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MASTER's Degree in MATHEMATICAL ENGINEERING



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Networked Public Good Games: Equilibria and Best Response Dynamics

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Abstract

This thesis investigates game theoretic models for public good provision. We consider network games defined on weighted and directed graphs, where agents are heterogeneous with respect to income level, preference structure, and effort cost. Public goods are here studied in their local form: the benefits of one agent's contributions are shared only among her direct out-neighbors. Each player allocates her income between private consumption and contributions to the public good. While private consumption provides purely individual benefits, public contributions yield payoffs that also depend on the allocations of neighboring agents, scaled by the intensity of their network connections.

We formalize the model, derive best response functions, and establish the existence of Nash equilibria. In addition, we provide the characterization of equilibrium profiles, with a focus on internal equilibria, where all agents contribute positively, and specialized equilibria, where only a subset contributes while others free-ride.

The main results are threefold. First, we derive a sufficient condition for the uniqueness of the Nash equilibrium in the general setting; provided the necessary assumptions, this condition reduces to a bound on the lowest eigenvalue of the symmetrized, per-row rescaled adjacency matrix. Second, we establish a Lipschitz condition ensuring contractivity of the synchronous best response function when the dominant eigenvalue of the adjacency matrix, suitably rescaled, is lower than one. This sufficient condition implies the contractivity of the discrete-time best response dynamics, which results in both the uniqueness of the Nash equilibrium and its global asymptotic stability for the discrete- and continuous-time dynamics. Moreover, under proper hypotheses, we characterize the stability of internal equilibria. Third, we study the limit behavior of the continuous-time best response dynamics within an appropriate framework where contractivity may not hold. We establish that, given suitable assumptions, global stability of the equilibrium is ensured also under the weaker uniqueness condition. In addition, when uniqueness is not guaranteed, we prove that the trajectories globally converge to the set of Nash equilibria and we characterize locally asymptotically stable equilibria as local maxima of an associated functional.

Finally, we investigate possible efficiency metrics to evaluate equilibrium outcomes and explore intervention strategies aimed at improving welfare. We propose a preliminary study of a subsidy mechanism that incentivizes higher contributions and a redistribution policy in which an external planner reallocates income to steer the system toward socially optimal equilibria.

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Chapter 1

Introduction and literature review

1.1 Definition of local public goods

The concept of public goods is of great philosophical and scientific interest, since their provision allows for the smooth functioning of our society, from an economic, political and cultural point of view [1]. Its meaning and importance is explained by Reiss and Julian in “Public Goods” [1], where the authors give an insight of how we have reached the current definition of this term. In 1954 Samuelson [2] is one of the first to define public goods as “[a good] which all enjoy in common in the sense that each individual’s consumption of such a good leads to no subtractions from any other individual’s consumption of that good”. Thus, Samuelson introduces the concept of non-rivalry, which implies that one person’s use of the good does not reduce its availability to others. Musgrave [3] proposes another criterion to characterize public goods in 1959, that of non-excludability, meaning that once the good has been provided, all or some members of the group directly benefit from it. Nowadays, in the economic context, public goods are expected to satisfy both, thus to be non-rivalrous and non-excludable [4]. Let us clarify this concept with a few examples.

In real-world settings, public goods assume a wide variety of forms at very different scales. At the national level, a common example is clean air: once efforts are made to reduce pollution in a given region, everyone in the affected area benefits from the reduced emissions, regardless of whether they contributed to its cost or not. Clean air is a typical example of non-excludable and non-rivalrous public good, since no individual can be excluded from breathing cleaner air, and one person’s enjoyment of it does not reduce the quality available to others. Public goods are also prevalent at local and social levels. For instance, when a homeowner renovates their property façade or plants a garden, neighbors benefit visually from the improved aesthetics, even though they did not participate in the cost; other common examples concern the public health field: actions such as vaccination protect not only the individual who receives it, but also others by reducing disease transmission. In education

and innovation, when an individual shares her knowledge, whether through formal research or informal social interactions, it can help others improve some aspects of their lives. For instance, a farmer who experiments with a new crop or technique may share results with neighboring farmers, who will benefit from the improvements too. Similarly, individuals often rely on the experiences of friends and family when deciding whether to try a new product or service, thus taking advantage of the informational public good generated by someone else's effort or risk.

Thus, public goods can be distinguished on the basis of who benefits from them. It is clear from the examples that, in a society, the provision of most public goods does not affect the whole population, but only a subset of it, often the one closer to the provider of the good. The term "local public good", coined by Tiebout [5], accounts for the fact that public goods are not equally enjoyed by the entire society, but only by the part of it which is directly or indirectly in contact with the source of the provision.

1.2 Network games

Because of their wide range of applicability in very different real-world scenarios, the study of how public goods are provided in a society or group is of strong interest. In the mathematical field, such kind of situations are studied by exploiting network games, which allow to model social networks, scenarios of interconnected agents that interact on the basis of some shared rules. This discipline merges two more general branches of mathematics: graph theory, that allows to formally define frameworks of connections between agents, and game theory, where the strategic interaction of individuals that make interdependent decisions are modeled.

The core idea of network games is to model a population of agents that are connected to each other, where the nature and strength of the connections are determined by a graph. Networked individuals interact to make decisions, represented by the played actions, that result in a numerical benefit, modeled by the utility functions. Games differentiate based on whether actions are chosen between two possibilities, which usually consist in doing the action or not doing it, or from a continuous domain, for example how much effort to exert. Each player's decision is directly affected only by the actions of agents to whom they are linked in the graph, called the neighbors. A key concept in the study of games defined on networks is the concept of equilibrium profile, an action configuration such that no agent has incentives to modify her action to improve its payoff, and so to escape that configuration; indeed, their central role is justified by the fact that equilibria represent end-points of the game dynamics.

In the past few decades, the study of games played on networks has become more and more relevant, as it is demonstrated by the large amount of literature on this topic. Jackson's "Social and economic network" [6], for example, offers an overview of models and techniques for analyzing social networks, in "Network games" Galeotti et al. [7] provide a detailed framework to analyze strategic interactions based on a

network of connections, “Games on networks” by Jackson and Zenou [8] contributes with a synthesis of the literature that analyzes games where players are connected via a network structure, and there are many more examples.

Jackson and Zenou [8] clarify that the main difficulty in the study of graphical games is the inherent complexity of networks. In particular, obtaining precise results while keeping the setting of interconnections general represents a tough challenge. As the authors underline, there are two main approaches, that are also of interest in this study, to navigate such a large and complex field. The first distinguishes two main classes of network games: strategic complements and strategic substitute games, that are characterized by the monotonic behavior of the agents utilities. In particular, in games exhibiting strategic complements, an increase in the action played by some agents results in a lower utility for individuals that are connected to them, who are thus encouraged to increase their action too. On the contrary, when a game exhibits strategic substitutes, each player’s utility is increasing in other players’ actions, so that choosing higher actions promotes the reduction of the played action for neighbors. The second approach exploits simpler models, such as linear or quadratic utility functions, to obtain results that can be generalized or directly apply also to more complex settings. Bramoullé and Kranton, for example, rely on both strategies for their papers “Strategic Interaction and Networks” [9] and “Games Played on Networks” [10]. They exploit the substitutes-complements classification when deriving more precise results from a general model and rely on quadratic utility functions, for which the theory of potential games is well known, to prove results that are also valid in less specific cases.

1.3 The public good game

Public good games are a widely used class of network games that allow to study how individuals contribute to goods or services that are shared in a group or society. As previously anticipated, public goods typically have two main properties: they are non-excludable and non-rivalrous. In this work we consider local public goods, since the benefits of a given player’s action are public only limitedly to her neighborhood. Because of these characteristics, individuals may prefer not to participate and to let others provide the good and pay the cost that comes with it, while still benefiting from their contributions. This gives rise to tensions between individual and collective interests, which makes it hard to predict the behavior of the members of the group.

As a consequence to the type of situations that they model, public good games fall under the category of strategic substitute games; indeed a given player is made better off by an increase in the effort of her neighbors and prefers to contribute less when she can benefit from the public good provision of contributing surrounding agents.

An important distinction that arises when modeling public goods through games on networks concerns the type of action that players can do, that is, their action space. In the so-called best-shot public good games players have binary discrete

action spaces, for example $\{0, 1\}$ (as in [7], [11]) or $\{T, F\}$ (as in [12]), and can choose between providing the public good or not contributing at all. Examples of binary scenarios include decisions on whether or not to vaccinate, report crime in a neighborhood, or purchase a shareable tool [11]. Continuous-action public good games, on the other hand, model situations in which agents have to choose how much contribution to provide. The action space can thus be the positive real line (see [13], [9], [14]) or a positive amount limited by a maximum level of effort (see [15], [16]). The action may represent, for example, the monetary amount invested by a State in the reduction of carbon emissions or by a firm in training for its employees, or the quantity of effort or time spent in activities from which a whole group benefits, like cleaning a neighborhood or acquiring knowledge that can be shared.

The features of public goods naturally lead to complex questions about individual behavior and collective outcomes. When benefits are shared among many, individuals often face a tension between contributing to the good and relying on the efforts of others. It is then interesting to understand whether and in what cases a cooperative behavior emerges, and when, in the opposite case, individuals tend to behave selfishly or as if they were not interconnected. In a setting of networked agents, a question could be how connections shape incentives. For example, is being surrounded by active contributors an encouragement to participation or does it make free-riding more tempting?

Apart from the nature of the outcomes that this kind of models leads to, attention could also turn to the type of interventions that might improve equilibria. Incentives, institutional planning, mechanisms for sharing information and network structure modification can all play a role in increasing overall contributions, enhancing efficiency, and promoting a fair distribution of responsibility throughout the network. Allouch [16], for example, following the lead of Bergstrom et al. [15], proposes a strategy of income redistribution via monetary transfers among players; similarly, Levit et al. [12] implement an algorithm to improve equilibrium efficiency by the use of transfer of payoffs. Bramoullé, Kranton and D’Amours [9] investigate how changes in the graph and payoff parameter affect equilibrium outcomes, while Kempe et al. [11] initiate an algorithmic study of network modifications aiming to induce equilibria of a particular form.

1.4 Related work

We now discuss the principal contributions of the main reference papers used in this work, following a chronological scheme to present them.

Among the main contributors to the study of public good games figure Y. Bramoullé and R. Kranton. Their 2007 paper, “Public goods in networks”, [13] gives an insight on a particular kind of Nash equilibrium profiles that are typical of the public good game setting, specialized equilibria. These action profiles are characterized by the extreme behavior of agents, who can be split into two groups based on their opposite behavior: specialized agents provide the individual maximal amount

of public good, while free-riders do not contribute at all and only benefit from the contributions of specialists. The authors analyze the game defined on an undirected and unweighted network, with positive and unbounded action spaces, representing the level of effort (public good) of each individual. They assume the cost of providing some effort to be equal for every player and linear in the invested effort; the definition of linear and concave utilities results in linear saturated best response functions. In this setting, an important finding is that specialization in public good provision at equilibrium is strongly related to the notion of maximal independent set; in fact, in any network there exists an equilibrium where some individuals contribute and others completely free-ride and specialized agents constitute a maximal independent set of nodes of the graph. Moreover, for the stability of a Nash equilibrium in the discrete-time dynamics, they find a sufficient and necessary condition related to the existence of maximal independent sets of order two: a Nash equilibrium is stable if and only if it is specialized and each free-rider has at least two specialist neighbors.

A generalization of the game is presented by Y. Bramoullé, R. Kranton and M. D'Amours in the 2014 paper “Strategic interaction and networks” [9]. The game is modified by adding differentiated costs for doing some effort for every agent and a (unique and positive) payoff parameter, allowing to weigh neighbors’ contribution to the public good. They still consider undirected and unweighted graphs and linear, concave utilities, which leads again to best response functions with a linear saturated form. One fundamental contribution of this paper to the study of public good games is methodological: due to the difficulty of dealing with the saturation (non-differentiability) of the best response, they derive results for a game with quadratic utilities leading to the same best response function; they use the well-known property of quadratic games to be exact potential games, which ensures that maximum and saddle points of the potential are equilibria, thus facilitating their characterization. With this approach, they find a precise characterization of Nash equilibria that avoids the direct saturation and is based on solving a linear system and verifying an inequality condition. Furthermore, they are the first to find that, for a large class of games, equilibria depend on a single network measure, the smallest eigenvalue. They prove a new sufficient condition for the uniqueness of the Nash equilibrium: the smallest eigenvalue of the adjacency matrix, in modulus, must be smaller than the inverse of the payoff parameter. If the Nash is unique, it is also stable. In addition, the paper provides two results concerning the stability of equilibria; first, the set of stable equilibria is the set of strict maximum points of the potential function. Second, if the uniqueness condition on the lowest eigenvalue is satisfied by the adjacency matrix restricted to contributors, then the equilibrium with those contributing agents is stable. Being the best response dynamics of the quadratic and public good game the same, the two games have the same set of equilibria, sharing the same behavior. Thus, the results are also valid in the second, more interesting setting of the public good games.

Similar results are published by Allouch in “On the private provision of public goods on networks” one year later [16]. The author analyzes the private provision of

public goods on unweighted and undirected networks employing a more economic approach. The study considers general quasi-concave utility functions, which still lead to the saturated form of the best response functions. The results are based on the notion of network normality, a property that relates the spectrum of the adjacency matrix to the Engel's curve related to the standard demand function of public good consumption. In particular, the derivative of the non linear part of the best reply must be lower-bounded by the inverse of the lowest eigenvalue of the adjacency matrix plus one and upper-bounded by one. It is proved that network normality ensures the uniqueness and local asymptotic stability (for the continuous-time dynamics) of the Nash equilibrium. Thus, coherently with the results of the previous work by Y. Bramoullé and R. Kranton, bounding from above the lowest eigenvalue of the adjacency matrix guarantees uniqueness and stability also in the more general setting of nonlinear utilities and despite the different approach.

In 2016 Y. Bramoullé and R. Kranton further generalize the game setting in “Games Played on Networks” [10] to the case of weighted, undirected networks and nonlinear, concave utilities. As in the 2014 paper, the results are obtained using the theory of potential games for undirected networks and quadratic utility functions having the same best reply as the one of the public good game. Their previous results are generalized to the case of weighted graphs and nonlinear utilities. In particular, Nash equilibria are maxima and saddle points of the potential function and only strict maxima are stable; uniqueness is again guaranteed upper bounding the absolute value of the lowest eigenvalue of the adjacency matrix.

Some of the findings are confirmed in the same year in the paper “On the Convergence of Piecewise Linear Strategic Interaction Dynamics on Networks” by B. Ghahesifard et al. [17], which aims at characterizing the asymptotic behavior of piecewise linear best-response dynamical systems of strategic interactions. In particular, they consider models where the payoff of each agent is positively enhanced by the investment of neighbors, of which public good games are a standard example. The main contribution is the proof that, on any weighted undirected graph, piecewise linear best-response dynamical systems of strategic interactions are asymptotically convergent to the set of equilibria. The proof relies again on the stability properties of potential games with component-wise concave potential, which serves as a Lyapunov function.

1.5 Thesis overview

This thesis work is primarily based on the presented papers, that are considered as the starting point of the research.

We consider a game defined on a general weighted and directed graph, that allows to model a very general setting of interconnections, where agents may not be mutually connected and externalities can be heterogeneous. In addition, individuals are assumed to be heterogeneous with respect to their income level, preference structure and effort cost. Each player has a total income that she shall entirely divide

among private and public good consumption. The two goods provide a payoff that is modeled by a utility function; private good provision is rewarded individually for each agent, while the payoff deriving from the public good concerns, for each player, both the individual allocation and the one of its direct out-neighbors in the graph, scaled by the weights of the connections. The thesis contributions are the following.

In the initial phase, we propose the general model definition, introduce the notion of linearity property of the value functions and derive best response functions. After observing that they present a saturated form and a monotonic behavior, we prove their equivalence to the best response functions of the game with generalized quadratic utilities. Afterwards, we concentrate on the game Nash equilibria and establish their existence in the general setting. A specific subclass of such value functions, those satisfying the linearity property, is thoroughly studied in this work; in particular, for these games we provide a characterization of Nash equilibrium profiles, with a focus on two specific types: internal equilibria, where all individuals contribute positively to the public good, and specialized equilibria, where a subset of agents contributes with their maximal amount and the others free-ride, not providing any contribution.

Subsequently, we present the most relevant results of this thesis work.

The first is a sufficient condition ensuring the uniqueness of the Nash equilibrium in the general setting of directed and weighted graphs. This technical result, applied to the case of linear value functions, relates uniqueness to the lowest eigenvalue of the per-row rescaled adjacency matrix.

The second relevant contribution is another sufficient condition, that guarantees the contractivity of the synchronous best response function. The uniqueness and global asymptotic stability of the Nash equilibrium both for the discrete- and continuous-time best response dynamics directly follows from this result.

Moreover, for the continuous-time dynamics, under the linearity of the value functions and the hypothesis of symmetric per-row rescaled adjacency matrix, we study the stability of Nash equilibrium profiles also in settings where contractivity may not be guaranteed. In particular, we prove that the weaker uniqueness condition ensures the global asymptotic stability of the unique Nash equilibrium under these hypothesis. In addition, we provide results on the stability of equilibrium profiles when uniqueness is violated. At this stage, we also characterize the asymptotic behavior of internal equilibria for both the continuous- and discrete-time dynamics, given the linearity property.

Finally, we define two efficiency metrics to evaluate the performance of different equilibrium profiles and we propose the initial framework for two potential interventions, opening the way for further study. In particular, following the lead of [18], we start investigating a strategy of subsidies to push agents to make their public good contributions higher and increase efficiency of equilibrium outcomes of the game. Moreover, we give the idea of a possible redistribution strategy, where an external planner allocates income to players in order to push the game towards a social optimum equilibrium.

A distinctive feature of this thesis is that it unifies and generalizes previous

results of Bramoullé and Kranton ([13], [9], [10]) and Allouch [16], generalizing them to directed and weighted networks with heterogeneous agents and nonlinear utilities. In this broader setting, we extend the characterization of equilibrium profiles and generalize uniqueness conditions. We also extend previous analysis of the asymptotic behavior of Nash equilibria, and introduce a novel contractivity condition for the synchronous best response dynamics, which guarantees uniqueness and global asymptotic stability of both discrete- and continuous-time dynamics.

The thesis is structured as follows. Chapter 2 represents an introductory and notational chapter, with a theoretical background on graph theory, game theory and games on networks. In Chapter 3 we provide the public good game model definition together with a few preliminary results on the best response function and Nash equilibrium existence. Chapter 4 presents the main contributions of the thesis research: two sufficient conditions, for the uniqueness of the Nash equilibrium and for the contractivity of the synchronous best response function, and a study of the limit behavior of equilibrium profiles for the discrete- and continuous-time dynamics. The last part, concerning the efficiency metrics and two intervention strategies, is presented in Chapter 5. The conclusive part, Chapter 6, summarizes findings and discusses future research avenues. Finally, in the Appendix A, we report some general mathematical results on the constrained maximization of concave functions that do not concern the game directly and that are used to make proofs, and in Appendix B we report some examples of value functions and particular forms they can take.

Chapter 2

Technical background

This chapter provides the notation used in the study and a theoretical background that focuses on introducing the main general topics that this work deals with. In particular, first comes a brief introduction on graph theory, followed by a more detailed part concerning game theory and some of its subtopics that will be central in the successive sections, strategic complements and substitutes and potential games; the two areas of study meet when dealing with games on networks, which is topic of the third part, together with a focus on benchmark quadratic games. Most of the theory concerning graphs, games and network games is taken from “Social and economic networks” by Jackson [6] and from “Games on networks” by Jackson and Zenou [8].

2.1 Graph theory

Graph theory is a very useful mathematical instrument that allows to model a set of agents that interact with each other. We give a formal definition of graph and recall the basics of graph theory.

Let us introduce some concepts that are fundamental to the definition of a graph. The set $\mathcal{N} = \{1, 2, \dots, n\}$ is the set of nodes that are involved in the network of relationships, often referred to as agents, players or individuals. $\mathcal{E} = \{(i, j) : G_{ij} > 0 \ \forall i, j = 1, \dots, n\}$ is the set of edges, or links, of the graph, representing the presence of connections between nodes: nodes $i, j \in \mathcal{N}$ are connected if and only if there exists an edge $(i, j) \in \mathcal{E}$. The weights of the connections between nodes of the graph are stored in the adjacency matrix $G \in \mathbb{R}_{\geq 0}^{n \times n}$ ¹, a real-valued $n \times n$ matrix with nonnegative entries such that G_{ij} represents the strength of the link between nodes $i, j \in \mathcal{N}$; $G_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$, while $G_{ij} = 0$ if nodes $i, j \in \mathcal{N}$ are not linked. We assume that the connection of a node to itself, called a self-loop, is not allowed, so that $G_{ii} = 0$ for every $i \in \mathcal{N}$.

Definition 2.1.1. A graph, or network, $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$ is defined by the triple

¹Concerning the notation, by $\mathbb{R}_{>0}^n$ we indicate vectors in $\mathbb{R}^n$ with strictly positive entries, and by $\mathbb{R}_{\geq 0}^n$ we indicate vectors in $\mathbb{R}^n$ with nonnegative entries. Similarly, we use $\mathbb{R}_{\geq 0}^{n \times n}$ to refer to $n \times n$ real matrices with nonnegative entries.

composed of $\mathcal{N} = \{1, 2, \dots, n\}$, the set of nodes of the graph, $G \in \mathbb{R}_{\geq 0}^{n \times n}$ the adjacency matrix and $\mathcal{E} = \{(i, j) : G_{ij} > 0 \ \forall \ i, j = 1, \dots, n\}$, the set of edges of the graph.

A graph is said to be *unweighted* if the adjacency matrix only takes two values, typically in $\{0, 1\}$, and $G_{ij} = 1$ if and only if nodes i and j are linked in the graph, otherwise $G_{ij} = 0$. If the entries of the adjacency matrix can take more than two values, they define the intensity of the relationships and they are called weights; such a graph is said to be *weighted*.

A graph is *undirected* if connections are always reciprocal, i.e., $G_{ij} = G_{ji}$. Otherwise, if non-mutual relationships can occur, meaning that it can happen that $G_{ij} \neq G_{ji}$, it is referred to as *directed*.

For a network $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$, given a subset of nodes $\mathcal{S} \subset \mathcal{N}$ denote $\mathcal{G}_{\mathcal{S}} = (\mathcal{S}, \mathcal{E}_{\mathcal{S}}, G_{\mathcal{S}})$ the graph restricted to the set of nodes of $\mathcal{S}$. $\mathcal{G}_{\mathcal{S}}$ is the network obtained from $\mathcal{G}$ by deleting all links except those between nodes in $\mathcal{S}$:

$$(G_{\mathcal{S}})_{ij} = \begin{cases} G_{ij} & \text{if } i, j \in \mathcal{S} \\ 0 & \text{otherwise} \end{cases}$$

so that $\mathcal{E}_{\mathcal{S}} = \{(i, j) : (G_{\mathcal{S}})_{ij} > 0 \ \forall \ i, j \in \mathcal{S}\}$.

The *out-neighborhood* $\mathcal{N}_i^o$ of a node $i \in \mathcal{N}$ is the set of nodes that i is linked to: $\mathcal{N}_i^o = \{j \in \mathcal{N} : G_{ij} > 0\}$ and the *out-degree* of i is its cardinality $d_i^o = |\mathcal{N}_i^o|$. Similarly, the *in-neighborhood* $\mathcal{N}_i^i$ is the set of nodes that are linked to i : $\mathcal{N}_i^i = \{j \in \mathcal{N} : G_{ji} > 0\}$ and the *in-degree* of i is its cardinality $d_i^i = |\mathcal{N}_i^i|$. These coincide in the case of a directed network and are referred to as neighborhood $\mathcal{N}_i$ and degree d_i of node i . A graph is said to be *out-regular* (or in-regular) if all nodes have the same out-degree (or in-degree).

A (*directed*) *path* of length $k \geq 1$ in a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$ is any sequence of distinct nodes $i_0, i_1, \dots, i_k \in \mathcal{N}$ such that $G_{i_j i_{j+1}} > 0$ for every $j = 1, \dots, k-1$. The graph is said to be *strongly connected* if there is a path in $\mathcal{G}$ between every couple of nodes $i, j \in \mathcal{N}$.

An *independent set* relative to a network $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$ is a subset of nodes $A \subset \mathcal{N}$ for which no two nodes are linked, i.e., if $i \in A$ and $j \in A$ then $(i, j) \notin \mathcal{E}$ and $(j, i) \notin \mathcal{E}$. A is a *maximal independent set* if it is not a proper subset of any other independent set of nodes.

Let us now introduce some notable graphs:

- the (undirected) complete graph is one where all possible links are present, except for self-loops, i.e., every node is connected to every other node: $G_{ij} > 0$ for every $i, j \in \mathcal{N}$ and $G_{ii} = 0$.
- the directed line graph is one with two extreme nodes with only one neighbor $G_{1i} > 0$ iff $i = 2$ and $G_{in} > 0$ iff $i = n-1$ and internal nodes only connected to the previous node $G_{ji} > 0$ iff $j = i-1$. In the undirected line also opposite links are present: $G_{21} > 0$, $G_{n-1n} > 0$ and $G_{ji} > 0$ iff $j \in \{i-1, i+1\}$

- the directed circle graph is one with a single cycle (a sequence of links connecting nodes that starts and ends at the same node) and such that every node has exactly two neighbors: $G_{ji} > 0$ iff $j = i - 1$ and $G_{ij} > 0$ iff $j = i + 1$; also, nodes 1 and n are connected, $G_{n1} > 0$. In the undirected circle also the opposite links are present: $G_{1n} > 0$ and $G_{ji} > 0$ iff $j \in \{i - 1, i + 1\}$.
- the (undirected) star graph is a network with a node i , called the center of the star, such that every edge involves i : $G_{ij} > 0$ and $G_{ji} > 0$ for every other node j .

2.1.1 Spectral theory for nonnegative matrices

Nonnegative square matrices are usually connected to the context of graph theory, since the adjacency matrix of a graph has nonnegative elements. Then, it is useful to recall some well known results concerning the spectral theory of nonnegative matrices. The following theoretical resume is derived from “Nonnegative Matrices in the Mathematical Sciences” by Berman and Plemmons [19].

Consider a nonnegative real square matrix $G \in \mathbb{R}_{\geq 0}^{n \times n}$, meaning that every entry of G is a real nonnegative value, $G_{ij} \in \mathbb{R}$ and $G_{ij} \geq 0$ for every $i, j = 1, \dots, n$. Denote the spectrum of G as $\rho(G) = \{\lambda_i : \lambda_i \text{ is an eigenvalue of } G, i = 1, \dots, n\}$, its spectral radius as $\lambda^G = \max_{i=1, \dots, n} \{|\lambda_i| : \lambda_i \in \rho(G)\}$ and the corresponding eigenvector as v^G .

$G \in \mathbb{R}_{\geq 0}^{n \times n}$ is said to be *reducible* if there exists a permutation matrix P such that the matrix $E = PGP^T$ is in block triangular form, i.e.,

$$E = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$$

with B, C square matrices. Otherwise, G is *irreducible*.

An equivalent definition of irreducibility links nonnegative matrices and graphs. Denote by $\mathcal{G}(G)$ the associated directed graph of $G \in \mathbb{R}_{\geq 0}^{n \times n}$, i.e., the graph whose adjacency matrix is G . Then, the matrix G is irreducible if and only if the associated directed graph $\mathcal{G}(G)$ is strongly connected.

For irreducible nonnegative matrices the *Perron Frobenius Theorem* can be stated: Let $G \in \mathbb{R}_{\geq 0}^{n \times n}$ nonnegative matrix. Then,

1. the spectral radius of G , λ^G , is an eigenvalue of G .
2. To λ^G is associated a nonnegative eigenvector $v^G \geq 0$.
3. if G is irreducible, λ^G is a simple eigenvalue of G and any eigenvalue of G of the same modulus is also simple. Moreover, G has a positive eigenvector $v^G > 0$ corresponding to λ^G , and any nonnegative eigenvector of G is a multiple of v^G .

A consequence of the previous result is the following bound for the spectral radius

λ^G of a nonnegative irreducible matrix $G \in \mathbb{R}_{\geq 0}^{n \times n}$:

$$\min_{i=1, \dots, n} \sum_{j=1}^n G_{ij} \leq \lambda^G \leq \max_{i=1, \dots, n} \sum_{j=1}^n G_{ij}$$

2.1.2 Matrix induced norm

We provide the definition of a vector norm related to an irreducible nonnegative square matrix $G \in \mathbb{R}_{\geq 0}^{n \times n}$, the G -norm. Then, we prove that the G -norm of the matrix G that defines it is exactly its dominant eigenvalue λ^G .

Given an irreducible nonnegative square matrix $G \in \mathbb{R}_{\geq 0}^{n \times n}$, denote the dominant eigenpair (λ^G, v^G) , where λ^G is the spectral radius and v^G is the corresponding eigenvector, i.e., such that $Gv^G = \lambda^G v^G$.

Definition 2.1.2. For a nonnegative irreducible square matrix $G \in \mathbb{R}_{\geq 0}^{n \times n}$, define the G -norm as the vector norm $\| \cdot \|_G : \mathbb{R}^n \rightarrow [0, +\infty)$ st

$$\|x\|_G = \sum_{i=1}^n v_i^G |x_i|$$

The induced matrix norm is then $\|G\|_G = \max_{x \in \mathbb{R}^n, \|x\|=1} \|Gx\|_G$.

Then, the following Lemma identifies the spectral radius of the matrix G with its G -norm.

Lemma 2.1.1. Given an irreducible matrix $G \in \mathbb{R}_{\geq 0}^{n \times n}$ for the G -norm it holds

$$\|G\|_G = \lambda^G$$

Proof. ($\geq$) In general, $\|G\|_G \geq |\lambda_i|$, so that $\|G\|_G \geq \lambda^G$.

($\leq$) $\forall x \in \mathbb{R}^n$

$$\begin{aligned} \|Gx\|_G &= \sum_{i=1}^n v_i^G |(Gx)_i| = \sum_{i=1}^n v_i^G \left| \sum_{j=1}^n G_{ij} x_j \right| \leq \sum_{i=1}^n v_i^G \sum_{j=1}^n G_{ij} |x_j| = \\ &= \sum_{i=1}^n \sum_{j=1}^n G_{ij} v_i^G |x_j| = \sum_{j=1}^n (Gv^G)_j |x_j| = \sum_{j=1}^n \lambda^G v_j^G |x_j| = \lambda^G \|x\|_G \end{aligned}$$

Then $\|G\|_G \leq \lambda^G$. □

2.2 Game Theory

Game theory provides a powerful tool to model the strategic interaction of agents that make decisions that are interdependent. Let us introduce some fundamental concepts to recall basic notions.

The set of players (or agents) is $\mathcal{N} = \{1, \dots, n\}$. In a strategic game, each agent has a set of actions (or strategies) χ_i , that can be discrete or continuous. We assume

that $\chi_i \subseteq \mathbb{R}$ is an interval of the real line. Let $\chi = \chi_1 \times \cdots \times \chi_n$, the action space, be the set of all profiles of actions $q = (q_1, \dots, q_n) \in \chi$. Each agent plays by choosing an action $q_i \in \chi_i$ and the reward of choosing such action is determined both by her action and the actions played by all players but i , $q_{-i} \in \chi_{-i}$. The payoff of player i as a function of the action profile $q \in \chi$ is described by a function $u_i : \chi \rightarrow \mathbb{R}$, $u_i(q) = u_i(q_i, q_{-i})$, called *utility function*.

Definition 2.2.1. A strategic game is determined by the set of players $\mathcal{N} = \{1, \dots, n\}$, the action space $\chi = \chi_1 \times \cdots \times \chi_n$ and the set of utility functions for every player $(u_i)_{i=1}^n$.

Agents are assumed to be rational, in the sense that they choose an action in order to maximize their utility. A strategy $q_i \in \chi_i$ is a *best response* (or best reply) of player i to a profile of strategies for the other players $q_{-i} \in \chi_{-i}$ if for every other action $q'_i \in \chi_i$

$$u_i(q_i, q_{-i}) \geq u_i(q'_i, q_{-i})$$

The *best response function* of player i , $\mathcal{B}_i : \chi_{-i} \rightarrow \chi_i$, is thus univocally defined by the solution to the optimization problem

$$\mathcal{B}_i(q_{-i}) = \operatorname{argmax}_{q_i \in \chi_i} u_i(q)$$

so it consists in choosing the best possible action q_i given q_{-i} .

For the set of players $\mathcal{N}$, the *synchronous best response function* $\mathcal{F} : \chi \rightarrow \chi$, is the unique vectorial function defined as $\mathcal{F}(q) = (\mathcal{B}_i(q_{-i}))_{i=1}^n$, in which every agent plays its best response at the same time.

Definition 2.2.2. An action profile $q^* \in \chi$ is said to be a *pure strategy Nash equilibrium* if, for every agent i , q_i^* is a best response to q_{-i}^* . That is, $q^* \in \chi$ is a Nash equilibrium if and only if $q_i^* \in \mathcal{B}_i(q_{-i}^*)$ for every i .

In a Nash equilibrium no player has any incentive to change her action. Thus, a Nash equilibrium is a fixed point of the synchronous best response function, that is $\mathcal{F}(q^*) = q^*$.

2.2.1 Concave games

As stated by Rosen in “Existence and uniqueness of equilibrium points for concave n-person games” [20], one of the main complications in the context of n-person games is the lack of uniqueness of the equilibrium profile. The author proposes a way to overcome this difficulty by exploiting the concept of concave games and deriving a theory to ensure the uniqueness of the Nash equilibrium in such games. Let us recall his paper’s main contributions.

A strategic game with set of players $\mathcal{N} = \{1, \dots, n\}$ and action space $\chi = \chi_1 \times \cdots \times \chi_n$ is a *concave game* if the utility function $u_i(q)$ for every player is continuous in the action profile $q \in \chi$ and concave in the agent’s action q_i , for each fixed value of q_{-i} .

Denote the positively weighted sum of the utilities, with weight vector $r \in \mathbb{R}_{\geq 0}^n$, as $\sigma(q, r) = \sum_{i=1}^n r_i u_i(q)$, and define the pseudogradient of $\sigma(q, r)$ as

$$h(q, r) = \begin{pmatrix} r_1 \frac{\partial u_1(q)}{\partial q_1} \\ \vdots \\ r_n \frac{\partial u_n(q)}{\partial q_n} \end{pmatrix}$$

$\sigma(q, r)$ is said to be *diagonally strictly concave* for a fixed weight vector $r \geq 0$ if, for every $q, \tilde{q} \in \chi$, it holds

$$(q - \tilde{q})^T h(\tilde{q}, r) + (\tilde{q} - q)^T h(q, r) > 0$$

Being such a property hard to verify, he provides a more analytical sufficient condition for it. Define the matrix $H(q, r)$ such that $(H(q, r))_{ij} = r_i \frac{\partial^2 u_i(q)}{\partial q_i \partial q_j}$ and its j -th column is

$$(H(q, r))_{\cdot j} = \frac{\partial h(q, r)}{\partial q_j} = \begin{pmatrix} r_1 \frac{\partial^2 u_1(q)}{\partial q_1 \partial q_j} \\ \vdots \\ r_n \frac{\partial^2 u_n(q)}{\partial q_n \partial q_j} \end{pmatrix}$$

Then, he proves that, if the symmetric matrix $H(q, \bar{r}) + H(q, \bar{r})^T$, for some $\bar{r} > 0$, is negative definite, then $\sigma(q, \bar{r})$ is diagonally strictly concave for $\bar{r}$.

Then, it is possible to state *Rosen's equilibrium criterion*.

Theorem 2.2.1. *If $\sigma(q, \bar{r})$ is diagonally strictly concave for some vector of weights $\bar{r} > 0$, then there exists a unique Nash equilibrium for the concave game.*

2.2.2 Game dynamics

Starting from an initial action profile $q \in \chi$, players will dynamically adjust their actions to singularly increase their utility. The game dynamics, or *best response dynamics*, tracks the evolution of the action profile $q \in \chi$ as players sequentially and individually choose their action according to the best response, after observing the actions played by the other players. To define it, assume that, for every action profile $q \in \chi$ and every player i , the best response $\mathcal{B}_i(q_{-i})$ is unique. Then, the game dynamics can be described by a dynamical system.

If the time variable is discrete, $t = 0, \dots, T$, it's possible to define the discrete-time best response dynamics as $q(t+1) = \mathcal{F}(q(t))$. For the continuous time variable $t \in [0, +\infty)$, recalling that the action spaces are intervals of the real line $\chi_i \subset [0, +\infty)$, the continuous-time best response dynamics is defined as $\dot{q}(t) = \mathcal{F}(q(t)) - q(t)$.

According to its definition, a Nash equilibrium profile is an equilibrium of the best response dynamics, $q^* = \mathcal{F}(q^*)$ for the discrete case, and $\dot{q}^* = 0$ in continuous time. An equilibrium q^* for the dynamical system $\dot{q}(t) = \mathcal{F}(q(t)) - q(t)$ (or $q(t+1) = \mathcal{F}(q(t))$) is said to be:

- *stable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that if at $t = 0$ the distance of the system from q^* is smaller than δ , i.e., $\|q(0) - q^*\| < \delta$, then for $t \rightarrow +\infty$

the system will stay ϵ -close to q^* , i.e., $\|q(t) - q^*\| < \epsilon$.

- *attractive* if there exists $\delta > 0$ such that if at $t = 0$ the distance of the system from q^* is smaller than δ , i.e., $\|q(0) - q^*\| < \delta$, then for $t \rightarrow +\infty$ the system will converge to q^* , i.e., $\|q(t) - q^*\| \rightarrow_{t \rightarrow +\infty} 0$.
- *locally asymptotically stable* if it is stable and attractive and *globally asymptotically stable* if it is stable and the dynamics converges to the equilibrium for any $\delta > 0$.
- *unstable* if it is not stable.

2.2.3 Strategic complements and strategic substitutes

In the context of strategic games, there are two relevant classes of games, whose main distinction lies on whether an increase in the action played by a given player pushes her neighbors to consequently increase or decrease their action.

It is necessary that the action space of each player χ_i is well-ordered, which is the case for χ_i intervals of the real line. Then also $\chi = \chi_1 \times \cdots \times \chi_n$ is associated an ordering $\geq$ ². It is then possible to define the two classes of games based on the property of increasing and decreasing differences.

Definition 2.2.3. *A game with players $\mathcal{N}$, action space χ and utilities $(u_i)_{i=1}^n$ is said to be a strategic complements game if it exhibits increasing differences, i.e., for every i , for every $q, q' \in \chi$ such that $q \geq q'$*

$$u_i(q_i, q_{-i}) - u_i(q'_i, q_{-i}) \geq u_i(q_i, q'_{-i}) - u_i(q'_i, q'_{-i})$$

An equivalent, analytical, definition is the following:

Definition 2.2.4. *A game with players $\mathcal{N}$, action space χ and utilities $(u_i)_{i=1}^n$ twice continuously differentiable is said to be a strategic complements game if for every action profile $q \in \chi$ and every couple of agents $i, j \in \mathcal{N}$, $i \neq j$ it holds*

$$\frac{\partial^2 u_i}{\partial q_i \partial q_j}(q) \geq 0$$

For a given agent, in games of strategic complements, an increase in the actions of her neighbors pushes her to also increase her action to have a higher utility. In strategic complements games with finite action space (or compact action space and continuous payoffs), the existence of an equilibrium is guaranteed; moreover, there exist algorithms to find maximal and minimal equilibria and the best response dynamics generally converges to equilibrium points.

Definition 2.2.5. *A game with players $\mathcal{N}$, action space χ and utilities $(u_i)_{i=1}^n$ is said to be a strategic substitutes game if it exhibits decreasing differences, i.e., for*

²Concerning the notation, inequalities between vectors are intended component-wise, i.e., $q \geq q'$ if $q_i \geq q'_i$ for every i . The same holds for $>, <, \leq$.

every i , for every $q, q' \in \chi$ such that $q \geq q'$

$$u_i(q_i, q_{-i}) - u_i(q'_i, q_{-i}) \leq u_i(q_i, q'_{-i}) - u_i(q'_i, q'_{-i})$$

An equivalent, analytical, definition is the following

Definition 2.2.6. *A game with players $\mathcal{N}$, action space χ and utilities $(u_i)_{i=1}^n$ twice continuously differentiable is said to be a strategic substitutes game if for every action profile $q \in \chi$ and every couple of agents $i, j \in \mathcal{N}$, $i \neq j$ it holds*

$$\frac{\partial^2 u_i}{\partial q_i \partial q_j}(q) \leq 0$$

On the contrary, in games of strategic substitutes, for a given player, an increase in the actions of her neighbors pushes her to diminish her action to have a higher utility. In strategic substitutes games existence of equilibrium points is guaranteed if the action spaces are non empty, compact and convex subsets of a Euclidian space and the utilities are continuous and quasi-concave for every player.

2.2.4 Potential games

Definition 2.2.7. *A game is an exact potential game if there exists a function $P : \chi \rightarrow \mathbb{R}$, called potential function, such that for every $q, \tilde{q} \in \chi$ such that $q_i \neq \tilde{q}_i$ and $q_{-i} = \tilde{q}_{-i}$ it holds*

$$u_i(q_i, q_{-i}) - u_i(\tilde{q}_i, q_{-i}) = P(q_i, q_{-i}) - P(\tilde{q}_i, q_{-i})$$

In an exact potential game, for any action profile $q \in \chi$, the utility variation incurred by player i when changing action unilaterally is the same as the corresponding variation in the potential function.

The main strength of potential games lies in the fact that every global maximum point of the potential function P is a pure strategy Nash equilibrium. Although equilibria that are not maximum points of the potential function may exist, the result guarantees the existence of a Nash equilibrium in at least two cases: when the action space is finite and when utilities are continuous on a compact action space.

2.3 Games on networks

In games defined on networks, players are connected through a graph, which determines the structure of relationships between them; the payoff of each individual depends on the actions taken by the neighbors, rather than on all other players, and the strength of the dependence is determined by the adjacency matrix of the graph.

In the field of network games, given a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$, the set of nodes $\mathcal{N}$ represents the agents involved in the game, connections between players are determined by the set of edges $\mathcal{E}$ and characterized by the adjacency matrix $G \in \mathbb{R}_{\geq 0}^{n \times n}$. Let the action space be $\chi = \chi_1 \times \cdots \times \chi_n$. The utility of each agent i now depends on

her action $q_i \in \chi_i$ and on the actions played by her neighbors in the graph $q_{\mathcal{N}_i} \in \chi_{\mathcal{N}_i}$, where $\chi_{\mathcal{N}_i} = \prod_{j \in \mathcal{N}_i} \chi_j \subset \chi$ denotes the action space restricted to neighbors of i , $u_i : \chi(G) \rightarrow \mathbb{R}$ $u_i(q(G)) = u_i(q_i, q_{\mathcal{N}_i})$.

Definition 2.3.1. *Given a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$, a network game, or graphical game, on $\mathcal{G}$ is a game with players $\mathcal{N} = \{1, \dots, n\}$, action space χ and utilities $(u_i)_{i=1}^n$ such that for every player $i \in \mathcal{N}$ and action profiles $q, \tilde{q} \in \chi$, if $q_i = \tilde{q}_i$ and $q_{\mathcal{N}_i} = \tilde{q}_{\mathcal{N}_i}$, then $u_i(q) = u_i(\tilde{q})$*

A (pure strategy) Nash equilibrium is thus a profile of actions $q \in \chi$ such that for every player i and for every other action of her neighbors $q'_{\mathcal{N}_i} \in \chi_{\mathcal{N}_i}$

$$u_i(q_i, q_{\mathcal{N}_i}) \geq u_i(q_i, q'_{\mathcal{N}_i})$$

To simplify the notation, for each agent i , we will drop the dependency on the graph and simply write q_{-i} , recalling that her payoff is only affected by agents to which i is linked in the graph.

2.3.1 Benchmark game with quadratic utilities

Quadratic utility functions are often used as a benchmark model to simplify the study of more complicated cases. In particular, when dealing with utility functions that are not explicitly defined but of which the main characteristics and best response are known, one can define a quadratic utility satisfying the assumptions or having the same best response dynamics. This strategy is adopted since the quadratic utility is often analytically tractable and easier to study: for example, defining a potential for such functions is usually a viable option. Then, results obtained for the quadratic model can be proved to be valid (or used as a starting point) for the more general case under study, as they share the hypothesis set or the underlying dynamical system.

Let us define a utility function with a quadratic term in each player's action and use it to prove some results that we will generalize to the game at the center of this study in the future chapters. In particular, we derive the best response for this game and prove, under suitable assumptions of linearity, that the game with quadratic utilities is exact potential and equilibria can be characterized exploiting the potential function.

Consider a graphical game on the graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$. Given a vector $w \in \mathbb{R}_{>0}^n$, take as action space $\chi = [0, w_1] \times \dots \times [0, w_n] \subset \mathbb{R}_{>0}^n$ and define the generalized quadratic utilities $\bar{u}_i : \chi \rightarrow \mathbb{R}$ with the form

$$\bar{u}_i(q) = -\frac{q_i^2}{2} + w_i q_i - k_i(w_i + (Gq)_i)q_i$$

where $k_i : (0, +\infty) \rightarrow (0, +\infty) \in \mathcal{C}^1$ for every i .

Proposition 2.3.1. *For the game with utilities $\bar{u}_i$, for every agent i the best response*

$\bar{\mathcal{B}}_i(q_{-i}) : \chi_{-i} \rightarrow \chi_i$ is uniquely defined and has the form

$$\bar{\mathcal{B}}_i(q_{-i}) = [w_i - k_i(w_i + (Gq)_i)]_+$$

Proof. The best response $\bar{\mathcal{B}}_i(q_{-i})$ for the game with utilities

$$\bar{u}_i(q) = -\frac{q_i^2}{2} + w_i q_i - k_i(w_i + (Gq)_i)q_i$$

with action space $\chi = [0, w_1] \times \dots \times [0, w_n] \subset \mathbb{R}_{\geq 0}^n$ is the solution to the maximization problem

$$\max_{q_i \in \chi_i} \bar{u}_i(q) = \max_{q_i \in [0, w_i]} -\frac{q_i^2}{2} + w_i q_i - k_i(w_i + (Gq)_i)q_i$$

Notice that $\bar{u}_i$ are strictly concave along every q_i :

$$\frac{\partial \bar{u}_i}{\partial q_i}(q) = -q_i + w_i - k_i(w_i + (Gq)_i)$$

and

$$\frac{\partial^2 \bar{u}_i}{\partial q_i^2}(q) = -1$$

So there exists a unique maximum for $\bar{u}_i$ with respect to q_i and it can be found by setting

$$\frac{\partial \bar{u}_i}{\partial q_i}(q) = -q_i + w_i - k_i(w_i + (Gq)_i) = 0$$

which is equivalent to

$$q_i = w_i - k_i(w_i + (Gq)_i)$$

adding the non negativity constraint $q_i \geq 0$ we obtain a best response with the saturated form

$$\bar{\mathcal{B}}_i(q_{-i}) = [w_i - k_i(w_i + (Gq)_i)]_+$$

□

Now assume that the function k_i is linear with $k'_i = \bar{k}_i \in (0, 1)$. Thus, the utilities $\bar{u}_i : \chi \rightarrow \mathbb{R}$ take the form:

$$\bar{u}_i(q) = -\frac{q_i^2}{2} + w_i q_i - \bar{k}_i q_i (w_i + (Gq)_i)$$

Rearranging the terms and defining the per-row rescaled matrix $\bar{G} = [\bar{k}]G = \text{diag}(\bar{k}_1, \dots, \bar{k}_n)G \in \mathbb{R}_{\geq 0}^{n \times n}$ such that $\bar{G}_{ij} = \bar{k}_i G_{ij}$ and the vector $b = (b_i)_{i=1}^n \in \mathbb{R}_{> 0}^n$ such that $b_i = w_i(1 - \bar{k}_i)$, the utilities $\bar{u}_i$ take the form

$$\bar{u}_i(q) = -\frac{q_i^2}{2} + b_i q_i - (\bar{G}q)_i q_i$$

and we refer to them as quadratic utilities. The best response function for every agent i can be written as

$$\bar{\mathcal{B}}_i(q_{-i}) = [b_i - (\bar{G}q)_i]_+$$

Let us state some results that hold for the game with these quadratic utilities $\bar{u}_i$ and that will be used in the subsequent part of the research to study the public good game. We begin by establishing that the game with quadratic utility functions exhibits strategic substitutes.

Proposition 2.3.2. *The game with quadratic utilities $\bar{u}_i(q) = -\frac{q_i^2}{2} + b_i q_i - (\bar{G}q)_i q_i$ exhibits strategic substitutes.*

Proof. Let us verify that for every i , $\frac{\partial^2 \bar{u}_i}{\partial q_i \partial q_j}(q) \leq 0$. Take the first derivative:

$$\frac{\partial \bar{u}_i}{\partial q_i}(q) = -q_i + b_i - (\bar{G}q)_i$$

then

$$\frac{\partial^2 \bar{u}_i}{\partial q_i \partial q_j}(q) = -\bar{G}_{ij} \leq 0$$

□

Let us define the functional $P : \chi \rightarrow \mathbb{R}$ st

$$P(q) = \sum_{i=1}^n \frac{1}{2} (-q_i^2 - q_i (\bar{G}q)_i + 2b_i q_i)$$

or, in vector form

$$P(q) = -\frac{1}{2} q^T (I + \bar{G}) q + b^T q$$

We exploit P to prove that the quadratic game is an exact potential game, which will be useful to find the characterization of Nash equilibrium profiles for such games.

Proposition 2.3.3. *Given a matrix $\bar{G} \in \mathbb{R}_{\geq 0}^{n \times n}$ and a vector $b \in \mathbb{R}_{> 0}^n$, the following properties of P hold:*

1. *P is strictly concave along each component*
2. *if $\bar{G}$ is symmetric and $\lambda_{\min}^{\bar{G}}$, smallest eigenvalue of $\bar{G}$, is st*

$$-\lambda_{\min}^{\bar{G}} < 1$$

then P is strictly concave

Proof. 1. The proof follows directly from the computations:

$$\begin{aligned} \frac{\partial P}{\partial q_i}(q) &= \frac{\partial}{\partial q_i} \left(\sum_{j \neq i} \frac{1}{2} (-q_j^2 - q_j (\bar{G}q)_j + 2b_j q_j) \right) + \frac{\partial}{\partial q_i} \left(\frac{1}{2} (-q_i^2 - q_i (\bar{G}q)_i + 2b_i q_i) \right) \\ &= -\frac{1}{2} \frac{\partial}{\partial q_i} \left(\sum_{j \neq i} q_j \left(\sum_{k=1}^n \bar{G}_{jk} q_k \right) - q_i - \frac{1}{2} (\bar{G}q)_i + b_i \right) = -q_i + b_i - \frac{1}{2} \sum_{j=1}^n q_j (\bar{G}_{ji} + \bar{G}_{ij}) \end{aligned}$$

so that

$$\frac{\partial^2 P}{\partial q_i^2}(q) = -1$$

2. If $\bar{G}$ is symmetric, its concavity is guaranteed by the negative definiteness of the hessian matrix. The gradient of P is $\nabla P(q) = -(I + \bar{G})q + b$ so that the hessian matrix is $H_P = -(I + \bar{G})$. It is negative definite if and only if for every $v \in \mathbb{R}^n$

$$v^T H_P v = -v^T (I + \bar{G}) v < 0$$

which is equivalent to having strictly negative eigenvalues, i.e., for every i $-1 - \lambda_i^{\bar{G}} < 0$. A sufficient condition is that the lowest eigenvalue is strictly negative, i.e., $-1 - \lambda_{min}^{\bar{G}} < 0$, which is equivalent to

$$-\lambda_{min}^{\bar{G}} < 1$$

Thus, if this condition is satisfied, P is strictly concave. □

Theorem 2.3.4. *If $\bar{G}$ is a symmetric matrix, the game with quadratic utility functions $\bar{u}_i(q) = -\frac{q_i^2}{2} + b_i q_i - (\bar{G}q)_i q_i$ is a potential game with potential function P .*

Proof. Recall that a game is said to be a potential game with potential function P if for every $q, \tilde{q} \in \chi$ such that $q_i \neq \tilde{q}_i$ and $q_{-i} = \tilde{q}_{-i}$ it holds

$$u_i(q) - u_i(\tilde{q}) = P(q) - P(\tilde{q})$$

The proof follows directly verifying the definition. Take $q, \tilde{q} \in \chi$ such that $q_i \neq \tilde{q}_i$ and $q_{-i} = \tilde{q}_{-i}$ and start by noticing that $(\bar{G}q)_i = (\bar{G}\tilde{q})_i$ since $\bar{G}_{ii} = 0$. Then for $\bar{u}_i$, it holds

$$\bar{u}_i(q) - \bar{u}_i(\tilde{q}) = -\frac{1}{2}(q_i^2 - \tilde{q}_i^2) + b_i(q_i - \tilde{q}_i) - (\bar{G}q)_i(q_i - \tilde{q}_i)$$

and for the potential

$$\begin{aligned} P(q) - P(\tilde{q}) &= \sum_{j \neq i} \frac{1}{2}(-q_j^2 - q_j(\bar{G}q)_j + 2b_j q_j) - \sum_{j \neq i} \frac{1}{2}(-\tilde{q}_j^2 - \tilde{q}_j(\bar{G}\tilde{q})_j + 2b_j \tilde{q}_j) \\ &\quad - \frac{1}{2}(q_i^2 - \tilde{q}_i^2) + b_i(q_i - \tilde{q}_i) - \frac{1}{2}(\bar{G}q)_i(q_i - \tilde{q}_i) \\ &= -\sum_{j \neq i} \frac{1}{2}q_j(\bar{G}(q - \tilde{q}))_j - \frac{1}{2}(q_i^2 - \tilde{q}_i^2) + b_i(q_i - \tilde{q}_i) - \frac{1}{2}(\bar{G}q)_i(q_i - \tilde{q}_i) \\ &= -\frac{1}{2}\sum_{j \neq i} q_j(\bar{G}_{ji}(q_i - \tilde{q}_i))_j - \frac{1}{2}(q_i^2 - \tilde{q}_i^2) + b_i(q_i - \tilde{q}_i) - \frac{1}{2}(\bar{G}q)_i(q_i - \tilde{q}_i) \end{aligned}$$

and for the symmetric property of $\bar{G}$, this is equal to

$$P(q) - P(\tilde{q}) = -\frac{1}{2}(\bar{G}q)_i(q_i - \tilde{q}_i) - \frac{1}{2}(q_i^2 - \tilde{q}_i^2) + b_i(q_i - \tilde{q}_i) - \frac{1}{2}(\bar{G}q)_i(q_i - \tilde{q}_i) = u_i(q) - u_i(\tilde{q})$$

□

Proposition 2.3.5. *For the game with quadratic utilities $\bar{u}_i(q) = -\frac{q_i^2}{2} + b_i q_i - (\bar{G}q)_i q_i$ and symmetric $\bar{G}$, if an action configuration $q^* \in \chi_w$ is a stationary point of the*

functional P , i.e.,

$$\nabla P(q^*) = 0$$

then q^* is a Nash equilibrium.

Proof. To prove the result, consider the derivatives of P along every q_i

$$\frac{\partial P}{\partial q_i}(q) = -q_i + b_i - \frac{1}{2} \sum_{j=1}^n q_j (\bar{G}_{ji} + \bar{G}_{ij}) = -q_i + b_i - (\bar{G}q)_i$$

and notice that $\frac{\partial P}{\partial q_i}(q) = \frac{\partial \bar{u}_i}{\partial q_i}(q)$.

If q^* is a stationary points of P , i.e., for every i

$$\frac{\partial P}{\partial q_i}(q^*) = 0$$

then

$$\frac{\partial \bar{u}_i}{\partial q_i}(q^*) = 0$$

Being $\bar{u}_i$ concave along every component, its derivative with respect to q_i equal to zero implies that agent i is playing his best response. Since it is true for every player if q^* is a stationary point of P , then q^* is a Nash equilibrium for the game with quadratic utilities $\bar{u}_i$. $\square$

Proposition 2.3.6. *For the game with quadratic utilities $\bar{u}_i(q) = -\frac{q_i^2}{2} + b_i q_i - (\bar{G}q)_i q_i$ and symmetric $\bar{G}$, an action configuration $q^* \in \chi_w$ is a Nash equilibrium if and only if for every i the following conditions on P hold:*

$$\frac{\partial P}{\partial q_i}(q^*) \leq 0$$

and

$$q_i^* \frac{\partial P}{\partial q_i}(q^*) = 0$$

Proof. Notice that

$$\frac{\partial P}{\partial q_i}(q) = -q_i + b_i - (\bar{G}q)_i = \frac{\partial \bar{u}_i}{\partial q_i}(q)$$

Thus the conditions can be rewritten for $\bar{u}_i$:

$$\frac{\partial \bar{u}_i}{\partial q_i}(q^*) \leq 0 \text{ and } q_i^* \frac{\partial \bar{u}_i}{\partial q_i}(q^*) = 0$$

Then, for every i :

- if $q_i^* > 0$, since $q_i^* \frac{\partial \bar{u}_i}{\partial q_i}(q^*) = 0$, it must hold

$$\frac{\partial \bar{u}_i}{\partial q_i}(q^*) = 0$$

being $\bar{u}_i$ strictly concave in q_i , it corresponds to a maximum of the utility. Thus, every agent i such that $q_i^* > 0$ is playing his best response.

- if $q_i^* = 0$, explicitly writing the condition $\frac{\partial \bar{u}_i}{\partial q_i}(q^*) \leq 0$ for $q_i^* = 0$, we get that it holds

$$b_i - (\bar{G}q^*)_i \leq 0$$

which is the condition for agent i to play zero as best response. Thus, also every agent i such that $q_i^* = 0$ is playing his best response.

Then the conditions are true if and only if every agent is playing his best response, i.e., for every i $q_i^* \in \mathcal{B}_i(q_{-i}^*)$, that is equivalent to q^* being a Nash equilibrium for the game with quadratic utilities. $\square$

Chapter 3

The public good game model and preliminary results

In this chapter, we provide our definition of the public good game and some initial results. The game draws on the specification of Allouch [16], but considers a more general setting (directed networks).

First, we introduce the game setting and the specifications of the model, presenting its main elements, such as the income vector and the value functions, two variable functions satisfying properties of smoothness and concavity.

Second, for the general game definition, we derive the best response functions and observe that they present a saturated form and a monotonic behavior. In addition, we prove that the best response functions coincide with those of the game with generalized quadratic utility. After introducing the notion of linearity property of the value functions, we notice that, under this assumption, the game exhibits linear saturated best response functions and propose the case of the Cobb-Douglas value functions, a standard example that will be recovered throughout the whole study.

Afterwards, we concentrate on the game Nash equilibria. We establish their existence in the general setting and provide their characterization when the value functions satisfy the linearity property.

In the last part, we focus on two particular types of Nash equilibria: internal equilibria, where all individuals provide a positive amount to the public good, and specialized equilibria, where a subset of agents contributes with their maximal amount and the others free-ride.

3.1 Definition of the game

Let us introduce the founding elements of the public good game, such as the income vector and the value functions, and state the main hypothesis.

To define a public good game, let us consider the graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$, a general directed and weighted graph without self-loops, meaning that $(i, j) \in \mathcal{E}$ if and only if $G_{ij} > 0$, otherwise $G_{ij} = 0$ and $G_{ii} = 0$ for every $i \in \mathcal{N}$, so that G is a matrix with nonnegative entries.

The game is characterized by an income vector $w \in \mathbb{R}_{>0}^n$, where for every i w_i is agent i 's total income, that he shall divide among private and public good consumption. Denote by $x \in \mathbb{R}_{\geq 0}^n$ the vector of private good provision and $q \in \mathbb{R}_{\geq 0}^n$ the vector of public good provision; then, for every player i the income division constraint $w_i = x_i + q_i$ must be verified.

Every agent i is associated a value function $\eta_i : \mathbb{R}_{\geq 0}^2 \setminus \{(0,0)\} \rightarrow \mathbb{R}$, a $\mathcal{C}^2$ function with hypothesis to be specified. We store them in an n -dimensional array $\eta = (\eta_i)_{i=1}^n \in \mathbb{R}^n$. The value function of each agent i is a two variable function, depending on her own private good provision x_i , on the sum of her public good provision q_i and the weighted sum of the provisions of her neighbors to the public good $(Gq)_i$: $\eta_i = \eta_i(x_i, q_i + (Gq)_i) = \eta_i(w_i - q_i, q_i + (Gq)_i)$, where the last equality is obtained using the income division constraint.

The action for every player is the amount of income reserved to the public good provision $q_i \in \chi_i = [0, w_i]$, and the private good contribution is derived as $x_i = w_i - q_i$. The game action space, denoted by χ_w for its dependence on the income vector, is then $\chi_w = [0, w_1] \times \cdots \times [0, w_n] \subset \mathbb{R}_{\geq 0}^n$.

Each agent benefits from her own provision of private good and from her and her neighbors provision of public good through the value function, which allows to define the utility function $u_i : \chi \rightarrow \mathbb{R}$, that depends on the action profile $q \in \chi_w$ and has the form:

$$u_i(q) = \eta_i(w_i - q_i, q_i + (Gq)_i) \quad (3.1)$$

For every i the following assumptions on the value function $\eta_i : \mathbb{R}_{\geq 0}^2 \setminus \{(0,0)\} \rightarrow \mathbb{R}$ must hold: η_i twice continuously differentiable, strictly concave and such that for every $x, y \geq 0$

$$\frac{\partial \eta_i(0, y)}{\partial x} \geq \frac{\partial \eta_i(0, y)}{\partial y} \quad (3.2)$$

$$\frac{\partial \eta_i(x, 0)}{\partial x} \leq \frac{\partial \eta_i(x, 0)}{\partial y} \quad (3.3)$$

Remark 3.1.1. A less restrictive set of hypothesis on the value functions η_i is also sufficient. In particular, the following assumptions must hold: $\eta_i : \mathbb{R}_{\geq 0}^2 \setminus \{(0,0)\} \rightarrow \mathbb{R}$ continuous function, $\mathcal{C}^2$ on $(0, +\infty)$, st

1. for every $(x, y) \in (0, +\infty)^2$

$$v^T \mathbf{H}_{\eta_i}(x, z - x)v < 0 \quad (3.4)$$

where $\mathbf{H}_{\eta_i}$ is the Hessian matrix of η_i and $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

2. for every $x > 0, y > 0$ the limit conditions hold:

$$\lim_{\epsilon \rightarrow 0^+} \frac{\partial \eta_i(\epsilon, y)}{\partial x} - \frac{\partial \eta_i(\epsilon, y)}{\partial y} \geq 0 \text{ (or } +\infty) \quad (3.5)$$

$$\lim_{\epsilon \rightarrow 0^+} \frac{\partial \eta_i(x, \epsilon)}{\partial x} - \frac{\partial \eta_i(x, \epsilon)}{\partial y} \leq 0 \text{ (or } -\infty) \quad (3.6)$$

Indeed, (3.4) is a relaxation of the strict concavity of η_i limited to the lines $(x, z - x)$, while (3.5) and (3.6) generalize (3.2) and (3.3). Thus, all the results are valid for both game formulations.

Remark 3.1.2. Notice that if

$$\eta_i(0, y) = \eta_i(x, 0) = 0 \quad (3.7)$$

for every $x, y > 0$ then (3.2) and (3.3) also hold and (3.7) is simpler to check.

In addition, for every value function η_i , for every $x, y \in \mathbb{R}_{\geq 0}^2 \setminus \{(0, 0)\}$, we assume

$$\frac{\partial^2 \eta_i}{\partial y^2}(x, y) \leq \frac{\partial^2 \eta_i}{\partial x \partial y}(x, y) \quad (3.8)$$

It is now possible to provide the definition of the public good game.

Definition 3.1.1. A public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$ is a strategic network game uniquely determined by the triple $(\mathcal{G}, w, \eta)$. $\mathcal{G}, w, \eta$ have the characteristics described above, the action space is $\chi_w = [0, w_1] \times \cdots \times [0, w_n]$ and the utilities are u_i of (3.1).

We now introduce the Cobb–Douglas value functions, which serve as the standard reference throughout this thesis. They allow us to demonstrate practical applications of the results on general graphs as well as parametric cases involving notable graphs.

Example 3.1.1. Cobb–Douglas value functions

Consider a game defined on a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$, an income vector $w \in \mathbb{R}_{>0}^n$, and value functions $\eta_i : \mathbb{R}_{\geq 0}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$ of the form:

$$\eta_i(x, y) = x^{\alpha_i} y^{\beta_i}$$

for $\alpha_i, \beta_i \in (0, 1)$. In general, η_i is not strictly concave in $\mathbb{R}_{\geq 0}^2 \setminus \{(0, 0)\}$, so let us verify that the less restrictive assumptions of Remark 3.1.1 hold.

- η_i continuous function, $\mathcal{C}^2$ on $(0, +\infty)$.
- The hessian matrix is $\mathbf{H}_{\eta_i}(x, y) = \begin{bmatrix} \alpha_i(\alpha_i - 1)x^{\alpha_i-2}y^{\beta_i} & \alpha_i\beta_i x^{\alpha_i-1}y^{\beta_i-1} \\ \alpha_i\beta_i x^{\alpha_i-1}y^{\beta_i-1} & \beta_i(\beta_i - 1)y^{\beta_i-2}x^{\alpha_i} \end{bmatrix}$,

so for every $(x, y) \in (0, +\infty)^2$ and for $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ it holds:

$$v^T \mathbf{H}_{\eta_i}(x, y) v = \alpha_i(\alpha_i - 1)x^{\alpha_i-2}y^{\beta_i} - 2\alpha_i\beta_i x^{\alpha_i-1}y^{\beta_i-1} + \beta_i(\beta_i - 1)y^{\beta_i-2}x^{\alpha_i} < 0$$

- hypothesis (3.5) and (3.6) are satisfied: for every $x, y > 0$

$$\lim_{\epsilon \rightarrow 0^+} \frac{\partial \eta_i(\epsilon, y)}{\partial x} - \frac{\partial \eta_i(\epsilon, y)}{\partial y} = \lim_{\epsilon \rightarrow 0^+} \alpha_i \epsilon^{\alpha_i-1} y^{\beta_i} - \beta_i \epsilon^{\alpha_i} y^{\beta_i-1} = +\infty$$

and

$$\lim_{\epsilon \rightarrow 0^+} \frac{\partial \eta_i(x, \epsilon)}{\partial x} - \frac{\partial \eta_i(x, \epsilon)}{\partial y} = \lim_{\epsilon \rightarrow 0^+} \alpha_i x^{\alpha_i-1} \epsilon^{\beta_i} - \beta_i x^{\alpha_i} \epsilon^{\beta_i-1} = -\infty$$

- finally, we verify (3.8): for every $(x, y) \in \mathbb{R}_{\geq 0}^2 \setminus \{(0, 0)\}$

$$\frac{\partial^2 \eta_i}{\partial y^2}(x, y) - \frac{\partial^2 \eta_i}{\partial x \partial y}(x, y) = \beta_i(\beta_i - 1)y^{\beta_i-2}x^{\alpha_i} - \alpha_i\beta_i x^{\alpha_i-1}y^{\beta_i-1} < 0$$

So, we have proved that the game with value functions $\eta_i(x, y) = x^{\alpha_i}y^{\beta_i}$ is a public good game.

3.2 Best response functions

After introducing a central characteristic of the value functions, the linearity property, we establish that the general version of the public good game has saturated best response functions. Moreover, we observe that the best response exhibits a monotone behavior and prove that it coincides with the best response of the game with generalized quadratic utilities. In the end, we apply the previous results and derive the best response functions under the linearity property.

Let us begin by proving that every value function is associated a continuous, strictly positive function k_i , specifying how much of her income an agent would optimally spend on the private good when facing the trade-off between private and public consumption.

Lemma 3.2.1. *For the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, for every agent i there exists the function $k_i : (0, +\infty) \rightarrow (0, +\infty) \in \mathcal{C}^1$ st*

$$k_i(z) = \underset{x \in [0, z], z > 0}{\operatorname{argmax}} \eta_i(x, z - x) \quad (3.9)$$

Proof. Being $\eta_i(x_i, z_i - x_i) \in \mathcal{C}^2$, strictly concave, under assumptions (3.2) and (3.3), for Lemma A.0.1, for every i there exists $k_i : (0, +\infty) \rightarrow (0, +\infty) \in \mathcal{C}^1$ solution of the maximization problem (3.9). $\square$

Thus, for each player, the function $k_i(z)$ captures the optimal rule that she follows when deciding how to split her income z between private and public consumption. To each possible income level $z > 0$, k_i associates the amount reserved to the private good, while the remainder, $z - k_i(z)$ is provided to the public good. The derivative $k'_i(z)$ then describes how this optimal allocation changes with the income. When $k'_i(z)$ is constant across all income levels— which defines the concept of the linearity property—the trade-off between private and public consumption does not depend on the amount of income. This property greatly simplifies the analysis of equilibria in the game. Let us provide the formal definition of this characteristic.

Definition 3.2.1. *The value functions $\eta = (\eta_i)_{i=1}^n$ of the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, are said to satisfy the linearity property $\mathbf{L}$ if for every i for k_i of (3.9) it holds $k'_i(z) = \bar{k}_i > 0$. If $\bar{k}_i = \bar{k} > 0$, then we say that the value functions η satisfy the strong linearity property $\mathbf{L}^+$.*

Let us derive the functions k_i in the Cobb-Douglas setting.

Example 3.2.1. *Cobb-Douglas value functions (cont.)*

For the game with Cobb-Douglas value functions, to derive the relation for k_i , recall that for every i it is the unique solution, for a given $z > 0$, to:

$$\eta_i(k_i(z), z - k_i(z)) = \max_{x \in [0, z], z > 0} \eta_i(x, z - x)$$

Then, $k_i(z)$ is obtained by solving

$$\frac{d\eta_i}{dx}(x, z - x) = \alpha_i x^{\alpha_i - 1} (z - x)^{\beta_i} - \beta_i x^{\alpha_i} (z - x)^{\beta_i - 1} = 0$$

which implies

$$x = k_i(z) = \frac{\alpha_i z}{\alpha_i + \beta_i}$$

Being k_i a linear function for every i , the Cobb-Douglas value functions satisfy the linearity property **L** with

$$\bar{k}_i = k'_i(z) = \frac{\alpha_i}{\alpha_i + \beta_i}$$

If the parameters are equal, i.e., $\alpha_i = \alpha$ and $\beta_i = \beta$ for every i , then the strong linearity property **L**⁺ holds with $\bar{k} = \frac{\alpha}{\alpha + \beta}$. Notice that, given the graph $\mathcal{G}$ and the income vector w , the Cobb-Douglas public good game is completely determined by the vectors of parameters $\alpha = (\alpha_i)_{i=1}^n$ and $\beta = (\beta_i)_{i=1}^n$, thus to stress this dependency, we refer to $\eta = (\eta_i)_{i=1}^n$ of $\mathcal{P}$ as $\eta(\alpha, \beta) = (\eta_i(\alpha_i, \beta_i))_{i=1}^n$.

For other examples of value functions and particular forms they can take, refer to Appendix B.

It is now possible to derive the best response functions. In order to maximize her individual utility, each agent aims to play the action that solves the constrained optimization problem:

$$\max_{q_i \in \chi_i} u_i(q) = \max_{0 \leq q_i \leq w_i} \eta_i(w_i - q_i, q_i + (Gq)_i) \quad (3.10)$$

Proposition 3.2.2. *Consider a public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$. Then, the best response function $\mathcal{B}_i : \chi_{-i} \rightarrow \chi_i$ is uniquely defined for every agent i and has the form:*

$$\mathcal{B}_i(q_{-i}) = [w_i - k_i(w_i + (Gq)_i)]_+$$

Proof. To prove the result, let us do a change of variable and reformulate the optimization problem for the best response: define $z_i = (Gq)_i + w_i$, so that $q_i + (Gq)_i = z_i - x_i$ since $x_i = w_i - q_i$, and reformulate problem (3.10) as:

$$\max_{0 \leq x_i \leq z_i - (Gq)_i} \eta_i(x_i, z_i - x_i) \quad (3.11)$$

Consider now the relaxation of problem (3.11):

$$\max_{0 \leq x_i \leq z_i} \eta_i(x_i, z_i - x_i) \quad (3.12)$$

For Lemma 3.2.1, for every i there exists $k_i : (0, +\infty) \rightarrow (0, +\infty) \in \mathcal{C}^1$ solution of (3.12). Then, since η_i strictly concave and, for hypothesis (3.2) and (3.3), increasing at $x_i = 0$:

$$x_i^* = \min\{k_i(z_i), w_i\}$$

is a solution to the initial problem (3.11). Finally, the uniquely determined best response function (solution of (3.10)) is:

$$\mathcal{B}_i(q_{-i}) = q_i^* = w_i - x_i^* = (w_i - \min\{k_i(z_i), w_i\}) = [w_i - k_i(z_i)]_+ = [w_i - k_i(w_i + (Gq)_i)]_+$$

□

Then, the synchronous best response function $\mathcal{F} : \chi_w \rightarrow \chi_w$, the vectorial function in which every agent plays her best response at the same time, is:

$$\mathcal{F}(q) = (\mathcal{B}_i(q_{-i}))_{i=1}^n = ([w_i - k_i(w_i + (Gq)_i)]_+)_{i=1}^n = [w - k(w - Gq)]_+$$

using the vector notation $(k_i(w_i + (Gq)_i))_{i=1}^n = k(w - Gq)$.

From the main hypothesis of the game, it is possible to derive the following monotonicity results for the best response of each individual and for the synchronous one.

Proposition 3.2.3. *The public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$ exhibits strategic substitutes and for every agent i the best response $\mathcal{B}_i(q_{-i}) : \chi_{-i} \rightarrow \chi_i$ is decreasing in the public good provision of her out-neighbors.*

Proof. The game exhibits strategic substitutes if and only if

$$\frac{\partial^2 u_i}{\partial q_i \partial q_j}(q) \leq 0$$

The second derivatives of the utilities are:

$$\frac{\partial^2 u_i}{\partial q_i \partial q_j}(q) = G_{ij} \left(\frac{\partial^2 \eta_i}{\partial y^2}(w_i - q_i, q_i + (Gq)_i) - \frac{\partial^2 \eta_i}{\partial x \partial y}(w_i - q_i, q_i + (Gq)_i) \right)$$

Then, from hypothesis (3.8) on the value functions follows the strategic substitutability. Moreover, from Remark A.0.1, we know that the condition that makes k_i an increasing function is exactly (3.8). Thus, since the best response has the form $\mathcal{B}_i(q_{-i}) = [w_i - k_i(w_i + (Gq)_i)]_+$ and k_i is increasing, then when neighbors contributions are higher, the best response of player i decreases. □

Proposition 3.2.4. *For the synchronous best response function $\mathcal{F} : \chi_w \rightarrow \chi_w$ of the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, for every couple of public good provision profiles $q, \tilde{q} \in \chi_w$ with $q \leq \tilde{q}$, the following properties of monotonicity hold:*

$$\mathcal{F}(q) \geq \mathcal{F}(\tilde{q})$$

and

$$\mathcal{F}^2(q) \leq \mathcal{F}^2(\tilde{q})$$

Proof. The thesis follows directly from the monotonicity of the best response function of every agent due to the strategic substitutes setting. For every couple of public good provision profiles $q, \tilde{q} \in \chi_w$ with $q \leq \tilde{q}$, for every agent i we have:

$$w_i + (Gq)_i \leq w_i + (G\tilde{q})_i$$

and, since k_i increasing,

$$k_i(w_i + (Gq)_i) \leq k_i(w_i + (G\tilde{q})_i)$$

then, for the decreasing behavior of $\mathcal{B}_i(q_{-i})$,

$$\mathcal{B}_i(q_{-i}) = [w_i - k_i(w_i + (Gq)_i)]_+ \geq \mathcal{B}_i(\tilde{q}_{-i}) = [w_i - k_i(w_i + (G\tilde{q})_i)]_+$$

In vector form

$$\mathcal{F}(q) \geq \mathcal{F}(\tilde{q})$$

Applying $\mathcal{F}$ on both sides again, we get

$$\mathcal{F}^2(q) \leq \mathcal{F}^2(\tilde{q})$$

□

Now consider a general graphical game defined on the graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$, with action space $\chi_w = [0, w_1] \times \dots \times [0, w_n] \subset \mathbb{R}_{\geq 0}^n$ and define for every i the generalized quadratic utilities $\bar{u}_i : \chi_w \rightarrow \mathbb{R}$ with the form

$$\bar{u}_i(q) = -\frac{q_i^2}{2} + w_i q_i - k_i(w_i + (Gq)_i) q_i$$

Proposition 3.2.5. *Given a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$ and a vector $w \in \mathbb{R}_{\geq 0}^n$, consider the generalized quadratic utility functions defined on $\chi_w = [0, w_1] \times \dots \times [0, w_n] \subset \mathbb{R}_{\geq 0}^n$, $\bar{u}_i : \chi_w \rightarrow \mathbb{R}$, having the form*

$$\bar{u}_i(q) = -\frac{q_i^2}{2} + w_i q_i - k_i(w_i + (Gq)_i) q_i$$

then the game with utilities $\bar{u}_i$ has the same best response function as the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$.

Proof. The result follows directly from the best response of the game with utilities $\bar{u}_i$. For the proof see Proposition 2.3.1. □

Finally, let us define a particular type of action configuration, called specialized, where players either provide their maximum amount of public good or do not contribute at all.

Definition 3.2.2. Consider the game $\mathcal{P} = (\mathcal{G}, w, \eta)$. A vector of public good contributions $q \in \chi_w$ is said to be a specialized profile if for every i $q_i > 0$ if and only if $q_i = \max_{q_i \in \chi_i} \mathcal{B}_i(q_{-i})$, otherwise $q_i = 0$.

The study repeatedly focuses on a particular subclass of public good games that are easily tractable, those where the value functions satisfy the linearity property. We propose a general notation that accounts for $\mathbf{L}$ and is based on the per-row rescaled adjacency matrix $\bar{G}$. Define the matrix $\bar{G} = [\bar{k}]G = \text{diag}(\bar{k}_1, \dots, \bar{k}_n)G \in \mathbb{R}_{\geq 0}^{n \times n}$ such that $\bar{G}_{ij} = \bar{k}_i G_{ij}$ and the vector $b = (b_i)_{i=1}^n \in \mathbb{R}_{\geq 0}^n$ such that $b_i = w_i(1 - \bar{k}_i)$.

For such games, let us derive the best response functions.

Proposition 3.2.6. Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, with value functions $\eta = (\eta_i)_{i=1}^n$ satisfying the linearity property $\mathbf{L}$ for some $\bar{k} = (\bar{k}_i)_{i=1}^n \in (0, 1)^n$. Then, the best response is uniquely defined for every agent i and has the linear saturated form:

$$\mathcal{B}_i(q_{-i}) = [b_i - (\bar{G}q)_i]_+$$

Proof. For Proposition 3.2.2, the best response is uniquely determined for every agent. Moreover, the definition of property $\mathbf{L}$ implies that k_i are linear functions and they can be written as

$$k_i(w_i + (Gq)_i) = \bar{k}_i(w_i + (Gq)_i)$$

then the best response functions have the form

$$\mathcal{B}_i(q_{-i}) = [w_i - \bar{k}_i(w_i + (Gq)_i)]_+ = [w_i(1 - \bar{k}_i) - \bar{k}_i(Gq)_i]_+ = [b_i - (\bar{G}q)_i]_+$$

□

The bound on the constants $\bar{k}_i$, i.e., $\bar{k}_i \in (0, 1)$, relies on the fact that b_i must be a positive value, otherwise the best response of agent i is always playing zero. Indeed $b_i = w_i(1 - \bar{k}_i) > 0$ if and only if $\bar{k}_i < 1$. Thus, we will assume that for the game $\mathcal{P} = (\mathcal{G}, w, \eta)$ it holds $\bar{k}_i < 1$ for every i .

The synchronous best response vectorial function is then $\mathcal{F} : \chi_w \rightarrow \chi_w$

$$\mathcal{F}(q) = [b - \bar{G}q]_+$$

Remark 3.2.1. Notice that, under the linearity property $\mathbf{L}$, for the corresponding quadratic game with the same best response as the public good game of Proposition 3.2.5, the utilities simplify to the quadratic form:

$$\bar{u}_i(q) = -\frac{q_i^2}{2} + b_i q_i - (\bar{G}q)_i q_i$$

It is interesting to notice that, under the linearity property of the value functions and symmetric matrix $\bar{G}$, the game with quadratic utilities is an exact potential game with potential function

$$P(q) = -\frac{1}{2}q^T(I + \bar{G})q + b^T q$$

for the proof see 2.3.4.

Example 3.2.2. *Cobb-Douglas value functions (cont.)*

Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$ with Cobb-Douglas value functions

$$\eta_i(w_i - q_i, q_i + (Gq)_i) = (w_i - q_i)^{\alpha_i} (q_i + (Gq)_i)^{\beta_i}$$

with $\alpha_i, \beta_i \in (0, 1)$.

As previously seen, the value functions satisfy property **L** with $\bar{k}_i = \frac{\alpha_i}{\alpha_i + \beta_i}$ and, if $\alpha_i = \alpha$ and $\beta_i = \beta$ for every i , property **L**⁺ holds with $\bar{k} = \frac{\alpha}{\alpha + \beta}$.

In this setting, $b_i = \frac{\beta_i w_i}{\alpha_i + \beta_i}$ and $\bar{G} = [\frac{\alpha}{\alpha + \beta}]G$. Then, for Proposition 3.2.6, the best response is uniquely defined for every agent and has the form

$$\mathcal{B}_i(q_{-i}) = [\frac{\beta_i w_i}{\alpha_i + \beta_i} - \frac{\alpha_i}{\alpha_i + \beta_i} (Gq)_i]_+$$

To simplify the notation, for the vectors of constants α, β, w , we will refer to the operations applied component-wise to the elements of the vectors writing the operations for the vectors, for example $\frac{\alpha}{\alpha + \beta} = (\frac{\alpha_i}{\alpha_i + \beta_i})_{i=1}^n$ and $\frac{\beta w}{\alpha + \beta} = (\frac{\beta_i w_i}{\alpha_i + \beta_i})_{i=1}^n$.

The synchronous best response vectorial function is $\mathcal{F} : \chi_w \rightarrow \chi_w$ st

$$\mathcal{F}(q) = [\frac{\beta w}{\alpha + \beta} - \frac{\alpha}{\alpha + \beta} Gq]_+$$

3.3 Nash equilibria

We prove that Nash equilibria always exist for the public good game in the general setting. In addition, under the linearity property of the value functions, we provide a characterization of the equilibrium profiles that relies on solving a linear system and verifying a system of inequalities.

For the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, a Nash equilibrium is an action profile $q^* \in \chi_w$ such that for every i it holds

$$q_i^* = [w_i - k_i(w_i + (Gq)_i)]_+$$

Theorem 3.3.1. *Consider a public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$. Then, there exists a Nash equilibrium $q^* \in \chi_w$ for the game.*

Proof. From Proposition 3.2.2 we know the best response function for every agent is $\mathcal{B}_i(q_{-i}) = [w_i - k_i(w_i + (Gq)_i)]_+$, and that it is single-valued and continuous.

A Nash equilibrium q^* is a fixed point of $\mathcal{F}$: $\mathcal{F}(q^*) = q^*$ and, being the synchronous best response function $\mathcal{F}$ a continuous function from a compact, convex set χ_w to itself, the existence of q^* follows from *Brouwer's fixed point theorem*. $\square$

Let us define two particular types of equilibrium profiles that will be at the center of the study of future sections.

Definition 3.3.1. For a general public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, consider $q^* \in \chi_w$ equilibrium for the game.

1. q^* is said to be an internal equilibrium if $q_i^* > 0$ for every $i = 1, \dots, n$.
2. q^* is said to be a specialized equilibrium if it is also a specialized profile.

We now provide a characterization of Nash equilibrium profiles under the linearity property of the value functions. In particular, we propose two equivalent ways to see if an action profile is a Nash equilibrium, both based on verifying a linear system and a system of inequalities. The first one is more general, while the second one highlights the partition of agents into contributors and non-contributors at equilibrium.

Let us first recall the definition of Nash equilibrium profile in the setting of value functions under **L** and derive some preliminary results. A Nash equilibrium for the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$ is a public good provision profile $q^* \in \chi_w$ st

$$q^* = [b - \bar{G}q^*]_+$$

Proposition 3.3.2. Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, with value functions satisfying the linearity property **L** for some $\bar{k} = (\bar{k}_i)_{i=1}^n \in (0, 1)^n$. $q^* \in \chi_w$ is a Nash equilibrium if and only if the following conditions hold:

$$q^{*T}(I + \bar{G})q^* = b^T q^*$$

and

$$q^* + \bar{G}q^* \geq b$$

Proof. Consider the quadratic utility function $\bar{u}_i(q) = -\frac{q_i^2}{2} + b_i q_i - (\bar{G}q)_i q_i$. It follows from Proposition 3.2.5 that the game with utilities $\bar{u}_i$ has the same best response functions as $\mathcal{P}$, i.e., $\mathcal{B}_i(q_{-i}) = [b_i - (\bar{G}q)_i]_+$.

Let us first prove that $q^* \in \chi_w$ is a Nash equilibrium if and only if for every i the following conditions for the quadratic utility $\bar{u}_i$ hold:

$$\frac{\partial \bar{u}_i}{\partial q_i}(q^*) \leq 0$$

and

$$q_i^* \frac{\partial \bar{u}_i}{\partial q_i}(q^*) = 0$$

For every i :

- if $q_i^* > 0$, since $q_i^* \frac{\partial \bar{u}_i}{\partial q_i}(q^*) = 0$, it must hold

$$\frac{\partial \bar{u}_i}{\partial q_i}(q^*) = 0$$

being $\bar{u}_i$ strictly concave in q_i , it corresponds to a maximum of the utility. Thus, every agent i such that $q_i^* > 0$ is playing his best response.

- if $q_i^* = 0$, explicitly writing the condition $\frac{\partial \bar{u}_i}{\partial q_i}(q^*) \leq 0$ for $q_i^* = 0$, we get that it holds

$$b_i - (\bar{G}q^*)_i \leq 0$$

which is the condition for agent i to play zero as best response. Thus, also every agent i such that $q_i^* = 0$ is playing his best response.

Then the conditions are true if and only if every agent is playing his best response, i.e., for every i $q_i^* \in \mathcal{B}_i(q_{-i}^*)$, that is equivalent to q^* being a Nash equilibrium for the game with quadratic utilities. Thus, q^* is a Nash equilibrium also for the public good game with linear saturated best response.

Explicitly writing these conditions, we get that $q^* \in \chi_w$ is a Nash equilibrium if and only if for every i it holds

$$\frac{\partial \bar{u}_i}{\partial q_i}(q^*) \leq 0 \iff b_i \leq q_i^* + (\bar{G}q^*)_i$$

and

$$q_i^* \frac{\partial \bar{u}_i}{\partial q_i}(q^*) = 0 \iff q_i^*(b_i - q_i^* - (\bar{G}q^*)_i) = 0$$

which is equivalent to

$$q_i^{*2} + (\bar{G}q^*)_i q_i^* = b_i q_i^*$$

The thesis follows writing the conditions in vector form:

$$q^{*T}(I + \bar{G})q^* = b^T q^*$$

and

$$q^* + \bar{G}q^* \geq b$$

□

Given a public good provision profile $q \in \chi$, let us define the set of contributors C as the set of agents providing a positive amount of public good, $C = \{i \in \mathcal{N} : q_i > 0\}$ and the set of non-contributors $-C$ as the set of agents that do not contribute to the public good, $-C = \{i \in \mathcal{N} : q_i = 0\}$. Denote by $q_C \in \mathbb{R}_{>0}^{|C|}$, $w_C \in \mathbb{R}_{>0}^{|C|}$, $G_C \in \mathbb{R}_{\geq 0}^{|C| \times |C|}$ the public good provision vector, income vector and adjacency matrix restricted to contributors, respectively, by $G_{-C,C} \in \mathbb{R}_{\geq 0}^{|-C| \times |C|}$ the adjacency matrix restricted to rows of contributors and columns of non contributors and $w_{-C} \in \mathbb{R}_{>0}^{|-C|}$ the income vector restricted to non-contributors.

Let us see an equivalent characterization that puts the accent on the distinction between contributing and non-contributing nodes at equilibrium.

Proposition 3.3.3. *Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, with value functions satisfying the linearity property **L** for some $\bar{k} = (\bar{k}_i)_{i=1}^n \in (0, 1)^n$. $q^* \in \chi_w$ is a Nash equilibrium if and only if there exists a partition of the agents $\mathcal{N} = C \cup -C$, C*

non empty, such that $q^*|_C = q_C$ and $q^*|_{-C} = 0$ and q_C satisfies

$$(I + \bar{G}_C)q_C = b_C \quad (3.13)$$

$$\bar{G}_{-C,C}q_C \geq b_{-C} \quad (3.14)$$

Proof. • ($\Rightarrow$) If $q^* \in \chi_w$ Nash equilibrium for the game, it is a fixed point of the best response function: $\mathcal{F}(q^*) = [b - \bar{G}q^*]_+ = q^*$. For every contributor $i \in C$, to be at equilibrium it must hold:

$$q_i^* = [b_i - (\bar{G}q^*)_i]_+ > 0$$

which means that the saturation is no longer necessary:

$$q_i^* = b_i - (\bar{G}q^*)_i$$

so that q_i^* satisfies the equation:

$$q_i^* + (\bar{G}q^*)_i = b_i \quad (3.15)$$

For every non-contributor $i \in -C$, to be at equilibrium it must hold:

$$q_i^* = [b_i - (\bar{G}q^*)_i]_+ = 0$$

which means that

$$b_i - (\bar{G}q^*)_i \leq 0$$

that is equivalent to

$$(\bar{G}q^*)_i - b_i \geq 0 \quad (3.16)$$

Then, writing (3.15) and (3.16) in vector form, we get the thesis.

• ($\Leftarrow$)

On the one hand, if q_C satisfies (3.13), then writing it component-wise, for every i it holds

$$(q_C)_i + (\bar{G}_C q_C)_i = b_i$$

that is equivalent to (3.15) restricted to C , which means that every agent $i \in C$ is playing his best response, thus is at equilibrium. On the other hand, since it also satisfies (3.14), for every $i \in \mathcal{N} \setminus C = -C$ it holds

$$(\bar{G}_{-C,C} q_C)_i \geq (b_{-C})_i$$

that is equivalent to the condition for non-contributors to be at equilibrium (3.16). Thus, all agents are at equilibrium and the equilibrium profile q^* is the extension of q_C to non contributors: $q^*|_C = q_C$ and $q^*|_{-C} = 0$.

□

Proposition 3.3.3 generalizes the Nash equilibria characterization of Bramoullé, Kranton and D'Amours [9] to the case of weighted directed graph $\mathcal{G}$ and heterogeneous payoff parameters $\bar{k}_i$ and incomes w_i .

Example 3.3.1. *Cobb-Douglas value functions (cont.)*

From the previous results, it is possible to derive the characterization of Nash equilibrium profiles for the public good game with Cobb-Douglas value functions $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$. For definition, a Nash equilibrium is a public good provision profile $q^* \in \chi_w$ st

$$\mathcal{F}(q^*) = [\frac{\beta}{\alpha + \beta}w - \frac{\alpha}{\alpha + \beta}Gq^*]_+ = q^*$$

For Proposition 3.3.2, $q^* \in \chi_w$ is a Nash equilibrium if and only if

$$q^{*T}(I + [\frac{\alpha}{\alpha + \beta}]G)q^* = (\frac{\beta w}{\alpha + \beta})^T q^*$$

and

$$q^* + [\frac{\alpha}{\alpha + \beta}]Gq^* \geq \frac{\beta w}{\alpha + \beta}$$

An equivalent characterization of Nash equilibria stressing the partition into contributing and non-contributing agents is given by Proposition 3.3.3. $q^* \in \chi_w$ is a Nash equilibrium if and only if there exists a partition of the agents $\mathcal{N} = C \cup -C$, C non empty, such that $q^*|_C = q_C$ and $q^*|_{-C} = 0$ and q_C satisfies

$$(I + [\frac{\alpha}{\alpha + \beta}]_C G_C)q_C = (\frac{\beta w}{\alpha + \beta})_C \quad (3.17)$$

and

$$[\frac{\alpha}{\alpha + \beta}]_{-C} G_{-C, C} q_C \geq (\frac{\beta w}{\alpha + \beta})_{-C} \quad (3.18)$$

3.4 Internal equilibria

Let us concentrate on the particular kind of equilibrium profiles, called internal, where every node actively participates in the public good provision at equilibrium. We prove that their existence is determined by the presence of a strictly positive solution to the linear system that characterizes equilibria. Moreover, we provide some sufficient conditions that ensure its presence for public good games with value functions satisfying the strong linearity property.

Throughout this section, consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, with value functions satisfying the linearity property **L** for some $\bar{k} = (\bar{k}_i)_{i=1}^n \in (0, 1)^n$ and recall that an internal equilibrium $q^* \in \chi_w$ is an equilibrium where all nodes are contributors: $q_i^* > 0$ for every $i \in \mathcal{N}$, i.e., $C = \mathcal{N}$.

Proposition 3.4.1. *Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, with value functions satisfying the linearity property **L** for some $\bar{k} = (\bar{k}_i)_{i=1}^n \in (0, 1)^n$. Then, there exists an internal equilibrium $q^* \in \chi_w$ for the game if and only if there exists a strictly*

positive solution of the linear system

$$(I + \bar{G})q^* = b \quad (3.19)$$

and q^* has the form

$$q^* = (I + \bar{G})^{-1}b$$

Proof. The result follows directly from the application of Proposition 3.3.3 with contributing nodes $C = \mathcal{N}$ and its shape is uniquely derived solving the linear system (3.19). It follows directly from the characterization of equilibria that, if there are no strictly positive solutions, then there are no internal equilibria. $\square$

Example 3.4.1. *Cobb-Douglas value functions (cont.)*

For the Cobb-Douglas public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, from Proposition 3.4.1, an internal equilibrium $q^* \in \chi_w$ is a positive solution to the linear system

$$(I + [\frac{\alpha}{\alpha + \beta}]G)q^* = \frac{\beta w}{\alpha + \beta} \quad (3.20)$$

meaning that it has the form

$$q^* = ([\frac{\alpha}{\alpha + \beta}]G + I)^{-1} \frac{\beta w}{\alpha + \beta}$$

For the game $\mathcal{P}$ where $\alpha_i = \alpha$ and $\beta_i = \beta$, the inverse matrix $(\frac{\alpha}{\alpha + \beta}G + I)^{-1}$ exists if its determinant is non zero, that is equivalent to $\bar{G}$ having non null eigenvalues, i.e., $\lambda_i^G \neq -\frac{\alpha + \beta}{\alpha}$.

Also, the internal solution $q = \frac{\beta}{\alpha + \beta}(\frac{\alpha}{\alpha + \beta}G + I)^{-1}w$ is positive if the inverse matrix is positive. Writing it as a power series, it is possible to see that it is not always the case:

$$(\frac{\alpha}{\alpha + \beta}G + I)^{-1} = \sum_{k=0}^{\infty} (-\frac{\alpha}{\alpha + \beta}G)^k$$

Notice that it is not always true that there exists an internal equilibrium for the game. Let us see a few examples.

Example 3.4.2. Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, defined on the unweighted undirected circle graph $\mathcal{G}$ of 4 nodes. The adjacency matrix is

$$G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Take $\alpha_i = \alpha = \frac{1}{3}, \beta_i = \beta = \frac{2}{3}$ and income vector $w = (1, 2, 1, 3)^T$.

When solving the linear system (3.20), the inverse matrix is not positive:

$$\left(\frac{\alpha}{\alpha+\beta}G + I\right)^{-1} = \begin{pmatrix} 1.4 & -0.6 & 0.4 & -0.6 \\ -0.6 & 1.4 & -0.6 & 0.4 \\ 0.4 & -0.6 & 1.4 & -0.6 \\ -0.6 & 0.4 & -0.6 & 1.4 \end{pmatrix}$$

and the solution is $q = (-0.8, 1.87, -0.8, 2.53)^T$. Thus, there exists no internal equilibrium for the game.

Example 3.4.3. Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$ defined on the unweighted undirected line graph $\mathcal{G}$ of three nodes. The adjacency matrix is

$$G = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Take $\alpha_i = \alpha = \frac{2}{3}, \beta_i = \beta = \frac{1}{3}$ and income vector $w = (2, 1, 2)^T$.

When solving the linear system (3.20), the inverse matrix is not positive:

$$\left(\frac{\alpha}{\alpha+\beta}G + I\right)^{-1} = \begin{pmatrix} 5 & -6 & 4 \\ -6 & 9 & -6 \\ 4 & -6 & 5 \end{pmatrix}$$

and the solution is $q = (4, -5, 4)^T$. Thus, there exist no internal equilibrium for the game.

It is thus interesting to study when there exists an internal equilibrium, i.e., when there exists a positive solution of (3.19). We focus now on the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, where the value functions satisfy the strong linearity property $\mathbf{L}^+$, meaning that $\bar{k}_i = \bar{k} \in (0, 1)$ for every i .

Proposition 3.4.2. Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$ with value functions satisfying the strong linearity property $\mathbf{L}^+$ for $\bar{k} \in (0, 1)$. If $I + \bar{G}$ is invertible and the income vector $w > 0$ is a positive eigenvector of G , associated to the eigenvalue λ^w , then there exists the internal equilibrium $q^* \in \chi_w$ of the form

$$q^* = (1 - \bar{k})(\bar{k}\lambda^w + 1)((I + \bar{G})^{-1})^2 w$$

Proof. If $w > 0$ eigenvector of G associated to the positive eigenvalue λ^w , then for definition G $Gw = \lambda^w w > 0$. Then, also

$$\bar{k}Gw = \bar{k}\lambda^w w > 0$$

that is equivalent to

$$(\bar{k}G + I)w = (\bar{G} + I)w = (\bar{k}\lambda^w + 1)w$$

which means that w is also the eigenvector of $\bar{G} + I$ associated to the eigenvalue $\bar{k}\lambda^w + 1$. Moreover notice that, being $I + \bar{G}$ invertible, w can be written as

$$w = (\bar{k}\lambda^w + 1)(\bar{G} + I)^{-1}w$$

and b is strictly positive since $b_i = (1 - \bar{k}_i)w_i = (1 - \bar{k})w_i > 0$ for $\bar{k} < 1$. Then, the solution of the linear system (3.19) is:

$$q^* = (I + \bar{G})^{-1}b = (1 - \bar{k})(I + \bar{G})^{-1}w$$

and it can be written substituting the expression for w as

$$q^* = (1 - \bar{k})(\bar{k}\lambda^w + 1)((I + \bar{G})^{-1})^2w > 0$$

for the positivity of $((I + \bar{G})^{-1})^2$, q^* is positive and so it is an internal Nash equilibrium for the game. $\square$

Example 3.4.4. *Cobb-Douglas value functions (cont.)*

For the game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, with $\alpha_i = \alpha, \beta_i = \beta$, from Proposition 3.4.2, if $\frac{\alpha}{\alpha + \beta}G + I$ is invertible and the income vector $w > 0$ is a positive eigenvector of G , associated to the eigenvalue λ^w , then there exists the internal equilibrium $q^* \in \chi_w$ of the form

$$q^* = \frac{\beta}{\alpha + \beta}(\frac{\alpha}{\alpha + \beta}\lambda^w + 1)((\frac{\alpha}{\alpha + \beta}G + I)^{-1})^2w$$

Let us see some examples.

Example 3.4.5. Consider again the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, with $\alpha_i = \alpha, \beta_i = \beta$, defined on the unweighted undirected circle graph $\mathcal{G}$ of 4 nodes of 3.4.2. The dominant eigenvalue is $\lambda^G = 2$ and the corresponding eigenvector is $v^G = (1, 1, 1, 1)^T > 0$. Take again $\alpha = \frac{1}{3}, \beta = \frac{2}{3}$; Now take as income vector $w = v^G$. Then, there exists an internal equilibrium for the game, that has the form

$$q^* = \frac{\beta}{\alpha + \beta}(\frac{\alpha}{\alpha + \beta}\lambda^w + 1)((\frac{\alpha}{\alpha + \beta}G + I)^{-1})^2 = (\frac{2}{5}, \dots, \frac{2}{5})^T$$

Example 3.4.6. Consider again the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$ defined on the unweighted undirected line graph $\mathcal{G}$ of 3 nodes of example 3.4.3. The dominant eigenvalue is $\lambda^G = \sqrt{2}$ and the corresponding eigenvector is $v^G = (\frac{1}{2}, \frac{7}{10}, \frac{1}{2})^T > 0$. Take $\alpha = \frac{2}{3}, \beta = \frac{1}{3}$; Now take as income vector $w = v^G$. Then, there exists an internal equilibrium and it has the form:

$$q^* = \frac{\beta}{\alpha + \beta}(\frac{\alpha}{\alpha + \beta}\lambda^w + 1)((\frac{\alpha}{\alpha + \beta}G + I)^{-1})^2 = (0.09, 0.12, 0.09)^T$$

Example 3.4.7. Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, with $\alpha_i = \alpha, \beta_i = \beta$, defined on the unweighted undirected star graph $\mathcal{G}$ of 5 nodes, where node

1 is the center of the star. The adjacency matrix is:

$$G = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The dominant eigenvalue is $\lambda^G = 2$ and the corresponding eigenvector is $v^G = (0.71, 0.35, 0.35, 0.35, 0.35)^T > 0$. Take $\alpha = \frac{1}{4}, \beta = \frac{2}{3}$ and income vector $w = v^G$. Then, there exists an internal Nash equilibrium, having the form

$$q^* = \frac{\beta}{\alpha + \beta} \left(\frac{\alpha}{\alpha + \beta} \lambda^w + 1 \right) \left(\left(\frac{\alpha}{\alpha + \beta} G + I \right)^{-1} \right)^2 = (0.33, 0.17, 0.17, 0.17, 0.17)^T$$

Let us now see a condition on the vector b that guarantees for the existence of an internal equilibrium.

Proposition 3.4.3. *Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$ with value functions satisfying the strong linearity property $\mathbf{L}^+$ for $\bar{k} \in (0, 1)$. If $\bar{G} + I$ is invertible and the vector $b > 0$ satisfies*

$$b > \bar{G}b \tag{3.21}$$

then there exists the internal equilibrium $q^* \in \chi_w$ of the form

$$q^* = (I + \bar{G})^{-1}b = \sum_{h=0}^{\infty} (\bar{G})^{2h} (I - \bar{G})b$$

Proof. Start by observing that the hypothesis $b > \bar{G}b$ is equivalent to $(I - \bar{G})b > 0$ and since $\bar{k} < 1$, $b_i = (1 - \bar{k})w_i > 0$ for every i ; this implies that also $I - \bar{G} > 0$.

Since $\bar{G} + I$ is invertible, it is possible to write the inverse matrix as a power series:

$$(I + \bar{G})^{-1} = \sum_{h=0}^{\infty} (-\bar{G})^h$$

Notice that the series can also be written as:

$$\sum_{h=0}^{\infty} (-\bar{G})^h = (I + \bar{G}^2 + \bar{G}^4 + \dots)(I - \bar{G}) = \sum_{h=0}^{\infty} (\bar{G})^{2h} (I - \bar{G})$$

Being it the sum of positive terms, we have that $(I + \bar{G})^{-1} > 0$. Then also

$$(I + \bar{G})^{-1}b = \sum_{h=0}^{\infty} (\bar{G})^{2h} (I - \bar{G})b > 0$$

so that the solution of the linear system (3.19)

$$q = (I + \bar{G})^{-1}b = \sum_{h=0}^{\infty} (\bar{G})^{2h} (I - \bar{G})b$$

is strictly positive, so it is an internal Nash equilibrium. $\square$

Example 3.4.8. *Cobb-Douglas value functions (cont.)*

For the game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, with $\alpha_i = \alpha, \beta_i = \beta$, from Proposition 3.4.3, if $\frac{\alpha}{\alpha+\beta}G + I$ is invertible and the income vector $w > 0$ satisfies

$$w > \frac{\alpha}{\alpha + \beta} Gw$$

then there exists the internal equilibrium $q^* \in \chi_w$ of the form

$$q^* = \frac{\beta}{\alpha + \beta} \left(\frac{\alpha}{\alpha + \beta} G + I \right)^{-1} w = \frac{\beta}{\alpha + \beta} \sum_{h=0}^{\infty} \left(\frac{\alpha}{\alpha + \beta} G \right)^{2h} \left(I - \frac{\alpha}{\alpha + \beta} G \right) w$$

Let us see a few examples.

Example 3.4.9. Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$ defined on the unweighted undirected circle graph $\mathcal{G}$ of 5 nodes. The adjacency matrix is

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Take $\alpha_i = \alpha = \frac{1}{4}, \beta_i = \beta = \frac{1}{3}$ and income vector $w = (5, 4, 4, 4, 5)^T$. In this case the condition

$$w > \frac{\alpha}{\alpha + \beta} Gw$$

is satisfied: $\frac{\alpha}{\alpha+\beta}G = 0.43G$ and

$$w - \frac{\alpha}{\alpha + \beta} Gw = \begin{pmatrix} 5 \\ 4 \\ 4 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 3.86 \\ 3.86 \\ 3.43 \\ 3.86 \\ 3.86 \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

thus there exists the internal equilibrium

$$q^* = \frac{\beta}{\alpha + \beta} \left(\frac{\alpha}{\alpha + \beta} G + I \right)^{-1} w = (1.73, 0.89, 1.52, 0.89, 1.73)^T$$

Example 3.4.10. Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, with $\alpha_i = \alpha, \beta_i = \beta$, defined on the unweighted undirected line graph $\mathcal{G}$ of 4 nodes. The adjacency

matrix is

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Take $\alpha = \frac{1}{4}, \beta = \frac{1}{3}$ and income vector $w = (2, 3, 3, 2)^T$. In this case the condition

$$w > \frac{\alpha}{\alpha + \beta} Gw$$

is satisfied: $\frac{\alpha}{\alpha + \beta} G = 0.43G$ and

$$w - \frac{\alpha}{\alpha + \beta} Gw = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1.23 \\ 2.14 \\ 2.14 \\ 1.23 \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

thus there exists the internal equilibrium

$$q^* = \frac{\beta}{\alpha + \beta} \left(\frac{\alpha}{\alpha + \beta} G + I \right)^{-1} w = (0.72, 0.98, 0.98, 0.72)^T$$

Example 3.4.11. Consider again the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, defined on the unweighted undirected star graph $\mathcal{G}$ of 5 nodes, where node 1 is the center of the star, of example 3.4.7. Take again $\alpha = \frac{1}{4}, \beta = \frac{2}{3}$ and income vector $w = (4, 2, 2, 2, 2)^T$.

In this case the condition

$$w > \frac{\alpha}{\alpha + \beta} Gw$$

is satisfied: $\frac{\alpha}{\alpha + \beta} G = 0.27G$ and

$$w - \frac{\alpha}{\alpha + \beta} Gw = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 2.18 \\ 1.09 \\ 1.09 \\ 1.09 \\ 1.09 \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

thus there exists the internal equilibrium

$$q^* = \frac{\beta}{\alpha + \beta} \left(\frac{\alpha}{\alpha + \beta} G + I \right)^{-1} w = (1.88, 0.94, 0.94, 0.94, 0.94)^T$$

Remark 3.4.1. In general, for the game with Cobb-Douglas value functions and equal parameters $\alpha_i = \alpha, \beta_i = \beta$, $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$ with players having equal income $w_i = w$, for $\mathcal{G}$ out-regular and equally weighted (of weight g), the following result on the internal Nash equilibrium holds:

There exists an internal equilibrium $q^* \in \chi_w$ where all nodes contribute to the public

good with the same positive amount

$$q_i^* = \frac{\beta w}{\alpha(1 + dg) + \beta}$$

Indeed, if there exists an internal equilibrium with equally contributing agents, then the contribution of every agent must solve

$$q_i = \left[\frac{\beta w}{\alpha + \beta} - \frac{\alpha}{\alpha + \beta} \sum_{j=1}^n G_{ij} q_j \right]_+ = \frac{\beta w}{\alpha + \beta} - \frac{\alpha}{\alpha + \beta} g d q_i > 0$$

since all nodes have d out-neighbors and same weight g . Then

$$q_i = \frac{\beta w}{\alpha(1 + dg) + \beta}$$

To see that it is an equilibrium, let us verify that it satisfies (3.13) with $C = \mathcal{N}$. The left hand side of the equation is

$$\begin{aligned} & \left(I + \frac{\alpha}{\alpha + \beta} G \right) \frac{\beta w}{\alpha(1 + dg) + \beta} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \\ &= \frac{\beta w}{\alpha(1 + dg) + \beta} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \frac{\alpha}{\alpha + \beta} \frac{\beta w}{\alpha(1 + dg) + \beta} G \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \\ &= \frac{\beta w}{\alpha(1 + dg) + \beta} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \frac{\alpha}{\alpha + \beta} \frac{\beta w d g}{\alpha(1 + dg) + \beta} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \end{aligned}$$

where the last equality is justified by the fact that the sum on the rows of G is $\sum_{i=1}^n G_{ij} = dg$ for every i . Then (3.13) simplifies to

$$\frac{\alpha + \beta + \alpha dg}{\alpha(1 + dg) + \beta} = 1$$

Then, the condition is satisfied and $q^* = (q_i)_{i=1}^n$ is a Nash equilibrium.

Remark 3.4.2. For the game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$ with $\alpha_i = \alpha, \beta_i = \beta$, and players having equal income $w_i = w$, for $\mathcal{G}$ unweighted directed line graph, the following result on the internal Nash equilibrium holds:

There exists the internal equilibrium $q^* \in \chi_w$ with the form

$$q_{n-i}^* = \frac{\beta w}{\alpha + \beta} \sum_{j=0}^i (-1)^j \left(\frac{\alpha}{\alpha + \beta} \right)^j$$

for $i = 0, \dots, n - 1$.

To derive the form of the internal equilibrium profile, let us consider nodes to be labeled in increasing order from 1 (left-most node) to n (right-most node). Then: Node n has no out-neighbors: $G_{nj} = 0$ for every j , which implies that its best response is always

$$q_n = \left[-\frac{\beta w}{\alpha + \beta} - \frac{\alpha}{\alpha + \beta} \sum_{j=1}^n G_{nj} q_j \right]_+ = \frac{\beta w}{\alpha + \beta}$$

Every other node i has as unique out-neighbor node $i + 1$: $G_{ij} = 0$ if $j \neq i + 1$. Then, it is possible to recursively compute the public good provision of every node starting from the last one:

$$q_{n-1} = \frac{\beta w}{\alpha + \beta} - \frac{\alpha}{\alpha + \beta} q_n = \frac{\beta w}{\alpha + \beta} \left(1 - \frac{\alpha}{\alpha + \beta} \right)$$

$$q_{n-2} = \frac{\beta w}{\alpha + \beta} - \frac{\alpha}{\alpha + \beta} q_{n-1} = \frac{\beta w}{\alpha + \beta} \left(1 - \frac{\alpha}{\alpha + \beta} + \frac{\alpha^2}{(\alpha + \beta)^2} \right)$$

In general for every $i = 0, \dots, n - 1$:

$$q_{n-i} = \frac{\beta w}{\alpha + \beta} \sum_{j=0}^i (-1)^j \left(\frac{\alpha}{\alpha + \beta} \right)^j$$

3.5 Specialized equilibria

Specialized equilibria involve a subset of contributing agents, which provide their maximal amount of public good, while the remaining nodes free-ride on the contribution of others. For the public good game with value functions satisfying the linearity property, the notion of specialized equilibria is linked to the concept of maximal independent sets of nodes: we prove that in this setting, under a hypothesis on the strength of the connection between agents, an action profile is a specialized equilibrium if and only if the set of contributors is a maximal independent set of nodes.

Throughout this section, consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, with value functions satisfying the linearity property **L** for some $\bar{k} = (\bar{k}_i)_{i=1}^n \in (0, 1)^n$ and recall that a vector of public good contributions $q \in \chi_w$ is said to be a specialized profile if for every i , $q_i > 0$ if and only if

$$q_i = \max_{q_i \in \chi_i} \mathcal{B}_i(q_{-i}) = \max_{q_i \in \chi_i} [b_i - (\bar{G}q)_i]_+ = b_i$$

otherwise $q_i = 0$. Thus, in a specialized profile agents can be partitioned into non contributors $-C$ and specialized (contributing) agents C , whose action is the maximum of their best response.

Proposition 3.5.1. *Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, with value functions satisfying the linearity property **L** for some $\bar{k} = (\bar{k}_i)_{i=1}^n \in (0, 1)^n$ and a specialized profile $q^* \in \chi_w$.*

- If q^* is a Nash equilibrium, then the set of specialists C is a maximal independent set of nodes of the graph $\mathcal{G}$.
- If the set of specialists C is a maximal independent set of nodes of the graph $\mathcal{G}$ and for every non contributor $j \in -C$ it holds

$$\sum_{h \in C} G_{jh} b_h \geq \frac{b_j}{\bar{k}_j}$$

then q^* is a Nash equilibrium.

Proof. • $(\Rightarrow)$:

If q^* is a specialized equilibrium, for every contributing node $i \in C$ $q_i^* = b_i > 0$. Since they play their best response, it must hold

$$(\bar{G}q^*)_i = \sum_{j=1}^n \bar{k}_i G_{ij} q_j^* = 0$$

that is true if every specialist's neighbor is playing zero, i.e., for every $j \in \mathcal{N}_i^o$ $q_j^* = 0$. So, every specialist has only non-contributing neighbors. Since no two specialists can be neighbors at equilibrium, we have proved that C is an independent set.

For every non-contributor $j \in -C$ $q_j^* = 0$. Since they play their best response at equilibrium, it must hold

$$b_j - \sum_{h=1}^n \bar{k}_j G_{jh} q_h^* \leq 0$$

which can happen only if

$$\sum_{h=1}^n \bar{k}_j G_{jh} q_h^* \geq b_j \geq 0$$

Thus

$$\sum_{h=1}^n \bar{k}_j G_{jh} q_h^* \geq 0$$

This means that every non-contributor must have at least one specialist neighbor. Thus, the set of contributors must be maximal. If it was not maximal, some non-contributor would have no specialist neighbors, which is not possible at equilibrium.

- $(\Leftarrow)$:

If C is a maximal independent set of specialists, the action of every specialized agent $i \in C$ must be

$$q_i^* = b_i > 0$$

and it is the best response since each $i \in C$ has no contributing neighbors for definition of independent set.

Let us see that every non-specialist $j \in -C$ is playing her best response by playing zero; since C is a maximal independent set, every non specialist $j \in -C$ must have at least one specialist neighbor. Then, her best response is

$$\mathcal{B}_j(q_{-j}) = [b_j - \sum_{h=1}^n \bar{k}_j G_{jh} q_h]_+ = [b_j - \sum_{h \in C} \bar{k}_j G_{jh} b_h]_+$$

For hypothesis the following condition holds

$$\sum_{h \in C} G_{jh} b_h \geq \frac{b_j}{\bar{k}_j}$$

and it is equivalent to

$$b_j - \sum_{h \in C} \bar{k}_j G_{jh} b_h \leq 0$$

thus the best response is playing zero:

$$\mathcal{B}_j(q_{-j}) = [b_j - \sum_{h \in C} \bar{k}_j G_{jh} b_h]_+ = 0$$

In conclusion, we have proved that both specialists and non contributors are playing their best response, which is equivalent to saying that q^* is a Nash equilibrium. □

The characterization of specialized equilibria is at the center of the 2007 paper by Bramoullé and Kranton [13]. Proposition 3.5.1 extends their main contribution that relates specialized equilibria and maximal independent sets of nodes to a more general game setting: nonlinear utilities, agents differing in $\bar{k}_i$ and w_i and $\mathcal{G}$ weighted and directed.

Corollary 3.5.2. *For the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, with value functions satisfying the linearity property $\mathbf{L}$ for some $\bar{k} = (\bar{k}_i)_{i=1}^n \in (0, 1)^n$, consider a specialized profile $q^* \in \chi_w$. If for every $j \in -C, h \in C$ it holds*

$$G_{jh} b_h \geq \frac{b_j}{\bar{k}_j}$$

then q^ is a Nash equilibrium if and only if C is a maximal independent set.*

Proof. If for every $j \in -C, h \in C$ it holds

$$G_{jh} b_h \geq \frac{b_j}{\bar{k}_j}$$

then

$$\sum_{h \in C} G_{jh} b_h \geq \frac{b_j}{\bar{k}_j}$$

and Proposition 3.5.1 holds. □

Notice that Corollary 3.5.2 hints that, if the strength of the connection between specialists and non-contributing agents is high enough, then it is sufficient to have one specialist neighbor to play zero as best response and maximal independent sets of specialists of order 1 are equilibrium profiles.

Example 3.5.1. *Cobb-Douglas value functions (cont.)*

For the game Cobb-Douglas public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, a specialized profile $q \in \chi_w$ is such that $q_i > 0$ if and only if

$$q_i = \max_{q_i \in [0, w_i]} \left[\frac{\beta w_i}{\alpha + \beta} - \frac{\alpha}{\alpha + \beta} (Gq)_i \right]_+ = \frac{\beta_i w_i}{\alpha_i + \beta_i}$$

For Proposition 3.5.1, given a specialized profile $q^* \in \chi_w$.

- If q^* is a Nash equilibrium, then the set of specialists C is a maximal independent set of nodes.
- If the set of specialists C is a maximal independent set and for every $j \in -C$ it holds $\sum_{h \in C} G_{jh} \frac{\beta_h w_h}{\alpha_h + \beta_h} \geq \frac{\beta_j w_j}{\alpha_j}$, then q^* is a Nash equilibrium.

Let us see some examples.

Example 3.5.2. Consider the Cobb-Douglas public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, defined on the weighted directed line of 4 nodes, with adjacency matrix

$$G = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

Take $\alpha(\frac{2}{5}, \frac{3}{4}, \frac{1}{6}, \frac{1}{2})^T = \beta(\frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3})^T$ and income vector $w = (3, 2, 2, 3)^T$.

Consider the specialized action profile $q = (1.67, 0, 1.5, 0)$. The set of specialist nodes is $C = \{1, 3\}$ and the non contributors are $-C = \{2, 4\}$. Thus, C constitutes a maximal independent set of nodes. To verify if q is an equilibrium, let us see that the condition is satisfied for every non contributor:

- for non contributor $j = 2 \in -C$ it holds:

$$\sum_{h \in C} G_{2,h} \frac{\beta_h w_h}{\alpha_h + \beta_h} - \frac{\beta_2 w_2}{\alpha_2} = G_{21} \frac{\beta_1 w_1}{\alpha_1 + \beta_1} + G_{23} \frac{\beta_3 w_3}{\alpha_3 + \beta_3} - \frac{\beta_2 w_2}{\alpha_2} = 2.28 \geq 0$$

- for non contributor $j = 4 \in -C$ it holds:

$$\sum_{h \in C} G_{4,h} \frac{\beta_h w_h}{\alpha_h + \beta_h} - \frac{\beta_4 w_4}{\alpha_4} = G_{41} \frac{\beta_1 w_1}{\alpha_1 + \beta_1} + G_{43} \frac{\beta_3 w_3}{\alpha_3 + \beta_3} - \frac{\beta_4 w_4}{\alpha_4} = 1 \geq 0$$

Thus,

$$q^* = (1.67, 0, 1.5, 0)$$

is a specialized equilibrium for the game.

Consider now the other specialized action profile $q = (0, 0.62, 0, 1.2)$. The set of specialist nodes is $C = \{2, 4\}$ and the non contributors are $-C = \{1, 3\}$. Thus, C constitutes a maximal independent set of nodes. To verify if q is an equilibrium, let us see that the condition is satisfied for every non contributor:

- for non contributor $j = 1 \in -C$ it holds:

$$\sum_{h \in C} G_{1,h} \frac{\beta_h w_h}{\alpha_h + \beta_h} - \frac{\beta_1 w_1}{\alpha_1} = G_{12} \frac{\beta_2 w_2}{\alpha_2 + \beta_2} + G_{14} \frac{\beta_4 w_4}{\alpha_4 + \beta_4} - \frac{\beta_1 w_1}{\alpha_1} = -2.52 < 0$$

- for non contributor $j = 3 \in -C$ it holds:

$$\sum_{h \in C} G_{3,h} \frac{\beta_h w_h}{\alpha_h + \beta_h} - \frac{\beta_3 w_3}{\alpha_3} = G_{32} \frac{\beta_2 w_2}{\alpha_2 + \beta_2} + G_{34} \frac{\beta_4 w_4}{\alpha_4 + \beta_4} - \frac{\beta_3 w_3}{\alpha_3} = -2.95 < 0$$

Thus the specialized profile $q = (0, 0.62, 0, 1.2)$ is not a Nash equilibrium for the game.

Example 3.5.3. Consider the game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, defined on the weighted directed star graph $\mathcal{G}$ of 5 nodes, where node 1 is the center of the star, with adjacency matrix

$$G = \begin{pmatrix} 0 & 2 & 2 & 2 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Take $\alpha = (\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$, $\beta = (\frac{2}{3}, \frac{5}{8}, \frac{3}{8}, \frac{5}{8}, \frac{3}{8})^T$ and income vector $w = (5, 3, 2, 3, 2)^T$. Consider the specialized action profile $q = (0, 2.14, 1.2, 2.14, 1.2)$. The set of specialist nodes is $C = \{2, 3, 4, 5\}$ and the non contributors are $-C = \{1\}$. Thus, C constitutes a maximal independent set of nodes. To verify if q is an equilibrium, let us see that the condition is satisfied for the non contributor $1 \in -C$:

$$\sum_{h \in C} G_{1,h} \frac{\beta_h w_h}{\alpha_h + \beta_h} - \frac{\beta_1 w_1}{\alpha_1} = 8.93 \geq 0$$

Thus,

$$q^* = (0, 2.14, 1.2, 2.14, 1.2)$$

is a specialized equilibrium for the game.

Consider now the other specialized action profile $q = (2.35, 0, 0, 0, 0)^T$. The set of specialist nodes is $C = \{1\}$ and the non contributors are $-C = \{2, 3, 4, 5\}$. Thus, C constitutes a maximal independent set of nodes. To verify if q is an equilibrium, let us see that the condition is satisfied for every non contributor: for $j = 2 \in -C$

$$\sum_{h \in C} G_{2,h} \frac{\beta_h w_h}{\alpha_h + \beta_h} - \frac{\beta_2 w_2}{\alpha_2} = -5.15 < 0$$

Thus the specialized profile $q = (2.35, 0, 0, 0, 0)^T$ is not a Nash equilibrium for the game.

Remark 3.5.1. Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$ with value functions satisfying the strong linearity property $\mathbf{L}^+$ for $\bar{k} \in (0, 1)$. For a specialized profile $q^* \in \chi_w$, if the set of specialists C is a maximal independent set, the condition for q^* to be a Nash equilibrium is: for every $j \in -C$ it must hold

$$\sum_{h \in C} G_{jh} b_h = \sum_{h \in C} G_{jh} (1 - \bar{k}) w_h \geq \frac{b_j}{\bar{k}} = \frac{(1 - \bar{k}) w_j}{\bar{k}}$$

which reduces to

$$\sum_{h \in C} G_{jh} w_h \geq \frac{w_j}{\bar{k}}$$

Example 3.5.4. Cobb-Douglas value functions (cont.)

Thus, for the game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, with $\alpha_i = \alpha, \beta_i = \beta$, a specialized profile $q^* \in \chi_w$ is a Nash equilibrium if and only if the set of specialists C is a maximal independent set and for every $j \in -C$ it holds $\sum_{h \in C} G_{jh} w_h \geq \frac{\alpha + \beta}{\alpha} w_j$.

Example 3.5.5. Consider again the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, defined on the unweighted undirected circle graph $\mathcal{G}$ of 4 nodes of example 3.4.2. Take $\alpha = \frac{1}{3}, \beta = \frac{2}{3}$ and income vector $w = (1, 2, 1, 3)^T$. Consider the specialized profile with contributors $C = \{2, 4\}$. C is a maximal independent set; let us verify that for every $j \in -C$ it holds

$$\sum_{h \in C} G_{jh} w_h \geq 3w_j$$

For $j = 1$ and $j = 3$ the condition is verified: $w_2 + w_4 = 5 \geq 3w_1 = 3w_3 = 3$. Thus the specialized profile

$$q^* = (0, \beta w_2, 0, \beta w_4)^T = (0, \frac{4}{3}, 0, 2)$$

is a Nash equilibrium for the game. On the contrary, for the specialized profile with contributors the maximal independent set $C = \{1, 3\}$, the condition is not verified $w_1 + w_3 = 2 \leq 3w_2 = 4$ and $w_1 + w_3 = 3 \leq 3w_4 = 9$, thus $q = (\beta w_1, 0, \beta w_3, 0)^T$ is not a Nash equilibrium for the game.

Example 3.5.6. Consider again the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, defined on the unweighted undirected line graph $\mathcal{G}$ of 3 nodes of example 3.4.3. Take $\alpha = \frac{2}{3}, \beta = \frac{1}{3}$ and income vector $w = (2, 1, 2)^T$. Consider the specialized profile with contributors $C = \{1, 3\}$. For $j = 2 \in -C$ it holds $w_1 + w_3 = 4 \geq \frac{w_2}{\bar{k}} = \frac{2}{3}$; thus

$$q^* = (\beta w_1, 0, \beta w_3)^T = (\frac{2}{3}, 0, \frac{2}{3})$$

is a specialized Nash equilibrium. On the other hand, taking $C = \{2\}$, the condition is not verified since $w_2 = 1 \leq \frac{w_1}{\bar{k}} = \frac{w_3}{\bar{k}} = 3$ and so $q = (0, \beta w_2, 0)^T$ is not a Nash equilibrium for the game.

Remark 3.5.2. Let us consider the game $\mathcal{P} = (\mathcal{G}, w, \eta)$, with $\alpha_i = \alpha, \beta_i = \beta$, defined on the equally weighted (of weight g) ring graph of n nodes, with n even. Assume

that players have equal income $w_i = w > 0$. It is interesting to study if there are specialized equilibria. For the topology of the graph, since the set of contributors C must be a maximal independent set, the only possible specialized equilibria are those where contributors and non-contributors alternate.

Let us consider, for example, $C = \{\text{even nodes}\}$ and $-C = \{\text{odd nodes}\}$. Then, contributors must behave as isolated nodes: $(q_C)_i = \frac{\beta w}{\alpha + \beta}$. Then for Proposition 3.5.1, the action profile $q \in \chi_w$ such that $q_i = \frac{\beta w}{\alpha + \beta}$ for every $i \in C$ is a Nash equilibrium if for every $j \in -C$ it holds

$$\sum_{h \in C} G_{jh} \frac{\beta_h w_h}{\alpha_h + \beta_h} \geq \frac{\beta_j w_j}{\alpha_j}$$

For the game $\mathcal{P}$, it is equivalent to

$$\sum_{h \in C} G_{jh} \frac{\beta w}{\alpha + \beta} \geq \frac{\beta w}{\alpha}$$

Then,

1. for the equally weighted directed ring graph, every contributor has exactly one specialist neighbor. Thus, that the condition becomes

$$g \frac{\beta w}{\alpha + \beta} \geq \frac{\beta w}{\alpha}$$

which is equivalent to

$$g \geq \frac{\alpha + \beta}{\alpha}$$

In conclusion, if $g \geq \frac{\alpha + \beta}{\alpha}$, then the specialized profile

$$q^* = (0, \frac{\beta w}{\alpha + \beta}, 0, \dots, \frac{\beta w}{\alpha + \beta})$$

is a Nash equilibrium for the game.

2. for the equally weighted undirected ring graph, every contributor has exactly two specialist neighbors. Thus, that the condition becomes

$$2g \frac{\beta w}{\alpha + \beta} \geq \frac{\beta w}{\alpha}$$

which is equivalent to

$$2g \geq \frac{\alpha + \beta}{\alpha}$$

In conclusion, if $2g \geq \frac{\alpha + \beta}{\alpha}$, then the specialized profile

$$q^* = (0, \frac{\beta w}{\alpha + \beta}, 0, \dots, \frac{\beta w}{\alpha + \beta})$$

is a Nash equilibrium for the game.

Since the result does not depend on what nodes contribute, it is possible to partition

the nodes as $C = \{\text{odd nodes}\}$ and $-C = \{\text{even nodes}\}$ and obtain, under the same condition, another specialized equilibrium:

$$q^* = (\frac{\beta w}{\alpha + \beta}, 0, \frac{\beta w}{\alpha + \beta}, \dots, 0)$$

Remark 3.5.3. Let us consider the game $\mathcal{P} = (\mathcal{G}, w, \eta)$, with $\alpha_i = \alpha, \beta_i = \beta$, defined on the equally weighted (of weight g) star graph of n nodes. For the topology of the graph, since the set of contributors C must be a maximal independent set, there are only two possible specialized equilibria: the one where the central node (node 1) contributes and the other free-ride and the one where the central node does not contribute and the others are specialists.

1. Consider the specialized profile $q \in \chi_W$ where $C = \{1\}$ and $-C = \{2, \dots, n\}$, so that $q_1 = \frac{\beta w}{\alpha + \beta}$. Then for Proposition 3.5.1, the action profile q is a Nash equilibrium if for every $j \in -C$ it holds

$$\sum_{h \in C} G_{jh} \frac{\beta_h w_h}{\alpha_h + \beta_h} \geq \frac{\beta_j w_j}{\alpha_j}$$

in the case where $C = \{1\}$, it is equivalent to

$$g \frac{\beta w}{\alpha + \beta} \geq \frac{\beta w}{\alpha}$$

which leads to the condition

$$g \frac{\alpha}{\alpha + \beta} \geq 1$$

If it is satisfied, then

$$q^* = (\frac{\beta w}{\alpha + \beta}, 0, \dots, 0)^T$$

is a specialized Nash equilibrium for the game.

2. Consider the specialized profile $q \in \chi_W$ where $C = \{2, \dots, n\}$ and $-C = \{1\}$, so that $q_i = \frac{\beta w}{\alpha + \beta}$ for every $i \in -C$. Then for Proposition 3.5.1, the action profile q is a Nash equilibrium if for $1 \in -C$ it holds

$$\sum_{h \in C} G_{1h} \frac{\beta_h w_h}{\alpha_h + \beta_h} \geq \frac{\beta_1 w_1}{\alpha_1}$$

in the case where $C = \{1\}$, it is equivalent to

$$(n-1)g \frac{\beta w}{\alpha + \beta} \geq \frac{\beta w}{\alpha}$$

which leads to the condition

$$(n-1)g \geq \frac{\alpha + \beta}{\alpha}$$

If it is satisfied, then

$$q^* = (0, \frac{\beta w}{\alpha + \beta}, \dots, \frac{\beta w}{\alpha + \beta})^T$$

is a specialized Nash equilibrium for the game.

Chapter 4

Uniqueness, contractivity and stability results

In this chapter, the main results of this work, concerning the uniqueness and asymptotic behavior of equilibrium profiles, are presented.

First, for the general version of the public good game, we prove a sufficient condition for the uniqueness of the Nash equilibrium. In the game with value functions satisfying $\mathbf{L}$, this condition relates uniqueness and the lowest eigenvalue of the symmetrization of the per-row rescaled adjacency matrix $\bar{G}$.

Afterwards, we study the limit behavior of the discrete and continuous-time best response dynamics of the game. In the general setting, we provide a sufficient condition that guarantees the contractivity of the discrete-time dynamics. This result ensures the uniqueness of the Nash equilibrium and the global asymptotical stability of the best response dynamics to it, both in discrete and continuous time.

Moreover, concerning the stability of the internal equilibrium for the game with value functions under $\mathbf{L}^+$ on undirected graphs, we establish that, if it exists, it is stable for the discrete-time dynamics if and only if contractivity holds, while for the continuous-time dynamics it is stable if and only if the uniqueness condition is true.

Finally, for the continuous-time dynamics, we study the stability of Nash equilibrium profiles when contractivity is not guaranteed. The results concern the stability of equilibria for the continuous-time dynamics under two main hypothesis: the value functions satisfy the linearity property $\mathbf{L}$ and the per-row rescaled adjacency matrix $\bar{G}$ is symmetric. We first prove that the system converges globally to the unique Nash equilibrium also when the weaker uniqueness condition is satisfied. In addition, when the equilibrium is not necessarily unique, we establish that the continuous-time best response dynamics globally converges to the set of Nash equilibria and action profiles that are local maximum points of the associated functional P are locally asymptotically stable equilibria.

Let us start with a preliminary example that serves as a motivation to the main results of the following sections. Some of the findings concerning the uniqueness and stability of equilibria that are empirically obtained, in fact, will be generalized later.

Example 4.0.1. *Complete graph of 3 nodes*

Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, where $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$ is the equally weighted complete graph of 3 nodes, $w \in \mathbb{R}_{>0}^n$ the income vector with players sharing equal income, i.e., $w_i = w > 0$ for every i , and η the Cobb-Douglas value functions with equal parameters $\alpha_i = \alpha, \beta_i = \beta \in (0, 1)$, $\eta_i(w - q_i, q_i + (Gq)_i) = (w - q_i)^\alpha (q_i + (Gq)_i)^\beta$. To simplify the calculations, let us assume that $\alpha + \beta = 1$ and $w_i = 1$ for every i .

We refer to the adjacency matrix G having with equal weights $G_{ij} = \frac{\gamma}{2\alpha}$:

$$G = \begin{pmatrix} 0 & \frac{\gamma}{2\alpha} & \frac{\gamma}{2\alpha} \\ \frac{\gamma}{2\alpha} & 0 & \frac{\gamma}{2\alpha} \\ \frac{\gamma}{2\alpha} & \frac{\gamma}{2\alpha} & 0 \end{pmatrix}$$

The synchronous best response function is

$$\mathcal{F}(q) = \begin{bmatrix} \beta \\ \beta \\ \beta \end{bmatrix} - \frac{\gamma}{2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} q \Big|_+$$

Analytically, we derive the following results on the equilibrium profiles:

- If $\gamma < 2$

There exists a unique fixed point of $\mathcal{F}$ and so a unique equilibrium, which is the internal one of equally contributing agents

$$q^* = \left(\frac{\beta}{1+\gamma}, \frac{\beta}{1+\gamma}, \frac{\beta}{1+\gamma} \right)$$

- If $\gamma > 2$

The internal equilibrium $q^* = (\frac{\beta}{1+\gamma}, \frac{\beta}{1+\gamma}, \frac{\beta}{1+\gamma})$ is still present, but others arise:
- specialized equilibria:

$$q_{s1}^* = (\beta, 0, 0), \quad q_{s2}^* = (0, \beta, 0), \quad q_{s3}^* = (0, 0, \beta)$$

- equilibria with 2 contributing nodes:

$$q_{12}^* = \left(\frac{2\beta}{2+\gamma}, \frac{2\beta}{2+\gamma}, 0 \right), \quad q_{23}^* = \left(0, \frac{2\beta}{2+\gamma}, \frac{2\beta}{2+\gamma} \right), \quad q_{13}^* = \left(\frac{2\beta}{2+\gamma}, 0, \frac{2\beta}{2+\gamma} \right)$$

Let us study the behavior of the best response dynamics to understand the nature of the Nash equilibria as γ varies. We rely on some simulations of the continuous and discrete-time dynamical systems. Recall their form:

$$q(t+1) = \mathcal{F}(q(t)) = \begin{bmatrix} \beta \\ \beta \\ \beta \end{bmatrix} - \frac{\gamma}{2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} q(t) \Big|_+$$

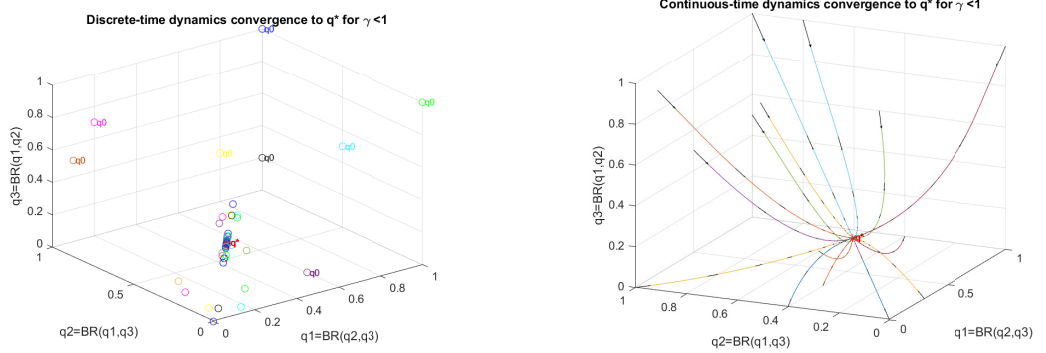
is the discrete-time best response dynamics, while

$$\dot{q}(t) = \mathcal{F}(q(t)) - q(t) = \left[\begin{pmatrix} \beta \\ \beta \\ \beta \end{pmatrix} - \frac{\gamma}{2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} q(t) \right]_+ - q(t)$$

is the one in continuous time. Empirically, we verify that the asymptotic behavior of the dynamical systems depends on the value of γ ; in particular, three phases are observed, corresponding to the intervals $\gamma < 1$, $1 < \gamma < 2$ and $\gamma > 2$.

- If $\gamma < 1$

The simulations, reported in Figure 4.1, show that the two dynamical systems converge to the unique Nash equilibrium $q^* = (\frac{\beta}{1+\gamma}, \frac{\beta}{1+\gamma}, \frac{\beta}{1+\gamma})$ from any starting point $q(0) \in \chi_w$. This indicates that q^* is globally asymptotically stable both for the discrete and continuous-time dynamics.



(a) discrete-time dynamics simulation for various random starting points q_0

(b) continuous-time dynamics simulation for various starting points q_0

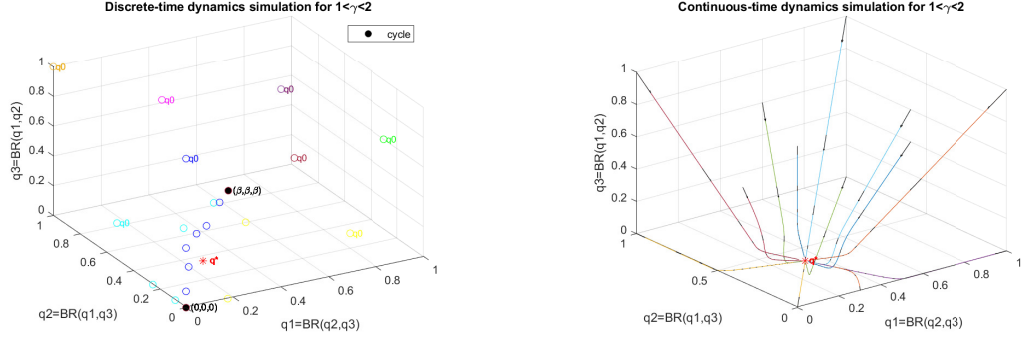
Figure 4.1: Best response dynamics simulations in discrete and continuous time for $\gamma < 1$: the system always converges to the unique equilibrium q^*

- If $1 < \gamma < 2$

The unique equilibrium is still $q^* = (\frac{\beta}{1+\gamma}, \frac{\beta}{1+\gamma}, \frac{\beta}{1+\gamma})$, but its asymptotic behavior is different in the two cases. Looking for the fixed points of $\mathcal{F}^2$, we find that a limit cycle of the form

$$C = (0, 0, 0) \leftrightarrow (\beta, \beta, \beta)$$

emerges for the discrete-time dynamics. In Figure 4.2, it is possible to see that, while q^* is still stable for the continuous-time dynamics, the discrete-time dynamical system does not converge to the Nash equilibrium, but rather to the limit cycle.



(a) discrete-time dynamics simulation for various random starting points q_0 : convergence to the limit cycle

(b) continuous-time dynamics simulation for various random starting points q_0 : convergence to the equilibrium

Figure 4.2: Best response dynamics simulations in discrete and continuous time for $1 < \gamma < 2$: a different behavior is observed in the discrete and continuous time cases

- If $\gamma > 2$

In this case, in addition to the internal $q^* = (\frac{\beta}{1+\gamma}, \frac{\beta}{1+\gamma}, \frac{\beta}{1+\gamma})$, other equilibrium profiles arise:

- specialized equilibria:

$$q_{s1}^* = (\beta, 0, 0), q_{s2}^* = (0, \beta, 0), q_{s3}^* = (0, 0, \beta)$$

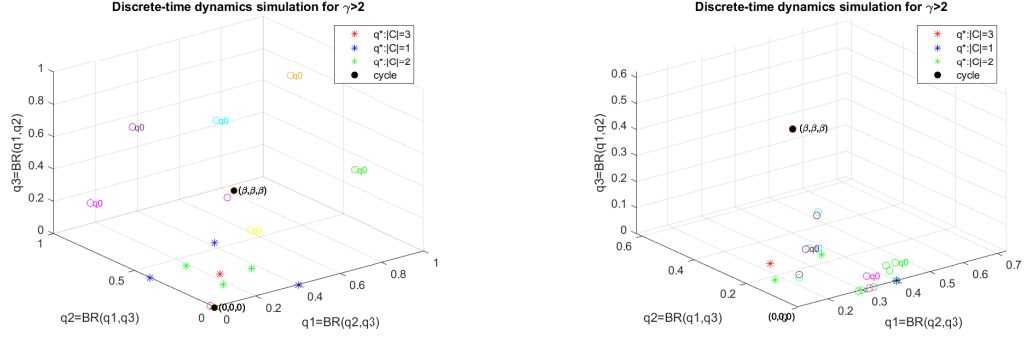
- equilibria with 2 contributing nodes:

$$q_{12}^* = (\frac{2\beta}{2+\gamma}, \frac{2\beta}{2+\gamma}, 0), q_{23}^* = (0, \frac{2\beta}{2+\gamma}, \frac{2\beta}{2+\gamma}), q_{13}^* = (\frac{2\beta}{2+\gamma}, 0, \frac{2\beta}{2+\gamma})$$

As before, we analytically derive the existence of the limit cycle in the discrete-time dynamics:

$$C : (0, 0, 0) \leftrightarrow (\beta, \beta, \beta)$$

The simulations help in understanding the nature of the equilibria and of the limit cycle. First, it is possible to observe in Figures 4.3, 4.4 that the internal equilibrium is unstable both in the discrete and continuous-time dynamical systems.

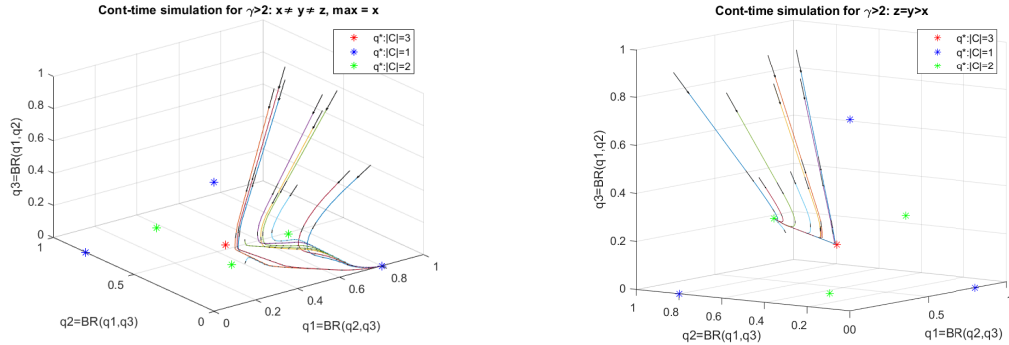


(a) discrete-time dynamics simulation for a random q_0 not close to the equilibria: convergence to the limit cycle

(b) discrete-time dynamics simulation for q_0 close to the equilibrium points: convergence to the specialized equilibria

Figure 4.3: Best response dynamics simulation in discrete time for $\gamma > 2$: convergence to the limit cycle or to a specialized equilibrium

Second, from the simulation of the discrete-time best response dynamics (in Figure 4.3), we observe that when the system starts from a random configuration $q(0)$ that is not close to one of the specialized equilibria, it will converge to the limit cycle, that attracts most of the domain (often in one step). If the dynamics begins close enough to a specialized equilibrium, it converges to it.



(a) continuous-time dynamics simulation for $q_0 = (q_1, q_2, q_3)$ and $q_1 > q_2, q_3$: convergence to the specialized equilibrium such that $q_1 = \beta$

(b) continuous-time dynamics simulation for $q_0 = (q_1, q_2, q_3)$ and $q_2 = q_3, q_1 < q_2, q_3$: convergence to the equilibrium such that $q_2 = q_3 = \frac{2\beta}{2+\gamma}$

Figure 4.4: Best response dynamics simulation in continuous time for $\gamma > 2$: convergence to a non internal equilibrium

Finally, the continuous-time dynamics simulation is visible in Figure 4.4: it shows that the internal equilibrium only attracts the bisector, that is an invariant set of null measure:

$$q^* \text{ attracts } \{q_1 = q_2 = q_3\}$$

which proves its instability. The attraction basins of the specialized equilibria

are hinted by the simulations:

$$q_{s1}^* = (\beta, 0, 0) \text{ attracts } \{(q_1, q_2, q_3) : q_1 > q_2, q_3\}$$

$$q_{s2}^* = (0, \beta, 0) \text{ attracts } \{(q_1, q_2, q_3) : q_2 > q_1, q_3\}$$

$$q_{s3}^* = (0, 0, \beta) \text{ attracts } \{(q_1, q_2, q_3) : q_3 > q_2, q_1\}$$

$$q_{23}^* = (0, \frac{2\beta}{2+\gamma}, \frac{2\beta}{2+\gamma}) \text{ attracts } \{(q_1, q_2, q_3) : q_1 < q_2 = q_3\}$$

$$q_{13}^* = (\frac{2\beta}{2+\gamma}, 0, \frac{2\beta}{2+\gamma}) \text{ attracts } \{(q_1, q_2, q_3) : q_2 < q_1 = q_3\}$$

$$q_{12}^* = (\frac{2\beta}{2+\gamma}, \frac{2\beta}{2+\gamma}, 0) \text{ attracts } \{(q_1, q_2, q_3) : q_3 < q_1 = q_2\}$$

The observed behavior suggests that the specialized equilibria $q_{s1}^*, q_{s2}^*, q_{s3}^*$ are locally asymptotically stable, while equilibria with 2 contributors $q_{12}^*, q_{13}^*, q_{23}^*$ attract measure zero sets, so are unstable.

We now prove that the uniqueness and asymptotic behavior of equilibrium profiles emerging from the example are also supported by a more general theory.

4.1 Uniqueness of the Nash equilibrium

Let us first provide a technical result that guarantees the uniqueness of the Nash equilibrium in the general game setting. The proof relies on the theory on uniqueness of equilibrium profiles for concave games by Rosen [20].

Lemma 4.1.1. *Consider a public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$. If for every $q \in \chi_w$ and every vector $v \in \mathbb{R}^n$, there exists a vector $r \in \mathbb{R}_{>0}^n$ such that it holds*

$$-\sum_{i=1}^n \sum_{j \neq i} v_i (r_i k'_i(w_i + (Gq)_i) G_{ij}) v_j - \sum_{i=1}^n \sum_{j \neq i} v_i (r_j k'_j(w_j + (Gq)_j) G_{ji}) v_j < \sum_{i=1}^n 2v_i^2 r_i \quad (4.1)$$

then the Nash equilibrium is unique.

Proof. Consider a game with generalized quadratic utilities of Proposition (3.2.5)

$$\bar{u}_i(q) = -\frac{q_i^2}{2} + w_i q_i - k_i(w_i + (Gq)_i) q_i$$

Given $q \in \chi_w$ and the vector $r \in \mathbb{R}_{>0}^n$, construct the matrix $H(q, r) \in \mathbb{R}_{\geq 0}^n$ defined as

$$H(q, r)_{i,i} = r_i \frac{\partial^2 \bar{u}_i}{\partial q_i^2}(q) = -r_i$$

and

$$H(q, r)_{i,j} = r_i \frac{\partial^2 \bar{u}_i}{\partial q_i \partial q_j}(q) = -r_i k'_i(w_i + (Gq)_i) G_{ij}$$

Denote its symmetrization as $H^s = H(q, r) + H(q, r)^T$. Then

$$H^s(q, r)_{ij} = -r_i k'_i(w_i + (Gq)_i)G_{ij} - r_j k'_j(w_j + (Gq)_j)G_{ji} \text{ and } H^s(q, r)_{ii} = -2r_i$$

For hypothesis it holds that for every $v \in \mathbb{R}^n$, there exists a vector $r \in \mathbb{R}_{>0}^n$ st

$$-\sum_{i=1}^n \sum_{j \neq i} v_i (r_i k'_i(w_i + (Gq)_i)G_{ij})v_j - \sum_{i=1}^n \sum_{j \neq i} v_i (r_j k'_j(w_j + (Gq)_j)G_{ji})v_j < \sum_{i=1}^n 2v_i^2 r_i$$

Notice that the left hand side is equivalent to

$$\sum_{i=1}^n \sum_{j \neq i} v_i (-r_i k'_i(w_i + (Gq)_i)G_{ij} - r_j k'_j(w_j + (Gq)_j)G_{ji})v_j = \sum_{i=1}^n \sum_{j \neq i} v_i H^s(q, r)_{ij}v_j$$

and the right hand side is equivalent to

$$\sum_{i=1}^n 2v_i^2 r_i = -\sum_{i=1}^n v_i H^s(q, r)_{ii}v_i$$

Which leads to

$$\sum_{i=1}^n \sum_{j \neq i} v_i H^s(q, r)_{ij}v_j + \sum_{i=1}^n v_i H^s(q, r)_{ii}v_i = \sum_{i=1}^n \sum_{j=1}^n v_i H^s(q, r)_{ij}v_j < 0$$

Writing it in vector form, we get that H^s is negative definite:

$$v H^s(q, r) v^T < 0$$

Then, for *Rosen's equilibrium criterion*, the equilibrium for the game with utilities $\bar{u}_i$ is unique. Since the best response functions are the same for Proposition 3.2.5, the Nash equilibrium is unique also for the public good games with utilities $u_i(q) = \eta_i(w_i - q_i, q_i + (Gq)_i)$.

□

Despite the difficulty in verifying the uniqueness condition of Lemma 4.1.1 for the game with general value functions, if property **L** is satisfied, the sufficient condition for the uniqueness of the Nash equilibrium simplifies to a bound on the lowest eigenvalue of the symmetrization of the per-row rescaled adjacency matrix $\bar{G}$.

Theorem 4.1.2. *Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, with value functions satisfying the linearity property **L** for some $\bar{k} = (\bar{k}_i)_{i=1}^n \in (0, 1)^n$. Denote the symmetrization of $\bar{G}$ as $\bar{G}^s = \frac{\bar{G} + \bar{G}^T}{2}$. If $-\lambda_{\min}^{\bar{G}^s} < 1$, then the Nash equilibrium is unique.*

Proof. For Lemma 4.1.1, if for every $q \in \chi_w$ and every vector $v \in \mathbb{R}^n$, there exists a vector $r \in \mathbb{R}_{>0}^n$ such that (4.1) holds, then the Nash equilibrium is unique. For **L**,

$k'_i(w_i + (Gq)_i) = \bar{k}_i > 0$, then the condition of the Lemma reduces to

$$-\sum_{i=1}^n \sum_{j \neq i} v_i(r_i \bar{k}_i G_{ij}) v_j - \sum_{i=1}^n \sum_{j \neq i} v_i(r_j \bar{k}_j G_{ji}) v_j < \sum_{i=1}^n 2v_i^2 r_i$$

Taking $r_i = 1$ for every i , we get

$$-\sum_{i=1}^n \sum_{j \neq i} v_i(\bar{k}_i G_{ij}) v_j - \sum_{i=1}^n \sum_{j \neq i} v_i(\bar{k}_j G_{ji}) v_j < \sum_{i=1}^n 2v_i^2$$

using the definition of $\bar{G}$, we get

$$-\sum_{i=1}^n \sum_{j \neq i} v_i(\bar{G}_{ij}) v_j - \sum_{i=1}^n \sum_{j \neq i} v_i(\bar{G}_{ji}) v_j < \sum_{i=1}^n 2v_i^2$$

writing it in vector form

$$-v^T \bar{G} v - v^T \bar{G}^T v < \sum_{i=1}^n 2v_i^2$$

Define the symmetrization of $\bar{G}$, $\bar{G}^s = \frac{\bar{G} + \bar{G}^T}{2}$. Then the uniqueness condition is equivalent to

$$-v^T \bar{G}^s v < \sum_{i=1}^n v_i^2 = \|v\|_2^2$$

Now notice that, in general, it holds

$$-v^T \bar{G}^s v \leq -\lambda_{\min}^{\bar{G}^s} \|v\|_2^2$$

Then, a sufficient condition for Lemma 4.1.1 to hold is $-\lambda_{\min}^{\bar{G}^s} < 1$, thus if it is satisfied, the Nash equilibrium is unique. $\square$

Theorem 4.1.2 provides a generalization to the uniqueness condition of Bramoullé and Kranton [10], to the case of directed graph $\mathcal{G}$ and heterogeneous $\bar{k}_i$. Moreover, it is interesting to notice that our condition also mirrors the uniqueness condition of Allouch [16] in the more general setting of weighted, directed graphs. In particular, the alternative, more restrictive condition $-\bar{k}_{\max} \lambda_{\min}^{G^s} < 1$, is exactly Allouch's condition extended to the weighted and directed case.

Corollary 4.1.3. *Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$. Assume $\mathcal{G}$ is an undirected graph and that the value functions satisfy the strong linearity property $\mathbf{L}^+$ for some $\bar{k} \in (0, 1)$. If*

$$-\lambda_{\min}^G \bar{k} < 1$$

then the Nash equilibrium is unique.

Proof. The strong linearity property $\mathbf{L}^+$ implies that

$$\bar{G} = \bar{k}G$$

so that

$$\bar{G}^s = \bar{k} \left(\frac{G + G^T}{2} \right)$$

Being the graph undirected, the adjacency matrix G is symmetric and $\bar{G}^s = \bar{G}$. Moreover, $\lambda_{\min}^{\bar{G}} = \bar{k} \lambda_{\min}^G$. Then, the condition for the equilibrium to be unique of Theorem 4.1.2 simplifies to

$$-\lambda_{\min}^G \bar{k} < 1$$

□

In Corollary 4.1.3 we derive exactly the sufficient condition for the uniqueness of the Nash equilibrium of Bramoullé and Kranton [10].

4.2 Contractivity of the discrete-time best response dynamics

We derive a sufficient condition that relies on the fact that the synchronous best response function is Lipschitz continuous in the $\bar{G}$ -norm. This guarantees the contractivity of the synchronous best response function if the dominant eigenvalue of $\bar{G}$ is bounded by 1. The uniqueness and global asymptotical stability of the Nash equilibrium both for the discrete and continuous-time best response dynamics directly follow from this result.

Moreover, for the game with value functions under $\mathbf{L}^+$ on undirected graphs, we prove that, if it exists, the internal equilibrium is stable for the discrete-time dynamics if and only if contractivity holds.

To derive the theory on the contractivity of the discrete-time best response dynamics, let us first notice that for every i , the function k_i is Lipschitz continuous on $(0, +\infty)$. This means that there exists a positive $\bar{k}_i > 0$ such that for every $z_1, z_2 \in (0, +\infty)$, with $z_1 \neq z_2$, it holds

$$|k_i(z_1) - k_i(z_2)| \leq \bar{k}_i |z_1 - z_2|$$

Using the formula for the derivative of k_i given by (A.5) of Lemma A.0.1, it is possible to explicitly derive the Lipschitz constant for every i :

$$\bar{k}_i = \max_{z>0} k'_i(z)$$

For the general game setting, define the matrix $\bar{G}$ as the adjacency matrix where every row is rescaled by the corresponding Lipschitz constant: $\bar{G}_{ij} = \bar{k}_i G_{ij}$, $\bar{G}_{ii} = 0$. Notice that when the linearity property holds, this definition of $\bar{G}$ is equivalent to the previous one.

Denote $(\lambda^{\bar{G}}, v^{\bar{G}})$ the dominant eigenpair of $\bar{G}$ and recall the definition of the $\bar{G}$ -norm for a general vector $x \in \mathbb{R}^n$: $\|x\|_{\bar{G}} = \sum_{i=1}^n v_i^{\bar{G}} |x_i|$

Proposition 4.2.1. *Given a public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, the synchronous best response function $\mathcal{F}$ is Lipschitz continuous in the $\bar{G}$ -norm, that is for every couple of public good provision vectors $q, \tilde{q} \in \chi_w$, $q \neq \tilde{q}$*

$$\|\mathcal{F}(q) - \mathcal{F}(\tilde{q})\|_{\bar{G}} \leq L \|q - \tilde{q}\|_{\bar{G}}$$

where $L = \lambda^{\bar{G}}$ is the Lipschitz constant.

Proof. Notice that, for Lemma 2.1.1, $\|\bar{G}\|_{\bar{G}} = \lambda^{\bar{G}}$. Then, for every couple of vectors of public good provisions $q, \tilde{q} \in \chi_w$ such that $q \neq \tilde{q}$, the following chain of equalities and inequalities holds

$$\begin{aligned} \|\mathcal{F}(q) - \mathcal{F}(\tilde{q})\|_{\bar{G}} &= \sum_{i=1}^n v_i^{\bar{G}} |\mathcal{B}_i(q_{-i}) - \mathcal{B}_i(\tilde{q}_{-i})| \\ &= \sum_{i=1}^n v_i^{\bar{G}} |k_i(w_i + (G\tilde{q})_i) - k_i(w_i + (Gq)_i)| \leq \sum_{i=1}^n v_i^{\bar{G}} \bar{k}_i |w_i + (G\tilde{q})_i - w_i - (Gq)_i| \\ &= \sum_{i=1}^n v_i^{\bar{G}} |(\bar{k}_i G(\tilde{q} - q))_i| = \sum_{i=1}^n v_i^{\bar{G}} \bar{k}_i \sum_{j=1}^n G_{ij} (\tilde{q} - q)_j \\ &= \sum_{i=1}^n v_i^{\bar{G}} \left| \sum_{j=1}^n \bar{G}_{ij} (\tilde{q} - q)_j \right| = \sum_{i=1}^n v_i^{\bar{G}} |(\bar{G}(\tilde{q} - q))_i| \\ &= \|\bar{G}(\tilde{q} - q)\|_{\bar{G}} \leq \|\bar{G}\|_{\bar{G}} \|q - \tilde{q}\|_{\bar{G}} = \lambda^{\bar{G}} \|q - \tilde{q}\|_{\bar{G}} \end{aligned}$$

□

Theorem 4.2.2. *Consider a public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$. If $\lambda^{\bar{G}} < 1$, then the synchronous best response function $\mathcal{F}$ is contractive in the $\bar{G}$ -norm, that is for every couple of public good provision vectors $q, \tilde{q} \in \chi_w$, $q \neq \tilde{q}$*

$$\|\mathcal{F}(q) - \mathcal{F}(\tilde{q})\|_{\bar{G}} \leq L \|q - \tilde{q}\|_{\bar{G}}$$

where $L = \lambda^{\bar{G}} < 1$ is the contraction constant.

Proof. The thesis follows directly from Proposition 3.2.3 and the hypothesis $\lambda^{\bar{G}} < 1$. □

Corollary 4.2.3. *For the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, if $\lambda^{\bar{G}} < 1$, then the Nash equilibrium is unique and the discrete-time best response dynamics globally converges to it.*

Proof. To prove the uniqueness, assume that both $q, \tilde{q} \in \chi_w$ are equilibrium profiles for the game. Then, they are fixed points of the discrete-time best response dynamics, that is $\mathcal{F}(q) = q$ and $\mathcal{F}(\tilde{q}) = \tilde{q}$. Thus, it must hold that

$$\|\mathcal{F}(q) - \mathcal{F}(\tilde{q})\|_{\bar{G}} = \|q - \tilde{q}\|_{\bar{G}}$$

which contradicts the contractivity result of Theorem 4.2.2, unless $q = \tilde{q}$.

For the global asymptotical convergence to the unique equilibrium, recall that for $\lambda^G < 1$, the synchronous best response function $\mathcal{F}$ is a contraction, thus the discrete-time best response dynamics $q(t+1) = \mathcal{F}(q(t))$ converges to its unique fixed point for the *Banach fixed-point theorem* for any starting point $q(0) \in \chi_w$. $\square$

Let us now focus on the game $\mathcal{P} = (\mathcal{G}, w, \eta)$ with value functions satisfying the linearity property **L** for some $\bar{k} = (\bar{k}_i)_{i=1}^n \in (0, 1)^n$ and derive some implications of the contractivity of the discrete-time best response dynamics.

Remark 4.2.1. Notice that the contractivity of the best response implies the uniqueness of the equilibrium, while vice versa does not hold in general. Moreover, if the value functions η_i satisfy the linearity property **L** for every i and $\bar{G}$ is symmetric, contractivity of the discrete-time dynamics implies the uniqueness condition of Theorem 4.1.2. To prove it, notice first that, in general, if an eigenvalue of $\bar{G}$ is positive $\lambda_i^{\bar{G}} > 0$, then it satisfies:

$$-\lambda_i^{\bar{G}} < 0 < 1$$

Assume now that the contractivity condition $\lambda^{\bar{G}} < 1$ holds; it is equivalent to

$$\max_{i=1, \dots, n} |\lambda_i^{\bar{G}}| < 1$$

then for every i

$$|\lambda_i^{\bar{G}}| < 1$$

If $\lambda_i^{\bar{G}} < 0$, then $|\lambda_i^{\bar{G}}| < 1$ implies

$$0 < -\lambda_i^{\bar{G}} < 1$$

Then we have proved that if the contractivity condition holds, then it is true that $-\lambda_i^{\bar{G}} < 1$ which implies that also $-\lambda_{\min}^{\bar{G}} < 1$, that is the uniqueness condition for linear k_i of Proposition 4.1.2.

Example 4.2.1. Cobb-Douglas value functions (cont.)

For the game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, with $\alpha_i = \alpha, \beta_i = \beta$, for Theorem 4.2.2, the condition for the contractivity of the synchronous best response function becomes

$$\lambda^G \frac{\alpha}{\alpha + \beta} < 1$$

if it is satisfied, it guarantees for the uniqueness and global asymptotical stability of the Nash equilibrium both in discrete and continuous time.

Example 4.2.2. Consider again the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, with equal parameters $\alpha_i = \alpha, \beta_i = \beta$, defined on the unweighted undirected star graph $\mathcal{G}$ of 5 nodes, of example 3.4.7. The contractivity condition is satisfied, since $\lambda^G \frac{\alpha}{\alpha + \beta} =$

$0.55 < 1$. Thus, the internal Nash equilibrium

$$q^* = \frac{\beta}{\alpha + \beta} \left(\frac{\alpha}{\alpha + \beta} \lambda^w + 1 \right) \left(\left(\frac{\alpha}{\alpha + \beta} G + I \right)^{-1} \right)^2 = (0.33, 0.17, 0.17, 0.17, 0.17)^T$$

is the unique equilibrium and it is globally asymptotically stable.

Example 4.2.3. Consider again the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, with equal parameters $\alpha_i = \alpha, \beta_i = \beta$, defined on the unweighted undirected line graph $\mathcal{G}$ of 4 nodes of example 3.4.10. Since the contractivity condition $\lambda^G \frac{\alpha}{\alpha + \beta} = 0.69 < 1$ is satisfied, the internal equilibrium is the unique equilibrium and it is globally asymptotically stable.

Remark 4.2.2. Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, with equal parameters $\alpha_i = \alpha, \beta_i = \beta$ and $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$ the unweighted directed line graph of n nodes. Assume that players have equal income $w_i = w$. Then, there exists always a unique globally asymptotically stable Nash equilibrium with the form:

$$q_{n-i}^* = \frac{\beta w}{\alpha + \beta} \sum_{j=0}^i (-1)^j \left(\frac{\alpha}{\alpha + \beta} \right)^j$$

for $i = 0, \dots, n - 1$

To prove it, it is enough to see that the contractivity condition is always verified. For the unweighted directed line graph, the adjacency matrix is the upper diagonal matrix

$$G = \begin{pmatrix} 0 & 1 & 0 & & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & & 1 \\ 0 & 0 & \dots & \dots & \dots & 0 \end{pmatrix}$$

For Perron Frobenius theorem, for the spectral radius it holds

$$0 \leq \lambda^G \leq 1$$

So that

$$0 \leq \lambda^G \frac{\alpha}{\alpha + \beta} \leq \frac{\alpha}{\alpha + \beta} < 1$$

Then for Corollary 4.2.3, there exists a unique Nash equilibrium for the game that is globally asymptotically stable. The form is given by Remark 3.4.2.

Remark 4.2.3. Let us focus on the condition for the global asymptotical stability of the unique equilibrium for regular graphs. Consider the game $\mathcal{P} = (\mathcal{G}, w, \eta)$, with $\alpha_i = \alpha, \beta_i = \beta$, defined on a general equally weighted (of weight g) regular graph, where all nodes have degree $d \geq 1$. For Perron-Frobenius theorem, the spectral radius has bounds

$$\min_i \sum_{j=1}^n G_{ij} \leq \lambda^G \leq \max_i \sum_{j=1}^n G_{ij}$$

Since the graph is equally weighted and regular of degree d , $\sum_{j=1}^n G_{ij} = dg$ for every i . Then, $\lambda^G = dg$ and the condition for the contractivity of the best response of Theorem 4.1.2 is

$$dg \frac{\alpha}{\alpha + \beta} < 1$$

For example,

1. For the equally weighted directed circle graph, the adjacency matrix is the upper diagonal matrix

$$G = \begin{pmatrix} 0 & g & 0 & & \dots & 0 \\ 0 & 0 & g & 0 & \dots & 0 \\ 0 & 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & & g \\ g & 0 & \dots & \dots & \dots & 0 \end{pmatrix}$$

Being the spectral radius $\lambda^G = g$, if $g \frac{\alpha}{\alpha + \beta} < 1$, then the unique Nash equilibrium is globally asymptotically stable. The unique equilibrium is found using Remark 3.4.1: $q^* \in \chi_w$ st

$$q_i^* = \frac{\beta w}{\alpha(1 + g) + \beta}$$

Notice that the contractivity condition is always guaranteed for the unweighted graph where $g = 1$, since $\frac{\alpha}{\alpha + \beta} < 1$ for $\alpha, \beta \in (0, 1)$.

2. For the equally weighted undirected circle graph, the adjacency matrix is

$$G = \begin{pmatrix} 0 & g & 0 & & & g \\ g & 0 & g & & & \\ 0 & g & 0 & \ddots & & \\ & & \ddots & \ddots & & g \\ g & & & & g & 0 \end{pmatrix}$$

Being the spectral radius $\lambda^G = 2g$, if $2g \frac{\alpha}{\alpha + \beta} < 1$, then contractivity implies the global stability of the unique equilibrium. The unique equilibrium can be found using Remark 3.4.1: $q^* \in \chi_w$ st

$$q_i^* = \frac{\beta w}{\alpha(1 + 2g) + \beta}$$

Thus, for examples 3.4.5 and 3.4.9, the internal equilibrium is the unique Nash and it is globally asymptotically stable. The same holds for the specialized equilibrium of example 3.5.5.

Considering the game defined on an undirected graph, with value functions satisfying $\mathbf{L}^+$, let us now prove another result on the stability of the internal equilibrium when the contractivity is not satisfied. In particular, for the discrete-time dynamics, the internal equilibrium is stable if and only if the dynamics is contractive.

Proposition 4.2.4. *Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$ with value functions satisfying the strong linearity property $\mathbf{L}^+$ for $\bar{k} \in (0, 1)$ and assume that $\mathcal{G}$ is an undirected graph. If there exists an internal equilibrium $q^* \in \chi_w$, then for the discrete-time dynamics, q^* is stable if and only if $\lambda^G \bar{k} < 1$. Otherwise, if $\lambda^G \bar{k} \geq 1$, it is unstable.*

Proof. Let us denote the spectrum of G $\sigma(G) = \{\lambda_1^G, \lambda_2^G, \dots, \lambda_n^G\} \subset \mathbb{R}$. Then the spectrum of $\bar{G}$ is $\sigma(\bar{G}) = \bar{k}\sigma(G) = \{\bar{k}\lambda_1^G, \bar{k}\lambda_2^G, \dots, \bar{k}\lambda_n^G\}$. To study the stability of the internal equilibrium it is possible to use a linearization of the best response dynamics, since the saturation is no longer necessary in a neighborhood of q^* . For the discrete-time dynamics

$$q(t+1) = \mathcal{F}(q(t)) = b - \bar{G}q(t)$$

the stability depends on the eigenvalues of $-\bar{G}$; recall that

$$\sigma(-\bar{G}) = \{-\bar{k}\lambda_i^G\}_{i=1}^n$$

The internal equilibrium is stable if and only if for every i it holds

$$|-\bar{k}\lambda_i^G| < 1$$

that is equivalent to

$$|\lambda_i^G| < \frac{1}{\bar{k}}$$

Since, for definition of dominant eigenvalue, for every i it holds

$$|\lambda_i^G| \leq \max_{i=1, \dots, n} |\lambda_i^G| = \lambda^G$$

then $q^* \in \chi_w$ is stable if and only if

$$\lambda^G \bar{k} < 1$$

q^* is unstable if there exists i such that

$$|-\bar{k}\lambda_i^G| \geq 1$$

that is equivalent to

$$\bar{k}|\lambda_i^G| \geq 1$$

so it is enough if it is true for the largest eigenvalue in modulus, i.e.,

$$\lambda^G \bar{k} \geq 1$$

□

Example 4.2.4. *Cobb-Douglas value functions (cont.)*

For the game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, with $\alpha_i = \alpha, \beta_i = \beta$, defined on a directed graph

$\mathcal{G}$, when contractivity is not guaranteed, i.e., $\lambda^G \frac{\alpha}{\alpha+\beta} \geq 1$, for Proposition 4.2.4, if there exists an internal equilibrium $q^* \in \chi_w$, then it is unstable for the discrete-time dynamics.

Let us see two examples.

Example 4.2.5. Consider again the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, with $\alpha_i = \alpha, \beta_i = \beta$, defined on the unweighted undirected circle graph $\mathcal{G}$ of 5 nodes of example 3.4.9. Now consider the equally weighted adjacency matrix

$$G = 2 \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Take again $\alpha = \frac{1}{4}, \beta = \frac{1}{3}$ and income vector $w = (5, 4, 4, 4, 5)^T$. In this case the uniqueness condition is not satisfied, since $-\lambda_{\min}^G \frac{\alpha}{\alpha+\beta} = 1.39 \geq 1$, but there exists the internal equilibrium

$$q^* = (1, 1.13, 0.32, 1.13, 1)^T$$

Since the contractivity condition is not satisfied, i.e., $\lambda^G \frac{\alpha}{\alpha+\beta} = 1.71 \geq 1$, for Proposition 4.2.4 q^* is unstable in discrete time. Taking $\alpha = \frac{1}{4}, \beta = \frac{2}{3}$, the uniqueness condition $-\lambda_{\min}^G \frac{\alpha}{\alpha+\beta} = 0.88 < 1$ holds and the unique equilibrium is the internal one:

$$q^* = (2.29, 0.18, 2.71, 0.18, 2.29)^T$$

Still, since:

$$\lambda^G \frac{\alpha}{\alpha+\beta} = 1.09 \geq 1$$

q^* is unstable for the discrete-time dynamics.

Example 4.2.6. Consider again the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, with $\alpha_i = \alpha, \beta_i = \beta$, defined on the unweighted undirected line graph of 4 nodes of example 3.4.10. Now consider the equally weighted adjacency matrix

$$G = 2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Take again $\alpha = \frac{1}{4}, \beta = \frac{1}{3}$ and income vector $w = (2, 3, 3, 2)^T$. In this case the uniqueness condition is not satisfied, as $-\lambda_{\min}^G \frac{\alpha}{\alpha+\beta} = 1.39 \geq 1$, but there exists the internal equilibrium

$$q^* = (0.58, 0.65, 0.65, 0.58)^T$$

Since the contractivity condition does not hold, i.e., $\lambda^G \frac{\alpha}{\alpha+\beta} = 1.39 \geq 1$, for Proposition 4.2.4 q^* is unstable for the discrete-time best response dynamics.

4.3 Stability of equilibria for the continuous-time dynamics

In this section we prove that the contractivity of the discrete-time best response dynamics also ensures uniqueness and global stability to the equilibrium for the continuous-time dynamics. In addition, we establish that, under the linearity property $\mathbf{L}$, global stability of the equilibrium is guaranteed also under the weaker uniqueness condition if $\bar{G}$ is symmetric.

In this setting, we further study the stability of equilibrium profiles when uniqueness may not hold and we demonstrate that the continuous-time best response dynamics globally converges to the set of Nash equilibria. In addition, action profiles that are local maximum points of an associated functional P are locally asymptotically stable equilibria for the game.

Finally, for the game with value functions under $\mathbf{L}^+$ on undirected graphs, we establish that, if there exists an internal equilibrium, for the continuous-time dynamics it is stable if and only if the uniqueness condition is true.

Theorem 4.3.1. *For the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, if $\lambda^{\bar{G}} < 1$, then the Nash equilibrium is unique and the continuous-time best response dynamics globally converges to it.*

Proof. From Corollary 4.2.3, the Nash equilibrium $q^* \in \chi_w$ is unique. To prove its stability, define the functional $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ having the form

$$L(q) = \|q^* - q\|_{\bar{G}} = \sum_{i=1}^n v_i^{\bar{G}} |q_i^* - q_i|$$

Then, L is continuous and positive definite with respect to q^* since $L(q) \geq 0$ and $L(q) = 0$ if and only if $q = q^*$. Let us prove that it is decreasing along the trajectories of the continuous-time best response dynamics. To do so notice that $L(q(t))$ is the composition of two Lipschitz continuous functions and thus it is Lipschitz continuous; this implies that, in particular, L is absolutely continuous in t . Define the function

$$\sigma_i = \text{sign}(q_i - q_i^*) = \begin{cases} +1 & \text{if } q_i - q_i^* > 0 \\ 0 & \text{if } q_i - q_i^* = 0 \\ -1 & \text{if } q_i - q_i^* < 0 \end{cases}$$

Then, almost everywhere the derivative of L with respect to t is defined and satisfies:

$$\begin{aligned} \dot{L}(q(t)) &= \frac{d}{dt} L(q(t)) = \sum_{i=1}^n v_i^{\bar{G}} \frac{d}{dt} |q_i(t) - q_i^*| \\ &= \sum_{i=1}^n v_i^{\bar{G}} \sigma_i \frac{d}{dt} (q_i(t) - q_i^*) = \sum_{i=1}^n v_i^{\bar{G}} \sigma_i (\mathcal{B}_i(q_{-i}) - q_i) \end{aligned}$$

Since q^* is an equilibrium, it holds for every i that $\mathcal{B}_i(q_{-i}^*) = q_i^*$, thus

$$\begin{aligned}\dot{L}(q(t)) &= \sum_{i=1}^n v_i^{\bar{G}} \sigma_i(\mathcal{B}_i(q_{-i}) - \mathcal{B}_i(q_{-i}^*) + q_i^* - q_i) \\ &= \sum_{i=1}^n v_i^{\bar{G}} (\sigma_i(\mathcal{B}_i(q_{-i}) - \mathcal{B}_i(q_{-i}^*)) - \sigma_i(q_i - q_i^*))\end{aligned}$$

Being $\sigma_i = \text{sign}(q_i - q_i^*) \in \{+1, 0, -1\}$ we have that $\sigma_i(q_i - q_i^*) = |q_i - q_i^*|$, thus we get

$$\dot{L}(q(t)) = \sum_{i=1}^n v_i^{\bar{G}} \sigma_i(\mathcal{B}_i(q_{-i}) - \mathcal{B}_i(q_{-i}^*)) - \sum_{i=1}^n v_i^{\bar{G}} |q_i - q_i^*|$$

Estimating the first term as

$$\sum_{i=1}^n v_i^{\bar{G}} \sigma_i(\mathcal{B}_i(q_{-i}) - \mathcal{B}_i(q_{-i}^*)) \leq \sum_{i=1}^n v_i^{\bar{G}} |\mathcal{B}_i(q_{-i}) - \mathcal{B}_i(q_{-i}^*)|$$

We obtain

$$\dot{L}(q(t)) \leq \sum_{i=1}^n v_i^{\bar{G}} |\mathcal{B}_i(q_{-i}) - \mathcal{B}_i(q_{-i}^*)| - \sum_{i=1}^n v_i^{\bar{G}} |q_i - q_i^*|$$

For the definition of the $\bar{G}$ -norm and for the contractivity of $\mathcal{F}$ it is true that

$$\dot{L}(q(t)) \leq \|\mathcal{F}(q) - \mathcal{F}(q^*)\|_{\bar{G}} - \|q - q^*\|_{\bar{G}} \leq \lambda^{\bar{G}} \|q - q^*\|_{\bar{G}} - \|q - q^*\|_{\bar{G}} = (\lambda^{\bar{G}} - 1) \|q - q^*\|_{\bar{G}}$$

Then it holds

$$\dot{L}(q(t)) \leq -(1 - \lambda^{\bar{G}}) L(q(t))$$

and, for the contractivity constraint, $1 - \lambda^{\bar{G}} > 0$. The convergence to q^* is global and exponential since for every $q \in \chi_w, q \neq q^*$

$$\frac{d}{dt} \log L(q(t)) = \frac{\dot{L}(q(t))}{L(q(t))} \leq -(1 - \lambda^{\bar{G}})$$

which implies

$$\log(L(q(t))) \leq \log(L(q(0))) - (1 - \lambda^{\bar{G}})t$$

and so

$$L(q(t)) \leq L(q(0)) e^{-(1 - \lambda^{\bar{G}})t}$$

Thus from every action profile $q \in \chi_w$, L converges exponentially to zero, and so the dynamics globally converges to the unique Nash equilibrium q^* . $\square$

Notice that our sufficient condition for the uniqueness and global stability of the Nash equilibrium of Theorem 4.3.1 implies the one derived by Ghahesifard et al. [17]. Moreover, we extend their framework to include best response functions that are non-necessarily linear saturated.

We now show that there is global asymptotical stability of the continuous-time best response dynamics also under weaker assumptions on the spectrum of $\bar{G}$. In

particular, if $\bar{G}$ is symmetric and the uniqueness condition of Theorem 4.1.2 holds, that is $-\lambda_{\min}^{\bar{G}} < 1$, then the Nash equilibrium is globally asymptotically stable for the continuous-time dynamics. To prove it, let us introduce the functional $P : \chi_w \rightarrow \mathbb{R}$ such that

$$P(q) = -\frac{1}{2}q^T(I + \bar{G})q + b^T q$$

and state the following preliminary Lemma.

Lemma 4.3.2. *Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, with value functions satisfying the linearity property **L** for some $\bar{k} = (\bar{k}_i)_{i=1}^n \in (0, 1)^n$. Assume $\bar{G}$ is a symmetric matrix. Then,*

$$\dot{P}(q(t)) \geq 0$$

and $q^* \in \chi_w$ is a Nash equilibrium for the game if and only if

$$\dot{P}(q^*) = 0$$

Proof. The derivative of P with respect to t is then

$$\dot{P}(q(t)) = \frac{dP(q(t))}{dt} = (b - \bar{G}q(t) - q(t))^T([b - \bar{G}q(t)]_+ - q(t))$$

which can be written as

$$\dot{P}(q(t)) = \sum_{i=1}^n \frac{\partial P(q(t))}{\partial q_i} \dot{q}_i(t) = \sum_{i=1}^n h_i(q)$$

with $h_i(q) = (-q_i + b_i - (\bar{G}q)_i)([b_i - (\bar{G}q)_i]_+ - q_i)$.

Let us see that for every i and for every $q \in \chi_w$, it holds $h_i(q) \geq 0$, and $h_i(q) = 0$ for every i if and only if q is a Nash equilibrium:

- If $b_i \leq (\bar{G}q)_i$, then $[b_i - (\bar{G}q)_i]_+ = 0$ and:

$$h_i(q) = q_i(q_i - b_i + (\bar{G}q)_i) \geq q_i^2 \geq 0$$

Moreover $h_i(q) = 0$ if and only if $q_i = 0$, i.e., if agent i is playing his best response.

- If $b_i \geq (\bar{G}q)_i$, then:

$$h_i(q) = (-q_i + b_i - (\bar{G}q)_i)^2 \geq 0$$

Moreover $h_i(q) = 0$ if and only if $q_i = b_i - (\bar{G}q)_i \geq 0$, i.e., if agent i is playing his best response.

Then $\dot{P}(q(t)) = \sum_{i=1}^n h_i(q) \geq 0$ and $\dot{P}(q(t)) = \sum_{i=1}^n h_i(q) = 0$ if and only if $h_i(q) = 0$ for every i , which is equivalent to q being a Nash equilibrium. $\square$

Theorem 4.3.3. *Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, with value functions satisfying the linearity property **L** for some $\bar{k} = (\bar{k}_i)_{i=1}^n \in (0, 1)^n$. Assume $\bar{G}$ is a*

symmetric matrix and $-\lambda_{\min}^{\bar{G}} < 1$. Then, the continuous-time best response dynamics globally converges to the unique Nash equilibrium.

Proof. For Proposition 4.1.2, the condition $-\lambda_{\min}^{\bar{G}} < 1$ ensures the uniqueness of the equilibrium $q^* \in \chi_w$. From Lemma 4.3.2, we know that $-P$ is decreasing along the trajectories of the dynamics, that is $-\dot{P}(q(t)) < 0$ for $q \neq q^*$. Then, being χ_w a compact set, from *LaSalle's Invariance Principle*¹ it follows that the continuous-time best response dynamics globally converges to the largest invariant set such that $\dot{P}(q(t)) = 0$, which is exactly q^* .

□

Theorem 4.3.3 generalizes the stability result for the unique Nash equilibrium of Bramoullé et al. [9] to weighted graphs. Moreover, it mirrors the contribution by Allouch [16] for the stability of the unique Nash equilibrium.

Remark 4.3.1. *Summing up, in some cases the behavior of the two best response dynamics for the game $\mathcal{P} = (\mathcal{G}, w, \eta)$ with value functions satisfying the linearity property $\mathbf{L}$ is different. As long as the contractivity condition $\lambda^{\bar{G}} < 1$ is satisfied, there exists a unique Nash equilibrium that is globally stable for both the discrete and continuous-time dynamics. If $\bar{G}$ is symmetric, when this condition is violated but uniqueness still holds, that is $-\lambda_{\min}^{\bar{G}} < 1$, then the Nash equilibrium is still stable for the continuous-time dynamics, while it can become unstable for the one in discrete time. Thus, if the lowest and greatest eigenvalue are different in modulus, there exists a "stability gap", that is a range of values of the lowest eigenvalue where the two dynamics behave differently. On the other hand, when the smallest eigenvalue, in modulus, coincides with the spectral radius, the two conditions become equivalent and the dynamics have the same asymptotic behavior. We provide more detailed examples of this gap considering the complete graph in Section 4.4.*

We further discuss the stability of equilibria for the continuous-time dynamics when uniqueness is not guaranteed and the matrix $\bar{G}$ is symmetric. In particular, we prove that the best response dynamics always converges to the set of Nash equilibria and that the set of strict maximum points of P are stable equilibria. Finally, we observe that, when the uniqueness condition is not satisfied, stable equilibria are non-internal.

Proposition 4.3.4. *Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, with value functions satisfying the linearity property $\mathbf{L}$ for some $\bar{k} = (\bar{k}_i)_{i=1}^n \in (0, 1)^n$. If $\bar{G}$ is a symmetric matrix, then, the continuous-time best response dynamics globally converges to the set of Nash equilibria.*

Proof. From Lemma 4.3.2, we know that $-P$ is decreasing along the trajectories of the dynamics, that is $-\dot{P}(q(t)) \leq 0$ and $\dot{P}(q^*) = 0$ if and only if q^* is an equilibrium.

¹Refer to *Teorema 3.5* of [21]

Then, the compact set $E = \{q \in \chi_w : \dot{P}(q) = 0\}$ coincides with the set of Nash equilibria of the game. It follows from *LaSalle's invariance principle* that the union of compact and invariant subsets of E , which is exactly the set of Nash equilibria, is attractive in χ_w . Thus, the continuous-time dynamics globally converges to the set of Nash equilibria of the game for any initial public good profile $q(0) \in \chi_w$. $\square$

With Proposition 4.3.4 we recover the result of Gharesifard et al. [17] on the convergence of piece-wise linear saturated best response dynamics to the set of equilibrium profiles.

Proposition 4.3.5. *Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$, with value functions satisfying the linearity property $\mathbf{L}$ for some $\bar{k} = (\bar{k}_i)_{i=1}^n \in (0, 1)^n$. Assume $\tilde{G}$ is a symmetric matrix and consider a public good provision profile $q^* \in \chi_w$ that is a strict local maximum of the functional P on χ_w , i.e., there exists $\epsilon > 0$ and a neighborhood of q^* , $B_{q^*}(\epsilon) = \{q \in \chi_w : \|q^* - q\| < \epsilon\}$, such that for every $q \in B_{q^*}(\epsilon)$ it holds $P(q^*) > P(q)$. Then, q^* is a locally asymptotically stable equilibrium for the continuous-time best response dynamics.*

Proof. Let us see that for Proposition 2.3.6 q^* is a Nash equilibrium: being it a maximum point of P , in every direction P can not increase, i.e., for every i ,

$$\frac{\partial P}{\partial q_i}(q^*) \leq 0$$

Moreover, along the positive components, i.e., where $q_i^* > 0$, stationarity must hold: $\frac{\partial P}{\partial q_i}(q^*) = 0$. Thus, also condition

$$q_i^* \frac{\partial P}{\partial q_i}(q^*) = 0$$

holds for every i . Thus, q^* is a Nash equilibrium for the game with quadratic utilities. Then, for 3.2.5 it is also a Nash equilibrium for the public good game under $\mathbf{L}$.

To prove its stability, define the functional $V : \chi_w \rightarrow \chi_w$ st

$$V(q) = P(q^*) - P(q)$$

Then, V is continuous and positive definite with respect to q^* in $B_{q^*}(\epsilon)$: in fact

$$V(q^*) = 0$$

and for $q \neq q^*$

$$V(q) = P(q^*) - P(q) > 0$$

for definition of strict local maximum of P . Moreover, for Lemma 4.3.2, V is non-increasing along the trajectories of the continuous-time best response dynamics, i.e., $\dot{V}(q(t)) \leq 0$. Then, V is a weak Lyapunov function for q^* . For the *1st Lyapunov*

theorem², q^* is a stable equilibrium. It follows from the *LaSalle's invariance principle*, that the union of compact and invariant subsets of $E = \{q \in \chi_w : \dot{V}(q(t)) = 0\}$, which is exactly q^* , is attractive in χ_w . Thus, the continuous-time dynamics converges to q^* for any initial public good profile $q(0) \in B_{q^*}(\epsilon)$ and q^* is locally asymptotically stable. $\square$

Proposition 4.3.5 recovers one implication of the equivalence for stable equilibria established in [10].

Considering the game defined on an undirected graph, with value functions satisfying $\mathbf{L}^+$, let us now prove another result on the stability of the internal equilibrium. In particular, for the continuous-time dynamics, the internal equilibrium is stable if and only if the uniqueness condition holds.

Proposition 4.3.6. *Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta)$ with value functions satisfying the strong linearity property $\mathbf{L}^+$ for $\bar{k} \in (0, 1)$ and assume that $\mathcal{G}$ is an undirected graph. If there exists an internal equilibrium $q^* \in \chi_w$, then for the continuous-time dynamics q^* is stable if and only if $-\lambda_{\min}^G \bar{k} < 1$. Otherwise, if $-\lambda_{\min}^G \bar{k} \geq 1$, it is unstable.*

Proof. Let us denote the spectrum of G $\sigma(G) = \{\lambda_1^G, \lambda_2^G, \dots, \lambda_n^G\}$. Then the spectrum of $\bar{G}$ is $\sigma(\bar{G}) = \bar{k}\sigma(G) = \{\bar{k}\lambda_1^G, \bar{k}\lambda_2^G, \dots, \bar{k}\lambda_n^G\}$. To study the stability of the internal equilibrium it is possible to use a linearization of the best response dynamics, since the saturation is no longer necessary in a neighborhood of q^* . For the continuous-time dynamics

$$\dot{q}(t) = \mathcal{F}(q(t)) - q(t) = b - \bar{G}q(t) - q(t) = b - (\bar{G} + I)q(t)$$

the stability depends on the eigenvalues of $-\bar{G} - I$; recall that

$$\sigma(-\bar{G} - I) = \{-\bar{k}\lambda_i^G - 1\}_{i=1}^n$$

The internal equilibrium is stable if and only if for every i it holds

$$-\bar{k}\lambda_i^G - 1 < 0$$

that is equivalent to

$$\lambda_i^G > -\frac{1}{\bar{k}}$$

Since for every i it holds

$$\lambda_{\min}^G = \min_{i=1, \dots, n} \lambda_i^G \leq \lambda_i^G$$

then $q^* \in \chi_w$ is stable if and only if

$$\lambda_{\min}^G > -\frac{1}{\bar{k}}$$

²Refer to *Teorema 3.1* of [21]

ie if and only if

$$-\lambda_{\min}^G \bar{k} < 1$$

q^* is unstable if there exists i such that

$$-\bar{k}\lambda_i^G - 1 \geq 0$$

that is equivalent to

$$\lambda_i^G \leq -\frac{1}{\bar{k}}$$

so it is enough if it is true for the smallest eigenvalue, i.e.,

$$-\lambda_{\min}^G \bar{k} \geq 1$$

□

Proposition 4.3.6 is consistent with the results concerning the stability of equilibria for the continuous-time dynamics established in [9] and [16].

Remark 4.3.2. *Given an internal equilibrium $q^* \in \chi_w$ for the game $\mathcal{P} = (\mathcal{G}, w, \eta)$ with value functions satisfying the strong linearity property $\mathbf{L}^+$ for $\bar{k} \in (0, 1)$, notice again that, if the contractivity condition is violated, i.e., $\lambda^G \bar{k} \geq 1$, and there exists an internal equilibrium $q^* \in \chi_w$, then for Proposition 4.2.4 q^* is unstable in the discrete-time dynamics. Nonetheless, if the uniqueness condition still holds, i.e., $-\lambda_{\min}^G \bar{k} < 1$, then for Proposition 4.3.6, it is stable in continuous time, since stability depends on the value of the lowest eigenvalue of G .*

Example 4.3.1. *Cobb-Douglas value functions (cont.)*

For the game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, with equal parameters $\alpha_i = \alpha, \beta_i = \beta$, defined on a directed graph $\mathcal{G}$, when contractivity is not guaranteed, i.e., $\lambda^G \frac{\alpha}{\alpha+\beta} \geq 1$, for Proposition 4.3.6, if there exists an internal equilibrium $q^ \in \chi_w$, then it is stable for the continuous-time dynamics as long as $-\lambda_{\min}^G \frac{\alpha}{\alpha+\beta} < 1$.*

Let us see two examples.

Example 4.3.2. *Consider again the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, with $\alpha_i = \alpha, \beta_i = \beta$, defined on the unweighted undirected circle graph $\mathcal{G}$ of 5 nodes of example 4.2.5. As previously seen, there exists the internal equilibrium*

$$q^* = (1, 1.13, 0.32, 1.13, 1)^T$$

which is unstable for the discrete-time dynamics. Since the uniqueness condition is not satisfied, i.e., $-\lambda_{\min}^G \frac{\alpha}{\alpha+\beta} = 1.39 \geq 1$, for Proposition 4.3.6 it is also unstable in continuous time.

Taking $\alpha = \frac{1}{4}, \beta = \frac{2}{3}$, the uniqueness condition now holds, as $-\lambda_{\min}^G \frac{\alpha}{\alpha+\beta} = 0.88 < 1$, and the unique equilibrium is the internal one:

$$q^* = (2.29, 0.18, 2.71, 0.18, 2.29)^T$$

In this case, the opposite behavior is observed in the two dynamics, since:

$$\lambda^G \frac{\alpha}{\alpha + \beta} = 1.09 \geq 1$$

implies the instability of q^* in discrete time, while

$$-\lambda_{\min}^G \frac{\alpha}{\alpha + \beta} = 0.88 < 1$$

allows to conclude that the internal equilibrium q^* is stable in continuous time.

Example 4.3.3. Consider again the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, with $\alpha_i = \alpha, \beta_i = \beta$, defined on the unweighted undirected line graph $\mathcal{G}$ of 4 nodes of example 4.2.6. We already noticed that the uniqueness condition is not satisfied, but there exists the internal equilibrium

$$q^* = (0.58, 0.65, 0.65, 0.58)^T$$

The internal equilibrium is unstable in both the dynamics, since:

$$\lambda^G \frac{\alpha}{\alpha + \beta} = 1.39 \geq 1$$

implies the instability in discrete time for Proposition 4.2.4, while

$$-\lambda_{\min}^G \frac{\alpha}{\alpha + \beta} = 1.39 \geq 1$$

implies the instability in continuous time for Proposition 4.3.6. Notice that in this case the lowest and greatest eigenvalue are equal in modulus, i.e., $\lambda^G = |\lambda_{\min}^G| = 3.24$, thus the two dynamics always have the same behavior: take for example, $\alpha = \frac{1}{4}, \beta = \frac{2}{3}$. The contractivity condition holds and the unique equilibrium is the internal one:

$$q^* = (0.85, 1.11, 1.11, 0.85)^T$$

which is globally asymptotically stable in both the dynamics.

4.4 Comparison of discrete- and continuous-time dynamics: the example of the complete graph

We present a final focus on the "stability gap" of the two dynamical systems, that is a range of values in which there exists a unique Nash equilibrium profile for the game, but the discrete and continuous-time dynamical systems may present opposite behaviors with respect to it. We concentrate on the study of the complete graph, which is often used in real-world applications. First, we recall the 3 nodes example 4.0.1 and motivate the previously observed behavior with the theoretical results; afterwards, we generalize the example to the complete graph of n nodes and finally we observe the particular behavior of the simpler graph of 2 nodes.

Example 4.4.1. *Complete graph of 3 nodes (cont.)*

Let us reconsider the game defined on the complete equally weighted graph of 3 nodes of example 4.0.1 and motivate the experimental results with the developed theory. Being the spectrum of G $\sigma(G) = \{\frac{\gamma}{\alpha}, -\frac{\gamma}{2\alpha}, -\frac{\gamma}{2\alpha}\}$, and $\bar{k} = \alpha$, the condition for uniqueness of the equilibrium of Theorem 4.1.2 is $\gamma < 2$, while contractivity of the synchronous best response, for Theorem 4.2.2, is guaranteed if $\gamma < 1$. The so-found intervals coincide with the ones that were observed through the simulations. In particular:

- If $\gamma < 1$:
The uniqueness condition is satisfied and there exists a unique Nash equilibrium $q^* = (\frac{\beta}{1+\gamma}, \frac{\beta}{1+\gamma}, \frac{\beta}{1+\gamma})$. It is globally asymptotically stable for the discrete-time dynamics for Corollary 4.2.3, and for the continuous-time dynamics for Theorem 4.3.1.
- for $1 < \gamma < 2$:
The equilibrium is still unique and its behavior is verified by the theory: for Proposition 4.3.3, q^* is still stable for the continuous-time dynamics, while, for Proposition 4.2.4, in discrete-time it is unstable.
- for $\gamma > 2$:
As the uniqueness condition is violated, stability of the internal equilibrium q^* is lost both for the discrete and continuous-time dynamical systems, according to Propositions 4.3.6 and 4.2.4. For these values of γ , together with the internal equilibrium, equilibria that are specialized and with 2 contributing nodes arise. Despite not being able to predict their asymptotic behavior for the discrete-time dynamical system using the main findings of this chapter, we know from Proposition 4.3.4 that the one in continuous time globally converges to the set of Nash equilibria.

Let us now turn to the more general $n \times n$ case.

Example 4.4.2. *Complete graph of n nodes*

Consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, where $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$ is the equally weighted complete graph of n nodes, $w \in \mathbb{R}_{>0}^n$ the income vector such that $w_i = w > 0$ for every i , and η the Cobb-Douglas value functions with equal parameters $\alpha_i = \alpha, \beta_i = \beta \in (0, 1)$, $\eta_i(w - q_i, q_i + (Gq)_i) = (w - q_i)^\alpha (q_i + (Gq)_i)^\beta$. To simplify the calculations, let us assume that $\alpha + \beta = 1$ and $w_i = 1$ for every i .

We refer to the adjacency matrix G having with equal weights $G_{ij} = g = \frac{\gamma}{\alpha(n-1)}$ for all $i \neq j$. The best response for every agent is

$$\mathcal{B}_i(q_{-i}) = [\beta - \alpha g \sum_{j \neq i} q_j]_+ = [\beta - \frac{\gamma}{n-1} \sum_{j \neq i} q_j]_+$$

and the synchronous best response function is:

$$\mathcal{F}(q) = \left[\begin{pmatrix} \beta \\ \vdots \\ \beta \end{pmatrix} - \alpha G q \right]_+ = \left[\begin{pmatrix} \beta \\ \vdots \\ \beta \end{pmatrix} - \frac{\gamma}{n-1} (\mathbf{1} - I) q \right]_+$$

where $\mathbf{1} = \mathbf{1}^{n \times n}$ is the $n \times n$ matrix of ones.

Being the spectrum of G $\sigma(G) = \{\frac{\gamma}{\alpha}, -\frac{\gamma}{\alpha(n-1)}, \dots, -\frac{\gamma}{\alpha(n-1)}\}$, and $\bar{k} = \alpha$, the condition for uniqueness of the equilibrium, for Theorem 4.1.2, is $\gamma < n-1$, while contractivity of the synchrnous best response dynamics is guaranteed, for Theorem 4.2.2, if $\gamma < 1$.

- If $\gamma < 1$

The unique Nash equilibrium is the internal one $q^* \in \chi_w$ where all nodes contribute equally to the public good, with provision:

$$q_i^* = \frac{\beta}{\alpha(1 + g(n-1)) + \beta} = \frac{\beta}{\alpha(1 + \frac{\gamma}{\alpha(n-1)}(n-1)) + \beta} = \frac{\beta}{1 + \gamma}$$

for Corollary 4.2.3 and Theorem 4.3.1 q^* is globally asymptotically stable both for the discrete and continuous-time dynamics.

- If $1 < \gamma < n-1$

The equilibrium is still unique and it is again the internal one with equally contributing nodes

$$q^* = (\frac{\beta}{1+\gamma}, \dots, \frac{\beta}{1+\gamma})$$

For Proposition 4.3.3, it is still stable for the continuous-time dynamics, while Proposition 4.2.4 implies that in discrete-time is unstable.

- If $\gamma > n-1$

Even though the internal equilibrium $q^* = (\frac{\beta}{1+\gamma}, \dots, \frac{\beta}{1+\gamma})$ of equally contributing nodes still exists, the uniqueness condition is violated. Thus, it is interesting to study whether equilibria with $C \neq \mathcal{N}$ are present.

- Being the graph complete, the only possible type of specialized equilibria are those where one node contributes and the others free-ride. Let us consider, for example, node 1 to be the only contributor, then $q_1 = \beta$ and the candidate equilibrium is the action profile

$$q = (\beta, 0, \dots, 0)$$

It is a Nash equilibrium if for every non-contributor the condition of Remark 3.5.1 is satisfied:

$$\frac{\gamma}{\alpha(n-1)} \geq \frac{1}{\alpha}$$

which is equivalent to

$$\gamma \geq n - 1$$

Then, for $\gamma > n - 1$ there are n specialized equilibria of the form $q_i^* = \beta$ and $q_j^* = 0$ for all $j \neq i$. Notice that specialized equilibria are also present if $\gamma = n - 1$.

- Let us now consider the case in which two nodes i and j contribute and the others free ride. Then,

$$q_i = \beta - \frac{\gamma}{n-1}q_j > 0 \text{ and } q_j = \beta - \frac{\gamma}{n-1}q_i > 0$$

It's easy to see that for $\gamma \neq n - 1$ then $q_i = q_j = \frac{\beta(n-1)}{n-1+\gamma}$. It is a Nash equilibrium if it satisfies (3.14):

$$\alpha \begin{pmatrix} g & g \\ \vdots & \vdots \\ g & g \end{pmatrix} \frac{\beta(n-1)}{n-1+\gamma} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \geq \beta \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

that is equivalent to

$$\alpha \frac{\gamma}{\alpha(n-1)} \frac{\beta(n-1)}{n-1+\gamma} \begin{pmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\gamma\beta}{n-1+\gamma} \begin{pmatrix} 2 \\ \vdots \\ 2 \end{pmatrix} \geq \beta \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

component-wise, it must hold:

$$\frac{2\gamma}{n-1+\gamma} \geq 1$$

which is equivalent to

$$\gamma \geq n - 1$$

Then, if $\gamma > n - 1$

$$q^* = (0, \dots, 0, \frac{\beta(n-1)}{n-1+\gamma}, 0, \dots, 0, \frac{\beta(n-1)}{n-1+\gamma}, 0, \dots, 0)$$

is a Nash equilibrium and there exist $\binom{n}{2}$ equilibria where 2 nodes contribute and the others free-ride on their provision.

- In general, if $1 < k < n$ nodes contribute and $n - k$ free-ride, then for two generic $i, l \in C$

$$q_i = \beta - \alpha g \sum_{j=1}^n q_j = \beta - \alpha g \sum_{j \in C} q_j + \alpha g q_i$$

and

$$q_l = \beta - \alpha g \sum_{j \in C} q_j + \alpha g q_l$$

then

$$q_i - q_l = \alpha g (q_i - q_l)$$

then for $\alpha g \neq 1$, i.e., $\gamma \neq n - 1$ all contributors must provide the same amount at equilibrium: $q_i = q_l$ for every $i, l \in C$ and the candidate equilibrium has the form, for every $i \in C$:

$$q_i = \frac{\beta(n-1)}{n-1+\gamma(k-1)}$$

For q to be an equilibrium, (3.14) must be verified:

$$\begin{aligned} & \alpha \begin{pmatrix} g & \dots & g \\ \vdots & \dots & \vdots \\ g & \dots & g \end{pmatrix} \frac{\beta(n-1)}{n-1+\gamma(k-1)} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \\ &= \alpha \frac{\gamma}{\alpha(n-1)} \frac{\beta(n-1)}{n-1+\gamma(k-1)} \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \dots & \vdots \\ 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \geq \beta \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \end{aligned}$$

component-wise:

$$\frac{\gamma k}{n-1+\gamma(k-1)} \geq 1$$

which is equivalent to

$$\gamma \geq n-1$$

Then, for $\gamma > n-1$

$$q^* = \left(\frac{\beta(n-1)}{n-1+\gamma(k-1)}, \dots, \frac{\beta(n-1)}{n-1+\gamma(k-1)}, 0, \dots, 0 \right)$$

is a Nash equilibrium and there are $\binom{n}{k}$ equilibria where $1 < k < n$ nodes contribute and all the others free-ride on their provisions.

To summarize, for the complete equally weighted graph of n nodes, there exist always an equilibrium of equally contributing nodes, which is unique for $\gamma < n-1$. Moreover, for $\gamma < 1$, it is globally asymptotically stable in both the discrete and continuous-time dynamics; for $1 < \gamma < n-1$ the unique equilibrium is still locally stable for the continuous-time dynamics, while it becomes unstable for the one in discrete time. If $\gamma > n-1$, the internal equilibrium is unstable in both the dynamics and new equilibrium profiles arise: n specialized equilibria where only one node contributes, and $\sum_{k=2}^n \binom{n}{k}$ internal equilibria where $k = 2, \dots, n-1$

nodes contribute and $n - k$ free-ride.

Focus now on the complete graph of 2 nodes. It is exemplifying of the case in which the greatest and smallest eigenvalue of the adjacency matrix are equal in modulus. As a consequence, the uniqueness and contractivity conditions coincide and the discrete and continuous-time dynamics do not exhibit the "stability gap". To enrich the study and hint a direction for further investigations, we provide also some simulations of the dynamical systems.

Example 4.4.3. *Complete graph of 2 nodes*

Let us consider the public good game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$ defined on the equally weighted complete graph of 2 nodes with adjacency matrix

$$G = \begin{pmatrix} 0 & \frac{\gamma}{\alpha} \\ \frac{\gamma}{\alpha} & 0 \end{pmatrix}$$

The synchronous best response function is

$$\mathcal{F}(q) = \mathcal{F}(q_1, q_2) = (\mathcal{B}_1(q_2), \mathcal{B}_2(q_1)) = ([\beta - \gamma q_2]_+, [\beta - \gamma q_1]_+)$$

Being the spectrum of G $\sigma(G) = \{\frac{\gamma}{\alpha}, -\frac{\gamma}{\alpha}\}$, the greatest and lowest eigenvalue are equal in modulus, thus the condition for uniqueness of the equilibrium of Theorem 4.1.2 and contractivity of the synchronous best response of Theorem 4.2.2 are equivalent; since $\bar{k} = \alpha$, it must be verified that $\gamma < 1$. Then:

- If $\gamma < 1$:
the unique Nash equilibrium is the internal one of equally contributing nodes

$$q^* = \left(\frac{\beta}{1 + \gamma}, \frac{\beta}{1 + \gamma} \right) > 0$$

as it can be seen in Figure 4.5a.

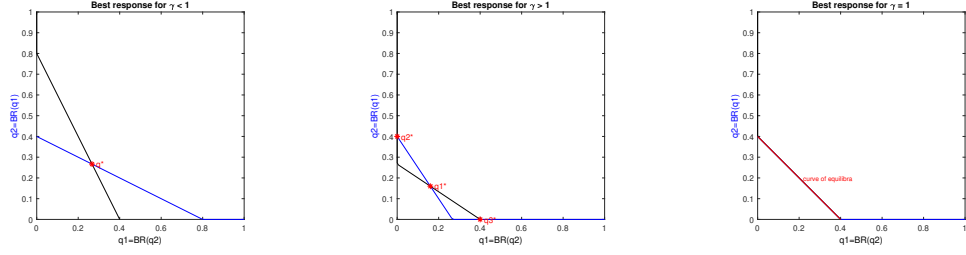
- If $\gamma > 1$:
there are 3 Nash equilibria:

$$q_1^* = \left(\frac{\beta}{1 + \gamma}, \frac{\beta}{1 + \gamma} \right) > 0, \quad q_2^* = (0, \beta), \quad q_3^* = (\beta, 0)$$

as we can see in Figure 4.5b. In particular, q_1^* is the internal equilibrium of equally contributing nodes and q_2^* and q_3^* are specialized equilibria.

- If $\gamma = 1$:
there are infinitely many equilibria, corresponding to the curve $q_2 = \beta - q_1$ for $0 \leq q_1 \leq \beta$ (see Figure 4.5c). The equilibria are then all those of the form

$$q^* = (\beta - c, c) \text{ for } 0 \leq c \leq \beta$$



(a) Best response curves for $\gamma < 1$ (b) Best response curves for $\gamma > 1$ (c) Best response curves for $\gamma = 1$

Figure 4.5: Comparison of best response curves as γ varies

Let us study the behavior of the best response dynamics to understand the nature of the Nash equilibria as γ varies. We rely on the results of Sections 4.2 and 4.3 and also report some simulations of the continuous- and discrete-time dynamical systems. Recall their form:

$$q(t+1) = \mathcal{F}(q(t)) = \left[\begin{pmatrix} \beta \\ \beta \end{pmatrix} - \gamma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} q(t) \right]_+$$

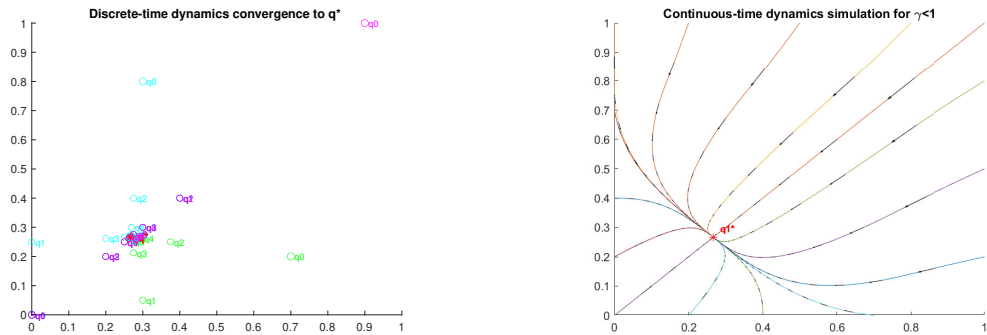
is the discrete-time best response dynamics, while

$$\dot{q}(t) = \mathcal{F}(q(t)) - q(t) = \left[\begin{pmatrix} \beta \\ \beta \end{pmatrix} - \gamma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} q(t) \right]_+ - q(t)$$

is the one in continuous time.

- If $\gamma < 1$:

Then, the unique Nash equilibrium $q^* = (\frac{\beta}{1+\gamma}, \frac{\beta}{1+\gamma})$ is globally asymptotically stable, for the discrete-time dynamics for Corollary 4.2.3, and for the continuous-time dynamics for Theorem 4.3.1. An example of simulation of the discrete-time and continuous-time dynamical systems is reported in Figures 4.6a and 4.6b: as expected, both the dynamics converge to q^* from any starting point $q(0)$.



(a) discrete-time dynamics simulation for various random starting points q_0

(b) continuous-time dynamics simulation for various starting points q_0

Figure 4.6: Best response dynamics simulations in discrete and continuous time for $\gamma < 1$: the dynamics always converges to the unique equilibrium q^*

It is also interesting to notice that the experimental speed of convergence of the discrete-time dynamical system to the equilibrium, as expected from the theory, is exponential. It can be seen in Figure 4.7 that the log distance of the sequence from the equilibrium decreases linearly.

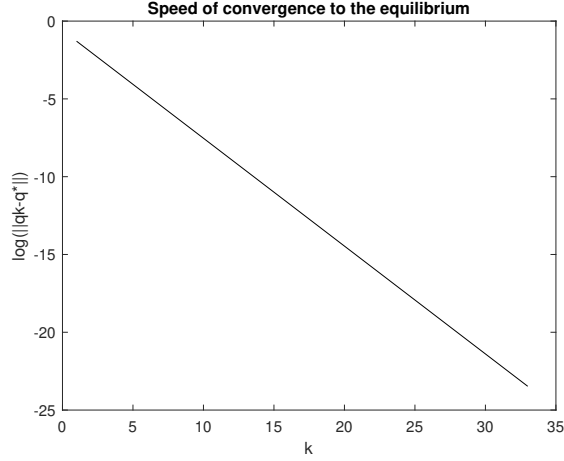


Figure 4.7: Experimental speed of convergence to the equilibrium of the discrete-time dynamics for $\gamma < 1$: linear decreasing of the log distance

- If $\gamma > 1$:

Since the lowest and greatest eigenvalue are equal in modulus, for Propositions 4.2.4 and 4.3.6, the internal equilibrium q_1^* is unstable in both the dynamics. Studying the fixed points of $\mathcal{F}^2$, we find that for the discrete-time dynamical system there are three limit cycles for $\mathcal{F}$ (see also Figure 4.8a):

$$C_1 : (0, 0) \leftrightarrow (\beta, \beta)$$

$$C_2 : (0, \frac{\beta}{1+\gamma}) \leftrightarrow (\frac{\beta}{1+\gamma}, \beta)$$

$$C_3 : (\frac{\beta}{1+\gamma}, 0) \leftrightarrow (\beta, \frac{\beta}{1+\gamma})$$

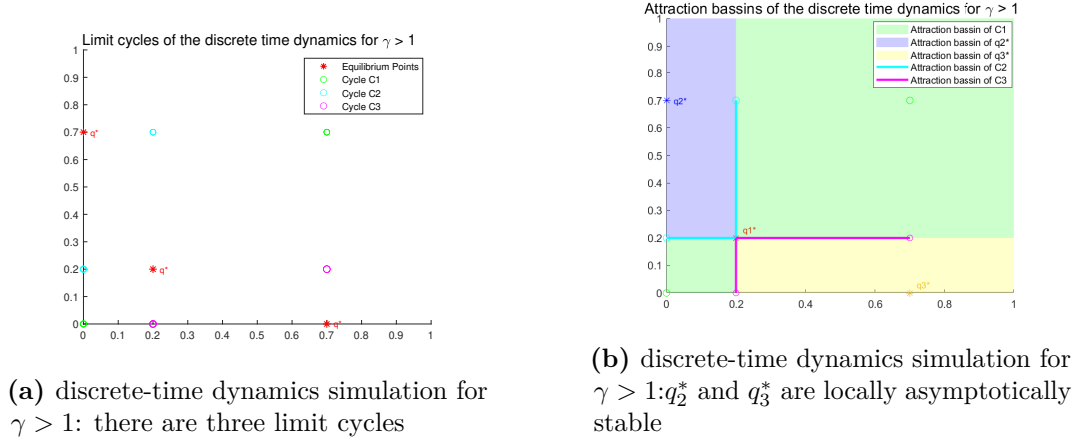


Figure 4.8: Best response dynamics simulations in discrete time for $\gamma > 1$: limit cycles in discrete time and attraction basins in continuous time

The simulations, reported in Figure 4.9b) of the discrete-time dynamical system, show the instability of the central equilibrium q_1^* , and reveal that the specialized equilibria q_2^* and q_3^* are locally asymptotically stable. In particular:

$$q_2^* \text{ attracts } [0, \frac{\beta}{1+\gamma}) \times (\frac{\beta}{1+\gamma}, 1]$$

and

$$q_3^* \text{ attracts } (\frac{\beta}{1+\gamma}, 1] \times [0, \frac{\beta}{1+\gamma})$$

From the simulations it is also possible to understand the limit behavior and attractivity of the limit cycles: it is interesting to notice that the attraction basin of C_1 represents a very large portion of the domain:

$$C_1 \text{ attracts } [0, \frac{\beta}{1+\gamma}) \times [0, \frac{\beta}{1+\gamma}) \cup (\frac{\beta}{1+\gamma}, 1] \times (\frac{\beta}{1+\gamma}, 1]$$

$$C_2 \text{ attracts } [0, \frac{\beta}{1+\gamma}) \times \{q_2 = \frac{\beta}{1+\gamma}\} \cup \{q_1 = \frac{\beta}{1+\gamma}\} \times (\frac{\beta}{1+\gamma}, 1]$$

$$C_3 \text{ attracts } \{q_1 = \frac{\beta}{1+\gamma}\} \times [0, \frac{\beta}{1+\gamma}) \cup (\frac{\beta}{1+\gamma}, 1] \times \{q_2 = \frac{\beta}{1+\gamma}\}$$

In the continuous-time framework, the limit cycles disappear, since the best response function is not synchronous anymore. This loss of synchronicity pushes again the system to a different behavior with respect to the discrete-time dynamics; the results of the simulations are reported in Figure 4.9a: as a consequence of the disappearing of the limit cycles, the attraction basin of the equilibria are larger:

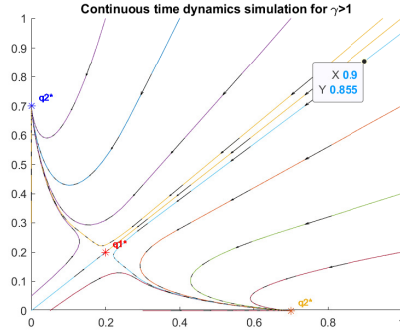
$$q_1^* \text{ attracts } \{(q_1, q_2) \in [0, 1]^2 : q_1 = q_2\}$$

thus proving its instability also in continuous time. While

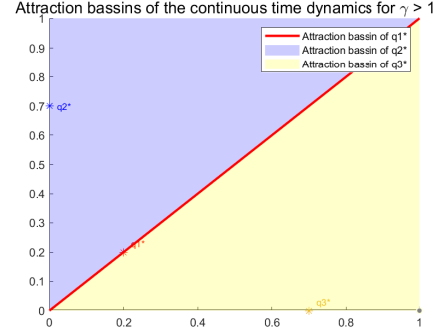
$$q_2^* \text{ attracts } \{(q_1, q_2) : q_1 \in [0, 1], q_2 > q_1\}$$

$$q_3^* \text{ attracts } \{(q_1, q_2) : q_1 \in [0, 1], q_2 < q_1\}$$

which hints the local asymptotical stability of the specialized equilibria q_2^* and q_3^* . These attraction basin also define three invariant sets of the dynamical system.



(a) Continuous-time dynamics simulation for $\gamma > 1$: there are no limit cycles



(b) Continuous-time dynamics simulation for $\gamma > 1$: q_2^* and q_3^* are locally asymptotically stable and q_1^* attracts the bisector

Figure 4.9: Best response dynamics simulations in continuous time for $\gamma > 1$: the internal equilibrium is unstable and the dynamics converges to the specialized equilibria

Chapter 5

Efficiency analysis and interventions

In this chapter, we introduce two efficiency metrics that may help evaluate the performance of action profiles of the game: the total aggregate welfare and the total aggregate public good provision.

We also discuss two potential avenues for modifying the game so that equilibrium profiles change according to some criterion. In the first case, one may consider using subsidies to shift equilibria toward higher total public good provision or a larger number of contributors. In the second case, a possible approach involving income redistribution suggests to modify the income vector in such a way that the unique equilibrium for the game coincides with the one maximizing total aggregate welfare. These approaches draw on the setting proposed by [18].

5.1 Efficiency metrics

It may be worth evaluating the efficiency of a strategic configuration of the game with respect to two main aspects: the satisfaction of agents and the amount of provided public good, when playing that action.

For the game $\mathcal{P} = (\mathcal{G}, w, \eta)$, the total aggregate welfare of a public good provision profile $q \in \chi_w$ is defined as

$$U(q) = \sum_{i=1}^n u_i(q)$$

According to this definition, an action $q^u \in \mathbb{R}_{\geq 0}^n$ is said to be a social optimum for the game $\mathcal{P}$ if it maximizes the total aggregate welfare U on the action space χ_w , i.e.,

$$q^u \in \operatorname{argmax}_{q \in \chi_w} U(q)$$

The total aggregate public good provision of a public good profile $q \in \chi_w$ for the

game $\mathcal{P} = (\mathcal{G}, w, \eta)$ is defined as

$$Q(q) = \sum_{i=1}^n q_i$$

Thus $U(q)$ measures the total utility of agents in the game when playing q , while $Q(q)$ is the sum of public good provisions of each agent in the game. One could be interested in measuring the goodness of a public good provision vector q in terms of both metrics: the first one evaluates how much the agents in the game are satisfied when playing q , while the second one focuses merely on the total public good provided, which could be more useful in terms of reaching a goal total provision.

We now consider the particular case of the game with Cobb-Douglas value functions $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$, with $\eta_i(w_i - q_i, q_i + (Gq)_i) = (w_i - q_i)^{\alpha_i} (q_i + (Gq)_i)^{\beta_i}$ and $\alpha_i, \beta_i \in (0, 1)$.

Even when there exist social optimum vectors, there is no guarantee that they will be Nash equilibria for the game. In the next sections, we will see how to modify the game in such a way that a desired social optimum is a Nash equilibrium.

5.2 Subsidies

In general, the Nash equilibria of the game are not guaranteed to be good in terms of the predefined metrics. When aiming to increase the total public good provision at equilibrium of the agents involved in the game or to increase the number of contributing agents at equilibrium, the following approach can be adopted.

Let us define a new game based on a strategy of subsidies that encourages agents to provide a larger amount of public good. Define, for every agent i , $P_q^i \in (0, 1]$, representing the incentive to her public good provision q_i . The constraint $x_i + q_i = w_i$ becomes $x_i + P_q^i q_i = w_i$. Then, writing the private good in terms of the public good, $x_i = w_i - P_q^i q_i$ the utility maximization can be reformulated as

$$\max_{0 \leq q_i \leq w_i^s} u_i^s(q) = \max_{0 \leq q_i \leq \frac{w_i}{P_q^i}} (w_i - P_q^i q_i)^{\alpha_i} (q_i + (Gq)_i)^{\beta_i}$$

where $u_i^s(q) = (w_i - P_q^i q_i)^{\alpha_i} (q_i + (Gq)_i)^{\beta_i}$ is the utility and $w_i^s = \frac{w_i}{P_q^i} > w_i$ is the maximum amount of public good in the subsidized game. The action space is $\chi_w^s = [0, w_1^s] \times \cdots \times [0, w_n^s]$

Proposition 5.2.1. *The subsidized game $\mathcal{P}_s = (\mathcal{G}, w^s, \eta^s(\alpha, \beta))$, with $w_i^s = \frac{w_i}{P_q^i}$ and value functions*

$$\eta_i^s(w_i^s - q_i, q_i + (Gq)_i) = (P_q^i (w_i^s - q_i))^{\alpha_i} (q_i + (Gq)_i)^{\beta_i}$$

*with $\alpha_i, \beta_i \in (0, 1)$ is a public good game. Moreover, the value functions of $\mathcal{P}_s$ satisfy property **L** with*

$$\bar{k}_i = \frac{\alpha_i}{\alpha_i + \beta_i}$$

Proof. The proof follows directly from the fact that the value functions of the subsidized game correspond to the Cobb-Douglas value functions of the original game, rescaled times a factor $(P_q^i)^{\alpha_i} > 0$ and with new income vector with components $w_i^s = \frac{w_i}{P_q^i}$:

$$\begin{aligned}\eta_i^s(w_i^s - q_i, q_i + (Gq)_i) &= (P_q^i(w_i^s - q_i))^{\alpha_i} (q_i + (Gq)_i)^{\beta_i} \\ &= (P_q^i)^{\alpha_i} (w_i - q_i)^{\alpha_i} (q_i + (Gq)_i)^{\beta_i} = (P_q^i)^{\alpha_i} \eta_i(w_i - q_i, q_i + (Gq)_i)\end{aligned}$$

Since the rescaling of the income vector and of the value functions does not affect the fulfillment of the game hypothesis, the subsidized game with value functions η_i^s is still a public good game.

To derive the relation for k_i , recall that for every i it is the unique solution, for a given $z > 0$, to:

$$\eta_i^s(k_i(z), z - k_i(z)) = \max_{0 \leq x \leq z} \eta_i^s(x, z - x)$$

Then, $k_i(z)$ is obtained by solving

$$\frac{d\eta_i}{dx}(x, z - x) = (P_q^i)^{\alpha_i} \alpha_i x^{\alpha_i - 1} (z - x)^{\beta_i} - (P_q^i)^{\alpha_i} \beta_i x^{\alpha_i} (z - x)^{\beta_i - 1} = 0$$

which implies

$$x = k_i(z) = \frac{\alpha_i z}{\alpha_i + \beta_i}$$

□

In the case of the subsidized game $\mathcal{P}_s$, $\bar{G} = [\frac{\alpha}{\alpha + \beta}]G$ as in $\mathcal{P}$, while $b_i^s = w_i^s(1 - \bar{k}_i) = \frac{\beta_i w_i}{(\alpha_i + \beta_i)P_q^i}$. Thus, from Proposition 3.2.6, the best response $\mathcal{B}_i^s : \chi_{-i}^s \rightarrow \chi_i^s$ is uniquely determined for every agent i and has the form:

$$\mathcal{B}_i(q_{-i}) = [\frac{\beta_i w_i}{(\alpha_i + \beta_i)P_q^i} - \frac{\alpha_i}{\alpha_i + \beta_i} (Gq)_i]_+$$

Denote $P_q = (P_q^i)_{i=1}^n$. As before, to simplify the notation, we use for operations on the vectors of constants $\alpha, \beta, w, P_q, P_x$ the scalar notation, referring to the operations as applied component-wise, for example $\frac{\beta w}{(\alpha + \beta)P_q} = (\frac{\beta_i w_i}{(\alpha_i + \beta_i)P_q^i})_{i=1}^n$. The synchronous best response vectorial function is then $\mathcal{F}(q) : \chi_w^s \rightarrow \chi_w^s$:

$$\mathcal{F}^s(q) = [\frac{\beta w}{(\alpha + \beta)P_q} - \frac{\alpha}{\alpha + \beta} Gq]_+$$

The characterization of the set of Nash equilibria is obtained from Proposition 3.3.2: $q^* \in \chi_w$ is a Nash equilibrium if and only if

$$q^{*T} (I + [\frac{\alpha}{\alpha + \beta}]G) q^* = (\frac{\beta w}{(\alpha + \beta)P_q})^T q^*$$

and

$$q^* + [\frac{\alpha}{\alpha + \beta}]Gq^* \geq \frac{\beta w}{(\alpha + \beta)P_q}$$

An equivalent characterization stressing the partition into contributing and non-contributing agents is given by Proposition 3.3.3: $q^* \in \chi_w^s$ is a Nash equilibrium if and only if there exists a partition of the agents $\mathcal{N} = C \cup -C$, C non empty, such that $q^*|_C = q_C$ and $q^*|_{-C} = 0$ and q_C satisfies

$$(I + [\frac{\alpha}{\alpha + \beta}]_C G_C) q_C = (\frac{\beta w}{(\alpha + \beta) P_q})_C$$

and

$$[\frac{\alpha}{\alpha + \beta}]_{-C} G_{-C,C} q_C \geq (\frac{\beta w}{(\alpha + \beta) P_q})_{-C}$$

It is clear then that the set of the equilibria of the subsidized game $\mathcal{P}_s$ is different from those of the original $\mathcal{P}$. In particular, in the subsidized game contributors provide a higher amount of public good at equilibrium thanks to the incentive $P_q^i \leq 1$. Let us see an example that highlights this increment.

Example 5.2.1. Consider, for example, the simple case of the complete unweighted graph of n nodes with agents sharing equal income $w_i = w$ and constants $\alpha_i = \alpha, \beta_i = \beta$. The unique Nash equilibrium q^* in the original game formulation consists of n equally contributing nodes, providing an amount of public good $q_i^* = \frac{\beta w}{\beta + \alpha n}$. Simply adding equal subsidies $P_q^i = P_q < 1$ to the provision of public good, the unique Nash equilibrium is shifted to q^{s*} such that $q_i^{s*} = \frac{\beta w}{(\beta + \alpha n) P_q}$. Since $P_q < 1$, $q_i^* < q_i^{s*}$, so for every agent the amount of public good at equilibrium is higher. Thus, $Q(q^*) = n \frac{\beta w}{\beta + \alpha n}$ and $Q(q^{s*}) = n \frac{\beta w}{(\beta + \alpha n) P_q}$, which implies that the total aggregate public good provision increased of a positive amount: $Q(q^{s*}) - Q(q^*) = n \frac{\beta w(1 - P_q)}{(\beta + \alpha n) P_q} > 0$

Subsidies can also be used to incentivize free-riding individuals to provide a positive amount of public good. Consider, for example, the case in which the unique equilibrium profile of the game is specialized. In some contexts, it may be preferable for the game to converge to an internal equilibrium, in which all agents contribute to the public good; to achieve such an equilibrium, a strategy could be subsidizing non-contributors. Let us see some examples.

Example 5.2.2. Consider the public good game defined on the undirected unweighted line graph of 3 nodes. Take $\alpha = (\frac{1}{4}, \frac{1}{3}, \frac{1}{6})^T, \beta = (\frac{1}{5}, \frac{1}{2}, \frac{1}{4})^T$ and income vector $w = (3, 2, 3)^T$. Then, the unique Nash equilibrium is the specialized profile $q^* = (1.33, 0, 1.8)^T$. How to subsidize node 2 to make it a contributor? A strategy could be the following. For the characterizing equation (3.18) of non-contributors, $2 \in -C$ since

$$\frac{\alpha_2}{\alpha_2 + \beta_2} G_{2,C} q_C \geq \frac{\beta_2 w_2}{\alpha_2 + \beta_2}$$

making $G_{2,C}$ explicit, we get

$$\frac{\alpha_2}{\alpha_2 + \beta_2} (G_{21} q_1^* + G_{23} q_3^*) \geq \frac{\beta_2 w_2}{\alpha_2 + \beta_2}$$

Thus, in the subsidized game we could take P_q^2 such that the best response of node 2

for q_{-2}^* is non zero, which means

$$\left[\frac{\beta_2 w_2}{(\alpha_2 + \beta_2) P_q^2} - \frac{\alpha_2}{\alpha_2 + \beta_2} (Gq^*)_2 \right]_+ > 0$$

doing the computations, we get the following condition for P_q^2 :

$$P_q^2 < \frac{\beta_2 w_2}{\alpha_2 (Gq^*)_2}$$

Substituting the values of the equilibrium of the non subsidized game, we get $P_q^2 < 0.96$. Thus, we can define the corresponding subsidized game, i.e., with same α, β, w , and $P_q = (1, 0.95, 1)^T$; the unique Nash equilibrium is now

$$q_s^* = (1.32, 0.02, 1.79)^T$$

which is very close to the previous q^* , but node 2 contributes. Increasing the subsidy, that is choosing a smaller P_q^2 , for example, $P_q^2 = (1, 0.7, 1)^T$, in the unique equilibrium $q_s^* = (0.92, 0.75, 1.5)^T$ also the provision of node 2 increases.

Example 5.2.3. Consider the public good game defined on the star graph of 5 nodes. Take $\alpha = (\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T, \beta = (\frac{2}{5}, \frac{5}{8}, \frac{3}{8}, \frac{5}{8}, \frac{3}{8})^T$ and income vector $w = (5, 3, 2, 3, 2)^T$. Then, the unique Nash equilibrium is the specialized profile $q^* = (0, 2.14, 1.2, 2.14, 1.2)^T$. To subsidize node 1 to make it a contributor, a strategy could be the following. For the characterizing equation (3.18) of non-contributors, $1 \in -C$ since

$$\frac{\alpha_1}{\alpha_1 + \beta_1} G_{1,C} q_C \geq \frac{\beta_1 w_1}{\alpha_1 + \beta_1}$$

Thus, in the subsidized game we could take P_q^1 such that the best response of node 1 for q_{-1}^* is non zero, which means

$$\left[\frac{\beta_1 w_1}{(\alpha_1 + \beta_1) P_q^1} - \frac{\alpha_1}{\alpha_1 + \beta_1} (Gq^*)_1 \right]_+ > 0$$

doing the computations, we get the following condition for P_q^1 :

$$P_q^1 < \frac{\beta_1 w_1}{\alpha_1 (Gq^*)_1}$$

Substituting the values of the equilibrium of the non subsidized game, we get $P_q^1 < 0.67$. Thus, we can define the corresponding subsidized game with $P_q = (0.66, 1, 1, 1, 1)^T$; the unique Nash equilibrium is now

$$q_s^* = (0.09, 2.12, 1.16, 2.12, 1.16)^T$$

in which node 1 contributes.

The previous examples hint that subsidizing non-contributors could move the game to a unique internal equilibrium in the case of one free-riding agent. It is

interesting to see what happens if there are more than one non-contributing nodes in the equilibrium profile of the non-subsidized game.

Example 5.2.4. Consider the public good game defined on the undirected circle of 4 nodes. Take $\alpha = (\frac{2}{2}, \frac{1}{4}, \frac{4}{5}, \frac{1}{3})^T$, $\beta = (\frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{2}{5})^T$ and income vector $w = (2, 1, 2, 1)^T$. Then, the unique Nash equilibrium is the specialized profile $q^* = (0, 0.67, 0, 0.55)^T$. How to choose P_q^1 and P_q^3 such that nodes 1 and 3 contribute? Let us follow the strategy of the previous examples and derive a bound for the subsidies of non-contributors, to make their best response for the previous equilibrium non zero. For node 1 we get

$$P_q^1 < \frac{\beta_1 w_1}{\alpha_1 (Gq^*)_1} = 0.82$$

while for node 3

$$P_q^3 < \frac{\beta_3 w_3}{\alpha_3 (Gq^*)_3} = 0.52$$

Replicating the approach of the previous examples, we can define the corresponding subsidized game with $P_q = (0.8, 1, 0.5, 1)^T$. However, for this game, the unique equilibrium is $q_s^* = (0.74, 0.14, 0.85, 0)^T$, thus the strategy of subsidizing all contributors is not efficient in this case, since it shifts too much the game equilibrium.

Let us try to only incentivize one of the two non-contributors. Subsidizing node 1, i.e., taking, $P_q = (0.8, 1, 1, 1)^T$, in the new equilibrium $q_s^* = (0.05, 0.65, 0, 0.51)^T$, node 1 contributes while agent 3 still does not. On the other hand, providing a subsidy only to node 3, i.e., taking $P_q = (1, 1, 0.5, 1)^T$, the new game equilibrium $q_s^* = (0.33, 0.37, 0.57, 0.14)^T$ is internal.

This example suggests that subsidizing only "the most non contributor", that is the agent who requires a higher incentive not to free-ride, may be enough to make everyone contribute.

Example 5.2.5. Consider the public good game defined on the undirected line of 5 nodes. Take $\alpha = (\frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{2}{5}, \frac{3}{4})^T$, $\beta = (\frac{1}{5}, \frac{1}{2}, \frac{1}{4}, \frac{2}{3}, \frac{1}{3})^T$ and income vector $w = (2, 3, 2, 3, 2)^T$. Then, the unique Nash equilibrium is the specialized profile $q^* = (0, 1.8, 0, 1.88, 0)^T$. How to choose P_q^1 , P_q^3 and P_q^5 such that nodes 1, 3, 5 contribute? Let us follow the strategy of the previous examples and derive a bound for the subsidies of non-contributors, to make their best response for the previous equilibrium non zero. For node 1 we get

$$P_q^1 < \frac{\beta_1 w_1}{\alpha_1 (Gq^*)_1} = 0.89$$

for node 3

$$P_q^3 < \frac{\beta_3 w_3}{\alpha_3 (Gq^*)_3} = 0.82$$

for node 5

$$P_q^5 < \frac{\beta_5 w_5}{\alpha_5 (Gq^*)_5} = 0.47$$

Let us see what is the outcome for different strategies of incentives: take $P_q^i = \text{bound} - 0.05$

- subsidizing agent 1: $P_q = (0.84, 1, 1, 1, 1)^T$ gives

$$q_s^* = (0.07, 1.77, 0, 1.88, 0)^T$$

- subsidizing agent 3: $P_q = (1, 1, 0.77, 1, 1)^T$ gives

$$q_s^* = (0, 1.75, 0.13, 1.83, 0)^T$$

- subsidizing agent 5: $P_q = (1, 1, 1, 1, 0.42)^T$ gives

$$q_s^* = (0, 1.8, 0, 1.8, 0.23)^T$$

- subsidizing agents 1, 3: $P_q = (0.84, 1, 0.77, 1, 1)^T$ gives

$$q_s^* = (0.12, 1.69, 0.16, 1.82, 0)^T$$

- subsidizing agents 1, 5: $P_q = (0.84, 1, 1, 1, 0.42)^T$ gives

$$q_s^* = (0.07, 1.77, 0, 1.79, 0.23)^T$$

- subsidizing agents 3, 5: $P_q = (1, 1, 0.77, 1, 0.42)^T$ gives

$$q_s^* = (0, 1.7, 1.19, 1.69, 0.29)^T$$

- subsidizing all agents 1, 3, 5: $P_q = (0.84, 1, 0.77, 1, 0.42)^T$ gives the internal equilibrium

$$q_s^* = (0.14, 1.65, 0.23, 1.68, 0.31)^T$$

On the contrary, in the last example providing an incentive only to some non-contributors is not enough, and all free-riders must be subsidized to obtain the internal equilibrium.

Thus, the naive strategies experimented in the examples do not provide a generalizable way to shift the unique specialized equilibrium of a game to an internal one through incentives to non-contributing nodes. The difficulty probably lies in the high dependency of the equilibrium profiles on the network of interconnections, as well as on the different parameters α_i, β_i .

Proposition 5.2.2. *For the subsidized game $\mathcal{P}_s$, at equilibrium, for every contributing agent $i \in C$ the ratio of private provision and aggregate public good consumption is linear if the incentive P_q^i :*

$$\frac{x_i^s}{q_i^s + (Gq^s)_i} = \frac{\alpha_i}{\beta_i} P_q^i$$

Proof. Every contributing agent $i \in C$ at equilibrium provides as amount of public

good $q_i^s > 0$ that is an internal maximum of his utility, which means that it holds

$$\frac{\partial u_i^s}{\partial q_i}(q^s) = 0$$

Since

$$\frac{\partial u_i^s}{\partial q_i}(q) = (w_i - P_q^i q_i)^{\alpha_i - 1} (q_i + (Gq)_i)^{\beta_i - 1} (-\alpha_i P_q^i (q_i + (Gq)_i) + \beta_i (w_i - P_q^i q_i))$$

It is equivalent to

$$\alpha_i P_q^i (q_i^s + (Gq^s)_i) = \beta_i (w_i - q_i^s P_q^i)$$

Recalling that the amount of private good in the subsidized game is $x_i^s = (w_i - q_i^s P_q^i)$, we get that the condition for contributing agents to be at equilibrium is exactly

$$\frac{x_i^s}{q_i^s + (Gq^s)_i} = \frac{\alpha_i}{\beta_i} P_q^i$$

□

Let us now see how this ratio is related to the social optimum for the non subsidized game.

Proposition 5.2.3. q^u is the social optimum of the non-subsidized game $\mathcal{P}$, if and only if the ratio of aggregate public allocation and private provision for contributing nodes $i \in C$ satisfies

$$\frac{q_i^u + (Gq^u)_i}{x_i^u} = M_i \frac{\beta_i}{\alpha_i}$$

where

$$M_i = 1 + \sum_{j=1}^n G_{j,i} \frac{\beta_j}{\beta_i} \frac{u_j(q^u)/(q_j^u + (Gq^u)_j)}{u_i(q^u)/(q_i^u + (Gq^u)_i)}$$

Proof. The social optimum maximizes the total aggregate welfare $U(q)$. This is equivalent for every $i \in C$ to:

$$\frac{\partial U}{\partial q_i}(q^u) = 0$$

Doing the computations

$$\frac{\partial U}{\partial q_i}(q) = \frac{\partial}{\partial q_i} \sum_{j=1}^n u_j(q) = \sum_{j \neq i} \frac{\partial u_j(q)}{\partial q_i} + \frac{\partial u_i(q)}{\partial q_i} = 0$$

It is equivalent to

$$\begin{aligned} & (w_i - q_i)^{\alpha_i - 1} (q_i + (Gq)_i)^{\beta_i - 1} (-\alpha_i (q_i + (Gq)_i) + \beta_i (w_i - q_i)) \\ &= \sum_{j=1}^n G_{j,i} \beta_j (w_j - q_j)^{\alpha_j} (q_j + (Gq)_j)^{\beta_j - 1} \end{aligned}$$

substituting the expression of the utility $u_i(q) = (w_i - q_i)^{\alpha_i} (q_i + (Gq)_i)^{\beta_i}$ and

$x_i = w_i - q_i$, we get:

$$-\alpha_i(q_i + (Gq)_i) + \beta_i x_i = \sum_{j=1}^n G_{j,i} \beta_j \frac{u_j(q^u)/(q_j^u + (Gq^u)_j)}{u_i(q^u)/(q_i^u + (Gq^u)_i)}$$

rearranging the terms, we get

$$-\frac{\alpha_i (q_i + (Gq)_i)}{\beta_i x_i} + 1 = \sum_{j=1}^n G_{j,i} \frac{\beta_j u_j(q^u)/(q_j^u + (Gq^u)_j)}{\beta_i u_i(q^u)/(q_i^u + (Gq^u)_i)}$$

from which we obtain the thesis. $\square$

Proposition 5.2.4. *If $q^u > 0$ social optimum of the non subsidized game $\mathcal{P}$, then $q^{s*} = q^u$ Nash equilibrium of the subsidized game with incentives:*

$$P_q^i = \frac{\beta_i w_i}{q_i^u(\alpha_i + \beta_i) + \alpha_i(Gq^u)_i}$$

Proof. Given $q^u \in \chi_w$ social optimum of the non subsidized game $\mathcal{P}$, we want that, for every agent i in the subsidized game, q_i^u is her best response. For every agent i , playing $q_i^u > 0$ as best response for q_{-i}^u in the subsidized game is equivalent to

$$q_i^u = [\frac{\beta_i w_i}{(\alpha_i + \beta_i)P_q^i} - \frac{\alpha_i}{\alpha_i + \beta_i}(Gq^u)_i]_+ = \frac{\beta_i w_i}{(\alpha_i + \beta_i)P_q^i} - \frac{\alpha_i}{\alpha_i + \beta_i}(Gq^u)_i > 0$$

From this relation we derive the form of P_q^i that makes agent i provide exactly q_i^u :

$$P_q^i(\alpha_i + \beta_i)q_i^u = \beta_i w_i - \alpha_i(Gq^u)_i P_q^i$$

which is equivalent to

$$P_q^i(q_i^u(\alpha_i + \beta_i) + \alpha_i(Gq^u)_i) = \beta_i w_i$$

from which we get the thesis. $\square$

Example 5.2.6. *Consider the simple case of the non-subsidized game $\mathcal{P} = (\mathcal{G}, w, \eta)$ defined on the unweighted undirected line of two nodes. The adjacency matrix is $G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Take $\alpha = (\frac{1}{4}, \frac{1}{3})^T, \beta = (\frac{1}{5}, \frac{1}{2})$ and income vector $w = (2, 3)^T$. The condition for the contractivity of the best response equilibrium is satisfied since $\lambda^{\bar{G}} = 0.47 < 1$, thus the equilibrium is unique and globally asymptotically stable. For the non subsidized game, it is the specialized profile*

$$q^* = (0, 1.8)^T$$

Let us now consider the aggregate welfare $U(q) = u_1(q) + u_2(q)$. For the unique equilibrium of the non subsidized game the aggregate welfare is

$$U(q^*) = 2.76$$

Plotting U on $\chi_w = [0, 2] \times [0, 3]$, it is evident that it is concave (see Figure 5.1), and thus has a unique maximum, the social optimum

$$q^u = (1.12, 1.68)^T$$

Using Proposition 5.2.4, we define the subsidized game $\mathcal{P}_s = (\mathcal{G}, w^s, \eta^s)$ with incentives

$$P_q^1 = \frac{\beta_1 w_1}{q_1^u(\alpha_1 + \beta_1) + \alpha_1(Gq^u)_1} = \frac{\beta_1 w_1}{q_1^u(\alpha_1 + \beta_1) + \alpha_1 q_2^u} = 0.43$$

and

$$P_q^2 = \frac{\beta_2 w_2}{q_2^u(\alpha_2 + \beta_2) + \alpha_2(Gq^u)_2} = \frac{\beta_2 w_2}{q_2^u(\alpha_2 + \beta_2) + \alpha_2 q_1^u} = 0.85$$

Then, the unique globally asymptotically stable equilibrium of the subsidized game is

$$q_s^* = q^u$$

having aggregate welfare

$$U(q_s^*) = U(q^u) = 3.03$$

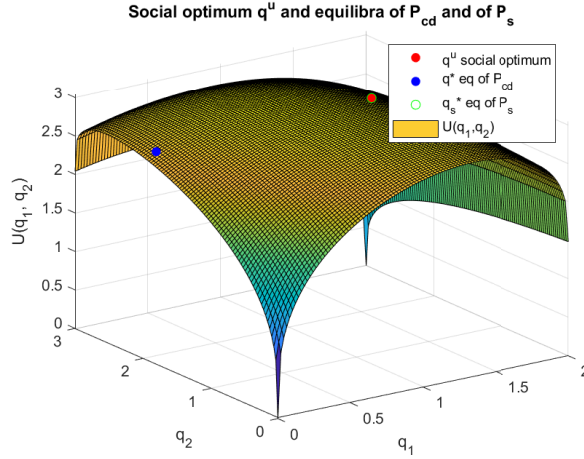


Figure 5.1: Aggregate welfare U for the line of two nodes, with social optimum q^u , equilibrium q^* of $\mathcal{P}$ and q_s^* of $\mathcal{P}_s$

5.3 Redistribution

Consider an external planner with redistributive capacity: given a total amount of income W , she must choose how to split it among nodes to maximize the total aggregate welfare. Since the utilities depend on w , U can be seen as a function of the income vector, $U(w) = \sum_{i=1}^n u_i(w)$. Then, the redistribution problem can be formulated as finding the optimal w^* st

$$w^* \in \operatorname{argmax}_{w \in \mathbb{R}_{>0}^n} U(w)$$

$$st \sum_{i=1}^n w_i = W$$

Assuming that the contractivity condition holds, $\lambda^{\bar{G}} < 1$, we know that for every income vector $w > 0$ there exists a unique globally asymptotically stable equilibrium. Then, if U is concave in w , a unique maximum w^* of the total aggregate welfare exists and choosing as income vector w^* the game globally converges to it. Let us see some simple examples for the game with Cobb-Douglas value functions $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$. We begin with the case of two nodes.

Example 5.3.1. *Consider the redistribution problem on the unweighted complete graph (undirected line) of two nodes, where the contractivity condition is satisfied. Thus, there exists a unique globally asymptotically stable equilibrium for the game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$ for every income vector $w \in \mathbb{R}_{>0}^2$. Given the total amount of income W , we are interested in the form of U as the income vector w varies. Under the constraint $w_1 + w_2 = W$, for the game with two agents, the aggregate welfare actually depends only on w_1 :*

$$U(w) = U(w_1, W - w_1) = u_1(w_1) + u_2(W - w_1)$$

So, for every vector $w = (w_1, W - w_1)$, we compute the unique Nash equilibrium for the game and evaluate the total aggregate welfare on the equilibrium point.

Let us take $W = 5$ and see some examples. Starting with the game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$ in which $\alpha_i = \alpha, \beta_i = \beta$, consider different cases; As $w_1 \in [0, W]$ varies, the unique equilibrium takes different forms. In Figure 5.2, it is possible to see the aggregate welfare U as a function of the income vector, colored depending on the set of contributors at equilibrium.

- For $\alpha = \beta = 0.5$ the maximum of aggregate welfare is $U^* = U(w^*) = 3.42$, reached for the two symmetric and imbalanced income vectors $w_1^* = (1.06, 3.94)^T$ and $w_2^* = (3.94, 1.06)$, where the Nash equilibrium is, respectively, the specialized profile $q_1^* = (0, 1.97)^T$ and $q_2^* = (1.97, 0)^T$.
- In the case $\alpha = 0.2 < \beta = 0.7$, $U^* = U(w^*) = 4.5$ is attained at income vectors $w_1^* = (0.52, 4.48)^T$ and $w_2^* = (4.48, 0.52)^T$, where the Nash equilibrium is, respectively, the specialized $q_1^* = (0, 3.48)^T$ and $q_2^* = (3.48, 0)^T$.
- For $\alpha = 0.8 > \beta = 0.1$, the maximum aggregate welfare is $U^* = U(w^*) = 3.52$, reached for $w_1^* = (1.88, 3.12)^T$ and $w_2^* = (3.12, 1.88)$, where the Nash equilibrium is, respectively, the specialized profile $q_1^* = (0, 0.35)^T$ and $q_2^* = (0.35, 0)^T$.

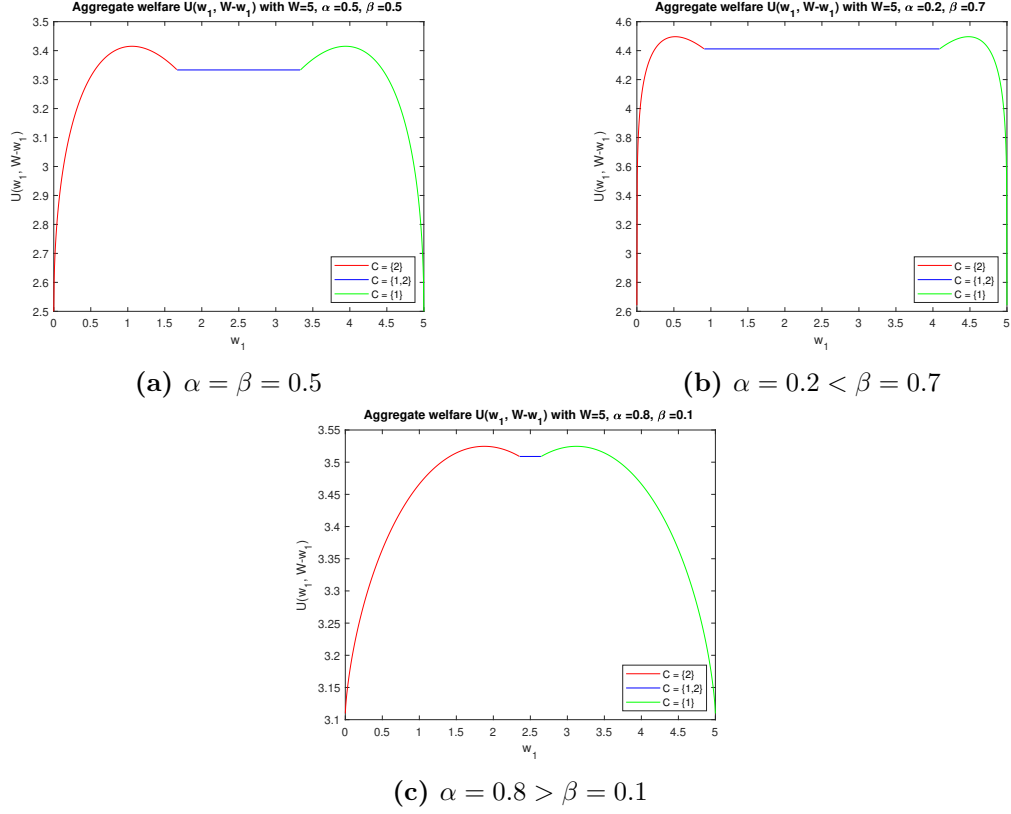


Figure 5.2: Aggregate welfare $U(w)$ as a function of $w_1, W - w_1$ for various values of the parameters α, β in the game $\mathcal{P}$ on the directed line of 2 nodes

A similar behavior is also observed for the game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$ with different values of α_i, β_i for the two nodes. The plot of the aggregate welfare in this more general case is reported in Figure 5.3.

- When $\alpha = (0.25, 0.33)^T, \beta = (0.2, 0.5)^T$, the unique maximum of the aggregate welfare is $U^* = U(q^*) = 3.03$, reached at $w^* = (0.64, 4.36)$, where the Nash equilibrium is the specialized $q^* = (0, 2.62)^T$.
- symmetrically, when $\alpha = (0.33, 0.25)^T, \beta = (0.5, 0.2)^T$, the unique maximum of the aggregate welfare is still $U^* = U(q^*) = 3.03$, reached at $w^* = (4.36, 0.64)$, where the Nash equilibrium is the specialized $q^* = (2.62, 0)^T$.

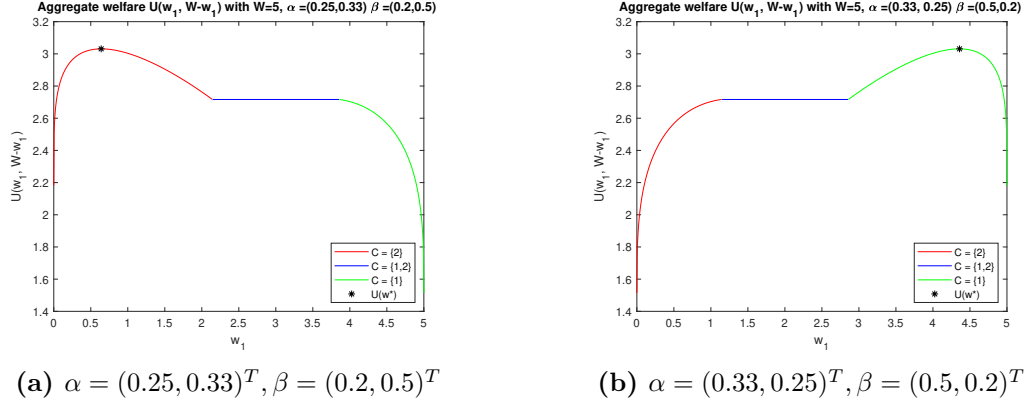


Figure 5.3: Aggregate welfare $U(w)$ as a function of $w_1, W - w_1$ for various values of the parameters α_i, β_i in the game $\mathcal{P}$ on the directed line of 2 nodes

In all the cases, U is concave on each set of contributors, but not globally. Moreover, the maximum of the total aggregate welfare is obtained when only one node contributes, thus when the unique Nash equilibrium is specialized.

Let us now extend the previous example to the complete graph of three nodes.

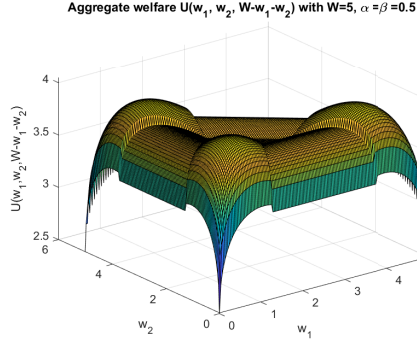
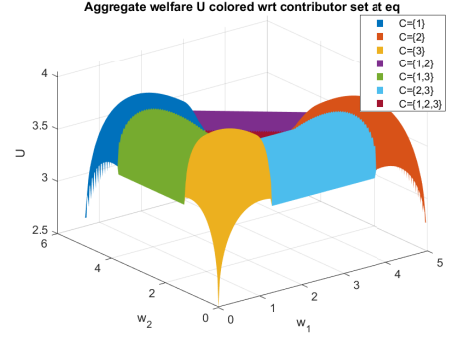
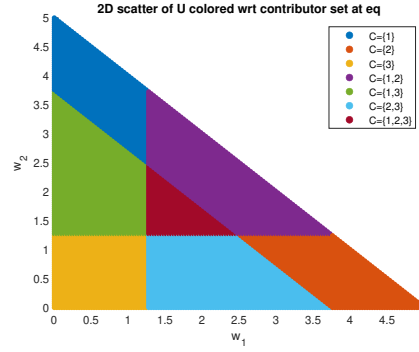
Example 5.3.2. Consider the redistribution problem on the unweighted complete graph of three nodes, where the contractivity condition is satisfied. Thus, there exists a unique globally asymptotically stable equilibrium for the game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$ for every income vector $w \in \mathbb{R}_{>0}^3$. Given the total amount of income W , we are interested in the form of U as the income vector w varies. Under the constraint $w_1 + w_2 + w_3 = W$, for the game with three agents, we have that the aggregate welfare actually depends only on w_1, w_2 :

$$U(w) = U(w_1, w_2, W - w_1 - w_2) = u_1(w_1) + u_2(w_2) + u_3(W - w_1 - w_2)$$

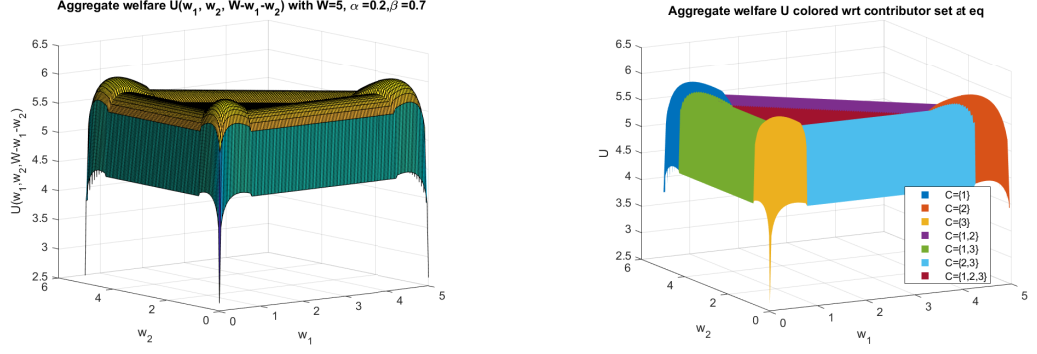
So, for every vector $w = (w_1, w_2, W - w_1 - w_2)$, we compute the unique Nash equilibrium for the game and evaluate the total aggregate welfare on the equilibrium point.

Let us take $W = 5$ and see some examples. Consider first the game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$ in which $\alpha_i = \alpha, \beta_i = \beta$; as $(w_1, w_2) \in [0, W] \times [0, W]$ varies, the unique equilibrium takes different forms. The plots show the aggregate welfare U as a function of the income vector under the constraint.

- In the symmetric case where $\alpha = \beta = 0.5$, reported in Figure 5.4, the unique maximum of the aggregate welfare is $U^* = U(w^*) = 4.04$, reached when the income vector is the imbalanced $w^* = (3.62, 0.70, 0.67)^T$ or one of its 6 perturbations, and the Nash is the specialized profile where the node with highest income is the only contributor and provides $q_i^* = 1.81$, in this case $q^* = (1.81, 0, 0)$.

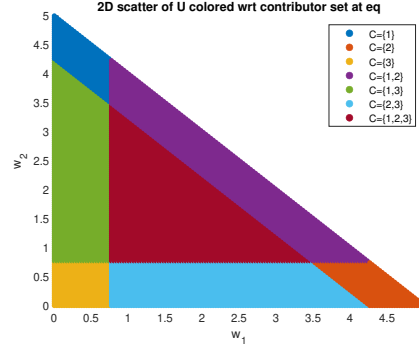

 (a) Aggregate welfare plot for $\alpha = \beta = 0.5$

 (b) Aggregate welfare plot colored wrt contributor sets for $\alpha = \beta = 0.5$

 (c) Aggregate welfare scatter plot colored wrt contributor sets for $\alpha = \beta = 0.5$
Figure 5.4: Aggregate welfare $U(w)$ as a function of $w_1, w_2, W - w_1 - w_2$ for $\alpha = \beta = 0.5$ in the game $\mathcal{P}$ on the directed line of 3 nodes

- In the case $\alpha = 0.2 < \beta = 0.7$, reported in Figure 5.5, the aggregate welfare has maximum value $U^* = U(w^*) = 6.08$, with imbalanced income vector $w^* = (0.34, 0.34, 4.33)^T$ (or one of its 3 perturbations) and Nash the specialized profile $q^* = (0, 0, 3.37)^T$ (or one of its 3 perturbations).



(a) Aggregate welfare plot for $\alpha = 0.2 < \beta = 0.7$

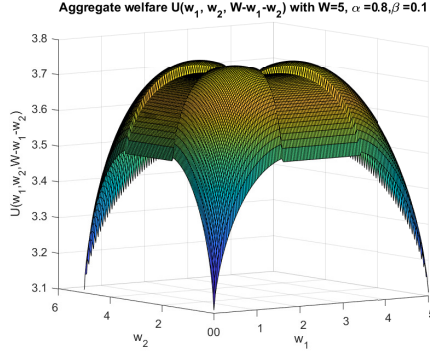
(b) Aggregate welfare plot colored wrt contributor sets for $\alpha = 0.2 < \beta = 0.7$



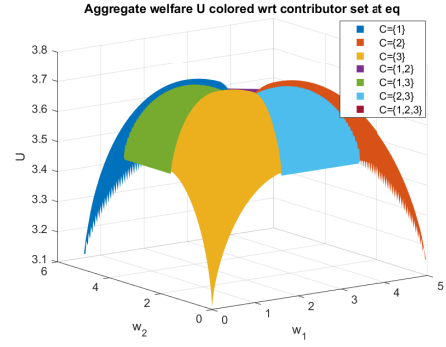
(c) Aggregate welfare scatter plot colored wrt contributor sets for $\alpha = 0.2 < \beta = 0.7$

Figure 5.5: Aggregate welfare $U(w)$ as a function of $w_1, w_2, W - w_1 - w_2$ for $\alpha = 0.2 < \beta = 0.7$ in the game $\mathcal{P}$ on the complete graph of 3 nodes

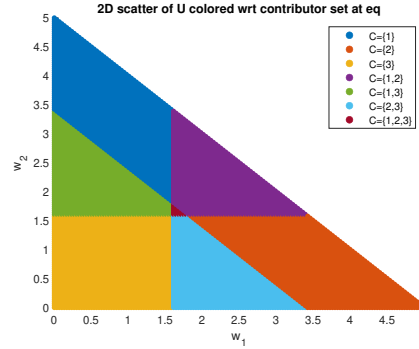
- In the case $\alpha = 0.8 > \beta = 0.1$, reported in Figure 5.6, the maximum of the aggregate welfare $U^* = U(w^*) = 3.77$ is obtained with income vector $w^* = (1.21, 1.21, 2.58)^T$ (or one of its 3 perturbations), where the Nash equilibrium is the specialized $q^* = (0, 0, 0.29)^T$ (or one of its 3 perturbations).



(a) Aggregate welfare plot for $\alpha = 0.8 > \beta = 0.1$



(b) Aggregate welfare plot colored wrt contributor sets for $\alpha = 0.8 > \beta = 0.1$

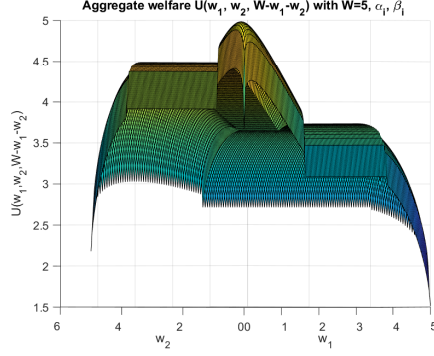


(c) Aggregate welfare scatter plot colored wrt contributor sets for $\alpha = 0.8 > \beta = 0.1$

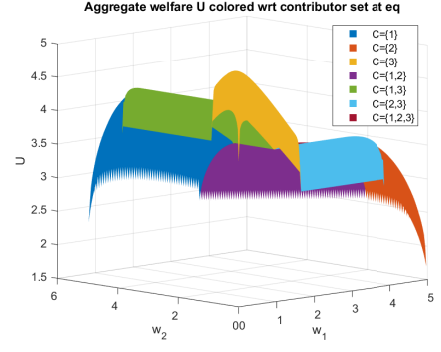
Figure 5.6: Aggregate welfare $U(w)$ as a function of $w_1, w_2, W - w_1 - w_2$ for $\alpha = 0.8 > \beta = 0.1$ in the game $\mathcal{P}$ on the complete graph of 3 nodes

In all the cases, U is concave on each set of contributors, but not globally. Moreover, the maximum of the total aggregate welfare is obtained with a strongly imbalanced income vector, which leads to only one contributing node at equilibrium.

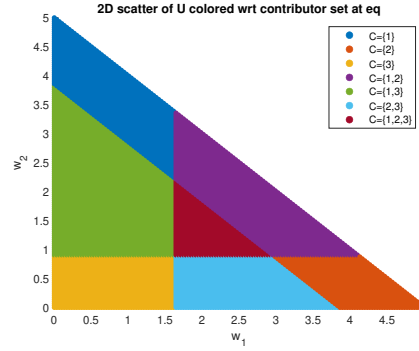
Let us also see what happens in the more general game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$ with different parameters α_i, β_i . Consider, for example, $\alpha = (0.33, 0.25, 0.66)^T; \beta = (0.5, 0.2, 0.75)^T$. The plot of the aggregate welfare is reported in Figure 5.7. In this case the maximum of U is unique and it is $U^* = U(q^*) = 4.99$, obtained with income vector $w^* = (0.30, 0.17, 4.53)^T$, where the unique Nash equilibrium is again specialized, $q^* = (0, 0, 2.4)^T$



(a) Aggregate welfare plot for $\alpha = (0.33, 0.25, 0.66)^T$; $\beta = (0.5, 0.2, 0.75)^T$;



(b) Aggregate welfare plot colored wrt contributor sets for $\alpha = (0.33, 0.25, 0.66)^T$; $\beta = (0.5, 0.2, 0.75)^T$;



(c) Aggregate welfare scatter plot colored wrt contributor sets for $\alpha = (0.33, 0.25, 0.66)^T$; $\beta = (0.5, 0.2, 0.75)^T$;

Figure 5.7: Aggregate welfare $U(w)$ as a function of $w_1, w_2, W - w_1 - w_2$ for $\alpha = (0.33, 0.25, 0.66)^T$; $\beta = (0.5, 0.2, 0.75)^T$; in the game $\mathcal{P}$ on the complete graph of 3 nodes

Let us now see a less symmetric example and consider the undirected line of three nodes.

Example 5.3.3. Consider the redistribution problem on the unweighted undirected line of three nodes, where the contractivity condition is satisfied. Thus, there exists a unique globally asymptotically stable equilibrium for the game $\mathcal{P} = (\mathcal{G}, w, \eta(\alpha, \beta))$ for every income vector $w \in \mathbb{R}_{>0}^3$. Given the total amount of income W , we are interested in the form of U as the income vector w varies. Under the constraint $w_1 + w_2 + w_3 = W$, the aggregate welfare actually depends only on w_1, w_2 :

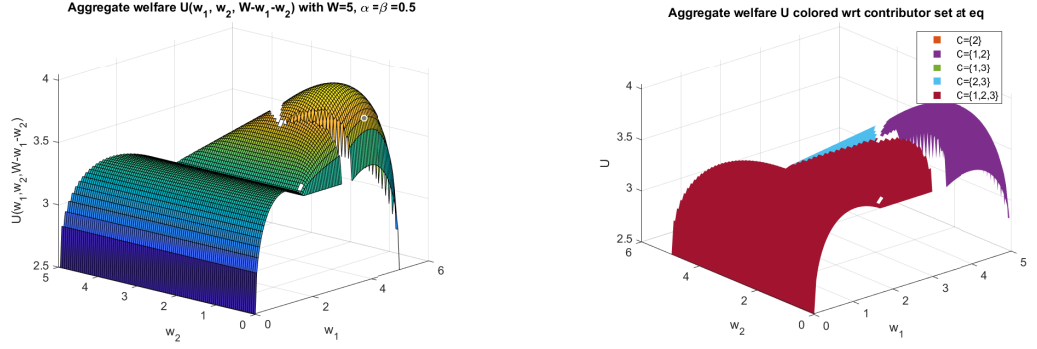
$$U(w) = U(w_1, w_2, W - w_1 - w_2) = u_1(w_1) + u_2(w_2) + u_3(W - w_1 - w_2)$$

So, for every vector $w = (w_1, w_2, W - w_1 - w_2)$, we compute the unique Nash equilibrium for the game and evaluate the total aggregate welfare on the equilibrium point.

Let us take $W = 5$ and see some examples. Consider the game $\mathcal{P} = (\mathcal{G}, w, \eta)$ in which $\alpha_i = \alpha, \beta_i = \beta$; as $(w_1, w_2) \in [0, W] \times [0, W]$ varies, the unique equilibrium takes different forms. The plots report the aggregate welfare U as a function of the

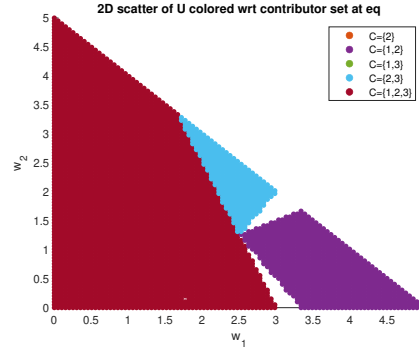
income vector under the constraint.

- In the symmetric case where $\alpha = \beta = 0.5$, reported in Figure 5.8, the maximum of aggregate welfare is $U^* = U(w^*) = 4.04$, reached for $w^* = (0.71, 3.59, 0.71)^T$ where the Nash equilibrium is the specialized $q^* = (0, 1.79, 0)^T$.



(a) Aggregate welfare plot for $\alpha = \beta = 0.5$

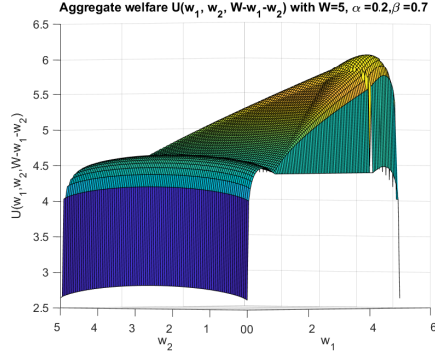
(b) Aggregate welfare plot colored wrt contributor sets for $\alpha = \beta = 0.5$



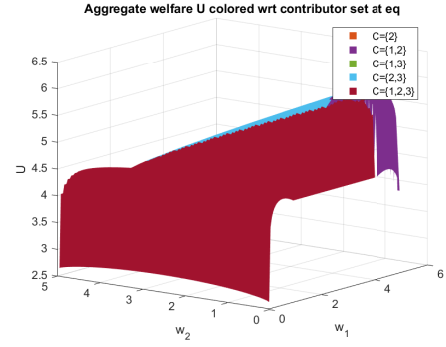
(c) Aggregate welfare scatter plot colored wrt contributor sets for $\alpha = \beta = 0.5$

Figure 5.8: Aggregate welfare $U(w)$ as a function of $w_1, w_2, W - w_1 - w_2$ for $\alpha = \beta = 0.5$ in the game $\mathcal{P}$ on the undirected line of 3 nodes

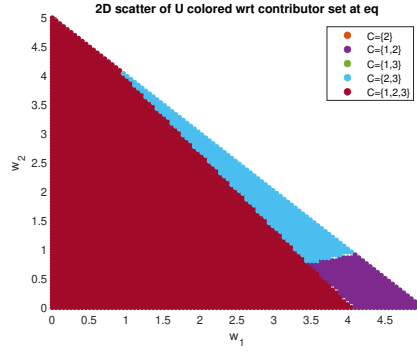
- In the case $\alpha = 0.2 < \beta = 0.7$, reported in Figure 5.9, the maximum of the aggregate welfare is $U^* = U(w^*) = 6.08$, reached for $w^* = (0.35, 4.29, 0.35)^T$ where the Nash equilibrium is the specialized $q^* = (0, 3.34, 0)^T$.



(a) Aggregate welfare plot for $\alpha = 0.2 < \beta = 0.7$



(b) Aggregate welfare plot colored wrt contributor sets for $\alpha = 0.2 < \beta = 0.7$



(c) Aggregate welfare scatter plot colored wrt contributor sets for $\alpha = 0.2 < \beta = 0.7$

Figure 5.9: Aggregate welfare $U(w)$ as a function of $w_1, w_2, W - w_1 - w_2$ for $\alpha = 0.2 < \beta = 0.7$ in the game $\mathcal{P}$ on the undirected line of 3 nodes

- In the case $\alpha = 0.8 > \beta = 0.1$, reported in Figure 5.10, the maximum of the aggregate welfare is $U^* = U(w^*) = 3.72$, reached for $w^* = (1.58, 1.81, 1.61)^T$ where the Nash equilibrium is the internal $q^* = (0.001, 0.2, 0.01)^T$.

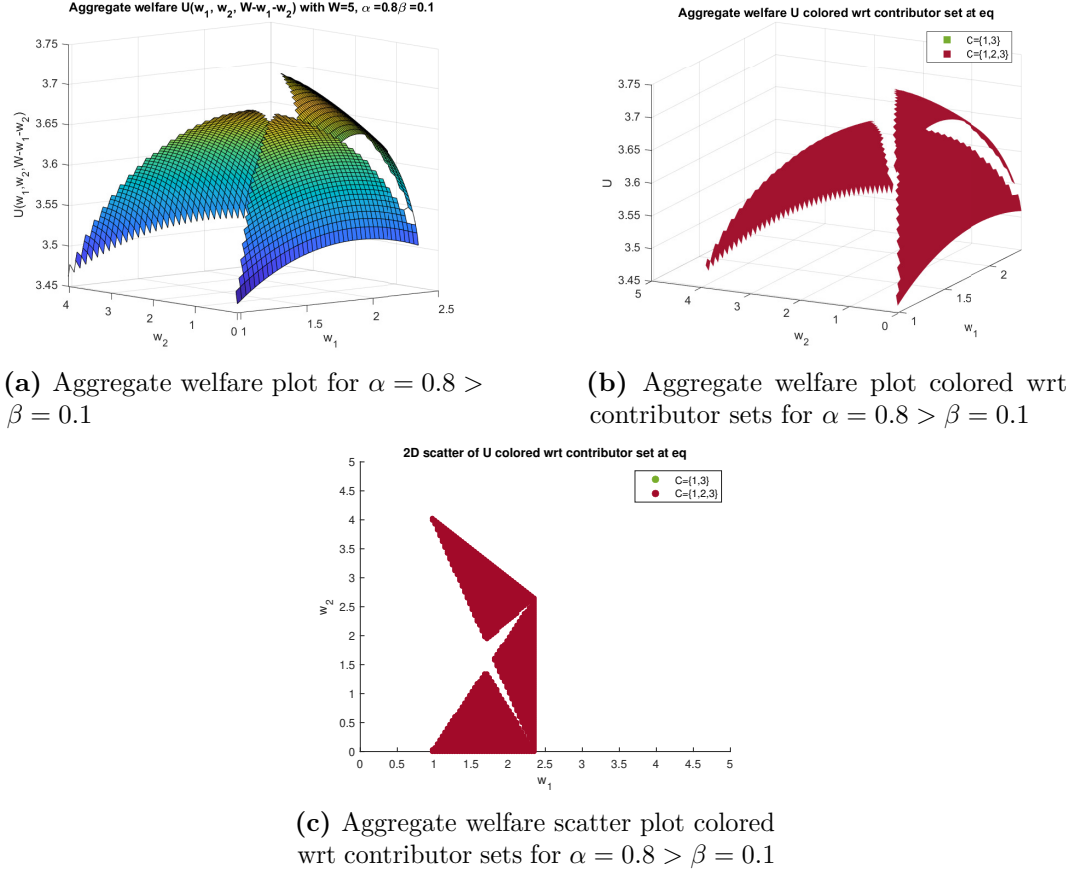


Figure 5.10: Aggregate welfare $U(w)$ as a function of $w_1, w_2, W - w_1 - w_2$ for $\alpha = 0.8 > \beta = 0.1$ in the game $\mathcal{P}$ on the undirected line of 3 nodes

In the first two cases, where $\alpha \leq \beta$, the maximum welfare at equilibrium is obtained with an imbalanced income vector w^* , that pushes the node with highest income to be the only contributor at equilibrium. In the third case, where $\alpha > \beta$, the total income W is more equally spread between the nodes at w^* ; indeed, even though the Nash equilibrium is internal in this case, the contributions to the public good at equilibrium are imbalanced.

The previous examples hint that the total aggregate welfare U is not concave of the set of possible income vectors, but it may be possible to prove that it is concave on each set of vectors w that results in a subset of contributing agents at equilibrium. Moreover, the initial investigation suggests that the income vectors maximizing the total aggregate welfare may be multiple for a given redistribution problem, are imbalanced in the amount of income provided to each individual and often correspond to a specialized equilibrium profile, where the only contributing node is the one with higher income.

It may be interesting to further study whether these preliminary observations can be established in general or under particular hypothesis on the graph and on the value function parameters α_i, β_i .

Chapter 6

Conclusions

Public goods lie at the core of many social, economic, and political processes. Defined as non-rivalrous and non-excludable resources, they are often enjoyed only by a subset of the population and are thus referred to as local. Since benefits are shared, each person faces the difficult choice of whether to contribute or to rely on others' efforts. This makes it relevant to ask under which conditions cooperation can be expected, and when instead people act selfishly, relying on others' contributions.

This thesis investigates local public good provision through the lens of networked game theory, aiming to understand how interconnections influence contributions, the properties of equilibrium outcomes, and the possibility to improve their efficiency through external interventions.

We consider network games defined on weighted and directed graphs. Heterogeneous agents, differing in income level, preference structure, and effort cost, must allocate income between private consumption and contributions to a public good. Every player benefits from her own provision of private good and from her and her neighbors' provision of public good through the value function. Of particular importance is the linearity property of value functions, which simplifies the analysis and allows for a tractable characterization of equilibria. The thesis establishes contributions concerning the existence, uniqueness, and stability of Nash equilibria in this framework. As a preliminary step, we derive and analyze best response functions and their monotonic, saturated structure.

First, we prove the existence of Nash equilibria for the general game and characterize equilibrium profiles when the linearity property holds. Also, we derive conditions ensuring the presence of two relevant types: internal and specialized equilibria.

Second, we obtain a sufficient condition for the uniqueness of the Nash equilibrium in the general game setting. Under the linearity property, it reduces to a bound on the lowest eigenvalue of the symmetrized per-row rescaled adjacency matrix.

Third, we study the asymptotic behavior of both the discrete- and continuous-time best response dynamics. A sufficient condition for the contractivity of the synchronous best response function, based on a Lipschitz property, ensures the uniqueness of the equilibrium and the global asymptotical stability of the two dynamical systems to it. This condition, consisting in constraining the dominant eigenvalue of the per-row

rescaled adjacency matrix, relates once again the game equilibrium and the spectrum of the graph. We then focus on the asymptotic behavior of the continuous-time dynamics under weaker assumptions and prove the global stability of the equilibrium when the less restrictive uniqueness condition holds.

The final part of the thesis explores the possible inefficiency of equilibrium outcomes. We propose two efficiency metrics -total aggregate welfare and aggregate public good provision- to evaluate equilibrium performance and suggest two potential intervention strategies: subsidies to incentivize higher provisions and number of contributing agents and redistribution to reallocate income across players to move the system towards socially optimal equilibria.

An important feature of the developed model is that it unifies and extends previous results, generalizing the analysis to directed and weighted networks with heterogeneous agents and nonlinear utilities. Building on strategies from [13], [10] and [16], it exploits quadratic utilities and exact potential games to characterize equilibrium profiles, study continuous-time dynamics, and formulate uniqueness and stability conditions. Overall, the results either generalize prior findings or confirm them from a broader framework. A key novelty lies in linking uniqueness and stability through the contractivity of the synchronous best response function, extending previous results to more complex network structures.

This thesis highlights the crucial role of network structure in shaping collective outcomes. Weighted and directed connections allow for heterogeneous and asymmetric externalities, reflecting many real-world situations. Spectral conditions for uniqueness and contractivity link mathematical properties of the network to agents' behavior, while applying to a broad framework with diverse agents and nonlinear utilities. From a practical perspective, the insights of this work help in understanding group behavior when facing problems such as vaccination campaigns, the coordination of community projects, or the diffusion of innovations and information in social networks. At the same time, the research presents limitations. The focus on value functions under the linearity property restricts the setting, but makes a more precise analysis possible. The study of symmetric networks in some cases limits the use of the results in more general directed settings. The assumption of perfectly rational agents with full information may not always be realistic. These restrictions suggest caution in interpreting the results, while also pointing to opportunities for further research.

Future work could investigate the discrete-time dynamics when not contractive, including the possibility of limit cycles arising. Another natural direction is the relaxation of the linearity property and the subsequent study of equilibria for more general classes of value functions. Additionally, the proposed intervention strategies could serve as starting points for developing practical methods to improve efficiency, and further studies could examine how network modifications, such as adding, removing, or reweighting links, affect equilibrium outcomes.

In conclusion, this thesis contributes to the understanding of local public good provision in networked environments by formalizing a general model, establishing

results on equilibrium existence, uniqueness, and stability, and proposing preliminary mechanisms to enhance efficiency. Beyond its technical contributions, the work highlights the deep connection between network structure and collective behavior, and opens the way to future studies at the intersection of game theory, network science, and public economics.

Appendix A

Constrained maximization of concave functions

In this section, we derive some general mathematical results that do not concern the game directly and that will be used to make proofs.

Initially, in a technical Lemma, we first prove that there exists a unique function solving a constrained maximization problem for the value function and then we characterize the first derivative of such function. Finally, we define the linearity property of the value function.

Lemma A.0.1. *Let $\eta : \mathbb{R}_{\geq 0}^2 \setminus \{(0,0)\} \rightarrow \mathbb{R}$ be a $\mathcal{C}^2$, strictly concave function st $\forall x, y \geq 0$*

$$\frac{\partial \eta(0, y)}{\partial x} \geq \frac{\partial \eta(0, y)}{\partial y} \quad (\text{A.1})$$

$$\frac{\partial \eta(x, 0)}{\partial x} \leq \frac{\partial \eta(x, 0)}{\partial y} \quad (\text{A.2})$$

then, there exists a unique function $k : (0, +\infty) \rightarrow (0, +\infty)$, $k \in \mathcal{C}^1$ st

$$\eta(k(z), z - k(z)) = \max_{(x,y): \text{ st } x+y=z, z>0} \eta(x, y) \quad (\text{A.3})$$

$$k(z) \leq z \quad \forall z > 0 \quad (\text{A.4})$$

$$k'(z) = \frac{\frac{\partial^2 \eta}{\partial y^2}(k(z), z - k(z)) - \frac{\partial^2 \eta}{\partial x \partial y}(k(z), z - k(z))}{\frac{\partial^2 \eta}{\partial x^2}(k(z), z - k(z)) - 2 \frac{\partial^2 \eta}{\partial x \partial y}(k(z), z - k(z)) + \frac{\partial^2 \eta}{\partial y^2}(k(z), z - k(z))} \quad (\text{A.5})$$

Proof. For $z > 0$, consider the function $\rho : [0, z] \rightarrow \mathbb{R}$, defined as $\rho(x) = \eta(x, z - x)$. It follows from the assumptions on η that ρ is of class $\mathcal{C}^2$ and strictly concave. Moreover,

$$\rho'(x) = \frac{\partial \eta}{\partial x}(x, z - x) - \frac{\partial \eta}{\partial y}(x, z - x)$$

Then, for assumptions (A.1) and (A.2) it holds $\rho'(0) \geq 0$ and $\rho'(z) \leq 0$. Standard calculus arguments imply that ρ has a unique maximum point, that is a stationary

point. We call such a point $k(z)$:

$$\rho(k(z)) = \max_{x \in [0, z]} \rho(x)$$

which, for the definition of ρ , is equivalent to the optimization problem (A.3). This defines the function $k : (0, +\infty) \rightarrow (0, +\infty)$, having the property $k(z) \leq z$.

We are now left with proving the regularity of k . To do this, we introduce the function of two variables $f : D \rightarrow \mathbb{R}$, with $D = \{(x, z) : z > 0, x \in [0, z]\} \subset \mathbb{R}^2$. f is defined as

$$f(x, z) = \frac{d\eta}{dx}(x, z - x) = \frac{\partial \eta}{\partial x}(x, z - x) - \frac{\partial \eta}{\partial y}(x, z - x)$$

Notice that for every $z > 0$, by previous considerations, it holds that $f(k(z), z) = 0$. Moreover, f is of class $\mathcal{C}^1$ and

$$\begin{aligned} \frac{\partial f}{\partial x}(x, z) &= \frac{\partial}{\partial x} \left(\frac{\partial \eta}{\partial x}(x, z - x) - \frac{\partial \eta}{\partial y}(x, z - x) \right) = \\ &= \frac{\partial^2 \eta}{\partial x^2}(x, z - x) - 2 \frac{\partial^2 \eta}{\partial x \partial y}(x, z - x) + \frac{\partial^2 \eta}{\partial y^2}(x, z - x) = v^T \mathbf{H}_\eta(x, z - x)v \end{aligned}$$

where $\mathbf{H}_\eta$ is the Hessian matrix of η and $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then, for the strict concavity of η , for every $(x, z) \in D$

$$\frac{\partial f}{\partial x}(x, z) < 0$$

We can apply the *Implicit function theorem* and deduce that k is $\mathcal{C}^1$ on its domain and

$$k'(z) = - \frac{\frac{\partial f}{\partial z}(k(z), z)}{\frac{\partial f}{\partial x}(k(z), z)}$$

since

$$\frac{\partial f}{\partial z}(x, z) = \frac{\partial}{\partial z} \left(\frac{\partial \eta}{\partial x}(x, z - x) - \frac{\partial \eta}{\partial y}(x, z - x) \right) = \frac{\partial^2 \eta}{\partial x \partial y}(x, z - x) - \frac{\partial^2 \eta}{\partial y^2}(x, z - x)$$

Explicitly writing k'

$$k'(z) = \frac{\frac{\partial^2 \eta}{\partial y^2}(k(z), z - k(z)) - \frac{\partial^2 \eta}{\partial x \partial y}(k(z), z - k(z))}{\frac{\partial^2 \eta}{\partial x^2}(k(z), z - k(z)) - 2 \frac{\partial^2 \eta}{\partial x \partial y}(k(z), z - k(z)) + \frac{\partial^2 \eta}{\partial y^2}(k(z), z - k(z))}$$

the proof is complete. $\square$

We now consider an extension of Lemma A.0.1 that is particularly useful in certain applications.

Lemma A.0.2. *Let $\eta : \mathbb{R}_{\geq 0}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$ be a continuous function, $\mathcal{C}^2$ on $(0, +\infty)$, satisfying the following properties:*

1. for every $(x, y) \in (0, +\infty)^2$

$$v^T \mathbf{H}_\eta(x, z - x)v < 0 \quad (\text{A.6})$$

where $\mathbf{H}_\eta$ is the Hessian matrix of η and $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

2. for every $x > 0, y > 0$ the following limit conditions hold:

$$\lim_{\epsilon \rightarrow 0^+} \frac{\partial \eta(\epsilon, y)}{\partial x} - \frac{\partial \eta(\epsilon, y)}{\partial y} \geq 0 \quad (\text{or } +\infty) \quad (\text{A.7})$$

$$\lim_{\epsilon \rightarrow 0^+} \frac{\partial \eta(x, \epsilon)}{\partial x} - \frac{\partial \eta(x, \epsilon)}{\partial y} \leq 0 \quad (\text{or } -\infty) \quad (\text{A.8})$$

then, there exists a unique function $k : (0, +\infty) \rightarrow (0, +\infty)$, $k \in \mathcal{C}^1$ st

$$\eta(k(z), z - k(z)) = \max_{(x, y): \text{ st } x+y=z, z>0} \eta(x, y)$$

$$k(z) \leq z \quad \forall \quad z > 0$$

$$k'(z) = \frac{\frac{\partial^2 \eta}{\partial y^2}(k(z), z - k(z)) - \frac{\partial^2 \eta}{\partial x \partial y}(k(z), z - k(z))}{\frac{\partial^2 \eta}{\partial x^2}(k(z), z - k(z)) - 2 \frac{\partial^2 \eta}{\partial x \partial y}(k(z), z - k(z)) + \frac{\partial^2 \eta}{\partial y^2}(k(z), z - k(z))}$$

Proof. The proof of Lemma A.0.1 still holds for the less restrictive hypothesis on η . In particular, assumption (A.6) is equivalent to the strict concavity of the function $x \mapsto \eta(x, z - x)$, while (A.7) and (A.8) are extensions of (A.1) and (A.2). $\square$

Remark A.0.1. Notice that if for every $x, y \geq 0$ it holds

$$\frac{\partial^2 \eta}{\partial y^2}(x, y) - \frac{\partial^2 \eta}{\partial x \partial y}(x, y) < 0$$

then $k'(z) > 0$, ie k is a strictly increasing function.

To prove it, recall that

$$k'(z) = \frac{\frac{\partial^2 \eta}{\partial y^2}(k(z), z - k(z)) - \frac{\partial^2 \eta}{\partial x \partial y}(k(z), z - k(z))}{\frac{\partial^2 \eta}{\partial x^2}(k(z), z - k(z)) - 2 \frac{\partial^2 \eta}{\partial x \partial y}(k(z), z - k(z)) + \frac{\partial^2 \eta}{\partial y^2}(k(z), z - k(z))}$$

and that the denominator is negative since it can be written as

$$v^T \mathbf{H}_\eta(x, z - x)v$$

for $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then $k'(z) \geq 0$ if the numerator is also negative. It is the case if for

every $x, y \in \mathbb{R}_{\geq 0}^2 \setminus \{(0, 0)\}$ it holds

$$\frac{\partial^2 \eta}{\partial y^2}(x, y) \leq \frac{\partial^2 \eta}{\partial x \partial y}(x, y)$$

Definition A.0.1. Let $\eta : \mathbb{R}_{\geq 0}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$ be a $\mathcal{C}^2$, strictly concave function st for every $x, y \geq 0$

$$\frac{\partial \eta(0, y)}{\partial x} \geq \frac{\partial \eta(0, y)}{\partial y}$$

and

$$\frac{\partial \eta(x, 0)}{\partial x} \leq \frac{\partial \eta(x, 0)}{\partial y}$$

Denote k the function $k : (0, +\infty) \rightarrow (0, +\infty)$, $k \in \mathcal{C}^1$ st

$$\eta(k(z), z - k(z)) = \max_{(x, y): \text{ st } x+y=z, z>0} \eta(x, y)$$

η is said to satisfy the linearity property **L** if there exists $\bar{k} > 0$ st $k'(z) = \bar{k}$.

Remark A.0.2. The same definition is also valid if η satisfies the less restrictive hypothesis of Lemma A.0.2.

Proposition A.0.3. η satisfies the linearity property **L** if and only if there exists $\bar{k} > 0$ st

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix}^T \mathbf{H}_\eta(x, y) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \bar{k} \begin{pmatrix} 1 \\ -1 \end{pmatrix}^T \mathbf{H}_\eta(x, y) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Proof. Recall the expression of $k'(z)$ (see Lemma A.0.1, eq (A.5)):

$$k'(z) = \frac{\frac{\partial^2 \eta}{\partial y^2}(k(z), z - k(z)) - \frac{\partial^2 \eta}{\partial x \partial y}(k(z), z - k(z))}{\frac{\partial^2 \eta}{\partial x^2}(k(z), z - k(z)) - 2 \frac{\partial^2 \eta}{\partial x \partial y}(k(z), z - k(z)) + \frac{\partial^2 \eta}{\partial y^2}(k(z), z - k(z))}$$

Then, if η satisfies **L** for $\bar{k} > 0$ then $k'(z) = \bar{k}$, which can be written as

$$\frac{\partial^2 \eta}{\partial y^2}(x, y) - \frac{\partial^2 \eta}{\partial x \partial y}(x, y) = \bar{k} \left(\frac{\partial^2 \eta}{\partial x^2}(x, y) - 2 \frac{\partial^2 \eta}{\partial x \partial y}(x, y) + \frac{\partial^2 \eta}{\partial y^2}(x, y) \right)$$

Using the vector notation and the Hessian matrix of η , we get an equivalent definition of the property **L**:

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix}^T \mathbf{H}_\eta(x, y) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \bar{k} \begin{pmatrix} 1 \\ -1 \end{pmatrix}^T \mathbf{H}_\eta(x, y) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

□

Appendix B

Examples of value functions

In this section we provide results on some particular forms that the value functions may assume and that satisfy the game hypothesis. In addition, we make a few examples of value functions using both the general hypothesis and these results and derive their best response functions.

Let us consider some particular cases of value functions η_i that guarantee that the game with utilities $u_i(q) = \eta_i(w_i - q_i, q_i + (Gq)_i)$ is a public good game. Together, we provide some practical examples.

Proposition B.0.1. *Consider a game defined on a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$, an income vector $w \in \mathbb{R}_{>0}^n$, and value functions $\eta_i : \mathbb{R}_{\geq 0}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$ of the form $\eta_i(x, y) = f_i(x)g_i(y)$, with $f_i, g_i : [0, +\infty) \rightarrow [0, +\infty)$ twice continuously differentiable, strictly increasing, strictly concave and st $f_i(0) = g_i(0) = 0$ for every i . Then, the game $\mathcal{P} = (\mathcal{G}, w, \eta)$ with utilities $u_i(q) = \eta_i(w_i - q_i, q_i + (Gq)_i)$ is a public good game. $k_i(z)$ for every i and for $z > 0$ is obtained through the relation*

$$f'_i(x)g_i(z - x) - f_i(x)g'_i(z - x) = 0$$

Proof. In general, η_i is not strictly concave in $\mathbb{R}_{\geq 0}^2 \setminus \{(0, 0)\}$, so let us verify that the less restrictive assumptions of Remark 3.1.1 hold. For every i

- η_i continuous function, $\mathcal{C}^2$ on $(0, +\infty)$, being f_i and g_i twice continuously differentiable and positive.
- the hessian matrix is $\mathbf{H}_{\eta_i}(x, y) = \begin{bmatrix} f''_i(x)g_i(y) & f'_i(x)g'_i(y) \\ f'_i(x)g'_i(y) & f_i(x)g''_i(y) \end{bmatrix}$, so for every $(x, y) \in (0, +\infty)^2$ and for $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ it holds:

$$v^T \mathbf{H}_{\eta_i}(x, y) v = f''_i(x)g_i(y) - 2f'_i(x)g'_i(y) + f_i(x)g''_i(y) < 0$$

- hypothesis (3.5) and (3.6) are satisfied: for every $x, y > 0$

$$\lim_{\epsilon \rightarrow 0^+} \frac{\partial \eta_i(\epsilon, y)}{\partial x} - \frac{\partial \eta_i(\epsilon, y)}{\partial y} = \frac{\partial \eta_i}{\partial x}(0, y) - \frac{\partial \eta_i}{\partial y}(0, y)$$

$$= g_i(y)f'_i(0) - f_i(0)g'_i(y) = g_i(y)f'_i(0) \geq 0$$

and

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0^+} \frac{\partial \eta_i(x, \epsilon)}{\partial x} - \frac{\partial \eta_i(x, \epsilon)}{\partial y} \\ &= \frac{\partial \eta_i}{\partial x}(x, 0) - \frac{\partial \eta_i}{\partial y}(x, 0) = g_i(0)f'_i(x) - f_i(x)g'_i(0) = -f_i(x)g'_i(0) \leq 0 \end{aligned}$$

- finally, we verify (3.8): for every $(x, y) \in \mathbb{R}_{\geq 0} \setminus \{(0, 0)\}$

$$\frac{\partial^2 \eta_i}{\partial y^2}(x, y) - \frac{\partial^2 \eta_i}{\partial x \partial y}(x, y) = f_i(x)g''_i(y) - f'_i(x)g'_i(y) < 0$$

So, we have proved that the game with value functions $\eta_i(x, y) = f_i(x)g_i(y)$ is a Public good game.

To derive the relation for k_i , recall that it is the unique solution, for a given $z > 0$, to:

$$\eta_i(k(z), z - k(z)) = \max_{0 \leq x \leq z} \eta_i(x, z - x)$$

Then, $k_i(z)$ is obtained by solving

$$\frac{d\eta_i}{dx}(x, z - x) = f'_i(x)g_i(z - x) - f_i(x)g'_i(z - x) = 0$$

□

Proposition B.0.2. Consider a game defined on a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$, an income vector $w \in \mathbb{R}_{>0}^n$, and value functions $\eta_i : \mathbb{R}_{\geq 0}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$ of the form $\eta_i(x, y) = f_i(x) + g_i(y)$, with $f_i, g_i : [0, +\infty) \rightarrow [0, +\infty)$ strictly increasing, strictly concave and st for every $x, y \geq 0$ $f'_i(0) \geq g'_i(y)$ and $g'_i(0) \geq f'_i(x)$. Then, the game the game $\mathcal{P} = (\mathcal{G}, w, \eta)$ with utilities $u_i(q) = \eta_i(w_i - q_i, q_i + (Gq)_i)$ is a public good game. $k_i(z)$ for every i and for $z > 0$ is obtained through the relation

$$f'_i(x) - g'_i(z - x) = 0$$

Proof. First, let us see that the value functions satisfy all the assumptions. For every i

- η_i is twice continuously differentiable and strictly concave: the hessian matrix is $\mathbf{H}_{\eta_i}(x, y) = \begin{bmatrix} \frac{\partial^2 \eta_i}{\partial x^2} & \frac{\partial^2 \eta_i}{\partial x \partial y} \\ \frac{\partial^2 \eta_i}{\partial x \partial y} & \frac{\partial^2 \eta_i}{\partial y^2} \end{bmatrix} = \begin{bmatrix} f''_i(x) & 0 \\ 0 & g''_i(y) \end{bmatrix}$, so that is eigenvalues are $f''_i(x) < 0$ and $g''_i(y) < 0$.
- hypothesis (3.2) and (3.3) are satisfied: for every $x, y > 0$

$$\frac{\partial \eta_i}{\partial x}(0, y) - \frac{\partial \eta_i}{\partial y}(0, y) = f'_i(0) - g'_i(y) \geq 0$$

and

$$\frac{\partial \eta_i}{\partial x}(x, 0) - \frac{\partial \eta_i}{\partial y}(x, 0) = f'_i(x) - g'_i(0) \leq 0$$

- finally, we verify (3.8): for every $(x, y) \in \mathbb{R}_{\geq 0}^2 \setminus \{(0, 0)\}$

$$\frac{\partial^2 \eta_i}{\partial y^2}(x, y) - \frac{\partial^2 \eta_i}{\partial x \partial y}(x, y) = g_i''(y) < 0$$

So, we have proved that the game with value functions $\eta_i(x, y) = f_i(x) + g_i(y)$ is a Public good game.

To derive the relation for k_i , recall that it is the unique solution, for a given $z > 0$, to:

$$\eta_i(k(z), z - k(z)) = \max_{0 \leq x \leq z} \eta_i(x, z - x)$$

Then, $k_i(z)$ is obtained by solving

$$\frac{d\eta_i}{dx}(x, z - x) = f_i'(x) - g_i'(z - x) = 0$$

□

Remark B.0.1. If $f_i(x) = g_i(x)$, then the assumptions $f_i'(0) \geq g_i'(y)$ and $g_i'(0) \geq f_i'(x)$ for every $x, y \geq 0$ follow directly from the concavity of the functions.

Example B.0.1. Sum of square roots value function

Consider a game defined on a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$, an income vector $w \in \mathbb{R}_{>0}^n$, and value functions $\eta_i : \mathbb{R}_{\geq 0}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$ of the form:

$$\eta(x, y) = \sqrt{x+1} + \sqrt{y+1}$$

The value function can be written as $\eta(x, y) = f(x) + g(y)$, with $f(x) = g(x) = \sqrt{x+1}$. Being f and g strictly increasing and concave and st $f'(0) \geq g'(y)$ and $g'(0) \geq f'(x)$ for every $x, y \geq 0$, for Proposition B.0.2, the game $\mathcal{P} = (\mathcal{G}, w, \eta)$ is a public good game. $k(z)$ is obtained by solving $f'(x) - g'(z - x) = 0$:

$$\frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{z-x+1}} = 0$$

that is equivalent to

$$z - x + 1 - (x + 1) = 0$$

which implies

$$x = k(z) = \frac{z}{2}$$

Example B.0.2. Sum of logarithms value function

Consider a game defined on a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$, an income vector $w \in \mathbb{R}_{>0}^n$, and value functions $\eta_i : \mathbb{R}_{\geq 0}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$ of the form:

$$\eta(x, y) = \log(x+1) + \log(y+1)$$

The value function can be written as $u(x, y) = f(x) + g(y)$, with $f(x) = g(x) =$

$\log(x+1)$. Being f and g strictly increasing and concave and st $f'(0) \geq g'(y)$ and $g'(0) \geq f'(x)$ for every $x, y \geq 0$, for Proposition B.0.2 the game $\mathcal{P} = (\mathcal{G}, w, \eta)$ is a public good game. $k(z)$ is obtained by solving $f'(x) - g'(z-x) = 0$:

$$\frac{1}{x+1} - \frac{1}{z-x+1} = 0$$

which implies

$$x = k(z) = \frac{z}{2}$$

Now, let us see two examples of value functions that do not take one of the forms presented above and for which we directly prove the fulfillment of the less restrictive game hypothesis of Remark 3.1.1.

Example B.0.3. *Logarithm of sum value function*

Consider a game defined on a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, G)$, an income vector $w \in \mathbb{R}_{>0}^n$, and value functions $\eta_i : \mathbb{R}_{\geq 0}^2 \setminus \{(0,0)\} \rightarrow \mathbb{R}$ of the form:

$$\eta_i(x, y) = \log(xy + 1)$$

In general, η_i is not strictly concave in $\mathbb{R}_{\geq 0}^2 \setminus \{(0,0)\}$, so let us verify that the less restrictive assumptions of Remark 3.1.1 hold.

- η_i continuous function, $\mathcal{C}^2$ on $(0, +\infty)$.
- The hessian matrix is $\mathbf{H}_{\eta_i}(x, y) = \begin{bmatrix} -\frac{y^2}{(xy+1)^2} & \frac{1}{(xy+1)^2} \\ \frac{1}{(xy+1)^2} & -\frac{x^2}{(xy+1)^2} \end{bmatrix}$, so for every $(x, y) \in (0, +\infty)^2$ and for $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ it holds:

$$v^T \mathbf{H}_{\eta_i}(x, y) v = -\frac{y^2}{(xy+1)^2} - \frac{2}{(xy+1)^2} - \frac{x^2}{(xy+1)^2} < 0$$

- hypothesis (3.5) and (3.6) are satisfied: for every $x, y > 0$

$$\lim_{\epsilon \rightarrow 0^+} \frac{\partial \eta_i(\epsilon, y)}{\partial x} - \frac{\partial \eta_i(\epsilon, y)}{\partial y} = \frac{\partial \eta_i(0, y)}{\partial x} - \frac{\partial \eta_i(0, y)}{\partial y} = y \geq 0$$

and

$$\lim_{\epsilon \rightarrow 0^+} \frac{\partial \eta_i(x, \epsilon)}{\partial x} - \frac{\partial \eta_i(x, \epsilon)}{\partial y} = \frac{\partial \eta_i(x, 0)}{\partial x} - \frac{\partial \eta_i(x, 0)}{\partial y} = -x \leq 0$$

- finally, we verify (3.8): for every $(x, y) \in \mathbb{R}_{\geq 0}^2 \setminus \{(0,0)\}$

$$\frac{\partial^2 \eta_i}{\partial y^2}(x, y) - \frac{\partial^2 \eta_i}{\partial x \partial y}(x, y) = -\frac{x^2}{(xy+1)^2} - \frac{1}{(xy+1)^2} < 0$$

So, we have proved that the game with value functions $\eta(x, y) = \log(xy + 1)$ is a Public good game.

To derive the relation for k_i , recall that for every i it is the unique solution, for a given $z > 0$, to:

$$\eta_i(k_i(z), z - k_i(z)) = \max_{0 \leq x \leq z} \eta_i(x, z - x)$$

Then, $k_i(z)$ is obtained by solving

$$\frac{d\eta_i}{dx}(x, z - x) = \frac{z - 2x}{zx - x^2 + 1} = 0$$

which implies

$$x = k_i(z) = \frac{z}{2}$$

Then, for Proposition 3.2.2, for every i the best response is uniquely defined and has the form:

$$\mathcal{B}_i(q_{-i}) = [w_i - k_i(w_i + (Gq)_i)]_+ = [w_i - \frac{w_i + (Gq)_i}{2}]_+ = [\frac{w_i}{2} - \frac{1}{2}(Gq)_i]_+$$

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