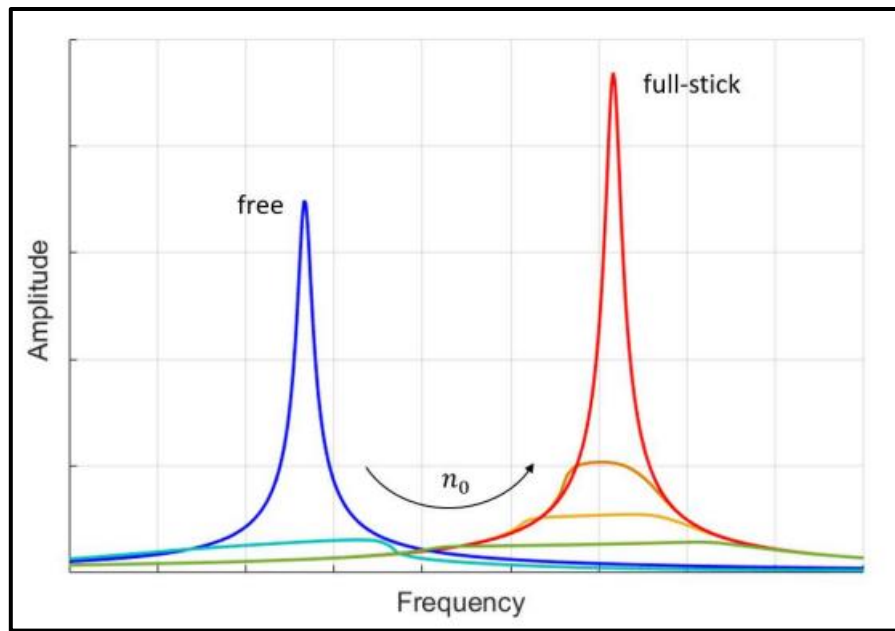


Figure 43: Typical stick-slip phenomenon in the frequency domain for a steady forced response [18]

Moreover, what is important to underline from Figure 42 is the impact of friction damping on the beats typically present in the transient response; it's possible to notice in fact that, the effectiveness of friction damping to limit, or remove completely, the beats from the response, depends on the applied Coulomb limit with respect to the external load.

If Cl is large enough to inhibit the activation of the friction damper, hence the generation of an hysteresis loop, the system will respond to the sweep as a linear system with an increased dynamic stiffness, ergo affected by beats; on the other hand, if the Cl is small compared to the applied load, the decay rate of the free response is not sufficiently influenced by the presence of the slider, and its additional damping effect, to prevent the onset of transient phenomena.

Along the entire spectrum of Coulomb limit values, this aspect of friction damping leads to a vast variety of transient responses.

2.3 Influence of Friction Damping on Transient Response

When moving from the free condition to the value of CI corresponding to the minimum of the resonance response it is possible to notice that, due to a progressive increase in the efficacy of the slider to damp the oscillation, the beating phenomena starts to degrade faster, ending at a lower number of revolutions per minute, compared to the linear system, until it disappears entirely. For lower values, the initial peaks, specifically the primary and secondary ones, allow the slider to slip and dissipate energy, thereby suppressing the growth of subsequent peaks; however, this dissipation is insufficient to fully eliminate the initial oscillations. This behavior changes at higher Coulomb limits.

After the minimum, approaching the full-stick condition, an opposite scenario occurs; the main peak is significantly reduced in height, while the following ones appear almost undisturbed by the influence of the friction damper. The reason is that, for higher values of f_f , the peaks following the first, e.g. x_2 etc. do not possess the sufficient ‘strength’ in terms of amplitude to activate the slippage, and thus, after a first series of cycles, the system continues to act as a linear one.

Variation of Sweep Rate

The second parameter considered in the sensitivity analysis is the RSV; distinct rotor accelerations are introduced thus introduced in the system, while all the other parameters are held constant.

The results presented refer to two values of RSV, respectively $RSV = 58$ and $RSV = 115$.

Parameters

Coulomb limit [N]	125
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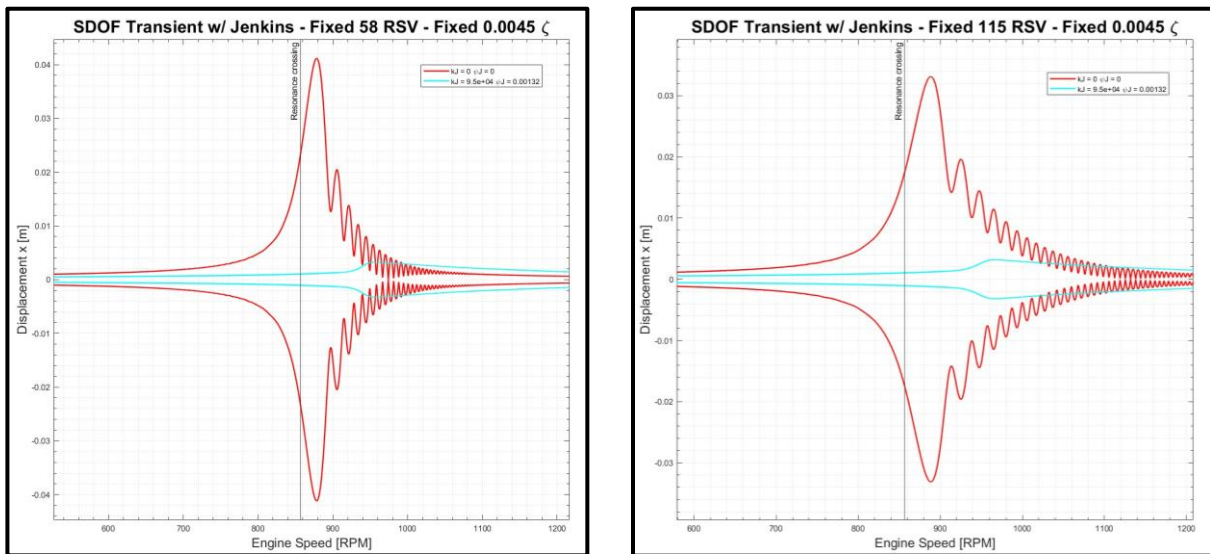


Figure 44: Linear and non-linear responses for different RSVs: (Left) $RSV = 58$, (Right) $RSV = 115$

The envelopes of the SDOF systems reflect the findings of section 2.2, with an increased frequency shift and a lower amplitude of displacement for a higher value of RSV; both these aspects are reduced, if not inhibited by the friction damper as shown by Figure 44 and the numerical results.

**Table 3: Main peak localization for linear and non-linear systems
subject to different RSVs**

Reduced Sweep Velocity		58
Amplitude (Linear)	[m]	0.0412
Amplitude (Non-Linear)	[m]	0.0032
Amplitude Ratio		0.078
Engine Speed (Linear)	[RPM]	877.9
Engine Speed (Non-Linear)	[RPM]	953.5
Engine Speed Ratio		1.086
Reduced Sweep Velocity		115
Amplitude (Linear)	[m]	0.0331
Amplitude (Non-Linear)	[m]	0.0032
Amplitude Ratio		0.097
Frequency (Linear)	[RPM]	888.9
Frequency (Non-Linear)	[RPM]	966.4
Frequency Ratio		1.087

As can be seen from Table 3, even though the nominal sweep rate has almost doubled, the non-linear responses present almost an identical amplitude, while the frequency shift is kept constant.

Variation of dry friction element's internal stiffness

Lastly, the stiffness k_J of the internal spring of the Jenkins element is varied from 4.75e4 N/m to 1.90e4 N/m, and its effect on the transient resonance crossing evaluated.

Being k_J part of the yield displacement ψ_J definition (eq. 2.29), an influence on the slider activation, for a given Coulomb limit, is expected.

Parameters

Reduced Sweep Rate	58
--------------------	----

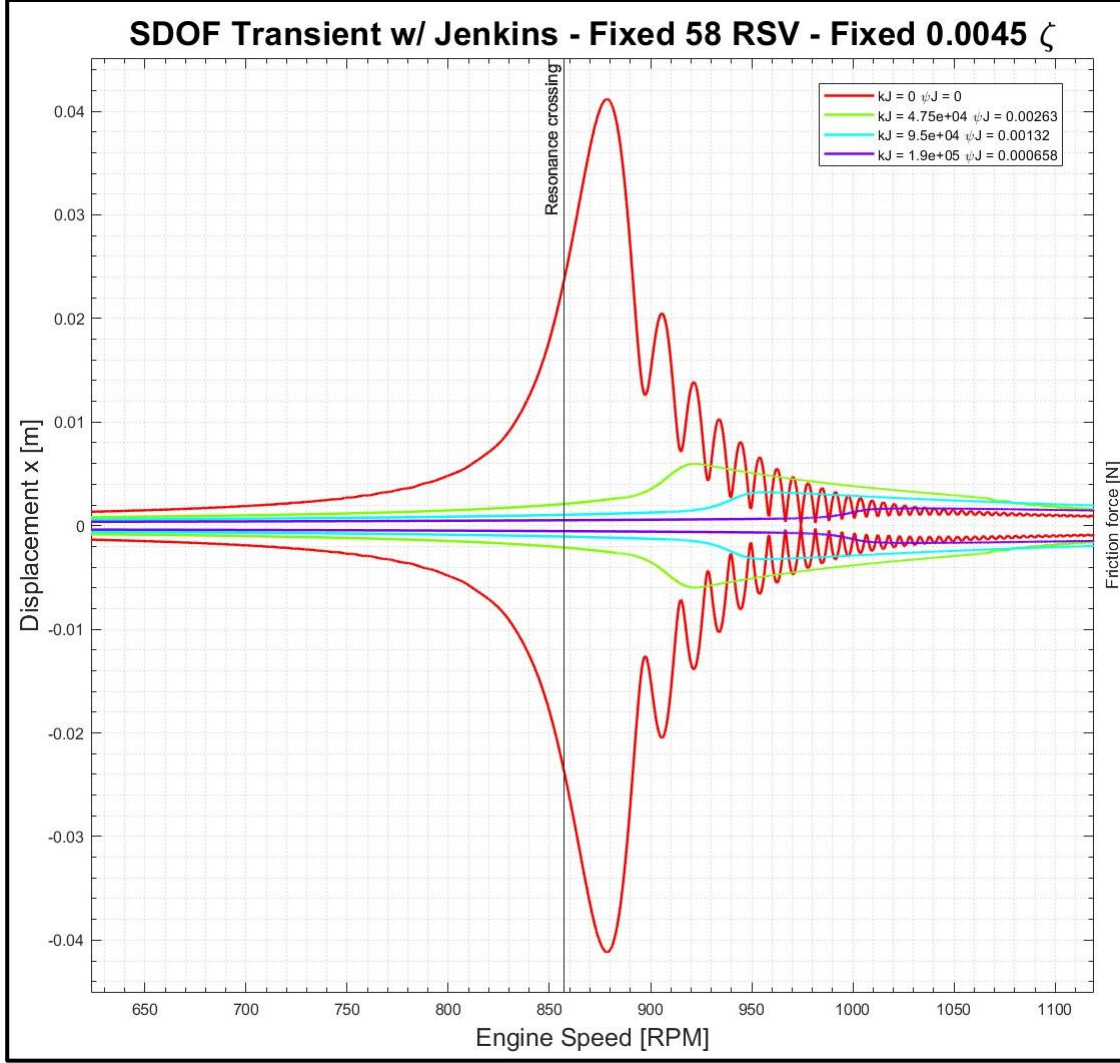


Figure 45: Transient resonance passage for different values of internal stiffness of the Jenkins element

As expected, the variation of the internal spring's stiffness influenced the activation and damping effect of the slider; from Figure 45 it can be observed that, with an increase in k_J , the friction damping contribution is more significant, resulting in a better performance of the damper.

This is accompanied by a reduction in the displacement amplitude and an increase in the frequency shift.

Chapter 3:

Finite Element Method Analysis of Transient Resonance Response

In this chapter, the analysis of the previously discussed SDOF model is extended to investigate the transient resonance crossing in a 3D Finite Element (FE) model of a dummy integral bladed disk. The objective is to evaluate the effects of a rapid sweep through a critical crossing on a system of higher complexity which allows also to consider a more realistic loading condition, characterized by a travelling wave excitation.

3.1 Test Article

The Test Article on which the FE model is based consists of a dummy integral bladed disk with twelve blades in total, where the term *dummy* is used to indicate a simplified non-functional geometry used as proof of concept and/or preliminary step of evaluation or testing.

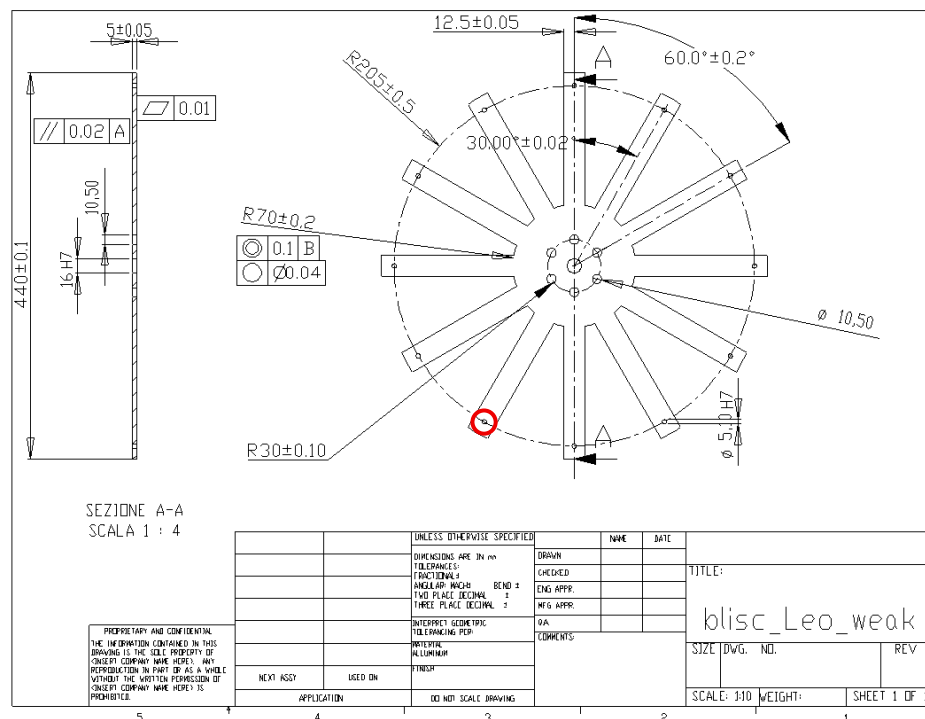


Figure 46: Technical drawing of the dummy integral bladed disk with indication of the magnets' placement [19]

The disk, as shown in Figure 46, presents a monolithic structure realized entirely in an Aluminum-Copper alloy series 2000, formerly referred to as Duralumin, except for the presence at each blade's tip of a cylindrical Neodymium (NdFeB) magnet for non-contact excitation during the rotation, which position is identified by the dashed circumference and highlighted by a red circle in the scheme.

During a test, a telemetry system is attached to the disk to transmit the measurements coming from the sensors, in this case strain gauges, to the acquisition system; for this reason, and to secure the disk to the test bench, a series of holes are present and will be used as a reference for the imposition of constraints in the FE model.

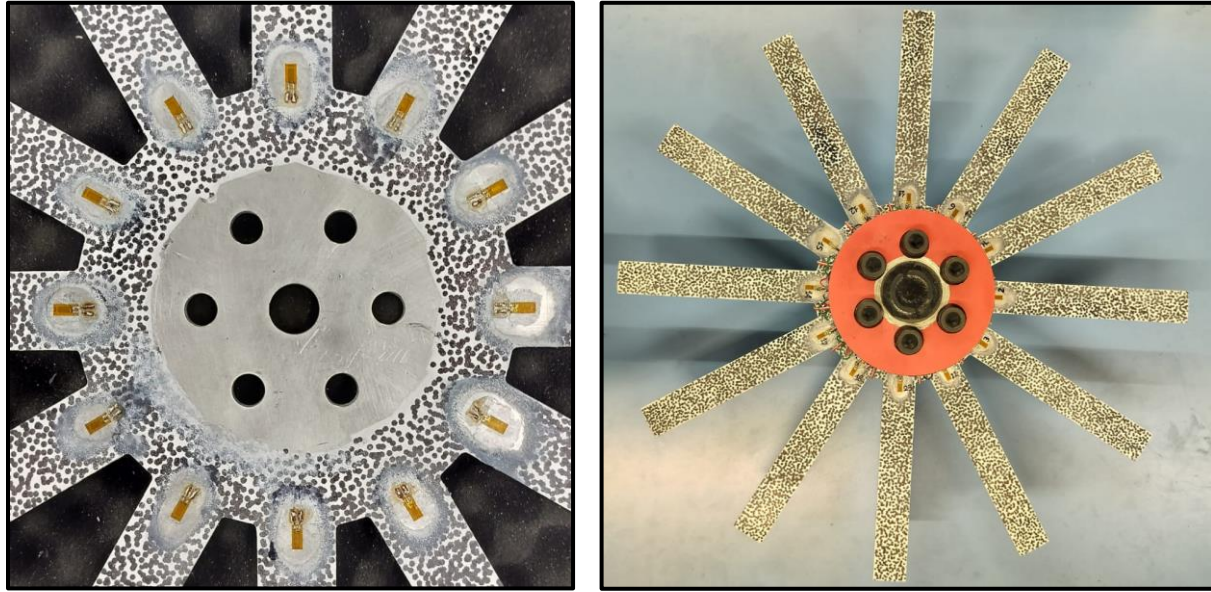


Figure 47: Location on the test article of the telemetry system and locking mechanism [19]

3.2 Finite Element Method

The *Finite Element Method* (FEM) aims to approximate the solution of a field variable distribution inside said problem domain by dividing it into smaller and simpler geometric pieces called *Finite Elements*. In case of structural analysis, the domain is usually represented by a CAD model of the geometry analyzed and the field variables are displacements, strains etc.

Each element is composed of a series of *nodes*, discrete points inside the domain in which the field variables are calculated; the results at the nodes are then interpolated in the entire region through the so-called *shape functions*.

The resulting grid is termed the *mesh*, and its construction is referred to as meshing. As a critical task in the pre-processing phase of FEM, mesh generation must reach an optimal balance between solution accuracy and computational cost, both depending not only on the mesh density in terms of elements, but also on the type of elements employed.

Each Finite Element Analysis (FEA) software in fact exhibits an archive of elements that differ in the number of nodes composing them and the number and type of degrees of freedom available per node.

3.2.1 Domain Discretization

During the meshing process, the geometry is divided into N_e elements and N_n nodes, connected consistently in the domain. It is possible to adopt elements of different types and different numbers of nodes, as long as they are *compatible*, that is, they connect without forming gaps or overlaps. Once the domain has been discretized, the pre-processor proceeds with the numbering of the elements and the nodes in a manner that guarantees the highest computational efficiency in the model construction.

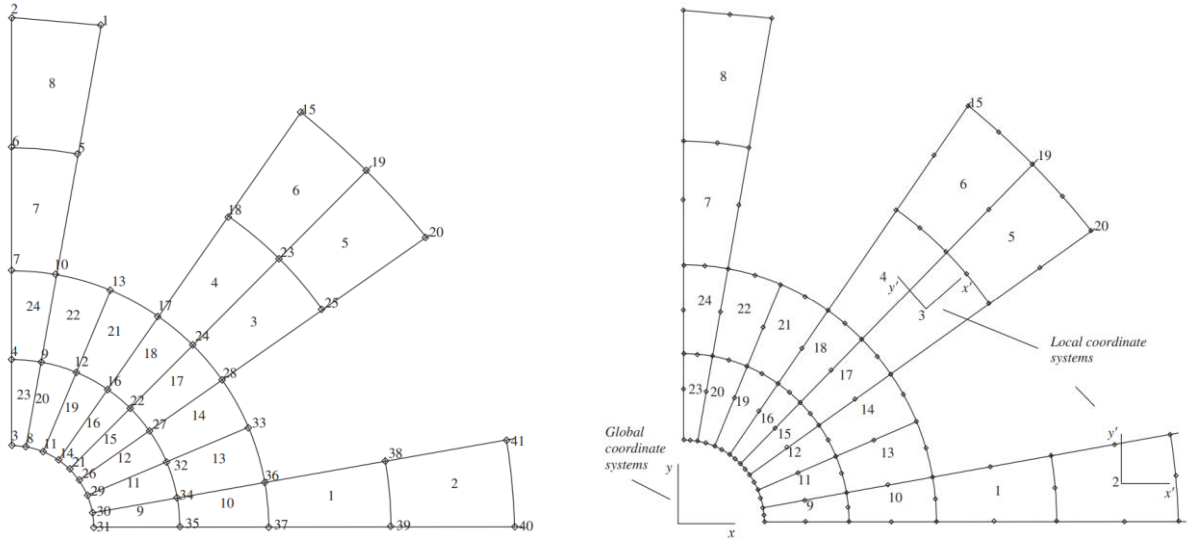


Figure 48: Example of a mesh with elements and nodes numbered (left). Local and Global coordinate systems indication (right) [20]

3.2.2 Finite Element Method equations

The displacement vector $\{U\}$ of a generic point inside a single element is defined as:

$$\{U(x, y, z, t)\} = [N(x, y, z)]\{d_e(t)\} \quad (3.1)$$

With:

- $[N]$ the Shape Function matrix
- $\{d_e\}$ the element nodal displacements vector, where the subscript e stands for *element*

The matrix $[N]$ contains functions linking the nodal displacements to the displacements inside the element, and has the general form:

$$[N(x, y, z)] = [N_1(x, y, z) \ N_2(x, y, z) \ \cdots \ N_n(x, y, z)]$$

Where N_i is a sub-matrix of shape functions for displacement components of the i -th node, arranged as:

$$N_i = \begin{bmatrix} N_{i1} & 0 & 0 & 0 \\ 0 & N_{i2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & N_{if} \end{bmatrix}$$

N_{ik} is the shape function for the k -th displacement component (DOF) at the i -th node, and N_f the number of DOFs for each node.

Once the shape function matrix has been defined, starting from eq. (3.1) it's possible to obtain the strains and the stresses in each point of the element, under the condition of linear behavior, with:

$$\{\varepsilon(x, y, z, t)\} = [B(x, y, z)]\{d_e(t)\} \quad (3.2)$$

$$\{\sigma(x, y, z, t)\} = [E(x, y, z)]\{\varepsilon(x, y, z, t)\} = [E(x, y, z)][B(x, y, z)]\{d_e(t)\} \quad (3.3)$$

With:

- $\{\varepsilon\}$ is the element's strain vector
- $[B]$ is the matrix containing the derivatives of the shape functions
- $[E]$ is the material stiffness matrix

Eq. (3.1), (3.2) and (3.3) are replaced into the *Virtual Work* equation (3.4); this is done in the FEM to convert the classic 3D elastic problem's set of differential equations into a single integral equation.

$$\int_A \{\delta u\}^T \{t\} dA = \int_V \{\delta \varepsilon\}^T \{\sigma\} dV - \int_V \{\delta u\}^T \{\phi\} dV \quad (3.4)$$

After the substitution it is possible to obtain the Element stiffness matrix $[K_e]$:

$$[k_e] = \int_V [B]^T [E] [B] dV \quad (3.5)$$

And the Element Mass matrix $[M_e]$:

$$[m_e] = \int_V \rho [N]^T [N] dV \quad (3.6)$$

The Mass matrix, as defined above, is called *consistent mass matrix*, because it is consistent with the formulation employed for the definition of the stiffness matrix.

The FE equation, in the local coordinate system, for the element is:

$$[m_e]\{\ddot{d}_e\} + [k_e]\{d_e\} = \{f_e\} \quad (3.7)$$

The last step involves the assembling of the full model; to do so eq. (3.7) needs to be translated into the global coordinate system how the structure, as seen in Figure 48, by applying some appropriate coordinate transformations.

The element nodal displacement vector $\{d_e\}$, in the local coordinate system, is linked to the global coordinate system displacement vector $\{D_e\}$ by a transformation matrix $[T]$ which form depends on the element's type:

$$\{d_e\} = [T]\{D_e\} \quad (3.8)$$

Similarly, for the nodal loads:

$$\{f_e\} = [T]\{F_e\} \quad (3.9)$$

The element's complete global equation is obtained thus by pre-multiplying and post-multiplying the element's matrices:

$$[M_e]\{\ddot{D}_e\} + [K_e]\{D_e\} = \{F_e\} \quad (3.10)$$

Where:

$$[M_e] = [T]^T [m_e] [T] \quad (3.11)$$

$$[K_e] = [T]^T [k_e] [T] \quad (3.12)$$

Finally, by adding up the contributions from all the elements connected to a common node, the full structure is assembled and the model's global equation written:

$$[M]\{\ddot{D}\} + [K]\{D\} = \{F\} \quad (3.13)$$

3.2.3 3D Solid Element selection

The analyses hereby presented have been conducted using ANSYS Mechanical APDL 2022 R2, the structural FEA software from ANSYS suite; the element chose for the discretization of the dummy blisk is the archived SOLID95, defined by 20 nodes, as shown in Figure 49, having three degrees of freedom each: translations in the nodal x, y and z directions.

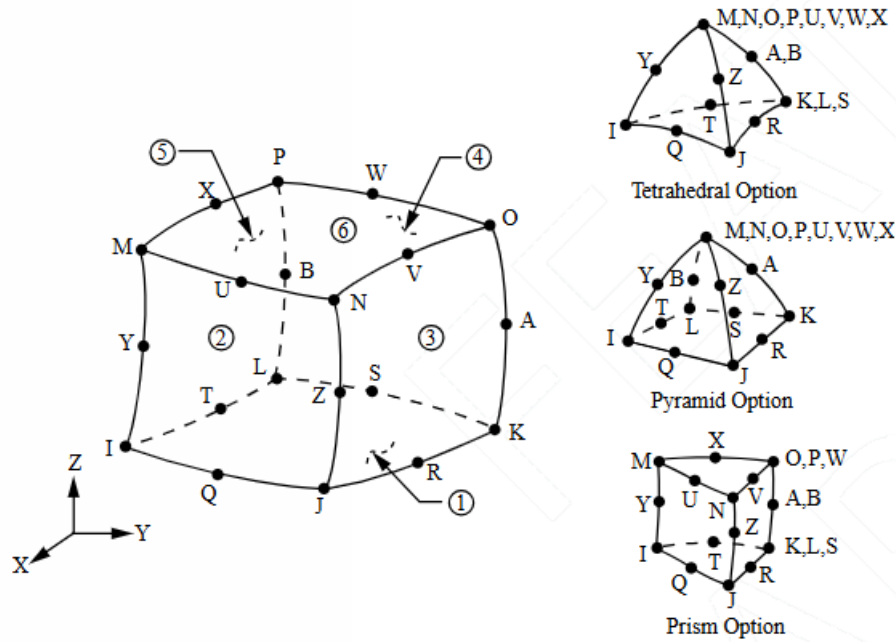


Figure 49: SOLID95 element geometry [21]

This element represents a higher-order version of the 3D 8-node solid element SOLID45 due to the addition of mid-edge nodes, which purpose is to increase the order of the shape functions' polynomials; this allows the element to tolerate irregular shapes without as much loss of accuracy.

3.3 FreND Diagram construction

Prior to analyzing the transient response of the disk, the resonance frequencies associated with each nodal diameter must be identified, to do so a modal analysis is performed. Material properties and boundary conditions will be tuned based on the results of an impact hammer test performed on the stationary test article.

3.3.1 Modal Analysis in Ansys Mechanical APDL

The geometry of the bladed disk, and the constraints applied to it, allows to exploit the cyclic symmetry to reduce the computational cost of the Modal Analysis, which will be done only a single sector (Base Sector) representative of the full structure, like showed previously in Figure 11.

The procedure differs from that outlined in Section 1.3 due to the inherent limitations of Ansys in handling complex numbers, which, thus, relies on the application of cyclic symmetry boundary conditions using real numbers and an identical Duplicate Sector of the base one.

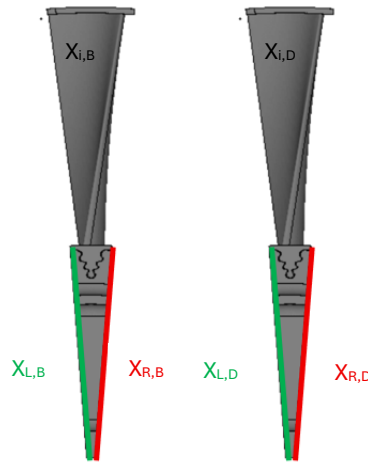


Figure 50: Example of Base sector (Left) and Duplicate sector (Right) with DOFs indication [3]

The degrees of freedom of base and duplicate sectors become the real (*Re*) and Imaginary (*Im*) parts, respectively, of the complex displacements.

Starting from the eigenvalue problem for the sector n with complex algebra:

$$(-\omega^2[m^{(n)}] + [k^{(n)}])\{\bar{x}^{(n)}\} = 0 \quad (3.14)$$

Expanding the complex vector:

$$\{\bar{x}^{(n)}\} = Re\{\bar{x}^{(n)}\} + iIm\{\bar{x}^{(n)}\} \quad (3.15)$$

Substituting eq. (3.15) into eq. (3.14):

$$(-\omega^2[m^{(n)}] + [k^{(n)}])\{Re\{\bar{x}^{(n)}\} + iIm\{\bar{x}^{(n)}\}\} \quad (3.16)$$

By defining:

$$Re\{\bar{x}^{(n)}\} = \{x_B\} \text{ and } Im\{\bar{x}^{(n)}\} = \{x_D\} \quad (3.17)$$

And decoupling the equilibrium equations:

$$\left(-\omega^2 \begin{bmatrix} m^{(n)} & 0 \\ 0 & m^{(n)} \end{bmatrix} + \begin{bmatrix} k^{(n)} & 0 \\ 0 & k^{(n)} \end{bmatrix}\right) \begin{Bmatrix} Re\{\bar{x}^{(n)}\} \\ Im\{\bar{x}^{(n)}\} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3.18)$$

It is obtained:

$$\left(-\omega^2 \begin{bmatrix} m^{(n)} & 0 \\ 0 & m^{(n)} \end{bmatrix} + \begin{bmatrix} k^{(n)} & 0 \\ 0 & k^{(n)} \end{bmatrix}\right) \begin{Bmatrix} \{x_B\} \\ \{x_D\} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3.19)$$

Applying the boundary conditions derived from cyclic symmetry:

$$\bar{x}_R^{(n+1)} = \bar{x}_R^{(n)} e^{ih\frac{2\pi}{N}} \Rightarrow \bar{x}_L^{(n)} = \bar{x}_L^{(n)} e^{ih\frac{2\pi}{N}} = \bar{x}_L^{(n)} e^{i\varphi}$$

Since:

$$\{x_B\} = \begin{Bmatrix} x_{R,B} \\ x_{i,B} \\ x_{L,B} \end{Bmatrix} \text{ base sector}$$

$$\{x_D\} = \begin{Bmatrix} x_{R,D} \\ x_{i,D} \\ x_{L,D} \end{Bmatrix} \text{ duplicate sector}$$

Using real notation:

$$Re\{\bar{x}_L^{(n)}\} + iIm\{\bar{x}_L^{(n)}\} = (Re\{\bar{x}_R^{(n)}\} + iIm\{\bar{x}_R^{(n)}\})(\cos(\varphi) + i\sin(\varphi))$$

The constraint equations found are:

$$\begin{Bmatrix} x_{L,B} \\ x_{L,D} \end{Bmatrix} = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \begin{Bmatrix} x_{R,B} \\ x_{R,D} \end{Bmatrix} \quad (3.20)$$

The complete mode shape, for every mode extracted, is then obtained through a cyclic expansion of the results of the base sector following eq. (3.21).

$$u_{i,n} = u_i^B \cos\{(n-1)k\alpha + \varphi\} - u_i^D \sin\{(n-1)k\alpha + \varphi\} \quad (3.21)$$

3.3.2 Base Sector

The Base Sector adopted is composed by a single blade and a portion of the flange, corresponding to a sector angle of 30 degrees of the full 360 model.

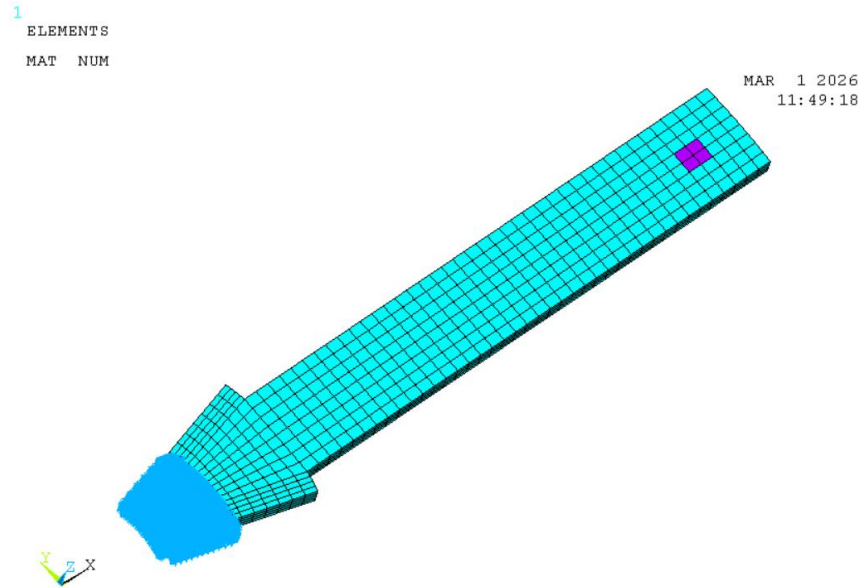


Figure 51: Base Sector of dummy bladed disk

The sector has been discretized using only the SOLID95 element, characterized by a material model based on the mechanical properties of the Aluminum alloy (blue elements) adopted for the disk, with the exception of the non-contact magnets. In order to accurately represent them, in fact, the elements at the coordinates specified by the technical drawing were characterized by a different material model based on the properties of Neodymium (purple elements).

Due to the shape of SOLID95, the magnets in FE model present a shape that slightly differs from their real counterparts, which are cylindrical and not prismatic; this is clearly visible in Figure 51.

To replicate the constraining effect of the bolting connections between the disk and the remote communication system, all the translations and rotations on the selected nodes locked on the selected node.

For the positioning, instead, there is no precise information, considering that the telemetry system has a contact surface with the flange that extends beyond the holes' location which cannot be exactly determined. For this reason, the constraints will be subjected, like the materials models, to a tuning process based only on the resonant frequencies identified with a hammer test on the static disk.

The amplitude of the response hasn't been considered for the tweaking of the parameters.

3.3.3 Tuning of the Finite Element model

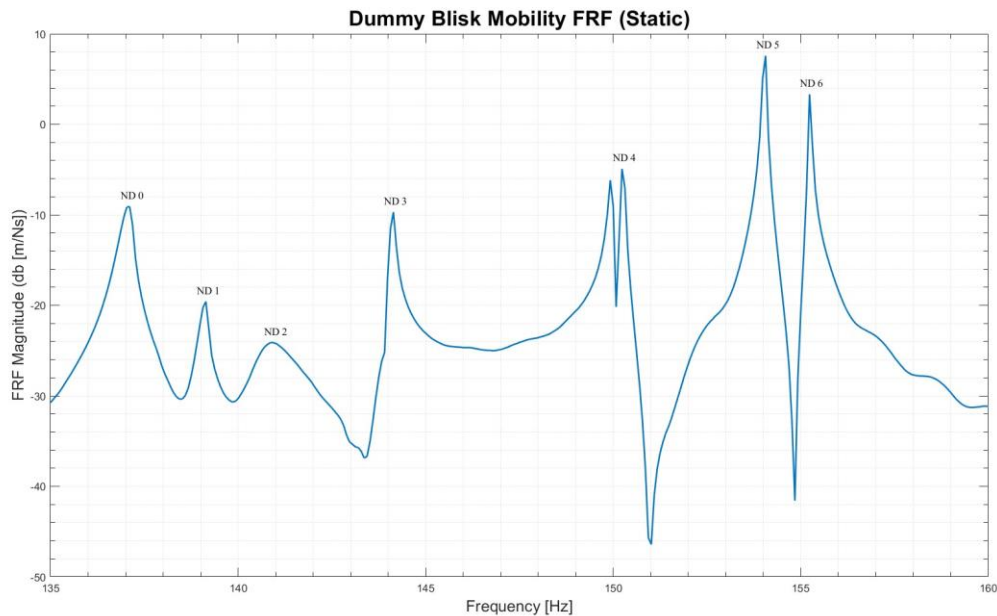


Figure 52: FRF for the first modal family of dummy blisk [19]

Figure 52 depicts the Frequency Response Function (FRF) for the first modal family of the dummy blisk that has been used as a reference for the tuning of the parameters characterizing the FE model. By examining the peaks labeled with their respective Nodal Diameter (ND) numbers, a previously unaddressed phenomenon becomes apparent, in fact, ND 4 presents a double peak at resonance.

This is due to the so-called *mistuning*, a phenomenon caused by inherent differences between sectors, due to, ad example:

- Machining tolerances
- Material inhomogeneity
- Assembling
- Wear
- Damages

The effects of mistuning on the dynamic behavior of bladed disks can be summarized into:

- Mode shapes distortion
- Response localization
- Complex modes' decoupling

The last one explains what can be seen in Figure 52, where the two stationary modes that contribute to the formation of a rotating mode, have different natural frequencies.

It's clear that a mistuned bladed disk does not possess the property of cyclic symmetry, and therefore it is not possible to exploit it to conduct the necessary dynamic analyses. It is essential thus to underline that the tuned Finite Element model hereby presented won't be able to exactly reproduce the real bladed disk, resulting in a limitation for the accuracy of the results of the analyses.

After several modal analysis, the tuned parameters for the materials models and the constraints' location are:

Table 4: FE model tuned parameters

Aluminum alloy		Values
Young's modulus	[GPa]	74
Poisson's ratio		0.33
Density	[kg/cm ³]	2780
Neodymium		
Young's modulus	[GPa]	160
Poisson's ratio		0.24
Density	[kg/cm ³]	7300
Constraints		
Radius	[m]	0.0408

The FRF of the first modal family (1st Flexural mode) of the tuned FE model of the dummy blisk, Figure 52, shows a good correlation with the experimental FRF, especially in the Nodal Diameters of major interest for this work.

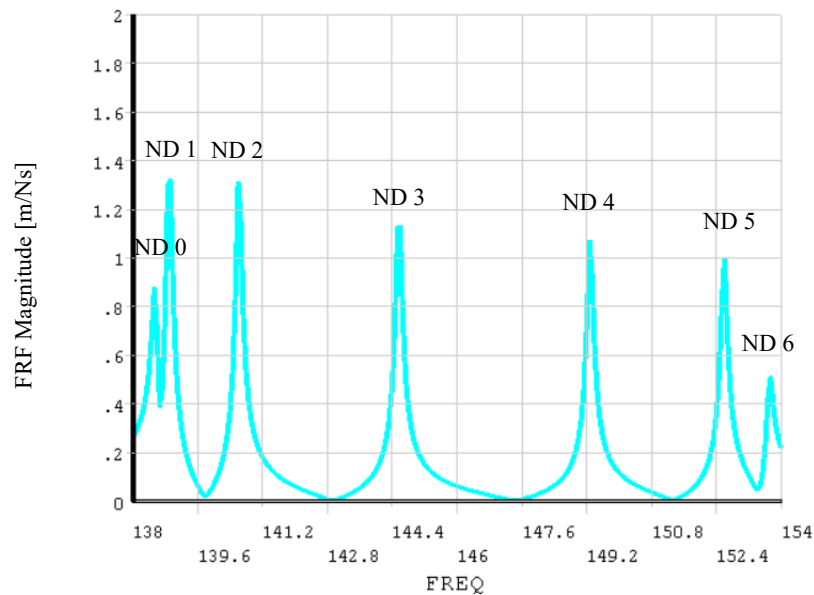


Figure 52: FRF of the first modal family from Ansys Mechanical APDL

Table 5: Resonance frequencies comparison Hammer vs FEM

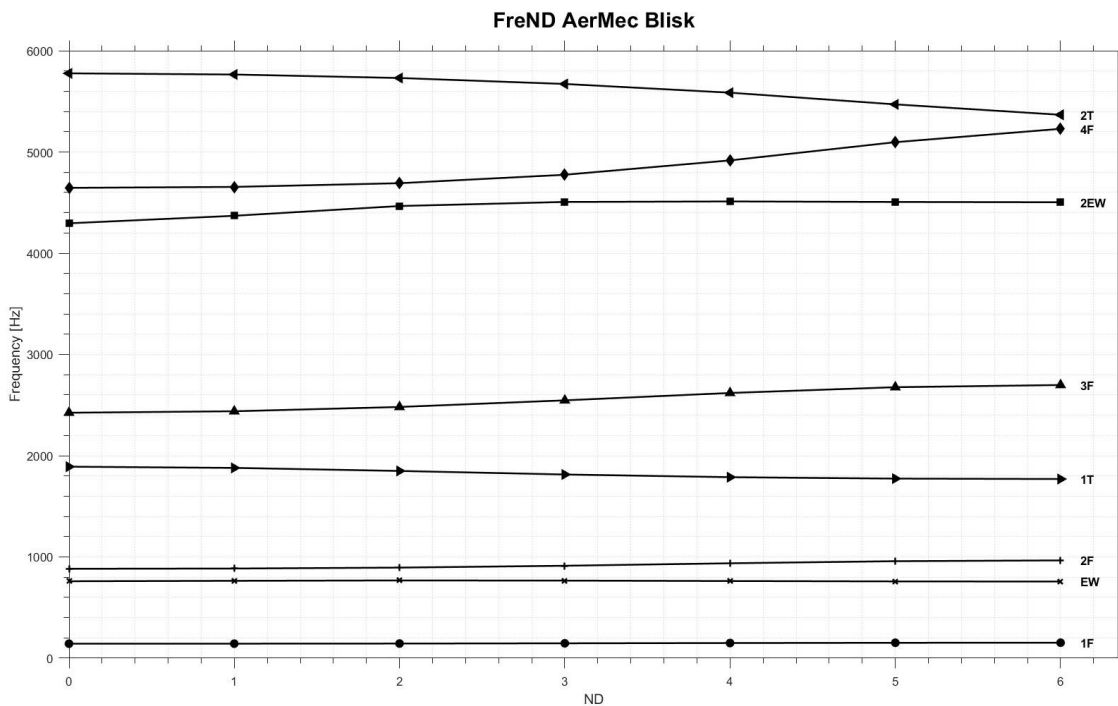
Nodal Diameter	Frequency – Hammer [Hz]	Frequency – FEM [Hz]	Delta [%]
3	144.14	144.57	-0.298
4	150.23	149.28	0.630
5	154.06	152.58	0.963

It's possible to notice also the absence of double peaks, and so mistuning, in the FE model response due to lack of differences between the sectors that compose the disk.

3.3.4 FreND Diagram

Once the tuning process is concluded, the FreND diagram can be created; to do a modal analysis is conducted to extract the first 96 modes, corresponding to 8 modal families going from ND 0 to ND 6. The first modal family extracted will be the one used for the tuning procedure.

To exclude any possible error in the reconstruction of the families it is necessary to proceed with the single mode shapes identification, especially in case of veering or crossings between adjacent families.


Figure 53: Dummy Blisk FreND diagram

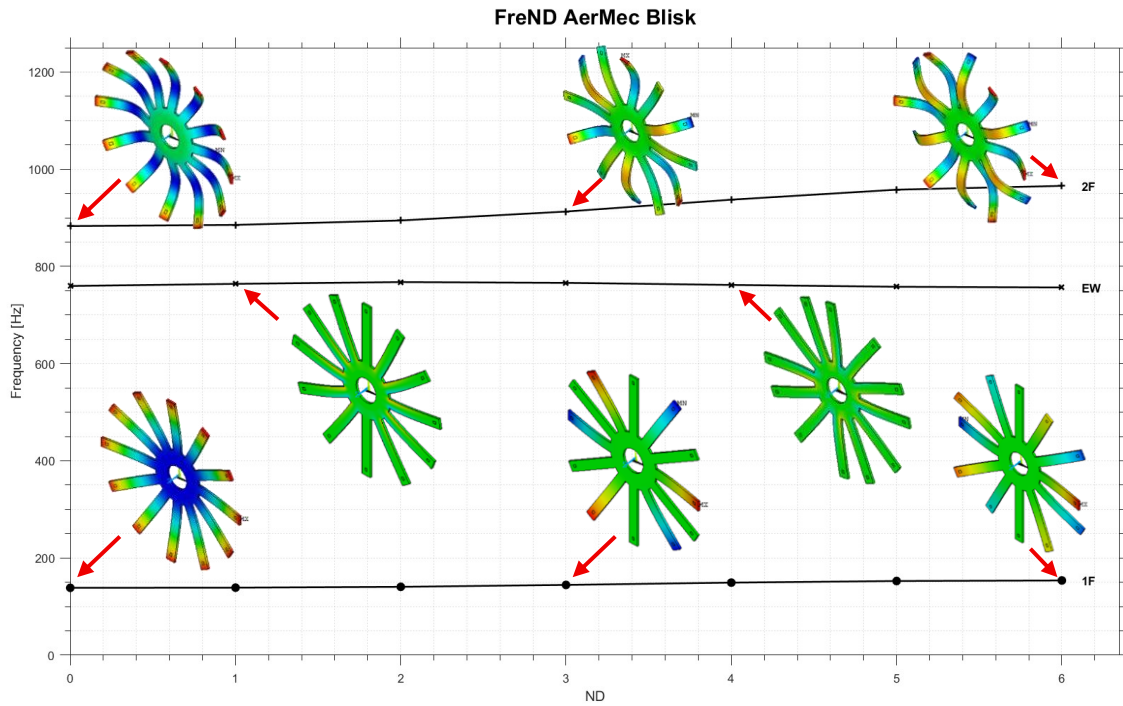


Figure 54: Mode shapes identification for 1st Flexural (1F), Edgewise (EW) and 2nd Flexural (2F) modal families

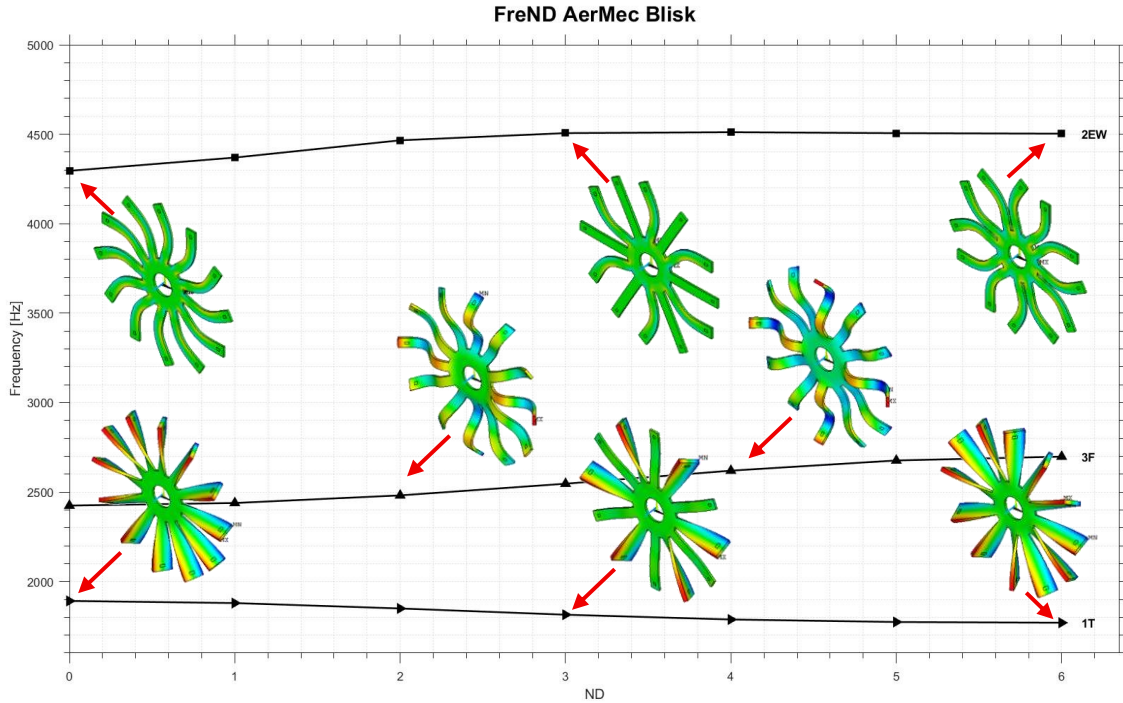


Figure 55: Mode shapes identification for 1st Torsional (1T), 3rd Flexural (3F) and 2nd Edgewise (2EW) modal families

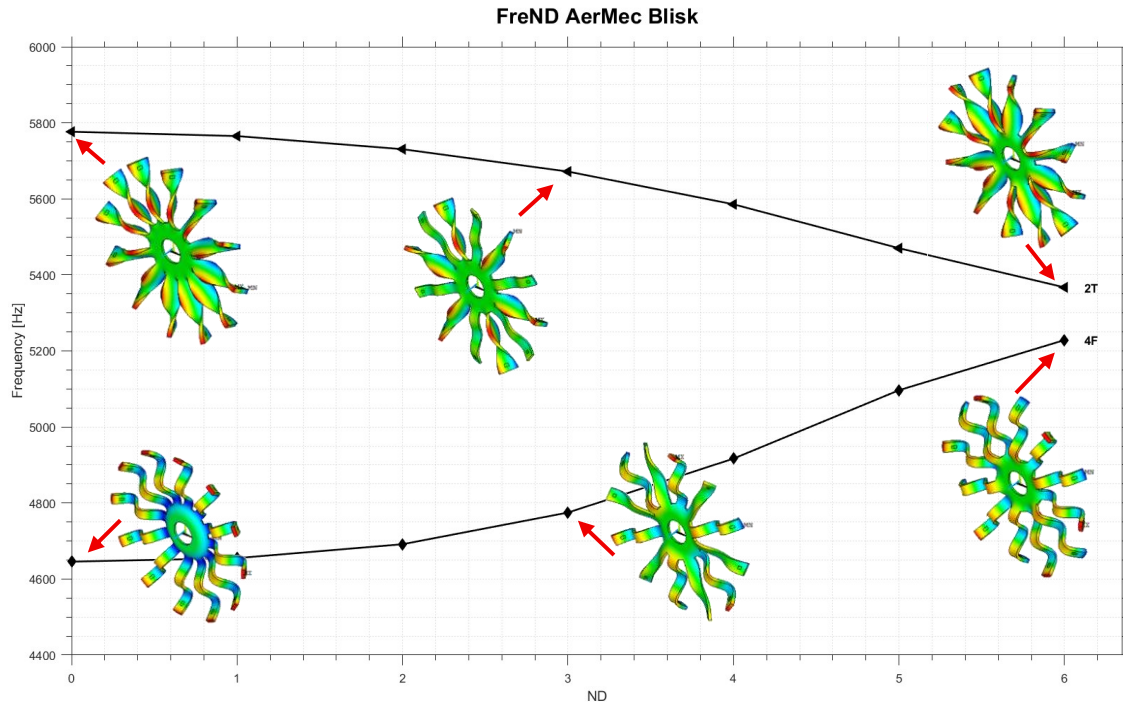


Figure 56: Mode shapes identification for 4th Flexural (4F) and 2nd Torsional (2T)

It is now possible to identify the Nodal Diameters, and thus the exciting Engine Orders, and the natural frequencies for the modes of interest for the transient analyses.

3.4 Transient Dynamic Analysis

In Ansys Mechanical APDL two methods are available to perform a transient dynamic analysis: full and mode-superposition; both will be evaluated in this study to determine which is more suitable for the task.

Regardless, the initial step remains the same for both approaches and involves the definition of the time-dependent loads on the model.

3.4.1 Travelling Wave excitation definition

Ansys Mechanical lacks the capability to perform transient analysis with cyclic symmetry; therefore, it was necessary to build a full 360-degrees model of the dummy bladed disk to apply the loading conditions. This was done with the help of Altair HyperMesh pre-processing software, by duplicating the base sector and then merging the common nodes between adjacent sectors.

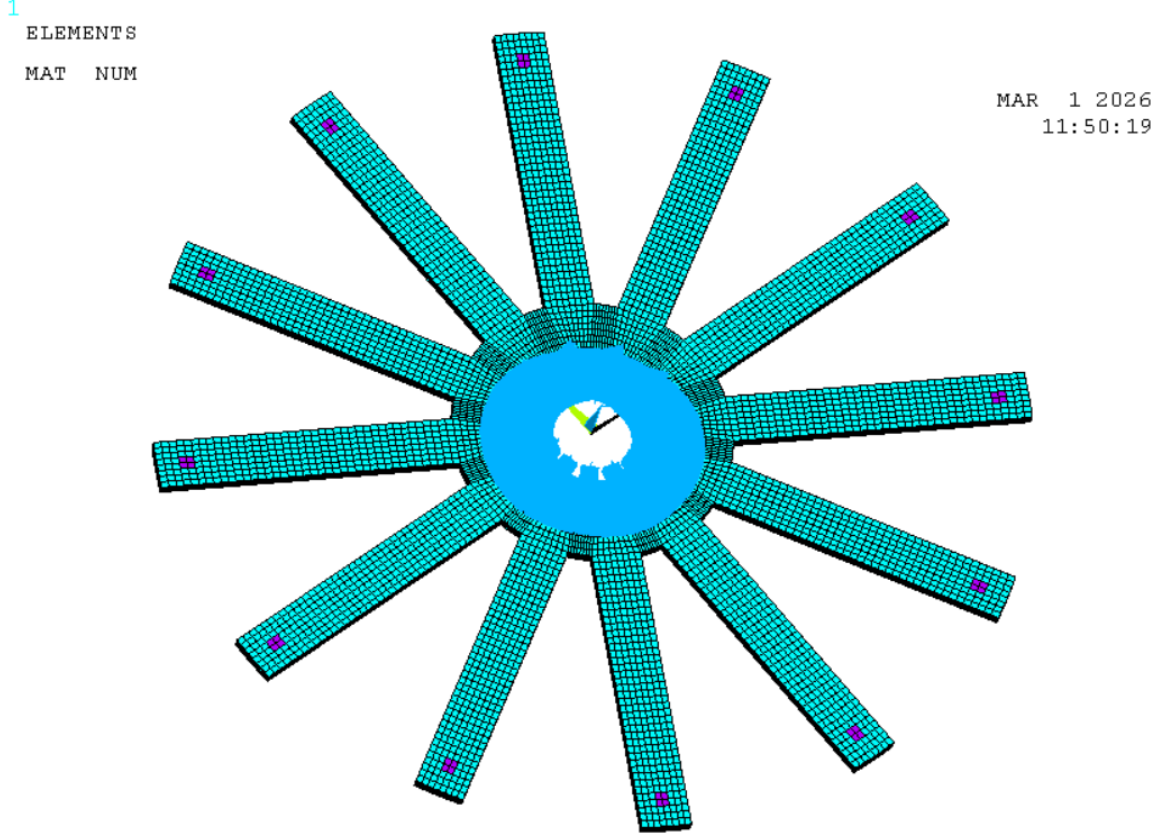


Figure 57: Full 360° dummy bladed disk model

To recreate a Travelling Wave type of excitation, in a cylindrical reference frame fixed on the disk, a series of sinusoidal functions have been applied on each blade tip; the phase angle between adjacent blades depends on the EO of the excitation, and follows what is discussed in section 1.3.4.

The sweep is instead simulated by introducing a constant acceleration, as explained in section 2.2.1.

The resulting harmonic loading for the i -th blade is:

$$F_{zi} = \widehat{F}_{zi} \cdot \sin \left(2\pi \cdot \left(\frac{1}{2} \dot{f} t^2 + f_0 t \right) + \psi_i \right) \quad (3.22)$$

Where:

- $\widehat{F}_{zi}$ is the amplitude of the forcing function in N
- $\dot{f}$ is the sweep rate in Hz/s, which derives from eq. (2.16)
- f_0 is the starting frequency, from eq. (2.15)
- ψ_i is the phase angle of the i -th blade

More precisely:

$$\psi_i = EO \frac{2\pi}{N_{tot}}(i - 1) \quad (3.23)$$

The time-history that follows is:

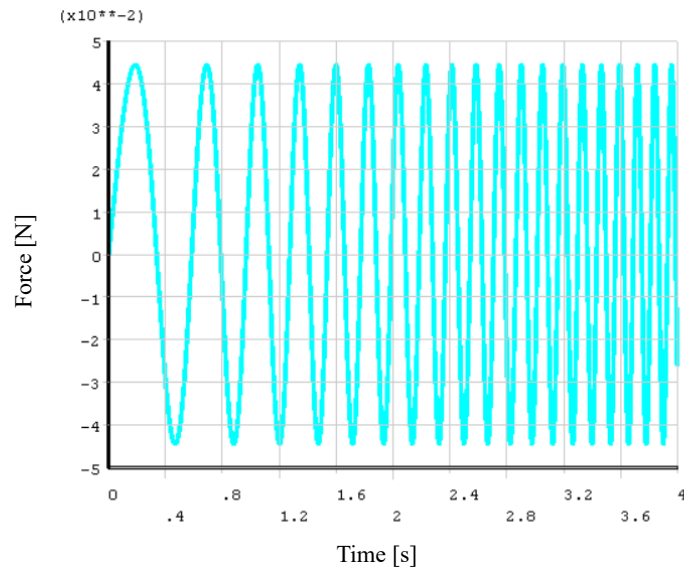


Figure 58: Applied sweep time history

These time-dependent loadings are applied as a nodal force via a tabular input by defining a one-dimensional table where the rows are identified by the time, and the single column present represents the force value in each time-step. The nodes chosen for the loading are those at the center of the magnets.

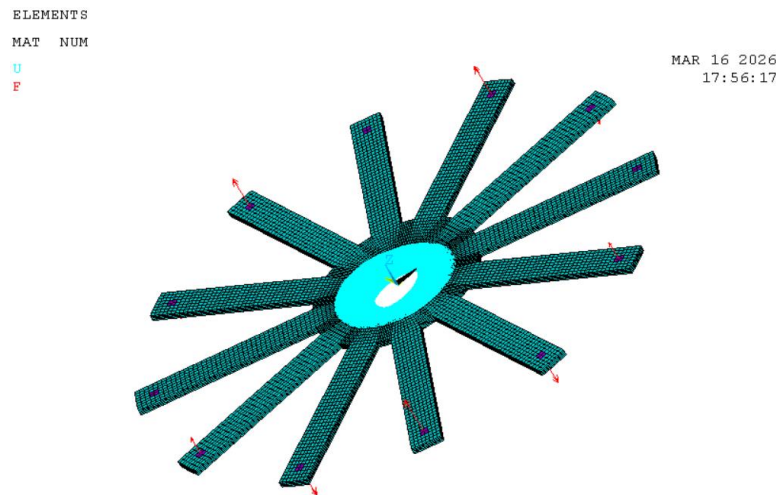


Figure 59: Nodal forces applied on each blade's tip forming a TW with EO 7

3.4.2 Time Integration Scheme for Linear Systems

The finite element equation of motion (3.13), given by the spatial discretization for the principle of the virtual work, can be expanded as follows:

$$[M]\{\ddot{D}(t)\} + [C]\{\dot{D}(t)\} + \{F^i(t)\} = \{F^a(t)\} \quad (3.24)$$

Where the new terms are:

- $[C]$ structural damping matrix
- $\{F^i(t)\}$ internal load vector
- $\{F^a(t)\}$ applied load vector

The structural dynamics problems described by this equation can be classified into two classes: linear and non-linear problems.

To solve eq. (3.24) three methods are available in Ansys Mechanical:

- *Newmark time integration method* - default method employed
- *HHT time integration method* – an extension of the Newmark method
- *Backward Euler time integration method*

At this stage of the work, contrary to previous discussions, typical non-linearities encountered in bladed disk dynamics, such as contact and friction damping, have been excluded. Consequently, the analyses belong to the class of linear problems, for which the stiffness matrix remains constant, and the internal load is linearly proportional to the nodal displacement, consequently eq. (3.24) can be rewritten as:

$$[M]\{\ddot{D}(t)\} + [C]\{\dot{D}(t)\} + [K]\{D(t)\} = \{F^a(t)\} \quad (3.25)$$

To solve it the Newmark time integration method was selected.

Newmark Method

The Newmark implicit time integration method is one of the most popular methods as a single step algorithm; the basis concept is to determine the temporal progression of the nodal displacements and then to solve the eq. (3.25) for the respective time step.

The time domain is in fact discretized into finite time intervals with a fixed, or variable, time increment Δt , thus only discrete time points t_n are considered, with $t_{n+1} = t_n + \Delta t$.

For the time step t_{n+1} , eq. (3.25) can be rewritten as:

$$[M]\{\ddot{D}_{n+1}\} + [C]\{\dot{D}_{n+1}\} + [K]\{D_{n+1}\} = \{F_{n+1}^a\} \quad (3.26)$$

The algorithm requires the displacement and velocity to be updated as follows, using the *Extended Mean Value* theorem:

$$\{\dot{D}_{n+1}\} = \{\dot{D}_n\} + [(1 - \delta)\{\ddot{D}_n\} + \delta\{\ddot{D}_{n+1}\}]\Delta t \quad (3.27)$$

$$\{D_{n+1}\} = \{D_n\} + \{\dot{D}_n\}\Delta t + \left[\left(\frac{1}{2} - \alpha\right)\{\ddot{D}_n\} + \alpha\{\ddot{D}_{n+1}\}\right]\Delta t^2 \quad (3.28)$$

Where α and δ are Newmark's integration parameters.

By making use of eq. (3.26) to eq. (3.28) and three known quantities $\{D_n\}$, $\{\dot{D}_n\}$ and $\{\ddot{D}_n\}$, the unknown $\{D_{n+1}\}$ can be written as:

$$(a_0[M] + a_1[C] + [K])\{D_{n+1}\} = \{F_{n+1}^a\} + [M](a_0\{D_n\} + a_2\{\dot{D}_n\} + a_3\{\ddot{D}_n\}) + [C](a_1\{D_n\} + a_4\{\dot{D}_n\} + a_5\{\ddot{D}_n\}) \quad (3.29)$$

a_1 to a_5 are a series of constant that depends on the integration parameters, which will be explored later.

Once $\{D_{n+1}\}$ it is determined, the algorithm computes:

$$\{\dot{D}_{n+1}\} = a_1(\{D_{n+1}\} - \{D_n\}) - a_4\{\dot{D}_n\} - a_5\{\ddot{D}_n\} \quad (3.30)$$

$$\{\ddot{D}_{n+1}\} = a_0(\{D_{n+1}\} - \{D_n\}) - a_2\{\dot{D}_n\} - a_3\{\ddot{D}_n\} \quad (3.31)$$

Integration Parameters

The factors that characterize a time integration method are accuracy, stability and dissipation. In conditionally stable time integration algorithms, stability is affected by a chosen size of the time step, whereas in unconditionally stable time integration algorithms, a time step size can be chosen independent of stability considerations.

In this case of Newmark's integration method, the introduced numerical dissipation is controlled by the *amplitude decay factor* γ :

- If there is no added damping, the values of the parameters which guarantee that the integration remains unconditionally stable are:

$$\delta = 0.5 \text{ and } \alpha = 0.25$$

This method, called the *constant mean acceleration method*, is possibly subject to instability to the contribution of high-frequency modes which do not decay to the missing numerical damping

- If $\gamma \geq 0$, it can be written:

$$\delta = \frac{1}{2} + \gamma$$
$$\alpha = \frac{1}{4}(1 + \gamma)^2$$

The value of γ , in Ansys Mechanical, depends on the application type for the analysis; in this case the *High-Speed Dynamic application* option was chosen, or *HISP*, to reduce the added numerical damping, but still avoid any possible instability. The amplitude decay factor is thus set to $\gamma = 0.005$.

The reason behind this choice is that the only source of damping in the FE model analyzed is the structural damping, with a $\zeta \sim 0.001$, or even lower as shown later, a higher numerical damping value could therefore interfere with the transient dynamics of the system.

3.4.3 Full Transient Dynamic Analysis

The full method for Transient Analysis uses the full system matrices to calculate the transient response; it is the more general method, and enables all types of non-linearities to be included like plasticity, large displacements, large strains etc. The main advantages are:

- No preliminary modal analysis required
- Accommodates all types of non-linearities
- No mass matrix approximation involved
- All components of the solution are calculated in a single pass, no expansion required
- Accepts all types of loads: nodal forces, imposed displacements and element loads

The main disadvantage of the full-method is that it is more computationally intensive than the mode-superposition analysis, since it solves eq. (3.29) directly without additional assumptions.

3.4.4 Mode-Superposition Dynamic Analysis

The mode-superposition (MSUP) method uses the natural frequencies and mode shapes of a linear structure, extracted through a previous modal analysis, to predict the response to transient loads.

The following restrictions and assumptions apply:

- Constant $[M]$ and $[K]$ matrices, this implies no large deflections, plasticity, creep or swelling
- Time varying imposed displacement not allowed
- Various types of system damping are available
- Constant time step size

It also requires, for the computation of element stresses, reaction forces etc. the expansion of the solution.

The main advantage of this method lies in the reduced computational time, at the cost of lower versatility and the approximations involved.

3.4.5 Comparison Full vs Mode-Superposition Transient Analysis

Both methods have been applied on a common test case in order to determine the most suitable approach for this work, in terms of accuracy, available analysis options, and computational time.

Test case parameters		
Starting Rotor Speed [RPM]		450
Final Rotor Speed [RPM]		640
Rotor Acceleration [RPM/s]		192
Engine Order		16
Damping Ratio ζ		0.001
Force Amplitude [N]		100
Solution Node ID		76250

This test case aims to excite the Nodal Diameter 4, with a theoretical reduced sweep velocity of $RSV = 2298$, ensuring the presence of significant transient effects.

Table 6: Full Vs MSUP Analysis setup options

	Full	Mode-Superposition
Time Increment [s]	5.0E-5	5.0E-5
Automatic Time Stepping	Yes	No
Damping model	Rayleigh damping	Damping ratio

Table 6 shows the similarities and the differences in the setup between the two methods.

First, the Time Increment, identical for both of them, has been chosen to guarantee accurate tracking of transient effects; to compute the minimum value of Δt that ensures a correct reconstruction of the response, the *Nyquist Theorem* can be adopted, which states that the *sampling period* T_s is given by:

$$T_s = \frac{1}{2f_{max}} \quad (3.32)$$

Where f_{max} is the highest frequency component of the signal to reconstruct; in this case, referring to the FrEND diagram, the highest frequency considered, is that of the last modal family extracted, which restitutes a sampling period of circa $T_s = 1E - 4$, that will be used as an upper limit for the analyses.

Regarding the time step, unlike the MSUP, the Full analysis allows for the use of Automatic Time Stepping (AUTOTS), which adapts the time step to the physics of the problem, in a range of user-defined values.

Lastly, concerning the structural damping, Ansys Mechanical supports different types of damping for each structural analysis type; more specifically, for the MSUP analysis both the Rayleigh damping and the definition of a Global Damping Ratio are permitted, while only the first one is available for the Full analysis.

The Rayleigh damping model for structural damping matrix is as follows:

$$[C] = \alpha_d[M] + \beta_d[K]$$

Where:

- α_d is the mass-proportional damping coefficient
- β_d is the stiffness-proportional damping coefficient

The values of α and β are not generally known directly but are calculated from the damping ratio; it is commonly assumed that the sum of the α and β terms is nearly constant over a range of frequencies for a given ζ . Therefore, known ζ and a frequency range ω_1 to ω_2 in rad/s, two simultaneous equations can be solved:

$$\alpha_d = 2\zeta \frac{\omega_1 \omega_2}{\omega_1 + \omega_2} \quad (3.33)$$

$$\beta_d = \frac{2\zeta}{\omega_1 + \omega_2} \quad (3.34)$$

Eq. (3.33) and (3.34) were applied to the frequency range containing the first modal family, which contributes most to the response.

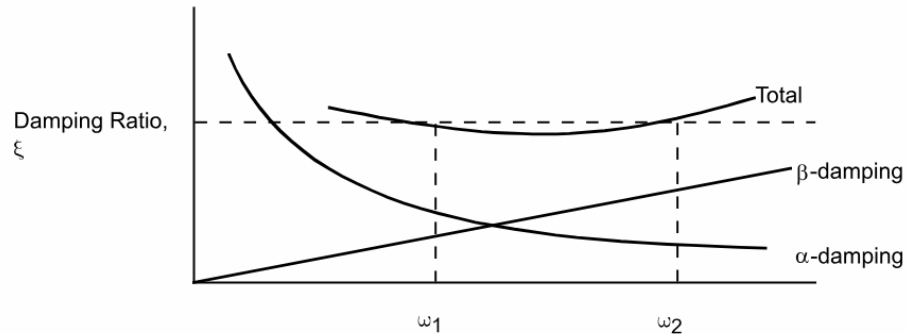


Figure 60: Rayleigh Damping coefficients over a frequency range [22]

The results of the two analyses are:

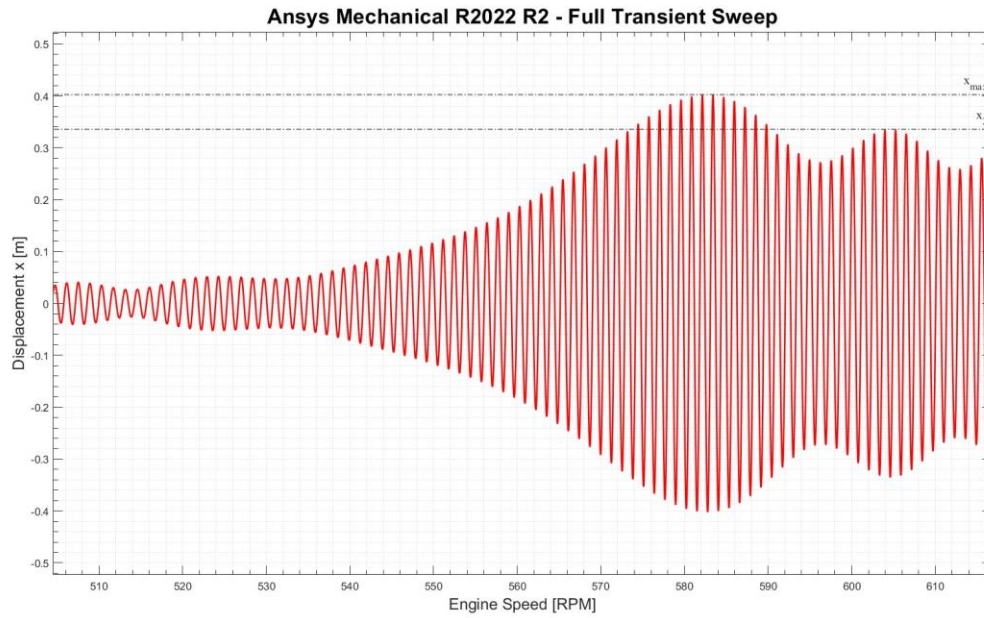


Figure 61: Full Transient Dynamic Analysis computed response for the test case

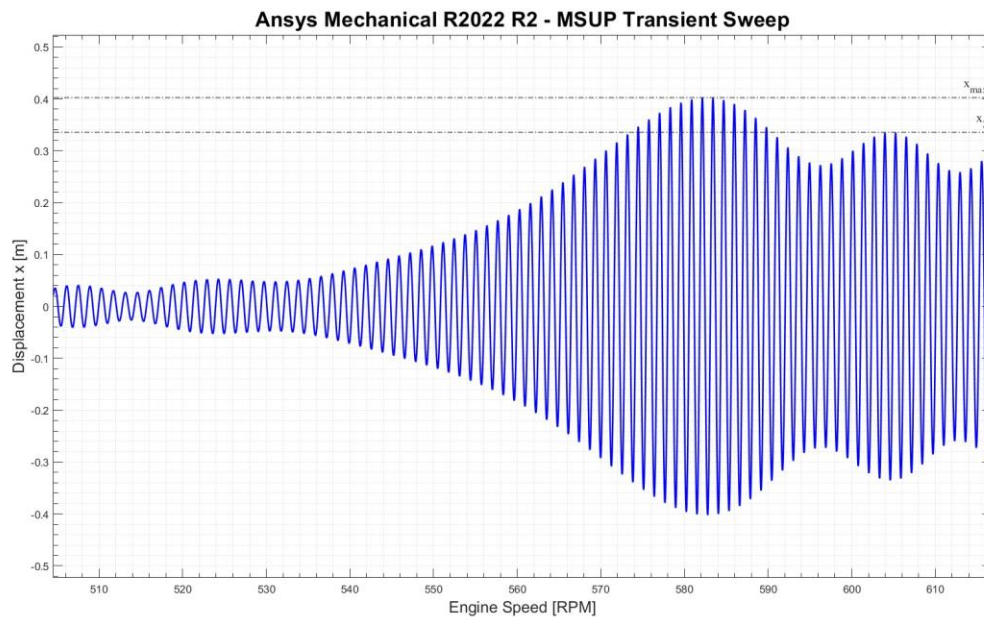


Figure 62: MSUP Transient Dynamic Analysis computed response for the test case

The transient effects described in Chapter 2 are clearly visible in the plot of the displacement at the selected node; the values defining the response, with particular reference to the two dominant peaks, are:

Table 7: Full Vs MSUP comparison results

		Full	Mode-Superposition	Difference %
Max peak amplitude	[m]	0.4027	0.4026	0.025
Max peak location	[RPM]	582.14	582.13	0.0017
Secondary peak amplitude	[m]	0.3358	0.3358	0.0
Secondary peak location	[RPM]	605.24	605.24	0.0

The transient response obtained, for the considered test case, is substantially identical between the two solution types as reported in Table 7, with the two time-histories that show a negligible relative error despite a substantial gap in runtimes: while the full analysis required several hours, the MSUP approach was completed in minutes on the same hardware.

These results, combined with the absence of nonlinearities (such as contact and friction damping) and the greater flexibility of a global damping model, lead to the selection of the MSUP solution as the standard for future analyses.

Chapter 4:

Experimental Evaluation of Transient Resonance Response

In this chapter, the last step of the investigation of the Transient Resonance Response in Bladed Disks will be presented. Following an introduction about the Blade Tip-Timing (BTT) technique and Strain Gauges, this section will discuss their application in the evaluation of the response of the previously presented test article to a series of frequency sweeps.

4.1 Fundamentals of the BTT technique

Blade Tip-Timing or BTT is a real-time measurement technique for blade health monitoring; it employs a set of non-intrusive sensors, mounted on the casing of a spinning test rig, that detect the passing time or *Time-of-Arrival* (ToA) of each rotating blade.

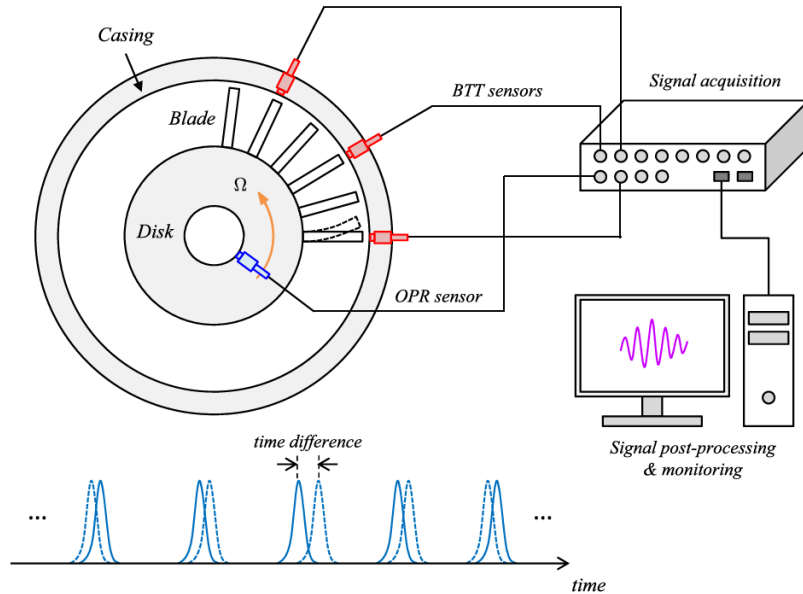


Figure 63: Schematic representation of a BTT measurement system [9]

The working principle of this method is based on the analysis of the differences between the theoretical and the actual ToA, caused by the blade's vibration. At a constant rotation speed, and in the absence of oscillations, a certain sensor would detect the blade passages with a constant time interval Δt between them; whereas, when the blade vibrates such intervals fluctuate since the actual ToA, and thus the Δt , depend on the vibration's amplitude.

To trace the blade vibration, the radial position of the measurement point on the blade, the rotation speed of the rotor and the time interval Δt need to be known.

The advantages of BTT over standard techniques are various:

- It's a *non-intrusive* measurement system; therefore, vibrations are detected without any mechanical interference by the sensors on the system; neither the air flow nor the disk dynamics are affected during the test. BTT uses optical, capacitive or eddy-current type of sensors; on the contrary traditional techniques adopt systems like strain gauges which have to adhere on the blade's airfoil, inevitably altering the blade's dynamics.
- Since every sensor is used to detect vibrations on each rotating blade, it is possible to identify whether a blade is vibrating more than the others, as in the case of typical mistuning problems. This feature does not hold, for example, when strain gauges are used since their application on bladed disks with more than a hundred blades is unfeasible for time and cost reasons; therefore, only the dynamics of a few blades can be measured, while the others are completely missed. A direct consequence is a reduced setup time for the BTT compared with traditional methods.

4.1.1 Working Principle

From the analysis of the time interval Δt between the expected and actual ToA, based on the BTT algorithm chosen, it is possible to extract the following outputs:

- Vibration's amplitude
- Resonant frequency
- Damping
- Engine Order
- Number of Nodal Diameters
- Operative Deformed Shape (ODS)

Where the time interval is given by:

$$\Delta t = t_e - t_a \quad (4.1)$$

With:

- t_e expected time of arrival
- t_a actual time of arrival

To estimate the expected TA, a preliminary calibration step is needed to determine the position of each blade with respect to the origin of the reference system used by the algorithm.

Time of Arrival in absence of vibration

The expected ToA of the i -th blade, in relation to the p -th sensor is given by:

$$t_p^{k,i} = \frac{\varepsilon + \beta}{\Omega} + \frac{2k\pi}{\Omega} \quad (4.2)$$

Where:

- ε defines the position of the i -th blade with respect to the origin of the reference system at the start of a new revolution
- β is the fixed angular position of the p -th sensor with respect to the reference system
- $k = 0, 1, 2, \dots$

To determine when a revolution is completed, a dedicated sensor called OPR or *Once Per Revolution*, is used; once this sensor receives an impulse the time is set to $t = 0$, and ε is traced.

This step is performed at low angular speeds to ensure sufficient distance from resonances, so that the vibration amplitude of each blade remains limited.

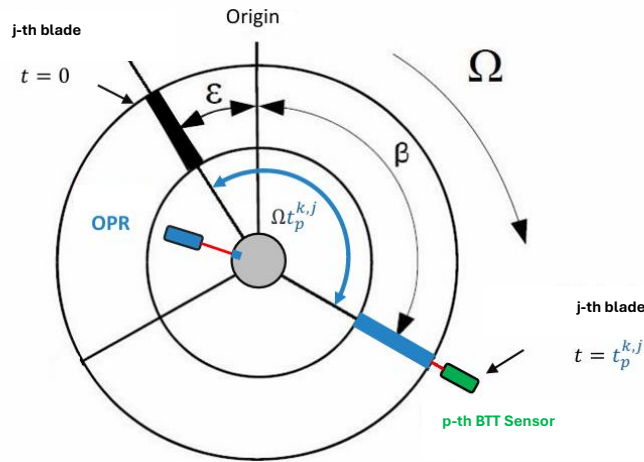


Figure 64: ToA in absence of vibration [24]

Time of Arrival in presence of vibration

In presence of vibration the deformed blade may reach the p -th sensor ahead or behind the expected t_e , thus covering a different length of arc.

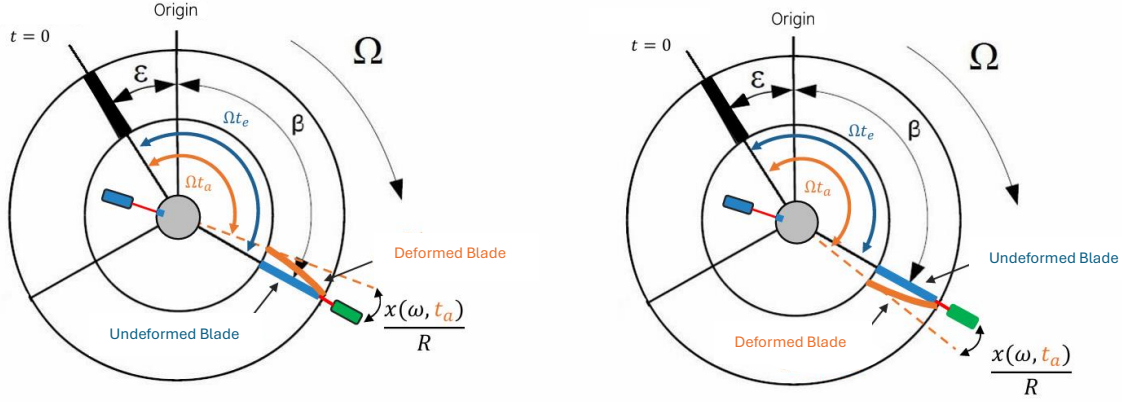


Figure 65: ToA in presence of vibration: t_a ahead of t_e (left), t_a behind t_e (right) [24]

The swept angle, as shown in Figure 65, can be written as:

$$\Omega(t_e - t_a) = \Omega \Delta t \quad (4.3)$$

That can be translated into displacements:

$$x(\omega, t_a) = R\Omega(t_e - t_a) \quad (4.4)$$

Where R is the radial position of the measured point on the blade, which is not necessarily the tip.

Every single BTT sensor detects the passage of the blades in the same position for each revolution, but, at different RPMs and thus excitation frequencies, the maximum amplitude of vibration is found in different circumferential positions; for this reason, it is possible that a specific sensor may never capture the maximum amplitude of a given blade. The Frequency Response Function obtained thus reflects the maximum response magnitude observed only by that sensor.

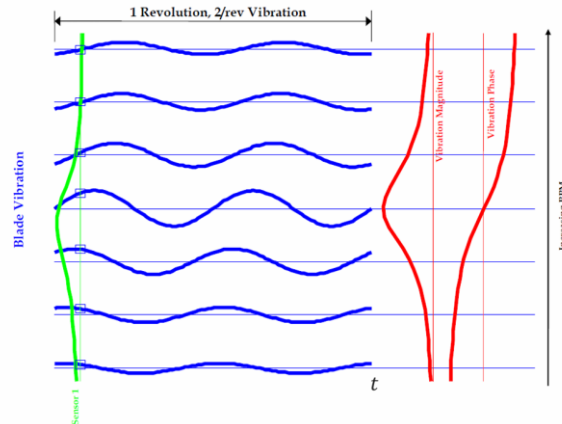


Figure 66: FRF tracing for a single BTT sensor [24]

Thus, to accurately estimate the response amplitude, resonance frequency and modal damping, a fitting of the data of various sensors is needed; to this end various BTT algorithms are available and are divided into *direct* and *indirect* methods.

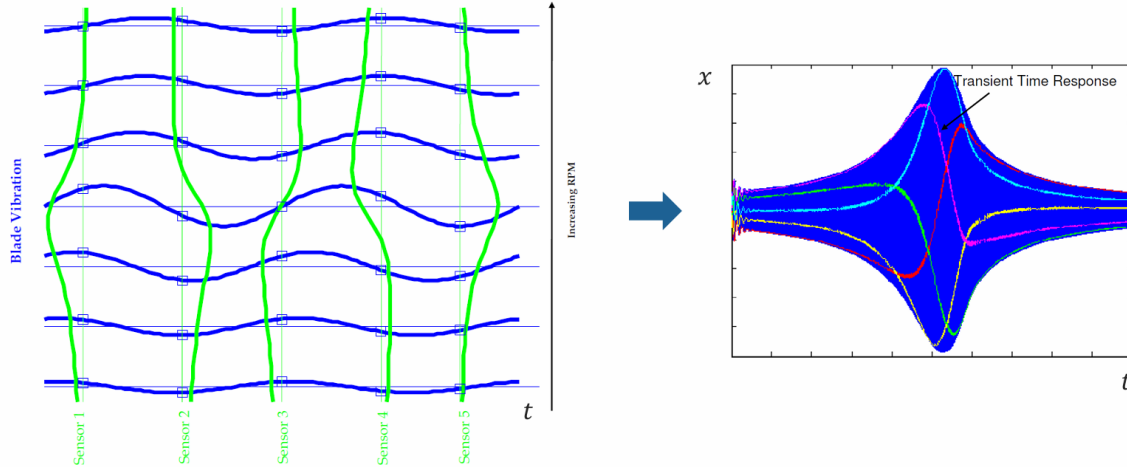


Figure 67: Fit of multiple BTT sensors data [24]

Direct methods allow to identify the forced response from which it is possible to process the modal parameters, they usually require at least three sensors and for each angular speed they require the fitting of a least squares function; some of these methods return the Engine Order as an output.

Indirect methods, on the other hand, require just one or two sensors, but do not allow to extract directly the forced response; in fact, their outputs are the modal parameters, and the EO if two sensors are adopted.

For the test campaign of which this work is part, both direct and indirect methods were employed for the post-processing of BTT measurements.

Direct Method: Circumferential Fourier Fit (CFF)

The *Circumferential Fourier Fit* is a direct method useful for cases where the resonance peaks of a bladed disk forced response are not well-separated, typical condition of mistuned phenomena or like in this case, transient response.

The method requires three or more sensors installed at the same chord-wise position. Assuming a certain response order, corresponding to the selected EO, for each averaged rpm a sinusoidal wave is fitted to the data, in other words, each blade is assumed to respond with a single harmonic.

The equation of the blade motion seen by the k -th sensor is:

$$x_k = c(\omega) + A(\omega) \sin(EO\Omega t_k + \varphi(\omega)) \quad (4.5)$$

With:

$$\theta_k = \Omega t_k \quad (4.6)$$

Substituting eq. (4.6) in eq. (4.5), and expanding:

$$x_k = c(\omega) + A(\omega) [\sin(E\theta_k) \cos(\varphi(\omega)) + \cos(E\theta_k) \sin(\gamma(\omega))] \quad (4.7)$$

With:

- A vibration amplitude
- φ phase
- c blade's static position between the *OPR* sensor and the k -th BTT sensor

At a certain frequency ω .

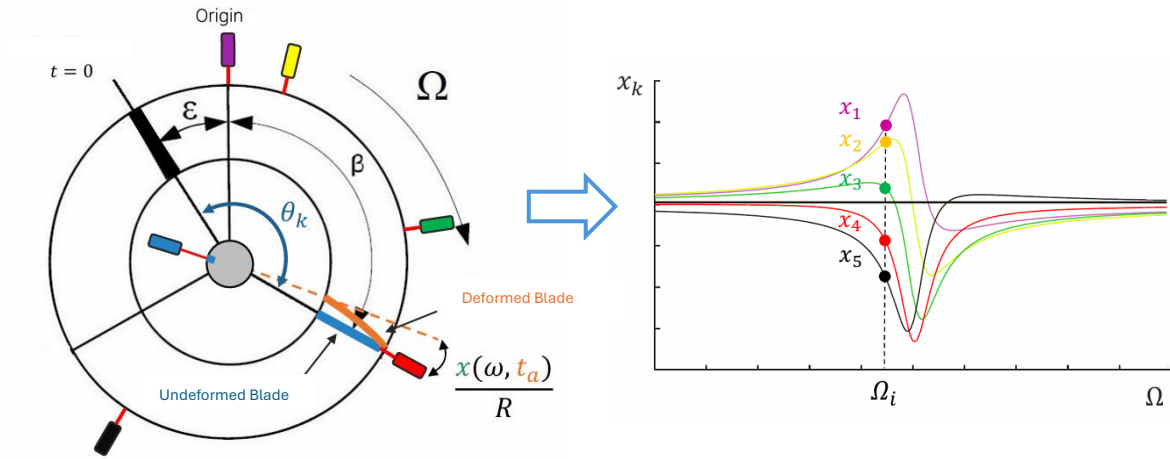


Figure 68: Example of Fit through CFF method [24]

Eq. (4.7) can be expanded in matrix form to consider all the sensors for a given frequency ω_i :

$$\begin{Bmatrix} x_1(\omega_i) \\ x_2(\omega_i) \\ x_3(\omega_i) \\ x_4(\omega_i) \\ x_5(\omega_i) \end{Bmatrix} = \begin{bmatrix} 1 & \sin(E\theta_1) & \cos(E\theta_1) \\ 1 & \sin(E\theta_2) & \cos(E\theta_2) \\ 1 & \sin(E\theta_3) & \cos(E\theta_3) \\ 1 & \sin(E\theta_4) & \cos(E\theta_4) \\ 1 & \sin(E\theta_5) & \cos(E\theta_5) \end{bmatrix} \begin{Bmatrix} c(\omega_i) \\ A(\omega_i) \cos(\varphi(\omega)) \\ A(\omega_i) \sin(\varphi(\omega)) \end{Bmatrix} \Rightarrow \mathbf{x}(\omega_i) = [\mathbf{M}] \begin{Bmatrix} \chi(\omega_i) \\ \delta(\omega_i) \\ \gamma(\omega_i) \end{Bmatrix} \quad (4.8)$$

Eq. (4.8) is solved in a least square sense in order to minimize the error ε defined as:

$$\varepsilon = [\mathbf{M}] \begin{Bmatrix} \chi(\omega_i) \\ \delta(\omega_i) \\ \gamma(\omega_i) \end{Bmatrix} - \mathbf{x}(\omega_i) \quad (4.9)$$

Where $\mathbf{x}(\omega_i)$ is the exact solution.

Finally, the amplitude and phase of the blade at the i -th frequency can be computed through the following relationships:

$$A(\omega_i) = \frac{A(\omega_i) \cos(\varphi(\omega))}{\cos(\varphi(\omega))} \text{ and } \varphi(\omega_i) = \tan^{-1} \left(\frac{A(\omega_i) \sin(\varphi(\omega))}{A(\omega_i) \cos(\varphi(\omega))} \right)$$

And be plotted against all the RPMs swept.

Indirect Method: Peak-to-peak value

The peak-to-peak value indirect method is based on the hypothesis that each blade acts like a single degree of freedom system and aims to identify the SDOF's parameters.

To achieve this, the fitting algorithm iterates the values of *static deflection* δ , *natural frequency* ω_n and *damping* ζ , searching for those that best approximate the experimental data through the equation:

$$\Re[x(\omega)] = A(\omega) \cos[E\Omega\Omega_a + \varphi(\omega)] \quad (4.10)$$

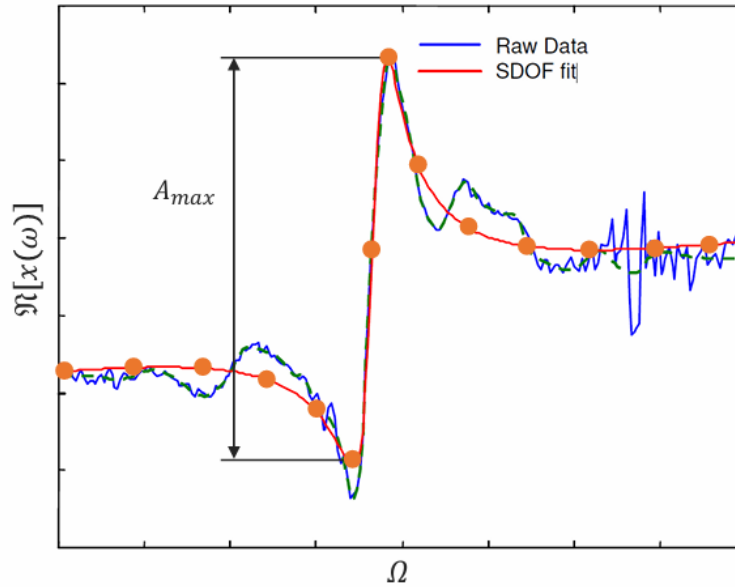


Figure 69: SDOF fitting of BTT data [24]

4.2 Strain Gauges

A Strain Gauge (SG) is a sensor that measures the amount of strain in an object by converting it into a measurable change in electrical resistance.

It consists of an electrical conductor, usually an encapsulated *constantan* foil, that is firmly bonded to the surface of the part whose strain state is to be evaluated, contrary to the BTT; it's functioning is based, in fact, on the assumption that the strains applied to the conductor's wire are the same as those of the workpiece; this is true if proper installation standards are followed.

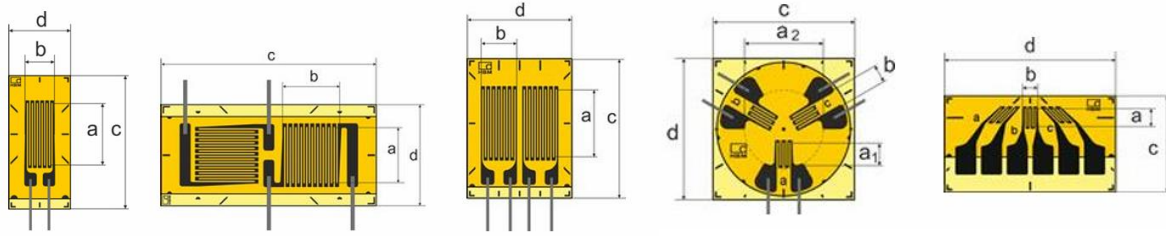


Figure 70: Different types of Strain Gauges, left to right: Linear, T Rosette, Double Linear, Rosette with 3 grids [24]

4.2.1 Working principle

Different factors contribute to the change in the electrical resistance of the conductor by modifying the wire length, section and/or resistivity, these are:

- $\left(\frac{\Delta R}{R}\right)_T$ Effect of temperature
- $\left(\frac{\Delta R}{R}\right)_\sigma$ Effect of stress
- $\left(\frac{\Delta R}{R}\right)_{\Delta\alpha}$ Effect of differential thermal expansion

These can be expressed into a single equation that captures the total variation:

$$\frac{\Delta R}{R} = \left(\frac{\Delta R}{R}\right)_\sigma + \left(\frac{\Delta R}{R}\right)_T + \left(\frac{\Delta R}{R}\right)_{\Delta\alpha} = k\varepsilon_\sigma + \beta\Delta T + k(\alpha - \alpha_{SG})\Delta T \quad (4.11)$$

Where k is the *Gauge Factor*, a measure of the SG's resistance sensitivity to strain.

Eq. (4.11) can be synthesized by dividing the stress induced longitudinal and transversal strains ε_σ , from the apparent strain ε_s , which is usually given as a function of temperature from the producer.

$$\frac{\Delta R}{R} = k\varepsilon_\sigma + [\beta + k(\alpha - \alpha_{SG})]\Delta T = k(\varepsilon_\sigma + \varepsilon_s) \quad (4.12)$$

The stress induced strains are the component of strain of interest and the objective of the measurement.

In practice, the strain measurements rarely involve quantities larger than a few millistrain ($\epsilon \times 10^{-3}$). Therefore, to measure the strain requires accurate measurement of very small changes in resistance. For this reason, and to compensate for the apparent strains previously discussed, strain gauges are almost always used in a bridge configuration with a voltage or current excitation source.

4.2.2 Wheatstone Bridge

A Wheatstone bridge is an electrical circuit used to measure an unknown electrical resistance by balancing two legs of a bridge circuit, one leg of which includes the unknown component.

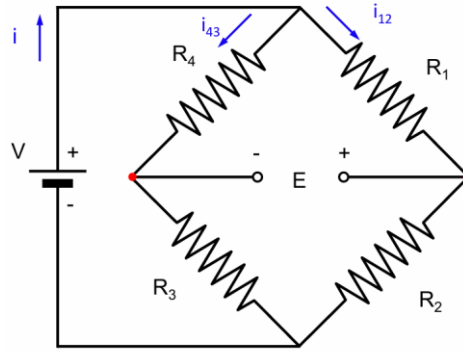


Figure 71: Wheatstone Bridge [24]

The unbalance E depends on the values of the resistance of the resistors R_1 , R_2 , R_3 and R_4 , and on their variations. If all resistors have the same value, the bridge is balanced, that is $E = 0$ for zero external loads.

The total unbalance is:

$$\Delta E = \frac{V}{4} \left[\frac{\Delta R_1}{R} - \frac{\Delta R_2}{R} + \frac{\Delta R_3}{R} - \frac{\Delta R_4}{R} \right] \quad (4.13)$$

Each resistor can be substituted by a strain gauge, in that case eq. (4.13) becomes:

$$\Delta E = \frac{V}{4} k [\epsilon_{\sigma 1} - \epsilon_{\sigma 2} + \epsilon_{\sigma 3} - \epsilon_{\sigma 4}] + \frac{V}{4} k [\epsilon_{s1}(T) - \epsilon_{s2}(T) + \epsilon_{s3}(T) - \epsilon_{s4}(T)] \quad (4.14)$$

Since, for the i -th SG:

$$\frac{\Delta R_i}{R} = k(\epsilon_{\sigma i} + \epsilon_{s i}(T))$$

Depending on the number and the disposition of strain gauges into the bridge, various configurations are possible, each one suitable for a different stress state:

- Quarter bridge
- Half bridge
- Full bridge

As will be discussed later, the tests were focused on the analysis of the non-synchronous vibrations occurring at the critical crossings of the first modal family, associated with the first bending mode vibrating out of plane. For this reason, the half bridge is the one adopted for this work since it's the proper configuration if strains are due to bending moment.

Half bridge configuration

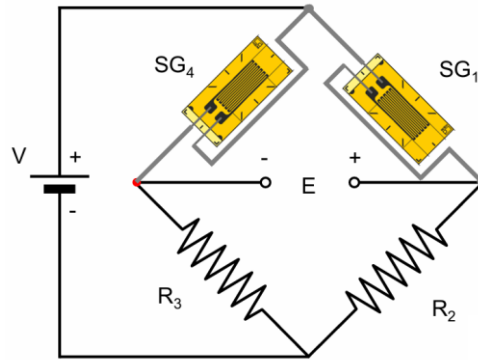


Figure 72: Half-bridge configuration [24]

The half bridge configuration employs two SGs, precisely the SG_1 and SG_4 , resulting in the following unbalance equation:

$$\Delta E = \frac{V}{4} k (\varepsilon_{\sigma 1} - \varepsilon_{\sigma 4} + \varepsilon_{s1}(T) - \varepsilon_{s2}(T)) \quad (4.15)$$

By cleverly positioning the strain gauges on the structure it is possible to take advantage of the terms of eq. (4.15) to:

- Compensate thermal effects
- Remove centrifugal stiffening effect
- Maximize the strain signal

In case bending moment is present, the most convenient placement for SG_1 and SG_4 , is on the opposite sides of the workpiece, in this case the blade, since $\varepsilon_{\sigma 4} = -\varepsilon_{\sigma 1}$.

Eq. (4.15) thus becomes:

$$\Delta E = 2 \frac{V}{4} k \varepsilon_{\sigma} \quad (4.16)$$

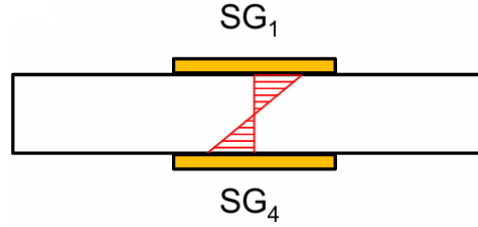


Figure 73: Strain gauges placement for bending modes [24]

Also, this configuration allows to eliminate the strain contribution given from normal loads, like centrifugal forces, since the strain is equal for both sides of the blade; this results in a measured strain that is already clear of centrifugal stiffening effects.

4.3 Spinning Test Rig

The dummy test article, object of this study, was tested in a spinning rig designed to integrate both the presented measurement techniques to evaluate the transient resonant response.

The rig is housed into a sealed pit capable of maintaining a vacuum environment in order to remove the fluid structure interaction during the analyses; two protective cylindrical structures, coaxial to a rotating shaft positioned under the floor, are present. The maximum rotating speed is 4000 RPMs.

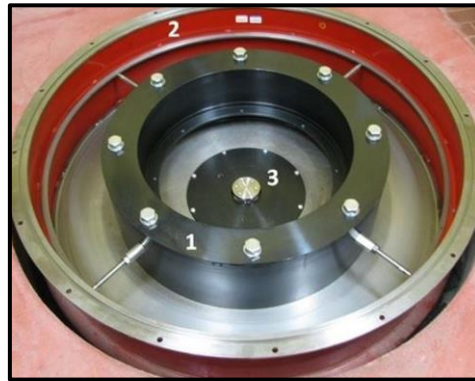


Figure 74: Protective cylinders of the spinning rig [23]

The cylinder 1 of Figure 74 also supports two static rings: the ring 1 is used to keep in a fixed position the excitation system, while the ring 2 holds the BTT sensors.

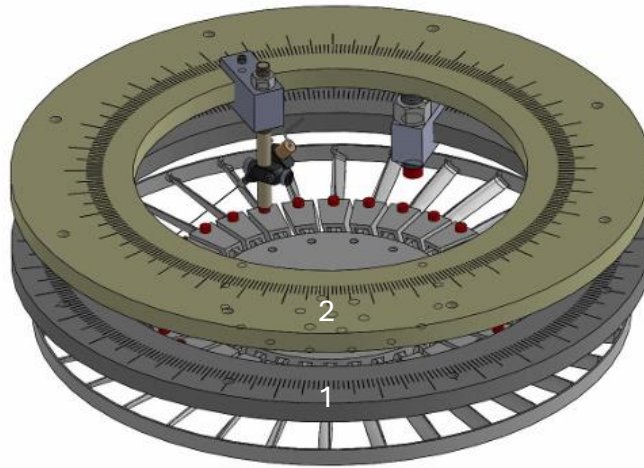


Figure 75: Spinning rig support rings [24]

4.3.1 Excitation System

To excite the dummy blisk, cylindrical permanent magnets are used. These magnets are mounted in equally spaced positions along the static ring 1 shown in Figure 75 and interact with those placed at the tip of each blade of the test article by generating a repulsive force. Up to a maximum of 28 can be installed, even though in this case only one was used to excite all the EOs.

Magnets specifications

Diameter [mm]	18
Height [mm]	10
Grade	N52

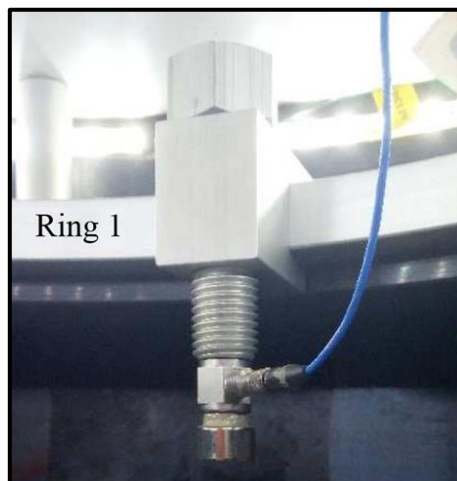


Figure 76: Fixed magnet on ring 1 [23]

4.3.2 BTT Sensors

The suite of sensors for the Blade-Tip Timing measurements is composed of:

- 5 transmitter-receiver laser sensors
- 1 *OPR* sensor

A sensor configuration called *Beam Shutter*, introduced in [23], was used to capture the OOP (out-of-plane) vibration modes of the blades.

Each sensor was mounted above the disk and produced a laser beam that was collimated through a lens on a reflective tape stuck on a fixed surface beneath the disk. During the disk rotation the passing blade acts as a shutter, blocking the returning light towards the sender-receiver sensor.

The advantage of this disposition, over a more standard one with the sensors in a radial position, consists in the capability to detect the ToA also in shrouded blades; useful for future tests which will interest more complex disks.

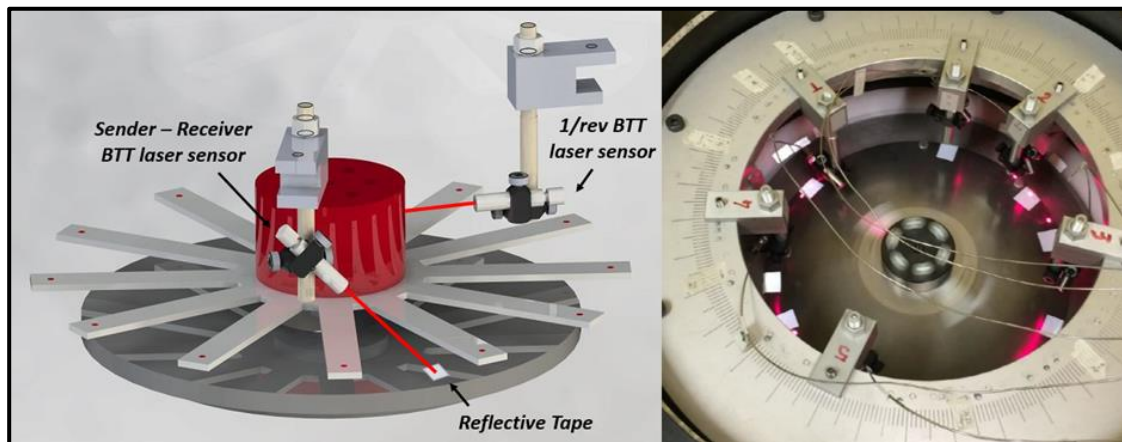


Figure 77: Beam Shutter configuration for the dummy disk [23]

Using the same algorithm presented for the CFF, the placement of the sensors was optimized to avoid the aliasing phenomena and to observe all the Nodal Diameters.

4.3.3 Strain measurement system

12 Wheatstone half bridges have been assembled on the blisk, one per blade, where strain values are the highest for all the NDs' mode shapes of the first modal family (1st Flexural OOP mode).

A telemetry system by *Manner* has been used to transmit strain gauges measurements to the acquisition system, a *National Instruments* card connected by means of BNC cables.

The telemetry has a voltage measurement range of $-/+10$ V, and the gain and offset on each half bridge has been set to fall into this range for a maximum tip displacement of ~ 1 mm; unfortunately it is not possible to know the gain applied after setting up the system, for this

reason an analytic approach to determine the strain calculated, for example through eq. (4.16), is unfeasible and a calibration step, explained in Appendix A, is necessary.

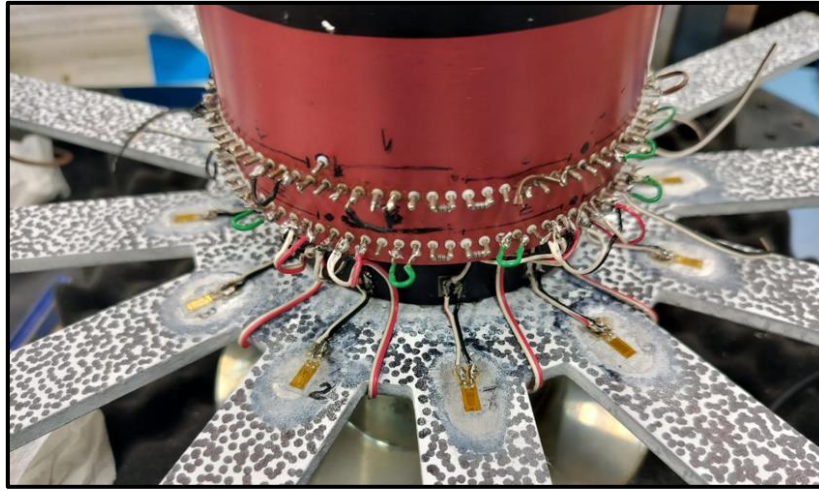


Figure 78: Strain Gauges placement and Manner telemetry system [19]

4.4 Test Campaign

A total of six tests were conducted with different sweep rates, Strain Gages sampling frequencies and directions of sweeps:

Table 8: December 2025 Test Campaign

Test name	Sweep rate [RPM/s]	SG sampling frequency [Hz]	Direction
1	1	1000	Up
2	16	1000	Up
3	16	14000	Up
4	16	14000	Down
5	64	14000	Up
6	64	14000	Down

Except for Test 1, data were recorded from both Strain Gauges (SGs) and Blade-Tip Timing (BTT) in all experimental runs. While the latter served primarily as a proof-of-concept to demonstrate its ability to detect beatings in the transient response, the SGs will be employed for the quantitative analysis of the dummy disk's dynamics and for comparison with numerical results.

Chapter 5:

Experimental Validation of FEM Transient Analyses and Determination of Stationary Response

In this final chapter, the transient resonance responses obtained via FEM numerical analyses are validated against experimental data from the previously described test campaign. The intent is to assess the FE model's capability to predict transient phenomena.

Finally, both numerical and experimental datasets are post-processed using the proposed methodology to extrapolate the stationary response under transient conditions.

5.1 Experimental Validation

In order to verify the capability of the FE Model of Chapter 3 to accurately predict the transient resonance response of the dummy blisk, the third test from the December 2025 test campaign, described in Table 8 (corresponding to *Test 3*), has been chosen.

This choice is motivated by the relatively high sweep rate, compared to the first two tests, that ensures significant transient effects, while still maintaining sufficient separation between adjacent engine order responses, unlike tests with faster ramps.

5.1.1 Test description and Modal Parameters identification

During Test 3, the shaft speed was ramped up from circa 400 RPM to 1600 RPM with a sweep rate of 16 RPM/s; in this speed range, due to the presence of a single magnet mounted on the static ring 1, all the Nodal Diameters of the disk were excited.

The harmonic content of the single impulse coming from the magnet consists in fact of all the possible EOs, but with a progressively lower energy content.

The EOs excited are given by the following equation:

$$EO_m^i = i \cdot m \quad (5.1)$$

With:

- $i = 0, 1, 2, 3, \dots$
- m is the number of stator-mounted magnets

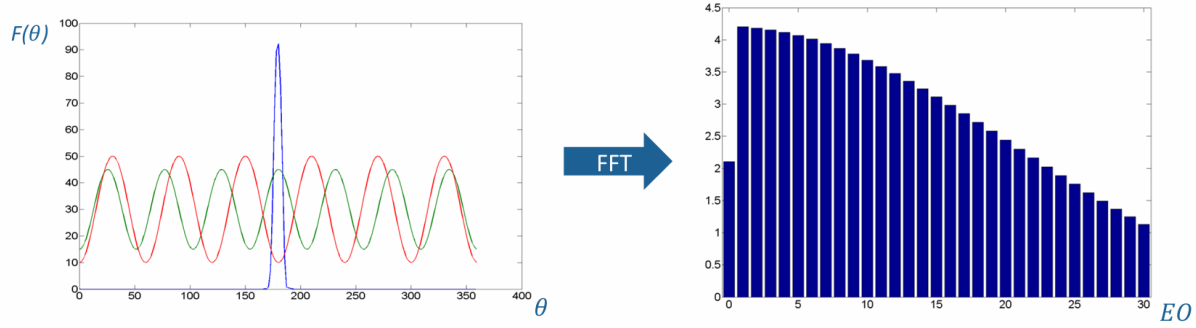


Figure 79: Single impulse harmonic content [24]

From the Campbell diagram of Test 3, the Engine Orders $EO = 9, 8, 7$ were selected since they show a clear Non-Synchronous Vibration, indicating a strong non-stationary response.

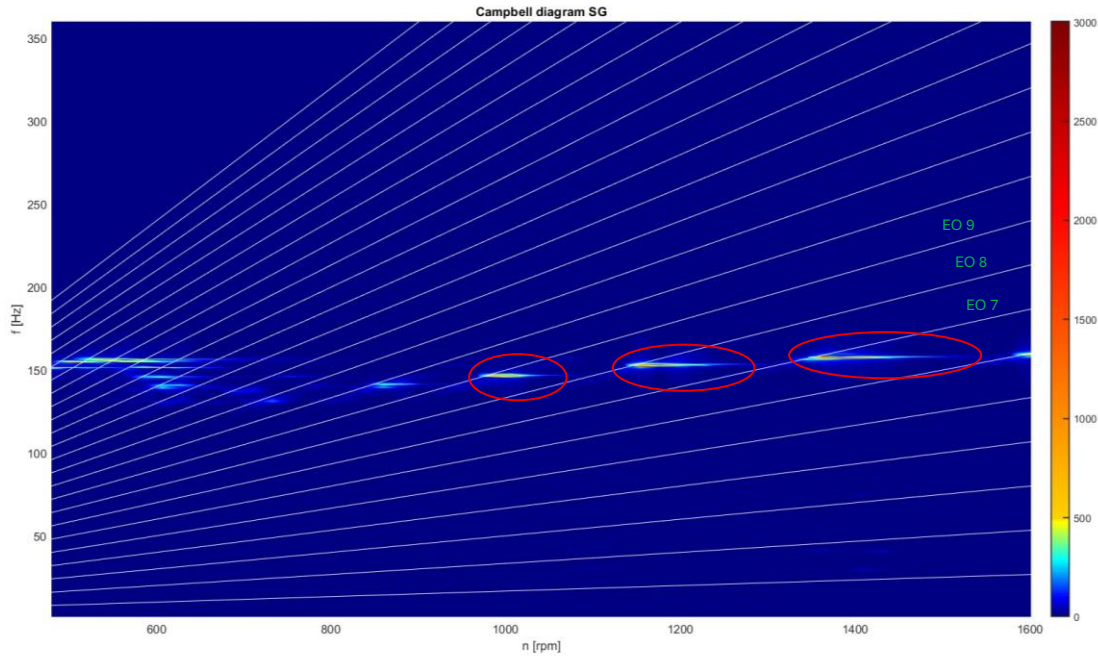


Figure 80: Test 3 Campbell Diagram with indication of selected Engine Order responses

Resonances occur at the intersections with three distinct modes, each characterized by a specific Nodal Diameter, in particular those interested are: $ND = 3, 4, 5$.

Since these crossings take place at different rotational speeds, the magnitude of the centrifugal stiffening effect varies for each excited mode. This leads to a mode-specific shift in the resonance frequencies compared to the static natural frequencies previously identified through impact testing.

It has to be remembered that another frequency shift is a direct consequence of the transient response, as discussed in Chapter 2; the one that is possible to obtain from the Campbell

diagram is thus an erroneous one, used due to the lack of available alternatives, and since the shifting effect of the sweep is generally small in scale.

The other modal parameter of interest for the modes excited, beyond natural frequencies, is the modal damping, however, its identification under operating conditions presents significant challenges.

The vacuum environment of the spinning test rig precludes the use of impact testing. Furthermore, traditional identification techniques (such as the Half-Power Bandwidth method) proved unreliable when applied to the strain gauge signals from the run-up tests; even at reduced sweep rates (*Test 1* and *Test 2*) in fact, the experimental data remained affected by transient phenomena, a consequence of the extremely low damping levels inherent to the system.

Consequently, the modal damping ratio was approximated using the value extracted from a static impact test performed in air (as detailed in Appendix A). While this approach provides a necessary baseline, it is important to note that this value does not account for operating effects, such as the absence of aerodynamic damping (due to the vacuum) or the influence of centrifugal stiffening on the structural response. Despite these inherent discrepancies, static modal damping remains the most consistent reference available for numerical validation.

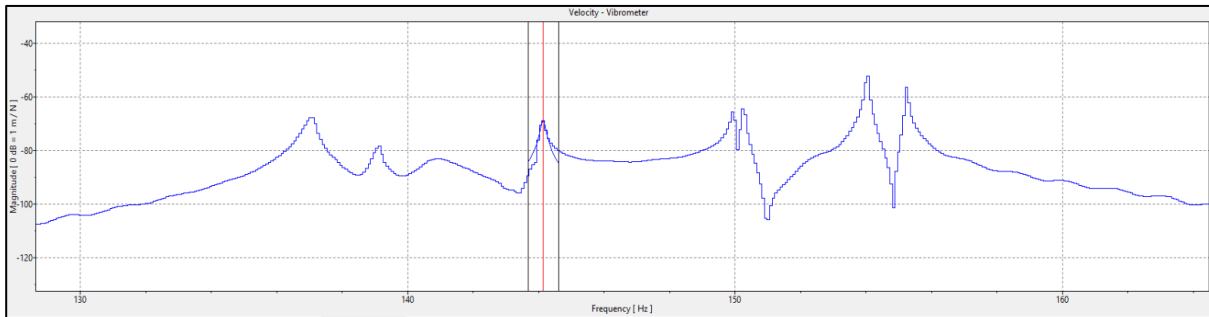


Figure 81: ND 3 Damping Ratio identification

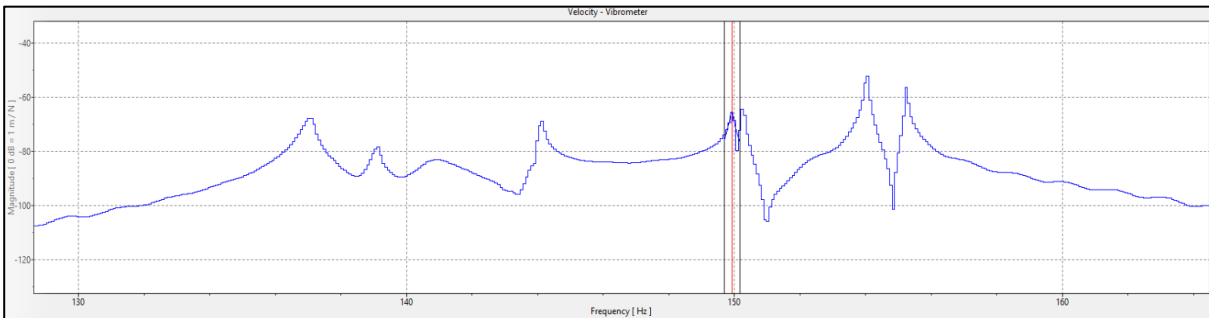


Figure 82: ND 4 Damping Ratio identification

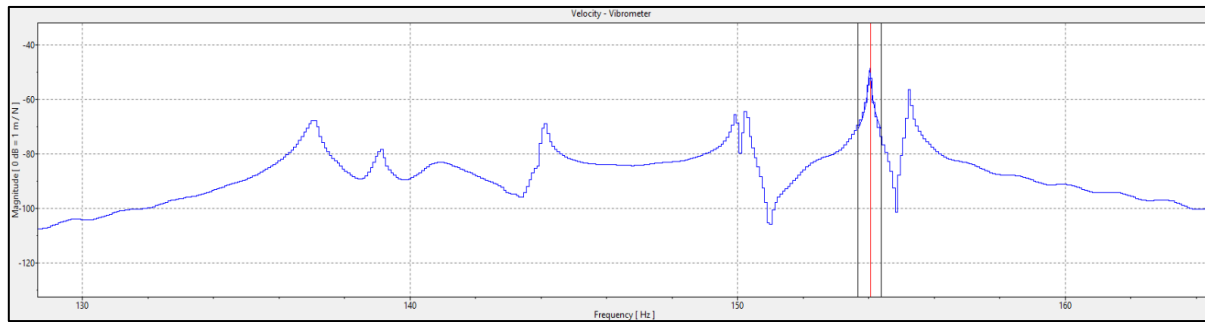


Figure 83: ND 5 Damping Ratio identification

Table 9: Test 3 NDs parameters

		ND 3	ND 4	ND 5
Engine Order		9	8	7
Rotor Speed	[RPM]	1004.28	1172.85	1372.60
Natural Frequency	[Hz]	146.5	153.0	157.5
Damping Ratio		5.32E-04	5.03E-04	1.75E-04

ND 4, as can be seen in the FRF of Figures 81-83, is a mistuned mode, thus the coexistence of two separate peaks at two different natural frequencies; this not only complicates the extrapolation of the damping ratio for that mode, but also introduces a series of effects under transient conditions, as later shown. The same problem applies to the ND 3, which appears tuned only due to the resolution in Hertz of the FFT applied to the signal; other tests in fact showed a mistuning similar in intensity to ND 4.

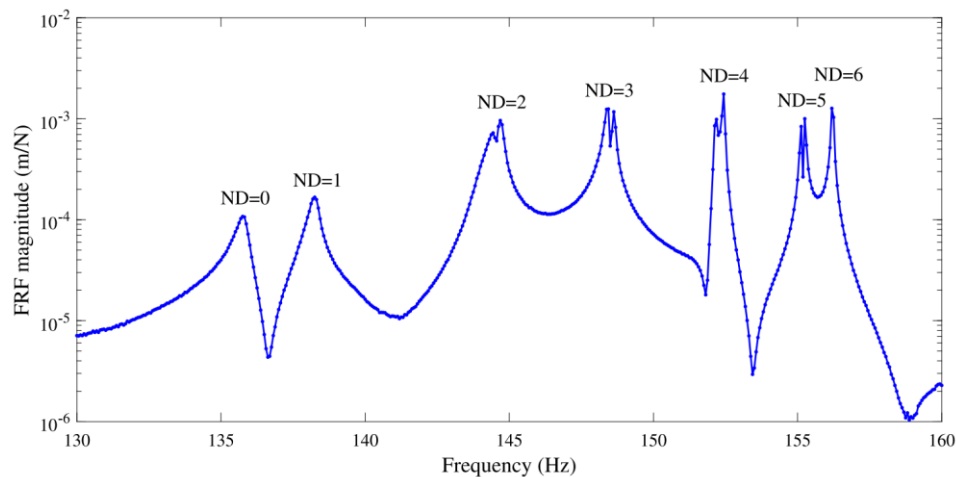


Figure 84: FRF from a previous test on the same dummy disk [23]

From Figure 84, ND 3 and ND 4 exhibit two well separated peaks, in contrast to ND 5 that instead is affected by a less intense mistuning of the complex mode, and thus, presents closer and sharper resonance peaks.

5.1.2 FEM Pre-Stressed Modal Analysis

Before performing the intended transient analyses, a preliminary step is required; the FE model built in Ansys Mechanical APDL has to be loaded with inertia loads to include the centrifugal stiffening effect into the sequent modal analysis.

This is achieved in Ansys Mechanical by conducting what is called a Pre-stressed Modal Analysis:

1. Obtain the static solution with an applied global angular velocity on the axis of rotation, Z in a cylindrical reference system; in this case the angular velocity is that of the single resonance crossing of the mode considered, as shown in Figure 80.
2. Start a new modal analysis including the prestress effects from the previous static solution; it is important to adopt the mass matrix configuration for both the analyses.
3. Expand the desired modes and calculate element results for later Transient MSUP Analysis.

For each NDs a different pre-stressed analysis has been conducted, with the rotor speeds indicated in Table 9.

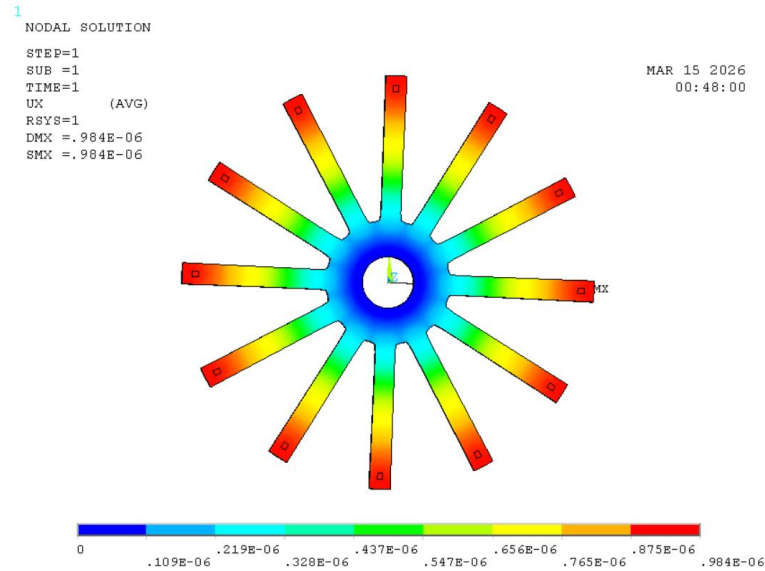


Figure 85: Displacement along the radial direction after static analysis with inertia loads

The stiffening effect, of the applied angular velocity, can be understood by looking at the radial displacements of Figure 85 in the dummy bladed disk after the static analysis was performed.

Table 10: Pre-Stressed Modal Analysis results

		ND 3	ND 4	ND 5
Campbell f_n	[Hz]	146.5	153.0	157.5
Ansysis Mechanical f_n	[Hz]	146.1	151.4	155.4
Difference	[%]	0.27	1.05	1.34

Table 10 shows an error between the computed stiffened natural frequencies for the modes of interest and the ones resulting from Test 3 measurements well under 2%, in line with what was found after the FE model tuning and sequent modal analysis.

5.1.3 Test results

The results of the transient mode-superposition analyses conducted for each of the three modes considered are compared with the data recorded by the strain gauges.

The experimental and numerical time-histories are overlaid on a global x-axis that can either represent: the rotor speed during the ramp in RPM, or the time since the start of the test; the idea is to evaluate the effectiveness of the FEM to predict the magnitude of transient effects in the response, in particular: time at which the resonance is crossed (thus the impact of frequency shifting), magnitude of the primary peak, ratio between the primary and secondary peaks and lastly, the duration of the beating effect, if present.

To ensure consistency with the FEM displacement, the Volts given by the strain gauges were converted into mm through an empirical calibration factor in m/V, as explained in Appendix A.

It is important to highlight that, while during the experimental run-up tests all the EOs excitation are present, in the numerical analysis, only one TW excitation with a fixed EO is introduced; thus, to accurately reproduce the test condition, the amplitude of the selected EO excitation should be calculated from the impulse induced by the permanent magnet. Unfortunately, the load cell used during the Test 3 reported a malfunction and, for this reason, the readings are intelligible.

To overcome this problem, a scaling of the TW amplitude has been applied; this approach is valid in linear conditions.

EO 7 – ND 5

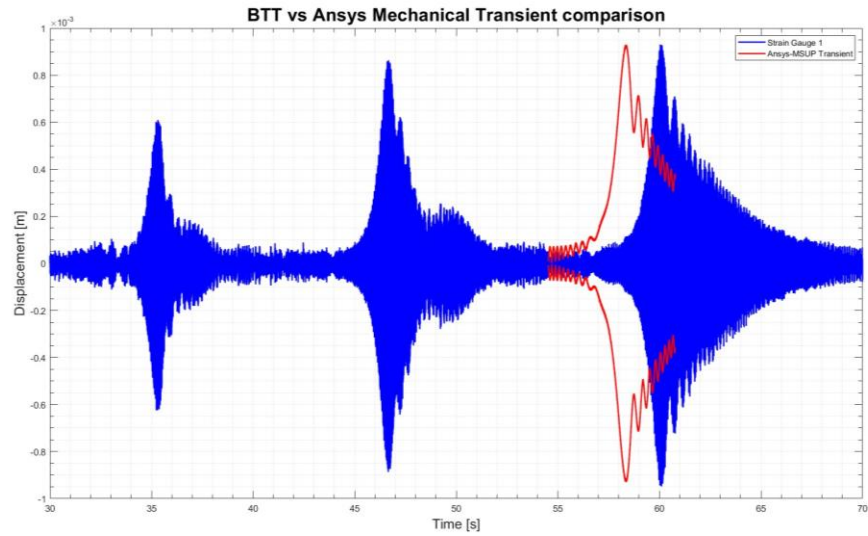


Figure 86: Time-History comparison between FEM and SG signal for EO 7 and ND 5

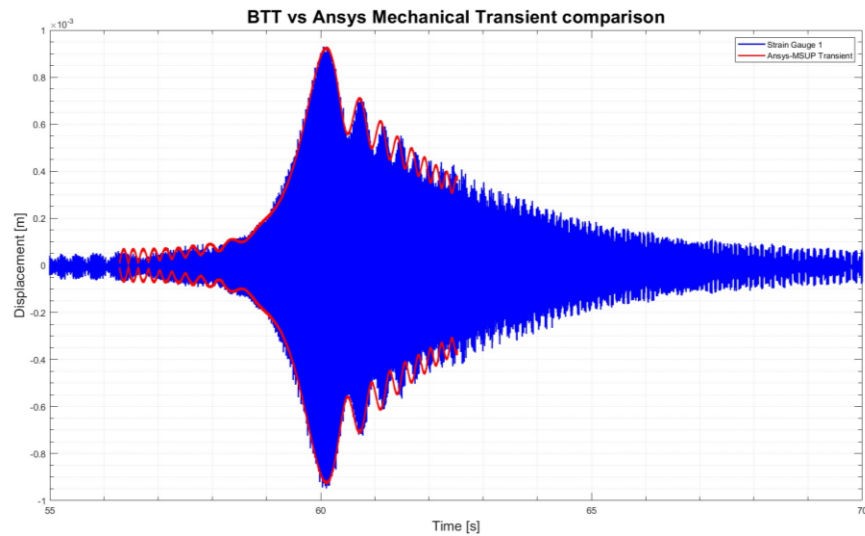


Figure 87: Fitting results for EO 7 and ND 5

Table 11: Comparison results for EO 7 and ND 5

		Strain Gauge 1	Numerical	Error [%]
Load Scale		-	22.5	-
Calibration factor	[m/V]	0.0003184	-	-
x_{max} position	[s]	60.1	58.4	2.80
x_2 position	[s]	60.8	59.0	2.96
x_{max} amplitude	[m]	9.28E-04	9.30E-04	0.21
x_2 amplitude	[m]	7.10E-04	7.14E-04	0.56
x_2/x_{max}		0.765	0.768	0.35

EO 8 – ND 4

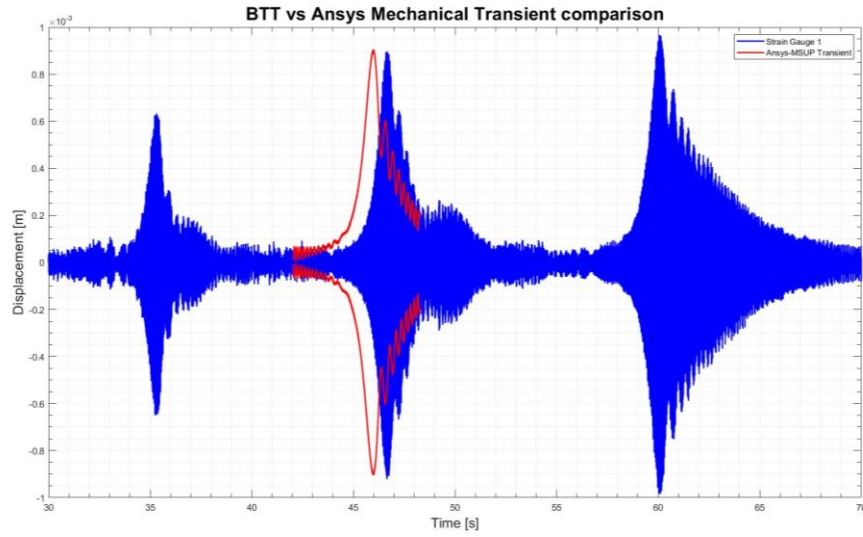


Figure 88: Time-History comparison between FEM and SG signal for EO 8 and ND 4

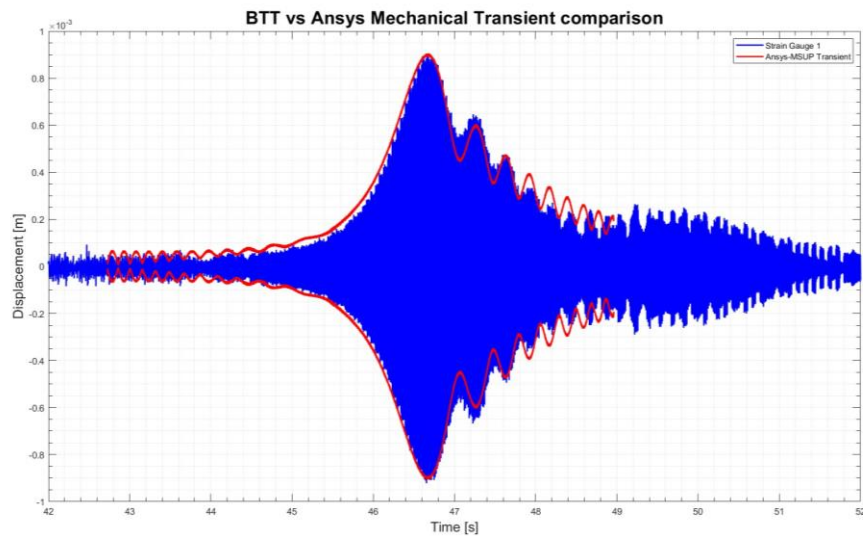


Figure 89: Fitting results for EO 8 and ND 4

Table 12: Comparison results for EO 8 and ND 4

		Strain Gauge 1	Numerical	Error [%]
Load Scale		-	19.8	-
Calibration factor [m/V]		0.0003309	-	-
x_{max} position	[s]	46.7	46.0	1.50
x_2 position	[s]	47.3	46.6	1.48
x_{max} amplitude	[m]	8.97E-04	9.05E-04	0.89
x_2 amplitude	[m]	6.47E-04	6.05E-04	6.49
x_2/x_{max}		0.721	0.669	7.21

EO 9 – ND 3

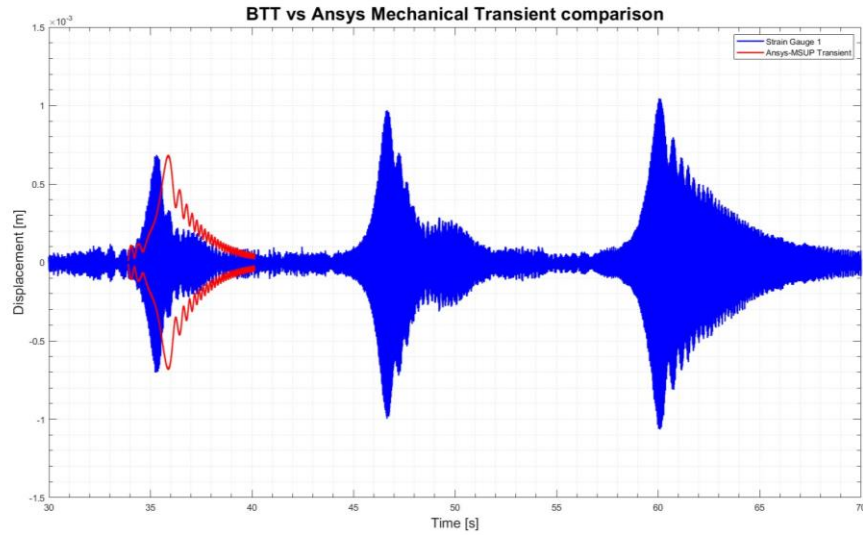


Figure 90: Time-History comparison between FEM and SG signal for EO 9 and ND 3

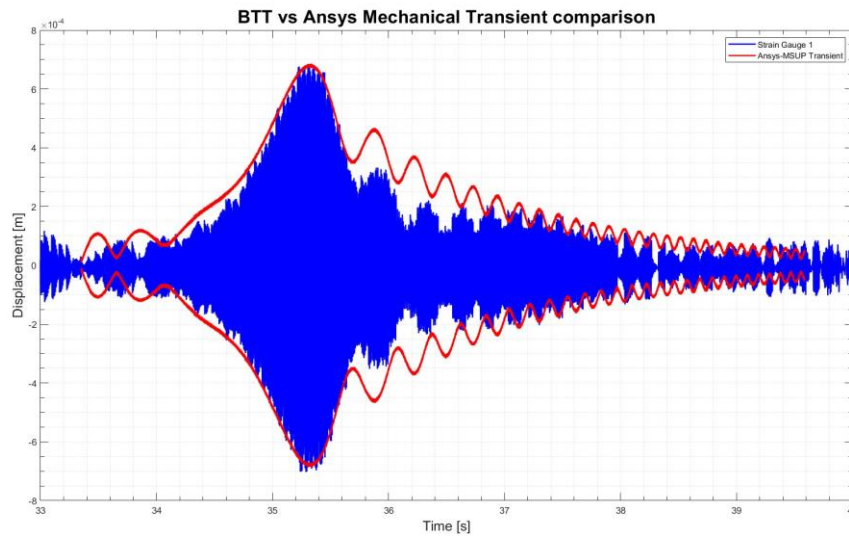


Figure 90: Fitting results for EO 9 and ND 3

Table 13: Comparison results for EO 9 and ND 3

		Strain Gauge 1	Numerical	Error [%]
Load Scale		-	25.5	-
Calibration factor [m/V]		0.0003578	-	-
x_{max} position	[s]	35.3	35.9	1.69
x_2 position	[s]	35.9	36.4	1.39
x_{max} amplitude	[m]	6.85E-04	6.85E-04	0.01
x_2 amplitude	[m]	3.32E-04	4.67E-04	40.7
x_2/x_{max}		0.485	0.682	40.6

The time-history extracted from SG 1's signal (blue harmonic) in the range of interest for the selected modes, and the ones resulting from three different MSUP Transient analyses (red envelope), are plotted on a common time axis in Figures 86, 88 and 90, in which transient effects are clearly present, as expected for the chosen sweep rate.

The most prominent aspect emerging from these plots is a mismatch in the time at which the resonance is crossed, and thus the position on the x-axis of the main peak; it can be explained by two concurrent factors:

- Difference in the natural frequency between the real disk and the FE counterpart: Table 10 testifies a difference of circa 0.3% to 1.5% between the natural frequencies identified in the pre-stressed modal analysis and the Campbell diagram of Figure 80. This variation in Hz translates directly into a variation in seconds for the reaching of the critical crossing when considering a run-up or run-down test. As deduced from eq. (2.15) and (2.16), a faster sweep reduces this distance.
- Approximation introduced in the modal damping: as already mentioned, the chosen modal damping for the numerical analyses differs from the real one acting on the blisk during Test 3. The damping is directly linked to the RSV (eq. (2.14)), and thus to the intensity of transient effects, one of which is the frequency shift of the response in the direction of the sweep. In conclusion, the missing knowledge of the correct real damping represents another factor that influences the localization of the response in time.

As written in Table 11, 12 and 13, the error in the position of x_{max} is always below 3%, a value that is certainly compatible with the order of magnitude of the delta in f_n , but still higher; it is then plausible that the remaining difference has to be attributed to the erroneous damping estimation.

The second step of the validation consists in the evaluation of the fit between the two datasets for each ND, without considering the time shift.

In this case, the ND 5 shows a very good fitting with the strain gauge signal and the ability to correctly predict the amplitude of the series of peaks, with an error lower than 1% for the primary and secondary peaks. A slight deviation in time is observed along the trail, probably also due to a flawed damping value.

The fitting is instead less convincing for the ND 4, and especially ND 3; they in fact show a faster decay in the amplitude of the response, compared to the FEM results, at least near the main peak, followed by a sequence of beats with an almost constant magnitude.

This shape of the response, observed in SG 1's signal for ND 4 and ND 3, is caused by the mistuning of these two complex modes. In fact, transient resonance crossings in a mistuned blisk can introduce a series of phenomena that weren't considered in this work, one of which is the so-called Transient Amplitude Amplification of Mistuned Structures (TAMS). The tuned FE model is unable to replicate these effects; therefore, this explains the higher error (over 40%) in the amplitude prediction of the primary and secondary peaks.

5.1.4 Concluding remarks

The Mode-Superposition Transient Analysis of Ansys Mechanical APDL, applied on the presented FE model of the dummy blisk, has proven to be able to accurately predict the magnitude of transient effects caused by a fast frequency sweep through resonance, under the conditions that: a correct setup is executed on the model in terms of modal parameters, and a tuned mode is considered; for further analysis it would be useful to exploit the full 360 model adopted here to artificially introduce a mistuning and extend the investigation to transient resonance crossings for a mistuned bladed disk.

5.2 Determination of Stationary Response from Transient Measurements

The procedure explained in Section 2.2.2 to determine the stationary response from the measured transient response, is now applied to the numerical and experimental datasets of Section 5.1, to evaluate if it's able to identify the modal parameters even in presence of transient effects for a real test case.

EO 7 – ND 5

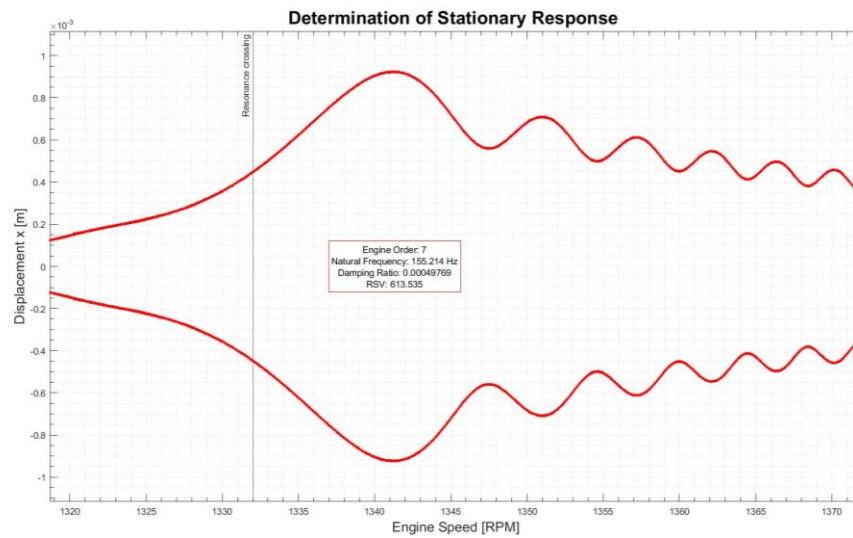


Figure 91: Modal parameters identification on numerical analysis for EO 7 and ND 5

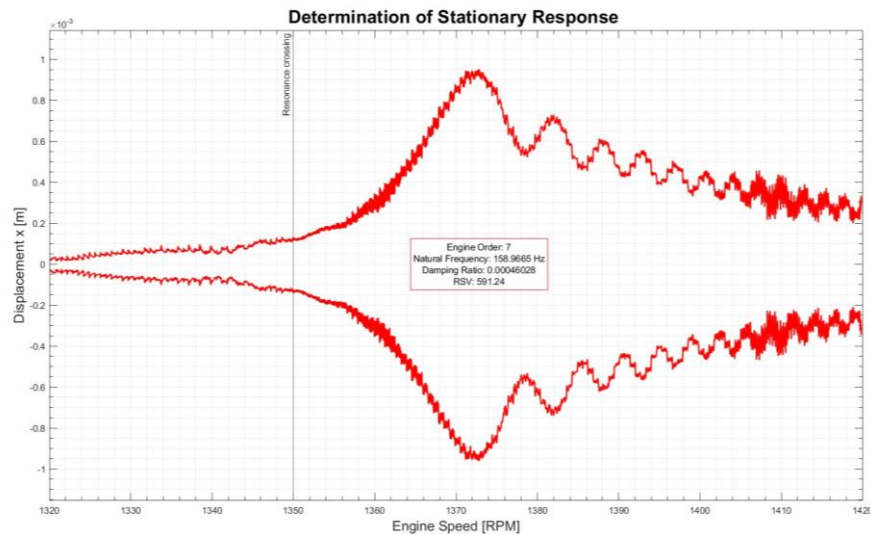


Figure 92: Modal parameters identification on SG signal for EO 7 and ND 5

Table 14: Modal parameter identification results for EO 7 and ND 5

		Strain Gauge 1	Difference [%]	Numerical	Difference [%]
Natural frequency					
f_n reference	[Hz]	157.5	-	155.40	-
f_n identified	[Hz]	158.97	0.92	155.21	0.12
f_n HPB	[Hz]	160.28	1.77	156.63	0.79
Damping ratio					
Damping ratio reference		1.75E-04	-	1.75E-04	-
Damping ratio identified		4.60E-04	162	4.98E-04	185
Damping ratio HPB		4.76E-03	2620	5.25E-03	2900
Reduced Sweep Velocity					
RSV reference		2524	-	2524	-
RSV identified		591	76.6	614	75.7
RSV HPB		3.21	99.9	2.77	99.9

EO 8 – ND 4

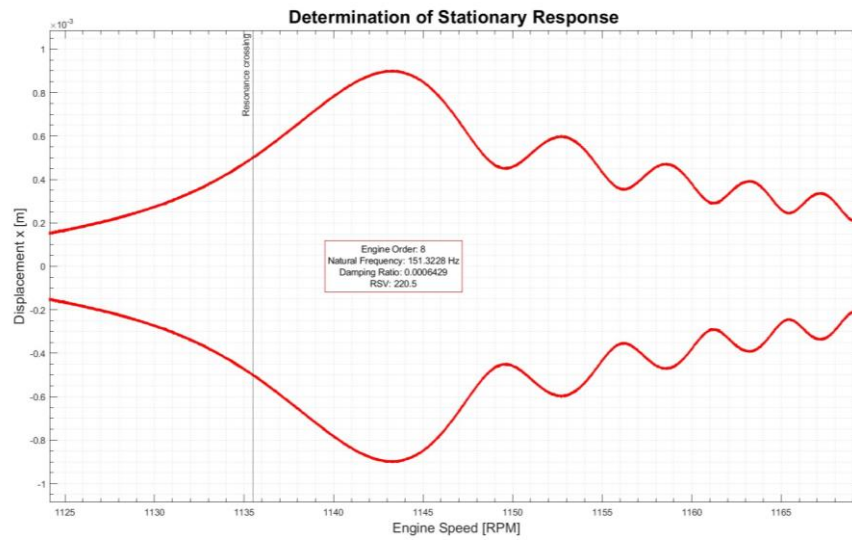


Figure 93: Modal parameters identification on numerical analysis for EO 8 and ND 4

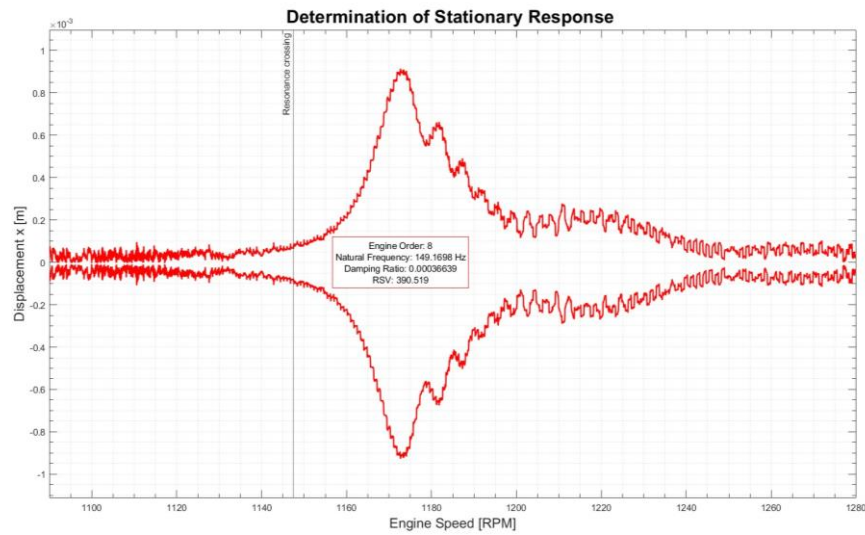


Figure 94: Modal parameters identification on SG signal for EO 8 and ND 4

Table 15: Modal parameter identification results for EO 8 and ND 4

		Strain Gauge 1	Difference [%]	Numerical	Difference [%]
Natural frequency					
f_n reference	[Hz]	153.00	-	151.40	-
f_n identified	[Hz]	149.17	2.51	151.32	0.05
f_n HPB	[Hz]	160.28	4.75	152.33	0.61
Damping ratio					
Damping ratio reference		5.03E-04	-	5.03E-04	-
Damping ratio identified		3.66E-04	27.23	6.43E-04	27.83
Damping ratio HPB		5.60E-03	1013	4.10E-03	715
Reduced Sweep Velocity					
RSV reference		367	-	367	-
RSV identified		391	6.54	221	66.1
RSV HPB		3.21	99.1	5.52	97.5

EO 9 – ND 3

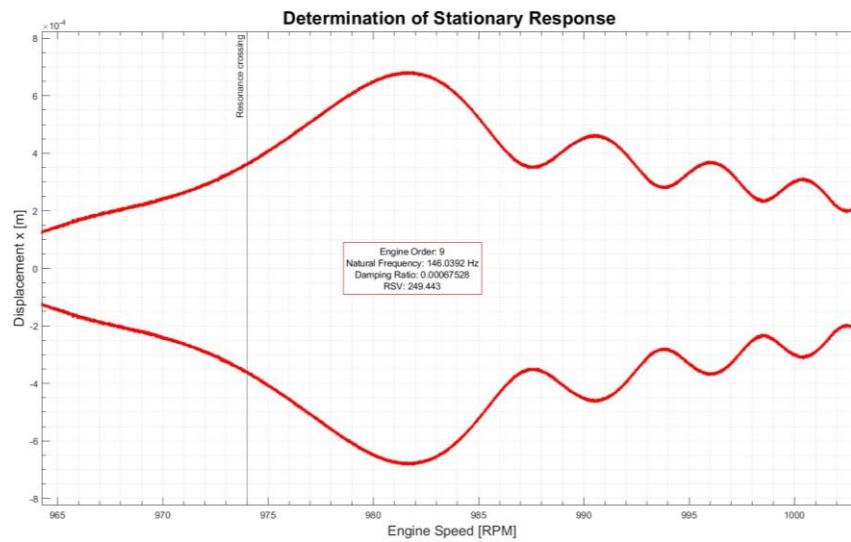


Figure 95: Modal parameters identification on numerical analysis for EO 9 and ND 3

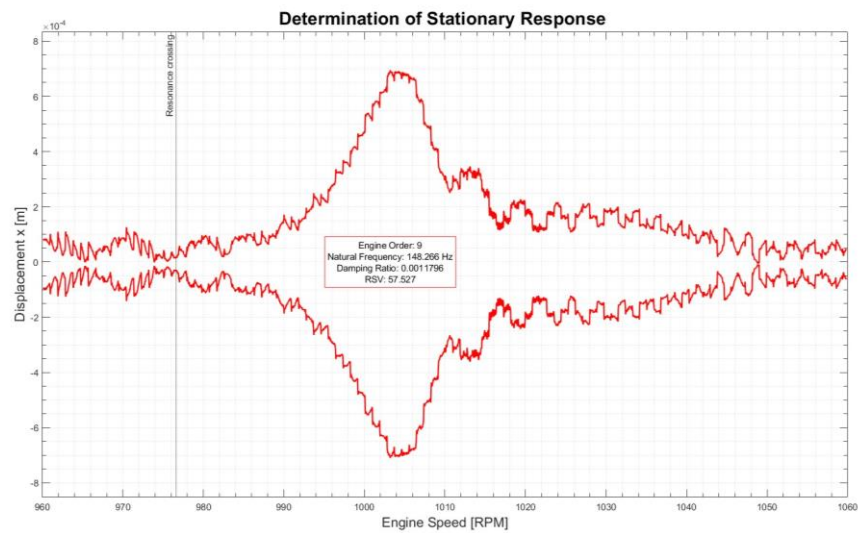


Figure 96: Modal parameters identification on SG signal for EO 9 and ND 3

Table 16: Modal parameter identification results for EO 9 and ND 3

		Strain Gauge 1	Difference [%]	Numerical	Difference [%]
Natural frequency					
f_n reference	[Hz]	146.50	-	146.10	-
f_n identified	[Hz]	148.27	1.21	146.03	0.05
f_n HPB	[Hz]	150.57	2.78	147.14	0.61
Damping ratio					
Damping ratio reference		5.32E-04	-	5.32E-04	-
Damping ratio identified		1.17E-03	119	6.75E-04	26.87
Damping ratio HPB		3.90E-03	633	4.60E-03	7.65
Reduced Sweep Velocity					
RSV reference		395	-	395	-
RSV identified		58	85.3	249	37.0
RSV HPB		6.92	98.2	5.34	98.6

5.2.1 Concluding remarks

The proposed identification procedure, as shown in Table 14, 15, 16 proved a greater accuracy in the modal identification in presence of transient effects in the dynamic response, both for numerical and experimental datasets; nevertheless, the difference with the reference data chosen for the comparison is still greater than expected.

This behavior can be explained by the various differences characterizing the dummy blisk and its 3D FE model, with the SDOF assumption on which the procedure is built:

- Modes coupling favored by a flat FreND diagram
- Simultaneous excitation of multiple EOs during the spin test
- Presence of mistuning in the test article, which is not accounted for either in the lumped parameter approach nor in the numerical model

These conclusions are backed by the fact that, by looking at the aforementioned tables, and Figures 28 from Chapter 2, a discrepancy is found: the ratio between the amplitudes of the primary and secondary peaks should reflect, according to Figure 28, an RSV way lower than the expected one from eq. (2.14).

5.3 Future works

The results of this validation show that commercial FEM tools, usually employed in the design process, are able to accurately predict the effects of transient conditions in the dynamics response of critical components like bladed disks, at least in absence of complications like mistuning or coupled modes.

Starting from these conclusions, future works may focus on the extension of this study with the inclusion of other aspects, here only briefly mentioned, like the effect of friction damping in transient state, or the implementation of artificially mistuned numerical models.

Appendix A:

Static and Dynamic Calibration of Strain Gauges

This Appendix serves the purpose of illustrating how calibration factors, needed to correlate the numerical results with the tests' data, were calculated.

A.1 Static Calibration of Strain Gauges

To correlate the Volts measured by the Strain Gauges, with the strains obtained from the numerical analysis, a calibration factor in ε/V has to be found.

The first step involves a linearity check to determine whether there is a linear relationship between the applied force and the Volts measured by the SGs (as it should).

To do so, a known has been applied on the tip of each blade, precisely on the permanent magnets that receive the repulsive force during the spin test.

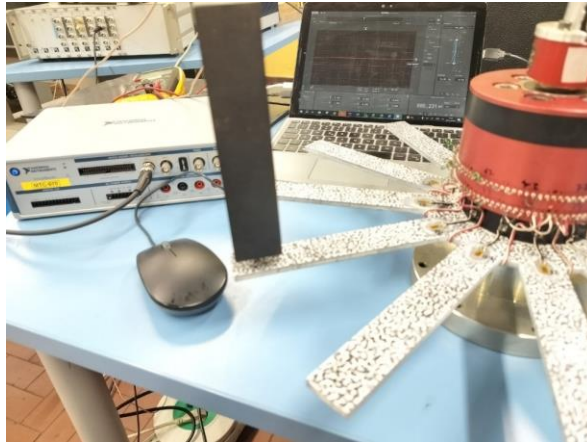


Figure 97: Static Calibration setup

The load is realized by a steel block of 0.99 kg, positioned using a magnet of 0.01 kg to ensure the application of a concentrated force in the tip.

At this point the delta between the signal amplitude with and without the applied load is measured for each blade; this allows to obtain the calibration factor expressed in V/N as:

$$C_f = \frac{\Delta V}{F} \quad (A.1)$$

Appendix A: Static and Dynamic Calibration of Strain Gauges

The voltage given in the free configuration is called *offset*, and is adjusted to have a maximum reading for a tip displacement of circa 1 mm.

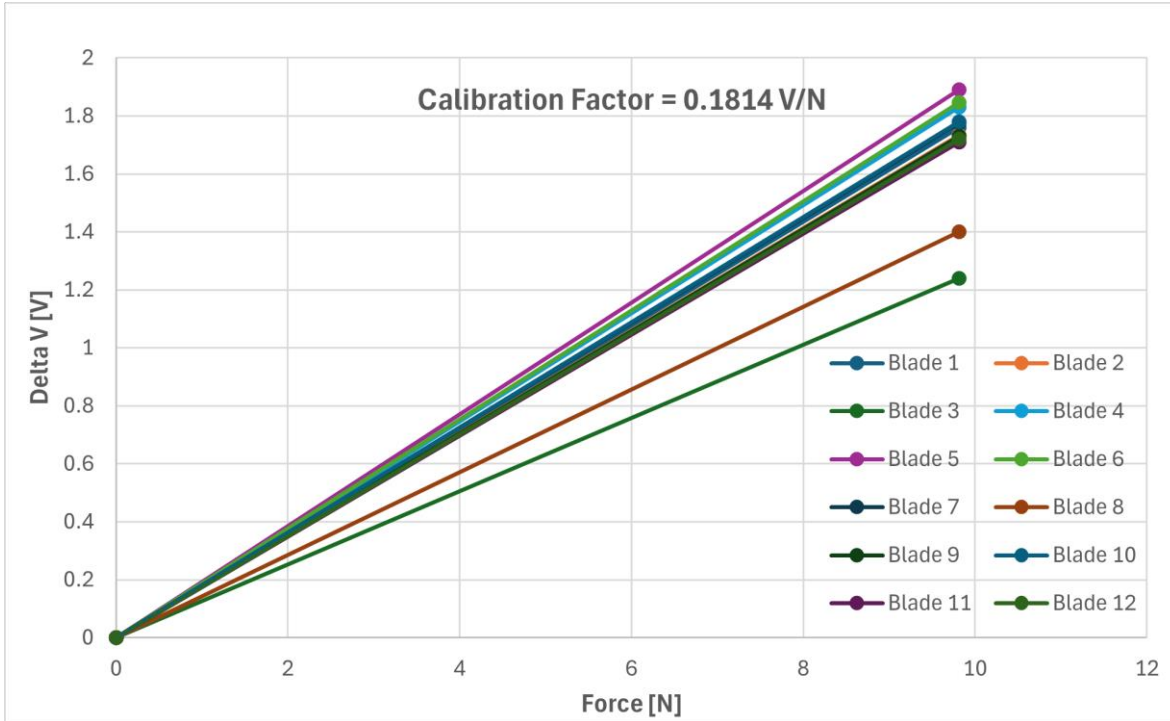


Figure 98: Static calibration results for each strain gauge

As visible in Figure 98, all the SGs show a consistent relationship between the force applied and the response, apart from those applied to blade 3 and 8; these two in fact, in the loaded configuration, reach the limit of the voltage measurement range resulting in a lower delta than the others. This is due to an excessively high starting voltage (offset), which, however, cannot be individually adjusted through the acquisition system.

The resulting calibration factor thus, excluding the two unreliable SGs is $C_{fN-V} = 5.513 \text{ N/V}$.

Blade	Channel	Start Volt [V]	End Volt [V]	Delta Volt [V]
1	14	3.37	5.13	1.76
2	19	-0.59	1.14	1.73
3	18	8.96	10.2	1.24
4	23	2.23	4.06	1.83
5	22	4.88	6.77	1.89
6	3	-2.58	-0.73	1.85
7	2	0.06	1.84	1.78
8	7	-8.8	10.2	1.40
9	6	3.94	5.67	1.73
10	11	-1.78	0.00	1.78
11	12	3.60	5.31	1.71
12	15	6.76	5.04	1.72

The next step consists in performing the same operation on the finite element model to evaluate, through static analysis, the strains calculated at the same locations where the strain gauges are positioned.

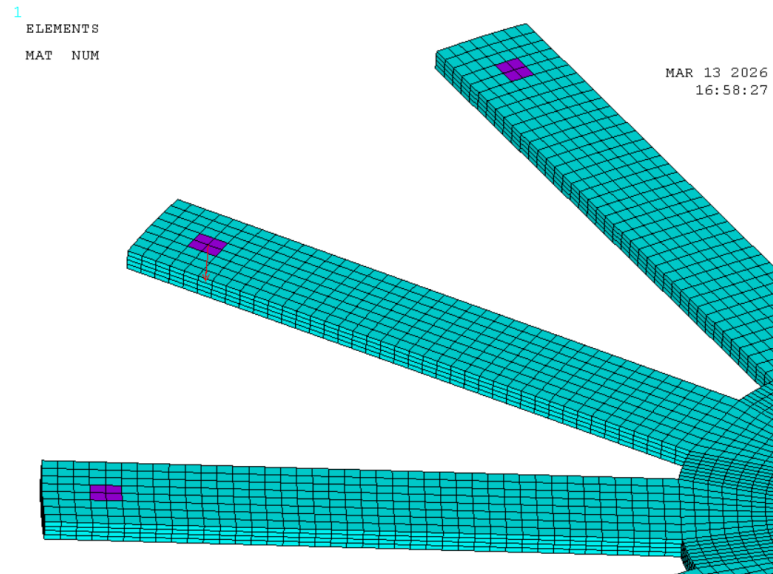


Figure 99: Nodal force of 9.81 N in the location of the permanent magnet on the FE model

A nodal force of 9.81 N is applied on a single blade in the node that represents the location of the magnet in the real dummy blisk, then the strains in the radial direction (X-axis in a circumferential reference frame) are evaluated.

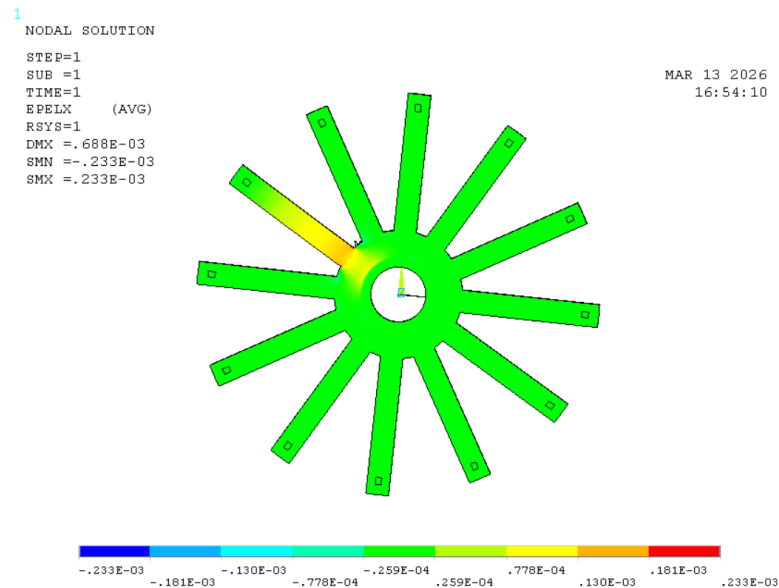


Figure 100: elastic strain in radial direction after static analysis

The numerical static analysis gives a calibration factor, expressed in N/ε of:

$$C_{f_{N-\varepsilon}} = 6.05E + 04$$

Finally, the desired parameter is calculated as a ratio of the two:

$$C_{f_{\varepsilon-V}} = \frac{C_{f_{N-V}}}{C_{f_{N-\varepsilon}}} = 9.11E - 05 \quad (A.2)$$

A.2 Dynamic Calibration of Strain Gauges

Once $C_{f_{\varepsilon-V}}$ has been determined statically, a Dynamic Calibration is executed to verify that, even in presence of vibrations this factor remains constant, for all the modes of interest for subsequent tests.

Since the dummy disk can be considered linear, and its response gives well separated modes, the following relationship should be satisfied:

$$K_{phis} = \frac{u_{z_{phis}}}{\varepsilon_{SG_{phis}}} = \frac{u_{z_{mod}}}{\varepsilon_{x_{mod}}} = K_{mod} \quad (A.3)$$

Where on the real disk:

- $u_{z_{phis}}$ physical displacement, in axial direction, of the blade at the measurement point
- $\varepsilon_{SG_{phis}}$ measured strain from the SG at the same frequency

On the FE model:

- $u_{z_{mod}}$ modal displacement along z-axis of the node in the same position of the measured point in the real disk
- $\varepsilon_{x_{mod}}$ modal radial strain in the position of the strain gauges

For each one of the modes considered, the relationship (A.3) is used to verify that the Calibration Factor $C_{f_{\varepsilon-V}}$ previously found is valid for the dynamic response.

Measurement of the axial displacement, for each blade, is done via a scanning laser vibrometer from the company Polytec.

To this end, for every blade, a reflective patch has been applied at the very tip and represents the reference point for the choice of the corresponding node on the FE model.

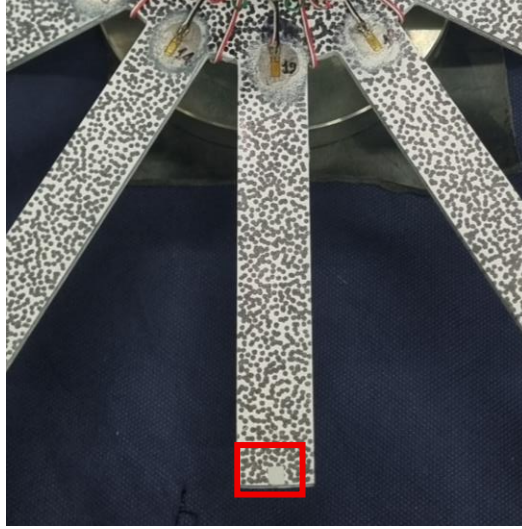


Figure 101: Laser vibrometer scanning point on the blade

The vibrometer is connected, together with the measurement system for the strain gauges signal, and an impact hammer to a *Digital Acquisition System* (DAQ).

The objective is to excite the vibrating modes of the disk through an impulse and extract two FRF by performing a FFT on the signal coming from the vibrometer and each SG; this permits to evaluate measure $u_{z_{phis}}$ and $\varepsilon_{SG_{phis}}$ at the frequencies corresponding to the Nodal Diameters considered in Chapter 5, but also for the ND = 0; this Nodal Diameter is also check to verify if the result vary between a stationary mode or a complex mode of the disk.

On the FE model instead, a modal analysis is conducted to evaluate $u_{z_{mod}}$ and $\varepsilon_{x_{mod}}$ in the location respectively of the reflective patch and SG.

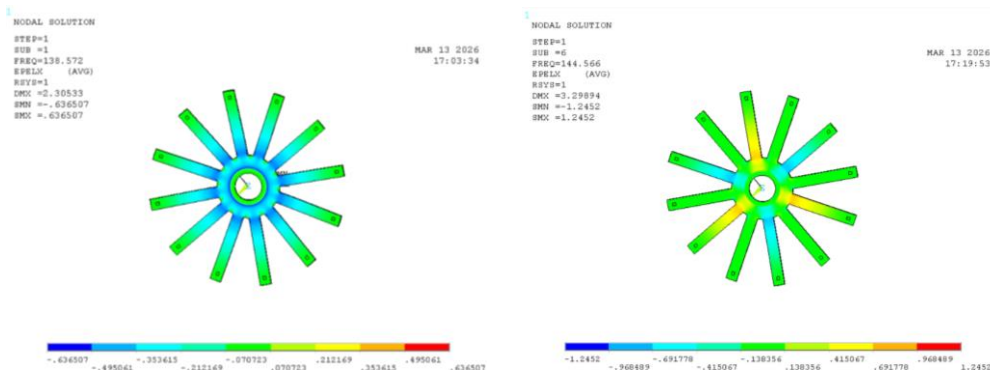


Figure 102: Modal Elastic Strain for ND 0 and ND 3

Appendix A: Static and Dynamic Calibration of Strain Gauges

To compare the two procedure, first $\varepsilon_{SG_{phis}}$ is calculated as:

$$\varepsilon_{SG_{phis}} = \frac{\varepsilon_{x_{mod}}}{u_{z_{mod}}} \cdot u_{z_{phis}} \quad (A.4)$$

Then it is compared with the static radial strain $\varepsilon_{SG_{static}}$ computed as:

$$\varepsilon_{SG_{static}} = C_{f_{\varepsilon-V}} \cdot V_{SG_{phis}} \quad (A.5)$$

The results are:

Table 17: Static vs Dynamic calibration results

		ND 0	ND 3	ND 5
$u_{z_{mod}}$	[m]	2.235	3.196	1.380
$\varepsilon_{x_{mod}}$		-0.502	-0.836	-0.405
$u_{z_{phis}}$	[m]	4.43E-06	3.90E-06	2.68E-05
$\varepsilon_{SG_{phis}}$		-9.96E-07	-1.02E-06	-7.86E-06
$V_{SG_{phis}}$	[V]	0.0108	0.0109	0.0839
$\varepsilon_{SG_{static}}$		9.84E-07	9.93E-07	7.64E-06
Delta	[%]	1.22	2.72	2.88

As Table 17 shows, the difference between the static and dynamic strains, calculated starting from the same Calibration Factor $C_{f_{\varepsilon-V}}$ is minimal, with an error that stays under the 3% also for NDs linked to complex modes, proving that a calibration executed on a static disk remains valid in dynamic conditions.

This comparison highlighted a useful aspect of the usage of the laser vibrometer coupled with strain gauges; it is in fact possible to retrieve, dividing the FRFs from the two measurement techniques, to directly obtain another Calibration Factor expressed this time in m/V that correlates the Voltage measurements of the strain gauges with the displacement in the laser scanning point for the entire frequency range.

It is obvious that, apart from a small error that is always present in experimental tests, this value should be constant over the frequency range.

	ND 0	ND 3	ND 5
$C_{f_{m-V}}$ [m/V]	3.578E-04	3.309E-04	3.184E-04

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