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Shocks and Defaults in Production Networks: a Theoretical and Simulation-Based Analysis

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Abstract

Modern economies are highly interconnected, as shocks in one sector can spill over across the entire network, generating aggregate fluctuations and increasing the systemic risk. In theory, such shocks would be less disruptive if industries could instantly adjust their decisions, optimizing their profits and avoiding defaults. In practice, however, production and investment choices cannot be costlessly or immediately adapted.

This thesis analyzes an economic equilibrium model in which firms make rigid, state-independent decisions prior to the realization of shocks. In the considered model, firms rely partly on debts financed by a bank and aim to maximize their expected profit. In this context, a unique rigid Walrasian equilibrium can be proven to exist, whereby defaults occur when equilibrium profit realizations are negative and loans remain partly unpaid.

The main contribution of this work is the use of the properties of the exponentially tilted distribution, its density function and moments, to characterize the default condition for a firm hit by a single node shock transmitted through one supplier. Sufficient moment based conditions are derived to ensure the existence of at most one default interval. We show that such an interval is always unique for exponential and Bernoulli shock distributions, while for gamma shock distributions the default interval is unique only for a range of the parameters that we characterize. Extending the analysis to a single node shock propagated through two suppliers, the results indicate that default intervals are still determined by the tilted moments of the distribution and coincide with those of the single supplier case.

Another key aspect is the implementation of numerical simulations on different network structures to compute equilibrium quantities, default probabilities and profit variances. Simulations show that shocks are more disruptive when they strike all nodes simultaneously, and may also affect firms not directly connected to the source. In simple structures, as line or DAG, shocks propagate only to consumers, with effects decreasing along the chain, while in cycles or real world networks they can also reach suppliers. Default probabilities remain unaffected by the presence of the bank, since they are only influenced by realized shocks. By contrast, profit variances depend on both firm centrality, determined by the debt cost, itself driven by the choice of the bank interest rates, and shock realizations. Consequently, industries with the highest centrality do not necessarily exhibit the largest profit variance.

Chapter 1

Introduction

The aim of this thesis is to analyze the effects of productivity shocks on production networks. We use an economic equilibrium model, in which firms make state-independent decisions prior to the realization of shocks, relying partly on debts financed by a bank. In this context, a generalized concept of the Walrasian equilibrium can be introduced and it is proved to be unique.

The considered model examines how shocks impact individual nodes and incorporates the possibility of default for them. The default for a given firm depends on the coefficients of the Leontief matrix, which represent the distance of the shock from that firm. A firm in the economy is in default if its equilibrium profit is negative, consequently its loans remain partly unpaid.

Hence, our objective is to analyze the profit and to characterize the default condition for a firm hit by a single node shock transmitted through one and two suppliers, depending on the shock distribution.

1.1 Disruptions in interconnected economies

The study of production networks is crucial because modern economies are highly interconnected [1]. Indeed, industries rarely operate in isolation, they depend on suppliers, consumers and financiers. This interdependence means that shocks affecting one firm or sector can propagate throughout the entire network [2]. In fact, a collapse in production not only affects the economic sector hit by the shock, but it also influences all the industries which are directly and indirectly linked with the shock's source.

An adverse event affecting a geographical area or an economic district directly disrupts the business activity of a set of firms that become incapable to fulfill their orders. This hits the productivity of firms downstream in the supply chain, but it might also affect the suppliers of the damaged firms that face an abrupt perturbation of their sales. This may trigger a domino effect [2] of canceled orders, delivery and payment delays affecting larger and larger pieces of the real economy. Hence, production networks can become an effective channel for propagating shocks that can hit firms far away from the disruption place.

In addition, the shock's effects can cause an unavoidable spillover on banks and financial intermediaries [3], [4], who provide credit to the real economy and are exposed to cascades of losses, since firms might be unable to repay debts to their financiers.

Globalization amplifies the systemic risk into production networks [5], which are threatened by natural catastrophes, pandemics and economic crisis. Although these phenomenons have always occurred, the undirected losses caused by worldwide rippling effects are particularly relevant and unexpected and show the inherent fragility of our interconnected economy.

1.2 Related literature

In 1941, Wassily Leontief demonstrated the interconnected structure of modern production systems with his input-output analysis, providing a framework to trace how shocks in one sector can ripple through the entire economy. Later, in 1983, Long and Plosser [1] showed how such shocks propagate along production chains in an interconnected economy.

After the 2008 crisis and its dramatic global implications, there was a significant renewed interest in the interconnected nature of economic and financial systems and on its role in propagating and amplifying shocks.

In 2012, Acemoglu [2] proposed a theoretical framework for production networks to analyze the effects of different exogenous shocks in interconnected economies. He studied a perfectly-competitive Cobb-Douglas economy, in which firms choose production levels contingent on the state of the world that occurs with probability one. Inputs of intermediate goods and labor depend on that state of the world, hence decisions are taken after the shock's realizations and are state-dependent, consequently there is no rigidity. The core of his theory is the concept of input-output matrix, which describes the transactions of goods and services among firms and leads to the concept of network. He also proved that the Bonacich centrality has a key role in production networks, showing that the effect of a productivity shock is proportional to its Bonacich centrality [6].

Acemoglu introduced the possibility that significant aggregate fluctuations may originate from idiosyncratic shocks due to the interconnections between different firms, leading to a shock propagation mechanism in the economy, called 'cascade effects'. He used the aggregate volatility, defined as the standard deviation of the log output, as a measure to assess the propagation of idiosyncratic shocks throughout the entire network.

The creation of macroeconomic fluctuations depends on the network structure. In fact, if the inter sectoral input-output linkages are symmetric, meaning that each sector relies equally on the outputs of all the other sectors, aggregate fluctuations cannot be produced. On the contrary, when the network is not symmetric, shocks to industries that are more important suppliers propagate more widely and thus do not wash out with the rest of the shocks upon aggregation. This happens for example when there are dominant sectors, meaning that a small number of sectors plays a disproportionately important role as input suppliers to others.

Furthermore, Acemoglu's model does not incorporate debt financing.

The need for external financing, still without rigidity, was introduced by Bigio and La'o (2020) [3], through a cash-in-advance mechanism. Inputs of intermediate goods and labor must be paid before revenues obtain, but after the occurrence of the shocks is revealed. Firms get loans from consumers themselves. The amount of the loan is state-dependent, so that there are never states of the world in which the cashed-in-advance outflow is greater than the optimal one, and cannot be refunded.

The cash-in-advance mechanism was used also by Huremovic (2020) [4], who studied the transmission of financial shocks to economic sectors.

Pellet and Tahbaz-Salehi (2023) [7] introduced the concepts of rigidity to capture the fact that, in the real world, decisions are state-independent. They must be taken before shocks are realized, when information about which shocks will occur is incomplete, and production processes require advance planning that cannot be immediately adjusted as the shock hits the economy.

Rigidity is different from the short versus long run adjustment to shocks embedded in Elliott and Jackson (2023) [5]. In that paper, indeed, shocks come totally unpredicted, hence firms do not readjust the input-output decisions, and their production is simply constrained by the input shortages generated by the primitive shocks. As these shortages are transmitted through the network, in the short run firms seek to minimize shock's effects by maximizing revenues.

1.3 Comparison between literature models and our approach

The economic literature regarding production networks, in particular Acemoglu models [2], [6] provides powerful insights, however it typically assumes ex-post adjustment. It means that firms adjust to shocks ex-post, hence after the shock is realized, knowing which state has occurred. Inputs of intermediate goods and labor depend on a specific state of the world, hence there is no rigidity. The consequences of real shocks are less disruptive, since producers can fully adapt their business decisions to shocks, reach zero profits at the equilibrium and pay back their debts, avoiding the default.

However, in the real world, decisions must be taken before shock's realizations and the shock's effect is not immediate, since production systems necessitate prior planning and there are lags in delivery of inputs. Furthermore, the speed of contagion depends on many variables, such as the nature of the original failures, the position of the defaulting nodes in the network and the global current state of the system. Fragilities emerge in specific critical patterns that depend on the topological structure of the network, on the distribution of disturbances, on the exchange flows among firms and on possible exposure against financial agents.

In order to capture these real world frictions, Pellet and Tahbaz-Salehi [7] assume ex-ante adjustment. It implies that firms adjust to shocks ex ante, meaning that decisions are taken before shock's realizations, when the information on which state of the world will occur is incomplete. In fact, industries are able to forecast which shocks may happen, nevertheless they do not know their actual realizations among the many possible ones. Then, they can revise their optimal input decisions accordingly, however rigidity prevents them from making state-contingent orders. Hence, they take decisions based on imperfect knowledge of possible shocks and they cannot change their investments and productions costlessly and instantaneously. They trade inputs at shock-consistent prices before the actual shock becomes true and they reach an unique Walrasian equilibrium, with well-defined quantities and prices, although this equilibrium is not attained instantaneously when the shock occurs.

The concept of rigidity is adopted also in our model [8], defined in detail in Chapter 2. However, while in Pellet and Tahbaz-Salehi [7] models there is no transmission of shocks from the real to the financial sector, since outside funding is absent and no banks are incorporated, in our model [8] it is present. In fact, our model includes a bank financing a fraction of firms' liabilities, who is paid back in full if profits are non-negative, and gets a partial recovery in case profits are negative. Hence, the financing does not depend on a cash-in-advance need, as in Bigio and La'o [3] and Huremovic [4].

In our model [8], when the shock actually hits, firms can end up with profits or losses. Indeed, in this scenario, profits have null mean, however they are not identically zero and can have any sign, hence the default might be realized. The default for a given firm depends on the coefficients of the Leontief matrix, which represent the distance of the shock from that firm. A firm in the economy is in default if its equilibrium profit is negative. The occurrence of default just depends on real shocks and on the network structure, while the magnitude of losses depends on the amplification provided by leverage, since firms rely on debts provided by the bank.

Hence, our approach differs from Pellet and Tahbaz-Salehi [7] model, because it incorporates the possibility of default and it enables to analyze the shock's effects on a single firm rather than on the global system.

1.4 Different types of shocks

Shocks can originate from multiple sources. They can be driven by political decisions like Brexit, terrorist acts like the 2001 September 11 suicide attacks in New York City, wars like the current Russian invasion of Ukraine, or pandemics like the COVID-19.

Shocks are generally classified in two types: productivity and demand [2].

Productivity shocks propagate downstream much more powerfully than upstream, meaning that downstream customers of directly hit industries are affected more strongly than their upstream suppliers. These shocks lead to an increase in the price of the sector's output hit by the shock, encouraging its customer industries to use this input less intensively and thus reduce their own production.

On the contrary, demand shocks propagate upstream mainly, implying that upstream suppliers are more heavily impacted than their downstream customers. These shocks have much more minor effects on prices as affected industries adjust their production levels and thus input demands.

The nature of propagation depends strongly on the structure of production functions. If production functions and consumer preferences are Cobb-Douglas [2], [6], there is a single factor of production and the assumption of constant return to scale, there is no upstream effect from supply-side shocks and no downstream effect from demand-side shocks.

On the contrary, when the production function is modeled through an elastic structure

[9], a negative productivity shock to industry i may not remain confined to i's downstream firms. Indeed, productivity shocks not only cause an increase in good i's price, which leads to a downstream propagation, but they may also result in reallocation of resources across different industries depending on the elasticities of substitution across various inputs. Hence, changes in industries' demand for intermediate productions induce also an upstream shock's propagation.

1.5 Network effects and shocks propagation

In order to understand shocks propagation, a distinction between first and higher order interconnections must be considered [6]. First-order effects capture the immediate transmission of shocks to directly connected industries, while higher-order effects account for propagation along longer chains of indirect connections. First order interconnections provide only partial information about the structure of the input-output relationships between different sectors, hence at least second order interconnections must be considered to evaluate the 'cascade effect'. These network effects are typically larger than the impact of sector-specific productivity shocks alone [2].

In fact, a negative productivity shock to industry j reduces its production and increases its price, adversely affecting all of the industries purchasing inputs from j. But this direct impact is further augmented in the competitive equilibrium because these first-round-affected industries change their production and prices, creating indirect negative effects on other customer industries, captured by the Leontief matrix. It can be expressed as the series of the power of A', where A is the input-output matrix: $L = (I - A')^{-1} = \sum_{h=0}^{+\infty} (A')^h$ and it shows how a firm depends on the rest of the network. In fact, the output of a node is affected not only by the exogenous shocks directly hitting it, but also by the exogenous shocks impacting its in-neighbors (suppliers) weighted by the effectiveness coefficient A_{ij}, as well as by those of the suppliers of its suppliers weighted by A^2_{ij}, and so on. In addition, the quantitative magnitude of shock's propagation through input-output networks is larger when different shocks are considered simultaneously.

1.6 Thesis structure

The rest of the thesis is structured as follows:

Chapter 2 presents the model, shows the existence of a unique Walrasian rigid equilibrium and derives explicit expressions for the equilibrium variables, illustrated with a simple example.

Chapter 3 defines the default event and analyzes the default condition for a firm affected by a single node shock transmitted through one supplier, using the properties of the exponentially tilted distribution. It contains sufficient moment-based conditions of the shock distribution to ensure the existence of at most one default interval, with explicit computations for exponential, gamma, and Bernoulli distributions. The chapter also extends the analysis to shocks propagated through two suppliers, focusing on simple examples and illustrating the default intervals for the same distributions of the single supplier case.

Chapter 4 contains model simulations on various network structures, including a line, a DAG, a cycle, and a real-world network. It focuses on the analysis of default probabilities, which depend only on realized shocks and studies how the profit variances are influenced by both shocks and firm centrality. It also examines how shocks affect both direct and indirect suppliers and consumers depending on network topology, highlighting the most critical nodes for large-scale propagation.

Chapter 5 summaries the main results of this thesis and proposes possible extensions for future works.

Appendix A contains the explicit results of default probabilities, profit variances and distorted Bonacich centralities for different network structures, while appendix B includes the code implemented in python to perform numerical simulations.

Chapter 2

Production network model

In this chapter, we present the economic model used for our analysis on production networks. We consider an economy composed of: firms, each producing a single good, used as input for the other firms and consumed by a representative household; a representative household, which provides labor and consumes the good and a bank, which finances firms through loans covering a fraction of their liabilities.

Our aim is to study the effects of productivity shocks on production networks in presence of debts, financed by a bank belonging to the economy [8]. The main assumption of our model is that firms can predict the distribution of shocks, anticipating the frequency and the severity of them, however they take decisions at time 0, before shock's realizations, independently from the scenario. The rigidity implies the inability to adapt ex ante to different states. In fact, the production cannot be modified immediately after the shock happens, because it requires advance planning and time to be realized and delivered; the productivity is revealed only at time 1. Hence, there is the risk to obtain negative profits at the equilibrium, leading to firms' defaults, which have also consequences on financiers because firms might not be able to repay the loans.

2.1 Model definition

We analyze a two-period economy with a set $\mathcal{V} := \{1, ..., n\}$ of $n \geq 1$ firms, they can be hit by production shocks and they produce a single homogeneous good, which is partially used as an intermediate input for other sectors and partially consumed by a representative household, which provides one unit of labor. All firms have the same production function and are substitutable. Industries borrow part of the capital they need to pay for their employed labor and consumption of intermediate goods from financiers, who are represented through a bank.

The production uncertainty is defined through a vector of non-positive random variables $\eta \in (\Omega, \mathcal{A}, \mathbb{P})$, whose entries $\eta_k \leq 0$ are the primitive log-productivity shocks on different sectors $k = 1, ..., n$. If η is deterministic there is no uncertainty.

2.1.1 Firms

The production of firm k is represented through a Cobb-Douglas function defined as:

$$y_k^\eta = e^{\eta_k} \varsigma_k l_k^{\beta_k} \prod_{j \in \mathcal{V}} (z_{jk}^\eta)^{A_{jk}}, \quad k = 1, ..., n \tag{2.1}$$

where y_k^η is the actual production output, l_k is the employed labor of sector k, z_{jk}^η is actual quantity produced by firm j and used in k's production, β_k is the labor share. A_{jk} is the importance of the good made by firm j in the production of firm k and ς is a positive normalization constant defined as $\varsigma_k = \beta_k^{-\beta_k} \prod_{j \in V} A_{jk}^{-A_{jk}}$. $A_{jk} > 0$ if and only if there is a direct link from node j to node k, meaning that the output of node j is used as input in the production of node k.

The output y_k^η and the intermediate goods z_{jk}^η are random, in fact they depend on the primitive shock η, while l_k is not directly affected by the shock. When $\eta = 0$, y_k^η, z_{jk}^η represent respectively the maximal output and the maximal intermediate quantity of good j for sector k.

Let w_k denote the unit cost of the employed labor and p_k the unit costs of goods produced by firm k. Therefore, the actual revenues and the actual liabilities, due to intermediate goods and labor, are respectively defined as:

$$\mathcal{A}_k(\eta) = p_k y_k^\eta = p_k e^{\eta_k} \varsigma_k l_k^{\beta_k} \prod_{j \in \mathcal{V}} (z_{jk}^\eta)^{A_{jk}} \tag{2.2}$$

$$\mathcal{L}_k(\eta) = \sum_{j \in \mathcal{V}} p_j z_{jk}^\eta + w l_k \tag{2.3}$$

Each industry borrows a fraction $\theta_k \in [0,1]$ of its liabilities $\mathcal{L}_k$ from the banks at an interest rate $r_k \geq 0$. The portion θ_k is exogenously determined, while r_k can be an exogenous or endogenous parameter as shown in section 2.1.3.

Firstly, firms pay for labor and intermediate goods, then, they give to the bank the minimum between the loan with its interests $((1 + r_k)\theta_k \mathcal{L}_k(\eta))$ and what remains to firms after the payment of labor and suppliers $(\mathcal{A}_k(\eta) - (1 - \theta_k)\mathcal{L}_k(\eta))$.

The firm's profit is the remainder of its assets once labor, other firms, and the bank have been paid:

$$\pi_k(\eta) = \mathcal{A}_k(\eta) - (1 + r_k \theta_k)\mathcal{L}_k(\eta) \tag{2.4}$$

If assets are not enough to pay other firms, labor and its debt to the bank plus interest, there is the default and the profit becomes negative.

Assuming risk neutrality, firms choose the maximal quantities of intermediate goods $(z_{jk})_{j \in \mathcal{V}}^0$ to order and the labor l_k to employ in order to maximize their expected profit:

$$\mathbb{E}[\pi_k(\eta)] \tag{2.5}$$

When a shock η hits the production, there is a proportional post-shock redistribution such that:

$$\frac{z_{jk}^\eta}{z_{jk}^0} = \frac{y_k^\eta}{y_k^0} \quad \forall j, k = 1, ..., n \ \text{ s.t. } \ z_{jk}^0 > 0 \tag{2.6}$$

The proportional mechanism in shock propagation implies that if the output of a good decreases of a given percentage with respect to its maximal value y_k^0, at time 1 the availability of that good for every firm j, that is a customer of sector k, is reduced of the same percentage with respect to the maximal value z_{jk}^0 ordered at time 0. This proportional mechanism depends on the homogeneity property of the model, meaning that the shock's effect is identical for every industry in the same sector k.

2.1.2 Representative household

The representative household consumes quantities c_k^η of the produced goods, its Cobb-Douglas utility function is defined as:

$$U(c^\eta) = \chi \prod_{k\in\mathcal{V}} (c_k^\eta)^{\gamma_k} \tag{2.7}$$

where $\gamma_k \geq 0$ are the consumer preference weight for the good produced by firm k, with the assumption that $\sum_{k\in\mathcal{V}} \gamma_k = 1$ and $\chi = \prod_k \gamma_k^{-\gamma_k}$ is a positive normalization constant.

Consumers make decisions at time 0 on the maximal quantities c_k^0 of different goods to order and solve an optimization problem, which consists in maximizing the expected utility:

$$\mathbb{E}[U(c^\eta)] \tag{2.8}$$

The actual consumptions are determined through a proportional rationing rule, similarly to the one described in equation (2.6) for firms, such that:

$$\frac{c_k^\eta}{c_k^0} = \frac{y_k^\eta}{y_k^0}, \quad \forall k = 1, ..., n \ \text{ s.t. } \ c_k^0 > 0 \tag{2.9}$$

The representative household receives wages as workers and gets profits as owners of both the firms and the bank, hence its total gain is:

$$\mathcal{E}(\eta) = w + \sum_{k\in\mathcal{V}} \pi_k(\eta) + \sum_{k\in\mathcal{V}} r_k\theta_k\mathcal{L}_k(\eta) \tag{2.10}$$

with the budget constraint:

$$\sum_{k\in\mathcal{V}} c_k^\eta p_k \leq \mathcal{E}(\eta) \tag{2.11}$$

2.1.3 Banks

Banks are a continuum, they operate in perfect competition and they are randomly matched with firms. We can consider a representative bank that finances firm k with a loan covering a fraction $\theta_k \in [0,1]$ of its actual liabilities.

We analyze two cases:

- The interest rates r_k and θ_k are fixed, since they are taken as exogenous parameters;

- the parameters θ_k are exogenous, while the interest rates r_k are endogenous, meaning that they are chosen by the banks, so that their expected profit from sector k is null.

  In order to describe the profit of a bank, we have to consider that the bank credit towards sector k is defined as $(1+r_k)\theta_k\mathcal{L}_k(\eta)$, while at the end of time 1, the available capital of firm k to pay back its debt to the bank is $[\mathcal{A}_k(\eta)-(1-\theta_k)\mathcal{L}_k(\eta)]_+$. Hence, the amount of credit the bank is able to cover from sector k is:

$$\min\{[\mathcal{A}_k(\eta) - (1-\theta_k)\mathcal{L}_k(\eta)]_+, (1+r_k)\theta_k\mathcal{L}_k(\eta)\} \tag{2.12}$$

  Consequently, the bank's profit is defined as:

$$\begin{aligned} \mathcal{I}_k(\eta) &= \min\{[\mathcal{A}_k(\eta) - (1-\theta_k)\mathcal{L}_k(\eta)]_+, (1+r_k)\theta_k\mathcal{L}_k(\eta)\} - \theta_k\mathcal{L}_k(\eta) \\ &= [\mathcal{A}_k(\eta) - \mathcal{L}_k(\eta)]_{-\theta_k\mathcal{L}_k(\eta)}^{r_k\theta_k\mathcal{L}_k(\eta)} \end{aligned} \tag{2.13}$$

  with the notation $[a]_b^c = \min\{\max\{a,b\},c\}$.

  Assuming risk neutrality, for computing the interest rates, the bank solves an optimization problem, which consists in imposing that its expected profit from sector k is null:

$$\mathbb{E}[\mathcal{I}_k(\eta)] = \mathbb{E}[[\mathcal{A}_k(\eta) - \mathcal{L}_k(\eta)]_{-\theta_k\mathcal{L}_k(\eta)}^{r_k\theta_k\mathcal{L}_k(\eta)}] = 0 \tag{2.14}$$

2.2 Analysis of the model at equilibrium

In the Cobb-Douglas economy (A,β,γ) described above, the constant-return-to-scale assumption implies that:

$$\alpha_k = \sum_{j\in\mathcal{V}} A_{jk}, \quad \alpha_k + \beta_k = 1,\ \forall k = 1,...n, \quad \sum_{k\in\mathcal{V}} \gamma_k = 1 \tag{2.15}$$

Definition 1. *A rigid Walrasian equilibrium in a constant-return-to-scale Cobb-Douglas economy (A,β,γ), with a financed fraction vector θ and a primitive log-production shock η, is a tuple $(y^0, z^0, c^0, l, r, p, w)$ such that:*

- *at time 0 every firm k chooses the employed labor l_k and the maximal quantities of intermediate goods $(z^0_{jk})_j$ in order to maximize its expected profit $\mathbb{E}[\pi_k(\eta)]$;*
- *the consumption vector c^0, chosen at time 0, maximizes the expected consumer utility $\mathbb{E}[U(c^\eta)]$ under the budget constraint (2.11),*
- *the banks financing sector k make zero expected profit $\mathbb{E}[\mathcal{I}_k(\eta)] = 0,\ \forall k = 1,...,n$;*
- *the clearing assumption for goods is:*

$$y_k^\eta = \sum_{j\in\mathcal{V}} z_{kj}^\eta + c_k^\eta \quad \forall k = 1,...,n; \tag{2.16}$$

- *the clearing assumption for labor is:*

$$\sum_{k\in\mathcal{V}} l_k = 1 \tag{2.17}$$

The constant returns-to-scale assumption (2.15) implies that the column sums of the matrix A are less or equal to one. Hence, A is a column-substochastic matrix and its spectral ray is strictly less than one. As a consequence, the Leontief inverse is well-defined, it can be expressed as the series of powers of A' and it has all non negative entries:

$$L = (I - A')^{-1} = \sum_{h=0}^{+\infty} (A')^h \tag{2.18}$$

A primary role in shocks' propagation in the economy is played by the random vector of the total log-production shocks defined as:

$$\rho_k = \sum_{j \in \mathcal{V}} L_{kj} \eta_j \tag{2.19}$$

Through the Leontief matrix L, it accounts for all indirect effects of the primary log-production shocks in the industries upstream to sector k in the production network. Hence, the total shock is made by the primitive shock η and by the network-induced shock generated from the propagation of the primitive shock in the network.

Proposition 1. *In a constant-returns-to-scale economy* (A, β, γ) *with* $\rho(A) < 1$ *and primitive shock* η*, the actual productions, intermediate quantities and household consumptions satisfy:*

$$y_k^\eta = e^{\rho_k} y_k^0, \quad z_{jk}^\eta = e^{\rho_j} z_{jk}^0, \quad c_k^\eta = e^{\rho_k} c_k^0, \quad \forall k, j = 1, ..., n \tag{2.20}$$

The equation (2.20) is compatible with proportional rules (2.6) and (2.9) and it implies that if equations for production (2.1), utility (2.7) and clearing condition for goods (2.16) are satisfied by the maximal quantities y_k^0, z_{jk}^0, c_k^0, then they are verified also by the actual quantities $y_k^\eta, z_{jk}^\eta, c_k^\eta$ for every realization of η.

We can introduce the sector normalized total shock, which is a random vector that measures the transmission of the shock to a specific sector because of the whole network:

$$\tau_k = \frac{e^{\rho_k}}{\mathbb{E}[e^{\rho_k}]} \tag{2.21}$$

We can also define the normalized suppliers' total shock weighted by its importance, it measures how the neighbors nodes hit firm k:

$$\varepsilon_k = \sum_{j \in \mathcal{V}} \tau_j A_{jk} + \beta_k \tag{2.22}$$

We have that:

$$\mathbb{E}[\tau_k] = 1, \ \forall k = 1, ..., n \tag{2.23}$$

consequently, using (2.15), we obtain that:

$$\mathbb{E}[\varepsilon_k] = 1, \ \forall k = 1, ..., n \tag{2.24}$$

Moreover, when the interest rates r_k are chosen exogenously, we can define the primitive cost of debt on sector k as:

$$\zeta_k = \log(1 + r_k \theta_k) \tag{2.25}$$

It is a deterministic quantity, which depends on the values of θ_k, r_k that are fixed.

On the contrary, when the interest rates are endogenous, the primitive cost of debt ζ_k can be derived from the following proposition.

Proposition 2. *For every sector k in $\mathcal{V}$ and leverage value θ_k in $[0,1]$, the equation:*

$$\mathbb{E}[[e^{\zeta_k}\tau_k]_{(1-\theta_k)\epsilon_k}^{e^{\zeta_k}\epsilon_k}] = \mathbb{E}[\max\{e^{\zeta_k}\min\{\tau_k,\epsilon_k\},(1-\theta_k)\epsilon_k\}] = 1 \tag{2.26}$$

admits a unique nonnegative solution $\zeta_k = \zeta_k(\theta_k)$.

Moreover, such solution is non-decreasing as a function of θ_k, with:

$$\zeta_k(0) = 0; \quad \zeta_k(1) = -\log \mathbb{E}[\min\{\tau_k,\epsilon_k\}] \tag{2.27}$$

Proposition 2 ensures that the cost of debt ζ_k for a sector k is well defined and it is a monotone non-decreasing function of the leverage θ_k, ranging from a minimum value $\zeta_k(0) = 0$ when $\theta_k = 0$, to a maximum value $\zeta_k(1) = -\log \mathbb{E}[\min\{\tau_k,\epsilon_k\}]$ achieved when $\theta_k = 1$.

In order to compute the values of ζ_k from equation (2.26), we need to find the zeros of the following function:

$$f(x,\theta_k) = \mathbb{E}[[x\tau_k]_{(1-\theta_k)\epsilon_k}^{x\epsilon_k}] - 1 = \mathbb{E}[\max\{x\min\{\tau_k,\epsilon_k\},(1-\theta_k)\epsilon_k\}] - 1 \tag{2.28}$$

with $x = e^{\zeta_k}$. Then, $\zeta_k = \log(x(\theta_k))$ satisfies $f(e^{\zeta_k},\theta_k) = 0$.

The total network induced cost of debt measures the additional cost to firms caused by the propagation of the leverage in the network and it represents the magnitude corresponding to $\rho = L\eta$. It is defined as:

$$\xi_k = \sum_{j\in\mathcal{V}} L_{kj}\zeta_j, \quad \forall k = 1,...,n \tag{2.29}$$

We can introduce the distorted Leontief matrix:

$$L^{\zeta} = (I - e^{-[\zeta]}A')^{-1} \tag{2.30}$$

where $e^{-[\zeta]}$ is the diagonal matrix with the discount factor $e^{-\zeta_k}$ for different sectors $k = 1,...,n$ on the main diagonal.

We consider also the normalized distorted centrality:

$$v_k^{\zeta} = \frac{1}{\psi(\zeta)}\sum_{j\in\mathcal{V}} L_{jk}^{\zeta}\gamma_j, \quad \forall k = 1,...,n \tag{2.31}$$

where $\psi(\zeta) = \sum_{j\in\mathcal{V}}\sum_{k\in\mathcal{V}}\gamma_j L_{jk}^{\zeta}\beta_k e^{-\zeta_k}$ is the normalization factor. When $\zeta = 0$, we obtain the standard Leontief matrix and v_k^{ζ} reduces to the Bonacich centrality:

$$v_k^0 = \sum_{j\in\mathcal{V}} L_{jk}\gamma_j \tag{2.32}$$

The following theorem establishes the existence of an unique Walrasian equilibrium.

Theorem 1. *Consider a constant-return-to-scale Cobb-Douglas economy* (A, β, γ)*, a financed fraction vector* θ *and a primitive log-production shock* η*. Then, there exists a unique rigid Walrasian equilibrium* $(y^0, z^0, c^0, l, r, p, w)$*.*

Moreover, let $\tau_k, \zeta_k, v_k^\zeta$ *be respectively the normalized total shock, the distortion and the distorted centrality of each sector* $k = 1, ..., n$ *defined respectively in* (2.21)*,* (2.25)*,* (2.31)*. Then, we obtain that the values of the following quantities at the equilibrium are:*

- *maximal productions:*
$$y_k^0 = v_k^\zeta e^{-\xi_k}; \tag{2.33}$$
- *maximal intermediate quantities:*
$$z_{jk}^0 = v_k^\zeta A_{jk} e^{-\zeta_k - \xi_j}; \tag{2.34}$$
- *employed labor:*
$$l_k = v_k^\zeta \beta_k e^{-\zeta_k}; \tag{2.35}$$
- *maximal household's consumption:*
$$c_k^0 = \frac{\gamma_k e^{-\xi_k}}{\psi(\zeta)}; \tag{2.36}$$
- *interest rates: if* $\theta_k > 0$*, then*
$$r_k = \frac{e^{\zeta_k} - 1}{\theta_k}; \tag{2.37}$$
- *prices over wage:*
$$\frac{p_k}{w} = \frac{e^{\xi_k}}{\mathbb{E}[e^{\rho_k}]} \tag{2.38}$$

Proof. the proof can be found in [8]. □

The term $e^{-\xi_k}$ represents the discount factor of sector k computed in correspondence of the total cost of debt, while the term $e^{-\zeta_k} = \frac{1}{1+r_k\theta_k}$ (obtained from equation (2.37)) is the discount factor of sector k computed in correspondence of the primitive cost of debt. We can observe that (2.37) is equivalent to (2.25), meaning that the discount factor on sector k is equivalent to the primitive cost of debt.

Observation 1. *Recall that the superscript* 0 *in the quantities* y_k^0, z_{jk}^0, c_k^0 *denotes the equilibrium values corresponding to the case* $\eta = 0$*. In this benchmark scenario, production, intermediate inputs, and consumption attain their maximal levels, since no shock is realized. When a shock* η *occurs, these quantities are rescaled by the factor* e^ρ *and we denote the resulting actual values by the superscript* η*. Hence, the notation distinguishes between the maximal equilibrium quantities (*$\eta = 0$*) and the shock-dependent equilibrium quantities (at realized* η*).*

Theorem 1 and Proposition 1 together imply the following result for actual quantities:

Corollary 1. *Consider a constant-return-to-scale Cobb-Douglas economy (A, β, γ), a financed fraction vector θ and a primitive log-production shock η. Then, at the rigid Walrasian equilibrium, we have that:*

- *actual productions:*
$$y_k^\eta = v_k^\zeta e^{\rho_k - \xi_k}; \tag{2.39}$$
- *actual intermediate quantities:*
$$z_{jk}^\eta = v_k^\zeta A_{jk} e^{\rho_j - \zeta_k - \xi_j}; \tag{2.40}$$
- *actual employed labor:*
$$l_k = v_k^\zeta \beta_k e^{-\zeta_k}; \tag{2.41}$$
- *actual household's consumption:*
$$c_k^\eta = \frac{\gamma_k e^{\rho_k - \xi_k}}{\psi(\zeta)}; \tag{2.42}$$
- *actual welfare:*
$$U(c^\eta) = \frac{e^{\sum_k v_k^0(\eta_k - \zeta_k)}}{\psi(\zeta)}; \tag{2.43}$$
- *actual profits:*
$$\pi_k(\eta) = w v_k^\zeta (\tau_k - \varepsilon_k) \tag{2.44}$$

Proof. the proof can be found in [8]. □

Corollary 1 shows that ,when the shock η is realized, productions, intermediate quantities and consumption are smaller than their maximal values because they are multiplied by the factor e^ρ since they are affected by the shock; they are called actual quantities. On the contrary, the labor is not influenced by shocks.

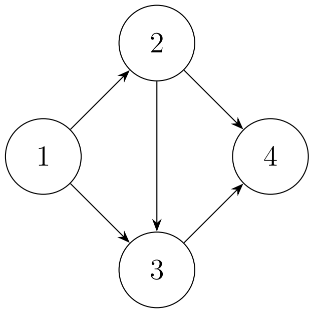
Corollary 1 contains the expression of equilibrium profits. It is important to notice that, although the expected profits are 0 for equation (2.23), (2.24), the random variable π_k can assume both positive or negative values (default event). Moreover, it can be split into two terms v_k^ζ and $\tau_k - \epsilon_k$.

The first one is always positive and it is the distorted Bonacich centrality, it depends on the bank's loans and it is not influenced by log-productions shocks. The presence of the bank amplifies profits or losses if and only if the distorted centrality index without debt is smaller than the same index with debts, meaning that $v_k^0 < v_k^\zeta$.

On the contrary, the second term can have any sign and it is linked exclusively to total shocks of the firm and its suppliers, in fact it does not depend on the interest rates and on the extent of the loans with the bank.

2.3 Example

We suppose to have the following network with independent exponential shocks $\eta = [\eta_1, \eta_2, \eta_3, \eta_4]$ on all nodes with parameters $\lambda = [\lambda_1, \lambda_2, \lambda_3, \lambda_4]$ respectively:



The adjacency matrix is:

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & 0 \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Consequently, the Leontief matrix $L = (I - A')^{-1}$ is:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{12} & 1 & 0 & 0 \\ a_{12}a_{23} + a_{13} & a_{23} & 1 & 0 \\ a_{12}(a_{23}a_{34} + a_{24}) + a_{13}a_{34} & a_{23}a_{34} + a_{24} & a_{34} & 1 \end{pmatrix}$$

For each node the total shock defined by (2.19) is obtained as:

$$\rho_1 = L_{11}\eta_1 + L_{12}\eta_2 + L_{13}\eta_3 + L_{14}\eta_4 = \eta_1$$

$$\rho_2 = L_{21}\eta_1 + L_{22}\eta_2 + L_{23}\eta_3 + L_{24}\eta_4 = a_{12}\eta_1 + \eta_2$$

$$\rho_3 = L_{31}\eta_1 + L_{32}\eta_2 + L_{33}\eta_3 + L_{34}\eta_4 = (a_{12}a_{23} + a_{13})\eta_1 + a_{23}\eta_2 + \eta_3$$

$$\rho_4 = L_{41}\eta_1 + L_{42}\eta_2 + L_{43}\eta_3 + L_{44}\eta_4 = (a_{12}a_{23}a_{34} + a_{12}a_{24} + a_{13}a_{34})\eta_1 + (a_{23}a_{34} + a_{24})\eta_2 + a_{34}\eta_3 + \eta_4$$

The normalized total shock defined in equation (2.21) is:

$$\tau_1 = \frac{e^{\rho_1}}{\mathbb{E}[e^{\rho_1}]} = \frac{e^{\eta_1}}{\mathbb{E}[e^{\eta_1}]} = \frac{e^{\eta_1}}{\frac{\lambda_1}{\lambda_1+1}}$$

$$\tau_2 = \frac{e^{\rho_2}}{\mathbb{E}[e^{\rho_2}]} = \frac{e^{a_{12}\eta_1+\eta_2}}{\mathbb{E}[e^{a_{12}\eta_1+\eta_2}]} = \frac{e^{a_{12}\eta_1+\eta_2}}{\mathbb{E}[e^{a_{12}\eta_1}]\mathbb{E}[e^{\eta_2}]} = \frac{e^{a_{12}\eta_1+\eta_2}}{\frac{\lambda_1}{\lambda_1+a_{12}}\frac{\lambda_2}{\lambda_2+1}}$$

$$\tau_3 = \frac{e^{\rho_3}}{\mathbb{E}[e^{\rho_3}]} = \frac{e^{(a_{12}a_{23}+a_{13})\eta_1+a_{23}\eta_2+\eta_3}}{\mathbb{E}[e^{(a_{12}a_{23}+a_{13})\eta_1+a_{23}\eta_2+\eta_3}]} = \frac{e^{(a_{12}a_{23}+a_{13})\eta_1+a_{23}\eta_2+\eta_3}}{\mathbb{E}[e^{(a_{12}a_{23}+a_{13})\eta_1}]\mathbb{E}[e^{a_{23}\eta_2}]\mathbb{E}[e^{\eta_3}]} =$$

$$\tau_4 = \frac{e^{\rho_4}}{\mathbb{E}[e^{\rho_4}]} = \frac{e^{(a_{12}a_{23}a_{34}+a_{12}a_{24}+a_{13}a_{34})\eta_1+(a_{23}a_{34}+a_{24})\eta_2+a_{34}\eta_3+\eta_4}}{\mathbb{E}[e^{(a_{12}a_{23}a_{34}+a_{12}a_{24}+a_{13}a_{34})\eta_1+(a_{23}a_{34}+a_{24})\eta_2+a_{34}\eta_3+\eta_4}]} =$$

$$\frac{e^{(a_{12}a_{23}a_{34}+a_{12}a_{24}+a_{13}a_{34})\eta_1+(a_{23}a_{34}+a_{24})\eta_2+a_{34}\eta_3+\eta_4}}{\mathbb{E}[e^{(a_{12}a_{23}a_{34}+a_{12}a_{24}+a_{13}a_{34})\eta_1}]\mathbb{E}[e^{(a_{23}a_{34}+a_{24})\eta_2}]\mathbb{E}[e^{a_{34}\eta_3}]\mathbb{E}[e^{\eta_4}]}$$

$$= \frac{e^{(a_{12}a_{23}a_{34}+a_{12}a_{24}+a_{13}a_{34})\eta_1+(a_{23}a_{34}+a_{24})\eta_2+a_{34}\eta_3+\eta_4}}{\frac{\lambda_1}{\lambda_1+a_{12}a_{23}a_{34}+a_{12}a_{24}+a_{13}a_{34}}\frac{\lambda_2}{\lambda_2+a_{23}a_{34}+a_{24}}\frac{\lambda_3}{\lambda_3+a_{34}}\frac{\lambda_4}{\lambda_4+1}}$$

The expected values in the denominator of τ vectors are computing exploiting the fact that the exponential shocks on all nodes are independent and using the result coming from Table 3.1 about the exponentially tilted density function of an exponential distribution.

The normalized suppliers shock defined in equation (2.22) is:

$$\epsilon_1 = \tau_1 A_{11} + \tau_2 A_{21} + \tau_3 A_{31} + \tau_4 A_{41} + \beta_1 = 1$$

$$\epsilon_2 = \tau_1 A_{12} + \tau_2 A_{22} + \tau_3 A_{32} + \tau_4 A_{42} + \beta_2 = 1 + a_{12}(\tau_1 - 1)$$

$$\epsilon_3 = \tau_1 A_{13} + \tau_2 A_{23} + \tau_3 A_{33} + \tau_4 A_{43} + \beta_3 = 1 + a_{13}(\tau_1 - 1) + a_{23}(\tau_2 - 1)$$

$$\epsilon_4 = \tau_1 A_{14} + \tau_2 A_{24} + \tau_3 A_{34} + \tau_4 A_{44} + \beta_4 = 1 + a_{24}(\tau_2 - 1) + a_{34}(\tau_3 - 1)$$

In this example we consider exogenous interest rates, then the primitive debt cost is deterministic for each node and it is expressed by equation (2.25):

$$\zeta_1 = \log(1 + r_1\theta_1)$$

$$\zeta_2 = \log(1 + r_2\theta_2)$$

$$\zeta_3 = \log(1 + r_3\theta_3)$$

$$\zeta_4 = \log(1 + r_4\theta_4)$$

As a consequence, the total debt cost defined in equation (2.29) is obtained as:

$$\xi_1 = L_{11}\zeta_1 + L_{12}\zeta_2 + L_{13}\zeta_3 + L_{14}\zeta_4 = \zeta_1$$

$$\xi_2 = L_{21}\zeta_1 + L_{22}\zeta_2 + L_{23}\zeta_3 + L_{24}\zeta_4 = a_{12}\zeta_1 + \zeta_2$$

$$\xi_3 = L_{31}\zeta_1 + L_{32}\zeta_2 + L_{33}\zeta_3 + L_{34}\zeta_4 = (a_{12}a_{23} + a_{13})\zeta_1 + a_{23}\zeta_2 + \zeta_3$$

$$\xi_4 = L_{41}\zeta_1 + L_{42}\zeta_2 + L_{43}\zeta_3 + L_{44}\zeta_4 = (a_{12}a_{23}a_{34} + a_{12}a_{24} + a_{13}a_{34})\zeta_1 + (a_{23}a_{34} + a_{24})\zeta_2 + a_{34}\zeta_3 + \zeta_4$$

Since the diagonal matrix $e^{-\zeta}$ can be written as:

$$e^{-\zeta} = \begin{pmatrix} e^{-\zeta_1} & 0 & 0 & 0 \\ 0 & e^{-\zeta_2} & 0 & 0 \\ 0 & 0 & e^{-\zeta_3} & 0 \\ 0 & 0 & 0 & e^{-\zeta_4} \end{pmatrix} = \begin{pmatrix} e^{-\log(1+r_1\theta_1)} & 0 & 0 & 0 \\ 0 & e^{-\log(1+r_2\theta_2)} & 0 & 0 \\ 0 & 0 & e^{-\log(1+r_3\theta_3)} & 0 \\ 0 & 0 & 0 & e^{-\log(1+r_4\theta_4)} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{1+r_1\theta_1} & 0 & 0 & 0 \\ 0 & \frac{1}{1+r_2\theta_2} & 0 & 0 \\ 0 & 0 & \frac{1}{1+r_3\theta_3} & 0 \\ 0 & 0 & 0 & \frac{1}{1+r_4\theta_4} \end{pmatrix}$$

The distorted Leontief matrix $L^\zeta = (I - e^{-\zeta}A')^{-1}$ obtained is:

$$L^\zeta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{a_{12}}{1+r_2\theta_2} & 1 & 0 & 0 \\ \frac{a_{12}a_{23}}{(1+r_2\theta_2)(1+r_3\theta_3)} + \frac{a_{13}}{1+r_3\theta_3} & \frac{a_{23}}{1+r_3\theta_3} & 1 & 0 \\ \frac{a_{12}a_{23}a_{34}}{(1+r_2\theta_2)(1+r_3\theta_3)(1+r_4\theta_4)} + \frac{a_{13}a_{34}}{(1+r_3\theta_3)(1+r_4\theta_4)} + \frac{a_{12}a_{24}}{(1+r_2\theta_2)(1+r_4\theta_4)} & \frac{a_{23}a_{34}}{(1+r_3\theta_3)(1+r_4\theta_4)} + \frac{a_{24}}{1+r_4\theta_4} & \frac{a_{34}}{1+r_4\theta_4} & 1 \end{pmatrix}$$

The Bonacich centrality defined in equation (2.32) is:

$$v_1^0 = L_{11}\gamma_1 + L_{21}\gamma_2 + L_{31}\gamma_3 + L_{41}\gamma_4 = \gamma_1 + a_{12}\gamma_2 + (a_{12}a_{23} + a_{13})\gamma_3 + (a_{12}a_{23}a_{34} + a_{12}a_{24} + a_{13}a_{34})\gamma_4$$

$$v_2^0 = L_{12}\gamma_1 + L_{22}\gamma_2 + L_{32}\gamma_3 + L_{42}\gamma_4 = \gamma_2 + a_{23}\gamma_3 + (a_{23}a_{34} + a_{24})\gamma_4$$

$$v_3^0 = L_{13}\gamma_1 + L_{23}\gamma_2 + L_{33}\gamma_3 + L_{43}\gamma_4 = \gamma_3 + a_{34}\gamma_4$$

$$v_4^0 = L_{14}\gamma_1 + L_{24}\gamma_2 + L_{34}\gamma_3 + L_{44}\gamma_4 = \gamma_4$$

In order to find the distorted Bonacich centrality expressed by equation (2.31), we need to compute the normalization factor $\psi(\zeta)$:

$$\begin{gathered} \psi(\zeta) = \gamma_1 L^\zeta_{11}\beta_1 e^{-\zeta_1} + \gamma_2 L^\zeta_{21}\beta_1 e^{-\zeta_1} + \gamma_3 L^\zeta_{31}\beta_1 e^{-\zeta_1} + \gamma_4 L^\zeta_{41}\beta_1 e^{-\zeta_1} + \gamma_1 L^\zeta_{12}\beta_2 e^{-\zeta_2} + \gamma_2 L^\zeta_{22}\beta_2 e^{-\zeta_2} + \\ \gamma_3 L^\zeta_{32}\beta_2 e^{-\zeta_2} + \gamma_4 L^\zeta_{42}\beta_2 e^{-\zeta_2} + \gamma_1 L^\zeta_{13}\beta_3 e^{-\zeta_3} + \gamma_2 L^\zeta_{23}\beta_3 e^{-\zeta_3} + \gamma_3 L^\zeta_{33}\beta_3 e^{-\zeta_3} + \\ \gamma_4 L^\zeta_{43}\beta_3 e^{-\zeta_3} + \gamma_1 L^\zeta_{14}\beta_4 e^{-\zeta_4} + \gamma_2 L^\zeta_{24}\beta_4 e^{-\zeta_4} + \gamma_3 L^\zeta_{34}\beta_4 e^{-\zeta_4} + \gamma_4 L^\zeta_{44}\beta_4 e^{-\zeta_4} = \\ \frac{1}{1+r_1\theta_1}(\gamma_1 + \gamma_2 L^\zeta_{21} + \gamma_3 L^\zeta_{31} + \gamma_4 L^\zeta_{41}) + \frac{(1-a_{12})}{1+r_2\theta_2}(\gamma_2 + \gamma_3 L^\zeta_{32} + \gamma_4 L^\zeta_{42}) + \\ \frac{(1-a_{13}-a_{23})}{1+r_3\theta_3}(\gamma_3 + \gamma_4 L^\zeta_{43}) + \frac{(1-a_{24}-a_{34})}{1+r_4\theta_4}\gamma_4 \end{gathered}$$

The distorted Bonacich centrality is:

$$v_1^\zeta = \frac{1}{\psi}(L^\zeta_{11}\gamma_1 + L^\zeta_{21}\gamma_2 + L^\zeta_{31}\gamma_3 + L^\zeta_{41}\gamma_4) = \frac{1}{\psi}(\gamma_1 + L^\zeta_{21}\gamma_2 + L^\zeta_{31}\gamma_3 + L^\zeta_{41}\gamma_4)$$

$$v_2^\zeta = \frac{1}{\psi}(L^\zeta_{12}\gamma_1 + L^\zeta_{22}\gamma_2 + L^\zeta_{32}\gamma_3 + L^\zeta_{42}\gamma_4) = \frac{1}{\psi}(\gamma_2 + L^\zeta_{32}\gamma_3 + L^\zeta_{42}\gamma_4)$$

$$v_3^\zeta = \frac{1}{\psi}(L^\zeta_{13}\gamma_1 + L^\zeta_{23}\gamma_2 + L^\zeta_{33}\gamma_3 + L^\zeta_{43}\gamma_4) = \frac{1}{\psi}(\gamma_3 + L^\zeta_{43}\gamma_4)$$

$$v_4^\zeta = \frac{1}{\psi}(L^\zeta_{14}\gamma_1 + L^\zeta_{24}\gamma_2 + L^\zeta_{34}\gamma_3 + L^\zeta_{44}\gamma_4) = \frac{1}{\psi}(\gamma_4)$$

We now want to compute the maximal quantities at the equilibrium:

- the production vector expressed by equation (2.33) is:

$$y_1^0 = v_1^\zeta e^{-\xi_1} = \frac{1}{\psi}(\gamma_1 + L_{21}^\zeta\gamma_2 + L_{31}^\zeta\gamma_3 + L_{41}^\zeta\gamma_4)e^{-\zeta_1}$$

$$y_2^0 = v_2^\zeta e^{-\xi_2} = \frac{1}{\psi}(\gamma_2 + L_{32}^\zeta\gamma_3 + L_{42}^\zeta\gamma_4)e^{-a_{12}\zeta_1-\zeta_2}$$

$$y_3^0 = v_3^\zeta e^{-\xi_3} = \frac{1}{\psi}(\gamma_3 + L_{43}^\zeta\gamma_4)e^{-(a_{12}a_{23}+a_{13})\zeta_1-a_{23}\zeta_2-\zeta_3}$$

$$y_4^0 = v_4^\zeta e^{-\xi_4} = \frac{1}{\psi}(\gamma_4)e^{-(a_{12}a_{23}a_{34}+a_{12}a_{24}+a_{13}a_{34})\zeta_1-(a_{23}a_{34}+a_{24})\zeta_2-a_{34}\zeta_3-\zeta_4}$$

- the intermediate quantities defined in equation (2.34) are expressed by the matrix:

$$z_{jk}^0 = \begin{pmatrix} 0 & v_2^\zeta a_{12}e^{-\zeta_2-\xi_1} & v_3^\zeta a_{13}e^{-\zeta_3-\xi_1} & 0 \\ 0 & 0 & v_3^\zeta a_{23}e^{-\zeta_3-\xi_2} & v_4^\zeta a_{24}e^{-\zeta_4-\xi_2} \\ 0 & 0 & 0 & v_4^\zeta a_{34}e^{-\zeta_4-\xi_3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{\psi}\frac{(\gamma_2+L_{32}^\zeta\gamma_3+L_{42}^\zeta\gamma_4)a_{12}}{1+r_2\theta_2}e^{-\zeta_1} & \frac{1}{\psi}\frac{(\gamma_3+L_{43}^\zeta\gamma_4)a_{13}}{1+r_3\theta_3}e^{-\zeta_1} & 0 \\ 0 & 0 & \frac{1}{\psi}\frac{(\gamma_3+L_{43}^\zeta\gamma_4)a_{23}}{1+r_3\theta_3}e^{-a_{12}\zeta_1-\zeta_2} & \frac{1}{\psi}\frac{\gamma_4 a_{24}}{1+r_4\theta_4}e^{-a_{12}\zeta_1-\zeta_2} \\ 0 & 0 & 0 & \frac{1}{\psi}\frac{\gamma_4 a_{34}}{1+r_4\theta_4}e^{-(a_{12}a_{23}+a_{13})\zeta_1-a_{23}\zeta_2-\zeta_3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- the labor expressed by equation (2.35) is:

$$l_1 = v_1^\zeta\beta_1 e^{-\zeta_1} = \frac{1}{\psi}\frac{(\gamma_1 + L_{21}^\zeta\gamma_2 + L_{31}^\zeta\gamma_3 + L_{41}^\zeta\gamma_4)}{1+r_1\theta_1}$$

$$l_2 = v_2^\zeta\beta_2 e^{-\zeta_2} = \frac{1}{\psi}\frac{(\gamma_2 + L_{32}^\zeta\gamma_3 + L_{42}^\zeta\gamma_4)(1-a_{12})}{1+r_2\theta_2}$$

$$l_3 = v_3^\zeta\beta_3 e^{-\zeta_3} = \frac{1}{\psi}\frac{(\gamma_3 + L_{43}^\zeta\gamma_4)(1-a_{13}-a_{23})}{1+r_3\theta_3}$$

$$l_4 = v_4^\zeta\beta_4 e^{-\zeta_4} = \frac{1}{\psi}\frac{\gamma_4(1-a_{24}-a_{34})}{1+r_4\theta_4}$$

- the consumption vector defined in equation (2.36) is:

$$c_1^0 = \frac{\gamma_1 e^{-\xi_1}}{\psi} = \frac{\gamma_1 e^{-\zeta_1}}{\psi}$$

$$c_2^0 = \frac{\gamma_2 e^{-\xi_2}}{\psi} = \frac{\gamma_2 e^{-a_{12}\zeta_1-\zeta_2}}{\psi}$$

$$c_3^0 = \frac{\gamma_3 e^{-\xi_3}}{\psi} = \frac{\gamma_3 e^{-(a_{12}a_{23}+a_{13})\zeta_1-a_{23}\zeta_2-\zeta_3}}{\psi}$$

$$c_4^0 = \frac{\gamma_4 e^{-\xi_4}}{\psi} = \frac{\gamma_4 e^{-(a_{12}a_{23}a_{34}+a_{12}a_{24}+a_{13}a_{34})\zeta_1-(a_{23}a_{34}+a_{24})\zeta_2-a_{34}\zeta_3-\zeta_4}}{\psi}$$

- prices over wages expressed by equation (2.38) are defined as:

$$\frac{p_1}{w} = \frac{e^{\xi_1}}{\mathbb{E}[e^{\rho_1}]} = \frac{e^{\zeta_1}}{\mathbb{E}[e^{\eta_1}]}$$

$$\frac{p_2}{w} = \frac{e^{\xi_2}}{\mathbb{E}[e^{\rho_2}]} = \frac{e^{a_{12}\zeta_1+\zeta_2}}{\mathbb{E}[e^{a_{12}\eta_1+\eta_2}]}$$

$$\frac{p_3}{w} = \frac{e^{\xi_3}}{\mathbb{E}[e^{\rho_3}]} = \frac{e^{(a_{12}a_{23}+a_{13})\zeta_1+a_{23}\zeta_2+\zeta_3}}{\mathbb{E}[e^{(a_{12}a_{23}+a_{13})\eta_1+a_{23}\eta_2+\eta_3}]}$$

$$\frac{p_4}{w} = \frac{e^{\xi_4}}{\mathbb{E}[e^{\rho_4}]} = \frac{e^{(a_{12}a_{23}a_{34}+a_{12}a_{24}+a_{13}a_{34})\zeta_1+(a_{23}a_{34}+a_{24})\zeta_2+a_{34}\zeta_3+\zeta_4}}{\mathbb{E}[e^{(a_{12}a_{23}a_{34}+a_{12}a_{24}+a_{13}a_{34})\eta_1+(a_{23}a_{34}+a_{24})\eta_2+a_{34}\eta_3+\eta_4}]}$$

The actual quantities at the equilibrium are found as:

- actual production:

$$y_1^\eta = y_1^0 e^{\rho_1};\; y_2^\eta = y_2^0 e^{\rho_2};\; y_3^\eta = y_3^0 e^{\rho_3};\; y_4^\eta = y_4^0 e^{\rho_4}$$

- actual consumption:

$$c_1^\eta = c_1^0 e^{\rho_1};\; c_2^\eta = c_2^0 e^{\rho_2};\; c_3^\eta = c_3^0 e^{\rho_3};\; c_4^\eta = c_4^0 e^{\rho_4}$$

- actual intermediate quantities:

$$z_{jk}^\eta = z_{jk}^0 e^{\rho_j}$$

In our example, for each firm, the profit, defined in equation (2.44), becomes:

$$\pi_1 = wv_1^\zeta(\tau_1 - \epsilon_1) = \frac{w}{\psi}(\gamma_1 + L_{21}^\zeta\gamma_2 + L_{31}^\zeta\gamma_3 + L_{41}^\zeta\gamma_4)(\tau_1 - 1)$$

$$\pi_2 = wv_2^\zeta(\tau_2 - \epsilon_2) = \frac{w}{\psi}(\gamma_2 + L_{32}^\zeta\gamma_3 + L_{42}^\zeta\gamma_4)(\tau_2 - 1 - a_{12}(\tau_1 - 1))$$

$$\pi_3 = wv_3^\zeta(\tau_3 - \epsilon_3) = \frac{w}{\psi}(\gamma_3 + L_{43}^\zeta\gamma_4)(\tau_3 - 1 - a_{13}(\tau_1 - 1) - a_{23}(\tau_2 - 1))$$

$$\pi_4 = wv_4^\zeta(\tau_4 - \epsilon_4) = \frac{w}{\psi}\gamma_4(\tau_4 - 1 - a_{24}(\tau_2 - 1) - a_{34}(\tau_3 - 1))$$

The results of this example are shown in section 4.1, where we perform a numerical simulation to verify the computations.

Chapter 3

Default analysis

In this chapter, we define the default condition and we analyze in detail the mechanism through which a firm k, that receives a negative production shock from a single node o, separated from firm k through just one or two of its suppliers, can experience default.

The default of a given firm k depends on the coefficients of the Leontief matrix, which capture the distance of the shock from that firm. The default interval for firm k is the range of Leontief coefficient values for which its profit becomes negative.

First, we consider a generic distribution searching for sufficient conditions of its exponentially tilted moments that guarantee the existence of an unique default interval. Then, we focus on exponential, gamma and Bernoulli density functions, finding their default intervals in the cases of a single and two suppliers.

3.1 Default

Given the model defined in Chapter 2, which admits a unique rigid Walrasian equilibrium, we aim to explain the situations in which default occurs and the conditions under which it can be identified.

Definition 2. *A firm $k \in \mathcal{V}$ is in default if its equilibrium profit is negative:*

$$\pi_k(\eta) < 0 \tag{3.1}$$

It depends on the network structure and on rigidity, it is not affected by the external bank through θ_k and r_k.

Proposition 3. *A firm k is in default if and only if:*

$$\epsilon_k > \tau_k \tag{3.2}$$

Proof. The proof follows from equation (2.44) and from definition (3.1), since w and v_k^ζ are positive quantities. □

Equation (3.2) shows that the default happens when shocks hitting neighbor nodes weighted by their importance (suppliers shocks) are greater than total shocks (both primitive and network-induced).

Example 1. *In the example defined in section 2.3, the default condition becomes:*

$\tau_1 < \epsilon_1$ *is equivalent to* $e^{\eta_1} < \frac{\lambda_1}{\lambda_1+1}$

$\tau_2 < \epsilon_2$ *is equivalent to* $\frac{e^{a_{12}\eta_1+\eta_2}}{\frac{\lambda_1}{\lambda_1+a_{12}}\frac{\lambda_2}{\lambda_2+1}} < 1 + a_{12}\left(\frac{e^{\eta_1}}{\frac{\lambda_1}{\lambda_1+1}} - 1\right)$

$\tau_3 < \epsilon_3$ *is equivalent to* $\frac{e^{(a_{12}a_{23}+a_{13})\eta_1+a_{23}\eta_2+\eta_3}}{\frac{\lambda_1}{\lambda_1+a_{12}a_{23}+a_{13}}\frac{\lambda_2}{\lambda_2+a_{23}}\frac{\lambda_3}{\lambda_3+1}} < 1+a_{13}\left(\frac{e^{\eta_1}}{\frac{\lambda_1}{\lambda_1+1}} - 1\right)+a_{23}\left(\frac{e^{a_{12}\eta_1+\eta_2}}{\frac{\lambda_1}{\lambda_1+a_{12}}\frac{\lambda_2}{\lambda_2+1}} - 1\right)$

$\tau_4 < \epsilon_4$ *is equivalent to* $\frac{e^{(a_{12}a_{23}a_{34}+a_{12}a_{24}+a_{13}a_{34})\eta_1+(a_{23}a_{34}+a_{24})\eta_2+a_{34}\eta_3+\eta_4}}{\frac{\lambda_1}{\lambda_1+a_{12}a_{23}a_{34}+a_{12}a_{24}+a_{13}a_{34}}\frac{\lambda_2}{\lambda_2+a_{23}a_{34}+a_{24}}\frac{\lambda_3}{\lambda_3+a_{34}}\frac{\lambda_4}{\lambda_4+1}} <$

$1 + a_{24}\left(\frac{e^{a_{12}\eta_1+\eta_2}}{\frac{\lambda_1}{\lambda_1+a_{12}}\frac{\lambda_2}{\lambda_2+1}} - 1\right) + a_{34}\left(\frac{e^{(a_{12}a_{23}+a_{13})\eta_1+a_{23}\eta_2+\eta_3}}{\frac{\lambda_1}{\lambda_1+a_{12}a_{23}+a_{13}}\frac{\lambda_2}{\lambda_2+a_{23}}\frac{\lambda_3}{\lambda_3+1}} - 1\right)$

In order to check if the condition is satisfied, we need to fix the samples and substitute the values of adjacency matrix's coefficients and λ parameters.

Remembering that the Leontief matrix is defined as $L = (I - A')^{-1}$, we can rewrite it as $L = I + A'L$. Then, we can denote the k-th row l^k of the matrix L as:

$$l^k = \delta_k + \sum_{j\in\mathcal{V}} A_{jk} l^j \tag{3.3}$$

From the equation (2.19) of the total shock ρ, we can write that:

$$\rho_k = (l^k)'\eta \tag{3.4}$$

We can introduce a function $f : \mathbb{R}^{\mathcal{V}} \times \mathbb{R}^{\mathcal{V}} \to \mathbb{R}$

$$f(x,t) = \frac{e^{t'x}}{\mathbb{E}[e^{t'\eta}]} \tag{3.5}$$

Exploiting the expressions (2.21), (2.22) of τ_k and ϵ_k and defining $\tau_k = f(\eta, l^k)$, the default condition $\tau_k < \epsilon_k$ becomes:

$$f(\eta, l^k) < \beta_k + \sum_{j\in\mathcal{V}} A_{jk} f(\eta, l^j) \tag{3.6}$$

Hence, the default occurs if and only if the density in correspondence of both primitive and network induced shocks is smaller than the weighted average of the density of the primitive and network-induced losses.

A special case is when firm k is hit only by a primitive shock, hence its suppliers are not exposed to shocks. It means that $\forall k \neq j,\ L_{jk} > 0$ implies that $\eta_j = 0\ \forall j$ suppliers of k. In this situation, we obtain:

$$\rho_k = \eta_k, \text{ hence } \tau_k = \frac{e^{\eta_k}}{\mathbb{E}[e^{\eta_k}]}, \tag{3.7}$$

then the default condition is simpler than (3.6) and it becomes:

$$f(\eta, l^k) < 1 \text{ or } e^{\eta_k} < \mathbb{E}[e^{\eta_k}] \tag{3.8}$$

The function $f(x,t)$ defined in (3.5) can represent the density of the random vector η under an exponential tilting of parameter t. Then, the occurrence of default can be evaluated under an exponentially tilted change of measure, for this reason it is useful to review the main properties of exponentially tilted distributions.

3.2 Properties of exponentially t-tilted distributions

Exponential tilting is a distribution shifting technique, it is known as the natural exponential family of a random vector η.

Given a random vector $\eta : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, taking values in $\mathbb{R}^n$, with density function p_0 and cumulant generating function [10]:

$$k_\eta(t) = \log \mathbb{E}[e^{t'\eta}] = \log \int_{\mathbb{R}^n} e^{t'x} p_0(x)dx, \quad \forall t \in \mathbb{R}^n_+, \tag{3.9}$$

we can define the exponential t-tilted probability measure as [11]:

$$p_t(\eta \in \mathcal{B}) = \mathbb{E}[e^{t'\eta}\mathbf{1}_{\mathcal{B}}(\eta)]e^{-k_\eta(t)} = \int_{\mathcal{B}} e^{t'x-k_\eta(t)} p_0(x)dx \tag{3.10}$$

We obtain that the exponentially t-tilted density can be written as:

$$p_t(x) = p_0(x)e^{t'x-k_\eta(t)} \tag{3.11}$$

When $t = 0$, $p_t(x)$ coincides with the original density function p_0.

In many cases, the tilted and the original distributions belong to the same parametric family. This is particularly true when the original density is in the exponentially family, as for example in the case of exponential, gamma and binomial distributions. In Table 3.1, we report the expressions of their exponentially t-tilted distributions in the scalar case ($n = 1$).

Original distribution	exponentially t-tilted distribution
$Binomial(m, p)$	$Binomial\left(m, \frac{pe^t}{1-p+pe^t}\right)$
$Exponential(\lambda)$	$Exponential(\lambda - t)$
$Gamma(\alpha, \beta)$	$Gamma(\alpha, \beta - t)$

Table 3.1: Exponentially t-tilted densities

The first, the second moment, the variance and the skewness of the exponentially tilted distribution have the following expressions:

$$m_1(t) = \frac{\mathbb{E}[\eta e^{t\eta}]}{\mathbb{E}[e^{t\eta}]}; \quad m_2(t) = \frac{\mathbb{E}[\eta^2 e^{t\eta}]}{\mathbb{E}[e^{t\eta}]}; \quad Var(t) = m_2(t) - m_1^2(t); \quad Sk(t) = \frac{\mathbb{E}[(\eta - m_1(t))^3]}{Var(t)^{\frac{3}{2}}} \tag{3.12}$$

Using $k^{(i)}(t)$ to represent the i-th cumulant of the tilted distribution, the expectation and the variance of the exponentially t-tilted distribution correspond to the first and the second cumulant respectively [10]:

$$m_1(t) = \frac{d}{dt}k_\eta(t) = k^{(1)}(t); \quad Var(t) = \frac{d^2}{dt^2}k_\eta(t) = k^{(2)}(t) \tag{3.13}$$

We can make some general considerations regarding the exponentially t-tilted moments. We can compute the derivative of the moments exploiting the following result:

$$\frac{d}{dl}(\mathbb{E}[e^{l\eta}]) = \mathbb{E}[\eta e^{l\eta}]; \quad \frac{d}{dl}(\mathbb{E}[\eta e^{l\eta}]) = \mathbb{E}[\eta^2 e^{l\eta}] \tag{3.14}$$

We obtain (3.14) using the definition of the expected value and of the density function of the exponentially tilted distribution:

$$\frac{d}{dl}(\mathbb{E}[e^{l\eta}]) = \frac{d}{dl}\int_0^{+\infty} \mathbb{P}(e^{l\eta} \geq x)dx = \int_0^{+\infty} \frac{d}{dl}\mathbb{P}\left(\eta \geq \frac{logx}{l}\right)dx = \int_0^{+\infty} f_\eta\left(\frac{logx}{l}\right)\frac{logx}{l^2}dx;$$

then imposing $y = \frac{logx}{l}; dy = \frac{dx}{xl}$, we have that:

$$\frac{d}{dl}(\mathbb{E}[e^{l\eta}]) = \int_0^{+\infty} f_\eta(y) y e^{ly} dy = \mathbb{E}[\eta e^{\eta l}]$$

$$\frac{d}{dl}(\mathbb{E}[\eta e^{l\eta}]) = \frac{d}{dl}\int_0^{+\infty} \eta e^{l\eta} f_\eta d_\eta = \int_0^{+\infty} \frac{d}{dl}\eta e^{l\eta} f_\eta d_\eta = \int_0^{+\infty} \eta^2 e^{l\eta} f_\eta d_\eta = \mathbb{E}[\eta^2 e^{l\eta}]$$

Note that we can differentiate under the integral sign because the integrand satisfies the conditions of Lebesgue theorem.

From (3.13), we obtain that [12]:

$$m_1'(t) = \frac{d}{dt}m_1(t) = \frac{d^2}{dt^2}k_\eta(t) = Var(t) > 0, \tag{3.15}$$

then $m_1(t)$ is an increasing function $\forall t > 0$.

Using (3.12), (3.14), we find that:

$$m_2'(t) = \frac{\mathbb{E}[\eta^2 e^{t\eta}]'\mathbb{E}[e^{t\eta}] - \mathbb{E}[\eta^2 e^{t\eta}]\mathbb{E}[e^{t\eta}]'}{(\mathbb{E}[e^{t\eta}])^2} = \frac{\mathbb{E}[\eta^3 e^{t\eta}]}{\mathbb{E}[e^{t\eta}]} - \frac{\mathbb{E}[\eta^2 e^{t\eta}]}{\mathbb{E}[e^{t\eta}]}\frac{\mathbb{E}[\eta e^{t\eta}]}{\mathbb{E}[e^{t\eta}]} = m_3(t) - m_2(t)m_1(t) \tag{3.16}$$

Moreover, we can observe that $m_1''(t)$ is the third cumulant and is equal to the derivative of the variance:

$$\begin{aligned} m_1''(t) = \frac{d^3}{dt^3}k_\eta(t) = k^{(3)}(t) = Var'(t) = m_2'(t) - 2m_1(t)m_1'(t) \\ = m_3(t) - m_2(t)m_1(t) - 2m_1(t)(m_2(t) - m_1^2(t)) = \\ m_3(t) - 3m_1(t)m_2(t) + 2m_1^3(t) = \mathbb{E}[(\eta - m_1(t))^3] = Sk(t)Var(t)^{\frac{3}{2}} \end{aligned} \tag{3.17}$$

We obtain that $m_1''(t)$ is linked with the skewness of an exponentially tilted distribution.

3.3 Default condition with a single node shock transmitted through one supplier using a generic distribution

We want to study the default condition of firm k in the case in which the following two assumptions hold:

1. The initial shock is concentrated only on a specific node (node o), meaning that:

$$\eta = \delta_o \eta_o, \text{ then } \rho_k = L_{ko}\eta_o \tag{3.18}$$

2. there exists only one supplier j of firm k which separates it from node o. It implies that all the paths from node o to node k pass through the supplier j, which is the sole intermediary between firm k and the source of the shock. Consequently, the total shock of firm k can be defined as:

$$\rho_k = A_{jk} L_{jo} \eta_o \tag{3.19}$$

with $L_{ko} = A_{jk} L_{jo}$

We can introduce the function $g(x, \alpha, l)$ defined as:

$$g(x, \alpha, l) = f(x, \alpha l) - (1 - \alpha) - \alpha f(x, l), \tag{3.20}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the one dimensional version of the function f introduced in (3.5) and it is defined as:

$$f(x, t) = \frac{e^{tx}}{\mathbb{E}[e^{t\eta}]} \tag{3.21}$$

where t is the exponential tilted parameter.

Substituting (3.21) in (3.20), we obtain that the function $g(x, \alpha, l)$ is:

$$g(x, \alpha, l) = \frac{e^{\alpha l x}}{\mathbb{E}[e^{\alpha l \eta}]} - (1 - \alpha) - \alpha \frac{e^{lx}}{\mathbb{E}[e^{l\eta}]} \tag{3.22}$$

Under the hypothesis expressed by equations (3.18), (3.19), the default condition (3.6) is equivalent to the condition $g(\eta_0, A_{jk}, L_{jo}) < 0$.

Now we want to study the function $g(x, \alpha, l)$. We decide to fix x, which is the realization of a single node shock and $\alpha < 1$, that measures the importance of supplier j in the production of firm k. Hence, we observe the behavior of $g(x, \alpha, l)$ with respect to variations in l, which represents the strength of the network from the shock's source to supplier j and the effect on shock's transmission.

In particular, we analyze the partial derivative of $g(x, \alpha, l)$ with respect to l. We obtain that it can be expressed through function $h(t)$, whose derivative depends on the moments of the exponentially tilted distribution.

Lemma 1. *The following expression holds:*

$$\frac{\partial g}{\partial l} = \alpha(h(\alpha l) - h(l)) \tag{3.23}$$

where

$$h(t) = \frac{e^{tx}(x\mathbb{E}[e^{t\eta}] - \mathbb{E}[\eta e^{t\eta}])}{(\mathbb{E}[e^{t\eta}])^2} \tag{3.24}$$

Proof. Starting from (3.22) and using (3.14), we can compute $\frac{\partial g}{\partial l}$:

$$\frac{\partial g}{\partial l} = \frac{\alpha x e^{\alpha l x}\mathbb{E}[e^{\alpha l \eta}] - e^{\alpha l x}\alpha \mathbb{E}[\eta e^{\alpha l \eta}]}{(\mathbb{E}[e^{\alpha l \eta}])^2} - \alpha \frac{x e^{l x}\mathbb{E}[e^{l \eta}] - e^{l x}\mathbb{E}[\eta e^{l \eta}]}{(\mathbb{E}[e^{l \eta}])^2}$$

$$\frac{\partial g}{\partial l} = \frac{\alpha e^{\alpha l x}(x\mathbb{E}[e^{\alpha l \eta}] - \mathbb{E}[\eta e^{\alpha l \eta}])}{(\mathbb{E}[e^{\alpha l \eta}])^2} - \frac{\alpha e^{l x}(x\mathbb{E}[e^{l \eta}] - \mathbb{E}[\eta e^{l \eta}]))}{(\mathbb{E}[e^{l \eta}])^2}$$

Using the definition (3.24) of $h(t)$, we find (3.23). □

Lemma 2. *$h'(t)$ can be written as a function of the moments of the exponentially tilted distribution:*

$$h'(t) = \frac{e^{tx}}{\mathbb{E}[e^{t\eta}]}((x - m_1(t))^2 - Var(t)) \tag{3.25}$$

Proof. Starting from (3.24) and using (3.14), we can compute $h'(t)$:

$$= \frac{[xe^{tx}(x\mathbb{E}[e^{t\eta}] - \mathbb{E}[\eta e^{t\eta}]) + e^{tx}(x\mathbb{E}[\eta e^{t\eta}] - \mathbb{E}[\eta^2 e^{t\eta}])](\mathbb{E}[e^{t\eta}])^2 - e^{tx}(x\mathbb{E}[e^{t\eta}] - \mathbb{E}[\eta e^{t\eta}])2\mathbb{E}[e^{t\eta}]\mathbb{E}[\eta e^{t\eta}]}{(\mathbb{E}[e^{t\eta}])^4}$$

$$= \frac{x^2 e^{tx}(\mathbb{E}[e^{t\eta}])^3 - xe^{tx}(\mathbb{E}[e^{t\eta}])^2\mathbb{E}[\eta e^{t\eta}] + xe^{tx}(\mathbb{E}[e^{t\eta}])^2\mathbb{E}[\eta e^{t\eta}]}{(\mathbb{E}[e^{t\eta}])^4}$$

$$\frac{-e^{tx}(\mathbb{E}[e^{t\eta}])^2\mathbb{E}[\eta^2 e^{t\eta}] - 2e^{tx}x(\mathbb{E}[e^{t\eta}])^2\mathbb{E}[\eta e^{t\eta}] + 2e^{tx}\mathbb{E}[e^{t\eta}](\mathbb{E}[\eta e^{t\eta}])^2}{(\mathbb{E}[e^{t\eta}])^4}$$

$$= \frac{e^{tx}\mathbb{E}[e^{t\eta}](x^2(\mathbb{E}[e^{t\eta}])^2 - \mathbb{E}[e^{t\eta}]\mathbb{E}[\eta^2 e^{t\eta}] - 2x\mathbb{E}[e^{t\eta}]\mathbb{E}[\eta e^{t\eta}] + 2(\mathbb{E}[\eta e^{t\eta}])^2)}{(\mathbb{E}[e^{t\eta}])^4}$$

$$h'(t) = \frac{e^{tx}}{(\mathbb{E}[e^{t\eta}])^3}(x^2(\mathbb{E}[e^{t\eta}])^2 - \mathbb{E}[e^{t\eta}]\mathbb{E}[\eta^2 e^{t\eta}] - 2x\mathbb{E}[e^{t\eta}]\mathbb{E}[\eta e^{t\eta}] + 2(\mathbb{E}[\eta e^{t\eta}])^2)$$

Rewriting the term in brackets as $(\mathbb{E}[e^{t\eta}])^2 \left(x^2 - \frac{\mathbb{E}[\eta^2 e^{t\eta}]}{\mathbb{E}[e^{t\eta}]} - 2x\frac{\mathbb{E}[\eta e^{t\eta}]}{\mathbb{E}[e^{t\eta}]} + 2\left(\frac{\mathbb{E}[\eta e^{t\eta}]}{\mathbb{E}[e^{t\eta}]}\right)^2\right)$ and using the definition of $m_1(t)$ and $m_2(t)$, the expression of $h'(t)$ becomes:

$$h'(t) = \frac{e^{tx}}{(\mathbb{E}[e^{t\eta}])^3}(\mathbb{E}[e^{t\eta}])^2(x^2 - m_2(t) - 2xm_1(t) + 2m_1^2(t)) = \frac{e^{tx}}{\mathbb{E}[e^{t\eta}]}((x - m_1(t))^2 - Var(t))$$

□

Given a network whose shock η satisfies the assumption (3.18), the objective is to determine the default condition as a function of the moments of the exponentially tilted distribution for all firms that comply with assumption (3.19).

In this setting, the default intervals for such firms correspond to the range of l for which the function $g(x, \alpha, l) < 0$.

We find the following result:

Proposition 4. *Given the realized shock x, the tilted first moment $m_1(t)$ and the tilted variance $Var(t)$ defined in (3.12),*

- *if* $(x - m_1(t))^2 > Var(t)\ \forall t > 0$*, then* $g(x, \alpha, l) < 0\ \forall l > 0$*;*
- *if* $(x - m_1(t))^2 \leq Var(t)\ \forall t \geq 0$*, then* $g(x, \alpha, l) \geq 0\ \forall l \geq 0$

Proof.

- $(x - m_1(t))^2 > Var(t)\ \forall t > 0$ implies $h'(t) > 0\ \forall t > 0$, hence $h(t)$ is an increasing function $\forall t > 0$. Using (3.23) and the fact that $\alpha < 1$, we obtain that $\frac{\partial g}{\partial l} < 0\ \forall l > 0$ and since $g(x, \alpha, 0) = 0$, then $g(x, \alpha, l) < 0\ \forall l > 0$;
- $(x - m_1(t))^2 \leq Var(t)\ \forall t \geq 0$ implies $h'(t) \leq 0\ \forall t \geq 0$, hence $h(t)$ is a non-increasing function $\forall t \geq 0$. Using (3.23) and the fact that $\alpha < 1$, we obtain that $\frac{\partial g}{\partial l} \geq 0\ \forall l \geq 0$ and since $g(x, \alpha, 0) = 0$, then $g(x, \alpha, l) \geq 0\ \forall l \geq 0$.

We obtain that if the realized shock x is far from its mean with respect to the variance there is the default, while if x is near to its mean there is not the default. Hence, the default condition depends on $m_1(t)$, $m_2(t)$, $Var(t)$ and on how the shock is far from the mean.

The following two propositions provide sufficient conditions to obtain an unique default interval, which are based on the properties of the exponentially tilted distribution.
In detail, proposition 5 establishes a condition on the sign of the derivative of function $h(t)$ in order to guarantee that function $g(x, \alpha, l)$ is negative over a single interval of l.
Proposition 6 directly employs the moments of the exponentially tilted distribution together with Proposition 5 to derive the condition ensuring the existence of a unique default interval, meaning that there is just one interval of l where $g(x, \alpha, l)$ is negative.

Proposition 5. *Let* $h'(t)$ *be the function defined in* (3.25)*, if* $\exists a \geq 0$ *such that* $h'(t) \leq 0\ \forall t : 0 < t \leq a$ *and* $h'(t) > 0\ \forall t > a$*, then* $g(x, \alpha, l)$ *is negative on at most one connected interval of* l*.*

Proof. We can distinguish two cases:

- if $a = 0$, $h'(t) > 0\ \forall t > 0$ implies that $h(t)$ is an increasing function, hence from equation (3.23) and from the fact that $\alpha < 1$, we obtain that $\frac{\partial g}{\partial l} < 0\ \forall l > 0$. Then, since $g(x, \alpha, 0) = 0$, then $g(x, \alpha, l) < 0\ \forall l > 0$, consequently there is one default interval;
- if $a > 0$, $h'(t) \leq 0$ in $(0, a]$, then $h(t)$ is a non-increasing function in $(0, a]$, from equation (3.23) and from the fact that $\alpha < 1$, we obtain that $\frac{\partial g}{\partial l} \geq 0$. Hence, since $g(x, \alpha, 0) = 0$, $g(x, \alpha, l) \geq 0$ in $(0, a]$. On the contrary, $h'(t) > 0\ \forall t > a$ implies that $h(t)$ is an increasing function on $(a, +\infty)$, hence from equation (3.23) and from the fact that $\alpha < 1$, we obtain that $\frac{\partial g}{\partial l} < 0$ in $(a, +\infty)$. Consequently, $g(x, \alpha, l)$ is a decreasing function in $(a, +\infty)$, it can remain positive or become negative generating a default interval, hence there is at most one interval where $g(x, \alpha, l) < 0$.

In both cases there is at most one default interval. □

Proposition 6. *Given a random variable η with support in $(-\infty, 0]$, if its tilted moments, defined in (3.12), satisfy the condition:*

$$2m_1(t) \geq Sk(t)Var(t)^{\frac{1}{2}} \quad \forall t \geq 0, \tag{3.26}$$

then $g(x, \alpha, l)$ is negative on at most one connected interval of l.

Proof. In order to simplify the notation, we can introduce:

$$f(t) = (x - m_1(t))^2 - Var(t)$$

Since $Var(t) \geq 0$ and using (3.12), (3.15), (3.17), condition (3.26) implies that

$$\forall x \leq 0, \quad 0 \leq 2(m_1(t) - x)Var(t) - Sk(t)Var(t)^{\frac{3}{2}} = 2(m_1(t) - x)m_1'(t) - Var(t)'$$

$$= \frac{d}{dt}((x - m_1(t))^2 - Var(t)) = f'(t)$$

We obtain that $f'(t) \geq 0$, which implies that $f(t)$ is a non decreasing function and consequently, also $h'(t) = \frac{e^{tx}}{\mathbb{E}[e^{t\eta}]} f(t)$ is a non decreasing function.

There are two cases:

- $h'(t) < 0 \; \forall t > 0$, then $h(t)$ is a decreasing function, hence from equation (3.23) and from the fact that $\alpha < 1$, we obtain that $\frac{\partial g}{\partial l} > 0$. Since $g(x, \alpha, 0) = 0$, $g(x, \alpha, l) > 0 \; \forall l > 0$, the default is never verified;
- $\exists a \geq 0$ such that $h'(t) \leq 0 \; \forall t : \; 0 < t \leq a$ and $h'(t) > 0 \; \forall t > a$. Using Proposition 5, we conclude that $g(x, \alpha, l)$ is negative in at most one interval of l.

□

Observation 2. *Since $\eta \in (-\infty, 0]$, $m_1(t)$ is negative while $Var(t)$ is positive, hence Proposition 6 suggests that the skewness of the distribution should be sufficiently negative in order to verify (3.26).*

3.3.1 Default condition with a single node exponential shock transmitted through one supplier

We consider the case in which η follows an exponential distribution with density function $f_x = \lambda e^{\lambda x}, x \leq 0$.

In order to understand the default condition, we start computing the tilted moments of the distribution.

Lemma 3. *The first moment, the variance and the skewness for the t-tilted exponential distribution are:*

$$m_1(t) = -\frac{1}{t + \lambda}; \quad Var(t) = \frac{1}{(t + \lambda)^2}; \quad Sk(t) = -2 \tag{3.27}$$

Proof.

- direct computations:

$$\mathbb{E}[e^{t\eta}] = \int_{-\infty}^{0} e^{ts} f_s ds = \int_{-\infty}^{0} e^{ts}\lambda e^{\lambda s} ds = \int_{-\infty}^{0} \lambda e^{s(t+\lambda)} ds =$$

$$\frac{\lambda}{(t+\lambda)} \int_{-\infty}^{0} (t+\lambda) e^{s(t+\lambda)} ds = \frac{\lambda}{t+\lambda}[e^{s(t+\lambda)}]_{-\infty}^{0} = \frac{\lambda}{t+\lambda}$$

$$\mathbb{E}[\eta e^{t\eta}] = \int_{-\infty}^{0} s e^{ts} f_s ds = \int_{-\infty}^{0} s e^{ts}\lambda e^{\lambda s} = \lambda \int_{-\infty}^{0} s e^{s(t+\lambda)} ds =$$

$$-\lambda \int_{-\infty}^{0} \frac{1}{t+\lambda} e^{s(t+\lambda)} ds = -\frac{\lambda}{(t+\lambda)^2}[e^{s(t+\lambda)}]_{-\infty}^{0} = -\frac{\lambda}{(t+\lambda)^2}$$

$$\mathbb{E}[\eta^2 e^{t\eta}] = \frac{2\lambda}{(t+\lambda)^3}; \quad \mathbb{E}[\eta^3 e^{t\eta}] = \frac{6\lambda}{(t+\lambda)^4} \quad \text{(found as previous formula)}$$

  Using (3.12) we obtain the result.

- From Table 3.1 we know that if the original distribution is a $Exp(\lambda)$, the exponentially t-tilted distribution is a $Exp(\lambda - t)$, however, if the exponential is defined only for negative values, the exponentially t-tilted distribution is a $Exp(\lambda + t)$.

  Since the expected value, the variance and the skewness of an exponential distribution defined for $x \leq 0$ with parameter λ are respectively $-\frac{1}{\lambda}$, $\frac{1}{\lambda^2}$ and -2, substituting λ with $\lambda + t$ in these expressions, we obtain the tilted first moment, variance and skewness.

□

Condition (3.26) of Proposition 6 is always verified as an equality in the exponential case. In fact, using (3.27), we obtain that:

$$2\left(-\frac{1}{t+\lambda}\right) + 2\left(\frac{1}{(t+\lambda)^2}\right)^{\frac{1}{2}} = 2m_1(t) - Sk(t)Var(t)^{\frac{1}{2}} = 0$$

This result implies that, in the case of an exponential shock, there cannot be more than one default interval.

In order to find it, we can derive the expression of $h'(t)$ in the exponential case substituting (3.27) in the generic expression (3.25):

$$h'(t) = \frac{e^{tx}}{\mathbb{E}[e^{t\eta}]}((x - m_1(t))^2 - Var(t))$$

$$h'(t) = \frac{e^{tx}}{\frac{\lambda}{t+\lambda}}\left[\left(x + \frac{1}{t+\lambda}\right)^2 - \frac{1}{(t+\lambda)^2}\right]$$

$$= \frac{e^{tx}(t+\lambda)}{\lambda}\left(x^2 + \frac{2x}{t+\lambda} + \frac{1}{(t+\lambda)^2} - \frac{1}{(t+\lambda)^2}\right) = \frac{e^{tx}}{\lambda}\left(x^2(t+\lambda) + 2x\right)$$

We obtain the following expression of $h'(t)$ for an exponential distribution:

$$h'(t) = \frac{e^{tx}}{\lambda}\left(x^2(t+\lambda) + 2x\right) \tag{3.28}$$

We can observe that $h'(t)$ has an unique zero ($t = t^*$):

$$h'(t) = 0 \text{ is equivalent to } x^2(t^*+\lambda) + 2x = 0, \text{ then the unique zero is } t^* = -\frac{2}{x} - \lambda$$

The following proposition shows the default interval, expressed through function $g(x, \alpha, l)$, when a generic firm k that verifies the hypothesis (3.19) is affected by an exponential shock which satisfies the assumption (3.18).

In the case of an exponential shock, the function $g(x, \alpha, l)$, defined in (3.20), becomes:

$$g(x, \alpha, l) = \frac{e^{\alpha l x}(\alpha l + \lambda)}{\lambda} - (1-\alpha) - \alpha\frac{e^{lx}(l+\lambda)}{\lambda}$$

Proposition 7. *Given the realized shock x and the exponential parameter λ,*

- *if $x < -\frac{2}{\lambda}, g(x, \alpha, l) < 0\ \forall l > 0$;*
- *if $x \geq -\frac{2}{\lambda},\ g(x, \alpha, l) \leq 0\ \forall l \geq l^*$, where l^* is the only positive solution of:*

$$g(x, \alpha, l^*) = \frac{e^{\alpha l^* x}(\alpha l^* + \lambda)}{\lambda} - (1-\alpha) - \alpha\frac{e^{l^* x}(l^*+\lambda)}{\lambda} = 0$$

Proof.

- $x < -\frac{2}{\lambda}$ implies that $t^* < 0$ then $t > t^*\ \forall t > 0$ hence $h'(t) > 0\ \forall t > 0$. Consequently $h(t)$ is an increasing function $\forall t > 0$; using (3.23) and the fact that $\alpha < 1$, we have that $\frac{\partial g}{\partial l} < 0\ \forall l > 0$ and since $g(x, \alpha, 0) = 0$, we find that $g(x, \alpha, l) < 0\ \forall l > 0$;
- $x \geq -\frac{2}{\lambda}$ implies that $t^* \geq 0$, then

$$\begin{cases} h'(t) < 0 & t < t^* \\ h'(t) = 0 & t = t^* \\ h'(t) > 0 & t > t^* \end{cases}$$

  Since $\alpha < 1,\ \alpha l < l\ \forall l > 0$, we have three cases:

  - if $\alpha l < l < t^*$, since $h(t)$ is a decreasing function in this interval, from (3.23) we obtain that $\frac{\partial g}{\partial l} > 0$ hence $g(x, \alpha, l) > 0$ since $g(x, \alpha, 0) = 0$;
  - if $t^* < \alpha l < l$, since $h(t)$ is a increasing function in this interval, from (3.23) we obtain that $\frac{\partial g}{\partial l} < 0$ hence $g(x, \alpha, l)$ is a decreasing function;
  - if $\alpha l < t^* < l$, $h(\alpha l)$ is going to decrease until the minimum point while $h(l)$ is going to increase after the minimum. Then, for the continuity of h, exists a point $\bar{l}$ such that $h(\alpha\bar{l}) = h(\bar{l})$ and $\forall l > \bar{l},\ h(\alpha l)$ is below $h(l)$. From (3.23) we obtain that $\frac{\partial g}{\partial l} < 0\ \forall l > \bar{l}$, then $g(x, \alpha, l)$ is a decreasing function.

The previous analysis is not conclusive for the last two cases because we cannot deduce the sign of $g(x, \alpha, l)$. However, we can use the definition of $g(x, \alpha, l)$ to show that for large values of l, $g(x, \alpha, l)$ becomes negative. In fact, computing the limit of $g(x, \alpha, l)$, we obtain that:

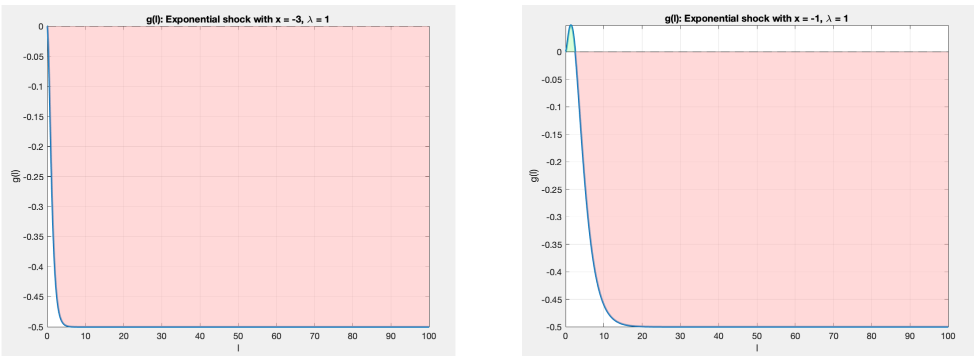
$$\lim_{l \to +\infty} g(x, \alpha, l) = \lim_{l \to +\infty} \frac{e^{\alpha l x}(\alpha l + \lambda)}{\lambda} - (1 - \alpha) - \alpha \frac{e^{lx}(l + \lambda)}{\lambda} = -1 + \alpha < 0$$

$\square$

Simulation with an exponential shock

Choice of parameters:

- $\alpha = 0.5$;
- $\lambda = 1$



(a) $g(x, \alpha, l)$ with $x = -3$ (b) $g(x, \alpha, l)$ with $x = -1$

Figure 3.1: Plots of $g(x, \alpha, l)$ with a negative exponential shock

From Figure 3.1 we can see that when $x < -\frac{2}{\lambda}$ (case (a)), $g(x, \alpha, l) < 0 \; \forall l > 0$, on the contrary when $x \geq -\frac{2}{\lambda}$ (case (b)), $g(x, \alpha, l)$ is positive for small values of l and then it becomes negative. These results are coherent with Proposition 7.

3.3.2 Default condition with a single node gamma shock transmitted through one supplier

Now we analyze the default condition in the case in which the shock comes from a gamma distribution ($Gamma(\bar{\alpha}, \beta)$) with density function $f_x = \frac{1}{\Gamma(\bar{\alpha})\beta^{\bar{\alpha}}}(-x)^{\bar{\alpha}-1}e^{\frac{x}{\beta}}$, $x \leq 0$.

Lemma 4. *The first moment, the variance and the skewness for the t-tilted gamma distribution are:*

$$m_1(t) = -\frac{\bar{\alpha}}{\beta + t}; \quad Var(t) = \frac{\bar{\alpha}}{(\beta + t)^2}; \quad Sk(t) = -\frac{2}{\sqrt{\bar{\alpha}}} \tag{3.29}$$

Proof. From Table 3.1 we know that if the original distribution is a $Gamma(\bar{\alpha}, \beta)$, the exponentially t-tilted distribution is a $Gamma(\bar{\alpha}, \beta - t)$, however if the gamma is defined only for negative values, the exponentially t-tilted distribution is a $Gamma(\bar{\alpha}, \beta + t)$.

Since the expected value, the variance and the skewness of a gamma distribution defined for $x \leq 0$ with parameters $\bar{\alpha}, \beta$ are respectively $-\frac{\bar{\alpha}}{\beta}$, $\frac{\bar{\alpha}}{\beta^2}$ and $-\frac{2}{\sqrt{\bar{\alpha}}}$, substituting β with $\beta + t$ in these expressions, we obtain (3.29). □

Differently from the exponential case, for a gamma distribution, condition (3.26) of Proposition 6 is satisfied only for some choices of parameters. In fact, using equation (3.29), we obtain that the condition:

$$2\left(\frac{-\bar{\alpha}}{\beta + t}\right) + \frac{2}{\sqrt{\bar{\alpha}}}\frac{\sqrt{\bar{\alpha}}}{\beta + t} = 2m_1(t) - Sk(t)Var(t)^{\frac{1}{2}} \geq 0$$

is verified if and only if $\bar{\alpha} \leq 1$; hence the default is at most in one interval of l only if $\bar{\alpha} \leq 1$.

Simulation with a gamma shock

In order to understand which is the default interval, we can perform numerical simulations.

We can find the expression of $h'(t)$ substituting (3.29) in the generic expression (3.25):

$$h'(t) = \frac{e^{tx}}{\left(\frac{\beta}{\beta+t}\right)^{\bar{\alpha}}}\left[\left(x + \frac{\bar{\alpha}}{\beta + t}\right)^2 - \frac{\bar{\alpha}}{(\beta + t)^2}\right]$$

Using (3.22) and the expression of the moment generating function of a gamma distribution defined for negative values ($\mathbb{E}[e^{t\eta}] = \left(\frac{\beta}{\beta+t}\right)^{\bar{\alpha}}$), we obtain that, with a gamma shock, the generic expression of $g(x, \alpha, l)$, defined in (3.20), becomes:

$$g(x, \alpha, l) = \frac{e^{\alpha l x}}{\left(\frac{\beta}{\beta+\alpha l}\right)^{\bar{\alpha}}} - (1 - \alpha) - \alpha\frac{e^{lx}}{\left(\frac{\beta}{\beta+l}\right)^{\bar{\alpha}}}$$

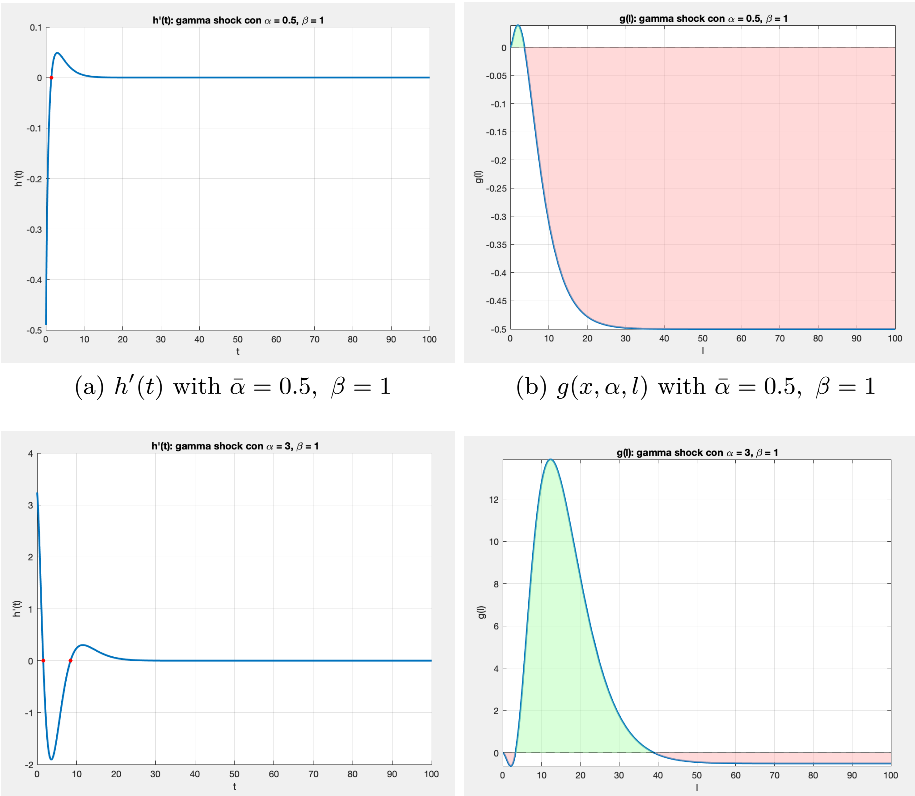
Choice of parameters:

- $x = -0.5$;
- $\alpha = 0.5$

From Figure 3.2, we can observe that when $\bar{\alpha} \leq 1$ (case (a), (b) with $\bar{\alpha} = 0.5, \beta = 1$) the default condition is an interval of l, in fact $g(x, \alpha, l) \geq 0$ for small values of l, then it becomes and remains negative for the other values of l. This is coherent with the fact that $h'(t)$ has an unique zero (it changes sign just one time).

On the contrary, when $\bar{\alpha} > 1$ (case (c), (d) with $\bar{\alpha} = 3, \beta = 1$), $h'(t)$ has two zeros, so it changes sign twice. As a consequence, $g(x, \alpha, l) \leq 0$ for small values of l, $g(x, \alpha, l) > 0$ for intermediate values and $g(x, \alpha, l) < 0$ for large values of l.

Numerical simulations confirm the the default is at most one interval of l when $\bar{\alpha} \leq 1$, in agreement with condition (3.26).



(a) $h'(t)$ with $\bar{\alpha} = 0.5,\ \beta = 1$ (b) $g(x, \alpha, l)$ with $\bar{\alpha} = 0.5,\ \beta = 1$

(c) $h'(t)$ with $\bar{\alpha} = 3,\ \beta = 1$ (d) $g(x, \alpha, l)$ with $\bar{\alpha} = 3,\ \beta = 1$

Figure 3.2: Plots of $h'(t)$ and $g(x, \alpha, l)$ for a negative gamma shock

3.3.3 Default condition with a single node Bernoulli shock transmitted through one supplier

Now we assume that $\eta = -\epsilon X$; where $X \sim Ber(p)$ and ϵ is a positive parameter representing the shock amplitude. Then, the initial shock follows a Bernoulli distribution, hence the shock realizations are $x = 0$ and $x = -\epsilon$.

In order to understand the condition for which the default is realized, we can compute the tilted moments in the case of a Bernoulli distribution.

Lemma 5. *The first moment, the variance and the skewness for a t-tilted Bernoulli distribution are:*

$$m_1(t) = \frac{-p\epsilon e^{-t\epsilon}}{1-p+pe^{-t\epsilon}}; \quad Var(t) = \frac{p\epsilon^2 e^{-t\epsilon}(1-p)}{(1-p+pe^{-t\epsilon})^2}; \quad Sk(t) = \frac{1-p-pe^{-t\epsilon}}{\sqrt{p(1-p)e^{-t\epsilon}}} \tag{3.30}$$

Proof.

- direct computation:

$$\mathbb{E}[e^{t\eta}] = e^{t(0)}\mathbb{P}(X=0) + e^{t(-\epsilon)}\mathbb{P}(X=-\epsilon) = 1-p+pe^{-t\epsilon}$$

$$\mathbb{E}[\eta e^{t\eta}] = 0e^{t(0)}\mathbb{P}(X=0) + (-\epsilon)e^{t(-\epsilon)}\mathbb{P}(X=-\epsilon) = -p\epsilon e^{-t\epsilon}$$
$$\mathbb{E}[\eta^2 e^{t\eta}] = 0e^{t(0)}\mathbb{P}(X=0) + (-\epsilon)^2 e^{t(-\epsilon)}\mathbb{P}(X=-\epsilon) = p\epsilon^2 e^{-t\epsilon}$$
$$\mathbb{E}[\eta^3 e^{t\eta}] = 0e^{t(0)}\mathbb{P}(X=0) + (-\epsilon)^3 e^{t(-\epsilon)}\mathbb{P}(X=-\epsilon) = -p\epsilon^3 e^{-t\epsilon}$$

using (3.12) we obtain the results.

- From Table 3.1(considering $m=1$ in the binomial case), we have that if the original distribution is a $Bern(p)$, the exponentially t-tilted distribution is a $Bern\left(\frac{pe^t}{1-p+pe^t}\right)$. Since the expected value, the variance and the skewness of a Bernoulli distribution with parameter p are respectively p, $p(1-p)$ and $\frac{1-2p}{\sqrt{p(1-p)}}$, substituting p with $\frac{pe^t}{1-p+pe^t}$ and $t=-\epsilon t$ in these expressions, we obtain the tilted first moment, variance and skewness.

□

We can observe that condition (3.26) of Proposition 6 is never satisfied in the Bernoulli case. In fact, using equation (3.30), we obtain that the condition:

$$2\left(-\frac{p\epsilon e^{-t\epsilon}}{1-p+pe^{-t\epsilon}}\right) - \frac{1-p-pe^{-t\epsilon}}{\sqrt{p(1-p)e^{-t\epsilon}}}\sqrt{\frac{p\epsilon^2 e^{-t\epsilon}(1-p)}{(1-p+pe^{-t\epsilon})^2}} = 2m_1(t) - Sk(t)Var(t)^{\frac{1}{2}} \geq 0$$

is satisfied if and only if $-\epsilon \geq 0$, that is not possible since ϵ is a positive parameter.

Given that the condition is sufficient but not necessary, the fact that it is never satisfied does not preclude the possibility that there is at most one default interval. Indeed, in the Bernoulli case, we can adopt an alternative approach to demonstrate that this is precisely what occurs.

We start considering the scenario in which the shock is realized ($x=-\epsilon$). Substituting (3.30) in the generic expression (3.25), we obtain that:

$$h'(t) = \frac{e^{tx}}{\mathbb{E}[e^{t\eta}]}((x-m_1(t))^2 - Var(t))$$

$$h'(t) = \frac{e^{-t\epsilon}}{1-p+pe^{-t\epsilon}}\left[\left(-\epsilon + \frac{p\epsilon e^{-t\epsilon}}{1-p+pe^{-t\epsilon}}\right)^2 - \frac{p\epsilon^2 e^{-t\epsilon}(1-p)}{(1-p+pe^{-t\epsilon})^2}\right]$$
$$= \frac{e^{-t\epsilon}}{1-p+pe^{-t\epsilon}}\left(\frac{(-\epsilon+p\epsilon)^2 - p\epsilon^2 e^{-t\epsilon}(1-p)}{(1-p+pe^{-t\epsilon})^2}\right) = \frac{e^{-t\epsilon}}{(1-p+pe^{-t\epsilon})^3}((\epsilon(1-p))^2 - p\epsilon^2 e^{-t\epsilon(1-p)})$$
$$= \frac{\epsilon^2(1-p)e^{-t\epsilon}}{(1-p+pe^{-t\epsilon})^3}(1-p-pe^{-t\epsilon})$$

Hence, the expression of $h'(t)$ in the Bernoulli case, when the shock is realized, is:

$$h'(t) = \frac{\epsilon^2(1-p)e^{-t\epsilon}}{(1-p+pe^{-t\epsilon})^3}(1-p-pe^{-t\epsilon}) \tag{3.31}$$

We can observe that $h'(t)$ has an unique zero ($t = t^*$):

$h'(t) = 0$ is equivalent to $1 - p - pe^{-t^*\epsilon} = 0$, then the unique zero is $t^* = \frac{1}{\epsilon} log\left(\frac{p}{1-p}\right)$

The following proposition shows the default interval, expressed through function $g(x, \alpha, l)$, when a generic firm k that satisfies the condition (3.19) is hit by a Bernoulli shock which verifies the assumption (3.18).

In the case of a realized Bernoulli shock, the function $g(x, \alpha, l)$, defined in (3.20), becomes:

$$g(x, \alpha, l) = \frac{e^{-l\epsilon\alpha}}{pe^{-l\epsilon\alpha} + 1 - p} - (1 - \alpha) - \alpha\frac{e^{-l\epsilon}}{pe^{-l\epsilon} + 1 - p}$$

Proposition 8. *Assuming that the shock is realized ($x = -\epsilon$) and given the Bernoulli parameter p,*

- *if $p < \frac{1}{2}$, $g(x, \alpha, l) < 0$ $\forall l > 0$,*
- *if $p \geq \frac{1}{2}$, $g(x, \alpha, l) \leq 0$ $\forall l \geq l^*$, where l^* is the only positive solution of:*

$$g(x, \alpha, l^*) = \frac{e^{-l^*\epsilon\alpha}}{pe^{-l^*\epsilon\alpha} + 1 - p} - (1 - \alpha) - \alpha\frac{e^{-l^*\epsilon}}{pe^{-l^*\epsilon} + 1 - p} = 0$$

Proof.

- $p < \frac{1}{2}$ implies that $t^* < 0$ then $t > t^*$ $\forall t > 0$, hence $h'(t) > 0$ $\forall t > 0$. Consequently $h(t)$ is an increasing function $\forall t > 0$, using (3.23) and the fact that $\alpha < 1$, we obtain that $\frac{\partial g}{\partial l} < 0$ $\forall l > 0$ and since $g(x, \alpha, 0) = 0$, then $g(x, \alpha, l) < 0$ $\forall l > 0$
- $p \geq \frac{1}{2}$, implies that $t^* > 0$, then:

$$\begin{cases} h'(t) < 0 & t < t^* \\ h'(t) = 0 & t = t^* \\ h'(t) > 0 & t > t^* \end{cases}$$

  Since $\alpha < 1$, we know that $\alpha l < l$ $\forall l > 0$, we have three cases:

  - if $\alpha l < l < t^*$, since $h(t)$ is a decreasing function in this interval, from (3.23) we obtain that $\frac{\partial g}{\partial l} > 0$ hence $g(x, \alpha, l) > 0$ since $g(x, \alpha, 0) = 0$;
  - if $t^* < \alpha l < l$, since $h(t)$ is a increasing function in this interval, from (3.23) we obtain that $\frac{\partial g}{\partial l} < 0$ hence $g(x, \alpha, l)$ is a decreasing function;
  - if $\alpha l < t^* < l$, $h(\alpha l)$ is going to decrease until the minimum point while $h(l)$ is going to increase after the minimum. Then, for the continuity of h, exists a point $\bar{l}$ such that $h(\alpha\bar{l}) = h(\bar{l})$ and $\forall l > \bar{l}$, $h(\alpha l)$ is below $h(l)$. From (3.23) we obtain that $\frac{\partial g}{\partial l} < 0$ $\forall l > \bar{l}$, then $g(x, \alpha, l)$ is a decreasing function.

As in the exponential case, the previous analysis is not conclusive for the last two setting because we cannot deduce the sign of $g(x, \alpha, l)$. However, we can exploit the definition of $g(x, \alpha, l)$ to show that for large values of $l, g(x, \alpha, l)$ becomes negative. In fact, computing the limit of $g(x, \alpha, l)$, we obtain that:

$$\lim_{l \to +\infty} g(x, \alpha, l) = \lim_{l \to +\infty} \frac{e^{-l\epsilon\alpha}}{pe^{-l\epsilon\alpha} + 1 - p} - (1 - \alpha) - \alpha \frac{e^{-l\epsilon}}{pe^{-l\epsilon} + 1 - p} = -1 + \alpha < 0$$

□

The same reasoning can be repeated considering the case in which the shock is not realized ($x = 0$). However, since the profit mean is zero for each firm and the Bernoulli distribution can have only two realizations ($x = 0$ and $x = -\epsilon$), the case in which the shock is not realized is symmetric to the one discussed above.

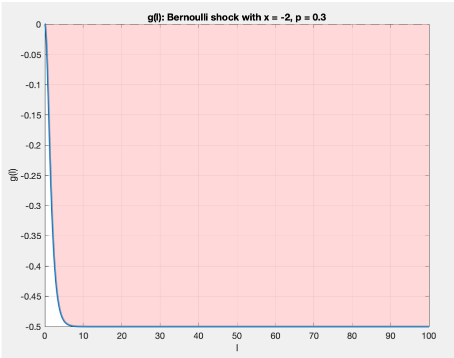
Simulation with a Bernoulli shock

Choice of parameters:

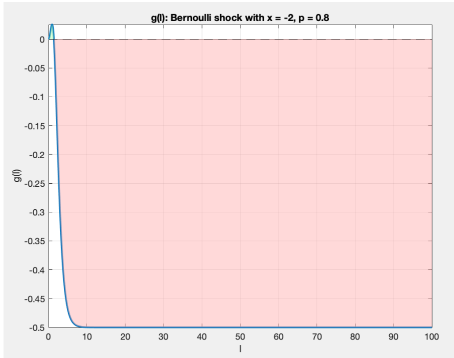
- $\alpha = 0.5$;
- $\epsilon = 2$

From Figure 3.3, we can observe that when the shock is realized ($x = -\epsilon$), if $p < \frac{1}{2}$ (case (a)), $g(x, \alpha, l)$ can assume only negative values, while if $p \geq \frac{1}{2}$ (case (b)), $g(x, \alpha, l)$ is positive for small values of l and then it becomes negative, as expected from Proposition 8.

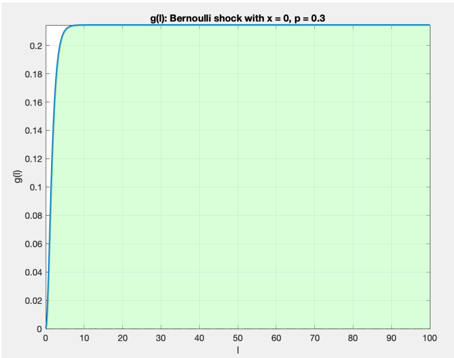
On the contrary, when the shock is not realized ($x = 0$), if $p < \frac{1}{2}$ (case (c)), $g(x, \alpha, l)$ can assume only positive values, while if $p \geq \frac{1}{2}$ (case (d)), $g(x, \alpha, l)$ is negative for small values of l and then it becomes positive.



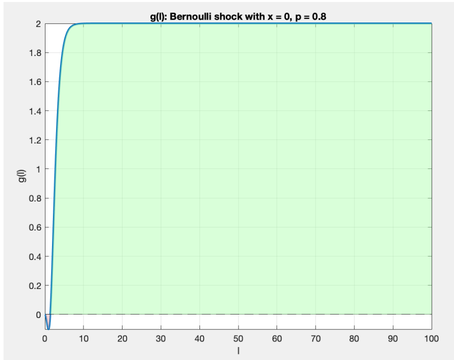
(a) $g(x, \alpha, l)$ with $p = 0.3,\ x = -\epsilon$



(b) $g(x, \alpha, l)$ with $p = 0.8,\ x = -\epsilon$



(c) $g(x, \alpha, l)$ with $p = 0.3,\ x = 0$



(d) $g(x, \alpha, l)$ with $p = 0.8,\ x = 0$

Figure 3.3: Plots of $g(x, \alpha, l)$ with a Bernoulli shock

3.4 Default condition with a single node shock transmitted through two suppliers and a generic distribution

We now consider the case in which a firm k receives the shock from node o through two suppliers. Hence, the following analysis is based on two assumptions:

1. The initial shock is concentrated on a specific node (node o), meaning that equation (3.18) is satisfied as in the case of one supplier,

2. there exist two suppliers j and h of firm k which separate it from node o. It implies that all the paths from node o to node k pass through the two suppliers of firm k. Consequently, the total shock of firm k can be defined as:

$$\rho_k = (A_{jk}L_{jo} + A_{hk}L_{ho})\eta_o \tag{3.32}$$

with $L_{ko} = A_{jk}L_{jo} + A_{hk}L_{ho}$

We can introduce $g(x, \alpha_1, \alpha_2, l_1, l_2)$ defined as:

$$g(x, \alpha_1, \alpha_2, l_1, l_2) = f(x, \alpha_1 l_1 + \alpha_2 l_2) - (1 - \alpha_1 - \alpha_2) - \alpha_1 f(x, l_1) - \alpha_2 f(x, l_2) \tag{3.33}$$

where $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is the two dimensional version of the function f introduced in (3.5). Substituting (3.5) in (3.33), we obtain that the function $g(x, \alpha_1, \alpha_2, l_1, l_2)$ is:

$$g(x, \alpha_1, \alpha_2, l_1, l_2) = \frac{e^{(\alpha_1 l_1 + \alpha_2 l_2)x}}{\mathbb{E}[e^{(\alpha_1 l_1 + \alpha_2 l_2)\eta}]} - (1 - \alpha_1 - \alpha_2) - \frac{\alpha_1 e^{l_1 x}}{\mathbb{E}[e^{l_1 \eta}]} - \frac{\alpha_2 e^{l_2 x}}{\mathbb{E}[e^{l_2 \eta}]} \tag{3.34}$$

Under the hypothesis expressed by equations (3.18), (3.32), the default condition (3.6) is equivalent to the condition $g(\eta_0, A_{jk}, A_{hk}, L_{jo}, L_{ho}) < 0$.

Now we want to study the function $g(x, \alpha_1, \alpha_2, l_1, l_2)$. We decide to fix x, which is the realization of a single node shock and α_1, α_2, that measure respectively the importance of suppliers j and h in the production of firm k and satisfy $\alpha_1 + \alpha_2 < 1$. Hence, we observe the behavior of $g(x, \alpha_1, \alpha_2, l_1, l_2)$ with respect to variations in l_1, l_2, which represent the strength of the network from the shock's source to suppliers j and h respectively.

In particular, we analyze the partial derivatives of $g(x, \alpha_1, \alpha_2, l_1, l_2)$ with respect to l_1 and l_2. We obtain that they can be expressed through function $h(t)$, which depends on the moments of the exponentially tilted distribution.

Lemma 6. *The following equalities hold:*

$$\frac{\partial g}{\partial l_1} = \alpha_1 (h(\alpha_1 l_1 + \alpha_2 l_2) - h(l_1)); \quad \frac{\partial g}{\partial l_2} = \alpha_2 (h(\alpha_1 l_1 + \alpha_2 l_2) - h(l_2)) \tag{3.35}$$

where

$$h(t) = \frac{e^{tx}(x\mathbb{E}[e^{t\eta}] - \mathbb{E}[\eta e^{t\eta}])}{(\mathbb{E}[e^{t\eta}])^2}$$

Proof. First, we need to find the partial derivatives with respect to l_1, l_2 of the expected value of the exponentially tilted distribution, using its definition:

$$\frac{\partial}{\partial l_1}\mathbb{E}[e^{(\alpha_1 l_1 + \alpha_2 l_2)\eta}] = \frac{\partial}{\partial l_1}\int_0^{+\infty} e^{(\alpha_1 l_1 + \alpha_2 l_2)\eta} f_\eta d_\eta = \int_0^{+\infty} \frac{\partial}{\partial l_1} e^{(\alpha_1 l_1 + \alpha_2 l_2)\eta} f_\eta d_\eta =$$

$$\alpha_1 \int_0^{+\infty} \eta e^{(\alpha_1 l_1 + \alpha_2 l_2)\eta} f_\eta d_\eta = \alpha_1 \mathbb{E}[\eta e^{(\alpha_1 l_1 + \alpha_2 l_2)\eta}]$$

$$\frac{\partial}{\partial l_2}\mathbb{E}[e^{(\alpha_1 l_1 + \alpha_2 l_2)\eta}] = \alpha_2 \mathbb{E}[\eta e^{(\alpha_1 l_1 + \alpha_2 l_2)\eta}]$$

Then, we compute the partial derivatives of $g(x, \alpha_1, \alpha_2, l_1, l_2)$ with respect to l_1, l_2 starting from (3.34). We obtain that:

$$\frac{\partial g}{\partial l_1} = \frac{\alpha_1 x e^{(\alpha_1 l_1 + \alpha_2 l_2)x}\mathbb{E}[e^{(\alpha_1 l_1 + \alpha_2 l_2)\eta}] - e^{(\alpha_1 l_1 + \alpha_2 l_2)x}\alpha_1 \mathbb{E}[\eta e^{(\alpha_1 l_1 + \alpha_2 l_2)\eta}]}{(\mathbb{E}[e^{(\alpha_1 l_1 + \alpha_2 l_2)\eta}])^2} - \frac{\alpha_1 x e^{l_1 x}\mathbb{E}[e^{l_1 \eta}] - \alpha_1 e^{l_1 x}\mathbb{E}[\eta e^{l_1 \eta}]}{(\mathbb{E}[e^{l_1 \eta}])^2}$$

$$\frac{\partial g}{\partial l_1} = \alpha_1 \left(\frac{e^{(\alpha_1 l_1 + \alpha_2 l_2)x}(x\mathbb{E}[e^{(\alpha_1 l_1 + \alpha_2 l_2)\eta}] - \mathbb{E}[\eta e^{(\alpha_1 l_1 + \alpha_2 l_2)\eta}])}{(\mathbb{E}[e^{(\alpha_1 l_1 + \alpha_2 l_2)\eta}])^2} - \frac{e^{l_1 x}(x\mathbb{E}[e^{l_1 \eta}] - \mathbb{E}[\eta e^{l_1 \eta}])}{(\mathbb{E}[e^{l_1 \eta}])^2} \right)$$

$$\frac{\partial g}{\partial l_2}=\frac{\alpha_2 x e^{(\alpha_1 l_1+\alpha_2 l_2)x}\mathbb{E}[e^{(\alpha_1 l_1+\alpha_2 l_2)\eta}]-e^{(\alpha_1 l_1+\alpha_2 l_2)x}\alpha_2\mathbb{E}[\eta e^{(\alpha_1 l_1+\alpha_2 l_2)\eta}]}{(\mathbb{E}[e^{(\alpha_1 l_1+\alpha_2 l_2)\eta}])^2}-\frac{\alpha_2 x e^{l_2 x}\mathbb{E}[e^{l_2\eta}]-\alpha_2 e^{l_2 x}\mathbb{E}[\eta e^{l_2\eta}]}{(\mathbb{E}[e^{l_2\eta}])^2}$$

$$\frac{\partial g}{\partial l_2}=\alpha_2\left(\frac{e^{(\alpha_1 l_1+\alpha_2 l_2)x}(x\mathbb{E}[e^{(\alpha_1 l_1+\alpha_2 l_2)\eta}]-\mathbb{E}[\eta e^{(\alpha_1 l_1+\alpha_2 l_2)\eta}])}{(\mathbb{E}[e^{(\alpha_1 l_1+\alpha_2 l_2)\eta}])^2}-\frac{e^{l_2 x}(x\mathbb{E}[e^{l_2\eta}]-\mathbb{E}[\eta e^{l_2\eta}])}{(\mathbb{E}[e^{l_2\eta}])^2}\right)$$

□

We can observe that $h(t)$ has the same expression of the one computed in the case of one supplier (3.24), hence the expression of $h'(t)$ is given by (3.25).

As a consequence, from Proposition 4, we already know the condition for which $h(t)$ is an increasing or a decreasing function. However, differently from the case with a single supplier, the monotony of $h(t)$ is not sufficient to determine the sign of the partial derivatives of $g(x,\alpha_1,\alpha_2,l_1,l_2)$, hence the behavior of $g(x,\alpha_1,\alpha_2,l_1,l_2)$ cannot be directly inferred, as it depends on the specific choice of the parameters $\alpha_1,\alpha_2,l_1,l_2$.

Despite that, we can try to analyze the behavior of $g(x,\alpha_1,\alpha_2,l_1,l_2)$ in some simple cases, using the fact that, given a network whose shock η satisfies the assumption (3.18), the default intervals for all the firms satisfying (3.32) correspond with the intervals of l_1,l_2 where $g(x,\alpha_1,\alpha_2,l_1,l_2)<0$.

- if $l_1=l_2$, the strength of the network from the shock's source to the two suppliers is the same, then we can write:

  $$\frac{\partial g}{\partial l_1}=\alpha_1(h((\alpha_1+\alpha_2)l_1)-h(l_1));\quad \frac{\partial g}{\partial l_2}=\alpha_2(h((\alpha_1+\alpha_2)l_1)-h(l_1))$$

  Since $\alpha_1+\alpha_2<1$, then $(\alpha_1+\alpha_2)l_1<l_1$, we obtain that:

  - if $h(t)$ is an increasing function $\forall t>0$ then $\frac{\partial g}{\partial l_1}<0$; $\frac{\partial g}{\partial l_2}<0\ \forall l_1,l_2>0$ then $g(x,\alpha_1,\alpha_2,l_1,l_2)<0\ \forall l_1,l_2>0$,
  - if $h(t)$ is an decreasing function $\forall t>0$ then $\frac{\partial g}{\partial l_1}>0$; $\frac{\partial g}{\partial l_2}>0\ \forall l_1,l_2>0$ then $g(x,\alpha_1,\alpha_2,l_1,l_2)>0\ \forall l_1,l_2>0$;

- if $l_1>>l_2$, the strength of the network from the shock's source to the first supplier is greater than the same quantity for the second supplier, hence $\alpha_1 l_1+\alpha_2 l_2\sim\alpha_1 l_1$, then

  $$\frac{\partial g}{\partial l_1}\sim\alpha_1(h(\alpha_1 l_1)-h(l_1));\quad \frac{\partial g}{\partial l_2}\sim\alpha_2(h(\alpha_1 l_1)-h(l_2))$$

  we can suppose that the behavior of g depends only on $\frac{\partial g}{\partial l_1}$, using the fact that $\alpha_1+\alpha_2<1$ and α_1,α_2 are positive quantities, consequently $\alpha_1<1$, then we obtain that:

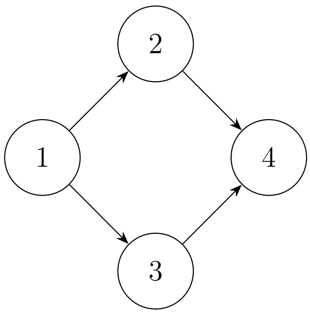
  - if $h(t)$ is increasing $\forall t>0$, $\frac{\partial g}{\partial l_1}<0\ \forall l_1>0$ then $g(x,\alpha_1,\alpha_2,l_1,l_2)<0\ \forall l_1,l_2>0$,
  - if $h(t)$ is decreasing $\forall t>0$, $\frac{\partial g}{\partial l_1}>0\ \forall l_1>0$ then $g(x,\alpha_1,\alpha_2,l_1,l_2)>0\ \forall l_1,l_2>0$;

- if $l_2 >> l_1$, the strength of the network from the shock's source to the second supplier is greater than the same quantity for the first supplier, hence $\alpha_1 l_1 + \alpha_2 l_2 \sim \alpha_2 l_2$, then
$$\frac{\partial g}{\partial l_1} \sim \alpha_1(h(\alpha_2 l_2) - h(l_1)); \quad \frac{\partial g}{\partial l_2} \sim \alpha_2(h(\alpha_2 l_2) - h(l_2))$$
we can repeat the same reasoning done above substituting $\frac{\partial g}{\partial l_1}$ with $\frac{\partial g}{\partial l_2}$ and we obtain the same result.

3.4.1 Example

This example shows that when a firm receives a single node shock through two suppliers with the same importance and the network is symmetric, the default condition for that industry is equivalent to the case in which the shock is transmitted through just one supplier.

We consider the case in which the shock is only on node 1, node 2 and 3 receive the shock from 1, while node 4 receives the shock from node 1 through both its suppliers 2 and 3.



We assume that the suppliers (node 2 and 3) of node 4 have the same importance in its production process, then the elements of the adjacency matrix for this network are defined as $A_{ij} = \frac{\alpha}{d_j}$, where $\alpha \in (0,1)$ and d_j is the in-degree of node j.

In our example $d_1 = 0, \; d_2 = 1, \; d_3 = 1, \; d_4 = 2$, hence $A_{12} = \alpha, \; A_{13} = \alpha, \; A_{24} = \frac{\alpha}{2}, \; A_{34} = \frac{\alpha}{2}$

$$A = \begin{pmatrix} 0 & \alpha & \alpha & 0 \\ 0 & 0 & 0 & \frac{\alpha}{2} \\ 0 & 0 & 0 & \frac{\alpha}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$L = (I - A')^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ \alpha & 0 & 1 & 0 \\ \alpha^2 & \frac{\alpha}{2} & \frac{\alpha}{2} & 1 \end{pmatrix}$$

Since the primitive shock is only on node 1, meaning that $\eta = [\eta_1, \eta_2, \eta_3, \eta_4] = [\eta_1, 0, 0, 0]$, then the total shock, defined in equation (2.19), becomes:

$$\rho_1 = \sum_{j=1}^{4} L_{1j}\eta_j = L_{11}\eta_1 = \eta_1$$

$$\rho_2 = \sum_{j=1}^{4} L_{2j}\eta_j = L_{21}\eta_1 = \alpha\eta_1$$

$$\rho_3 = \sum_{j=1}^{4} L_{3j}\eta_j = L_{31}\eta_1 = \alpha\eta_1$$

$$\rho_4 = \sum_{j=1}^{4} L_{4j}\eta_j = L_{41}\eta_1 = \alpha^2\eta_1$$

Using the definition of the normalized total shock and the suppliers' total shock expressed by equations (2.21), (2.22) respectively, for our example we obtain that:

$$\tau_1 = \frac{e^{\rho_1}}{\mathbb{E}[e^{\rho_1}]} = \frac{e^{\eta_1}}{\mathbb{E}[e^{\eta_1}]}; \qquad \epsilon_1 = 1$$

$$\tau_2 = \frac{e^{\rho_2}}{\mathbb{E}[e^{\rho_2}]} = \frac{e^{\alpha\eta_1}}{\mathbb{E}[e^{\alpha\eta_1}]}; \qquad \epsilon_2 = 1 - \sum_j A_{j2} + \sum_j A_{j2}\tau_j = 1 - \alpha + \alpha\tau_1$$

$$\tau_3 = \frac{e^{\rho_3}}{\mathbb{E}[e^{\rho_3}]} = \frac{e^{\alpha\eta_1}}{\mathbb{E}[e^{\alpha\eta_1}]}; \qquad \epsilon_3 = 1 - \sum_j A_{j3} + \sum_j A_{j3}\tau_j = 1 - \alpha + \alpha\tau_1$$

$$\tau_4 = \frac{e^{\rho_4}}{\mathbb{E}[e^{\rho_4}]} = \frac{e^{\alpha^2\eta_1}}{\mathbb{E}[e^{\alpha^2\eta_1}]}; \qquad \epsilon_4 = 1 - \sum_j A_{j4} + \sum_j A_{j4}\tau_j = 1 - \frac{\alpha}{2} - \frac{\alpha}{2} + \frac{\alpha}{2}\tau_2 + \frac{\alpha}{2}\tau_3 = 1 - \alpha + \alpha\tau_2$$

From equation (3.2), we know that the default is realized when $\epsilon_k > \tau_k$, then in our example we have that:

$$\epsilon_1 > \tau_1 \text{ is equivalent to } e^{\eta_1} < \mathbb{E}[e^{\eta_1}] \tag{3.36}$$

$$\epsilon_2 > \tau_2 \text{ and } \epsilon_3 > \tau_3 \text{ is equivalent to } \frac{e^{\alpha\eta_1}}{\mathbb{E}[e^{\alpha\eta_1}]} - (1-\alpha) - \alpha\frac{e^{\eta_1}}{\mathbb{E}[e^{\eta_1}]} < 0 \tag{3.37}$$

$$\epsilon_4 > \tau_4 \text{ is equivalent to } \frac{e^{\alpha^2\eta_1}}{\mathbb{E}[e^{\alpha^2\eta_1}]} - (1-\alpha) - \alpha\frac{e^{\alpha\eta_1}}{\mathbb{E}[e^{\alpha\eta_1}]} < 0 \tag{3.38}$$

For nodes $2, 3$, the condition $\tau_k - \epsilon_k < 0$, expressed by (3.37), is equivalent to $g(x, \alpha, l) < 0$, where $g(x, \alpha, l)$ is defined as in (3.22).

For node 4, the default condition, expressed by (3.38), is equivalent to $g(x, \alpha_1, \alpha_2, l_1, l_2) < 0$, which is defined as in (3.34), taking, as in our example, $l_1 = l_2$ and $\alpha_1 + \alpha_2 = \alpha$.

It is worth to notice that it is the same condition we would have found if we had considered 4 as the third node of a line with the shock on node 1, that is the case in which there is only one supplier which separates node 4 from the shock's source. Hence, the default condition for node 4 that receives the single node shock through two suppliers with the same importance is equivalent to the case in which the shock is transmitted through just one supplier.

3.4.2 Single node exponential shock transmitted through two suppliers

We decide to use numerical simulations to show that the default condition, in the case in which the single-node shock comes from an exponential distribution and it is transmitted through two suppliers, is similar to one found when a single-node exponential shock is propagated through just one supplier.

For an exponential distribution, the general expression of $g(x, \alpha_1, \alpha_2, l_1, l_2)$, as given in (3.34), takes the following form:

$$g(x, \alpha_1, \alpha_2, l_1, l_2) = \frac{(e^{x(\alpha_1 l_1 + \alpha_2 l_2)}((\alpha_1 l_1 + \alpha_2 l_2) + \lambda))}{\lambda} - (1 - \alpha_1 - \alpha_2) - \frac{\alpha_1 e^{x l_1}(l_1 + \lambda)}{\lambda} - \frac{\alpha_2 e^{x l_2}(l_2 + \lambda)}{\lambda}$$

Choice of parameters (changes in α_1, α_2 do not modify the general behavior of $g(x, \alpha_1, \alpha_2, l_1, l_2)$):

- $\lambda = 1$ (we decide to fix λ and try different values of x)
- $\alpha_1 = 0.3$
- $\alpha_2 = 0.5$

x	g_{min}	g_{max}
-4	-0.70000	-5.5511e-17
-3	-0.69999	-5.5511e-17
-2.1	-0.69957	-5.5511e-17
-1.9	-0.69894	3.5377e-05
-1	-0.63901	0.058775
-0.5	-0.3926	0.33666
-0.1	-5.5511e-17	1.7359

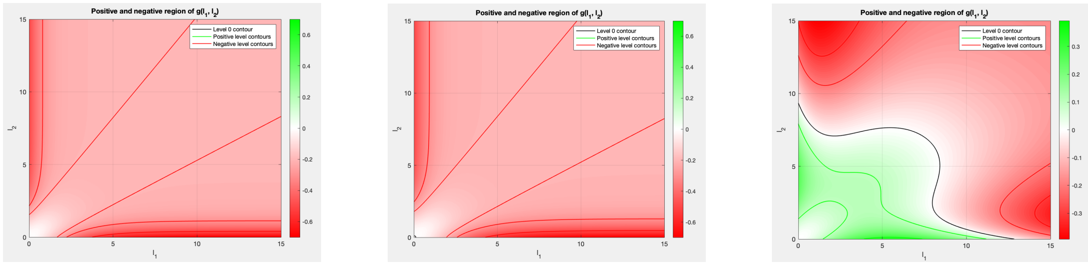
Table 3.2: Exponential results with two suppliers

In Table 3.2, we underline in red the chance of behavior of the function $g(x, \alpha_1, \alpha_2, l_1, l_2)$ near to the point $x = -\frac{2}{\lambda}$ ($x = -2$ in our example). In fact, we observe that:

- if $x < -\frac{2}{\lambda}$, then $g(x, \alpha_1, \alpha_2, l_1, l_2) < 0 \ \forall l_1, l_2 > 0$
- if $x \geq -\frac{2}{\lambda}$, $g(x, \alpha_1, \alpha_2, l_1, l_2)$ can also assume positive values. In particular, for small values of l_1, l_2, $g(x, \alpha_1, \alpha_2, l_1, l_2)$ is positive, while for large values of l_1, l_2, it becomes negative (contour lines in Figure 3.4).

3.4.3 Single node gamma shock transmitted through two suppliers

We perform numerical simulations to analyze the default condition in the case in which the single-node shock comes from a gamma distribution and it is transmitted through two suppliers.



(a) $x = -2.1$ (b) $x = -1.9$ (c) $x = -0.5$

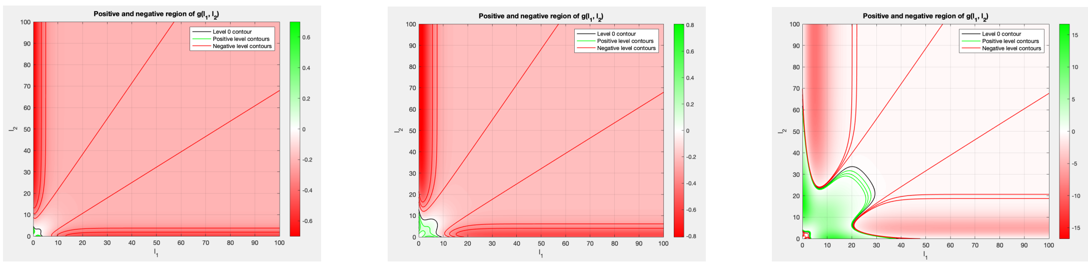
Figure 3.4: Positive and negative regions of $g(x, \alpha_1, \alpha_2, l_1, l_2)$ with an exponential shock

For a gamma distribution $Gamma(\alpha, \beta)$, the general expression of $g(x, \alpha_1, \alpha_2, l_1, l_2)$, as given in (3.34), becomes:

$$g(x, \alpha_1, \alpha_2, l_1, l_2) = \frac{e^{x(\alpha_1 l_1 + \alpha_2 l_2)}}{\left(\frac{\beta}{\beta + (\alpha_1 l_1 + \alpha_2 l_2)}\right)^{\alpha}} - (1 - \alpha_1 - \alpha_2) - \frac{\alpha_1 e^{x l_1}}{\left(\frac{\beta}{\beta + l_1}\right)^{\alpha}} - \frac{\alpha_2 e^{x l_2}}{\left(\frac{\beta}{\beta + l_2}\right)^{\alpha}}$$

Choice of parameters:

- $x = -0.5$
- $\alpha_1 = 0.5$
- $\alpha_2 = 0.3$



(a) $\alpha = 0.5, \beta = 1$ (b) $\alpha = 1, \beta = 1$ (c) $\alpha = 3, \beta = 1$

Figure 3.5: Positive and negative regions of $g(x, \alpha_1, \alpha_2, l_1, l_2)$ with a gamma shock

From Figure 3.5 it is possible to observe that if $\alpha \leq 1$, the default condition is an interval of l_1, l_2, in fact $g(x, \alpha_1, \alpha_2, l_1, l_2) \geq 0$ for small values of l_1, l_2, then it becomes and remains negative for the other values of l_1, l_2. On the contrary, when $\alpha > 1$, $g(x, \alpha_1, \alpha_2, l_1, l_2) \leq 0$ for small values of l_1, l_2, $g(x, \alpha_1, \alpha_2, l_1, l_2) > 0$ for intermediate values and $g(x, \alpha_1, \alpha_2, l_1, l_2) < 0$ for large values of l_1, l_2; hence more than one default interval might be generated.

Numerical simulations show that the default condition for a single node gamma shock transmitted through two suppliers depends on the value of the parameter α and it is equivalent to the analysis with a single node gamma shock propagated through just one supplier.

3.4.4 Single node Bernoulli shock transmitted through two suppliers

We can show that the default condition, in the case in which the single node shock comes from a Bernoulli distribution and it is transmitted through two suppliers, is similar to one found with a single node Bernoulli shock propagated through just one supplier.

Case in which the shock is realized ($x = -\epsilon$):

For a Bernoulli distribution, if the shock is realized, the general expression of $g(x, \alpha_1, \alpha_2, l_1, l_2)$, as given in (3.34), reduces to:

$$g(x, \alpha_1, \alpha_2, l_1, l_2) = \frac{e^{-\epsilon(\alpha_1 l_1 + \alpha_2 l_2)}}{1 - p + pe^{-\epsilon(\alpha_1 l_1 + \alpha_2 l_2)}} - (1 - \alpha_1 - \alpha_2) - \frac{\alpha_1 e^{-\epsilon l_1}}{1 - p + pe^{-\epsilon l_1}} - \frac{\alpha_2 e^{-\epsilon l_2}}{1 - p + pe^{-\epsilon l_2}}$$

Choice of parameters (changes in α_1, α_2 do not modify the general behavior of $g(l_1, l_2)$):

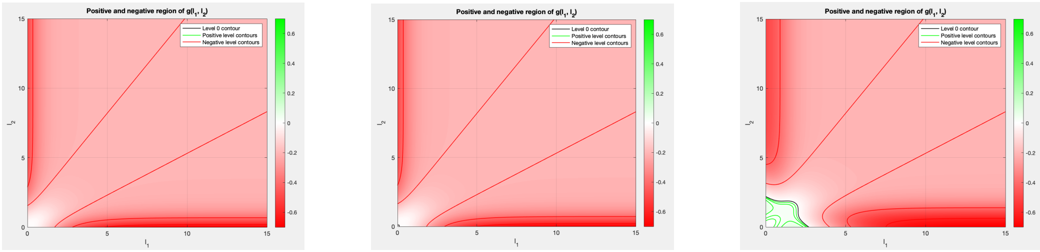
- $\alpha_1 = 0.3$
- $\alpha_2 = 0.5$
- $\epsilon = 2$

p	g_{min}	g_{max}
0.1	-0.69986	-5.5511e-17
0.3	-0.69982	-5.5511e-17
0.49	-0.69976	-5.5511e-17
0.53	-0.69974	3.31e-05
0.8	-0.69938	0.027326
0.9	-0.69877	0.072393

Table 3.3: Bernoulli results with two suppliers and $x = -\epsilon$

In Table 3.3, we underline in red the chance of behavior of the function $g(x, \alpha_1, \alpha_2, l_1, l_2)$ near to the point $p = \frac{1}{2}$, in the case in which the Bernoulli shock is realized. In fact, we observe that:

- if $p < \frac{1}{2}$, $g(x, \alpha_1, \alpha_2, l_1, l_2) < 0 \; \forall l_1, l_2 > 0$
- if $p > \frac{1}{2}$, $g(x, \alpha_1, \alpha_2, l_1, l_2)$ can also assume positive values. In particular, for small values of l, $g(x, \alpha_1, \alpha_2, l_1, l_2)$ is positive, while for large values of l, it becomes negative (contour lines in Figure 3.6).



(a) $p = 0.48$ (b) $p = 0.53$ (c) $p = 0.9$

Figure 3.6: Positive and negative regions of $g(x, \alpha_1, \alpha_2, l_1, l_2)$ with a realized Bernoulli shock

Case in which the shock is not realized ($x = 0$):

For a Bernoulli distribution, if the shock is not present, the general expression of $g(x, \alpha_1, \alpha_2, l_1, l_2)$, as given in (3.34), simplifies to:

$$g(x, \alpha_1, \alpha_2, l_1, l_2) = \frac{1}{1 - p + pe^{-\epsilon(\alpha_1 l_1 + \alpha_2 l_2)}} - (1 - \alpha_1 - \alpha_2) - \frac{1}{1 - p + pe^{-\epsilon l_1}} - \frac{1}{1 - p + pe^{-\epsilon l_2}}$$

Choice of parameters (changes in α_1, α_2 do not modify the general behavior of $g(l_1, l_2)$):

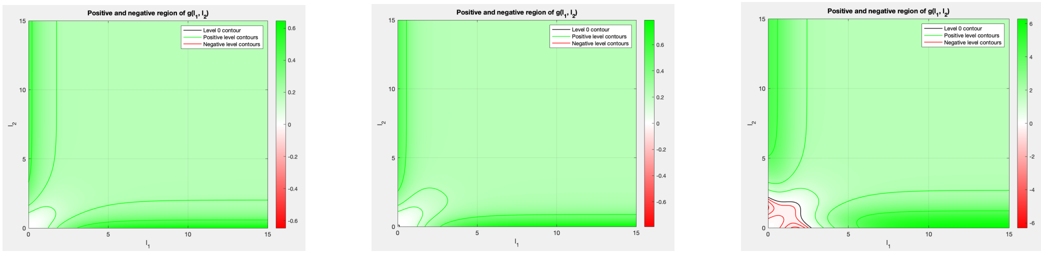
- $\alpha_1 = 0.3$
- $\alpha_2 = 0.5$
- $\epsilon = 2$

p	g_{min}	g_{max}
0.1	-5.5511e-17	0.077763
0.3	-5.5511e-17	0.29992
0.49	-5.5511e-17	0.67232
0.53	-3.7325e-05	0.78907
0.8	-0.10931	2.7975
0.9	-0.65154	6.2889

Table 3.4: Bernoulli results with $x = 0$

In Table 3.4, we underline in red the chance of behavior of the function $g(x, \alpha_1, \alpha_2, l_1, l_2)$ near to the point $p = \frac{1}{2}$, in the case in which the Bernoulli shock is not realized. In fact, we observe that:

- if $p < \frac{1}{2}$, $g(x, \alpha_1, \alpha_2, l_1, l_2) > 0 \ \forall l_1, l_2 > 0$
- if $p > \frac{1}{2}$, $g(x, \alpha_1, \alpha_2, l_1, l_2)$ can also assume negative values. In particular, for small values of l_1, l_2, $g(x, \alpha_1, \alpha_2, l_1, l_2)$ is negative, while for large values of l_1, l_2, it becomes positive (contour lines in Figure 3.7).



(a) $p = 0.48$ (b) $p = 0.53$ (c) $p = 0.53$

Figure 3.7: Positive and negative regions of $g(x, \alpha_1, \alpha_2, l_1, l_2)$ with a non-realized Bernoulli shock

We have shown that in the case in which the single node shock is transmitted through two suppliers, the function $h(t)$, used to understand the default condition, is equal to the one used in the case in which there is just one supplier.

However, since $g(x, \alpha_1, \alpha_2, l_1, l_2)$ is a complex two dimensional function to analyze, it is hard to understand the behavior of it starting from $h'(t)$. Despite that, from numerical simulations, we can deduce that $g(x, \alpha_1, \alpha_2, l_1, l_2)$ has a similar behavior of $g(x, \alpha, l)$ in all exponential, gamma and Bernoulli cases.

Chapter 4

Model's simulations

The aim of this chapter is to show and to explain the results obtained from the implementation of the model.

Simulations are performed using a MonteCarlo method to generate shock's samples and to compute the mean of their exponential transformation and selecting exponential, gamma and Bernoulli density functions as shock's distributions.

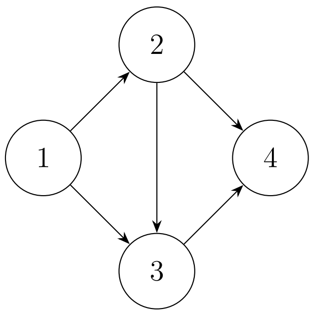
First, we verify the results obtained in the example made in section 2.3.

Then, we test the model choosing as network structures a line, a DAG and a cycle.

Finally, we analyze a real world network composed of 62 firms extracting its adjacency matrix from a symmetric table 'branch by branch'.

4.1 Example

We show the results of the example presented in section 2.3.



Fixing $a_{12} = 0.3$, $a_{13} = 0.4$, $a_{23} = 0.2$, $a_{24} = 0.5$, $a_{34} = 0.3$, we obtain that the adjacency matrix is:

$$A = \begin{pmatrix} 0 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0.2 & 0.5 \\ 0 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Using the fact that $L = (I - A')^{-1}$, the Leontief matrix is:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 1 & 0 & 0 \\ 0.46 & 0.2 & 1 & 0 \\ 0.288 & 0.56 & 0.3 & 1 \end{pmatrix}$$

We fix the following quantities:

- wage: $w = 1$,
- consumer preference weight: $\gamma = [0.25, 0.25, 0.25, 0.25]$,
- fraction of each firm's liabilities financed by the bank: $\theta = [0.2, 0.5, 0.4, 0.7]$,
- interest rate: $r = [0.1, 0.3, 0.2, 0.4]$, they are fixed since we use exogenous interest rates in this example,
- exponential parameter: $\lambda = [2, 0.5, 1, 3]$.

We obtain the following results:

- debt cost: $\zeta = [0.0198, 0.1398, 0.0770, 0.2469]$,
- total cost: $\xi = [0.0198, 0.1457, 0.1140, 0.3539]$.

  The debt cost has all positive components since the fraction of liabilities financed by the bank is different from 0 for all the nodes involved in our example. Moreover, the total cost is larger or equal than the debt cost for each firm because it takes into account the leverage which is propagated through the network.

- Distorted Leontief matrix:

$$L^{\zeta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.261 & 1 & 0 & 0 \\ 0.419 & 0.185 & 1 & 0 \\ 0.200 & 0.434 & 0.234 & 1 \end{pmatrix}$$

- normalization factor: $\psi = 0.8604$,
- Bonacich centrality: $v^0 = [0.512, 0.44, 0.325, 0.25]$,
- distorted Bonacich centrality: $v^{\zeta} = [0.5461, 0.4705, 0.3586, 0.2905]$.

Since the debt cost is different from 0 for all the firms, the distorted centrality differs from the Bonacich centrality, however the centralities' order is invariant. In fact, for instance, the first firm is the one with the highest centrality both before and after the distortion is realized.

The quantities at the equilibrium are:

- maximal production: $y^0 = [0.5354, 0.4067, 0.3200, 0.2039]$,

- actual production: $y^\eta = [0.3570, 0.1168, 0.0932, 0.0486]$;
- maximal intermediate quantities:

$$z_{jk}^0 = \begin{pmatrix} 0 & 0.1203 & 0.1302 & 0 \\ 0 & 0.5741 & 0.0981 & 0 \\ 0 & 0 & 0 & 0.0608 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- actual intermediate quantities:

$$z_{jk}^\eta = \begin{pmatrix} 0 & 0.0802 & 0.0868 & 0 \\ 0 & 0 & 0.0165 & 0.0282 \\ 0 & 0 & 0 & 0.0177 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- labor: $l = [0.5354, 0.2864, 0.1328, 0.0454]$;
- maximal household consumption: $c^0 = [0.2849, 0.2512, 0.2592, 0.2039]$,
- actual household consumption: $c^\eta = [0.1899, 0.0722, 0.0755, 0.0486]$;
- prices over wages: $\frac{p}{w} = [1.5298, 4.0266, 3.8468, 5.9771]$.

We can notice that actual intermediate quantities, household consumption and consequently actual production are smaller then their maximal values, as expected from the theory, because they are affected by a negative exponential shock.

The variances and covariances of profits and the default probabilities for each industry are:

- $\tau_k - \epsilon_k$ variances: $[0.1235, 0.8041, 0.4116, 0.1406]$

  Since the distorted centrality is a deterministic quantity and the profit is defined as in equation (2.44), if we multiply the variance of $\tau_k - \epsilon_k$ by the distorted Bonacich centrality squared, we obtain the profit variance:

- profit variances: $[0.0374, 0.1803, 0.0526, 0.0114]$,
- total variance of profits: 0.2816;
- profit covariances:

$$\begin{pmatrix} 0.0374 & 0.0027 & 0.0015 & 0.0002 \\ 0.0030 & 0.1803 & 0.0136 & 0.0102 \\ 0.0015 & 0.0136 & 0.0526 & 0.0071 \\ 0.0002 & 0.0100 & 0.0071 & 0.0114 \end{pmatrix}$$

- default probabilities: $[0.4443, 0.5766, 0.5355, 0.5474]$

We can notice that shock on node 2 has a greater impact on the corresponding firm than shocks on the other nodes. In fact, its profit variance and its default probability are larger than the same quantities on the other nodes. This effect is coherent with the fact that the second firm is hit by an exponential shock with the smallest λ parameter, hence the distribution is more concentrated on negative values, as a consequence shocks are more intense and likely.

The profit variance of node 4 is the lowest, as it has the highest λ parameter, and is therefore subject to the weakest shock. However, this does not imply that it has the lowest default probability. Indeed, for the chosen value of θ, this node receives the largest amount of financing from the bank; consequently, its debt cost is high, which may make loan repayment more difficult.

We also observe that profit variance is determined not only by the distorted Bonacich centrality, but also by the term $\tau_k - \epsilon_k$, which reflects the strength of the shock. Indeed, although node 1 exhibits the highest distorted centrality, it does not correspond to the maximum profit variance, which is observed in node 2.

4.2 Line, DAG and cycle structures tested on different shock distributions

The results obtained in the following subsections show that the default probabilities and profit variances decrease moving away from the node where the shock occurs.

When the shock affects all nodes rather than a single one, the profit variances across the different nodes become similar and the total profit variance is larger than the one obtained when the shock hits a single node.

Moreover, when the shock is restricted to a specific node and the network structure is a line or a DAG, only its direct and indirect consumers are influenced by the shock's effects, while its suppliers are not affected by them. In fact, their default probabilities and their profit variances are equal to 0.

Conversely, in the presence of a cycle, a node may still receive the shock even if it is a supplier of the initially affected node, as the propagation can reach it through another node in the cycle that acts as a consumer of the shocked firm.

Another observation is the fact that when the shock hits a single node and the network structure is a line or a DAG, the total variance of profits is greater when the impacted node is the first in the production line. On the contrary, in the case of a cycle, the profit variance is larger when the node affected by the shock belongs to the cycle.

4.2.1 Fixed parameters

The parameters fixed in all the following simulations are:

- $N = 6$ nodes,
- $A_{ij} = \frac{\alpha}{d_j}$, with $\alpha = 0.5$ and d_j is the in-degree of node j,
- wage: $w = 1$,

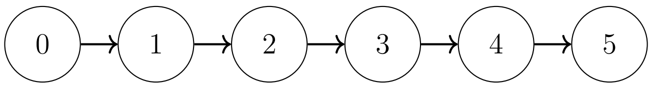
- consumers preference weight: $\gamma_k = \frac{1}{N}$ $\forall k = 0, ..,5$,
- fraction of the liabilities financed by the bank: $\theta_k = 0.25$, $\forall k = 0, ..,5$,
- interest rate: $r_k = 1$, $\forall k = 0, ..,5$, they are fixed since we use exogenous interest rates in these examples.

4.2.2 Exponential shock

The parameters of the exponential distribution chosen for these experiments are:

- exponential parameter: $\lambda = 2$,
- amplitude of the negative exponential shock: scale $= -1$.

Line



$$A = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From Tables 4.1 and 4.2 we can observe that, in the case of a line structure, the default probabilities and the profit variances decrease as firms are further away from the shock's source and they are null for the suppliers of the node affected by the shock. In fact, when the shock hits node 3, firms 0, 1, 2 are not affected by it, since they are suppliers of node 3 and the propagation is downstream.

Shocked node	**Default probabilities**
Node 0	$[0.58, 0.34, 0.26, 0.20, 0.17, 0.16]$
Node 3	$[0, 0, 0, 0.58, 0.34, 0.25]$
All nodes	$[0.58, 0.63, 0.64, 0.64, 0.65, 0.64]$

Table 4.1: Default probabilities under an exponential shock in a line network

From Table 4.3, we can notice that when the shock is restricted to an individual node, the total profit variance reaches its maximum if the first node in the network is impacted. However, the effect of the shocks is stronger when all nodes are hit, as the total profit variance in this case exceeds that observed when only a single node is affected.

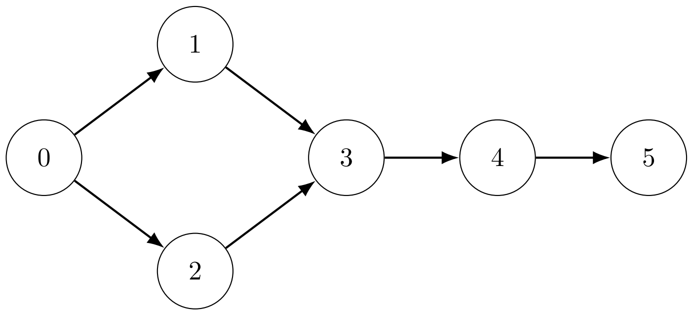
Shocked node	**Profit variances**
Node 0	[1.28e-01, 5.34e-03, 1.31e-03, 2.16e-04, 2.21e-05, 1.08e-06]
Node 3	[0, 0, 0, 1.10e-01, 3.76e-03, 4.81e-04]
All nodes	[1.22e-01, 1.67e-01, 1.85e-01, 1.76e-01, 1.49e-01, 7.53e-02]

Table 4.2: Profit variances under an exponential shock in a line network

Shocked node	**Total variance of profits**
Node 0	0.135
Node 3	0.114
All nodes	0.873

Table 4.3: Total variance of profits under an exponential shock in a line network

DAG



$$A = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Differently from the line case, in a DAG structure, nodes 1 and 2 are both directly connected with node 0 and they are both the suppliers of node 3. When the shock's source is node 0, they have the same default probability and profit variance.

From Tables 4.4, 4.5, we can see that both default probabilities and profit variances decrease as firms are located further from the shock origin and they become zero for the suppliers of the affected node, as in a line structure.

Shocked node	**Default probabilities**
Node 0	$[0.58, 0.34, 0.34, 0.26, 0.20, 0.17]$
Node 3	$[0, 0, 0, 0.58, 0.34, 0.25]$
All nodes	$[0.58, 0.63, 0.63, 0.64, 0.64, 0.64]$

Table 4.4: Default probabilities under an exponential shock in a DAG network

Shocked node	**Profit variances**
Node 0	[1.90e-01, 3.29e-03, 3.29e-03, 1.17e-03, 1.70e-04, 1.10e-05]
Node 3	[0, 0, 0, 1.08e-01, 3.67e-03, 4.70e-04]
All nodes	[1.82e-01, 1.03e-01, 1.04e-01, 1.63e-01, 1.37e-01, 7.23e-02]

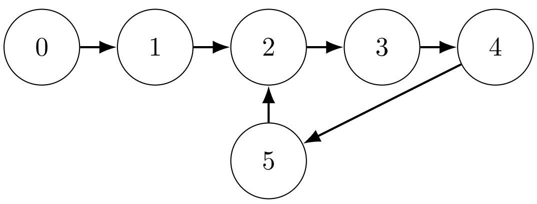
Table 4.5: Profit variances under an exponential shock in a DAG network

From Table 4.6, we observe that, while the total variance is larger when the shock affects all nodes simultaneously, in the case of a single node shock, it is maximized when the first node in the network is impacted, as in a line structure.

Shocked node	**Total variance of profits**
Node 0	0.198
Node 3	0.112
All nodes	0.760

Table 4.6: Total variance of profits under an exponential shock in a DAG network

Cycle



$$A = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.25 & 0 & 0 & 0 \end{pmatrix}$$

The aspects that remain unchanged also in the case of a cycle are the fact that the total variance is greater when the shock affects all nodes simultaneously rather than a single node as shown in Table 4.9. Furthermore, both default probabilities and profit variances decrease as firms are located further from the shock origin as reported in Tables 4.7, 4.8.

Shocked node	**Default probabilities**
Node 0	$[0.58, 0.34, 0.24, 0.17, 0.16, 0.15]$
Node 3	$[0, 0, 0.19, 0.60, 0.34, 0.26]$
All nodes	$[0.58, 0.63, 0.63, 0.65, 0.65, 0.65]$

Table 4.7: Default probabilities under an exponential shock in a cycle network

Both default probabilities and profit variances might not be zero for the suppliers of the affected node. In fact, from Tables 4.7, 4.8, we can observe that, when the shock is located on node 3, although node 2 is its supplier, its profit variance and default probability are different from 0. The reason of this behavior is the fact that node 2 belongs to the cycle, hence it receives the shock's propagation from node 5, which is an indirect consumer of node 3.

Shocked node	**Profit variances**
Node 0	[1.10e-01, 3.49e-03, 1.04e-03, 3.28e-05, 2.86e-06, 1.71e-07]
Node 3	[0, 0, 1.93e-04, 1.14e-01, 4.91e-03, 9.42e-04]
All nodes	[1.08e-01, 1.07e-01, 1.74e-01, 1.75e-01, 1.60e-01, 1.22e-01]

Table 4.8: Profit variances under an exponential shock in a cycle network

Differently from the other topologies, from Table 4.9, we can see that when the shock is confined to a single node, the total variance reaches its maximum value when a node within the cycle is impacted, rather than the first node.

Shocked node	Total variance of profits
Node 0	0.114
Node 3	0.120
All nodes	0.846

Table 4.9: Total variance of profits under an exponential shock in a cycle network

The results obtained with a gamma and a Bernoulli shock are shown in the appendix A.1, A.2 since they are similar to the exponential case.

4.2.3 Combination of two Bernoulli shocks

We consider two Bernoulli shock distributions, the first one on node 0 and the second one on node 3.

The parameters of the Bernoulli distributions chosen for these experiments are:

- amplitude of the two negative Bernoulli shocks: $\epsilon = -2$,
- shock's probability on node 0 : $p_1 = 0.65$, shock's probability on node 3 : $p_2 = 0.45$.

Line

In a line structure, we obtain that the default probabilities are:

$$[0.65, 0.65, 0.35, 0.45, 0.45, 0.45]$$

with profit variances:

$$[1.39e-01, 2.32e-04, 2.86e-06, 6.86e-02, 8.75e-04, 2.06e-05]$$

leading to a total variance of profits of 0.209.

DAG

In a DAG structure, we have that the default probabilities are:

$$[0.65, 0.65, 0.65, 0.45, 0.45, 0.65]$$

with profit variances:

$$[2.05e-01, 1.50e-04, 1.50e-04, 7.03e-02, 9.54e-04, 2.68e-05]$$

implying a total variance of profits equal to 0.277.

In general, when the shock comes from a Bernoulli with parameter p, the positive default probabilities correspond with p or $1-p$ for all nodes.

Both in the case of the line and in that of the DAG, we can observe that the default probabilities are equal to p_1 or $1 - p_1$ and p_2 or $1 - p_2$, in particular nodes 1 and 2 are influenced by the shock on node 0, while nodes 4 and 5 are affected by the shock on node 3, as it is closer to them than node 0. Then, the variation in the profit variance of node 4 and 5 is greater than the one obtained when the shock is confined solely to node 0.

Cycle

In a cycle structure, we obtain that the default probabilities are:

$$[0.65, 0.65, 0.15, 0.45, 0.45, 0.45]$$

with profit variances:

$$[1.19e - 01, 1.59e - 04, 1.21e - 05, 6.92e - 02, 1.16e - 03, 3.90e - 05]$$

which lead to a total variance of profits equal to 0.189.

We can notice that node 2 is affected both by the shock originating from node 0, as it acts as its supplier, and by the shock transmitted from node 3 through the propagation coming from node 5. Consequently, the default probability of node 2 does not coincide with p_1 or p_2, as it does in the linear and DAG configurations.

For all the three structures, the total variance of profits lies between the values obtained when a Bernoulli shock affects a single node and when it impacts all nodes (cases shown in the appendix A.2).

4.3 Simulations on a real world network

Now we want to test the model on a real network structure, shown in Figure 4.1, which is composed of 62 firms that are highly interconnected. We extract its adjacency matrix from a symmetric table 'branch by branch'.

4.3.1 Extraction of the adjacency matrix from a symmetric table

A symmetric table 'branch by branch' [13] describes the industrial relationships and, for each branch, the use of goods produced by other branches.

The branch technology assumption implies that each product has its specific sale structure, meaning that the proportion of output sold to intermediate and final uses is the same, independently from the branch. The matrix of direct coefficients 'branch by branch' is:

$$A = DB \tag{4.1}$$

where B is the matrix of intermediate coefficients and D is the matrix of market shares.

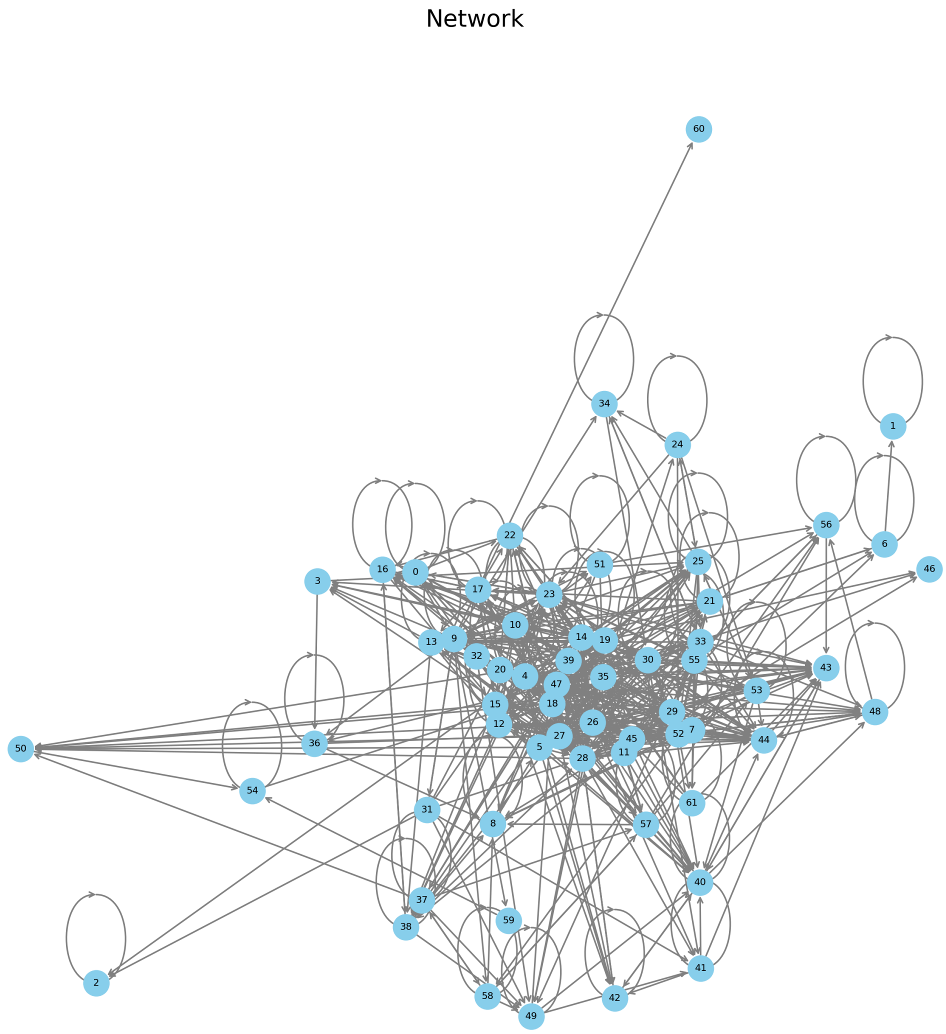


Figure 4.1: Network structure

The element a_{ij} of matrix A is given by the ratio between the input of sector i used in sector j and the total output given by the intermediate production of the various sectors plus the value added which contains wages, profits, and taxes:

$$a_{ij} = \frac{\text{input of sector } i \text{ used in sector } j}{\text{total output of sector } j}$$

With this procedure we compute the coefficients of the adjacency matrix for the network shown in Figure 4.1 and use them in the following examples.

4.3.2 Experiments with a real network structure

The parameters chosen for all the following simulations are:

- $N = 62$ nodes,
- wage: $w = 1$,
- consumers preference weight: $\gamma = \frac{1}{N}$ $\forall k = 0, ..,61$,
- negative exponential shock: $\lambda = 2$, shock's amplitude : scale $= -1$.

We perform the following simulations using only an exponential shock since the choice of the shock's distribution does not influence the results. We examine different cases:

1. the fractions of liabilities financed by the bank are null for all the firms in the network: $\theta_k = 0$ $\forall k = 0, ..,61$;
2. the fractions of liabilities financed by the bank and the interest rates are exogenous, meaning that they are fixed: $\theta_k = 0.25$, $r_k = 0.2$ $\forall k = 0, ...,61$ and the debt cost is a deterministic quantity computed through (2.25);
3. the fractions of liabilities financed by the bank are exogenous, in particular we choose $\theta_k = 0.25$, $\forall k = 0, ...,61$, while the interest rates are endogenous, meaning that the debt cost is found from proposition (2).

The results presented below suggest several observations:

- The Bonacich centrality assumes the same values in the three examples, since it depends only on the network structure and consumers preference.
- The distorted Bonacich centrality is different in the three cases, as it is influenced by the values of the debt cost. In fact, in the first example, the debt cost is null since the fractions of firms' liabilities financed by the bank are zero, in the second setting, it is deterministic due to the fact that the interest rates are exogenous. Finally, in the last case, the debt cost depends on the shock's samples because the interest rates are endogenous.
- The total variance of profits takes higher values when the shock is on all nodes rather than on a single node.
- The default probabilities are identical in the three examples for each node, which is consistent with the fact that they do not depend on the choice of θ and r. Conversely, although the profit variances are of the same order of magnitude, they differ across the three cases, as profits (2.44) are also affected by the distorted centrality, which depends on the debt cost, determined by the choice of θ and r.

Case 1.

The Bonacich centrality is equal to the distorted Bonacich centrality when the bank does not finance firms, since $\theta_k = 0$ $\forall k = 0, ..,61$, hence the debt cost $\zeta_k = 0$, $\forall k = 0, ...,61$.

Case 2.

The Bonacich centrality is different from the distorted Bonacich centrality when the bank finances a fraction of firms' liabilities, since θ_k and consequently ζ_k are not null for all the firms.

The centrality does not depend on which node the shock is located, meaning that it assumes the same values when the shock is on node 0 or on all the nodes, because the interest rates are chosen exogenously. Then, the debt cost is not influenced by the shock's samples and it is deterministic.

Case 3.

As in the second case, the Bonacich centrality is different from the distorted Bonacich centrality since the bank finances a fraction of firms' liabilities, in fact $\theta_k = 0.25\ \forall k = 0, ..., 61$.

However, differently from the previous cases, the centrality depends on which node the shock is located because the interest rates are chosen endogenously, hence the debt cost changes as the shock samples vary.

For all the three cases, the explicit computations of the distorted Bonacich centrality, default probabilities, profit variances and total profits variance, when the shock is on node 0 or on all the nodes, are reported in the appendix A.3.

Total variance of profits in relation with firms' centrality

For this experiment we choose exogenous interest rates, as in case 2. above.

We observe which is the node that, when it is affected by the shock, maximizes the total variance of profits.

We might aspect that the total variance of profits assumes the maximum and the minimum values when the shock hits node 14 and node 2 respectively, since they are the ones with the highest and the lowest distorted Bonacich centrality (Figure 4.2).

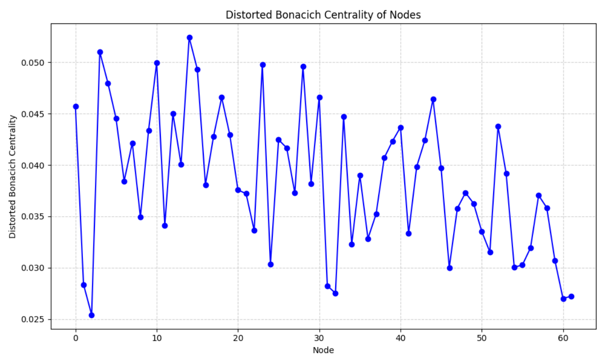


Figure 4.2: Distorted Bonacich centrality

From Table 4.10 we can observe that, although node 14 is the one with the highest distorted centrality, shock on it is not associated with the largest total profit variance. In fact node 3, 10, 28 have a larger total profit variance than node 14. It can happen because the profit (2.44) does not depend only on the distorted centrality, but it also is influenced by the shock samples, through the difference $\tau_k - \epsilon_k$. On the contrary, node 2, which is the one with the lowest distorted centrality, has also the smallest total profit variance.

Shock on node	0	1	2	3	4	5
Total profit variance	1.55e-03	6.25e-04	5.12e-04	2.13e-03	1.24e-03	9.93e-04

6	7	8	9	10	11	12	13
1.06e-03	1.21e-03	9.56e-04	1.29e-03	1.69e-03	8.61e-04	1.45e-03	1.15e-03
14	15	16	17	18	19	20	21
1.68e-03	1.66e-03	1.08e-03	1.28e-03	1.37e-03	1.23e-03	9.30e-04	9.47e-04
22	23	24	25	26	27	28	29
8.92e-04	1.43e-03	7.09e-04	1.34e-03	1.04e-03	1.06e-03	1.83e-03	1.16e-03
30	31	32	33	34	35	36	37
1.60e-03	6.21e-04	6.16e-04	1.49e-03	8.15e-04	1.19e-03	8.83e-04	8.90e-04
38	39	40	41	42	43	44	45
1.14e-03	1.30e-03	1.46e-03	8.36e-04	1.25e-03	1.44e-03	1.67e-03	1.21e-03
46	47	48	49	50	51	52	53
7.32e-04	1.04e-03	1.08e-03	1.01e-03	8.82e-04	7.83e-04	1.46e-03	1.22e-03
54	55	56	57	58	59	60	61
7.28e-04	6.74e-04	7.64e-04	9.55e-04	9.43e-04	7.78e-04	5.94e-04	5.96e-04

Table 4.10: Total profit variance varying the shock's source

We then fix the shock on node 0 and analyze which nodes exert the greatest influence on it, in terms of default probability and total profit variance. Our aim is to determine whether the greatest risk to firm 0 arises when the shock affects its direct or its indirect suppliers and if it is also influenced by shocks on its consumers.

In Figure 4.3, we can observe that the direct suppliers of node 0 are firms $4, 5, 23, 28, 29, 35$, while its direct consumers are nodes $4, 9, 10, 22, 23, 28$.

From Table 4.11, we can notice that nodes which influence most firm 0 are $4, 5, 9, 10, 23, 26$, since, when the shock hits those nodes, the default probability and the total profit variance of node 0 are higher than the ones obtained when the shock affects other nodes. On the other hand, node 0 is weakly influenced by shock on firms $1, 2, 54, 55, 56, 60, 61$, in fact its default probabilities are low and its total profit variances are almost negligible.

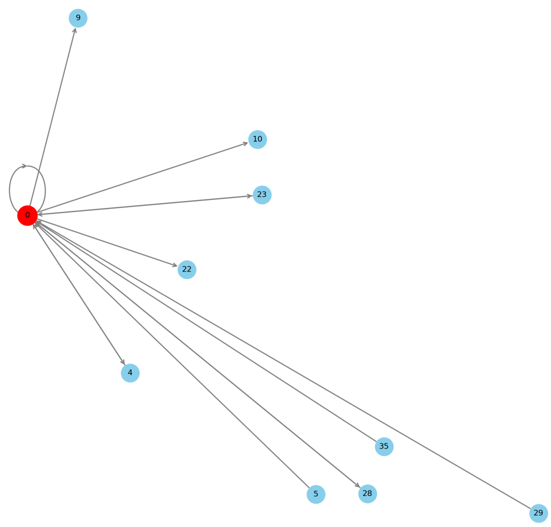


Figure 4.3: Nodes linked with firm 0

Hence, combining the observations made from Table 4.11 and Figure 4.3, we can affirm that node 0 is mainly influenced by shocks on its direct suppliers (nodes $4, 5, 23$), however it is also affected by nodes which are not directly linked with it, as node 26. Moreover, node 0 is surprisingly impacted also by shocks on firms $9, 10$, which are its consumers.

Shock on node	0	1	2	3	4	5
Total profit variance	1.47e-03	2.47e-11	1.25e-11	1.16e-06	1.99e-05	1.89e-07
Default probability	0.592	0.145	0.155	0.230	0.298	0.280

6	7	8	9	10	11	12	13
3.52e-08	5.22e-08	1.37e-09	3.92e-06	5.61e-06	2.06e-07	2.29e-07	3.92e-07
0.243	0.244	0.239	0.273	0.273	0.254	0.245	0.250

14	15	16	17	18	19	20	21
2.24e-08	9.55e-08	5.32e-09	4.63e-08	6.00e-08	8.50e-09	8.86e-09	8.18e-08
0.211	0.242	0.243	0.252	0.257	0.225	0.252	0.255

22	23	24	25	26	27	28	29
3.99e-07	1.45e-06	2.69e-07	9.60e-08	8.21e-05	1.39e-08	1.02e-06	2.56e-07
0.257	0.278	0.248	0.237	0.272	0.234	0.250	0.235

30	31	32	33	34	35	36	37
5.17e-07	4.40e-09	2.42e-10	6.23e-08	1.54e-09	1.04e-07	5.38e-10	4.12e-10
0.241	0.232	0.231	0.201	0.211	0.235	0.163	0.217

38	39	40	41	42	43	44	45
3.58e-09	1.02e-08	1.07e-07	2.03e-07	9.58e-08	1.26e-08	1.51e-07	5.03e-09
0.204	0.222	0.242	0.260	0.244	0.208	0.240	0.206

46	47	48	49	50	51	52	53
3.30e-09	1.66e-08	1.16e-07	5.10e-09	1.69e-10	7.52e-11	6.74e-08	4.29e-08
0.241	0.205	0.253	0.225	0.145	0.148	0.238	0.243

54	55	56	57	58	59	60	61
2.99e-11	3.45e-11	3.35e-11	1.59e-09	2.73e-09	2.69e-07	6.96e-11	1.77e-11
0.156	0.221	0.165	0.173	0.207	0.247	0.192	0.226

Table 4.11: Total profit variance and default probability for node 0 varying the shock's source

Chapter 5

Conclusions

In this work we studied the propagation of microscopic productivity shocks, which might generate aggregate fluctuations in highly interconnected and asymmetric networks.

Our model assumes that firms make rigid, state-independent decisions prior to the realization of shocks. Firms partly rely on debts financed by a bank, whose interest rates can be exogenous or endogenous. While the model guarantees the existence of a unique equilibrium, profits are not zero at the equilibrium, hence the default may occur and loans might remain unpaid.

The analysis of default events constitutes the central aspect of this thesis. We proved that for a generic firm affected by a single-node shock transmitted through one supplier, the default condition depends on the moments of the exponentially tilted distribution of the shock. Using the function $g(x, \alpha, l)$ defined in (3.20), we characterized the default intervals and we derived sufficient moment-based conditions ensuring the existence of at most one default interval. We proved that a single default interval exists for exponential and Bernoulli distributions, while for a $Gamma(\alpha, \beta)$ distribution, the interval is unique only if $\alpha \leq 1$. The analysis was then extended to a single-node shock transmitted through two suppliers, using the function $g(x, \alpha_1, \alpha_2, l_1, l_2)$ defined in (3.33). In this case, the default condition remains determined by the tilted moments and the default intervals coincide with those obtained for a single supplier across all considered distributions.

Moreover, numerical simulations complement the theoretical results, demonstrating how shocks can propagate across different network structures. They highlight that shocks have stronger effects when they hit all nodes simultaneously, and they can propagate to firms not directly connected to the shock source. In simple structures, as a line or a DAG, shocks primarily affect downstream consumers, with impacts diminishing along the supply chain. By contrast, in a cycle or in a real world network, shocks might also reach upstream suppliers. Furthermore, default probabilities are independent of the bank's presence, depending only on realized shocks. Conversely, profit variances are influenced by both distorted Bonacich centrality, determined by the debt cost, itself driven by the choice of interest rates, and on shock realizations. As a result, firms with the highest distorted Bonacich centrality do not necessarily exhibit the largest profit variance.

The main contribution of this thesis is therefore twofold. First, it highlights how network topology and the choice of shock distribution are decisive in shaping firms' profits and default conditions. Second, it introduces the use of the moments of exponentially tilted distributions as a novel analytical tool to explicitly characterize default intervals, an approach not previously applied in this context. These theoretical insights are strengthened by explicit derivations for specific distributions and by simulations that quantify the interaction between network structures, distributional choices, and firm-level risk.

Equally important, the adopted model captures a realistic decision-making environment, in which firms must choose their strategies before shocks are realized.

Finally, the analysis suggests several promising avenues for future work.

A natural extension would be to investigate analytically cases in which shocks affect multiple nodes simultaneously rather than a single one, thereby introducing correlations among different shocks.

Another promising direction concerns the incorporation of demand shocks, which could be modeled, for instance, through a lump-sum tax imposed by the government.

A further aspect that might be considered is the geographic collocation of industries [2], which might be a powerful transmitter of shocks from one industry to others, distinct from network effects. It reflects the importance of localized networks, as industries with substantial exchanges frequently locate near each other to reduce transportation costs and facilitate information transfer.

Appendix A

Results of the experiments

A.1 Gamma shock

The parameters of the gamma distribution chosen for these experiments are:

- gamma parameters: $\alpha = 0.5$, $\beta = 1$,
- amplitude of the negative gamma shock: scale $= -1$.

A.1.1 Line

Shocked node	**Default probabilities**
Node 0	$[0.41, 0.20, 0.17, 0.14, 0.13, 0.13]$
Node 3	$[0, 0, 0, 0.40, 0.19, 0.16]$
All nodes	$[0.41, 0.43, 0.44, 0.44, 0.44, 0.44]$

Table A.1: Default probabilities under a gamma shock in a line network

Shocked node	**Profit variances**
Node 0	[2.40e-02, 5.96e-04, 1.01e-04, 1.13e-05, 8.38e-07, 3.31e-08]
Node 3	[0, 0, 0, 2.10e-02, 4.36e-04, 3.91e-05]
All nodes	[2.31e-02, 2.51e-02, 2.45e-02, 2.45e-02, 1.94e-02, 1.01e-02]

Table A.2: Profit variances under a gamma shock in a line network

Shocked node	Total variance of profits
Node 0	0.025
Node 3	0.021
All nodes	0.127

Table A.3: Total variance of profits under a gamma shock in a line network

A.1.2 DAG

Shocked node	Default probabilities
Node 0	$[0.41, 0.20, 0.20, 0.16, 0.14, 0.13]$
Node 3	$[0, 0, 0, 0.40, 0.19, 0.16]$
All nodes	$[0.41, 0.43, 0.43, 0.44, 0.43, 0.44]$

Table A.4: Default probabilities under a gamma shock in a DAG network

Shocked node	Profit variances
Node 0	[3.56e-02, 3.68e-04, 3.68e-04, 9.06e-05, 8.86e-06, 4.17e-07]
Node 3	[0, 0, 0, 2.05e-02, 4.26e-04, 3.82e-05]
All nodes	[3.44e-02, 1.55e-02, 1.53e-02, 2.37e-02, 1.86e-02, 9.79e-03]

Table A.5: Profit variances under a gamma shock in a DAG network

Shocked node	Total variance of profits
Node 0	0.036
Node 3	0.021
All nodes	0.117

Table A.6: Total variance of profits under a gamma shock in a DAG network

A.1.3 Cycle

Shocked node	Default probabilities
Node 0	$[0.41, 0.20, 0.16, 0.13, 0.13, 0.12]$
Node 3	$[0, 0, 0.14, 0.41, 0.19, 0.16]$
All nodes	$[0.41, 0.43, 0.44, 0.44, 0.44, 0.45]$

Table A.7: Default probabilities under a gamma shock in a cycle network

Shocked node	Profit variances
Node 0	[2.06e-02, 3.90e-04, 7.01e-05, 1.26e-06, 8.81e-08, 4.65e-09]
Node 3	[0, 0, 1.00e-05, 2.12e-02, 5.65e-04, 7.60e-05]
All nodes	[1.99e-02, 1.64e-02, 2.42e-02, 2.45e-02, 2.20e-02, 1.69e-02]

Table A.8: Profit variances under a gamma shock in a cycle network

Shocked node	Total variance of profits
Node 0	0.021
Node 3	0.022
All nodes	0.124

Table A.9: Total variance of profits under a gamma shock in a cycle network

A.2 Bernoulli shock

The parameters of the Bernoulli distribution chosen for these experiments are:

- amplitude of the negative Bernoulli shock: $\epsilon = -2$,
- shock frequency: $p = 0.65$.

We can observe that, in the case of a Bernoulli distribution, the positive default probabilities correspond with the Bernoulli parameters p or $1 - p$ for all nodes.

A.2.1 Line

Shocked node	**Default probabilities**
Node 0	$[0.65, 0.65, 0.35, 0.35, 0.35, 0.35]$
Node 3	$[0, 0, 0, 0.65, 0.65, 0.35]$
All nodes	$[0.65, 0.65, 0.65, 0.65, 0.65, 0.65]$

Table A.10: Default probabilities under a Bernoulli shock in a line network

Shocked node	**Profit variances**
Node 0	[1.37e-01, 2.56e-04, 2.05e-06, 1.12e-06, 9.54e-08, 3.71e-09]
Node 3	[0, 0, 0, 1.21e-01, 1.84e-04, 7.79e-07]
All nodes	[1.38e-01, 1.70e-01, 1.76e-01, 1.69e-01, 1.34e-01, 6.70e-02]

Table A.11: Profit variances under a Bernoulli shock in a line network

Shocked node	**Total variance of profits**
Node 0	0.137
Node 3	0.121
All nodes	0.859

Table A.12: Total variance of profits under a Bernoulli shock in a line network

A.2.2 DAG

Shocked node	**Default probabilities**
Node 0	$[0.65, 0.65, 0.65, 0.35, 0.35, 0.35]$
Node 3	$[0, 0, 0, 0.65, 0.65, 0.35]$
All nodes	$[0.65, 0.65, 0.65, 0.65, 0.65, 0.65]$

Table A.13: Default probabilities under a Bernoulli shock in a DAG network

Shocked node	Profit variances
Node 0	[2.04e-01, 1.58e-04, 1.58e-04, 1.85e-06, 8.78e-07, 4.75e-08]
Node 3	[0, 0, 0, 1.19e-01, 1.70e-04, 9.34e-07]
All nodes	[2.02e-01, 1.06e-01, 1.05e-01, 1.42e-01, 1.29e-01, 6.67e-02]

Table A.14: Profit variances under a Bernoulli shock in a DAG network

Shocked node	Total variance of profits
Node 0	0.204
Node 3	0.119
All nodes	0.751

Table A.15: Total variance of profits under a Bernoulli shock in a DAG network

A.2.3 Cycle

Shocked node	Default probabilities
Node 0	[0.65, 0.65, 0.35, 0.35, 0.35, 0.35]
Node 3	[0, 0, 0.35, 0.65, 0.65, 0.35]
All nodes	[0.65, 0.65, 0.65, 0.65, 0.65, 0.65]

Table A.16: Default probabilities under a Bernoulli shock in a cycle network

Shocked node	Profit variances
Node 0	[1.18e-01, 1.68e-04, 3.10e-06, 1.43e-07, 9.91e-09, 5.10e-10]
Node 3	[0, 0, 9.28e-07, 1.27e-01, 2.91e-04, 1.11e-06]
All nodes	[1.15e-01, 1.14e-01, 1.61e-01, 1.67e-01, 1.61e-01, 1.19e-01]

Table A.17: Profit variances under a Bernoulli shock in a cycle network

Shocked node	Total variance of profits
Node 0	0.118
Node 3	0.128
All nodes	0.837

Table A.18: Total variance of profits under a Bernoulli shock in a cycle network

A.3 Real-world network results

In this section we show the results of the three cases presented in 4.3.2.

The Bonacich centrality vector for all the three cases is:

$[3.99e-02, 2.06e-02, 1.77e-02, 4.50e-02, 4.21e-02, 3.80e-02, 3.10e-02, 3.52e-02, 2.74e-02, 3.67e-02, 4.39e-02, 2.65e-02, 3.85e-02, 3.32e-02, 4.72e-02, 4.34e-02, 3.07e-02, 3.59e-02, 4.02e-02, 3.60e-02, 3.00e-02, 2.98e-02, 2.63e-02, 4.42e-02, 2.28e-02, 3.59e-02, 3.47e-02, 3.01e-02, 4.33e-02, 3.12e-02, 4.04e-02, 2.07e-02, 1.99e-02, 3.83e-02, 2.48e-02, 3.19e-02, 2.53e-02, 2.75e-02, 3.36e-02, 3.53e-02, 3.67e-02, 2.58e-02, 3.25e-02, 3.54e-02, 3.97e-02, 3.26e-02, 2.25e-02, 2.86e-02, 3.01e-02, 2.89e-02, 2.62e-02, 2.36e-02, 3.69e-02, 3.22e-02, 2.24e-02, 2.24e-02, 2.40e-02, 2.95e-02, 2.83e-02, 2.33e-02, 1.94e-02, 1.95e-02]$

A.3.1 Case 1.

The distorted Bonacich centrality vector is:
$[3.99e-02, 2.06e-02, 1.77e-02, 4.50e-02, 4.21e-02, 3.80e-02, 3.10e-02, 3.52e-02, 2.74e-02, 3.67e-02, 4.39e-02, 2.65e-02, 3.85e-02, 3.32e-02, 4.72e-02, 4.34e-02, 3.07e-02, 3.59e-02, 4.02e-02, 3.60e-02, 3.00e-02, 2.98e-02, 2.63e-02, 4.42e-02, 2.28e-02, 3.59e-02, 3.47e-02, 3.01e-02, 4.33e-02, 3.12e-02, 4.04e-02, 2.07e-02, 1.99e-02, 3.83e-02, 2.48e-02, 3.19e-02, 2.53e-02, 2.75e-02, 3.36e-02, 3.53e-02, 3.67e-02, 2.58e-02, 3.25e-02, 3.54e-02, 3.97e-02, 3.26e-02, 2.25e-02, 2.86e-02, 3.01e-02, 2.89e-02, 2.62e-02, 2.36e-02, 3.69e-02, 3.22e-02, 2.24e-02, 2.24e-02, 2.40e-02, 2.95e-02, 2.83e-02, 2.33e-02, 1.94e-02, 1.95e-02]$

Shock on node 0:

- default probabilities:

  $[0.60, 0.22, 0.22, 0.22, 0.37, 0.26, 0.24, 0.23, 0.21, 0.20, 0.23, 0.23, 0.25, 0.23, 0.23, 0.23, 0.24, 0.22, 0.21, 0.22, 0.20, 0.23, 0.19, 0.26, 0.22, 0.22, 0.21, 0.20, 0.26, 0.25, 0.21, 0.22, 0.21, 0.21, 0.20, 0.26, 0.21, 0.23, 0.22, 0.21, 0.22, 0.25, 0.19, 0.21, 0.22, 0.24, 0.24, 0.20, 0.22, 0.19, 0.21, 0.19, 0.25, 0.23, 0.21, 0.22, 0.23, 0.23, 0.25, 0.22, 0.22, 0.23]$,

- profit variances:

$[1.14e-03, 2.42e-13, 1.21e-09, 4.87e-09, 1.84e-05, 1.96e-06, 3.07e-09, 1.07e-08, 5.38e-10, 1.70e-08, 4.12e-07, 1.06e-07, 3.31e-07, 1.08e-08, 1.34e-07, 8.65e-09, 1.33e-08, 4.12e-09, 9.96e-09, 4.14e-09, 1.37e-09, 7.97e-09, 1.50e-09, 8.82e-07, 1.26e-09, 3.04e-09, 1.25e-08, 5.84e-09, 3.92e-06, 5.60e-07, 1.04e-07, 1.49e-08, 1.59e-10, 2.23e-08, 1.48e-10, 1.18e-05, 4.34e-10, 7.59e-10, 1.63e-09, 2.23e-09, 8.59e-11, 5.36e-10, 4.94e-10, 8.57e-10, 9.86e-09, 2.68e-08, 6.91e-09, 4.04e-10, 1.59e-09, 4.76e-10, 8.67e-12, 1.88e-09, 3.03e-07, 7.11e-09, 1.33e-09, 1.43e-08, 9.35e-09, 5.72e-09, 1.45e-08, 1.70e-09, 2.83e-11, 2.87e-09]$,

- Total variance of profits: $1.18e-03$.

Shock on all nodes:

- default probabilities:

  $[0.60, 0.58, 0.58, 0.58, 0.69, 0.66, 0.61, 0.62, 0.58, 0.67, 0.65, 0.60, 0.62, 0.60, 0.66, 0.64, 0.60, 0.62, 0.68, 0.67, 0.62, 0.63, 0.58, 0.64, 0.59, 0.60, 0.65, 0.60, 0.62, 0.58, 0.61, 0.58, 0.59, 0.61, 0.58, 0.62, 0.58, 0.61, 0.62, 0.60, 0.60, 0.61, 0.59, 0.58, 0.59, 0.58, 0.58, 0.61, 0.58, 0.58, 0.58, 0.58, 0.59, 0.59, 0.58, 0.61, 0.59, 0.61, 0.59, 0.58, 0.58, 0.58]$,

- profit variances:

  $[1.22e-03, 3.20e-04, 2.51e-04, 1.63e-03, 2.69e-03, 8.97e-04, 7.03e-04, 9.09e-04, 5.83e-04, 1.66e-03, 1.69e-03, 5.60e-04, 1.30e-03, 8.29e-04, 1.79e-03, 1.56e-03, 7.35e-04, 1.01e-03, 1.84e-03, 1.46e-03, 6.88e-04, 7.96e-04, 5.77e-04, 1.22e-03, 4.20e-04, 1.02e-03, 1.04e-03, 7.79e-04, 1.75e-03, 8.11e-04, 1.37e-03, 3.71e-04, 3.36e-04, 1.18e-03, 4.77e-04, 1.08e-03, 5.16e-04, 5.52e-04, 8.11e-04, 9.61e-04, 1.06e-03, 5.64e-04, 8.42e-04, 9.90e-04, 1.23e-03, 8.62e-04, 4.19e-04, 7.62e-04, 7.22e-04, 6.63e-04, 5.50e-04, 4.41e-04, 1.14e-03, 8.40e-04, 4.08e-04, 4.22e-04, 4.40e-04, 6.17e-04, 6.10e-04, 4.36e-04, 2.98e-04, 3.01e-04]$,

- total variance of profits: $5.50e-02$.

A.3.2 Case 2.

The distorted Bonacich centrality vector is:
$[4.10e-02, 2.22e-02, 1.92e-02, 4.61e-02, 4.32e-02, 3.93e-02, 3.25e-02, 3.66e-02, 2.89e-02, 3.80e-02, 4.51e-02, 2.81e-02, 3.97e-02, 3.46e-02, 4.82e-02, 4.45e-02, 3.22e-02, 3.73e-02, 4.14e-02, 3.74e-02, 3.15e-02, 3.13e-02, 2.78e-02, 4.52e-02, 2.44e-02, 3.72e-02, 3.61e-02, 3.15e-02, 4.45e-02, 3.26e-02, 4.15e-02, 2.22e-02, 2.15e-02, 3.95e-02, 2.63e-02, 3.33e-02, 2.69e-02, 2.91e-02, 3.50e-02, 3.67e-02, 3.80e-02, 2.73e-02, 3.40e-02, 3.68e-02, 4.10e-02, 3.40e-02, 2.40e-02, 3.00e-02, 3.16e-02, 3.04e-02, 2.77e-02, 2.53e-02, 3.82e-02, 3.36e-02, 2.40e-02, 2.40e-02, 2.56e-02, 3.10e-02, 2.98e-02, 2.48e-02, 2.09e-02, 2.11e-02]$.

Shock on node 0:

- default probabilities:

[0.60, 0.22, 0.22, 0.22, 0.37, 0.27, 0.23, 0.22, 0.20, 0.20, 0.23, 0.23, 0.25, 0.23, 0.23, 0.22, 0.24, 0.22, 0.21, 0.22, 0.20, 0.22, 0.19, 0.26, 0.22, 0.22, 0.21, 0.20, 0.26, 0.25, 0.21, 0.22, 0.20, 0.21, 0.20, 0.26, 0.20, 0.23, 0.22, 0.21, 0.21, 0.25, 0.18, 0.21, 0.22, 0.24, 0.23, 0.20, 0.22, 0.18, 0.21, 0.19, 0.25, 0.23, 0.21, 0.22, 0.22, 0.23, 0.25, 0.22, 0.22, 0.22],

- profit variances:

  $[1.18e-03, 3.03e-13, 1.46e-09, 5.32e-09, 1.86e-05, 2.03e-06, 3.50e-09, 1.19e-08, 6.58e-10, 1.95e-08, 4.32e-07, 1.20e-07, 3.51e-07, 1.20e-08, 1.40e-07, 9.54e-09, 1.49e-08, 4.70e-09, 1.11e-08, 4.64e-09, 1.65e-09, 9.23e-09, 1.85e-09, 9.09e-07, 1.51e-09, 3.43e-09, 1.42e-08, 6.85e-09, 4.05e-06, 6.06e-07, 1.15e-07, 1.74e-08, 2.01e-10, 2.51e-08, 1.82e-10, 1.23e-05, 5.31e-10, 8.80e-10, 1.89e-09, 2.59e-09, 1.00e-10, 6.07e-10, 6.11e-10, 9.82e-10, 1.10e-08, 2.98e-08, 7.99e-09, 4.84e-10, 1.83e-09, 5.90e-10, 1.05e-11, 2.35e-09, 3.27e-07, 8.00e-09, 1.62e-09, 1.68e-08, 1.09e-08, 6.51e-09, 1.62e-08, 1.97e-09, 3.51e-11, 3.52e-09]$,

- total variance of profits: $1.22e-03$.

Shock on all nodes:

- default probabilities:

  [0.61, 0.58, 0.58, 0.58, 0.68, 0.67, 0.61, 0.61, 0.59, 0.67, 0.64, 0.60, 0.62, 0.60, 0.66, 0.64, 0.59, 0.62, 0.68, 0.67, 0.63, 0.63, 0.58, 0.65, 0.58, 0.61, 0.67, 0.61, 0.61, 0.59, 0.61, 0.59, 0.59, 0.61, 0.58, 0.62, 0.58, 0.60, 0.61, 0.61, 0.60, 0.60, 0.59, 0.58, 0.59, 0.59, 0.58, 0.60, 0.59, 0.58, 0.58, 0.59, 0.60, 0.59, 0.58, 0.61, 0.59, 0.61, 0.60, 0.58, 0.58, 0.58],

- profit variances:

  $[1.31e-03, 3.81e-04, 2.88e-04, 1.70e-03, 2.69e-03, 9.96e-04, 7.53e-04, 9.58e-04, 6.93e-04, 1.78e-03, 1.78e-03, 6.20e-04, 1.38e-03, 8.92e-04, 1.91e-03, 1.65e-03, 7.90e-04, 1.12e-03, 1.95e-03, 1.56e-03, 7.91e-04, 8.49e-04, 6.44e-04, 1.28e-03, 4.77e-04, 1.11e-03, 1.21e-03, 8.73e-04, 1.83e-03, 8.97e-04, 1.51e-03, 4.20e-04, 3.92e-04, 1.23e-03, 5.49e-04, 1.15e-03, 5.69e-04, 6.23e-04, 8.71e-04, 1.06e-03, 1.12e-03, 6.26e-04, 9.14e-04, 1.07e-03, 1.30e-03, 9.49e-04, 4.75e-04, 8.37e-04, 8.10e-04, 7.29e-04, 6.17e-04, 4.99e-04, 1.24e-03, 9.21e-04, 4.61e-04, 4.87e-04, 4.96e-04, 6.95e-04, 6.65e-04, 4.95e-04, 3.51e-04, 3.58e-04]$,

- total variance of profits: $5.97e-02$.

A.3.3 Case 3.

In this case, the distorted Bonacich centrality vector depends on which node the shock is realized.
Shock on node 0:

- distorted Bonacich centrality:

  $[3.85e-02, 2.09e-02, 1.78e-02, 4.51e-02, 4.10e-02, 3.80e-02, 3.13e-02, 3.54e-02, 2.76e-02, 3.67e-02, 4.38e-02, 2.66e-02, 3.86e-02, 3.33e-02, 4.76e-02, 4.38e-$

$02, 3.10e{-}02, 3.62e{-}02, 4.05e{-}02, 3.63e{-}02, 3.03e{-}02, 2.998e{-}02, 2.65e{-}02, 4.42e{-}02, 2.30e{-}02, 3.61e{-}02, 3.49e{-}02, 3.03e{-}02, 4.33e{-}02, 3.12e{-}02, 4.03e{-}02, 2.09e{-}02, 2.01e{-}02, 3.84e{-}02, 2.50e{-}02, 3.21e{-}02, 2.56e{-}02, 2.78e{-}02, 3.38e{-}02, 3.55e{-}02, 3.68e{-}02, 2.60e{-}02, 3.27e{-}02, 3.55e{-}02, 3.99e{-}02, 3.29e{-}02, 2.27e{-}02, 2.87e{-}02, 3.03e{-}02, 2.91e{-}02, 2.64e{-}02, 2.39e{-}02, 3.71e{-}02, 3.24e{-}02, 2.26e{-}02, 2.26e{-}02, 2.43e-02, 2.98e-02, 2.85e-02, 2.34e-02, 1.96e-02, 1.97e-02]$,

- default probabilities:

  $[0.60, 0.22, 0.22, 0.21, 0.38, 0.26, 0.23, 0.22, 0.20, 0.20, 0.23, 0.23, 0.25, 0.22, 0.23, 0.22, 0.24, 0.22, 0.21, 0.22, 0.20, 0.22, 0.19, 0.25, 0.22, 0.22, 0.21, 0.20, 0.25, 0.25, 0.21, 0.22, 0.20, 0.21, 0.20, 0.26, 0.20, 0.23, 0.21, 0.21, 0.21, 0.25, 0.19, 0.21, 0.22, 0.24, 0.23, 0.20, 0.21, 0.19, 0.20, 0.19, 0.24, 0.23, 0.21, 0.22, 0.22, 0.23, 0.24, 0.22, 0.22, 0.22]$,

- profit variances:

  $[1.09e-03, 2.63e-13, 1.28e-09, 5.13e-09, 1.76e-05, 2.00e-06, 3.26e-09, 1.13e{-}08, 5.85e{-}10, 1.80e{-}08, 4.21e{-}07, 1.11e{-}07, 3.42e{-}07, 1.13e{-}08, 1.40e{-}07, 9.21e{-}09, 1.41e{-}08, 4.41e{-}09, 1.06e{-}08, 4.40e{-}09, 1.49e{-}09, 8.50e{-}09, 1.63e{-}09, 9.01e{-}07, 1.34e{-}09, 3.23e{-}09, 1.33e{-}08, 6.28e{-}09, 3.99e{-}06, 5.76e{-}07, 1.09e{-}07, 1.57e{-}08, 1.73e{-}10, 2.36e{-}08, 1.61e{-}10, 1.21e{-}05, 4.71e{-}10, 8.08e{-}10, 1.74e{-}09, 2.40e{-}09, 9.20e{-}11, 5.61e{-}10, 5.41e{-}10, 9.11e{-}10, 1.04e{-}08, 2.82e{-}08, 7.30e{-}09, 4.33e{-}10, 1.69e{-}09, 5.22e{-}10, 9.37e{-}12, 2.05e{-}09, 3.16e{-}07, 7.51e{-}09, 1.44e{-}09, 1.52e{-}08, 9.92e-09, 6.05e-09, 1.52e-08, 1.79e-09, 3.06e-11, 3.08e-09]$,

- total variance of profits: $1.13e-03$.

Shock on all nodes:

- distorted Bonacich centrality:

  $[4.69e-02, 2.82e-02, 2.52e-02, 5.14e-02, 4.90e-02, 4.62e-02, 3.86e-02, 4.23e{-}02, 3.48e{-}02, 4.35e{-}02, 5.05e{-}02, 3.40e{-}02, 4.53e{-}02, 4.04e{-}02, 5.35e{-}02, 4.99e{-}02, 3.81e{-}02, 4.30e{-}02, 4.70e{-}02, 4.30e{-}02, 3.76e{-}02, 3.72e{-}02, 3.36e{-}02, 5.10e{-}02, 3.02e{-}02, 4.28e{-}02, 4.21e{-}02, 3.73e{-}02, 5.01e{-}02, 3.84e{-}02, 4.70e{-}02, 2.81e{-}02, 2.73e{-}02, 4.49e{-}02, 3.21e{-}02, 3.90e{-}02, 3.26e{-}02, 3.51e{-}02, 4.09e{-}02, 4.23e{-}02, 4.37e{-}02, 3.32e{-}02, 3.98e{-}02, 4.24e{-}02, 4.65e{-}02, 3.97e{-}02, 2.99e{-}02, 3.58e{-}02, 3.73e{-}02, 3.62e{-}02, 3.35e{-}02, 3.13e{-}02, 4.39e{-}02, 3.93e{-}02, 2.99e{-}02, 3.00e{-}02, 3.16e-02, 3.70e-02, 3.58e-02, 3.06e-02, 2.68e-02, 2.70e-02]$,

- default probabilities:

  $[0.60, 0.58, 0.58, 0.58, 0.67, 0.67, 0.61, 0.62, 0.58, 0.67, 0.64, 0.60, 0.62, 0.61, 0.66, 0.64, 0.60, 0.62, 0.68, 0.67, 0.62, 0.63, 0.58, 0.64, 0.58, 0.60, 0.66, 0.61, 0.62, 0.58, 0.61, 0.59, 0.58, 0.61, 0.58, 0.62, 0.58, 0.61, 0.61, 0.60, 0.59, 0.61, 0.58, 0.58, 0.59, 0.59, 0.58, 0.60, 0.59, 0.58, 0.58, 0.58, 0.59, 0.58, 0.57, 0.61, 0.59, 0.61, 0.59, 0.58, 0.58, 0.58]$,

- profit variances:

  $[1.70e-03, 6.08e-04, 5.17e-04, 2.12e-03, 3.22e-03, 1.34e-03, 1.05e-03, 1.29e{-}03, 9.78e{-}04, 2.32e{-}03, 2.23e{-}03, 9.05e{-}04, 1.81e{-}03, 1.24e{-}03, 2.31e{-}03, 2.05e{-}$

$03, 1.10e-03, 1.39e-03, 2.57e-03, 2.18e-03, 1.09e-03, 1.23e-03, 9.16e-04, 1.59e-03, 7.06e-04, 1.44e-03, 1.62e-03, 1.23e-03, 2.34e-03, 1.26e-03, 1.92e-03, 7.11e-04, 6.23e-04, 1.62e-03, 8.20e-04, 1.59e-03, 8.61e-04, 9.04e-04, 1.17e-03, 1.36e-03, 1.48e-03, 9.40e-04, 1.23e-03, 1.45e-03, 1.69e-03, 1.30e-03, 7.13e-04, 1.16e-03, 1.12e-03, 1.05e-03, 9.01e-04, 7.61e-04, 1.62e-03, 1.22e-03, 6.78e-04, 7.68e-04, 7.65e-04, 9.85e-04, 9.50e-04, 7.75e-04, 5.73e-04, 5.95e-04]$,

- total variance of profits: $8.07e-02$.

Appendix B

Code

```
import numpy as np
import scipy.stats as stats
import pandas as pd
import matplotlib.pyplot as plt

np.random.seed(42)
#DEFINITION OF THE ADJACENCY MATRIX A
#N=6
def A_line(N, alpha):
    A = np.zeros((N, N))
    indices = [(0,1), (1,2), (2,3), (3,4), (4,5)]
    for i, j in indices:
        A[i, j] = alpha
    return A

def A_DAG(N, alpha):
    A = np.zeros((N, N))
    edges = [(0, 1), (0, 2), (1, 3), (2, 3), (3, 4), (4, 5)]
    in_deg = {0: 0, 1: 1, 2: 1, 3: 2, 4: 1, 5: 1}
    for i, j in edges:
        if in_deg[j] > 0:
            A[i, j] = alpha / in_deg[j]
    return A

def A_cicle(N, alpha):
    A = np.zeros((N, N))
    edges = [(0, 1), (1, 2), (2, 3), (3, 4), (4, 5), (5, 2)]
    in_deg = {i: 0 for i in range(N)}
    for j, k in edges:
        in_deg[k] += 1
    for j, k in edges:
        A[j, k] = alpha / in_deg[k]
    return A
```

```
N=4
def example(N):
    A_example=np.zeros((N, N))
    A_example[0,1]=0.3
    A_example[0,2]=0.4
    A_example[1,2]=0.2
    A_example[1,3]=0.5
    A_example[2,3]=0.3
    return A_example

A=example(N)
#A=A_line(N,0.5)
#A=A_DAG(N, 0.5)
#A=A_cicle(N, 0.5)
print('A:\n', A)

beta = (np.ones((N, )) - np.sum(A, axis=0))
L = np.linalg.solve(np.eye(N) - A.T, np.eye(N))
print('L:\n',L)

#EXTRACTION OF THE ADJACENCY MATRIX FROM A SYMMETRIC TABLE
N=62
df = pd.read_excel('SIMM_TOT_63BxB_v2.xlsx',
                    index_col=0, usecols='B:BZ', skiprows
                        =[0,1,2,3,5], engine='openpyxl',
                        sheet_name="STOTBB_2020")
df = df.drop(index=df.index[-1])
temporaryA = df.iloc[0:62, 0:62].values
industries = df.iloc[0:62, 0:62].index
A = temporaryA / (np.sum(temporaryA, axis=1, keepdims=True) + df.
    iloc[-6, :62].values)

A = np.array(A, dtype=float)
print('A:\n', np.round(A, 5))
beta = (np.ones((N, )) - np.sum(A, axis=0))
L = np.linalg.solve(np.eye(N) - A.T, np.eye(N))
print('L:\n', np.round(L, 5))

industries=[i for i in range(N)]
import networkx as nx
G = nx.DiGraph()
for i in range(N):
    G.add_node(industries[i])

for i in range(N):
    for j in range(N):
        if A[i, j] > 0.01:
```

```
            G.add_edge(industries[j], industries[i], weight=A[i, j
                ])

pos = nx.spring_layout(G, seed=42)
plt.figure(figsize=(10, 10))
nx.draw_networkx_nodes(G, pos, node_size=250, node_color='skyblue'
    )
nx.draw_networkx_edges(G, pos, arrowstyle='->', arrowsize=8,
    edge_color='grey')
nx.draw_networkx_labels(G, pos, font_size=6)
plt.title('Network', fontsize=15)
plt.axis('off')
plt.tight_layout()
plt.savefig("network_graph.png", dpi=300)
plt.show()

#GENERATION OF SAMPLES
def sample_from_distribution(distribution_name, params, industries
    , size, affected_industries, scale):

    affected_industries = set(affected_industries)
    samples = np.zeros((len(industries), size), dtype=float)

    distribution = getattr(stats, distribution_name)

    for i, industry in enumerate(industries):
        if industry in affected_industries:
            samples[i] = scale * distribution.rvs(**params, size=
                size)
return samples

industries=[i for i in range(N)]
size=int(1e4)

bern_samples = sample_from_distribution("bernoulli", {"p": 0.65},
    industries, size, affected_industries=[0], scale=-2)
bern_samples_2 = sample_from_distribution("bernoulli", {"p":
    0.45}, industries, size, affected_industries=[3], scale=-2)
total_bern_samples=bern_samples+bern_samples_2

exp_samples=sample_from_distribution("expon", {"scale": 2},
    industries, size, affected_industries=[0], scale=-1)

gamma_samples=sample_from_distribution("gamma", {"a": 0.5, "scale"
    : 1}  , industries, size, affected_industries=[0], scale=-1)

shock1=sample_from_distribution("expon", {"scale": 0.5},
    industries, size, affected_industries=[0], scale=-1)
```

```
shock2=sample_from_distribution("expon", {"scale": 2}, industries,
     size, affected_industries=[1], scale=-1)
shock3=sample_from_distribution("expon", {"scale": 1}, industries,
     size, affected_industries=[2], scale=-1)
shock4=sample_from_distribution("expon", {"scale": 1/3},
    industries, size, affected_industries=[3], scale=-1)
example_exp_shock_tot=shock1+shock2+shock3+shock4

#MONTECARLO MEAN
def monte_carlo_expectation(samples, L):
    rho= np.dot(L,samples)
    exp_rho=np.exp(rho)
    expectation = np.mean(exp_rho, axis=1)
    return expectation
monteCarlo_mean=monte_carlo_expectation(example_exp_shock_tot, L)
print('MonteCarlo expectation: \n', monteCarlo_mean)

#DEBT COST FROM PROPOSITION 2 (when the interest rates are
    endogenous)
from scipy.optimize import root_scalar

rho= np.dot(L,exp_samples)
exp_rho=np.exp(rho)
tau = exp_rho / monteCarlo_mean[:, None]
epsilon = A.T @ tau + beta[:, None]
theta=0.25*np.ones((N))

from scipy.optimize import bisect

def f_k(x, tau_k, eps_k, theta_k):
     vals = np.maximum(x * np.minimum(tau_k, eps_k),
                       (1 - theta_k) * eps_k)
     return vals.mean() - 1

def solve_zeta(tau, eps, theta, x_min=1.0, x_max=100.0, tol=1e-8):

     zeta = np.zeros(N)
     for k in range(N):
         tau_k = tau[k]
         eps_k = eps[k]
         th_k = theta[k]

         if f_k(1.0, tau_k, eps_k, th_k) >= 0:
             zeta[k] = 0.0
             continue

         root = bisect(f_k, x_min, x_max, args=(tau_k, eps_k, th_k)
             , xtol=tol)
         zeta[k] = np.log(root)
```

```

    return zeta
debt_cost = solve_zeta(tau, epsilon, theta)
print('Debt cost:', debt_cost)

#DETERMINISTIC DEBT COST (when the interest rates are exogenous)
#theta = 0.25 * np.ones((N, ))
#r = np.ones((N, ))
theta=np.array([0.2, 0.5, 0.4, 0.7])   #for example 1.3
r=np.array([0.1, 0.3, 0.2, 0.4])       #for example 1.3

debt_cost = np.zeros((N,))
for i in range(N):
    if theta[i]!=0:
        debt_cost[i]=np.log(1+r[i]*theta[i])
print('Deterministic debt cost:', debt_cost)

#QUANTITIES AT THE EQUILIBRIUM
w=1
gamma=1/N * np.ones((N, ))
v=np.dot(L.T, gamma)
print('Bonacich centrality:', v)

total_cost=np.zeros((N,))
total_cost=np.dot(L, debt_cost)
print('Total cost:', total_cost)

D = np.diag(np.exp(-debt_cost))
L_dist=np.linalg.solve(np.eye(N) - D@A.T, np.eye(N))
print('Distorted Leontief matrix: \n', L_dist)
chi=gamma @ L_dist @ (beta*np.exp(-debt_cost))
print('Chi:', chi)
v_dist = np.dot(L_dist.T, gamma) / chi
print('Distorted Bonacich centrality:', v_dist)

plt.figure(figsize=(10, 6))
plt.plot(v_dist, marker='o', linestyle='-', color='blue')
plt.xlabel('Node')
plt.ylabel('Distorted Bonacich Centrality')
plt.title('Distorted Bonacich Centrality of Nodes')
plt.grid(True, linestyle='--', alpha=0.6)
plt.tight_layout()
plt.savefig("bon_cen.png")
plt.show()

#maximal and actual production
y=v_dist*np.exp(-total_cost)
actual_production=monteCarlo_mean*y
print('Maximal production:', y)
```

```
print('Actual production:', actual_production)

#maximal and actual intermediate quantities
z = np.zeros((N, N))
actual_zeta = np.zeros((N, N))
for j in range(N):
    for k in range(N):
        z[j, k] = v_dist[k] * A[j, k] * np.exp(-debt_cost[k] -
            total_cost[j])
        actual_zeta[j, k] =z[j, k] * monteCarlo_mean[j]
print('Maximal intermediate quantities: \n', z)
print('Actual intermediate quantities: \n', actual_zeta)

#labor
l=v_dist*beta*np.exp(-debt_cost)
print('Labor:', l)

#maximal and actual household's consumption
c=(gamma*np.exp(-total_cost))/chi
actual_c=monteCarlo_mean*c
print('Maximal household consumption:', c)
print('Actual household consumption:', actual_c)

#prices over wages
p=w*(np.exp(total_cost)/monteCarlo_mean)
print('Prices over wages:', p)

#utility
s = np.sum(v[:, None] * (example_exp_shock_tot  - debt_cost[:,
    None]), axis=0)
utility=(np.exp(s)) / chi
print('Utility:', utility)
ut_mean=np.mean(utility)
print('Utility mean:', ut_mean)

#RESAMPLING FOR PROFIT COMPUTATIONS
new_bern_samples = sample_from_distribution("bernoulli", {"p":
    0.65}, industries, size,affected_industries=[0], scale= -2)
new_bern_samples_2 = sample_from_distribution("bernoulli", {"p":
    0.45}, industries, size, affected_industries=[3], scale= -2)
new_bern_total_samples=new_bern_samples+new_bern_samples_2

new_exp_samples = sample_from_distribution("expon", {"scale": 2},
    industries, size, affected_industries=[0], scale=-1)

new_gamma_samples=sample_from_distribution("gamma", {"a": 0.5, "
    scale": 1}, industries, size, affected_industries=industries,
    scale=-1)

```

```
new_shock1=sample_from_distribution("expon", {"scale": 0.5},
    industries, size, affected_industries=[0], scale=-1)
new_shock2=sample_from_distribution("expon", {"scale": 2},
    industries, size, affected_industries=[1], scale=-1)
new_shock3=sample_from_distribution("expon", {"scale": 1},
    industries, size, affected_industries=[2], scale=-1)
new_shock4=sample_from_distribution("expon", {"scale": 1/3},
    industries, size, affected_industries=[3], scale=-1)
new_example_exp_shock_tot=new_shock1+new_shock2+new_shock3+
    new_shock4

new_expectation=monte_carlo_expectation(new_example_exp_shock_tot,
     L)

#DEFAULT PROBABILITY AND PROFIT VARIANCE
def profit_distribution(L, beta, w, v_dist, A):

     rho= np.dot(L,new_example_exp_shock_tot)
     exp_rho=np.exp(rho)

     tau = exp_rho / new_expectation[:, None]
     epsilon = A.T @ tau + beta[:, None]
     #print(tau)
     #print(epsilon)
     diff = tau - epsilon
     print('Tau-epsilon: \n', diff)
     diff_var=np.var(diff, axis=1)
     print('tau-epsilon variances:', diff_var)
     print('tau-epsilon variances* v_dist^2:', diff_var*v_dist**2)
     tau_mean=np.mean(tau, axis=1)
     print('mean of tau:', tau_mean)
     epsilon_mean=np.mean(epsilon, axis=1)
     print('mean of epsilon:', epsilon_mean)

     profit = w * v_dist[:, None] * diff
     print('Profit: \n', profit)
     profit_mean = np.mean(profit, axis=1)
     profit_mean = np.where(np.abs(profit_mean) < 1e-15, 0,
         profit_mean)
     print('mean profit:', profit_mean)
     profit_var = np.var(profit, axis=1)
     print('Profit variances:', profit_var)
     total_profit_variance = np.sum(profit_var)
     print('Total variance of profits:', total_profit_variance)
     profit_cov = np.cov(profit)
     print('Profit covariances:\n', profit_cov)

     default=epsilon > tau
     default_int = default.astype(int)
```

```
    corr_matrix = np.corrcoef(default_int)
    #print(corr_matrix)

    prob_default = np.mean(default, axis=1)
    print('Default probability: \n', prob_default)

    return tau_mean, epsilon_mean, profit_mean, prob_default,
       profit_var, total_profit_variance, corr_matrix, profit_cov
       ,diff, profit
tau_mean, epsilon_mean, profit_mean, prob_default, profit_var,
   total_profit_variance, corr_matrix, profit_cov, diff, profit=
   profit_distribution(L, beta, w, v_dist, A)
```

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