POLITECNICO DI TORINO

Master's Degree in Computer Engineering

Politecnico di Torino

EURECOM
Sophia Antipolis

Master's Degree Thesis

Information, Entropy and Time in Quantum Systems

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April 2025

A i' mi nonno

Abstract

Information theory has been crucial to develop modern digital communications. As quantum computing is evolving and getting increasing attention, the development of communication between quantum devices appears as a natural consequence. Researchers around the world have defined quantum versions of classical information theory concepts and explored their differences. Not only the community has found new version of the classical concepts, but has also defined many new concept that arise from the laws of quantum mechanics— and so have no classical analogous. In this work, we present two main topics: a study of the evolution of quantum entropies during a decoherence process and the analysis of entanglement fidelity for different quantum channels acting on one qubit messages, both in a general case and a concrete example. We show how relaxing the Markovianity assumption on the evolution of a system complicates the study of long time bounds of its von Neumann entropy. We report our attempts in generalizing such bounds to quantum Rényi and Tsallis entropies assuming Markovian evolution— even though they did not lead to crisp results. We compute the Entanglement fidelity for Random Pauli-X, Dephasing, Depolarizing, Werner-Holevo, Generalized Pauli and Amplitude damping channels. The quantity is computed first for a general density matrix and then for a particular two letter message. This computation allows to find guidelines on how to choose the optimal letters or channel in a real case scenario. To conclude, we obtain the time derivative of *Entanglement fidelity* assuming that the letter undergoes different types of *decoherence*, both Markovian and non-Markovian. For the case of *pure decoherence* we also compute the time derivative for all the studied channels.

Abstract

La théorie de l'information a joué un rôle crucial dans le développement des communications numériques modernes. Avec l'essor de l'informatique quantique et l'attention croissante qu'elle suscite, le développement de la communication entre dispositifs quantiques apparaît comme une conséquence naturelle. Des chercheurs du monde entier ont défini des versions quantiques des concepts classiques de la théorie de l'information et ont exploré leurs différences. Non seulement la communauté a découvert de nouvelles versions des concepts classiques, mais elle a également défini de nombreux nouveaux concepts issus des lois de la mécanique quantique—et qui n'ont donc pas d'analogues classiques. Dans ce travail, nous présentons deux thèmes principaux : une étude de l'évolution des entropies quantiques au cours d'un processus de décohérence et l'analyse de la fidélité d'intrication pour différents canaux quantiques agissant sur des messages d'un qubit, à la fois dans un cas général et dans un exemple concret. Nous montrons comment la relaxation de l'hypothèse de markovianité sur l'évolution d'un système complique l'étude des bornes à long terme de son entropie de von Neumann. Nous rapportons nos tentatives de généralisation de ces bornes aux entropies de Rényi et de Tsallis quantiques sous l'hypothèse d'une évolution markovienne—même si elles n'ont pas abouti à des résultats précis. Nous calculons la fidélité d'intrication pour les canaux de Pauli-X aléatoire, de déphasage, de dépolarisation, de Werner-Holevo, de Pauli généralisé et d'amortissement d'amplitude. Cette quantité est d'abord calculée pour une matrice de densité générale, puis pour un message particulier à deux lettres. Ce calcul permet d'établir des lignes directrices sur le choix optimal des lettres ou du canal dans un scénario réel. Enfin, nous obtenons la dérivée temporelle de la *fidélité d'intrication* en supposant que la lettre subit différents types de *décohérence*, tant markovienne que non markovienne. Pour le cas de la *décohérence pure*, nous calculons également la dérivée temporelle pour tous les canaux étudiés.

Summary

Nowadays, *Quantum technologies* are one of the most popular topics in the field of information technologies. The promise for computations that would not be possible in reasonable time with classical computers and recent progress in the hardware are getting them more and more attention. Nevertheless, there are still many limits we have to overcome before this devices reach their potential. One of the biggest problem to be solved is the interaction of the devices with the environment that causes them to degenerate to a classical behavior.

In the attempt to understand and overcome this problem we need to study the interaction between the devices and the external word. This is part of the study of *Open quantum systems*. Quantum computers are nothing but systems that cannot be perfectly isolated, so that studying how these behave is strictly related to the general study of quantum systems.

One of the topics we studied in this thesis is the modeling of interaction between quantum systems. Many models have been proposed to describe different scenarios and they can be divided in two main groups: Markovian [1, 2] and non-Markovian processes [3]. Both types of evolution describe the presence of system-environment correlations that lead to an irreversible loss of coherence. The main difference stands in the presence of memory effects and backflow of information from the environment to the system. These last properties are used to define the non-Markovianity of a process [3].

The first two chapters serve as an introduction to the subject of *Quantum Shannon theory* and present some mathematical background useful for the following work.

In Chapter 3, we tried to replicate the results obtained by Kohei Kobayashi in [2] removing the assumption of Markovian evolution. Kobayashi found a lower bound on the *von Neumann entropy* S of a quantum system with density matrix $\rho_S(t)$ in the limit $t \to \infty$:

$$S_\infty \geq \frac{\mathrm{Tr}(L^\dagger L \rho_\infty) - \mathrm{Tr}(L \rho_\infty L^\dagger \rho_\infty)}{||L||_F^2}, \tag{0.0.1}$$

where $||A||_F^2 := \sqrt{\mathrm{Tr}\,(A^\dagger A)}$ is the Frobenius norm of the operator A, Tr is the matrix trace and L is the *Lindblad operator* representing the decoherence process.

Von Neumann entropy is a generalization of *Shannon entropy* for random variables. It measures the uncertainty of in quantum state and is therefore associated to its pureness: a pure quantum

state will have zero entropy while a maximally mixed one has $S(\rho) = \ln d$, if d is the dimension of the associated Hilbert space. This quantity is also associated to *Entanglement*, a property of quantum systems that will be discussed later.

In Kobayashi's work, the evolution is assumed to be Markovian and described by *Lindblad master equation*:

$$\frac{d\rho_S}{dt} = -i\left[H_S, \rho_S\right] + \mathcal{D}[L](\rho_S), \tag{0.0.2}$$

where H_S represents the Hamiltonian of the system of interest.

To find the bound, Kobayashi exploited the behavior of the derivative of *von Neumann entropy*. He showed that this derivative can be lower bounded by a monotonically decreasing function $f(x)$ of *von Neumann entropy* itself that has alternated sign when computed in the extreme values of S— namely $f(0) \geq 0$ and $f(\ln d) < 0$. Because the limit $t \to \infty$ implies that the decoherence process must have stopped and thanks to *Bolzano's theorem* on uniqueness of the root, we can conclude that the lower bound presented above must hold. If this was not the case, we could prove that *von Neumann entropy* is increasing as its derivative would be positive.

We tried to apply the same procedure to two non-Markovian models described by Breuer et al. in [3]. The first is called *Pure decoherence model* and described by the equation:

$$\frac{d}{dt}\rho_S = \gamma(t)\left[\sigma_z \rho_S \sigma_z - \rho_S\right]. \tag{0.0.3}$$

The second represents a *Two-level system in a dissipative environment* and the evolution of the state is given by:

$$\frac{d}{dt}\rho_S = -\frac{i}{4}S(t)[\sigma_z, \rho_S] \quad + \gamma(t)\left[\sigma_- \rho_S \sigma_+ - \frac{1}{2}\left\{\sigma_+\sigma_-, \rho_S\right\}\right]. \tag{0.0.4}$$

Both models led to constraints that did not allow us to apply Kobayashi's procedure. In the first case, the derivative is a function of S but the dependence on the decay rate $\gamma(t)$, whose sign is not constant in time, leads to a derivative that is not monotonic with respect to *von Neumann entropy*. For the second case, the derivative is not even a function of S and the dependence on $\gamma(t)$ still appears. Again, the sign of the derivative is not well defined and we cannot conclude anything.

Our work does not rule out the existence of lower bounds for the *von Neumann entropy* under non-Markovian evolution but shows how Kobayashi's procedure cannot be applied to any possible type of evolution of a quantum state.

In Chapter 4, we tried to apply Kobayashi's idea while preserving the Markovian assumption. In this case, we wanted to find a lower bound on two generalizations of *von Neumann entropy*: the quantum versions of *Rényi* and *Tsallis* entropies. These two quantities are defined respectively by

$$R_\alpha(\rho) := \frac{1}{1-\alpha} \ln \mathrm{Tr}(\rho^\alpha) \tag{0.0.5}$$

and

$$T_\alpha(\rho) := \frac{1}{1-\alpha} \left[\mathrm{Tr}(\rho^\alpha) - 1\right]. \tag{0.0.6}$$

Unfortunately, deriving these quantities lead to functions that do not directly depend on the associated entropies making it difficult to apply directly Kobayashi's idea. We have not found analogous bounds but we still present in this thesis some of the inequalities we applied to try to solve the problem.

One of the new quantities that were introduced to study quantum communications is the *Entanglement fidelity*. Introduced by Schumacher in 1996 [4], is a measure of how well a quantum channel preserves the entanglement between a system of interest and a reference system we do not have access to. *Entanglement fidelity* has applications in many fields of *Quantum information theory*: it appears in the definition of *distortion* in the *Quantum rate distortion* problem [5], is involved in the study of *Quantum error correction* [6], in the description of desirable quantum communication scenarios and the study of other entanglement related quantities [7, 8].

In Chapter 5, we studied the value of *Entanglement fidelity* for some common quantum channel. Here, we considered the general scenario of sending a general one-qubit state over a quantum channel. The density matrix associated to such state is

$$\rho = \begin{bmatrix} a & c \\ \overline{c} & b \end{bmatrix}, \tag{0.0.7}$$

where $a, b \in \mathbb{R}$ and $c \in \mathbb{C}$. The number $\overline{c}$ is the complex conjugate of c.

In order to compute the *Entanglement fidelity* it is useful to know its *operator-sum representation*, we give here the ones for the channels we studied:

- Random Pauli-X channel: $N_u(\rho) = u \cdot X\rho X^\dagger + (1-u)\rho \ ; \ u \in [0,1]$
- Dephasing channel: $N_u(\rho) = u \cdot Z\rho Z^\dagger + (1-u)\rho \ ; \ u \in [0,1]$
- Depolarizing channel: $N_u(\rho) = (1-u)\rho + u \cdot \frac{I}{2} \ ; \ u \in [0,1]$
- Werner-Holevo channel: $\Lambda_{WH}(\rho) = Y\rho Y$
- Generalized Pauli channel: $p_0 0 \cdot \rho + p_{01} \cdot Z\rho Z + p_{10} \cdot X\rho X + p_{11} \cdot XZ\rho ZX$
- Amplitude damping channel: $\mathcal{N}(\rho) = A_0\rho A_0^\dagger + A_1\rho A_1^\dagger$

Where the matrices for the *Amplitude damping channel* are defined as follows: $A_0 = \sqrt{\gamma} \cdot |0\rangle\langle 1|$ and $A_1 = |0\rangle\langle 0| + \sqrt{1-\gamma} \cdot |1\rangle\langle 1|$.

Schumacher proved that *Entanglement fidelity* is a function of the density matrix of the system and the quantum channel, that is, it does not depend on the state of the reference system. By exploiting this, we computed the following entanglement fidelities:

- Random Pauli-X channel: $F_e(\rho, \mathcal{N}_u) = 1 + u\left[4 \cdot \Re^2(c) - 1\right]$
- Dephasing channel: $F_e(\rho, \mathcal{N}_u) = 1 + u\left[(a-b)^2 - 1\right]$
- Depolarizing channel: $F_e(\rho, \mathcal{N}_u) = 1 + u \cdot \left[\Re^2(c) + \Im^2(c) + \frac{(a-b)^2}{4} - \frac{3}{4}\right]$
- Werner-Holevo channel: $F_e(\rho, \Delta_{WH}) = 4 \cdot \Im^2(c)$
- Generalized Pauli channel: $F_e(\rho, \mathcal{N}) = p_{00} + p_{10} \cdot 4 \cdot \Re^2(c) + p_{01}(a-b)^2 - p_{11} \cdot \Im^2(c)$
- Amplitude damping channel: $F_e(\rho, \mathcal{N}) = \gamma \cdot |c|^2 + \left(a + b \cdot \sqrt{1-\gamma}\right)^2$

In Chapter 6, we applied these results to a case of a two letter message. The two letters are parametrized by $p \in [0, 1]$ and defined as

$$\begin{aligned} |\psi^+\rangle &= \sqrt{p}\,|0\rangle + \sqrt{1-p}\,|1\rangle \\ |\psi^-\rangle &= \sqrt{p}\,|0\rangle - \sqrt{1-p}\,|1\rangle\,. \end{aligned} \tag{0.0.8}$$

The state of the letter sent through a channel is a probability mixture of the two given by the density matrix

$$\rho = q \cdot |\psi^+\rangle\langle\psi^+| + (1-q) \cdot |\psi^-\rangle\langle\psi^-|\,. \tag{0.0.9}$$

That is, the sent letter is $|\psi^+\rangle$ with probability q or $|\psi^-\rangle$ with probability $1-q$.

Observing that in this case the values of $a, b and c$ are given by

$$\begin{aligned} a &= p \\ b &= 1-p \\ c &= (2q-1) \cdot \sqrt{p(1-p)}. \end{aligned} \tag{0.0.10}$$

The obtained fidelities are summarized in Table 1.
Since we have some control on the value of p but not on the other parameter, the table above can be used to choose the set of letters that works best for the problem at hand. Moreover, because the channels *Random Pauli-X, Dephasing* and Depolarizing depend on the same parameters, we ranked them based on the relationship between such values. To complete the discussion, we added the ranking of these entanglement fidelities when sending commonly used states like $|0\rangle, |1\rangle, |+\rangle, |-\rangle$. As these last results cannot be briefly summarized we invite the interested reader to read Section 4.5.

Channel $\mathcal{N}$	$F_e(\rho, \mathcal{N})$
Pauli-X	$1 + u \cdot \left[4 \cdot (2q-1)^2 \cdot p(1-p) - 1\right]$
Dephasing	$1 + u\left[4 \cdot p(p-1)\right]$
Depolarizing	$1 + \frac{u}{4} \cdot \left\{\left[4 \cdot (2q-1)^2 \cdot p(1-p) - 1\right] + \left[4 \cdot p(p-1)\right] - 1\right\}$
Generalized Pauli	$p_{00} + p_{10} \cdot [4 \cdot (2q-1)^2 \cdot p(1-p) - 1] + p_{01} \cdot (2p-1)^2$
Amplitude damping	$p^2 + (1-\gamma)(1-p)^2 + p(1-p)\left[2\sqrt{1-\gamma} + \gamma(2q-1)^2\right]$
Werner-Holevo	0

Table 1: *Entanglement fidelity* for the studied quantum channels and ρ as defined in Equation 0.0.9.

In Chapter 7 we presented a discussion on the meaning of "entanglement preservation". Some objections to the definition given by Schumacher in [4] were raised in two articles [9, 10]. We described our take on the topic and presented some counter-arguments to the objections. To conclude the discussion we pointed out how *Entanglement fidelity* does not capture the simple preservation fo the amount of entanglement between two systems, but seems to take into account the difference in the structure of the initial and final state.

Our last work consists in the computation of the time derivative of *Entanglement fidelity* under the assumption of different type of decoherence processes. The quantity is computed for a general channel with *operator sum representation*

$$\mathcal{N}(\rho) = \sum_{\mu} A_\mu \rho A_\mu^\dagger. \tag{0.0.11}$$

Because of the length of the resulting equations, we omit them here but we invite the reader to take view of the results in Chapter 8 of this thesis. The computation of this derivatives allows us to study how the information measure changes in time if the associated letter is a *open quantum system* interacting with a reference system that we cannot access.

For the special case of the *Pure decoherence* model, we proceeded to compute the time derivative of *Entanglement fidelity* for the different quantum channels studied in the previous chapters.

Acknowledgements

Ringrazio i mi' genitori per il costante supporto ed entusiasmo. Ringrazio i' mi' fratello, senza i cui onesti consigli probabilmente starei facendo qualcos'altro al posto di scrivere questa tesi. Ringrazio i' mi' nonno che è sempre stato, e sempre sarà, il mio più grande sostenitore. Ringrazio i mi' amici che hanno reso questi cinque anni più leggeri e felici. Ringrazio tutta la mi' famiglia, quella che non ho potuto scegliere, e quella che mi sono scelto. Un grande grazie va a tutti i veri insegnanti che ho trovato lungo la strada, purtroppo sono molti meno di quel che si pensi. Ringrazio il professor Marios Kountouris per la possibilità di lavorare a questa tesi e l'interesse e impegno messo in questi mesi. Ringrazio EURECOM e il Politecnico di torino per avermi permesso di completare questo progetto. Ringrazio i professori Bartolomeo Montrucchio e Roberto Morabito per la loro disponibilità e supervisione.

I thank my parents for their constant support and enthusiasm. I thank my brother, for without his sincere advice I would probably be doing something else instead of writing this thesis. I thank my grandfather who always was, and always will be, my biggest supporter. I thank all my friend for making these five years lighter and happier. I thank my whole family, the one I could not choose, and the one I chose for my self. A very big thank you goes to all the true teachers I found along the way, unfortunately they are way less than we think. I thank professor Marios Kountouris for the chance of working on this thesis and the interest and effort he put in the past months. I thank EURECOM and the Polytechnic of Turin for the possibility of even starting this work. I thank professors Bartolomeo Montrucchio and Roberto Morabito for their availability and supervision.

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Chapter 1

Introduction

The interest for *Quantum computing* is constantly growing. Many countries and companies have devolved several projects to the study of *Quantum devices*, *Quantum algorithms* and *Quantum communication*. Since the first ideas about quantum computers, the supremacy over classical computing and a wide range of new applications to exact sciences have kept the hope alive. The debate on the practical usefulness of this technology and its real potential is still very active. Nonetheless, the international community is making more and more progress in all of such aspects. It is in this spirit of research and innovation that this work was conducted.

One of the key concept that differentiates *Quantum computing* from the classical one is the information unit: the *Qubit* is the most elementary form of quantum information used in modern quantum devices and it differs greatly from a classical *bit*. Not only the qubit presents new properties but it also contains the ones already present in classical *bits*. That is, we can use qubits to transmit both classical and quantum information. They can be used to send one or the other and studies about simultaneous transmission of both types of information [8] were developed. Moreover, the usage of qubits presents the possibility of communication protocols that cannot be achieved with classical bits: *Quantum teleportation* [11] and *Superdense coding* [12] are two of the most popular examples.

Since this novelty promises a more powerful type of communication, it is natural to believe in the possibility of building connections between quantum computers. In fact, many have already made speculations about the road to a *Quantum internet* [13, 14, 15].

Most of modern classical communication is built on top of the revolutionary work from Claude Shannon "A Mathematical Theory of Communication" (1948) [16]. This article set the groundings for the field of *Information theory*, a theory that studies the limits of reliable communication and strategies to achieve the best communication possible using limited resources. Since the 1970s, researchers have been studying an equivalent of this theory for quantum

information. Many of the results where adapted and extended to the usage of qubits and a lot of new properties and measures arose from the laws of quantum mechanics. This new theory is generally called *Quantum Shannon theory*.

Even though very promising, modern *Quantum computers* still have many limitations. One of the biggest issues that scientists are trying to overcome is the degeneration of physical qubits to classical ones. Because of the inevitable interaction with the environment these computers tend to lose their quantum properties. In order to address this problem is therefore necessary to study how quantum devices interact with the environment and take this interaction into account. This is the purpose of the study of *Open quantum systems* that will be presented in following chapters.

During the years, many models describing the evolution of a quantum system were presented. In analogy to the classical theory of *Random processes* this models were divided in *Markovian* and *non-Markovian* models [3]. Although the distinction between the two in the quantum scenario is not uniquely defined, the second class involves memory effects in the interaction and a relaxation of assumptions made in the Markovian case.

The main goal of this thesis work is to explore the behavior of *Quantum information measures*, like *von Neumann entropy* and *Entanglement fidelity*, when complex interaction with the environment is present. We started our work based on an article from Kohei Kobayashi [2] about the evolution of *von Neumann entropy* under Markovian decoherence. We tried to obtain analogous results when considering more complex interactions and to extend the result to *quantum Rényi* and *Tsallis entropies*.

Expanding our research on quantum information measures we decided to dive deeper in the analysis of *Entanglement fidelity* [4]: a key measure in the description of interaction between system and environment when transmitting information with quantum channels. We encountered this measure while studying the quantum version of the *Rate distortion theory* defined by Datta et al. [5]. The goal of this study was to analyze the value of the measure for different quantum channels and compare them. Moreover, it led us to investigate its meaning and the wider concept of *Entanglement preservation*: a notion that does not have a single accepted definition.

To conclude our work, we decided to put together the two topics by studying the evolution of *Entanglement fidelity* taking into account the interaction between the system of interest and the environment. To do this, we studied the time derivative of the quantity using different models for the evolution of a quantum state, covering both Markovian and non-Markovian cases. For the case of *pure decoherence* we computed the derivative also for the specific channels previously studied.

Chapter 2

Preliminaries

We present in this section some necessary concepts to understand our work. We tried to present the concepts that cannot be overlooked and that are the foundations for the theories studied in this thesis. This section is not a complete description of the theories and we do not present here all the concepts use throughout our work. All those definitions and notations that are specific to a smaller part of our study are introduced as soon as they become necessary.

The laws and the postulates of *Quantum mechanics*, together with the mathematical tools associated to them, have many formulations, in the following discussion we will use and refer to those given by Nielsen and Chuang in their book *"Quantum Computation and Quantum Information"* [17] and by Jhon Preskill in his lecture notes [18].

2.1 Mathematical prelude

2.1.1 Bra-Ket notation

In *Quantum mechanics* is a common practice to indicate the column vectors of an n-dimensional vector space V on $\mathbb{C}$ with a *ket*:

$$|\psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix}. \tag{2.1.1}$$

The vector $\langle\psi|$ is called a *bra*, is the dual of the *ket* $|\psi\rangle$ and is defined as the row vector

$$\langle\psi| = \begin{bmatrix} \overline{\psi_1} & \overline{\psi_2} & \cdots & \overline{\psi_n} \end{bmatrix}. \tag{2.1.2}$$

2.1.2 Hilbert spaces

An *inner product* is a function that maps two vectors of a *vector space* $|\psi\rangle, |\varphi\rangle \in V$ to a number in $\mathbb{C}$ and satisfies the following properties:

1. $\langle\psi|\psi\rangle > 0 \quad \forall\, |\psi\rangle \neq 0_{\mathcal{H}}$
2. $\langle\varphi| (a\,|\psi_1\rangle + b\,|\psi_2\rangle) = a\,\langle\varphi|\psi_1\rangle + b\,\langle\varphi|\psi_2\rangle$
3. $\langle\varphi|\psi\rangle = \overline{\langle\varphi|\psi\rangle}$

where with $\bar{c}$ we indicate the complex conjugate of a number $c \in \mathbb{C}$. In this work, we will use $\Re(c)$ to refer to the *real part* of c and $\Im(c)$ to refer to its *imaginary part*.

A vector space on the complex numbers $\mathbb{C}$ equipped with an *inner product* is called an *Hilbert space*.

2.1.3 Hermitian and Unitary operators

It is a known fact from *Linear algebra* that linear operators can be described by matrices and any matrix represents a linear operator. In *Quantum mechanics*, there are two classes of operators that are particularly useful: *Hermitian operators* and *Unitary operators*.

Given an operator A on an Hilbert space $\mathcal{H}$, we define the *adjoint* of A as the operator $A^\dagger$ such that $\forall\, |\psi\rangle, |\varphi\rangle \in \mathcal{H}$

$$\langle\psi|A\varphi\rangle = \langle A^\dagger\psi|\varphi\rangle\,. \tag{2.1.3}$$

If the operator A is equal to its *adjoint* $A^\dagger$, then A is called *Hermitian*.

Given a matrix U, this is said to be *unitary* if and only if $U^\dagger U = UU^\dagger = I$, where with I we indicate the identity matrix.

2.1.4 Operator functions

Another interesting class of operators is the one of *Normal operators*. An operator A is normal if and only if $AA^\dagger = A^\dagger A$. Any *normal operator* A can be written using its *spectral decomposition*

$$A = \sum_i a_i\,|a_i\rangle\,\langle a_i|\,, \tag{2.1.4}$$

where a_i represents the i-th eigenvalue of A and $|a_i\rangle$ the associated eigenvector.

For normal operators we can define the result of the application of a function $f : \mathbb{C} \to \mathbb{C}$. With the notation $f(A)$ we refer to a new operator obtained as follows:

$$\sum_i f(a_i) \left|a_i\right\rangle \left\langle a_i\right| . \tag{2.1.5}$$

2.1.5 Tensor product

Given two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with orthonormal bases respectively $\{\left|i\right\rangle_A\}$ and $\{\left|\mu\right\rangle_B\}$, the tensor product $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$ is an Hilbert space with dimension
$dim(\mathcal{H}_A \otimes \mathcal{H}_B) = dim(\mathcal{H}_A) \cdot dim(\mathcal{H}_B)$ and with basis

$$\{\left|i, \mu\right\rangle_{AB} := \left|i\right\rangle_A \otimes \left|\mu\right\rangle_B\} . \tag{2.1.6}$$

Given an operator $\mathbf{M}_A$ acting on the space $\mathcal{H}_\mathcal{A}$ and an operator $\mathbf{N}_B$ acting on $\mathcal{H}_B$, the tensor product operator $\mathbf{M}_A \otimes \mathbf{N}_B$ on $\mathcal{H}_A \otimes \mathcal{H}_B$, is an operator that applies $\mathbf{M}_A$ on system A and $\mathbf{N}_B$ to system B.

2.2 Quantum systems

Quantum mechanics is a mathematical framework that allows to describe a physical system. For physical system we generally refer to a portion of the universe we want to study. Loosely speaking, describing a system means that we have a way to quantify all its properties we are interested in, define how it evolves in time and make measurements to get to know more about it. For *Quantum mechanics*, the way this is achieved is defined by the *Axioms* or *Postulates of Quantum mechanics*.

The *State* of a system is a mathematical object that describes everything we know about such system. In Quantum mechanics, this is described by the following postulate:

Postulate 1. The state of any isolated physical system is described by a unitary norm *state vector*. Such vector belongs to a Hilbert space known as *state space* of the system.

We will refer to the *state space* of a system with the letter $\mathcal{H}$ unless stated otherwise. The letter H instead is used to describe the *Hamiltonian* of a system, an operator involved in the quantification of the total energy of a system. We will not add details about it as it is not in the scope of our work. Anyway, attention must be paid to not confuse these two notations.

This postulate only applies to isolated quantum systems and these *state vectors* are called *pure states*. However, if a system interacts with another one we cannot generally describe its state with a single vector. These interactions or loss of information about the exact state of a system lead to a scenario where the system could be in one of many possible pure states.

Whenever this happens, we define an *ensemble of pure states* $\{p_i, \left|\psi_i\right\rangle\}$ and describe the state of the system with a matrix

$$\rho := \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| . \tag{2.2.1}$$

This matrix is called *density operator* of *density matrix* and we can interpret it by saying that the system is in the state $|\psi_i\rangle$ with probability p_i. If the sum is composed by just one element, than we know for sure the state of the system and we say it is in a *pure state*. Otherwise, the system is said to be in a *mixed state*.

Not all matrices can represent the state of a quantum system. In fact, an operator ρ is a density matrix only if it satisfies the following properties:

1. Trace unitarity: $\mathrm{Tr}(\rho) = 1$
2. Positivity: $\langle\psi|\,\rho\,|\psi\rangle \geq 0 \quad \forall\,|\psi\rangle \in \mathcal{H}$

These two properties are of fundamental importance and will be widely used in the following chapters.

2.3 State evolution

Once we know how to describe the state of a system, we want to know how it evolves in time. If we assume our system to be *closed*, we can define the following postulate:

Postulate 2. The evolution of a *closed* quantum system is described by a unitary transformation. If $|\psi_1\rangle$ the state of the system at time t_1 and $|\psi_2\rangle$ is the state at time t_2, then there exist a unitary operator $U(t_1, t_2)$ such that

$$|\psi_2\rangle = U(t_1, t_2)\,|\psi_1\rangle \tag{2.3.1}$$

.

This defines how a quantum system evolves under the assumption of being *closed*, i.e. when it exchanges energy with other systems but does not exchange matter or information. As we mentioned in Chapter 1, this assumption is often too strong and we are interested in understanding what happens to an *open quantum system*.

A system is said to be *open* when it can exchange both energy and information with its surroundings. When this happens, the definition of evolution given in Postulate 2 does not hold anymore: the evolution of the system is not unitary but it can be described with the derivative of the associated *density matrix*. Many models for this evolution exist and a deeper discussion about the topic can be found in Chapter 3.

2.4 Composite systems

So far, we have presented some of the most important aspects of the description of a single quantum system. Anyway, in most applications this is too limiting and we need to be able to describe the state of a composite system.

As described by Nielsen and Chuang [17], the postulate of quantum mechanics can be presented both in terms of *state vectors* and *density matrices*. In this case, it is more useful to present the postulate describing composite systems with the second approach. This postulate is generally considered the *Fourth postulate of quantum mechanics* and we decided to preserve this numbering.

Postulate 4. The state space of a composite system is the tensor product of the state spaces of the component sub-systems. Moreover, if we have sub-systems numbered by 1 through n, and the system i is prepared in the state ρ_i, then the joint state of the total system is $\rho_1 \otimes \rho_2 \otimes ... \otimes \rho_n$

Once we know how to describe the state a composite system, it is important to be able to retrieve from it the state of one of its subsystems. Given two systems A, B and indicating the state of the composite system with ρ_{AB}, the state of the sub-system A is obtained through the *partial trace over B*:

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}). \tag{2.4.1}$$

If we define $\mathrm{Tr}(\cdot)$ as the trace of a matrix and consider any two vectors $|a_1\rangle, |a_2\rangle \in \mathcal{H}_A$ and $|b_1\rangle, |b_2\rangle \in \mathcal{H}_B$, the *partial trace over B* is defined as follows:

$$\mathrm{Tr}_B\left(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|\right) := |a_1\rangle\langle a_2| \,\mathrm{Tr}\left(|b_1\rangle\langle b_2|\right). \tag{2.4.2}$$

Thanks to the linearity of the matrix trace, the *partial trace* is linear and its computation for more complex density matrices is easily obtained.

2.5 Entanglement

One of the most peculiar and important properties of composite quantum systems is *Entanglement*. This concept is related to the presence of correlations between the sub-systems of a composite system that causes many of the non-intuitive behaviors in *Quantum mechanics*. *Entanglement* between two systems is a desirable property and is exploited in many applications like *Quantum teleportation* and *Superdense coding*[11, 12]. Moreover, it arises in the interaction between systems and their environment complicating the study of *open quantum systems.* Preserving this property is one of the goals of quantum communication as it is regarded as an

additional source of information. A more detailed explanation of this concepts is given in Chapters 5, 6 and 7.

A way of defining *Entanglement* is through the concept of *separability* of a quantum state. Given a bipartite system AB, its state is said to be *separable* if it can be written in the form

$$|\psi\rangle_{AB} = |\varphi\rangle_A \otimes |\chi\rangle_B . \tag{2.5.1}$$

In this case, the reduced density matrices of the sub-systems A and B can be written simply as

$$\rho_A = |\varphi\rangle \langle\varphi| \quad \rho_B = |\chi\rangle \langle\chi| . \tag{2.5.2}$$

Any state that cannot be written this way is called *Entangled*.

2.6 Quantum Shannon theory

As introduced in Chapter 1, *Quantum Shannon theory* is an extension of classical *Information theory* to the framework of *Quantum mechanics*. We present in this section only the necessary concepts needed to understand the following chapters.

2.6.1 Qubits

The fundamental unit of *Quantum information* is the *Qubit*. We call *Qubit* a two-dimensional quantum system whose basis is generally defined by the pair

$$|0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \tag{2.6.1}$$

Many parametrization for the state of a *Qubit* appear in the literature, for our purpose it is enough to know that any state can be written in terms of a *density matrix* of the form

$$\rho = \begin{bmatrix} a & c \\ \overline{c} & b \end{bmatrix} , \tag{2.6.2}$$

where $a, b \in \mathbb{R}$ must satisfy $a + b = 1$ and c is a complex number.

2.6.2 Quantum channels

We have already defined in Section 2.2 how the state of a quantum system can in general expressed by a *density matrix* defined as in Equation 2.2.1. If the state of the system is not *pure*, its evolution cannot be described by Postulate 2. The most general way to describe the evolution of a quantum system is through *Quantum channels*.

A simple description of *Quantum channels* was given by M. Wilde in his book *"From Classical to Quantum Shannon Theory"* [19], we will summarize it in this section.

Let denote the space of density operators acting on a Hilbert space $\mathcal{H}$ with $\mathcal{D}(\mathcal{H})$, let $\mathcal{L}(\mathcal{H})$ denote the space of linear operators acting on $\mathcal{H}$ and $\mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)$ denote the space of linear operators taking vectors of the Hilbert space $\mathcal{H}_A$ to a Hilbert space $\mathcal{H}_B$.

A linear map $\mathcal{M} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ is said to be *positive* if $\mathcal{M}(\rho)$ is *positive semi-definite* for all positive semi-definite $\rho \in \mathcal{L}(\mathcal{H}_A)$. Where a matrix ρ is *positive semi-definite* if and only if

$$\langle\psi|\,\rho\,|\psi\rangle \geq 0 \quad \forall\,|\psi\rangle \in \mathcal{H}. \tag{2.6.3}$$

A *Quantum channel* is a linear map $\mathcal{N} : \mathcal{D}(\mathcal{H}_A) \to \mathcal{D}(\mathcal{H}_B)$ that is also:

1. Trace preserving: $\mathrm{Tr}(\rho) = \mathrm{Tr}[\mathcal{N}(\rho)] \quad \forall \rho \in \mathcal{D}(\mathcal{H}_A)$
2. Completely positive: $\mathrm{Id}_R \otimes \mathcal{N}$ is a *positive map* for any reference system R of arbitrary dimension

where Id_R is the identity channel that leaves the system R unchanged. This two properties are necessary to guarantee that the map $\mathcal{N}$ describes a reasonable physical evolution.

A fundamental result for our work is the *Choi-Kraus decomposition* of a quantum channel, also know as *operator-sum representation*. It can be proven that a map $\mathcal{N} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ is a *quantum channel* if and only if it can be written as follows:

$$\mathcal{N}(\rho) = \sum_{\mu} A_\mu \rho A_\mu^\dagger, \tag{2.6.4}$$

where $\rho \in \mathcal{D}(\mathcal{H})$, $A_\mu \in \mathcal{L}(\mathcal{H}_A, \mathcal{H}_B) \quad \forall \mu$ and the following equality is satisfied:

$$\sum_{\mu} A_\mu^\dagger A_\mu = I_A. \tag{2.6.5}$$

The operators A_μ are called Kraus operators of the channel $\mathcal{N}$.

2.6.3 Von Neumann Entropy

In classical *Information theory*, the *Entropy* of a random variable $X : \Omega \to \mathcal{X}$ is defined as

$$H(X) := -\sum_{x \in \mathcal{X}} p(x) \log_2(x), \tag{2.6.6}$$

where $p(x)$ represents the probability of X having value x. The function $H(X)$ quantifies the amount of *information* carried by the random variable X.

An analogous concept was defined for a quantum state ρ, describing the amount of information carried by it— both *classical* and *quantum*. Moreover, it is important to quantify the *Entanglement* of a bipartite system as described by Preskill in [18]. The *von Neumann Entropy* of a state ρ is defined as

$$S(\rho) = -\mathrm{Tr}(\rho \ln \rho). \tag{2.6.7}$$

Sometimes we will refer to it as *VNE*.

In our work, the most important properties of this metric are the following:

- A *pure state* $\rho = |\varphi\rangle \langle\varphi|$ have $S(\rho) = 0$
- If $\rho = \frac{I}{d}$, i.e. is *maximally mixed*, than its S is maximal and given by $S(\rho) = \ln d$.

where d is the dimension of the Hilbert space associated to ρ.

Chapter 3

Evolution of Quantum Entropies under non-Markovian interaction

3.1 Introduction

In this chapter we study the evolution of *von Neumann Entropy* under the assumption of non-Markovian interaction. K. Kobayashi proved the existence of a lower bound to this quantity for a particular case: a quantum system subject to a specific Markovian evolution [2]. The lower bound is valid for the *VNE* of the state when time approaches infinity. This time limit guarantees that the decoherence process reached a stable point.

In this chapter, we will mostly use the notation adopted by Breuer et al. in [3]. We will call the subsystem of interest S and the environment E, the Hilbert space of the total system $S + E$ is

$$\mathcal{H}_{SE} = \mathcal{H}_S \otimes \mathcal{H}_E. \tag{3.1.1}$$

A state of the total system can be described by a density matrix $\rho_{SE} \in \mathcal{S}(\mathcal{H}_{SE})$, the set of physical states of $S + E$. Such state is a function of time, we will not explicit this relationship unless necessary.

The corresponding state of a subsystem is obtained by partial trace over the other subsystem:

$$\rho_S = \mathrm{Tr}_E\left(\rho_{SE}\right) \qquad \rho_E = \mathrm{Tr}_S\left(\rho_{SE}\right). \tag{3.1.2}$$

The *von Neumann Entropy* of the open system at time t is

$$S_t = -\mathrm{Tr}\left[\rho_S(t) \ln \rho_S(t)\right]. \tag{3.1.3}$$

We suppose that the evolution of the total system is governed by the Hamiltonian

$$H = H_s \otimes I_E + I_S \otimes H_E + H_I, \tag{3.1.4}$$

where H_S and H_E are the Hamiltonians of the system and the environment while H_I is an interaction Hamiltonian.

In Kobayashi's example, the evolution of the open system is governed by the *Lindblad master equation* [1, 20, 21]

$$\frac{d\rho_S}{dt} = -i\,[H_S, \rho_S] + \mathcal{D}[L](\rho_S). \tag{3.1.5}$$

L is called the *Lindblad operator* and it represents the decoherence process. Setting the reduced Planck constant $\hbar = 1$, we define

$$\mathcal{D}[L](\rho_S) = L\rho_S L^\dagger - \frac{1}{2}L^\dagger L\rho_S - \frac{1}{2}\rho_S L^\dagger L. \tag{3.1.6}$$

Kobayashi's result is the following:

$$S_\infty \geq \frac{\mathrm{Tr}(L^\dagger L\rho_\infty) - \mathrm{Tr}(L\rho_\infty L^\dagger \rho_\infty)}{||L||_F^2}, \tag{3.1.7}$$

where

$$S_\infty = \lim_{t\to\infty} S_t \qquad \rho_\infty = \lim_{t\to\infty} \rho_S(t).$$

In the cases where $\mathrm{Tr}(L^\dagger L\rho_\infty) - \mathrm{Tr}(L\rho_\infty L^\dagger\rho_\infty) > 0$, Equation 3.1.7 states that the *von Neumann Entropy* of the open system does not vanish in the decoherence process.

Even if interesting, Markovian evolution is often the wrong model to describe real world applications [3]. Our goal is to verify if the techniques used by Kobayashi in [2] can be useful to get similar results for non-Markovian evolution.

3.2 Non-Markovian evolution

According to the *Second postulate* of quantum mechanics [17], the unitary evolution of ρ_{SE} is given by

$$\rho_{SE}(t) = U(t)\rho_{SE}(0)U^\dagger(t), \tag{3.2.1}$$

where the unitary operator is $U(t) = \exp(-iHt)$, $\hbar = 1$ and $\rho_{SE} = \rho_S(0) \otimes \rho_E(0)$ by assumption.

For every $t \geq 0$, the open system evolution is given by

$$\rho_S = \mathrm{Tr}_E \left\{ U(t)\rho_S(0) \otimes \rho_E(0)U^\dagger(t) \right\}. \tag{3.2.2}$$

Given an initial environment state $\rho_E(0)$, Equation 3.2.2 defines a quantum dynamical map

$$\Phi_t : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H}). \tag{3.2.3}$$

This map describe the time evolution of an initial state of the open system:

$$\rho_S(0) \mapsto \rho_S = \Phi_t \rho_S(0). \tag{3.2.4}$$

We decided to focus on the set of dynamical maps Φ_t for which the inverse Φ_t^{-1} exists for each $t \geq 0$. If the evolution of the open system is described by one of these maps, its quantum master equation has the general structure

$$\frac{d}{dt}\rho_S = -i\left[H_S(t), \rho_S\right] + \sum_i \gamma_i(t) \left[A_i(t)\rho_S A_i^\dagger(t) - \frac{1}{2}\left\{ A_i^\dagger(t)A_i(t), \rho_S \right\} \right]. \tag{3.2.5}$$

If A, B are two operators: $[A, B] = AB - BA$ represents their *Commutator* and $\{A, B\} = AB + BA$ represents their *Anti-Commutator* [17].

The evolution described by such maps can be either *Markovian* or *non-Markovian*. Lindblad master equation described in 3.1.5 is a particular case of Equation 3.2.5 that describes a Markovian evolution.

Our idea is to substitute Lindblad equation with versions of Equation 3.2.5 describing non-Markovian evolution to see if bounds analogous to Equation 3.1.7 can be found.

3.2.1 Non-Markovianity criteria

Many criteria to distinguish Markovian and non-Markovian evolution have been proposed. This work is based on the definition given by Breuer et al. in [3]: if Φ_t is always invertible, the evolution is Markovian if and only if Φ_t is P-divisible.

An interesting subset of the P-divisible maps is the set of CP-divisible ones. For maps with the form of Equation 3.2.5, a simple criteria for CP-divisibility is introduced by Breuer et al.: Φ_t is CP-divisible if and only if $\gamma_i(t) \geq 0 \ \forall t$.

The authors also introduce criteria based on the physical interpretation of quantum non-Markovianity in Section C. The definition of P-divisibility and the other criteria is outside the scope of our work, we invite the reader to refer to the cited references for a detailed explanation.

3.3 Analysis of some non-Markovian models

All the master equations studied in this section are associated to non-Markovian evolution. In fact, it can be proved that the related dynamical maps are not P-divisible. The definition of the models and the study of their non-Markovianity can be found in Section III of Breuer et al. [3].

3.3.1 Pure decoherence model

This model is associated to a single Lindblad operator and has the following master equation

$$\frac{d}{dt}\rho_S = \gamma(t)\left[\sigma_z \rho_S \sigma_z - \rho_S\right], \tag{3.3.1}$$

where the time dependent decay rate

$$\gamma(t) = -\frac{1}{G(t)}\frac{d}{dt}G(t) \tag{3.3.2}$$

is related to the decoherence function

$$G(t) = \exp\left[-\int_0^\infty d\omega J(\omega)\coth\left(\frac{\beta\omega}{2}\right)\frac{1-\cos(\omega t)}{\omega^2}\right]. \tag{3.3.3}$$

The matrix σ_z is the Pauli-Z matrix:

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.3.4}$$

The first step to apply Kobayashi's idea is to find the derivative of VNE with respect to time. This function is independent from the Markovianity of the process and we can prove that it has the form

$$\frac{dS_t}{dt} = -\mathrm{Tr}\left(\frac{d\rho_S}{dt}\ln\rho_S\right). \tag{3.3.5}$$

Then, we substitute $\frac{d\rho_S}{dt}$ with the master equation in Equation 3.3.1 and we obtain

$$\begin{aligned} \frac{dS_t}{dt} &= -\gamma(t)\cdot\left[\mathrm{Tr}\left(\sigma_z\rho_S\sigma_z\ln\rho_S\right) + S_t\right] = \\ &= -\gamma(t)S_t - \gamma(t)\cdot\mathrm{Tr}\left(\sigma_z\rho_S\sigma_z\ln\rho_S\right). \end{aligned} \tag{3.3.6}$$

A fundamental step in Kobayashi's work is to study the sign of $\frac{dSt}{dt}$ to find constraints on the possible values of S_t when the decoherence process reaches a stable point. Following his

procedure we observe that the derivative is a linear function of S_t. Kobayashi exploits the monotonicity of such function to derive the lower bound on VNE.

In the case of Equation 3.3.6, the monotonicity is strictly related to the sign of $\gamma(t)$. Unfortunately, the sign of $\gamma(t)$ oscillates because of the sinusoidal nature of the decoherence function. Such behavior is discussed in Section III.A.1 of Breuer et al. [3] and is associated to the non-Markovianity of the process. It follows that $\frac{dS_t}{dt}$ is not a monotonic function of S_t. Therefore, the procedure that Kobayashi uses to find a lower bound on VNE cannot be applied to this model.

The only case where we can be sure of the sign of $\gamma(t)$ is when time tends to infinity. The decay rate $\gamma(t)$ has already been studied by Addis et al. in [22]. From their work it emerges that $\gamma(t)$ only has two possible behaviors when $t \to \infty$:

- $\lim_{t\to\infty} \gamma(t) = \lambda > 0$
- $\lim_{t\to\infty} \gamma(t) = 0$

In the first case, the decoherence process is Markovian and is not interesting for our study.
In the second case, the convergence of the decay rate implies the convergence of $\frac{dSt}{dt}$ to zero. We can interpret this by saying that the VNE converges to a fixed value in the long time limit, but we cannot tell anything about this value. As $\gamma(t) \to 0$, the second factor in Equation 3.3.6 could have any value.

Because of this, even when $t \to \infty$, we can state that the procedure used by Kobayashi in [2] is not useful to find a lower bound on the VNE when the evolution of the state is governed by Equation 3.3.1.

3.3.2 Two-level system in a dissipative environment

The second model we studied has the following master equation:

$$\frac{d}{dt}\rho_S = -\frac{i}{4}S(t)[\sigma_z, \rho_S] + \gamma(t)\left[\sigma_-\rho_S\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho_S\}\right], \tag{3.3.7}$$

where $S(t)$ represents the time dependent Lamb shift and is given by $S(t) = -2\Im(\dot{G}/G(t))$.

The matrices σ_+ and σ_- have the following forms:

$$\sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{3.3.8}$$

The operators $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$ represent the *Commutator* and *Anti-Commutator* described in Section 3.2.

For this model, the decay rate is defined as

$$\gamma(t) = -\frac{2}{|G(t)|} \cdot \frac{d}{dt}|G(t)|. \tag{3.3.9}$$

The form of the decoherence function $G(t)$ depends on the studied scenario. In particular, it depends on the value of a parameter γ_0 that quantifies the strength of the system-environment coupling. Its behavior has been studied in Section III.A.2 of Breuer et al. [3]. The case we are interested in is the one where $\gamma_0 > \frac{\lambda}{2}$. Here, λ represents the width of the spectral density function $J(\omega)$. Such function is fundamental in determining the decoherence function $G(t)$.

It turns out that when $\gamma_0 > \frac{\lambda}{2}$, the decoherence process is non-Markovian and the sign of $\gamma(t)$ oscillates. Moreover, in this case, the resulting dynamical map Φ_t is not always invertible. This means that the previously used criteria are not strictly applicable. Nevertheless, the non-Markovianity can be proved using the measure-based criteria we cited in Section 3.2.1.

As we did for the previous model, it is necessary to compute the derivative of the VNE. Substituting Equation 3.3.7 inside Equation 3.3.5 we obtain

$$\frac{dS_t}{dt} = \gamma(t) \cdot \left[\mathrm{Tr}\left(\sigma_+\sigma_-\rho_S \ln \rho_S\right) - \mathrm{Tr}\left(\sigma_-\rho_S\sigma_+ \ln \rho_S\right)\right]. \tag{3.3.10}$$

None of the trace terms is a monotonic function of VNE. Even if they were, because of the oscillation of the sign of $\gamma(t)$, $\frac{dS_t}{dt}$ would not be a monotonic function of S_t. Again, Kobayashi's procedure could not be applied as it relies on the monotonicity with respect to VNE.

When considering the limit for $t \to \infty$, even if $\gamma(t) \to \lambda \neq 0$, the lack of relationship with S_t would not allow us to draw any constraint on the value of VNE. If $\gamma(t) \to 0$ instead, the analysis would be analogous to the one in Section 3.3.1.

Chapter 4

Time evolution of Quantum Rényi and Tsallis Entropies

4.1 Introduction

Quantum Rényi entropies are a family of function that generalize *von Neumann entropy*. This functions were defined by Müller-Lennert et al. in [23] and the *Rényi entropy* of order $\alpha \in (0,1) \cup (1,+\infty)$ for the density matrix ρ is defined as follows:

$$R_\alpha(\rho) := \frac{1}{1-\alpha} \ln \mathrm{Tr}(\rho^\alpha). \tag{4.1.1}$$

A quantity that is strictly related to this family of functions is the *Quantum Tsallis entropy* of order $\alpha \in (0,1) \cup (1,+\infty)$:

$$T_\alpha(\rho) := \frac{1}{1-\alpha} \left[\mathrm{Tr}(\rho^\alpha) - 1\right]. \tag{4.1.2}$$

Both functions can be expressed in terms of the other:

$$\begin{aligned} R_\alpha(\rho) &= \frac{1}{1-\alpha} \ln \left[(1-\alpha) \cdot T_\alpha(\rho) + 1\right] \\ T_\alpha(\rho) &= \frac{1}{1-\alpha} \left(e^{(1-\alpha)\cdot R_\alpha} - 1\right). \end{aligned} \tag{4.1.3}$$

Thanks to the non-negativity of R_α and T_α and to the monotonicity of the logarithm and exponential functions, we can show that any bound found for *Rényi entropy* results in a bound for *Tsallis entropy*— and vice-versa. This allows us to study one of them to find bounds for both.

Some bounds for these quantities are known [24], but we have not found any result about a

system undergoing Markovian evolution and time approaching infinity. Hence, we decided to try if the methodology applied by Kobayashi to *von Neumann entropy* [2], and discussed in Chapter 3, could be useful to find analogous results. Unfortunately, the derivatives of *Renyi* and *Tsallis* entropy are not as simple to deal with as the one for *von Neumann entropy*, and we have not found a useful lower bound for these quantities. Nevertheless, in the following sections we present the tests we devised in the hope of laying the groundwork for a generalization of Kobayashi's result.

4.2 Rényi and Tsallis time derivatives

The first step in Kobayashi's process is the derivation of the *von Neumann Entropy* with respect to time and the application of Lindblad master equation to it. We proceed here to do the same for *quantum Rényi entropy*.

The first derivative of *Rényi entropy* is

$$\frac{dR_\alpha(\rho)}{dt} = \frac{d}{dt}\left[\frac{1}{1-\alpha}\ln \mathrm{Tr}(\rho^\alpha)\right]. \tag{4.2.1}$$

Thanks to the linearity of the trace and of the derivative

$$\frac{dR_\alpha(\rho)}{dt} = \frac{1}{1-\alpha}\cdot\frac{d}{dt}\ln \mathrm{Tr}(\rho^\alpha). \tag{4.2.2}$$

By applying twice the chain rule of calculus we get

$$\begin{aligned}\frac{dR_\alpha(\rho)}{dt} &= \frac{1}{1-\alpha}\cdot\frac{1}{\mathrm{Tr}(\rho^\alpha)}\cdot\frac{d}{dt}\mathrm{Tr}(\rho^\alpha) = \\ &= \frac{1}{1-\alpha}\cdot\frac{1}{\mathrm{Tr}(\rho^\alpha)}\cdot\mathrm{Tr}\left(\frac{d\rho^\alpha}{dt}\right) = \\ &= \frac{\alpha}{1-\alpha}\cdot\frac{1}{\mathrm{Tr}(\rho^\alpha)}\cdot\mathrm{Tr}\left(\rho^{\alpha-1}\frac{d\rho}{dt}\right).\end{aligned} \tag{4.2.3}$$

Now, we substitute Equation 3.1.5 and obtain

$$\frac{dR_\alpha(\rho)}{dt} = \frac{\alpha}{1-\alpha}\cdot\frac{1}{\mathrm{Tr}(\rho^\alpha)}\cdot\mathrm{Tr}\left[\rho^{\alpha-1}\left(-i[H,\rho] + L\rho L^\dagger - \frac{1}{2}L^\dagger L\rho - \frac{1}{2}\rho L^\dagger L\right)\right]. \tag{4.2.4}$$

Because of the linearity of the trace and the fact that $\mathrm{Tr}\left[\rho^{\alpha-1}[H,\rho]\right] = 0$, we can write

$$\frac{dR_\alpha(\rho)}{dt} = \frac{\alpha}{1-\alpha}\cdot\frac{1}{\mathrm{Tr}(\rho^\alpha)}\cdot\left[\mathrm{Tr}(\rho^{\alpha-1}L\rho L^\dagger) - \frac{1}{2}\mathrm{Tr}(\rho^{\alpha-1}L^\dagger L\rho) - \frac{1}{2}\mathrm{Tr}(\rho^\alpha L^\dagger L)\right]. \tag{4.2.5}$$

Finally, thanks to the cyclic property of the matrix trace, the last two terms in the parenthesis sum together to give

$$\frac{dR_\alpha(\rho)}{dt} = \frac{\alpha}{1-\alpha} \cdot \frac{1}{\mathrm{Tr}(\rho^\alpha)} \cdot \left[\mathrm{Tr}(\rho^{\alpha-1} L\rho L^\dagger) - \mathrm{Tr}(\rho^\alpha L^\dagger L)\right]. \tag{4.2.6}$$

With analogous steps, we can show that the time derivative for *Tsallis entropy* is:

$$\frac{dT_\alpha(\rho)}{dt} = \frac{\alpha}{1-\alpha} \cdot \left[\mathrm{Tr}(\rho^{\alpha-1} L\rho L^\dagger) - \mathrm{Tr}(\rho^\alpha L^\dagger L)\right]. \tag{4.2.7}$$

The second step in Kobayashi's procedure is to find a lower bound on the derivative. This lower bound has a particular structure: it is a monotonic decreasing function that is non-negative when *von Neumann entropy* is zero. Moreover, the bounding function must also be negative when computed in $\ln(d)$— $d \geq 3$ is the rank of the quantum system. The final lower bound on *VNE* is found thanks to this property and *Bolzano's theorem*. These conditions enforce the lower bound when $t \to \infty$ for otherwise the entropy would be monotonically increasing and the decoherence process would not be stable— condition that is assumed in the limit condition.

Differently for the case of *von Neumann entropy*, the derivatives in Equations 4.2.6 and 4.2.7 present two problems. The derivative for Rényi does not contain $\ln(\rho^\alpha)$ that makes it harder to find an explicit lower bound that depends on *Rényi entropy*. Similarly, the derivative for Tsallis does not contain a "-1" term as in 4.1.2. To complicate things, the derivatives are multiplied by the term $\frac{\alpha}{1-\alpha}$ that changes sign depending on the interval in which α is. This implies that inequalities that allow to lower bound the derivatives when $\alpha \in (0,1)$ cannot be used when $\alpha \in (0,+\infty)$ and vice-versa.

4.3 $\ln x$ inequality

As described in the previous section, when using inequalities for the derivatives we have to be careful in considering the value of α for which the right inequality sign holds. In this section, we assume that $\alpha \in (0,1)$.

It is a known result that

$$\frac{1}{x} \geq 1 - \ln x \quad \forall x > 0. \tag{4.3.1}$$

Since $\mathrm{Tr}(\rho^\alpha) > 0$, we can apply the inequality to Equation 4.2.6 to get

$$\frac{dR_\alpha(\rho)}{dt} \geq \frac{\alpha}{1-\alpha} \cdot \left[1 - \ln\left(\frac{1}{\mathrm{Tr}(\rho^\alpha)}\right)\right] \cdot \left[\mathrm{Tr}(\rho^{\alpha-1} L\rho L^\dagger) - \mathrm{Tr}(\rho^\alpha L^\dagger L)\right]. \tag{4.3.2}$$

By computing the product and recalling the definition of *Rényi entropy* given in Equation 4.1.1, we obtain the following lower bound

$$\begin{aligned}\frac{dR_\alpha(\rho)}{dt} \geq &- \alpha \cdot R_\alpha(\rho) \cdot \left[\mathrm{Tr}(\rho^{\alpha-1} L\rho L^\dagger) - \mathrm{Tr}(\rho^\alpha L^\dagger L)\right] + \\ &+ \frac{\alpha}{1-\alpha} \cdot \left[\mathrm{Tr}(\rho^{\alpha-1} L\rho L^\dagger) - \mathrm{Tr}(\rho^\alpha L^\dagger L)\right].\end{aligned} \tag{4.3.3}$$

We have obtained a lower bound that is a monotonically decreasing function of $R_\alpha(\rho)$ as described in Section 4.2. As in Kobayashi's procedure, we now define a lower bounding function and compute it in 0 and $\ln d$. The author explains that these two values are associated respectively to ρ being a pure state and ρ being maximally mixed.

$$\begin{aligned}f(x) = &- \alpha \cdot x \cdot \left[\mathrm{Tr}(\rho^{\alpha-1} L\rho L^\dagger) - \mathrm{Tr}(\rho^\alpha L^\dagger L)\right] + \\ &+ \frac{\alpha}{1-\alpha} \cdot \left[\mathrm{Tr}(\rho^{\alpha-1} L\rho L^\dagger) - \mathrm{Tr}(\rho^\alpha L^\dagger L)\right].\end{aligned} \tag{4.3.4}$$

Since $x = 0$ means that ρ is a pure state, and if ρ is pure than it is trivial to show that $\rho^\alpha = \rho$, the previous equation simplifies to

$$f(0) = \frac{\alpha}{1-\alpha}\left[\mathrm{Tr}(L\rho L^\dagger\rho) - \mathrm{Tr}(L^\dagger L\rho)\right]. \tag{4.3.5}$$

Unfortunately, in his equation (7), Kobayashi proved that the term in the parenthesis is non-positive and so the whole function is. Because $f(0) \leq 0$, it does not matter what the value of $f(\ln d)$ is, we cannot conclude the anything about the existence of a single root and the sign of $f(x)$. For this reason, Kobayashi's argument cannot be applied to this lower bounding function. Anyway, this result does not rule out the existence of a possible lower bound on $R_\alpha(\rho)$ when $t \to \infty$.

4.4 Adding and subtracting $||L||_F^2$

We mentioned in Section 4.1 that finding a lower bound on *Tsallis entropy* is the same as finding a lower bound on *Rényi entropy*. In this section, we tried to manipulate the derivative of *Tsallis entropy* to find such bound. Again, we assume $\alpha \in (0, 1)$.

By definition, the density matrix ρ is positive semi-definite. Writing its spectral decomposition and taking the power α of every eigenvalue we obtain the matrix ρ^α. Because the power of a positive number is another positive number, the matrix ρ^α has only positive eigenvalues and is by definition positive. Moreover, the product $L^\dagger L$ is a positive operator for every possible operator L as mentioned in Nielsen and Chuang [17].

Thanks to the positivity of these two matrices, we can write the following inequalities

$$\mathrm{Tr}(L^\dagger L\rho^\alpha) \leq ||L||_F^2 \cdot \mathrm{Tr}(\rho^\alpha) \implies -\mathrm{Tr}(L^\dagger L\rho^\alpha) \geq -||L||_F^2 \cdot \mathrm{Tr}(\rho^\alpha), \tag{4.4.1}$$

where $||A||_F^2 := \sqrt{\mathrm{Tr}\,(A^\dagger A)}$ is the Frobenius norm of the operator A. The fact that $\mathrm{Tr}(AB) \leq \mathrm{Tr}(A)\mathrm{Tr}(B)$ for all $A, B \geq 0$ was used.

If we add and subtract $||L||_F^2$ to Equation 4.4.1 we obtain

$$\begin{aligned} -\mathrm{Tr}(L^\dagger L\rho^\alpha) &\geq -||L||_F^2 \cdot \mathrm{Tr}(\rho^\alpha) + ||L||_F^2 - ||L||_F^2 \\ &\geq -||L||_F^2 \left[\mathrm{Tr}(\rho^\alpha) - 1\right] - ||L||_F^2. \end{aligned} \tag{4.4.2}$$

We substitute inside Equation 4.2.7 and obtain the lower bound

$$\frac{dT_\alpha(\rho)}{dt} \geq \frac{\alpha}{1-\alpha} \cdot \left\{\mathrm{Tr}(\rho^{\alpha-1} L\rho L^\dagger) - ||L||_F^2 \left[\mathrm{Tr}(\rho^\alpha) - 1\right] - ||L||_F^2\right\}. \tag{4.4.3}$$

Rearranging the terms we are able to obtain a lower bounding function that is monotonically decreasing in $T_\alpha(\rho)$:

$$\frac{dT_\alpha(\rho)}{dt} \geq -\alpha||L||_F^2 \cdot T_\alpha(\rho) + \frac{\alpha}{1-\alpha}\left[\mathrm{Tr}(\rho^{\alpha-1} L\rho L^\dagger) - ||L||_F^2\right]. \tag{4.4.4}$$

Again, we define the lower bounding function

$$f(x) = -\alpha||L||_F^2 \cdot x + \frac{\alpha}{1-\alpha}\left[\mathrm{Tr}(\rho^{\alpha-1} L\rho L^\dagger) - ||L||_F^2\right]. \tag{4.4.5}$$

From the observations in Section 4.3 about the case $x = 0$ it follows that

$$f(0) = \frac{\alpha}{1-\alpha}\left[\mathrm{Tr}(L\rho L^\dagger\rho) - ||L||_F^2\right]. \tag{4.4.6}$$

Moreover, using the aforementioned property of the product of the trace of positive definite matrices and Equation (7) from [2], we can prove

$$\mathrm{Tr}(L\rho L^\dagger\rho) \leq \mathrm{Tr}(L^\dagger L\rho) \leq \mathrm{Tr}(L^\dagger L) \cdot \mathrm{Tr}(\rho) = ||L||_F^2, \tag{4.4.7}$$

where the last equation follows from the definition of density matrix.

From this inequality it follows that $f(0) \leq 0$ and a conclusion analogous to the one in Section 4.3 can be drawn.

4.5 $\rho^\alpha > \rho - I$

In this case, we analyze a possible lower bound on *Tsallis entropy* but we consider the interval $\alpha \in (1, +\infty)$. This implies that the multiplicative term $\frac{\alpha}{1-\alpha}$ is negative. We start by presenting two results that are then applied to Equation 4.2.7.

Using analogous steps as done by Kobayashi in his Appendix A of [2], we can prove that

$$\rho^\alpha > \rho - I \quad \forall \alpha \in \mathbb{R}. \tag{4.5.1}$$

The relationship follows from the trivial proof that if $x \in (0, 1)$, then $x^\alpha > x - I$ no matter what the value of α is. From this, we can prove the following chain of inequalities

$$\begin{aligned} & \mathrm{Tr}(L^\dagger L \rho^\alpha) \geq \mathrm{Tr}\left[L^\dagger L(\rho - I)\right] \\ \implies & -\mathrm{Tr}(L^\dagger L \rho^\alpha) \leq -\left[\mathrm{Tr}(L^\dagger L \rho) - ||L||_F^2\right] \\ \implies & -\frac{\alpha}{1-\alpha}\mathrm{Tr}(L^\dagger L \rho^\alpha) \geq -\frac{\alpha}{1-\alpha}\left[\mathrm{Tr}(L^\dagger L \rho) - ||L||_F^2\right]. \end{aligned} \tag{4.5.2}$$

For the second result, we exploit the positivity of $AA^\dagger$ for an operator A to prove the positivity of the matrix $L\rho L^\dagger$. We can write the density matrix as

$$\rho = \rho^{\frac{1}{2}}\rho^{\frac{1}{2}}. \tag{4.5.3}$$

Substituting in the product matrix and observing that $(\rho^{\frac{1}{2}})^\dagger = \rho^{\frac{1}{2}}$ we obtain

$$L\rho L^\dagger = (L\rho^{\frac{1}{2}})(\rho^{\frac{1}{2}}L^\dagger) = (L\rho^{\frac{1}{2}})(L\rho^{\frac{1}{2}})^\dagger \geq 0, \tag{4.5.4}$$

where the property of the Hermitian $(AB)^\dagger = B^\dagger A^\dagger$ was used.

Applying this and the inequality for the product of positive matrices we get

$$\begin{aligned} \mathrm{Tr}(\rho^{\alpha-1} L\rho L^\dagger) &\leq \mathrm{Tr}(L\rho L^\dagger) \cdot \mathrm{Tr}(\rho^{\alpha-1}) \\ &\leq ||L||_F^2 \cdot \mathrm{Tr}(\rho^{\alpha-1}) \\ &\leq ||L||_F^2 \cdot \left[\mathrm{Tr}(\rho^{\alpha-1}) - 1\right] + ||L||_F^2. \end{aligned} \tag{4.5.5}$$

In the last inequality we added and subtracted $||L||_F^2$.

Recalling that $\frac{\alpha}{1-\alpha} < 0$ and multiplying on both sides:

$$\begin{aligned}\frac{\alpha}{1-\alpha}\mathrm{Tr}(\rho^{\alpha-1}L\rho L^\dagger) &\geq \frac{\alpha}{1-\alpha}\cdot\left\{||L||_F^2\cdot\left[\mathrm{Tr}(\rho^{\alpha-1})-1\right]+||L||_F^2\right\}\\ &\geq \frac{\alpha}{1-\alpha}\cdot\frac{1-(\alpha-1)}{1-(\alpha-1)}\cdot\left\{||L||_F^2\cdot\left[\mathrm{Tr}(\rho^{\alpha-1})-1\right]+||L||_F^2\right\} \qquad (4.5.6)\\ &\geq \frac{\alpha(2-\alpha)}{1-\alpha}\cdot||L||_F^2\cdot T_{\alpha-1}(\rho)+\frac{\alpha}{1-\alpha}||L||_F^2.\end{aligned}$$

Substituting the results of Equations 4.5.2 and 4.5.6 in *Tsallis entropy*'s derivative we find the inequality

$$\begin{aligned}\frac{dT_\alpha(\rho)}{dt} &\geq \frac{\alpha(2-\alpha)}{1-\alpha}||L||_F^2\cdot T_{\alpha-1}(\rho)+\frac{\alpha}{1-\alpha}||L||_F^2-\frac{\alpha}{1-\alpha}\left[\mathrm{Tr}(L^\dagger L\rho)-||L||_F^2\right]\\ &\geq \frac{\alpha(2-\alpha)}{1-\alpha}||L||_F^2\cdot T_{\alpha-1}(\rho)+\frac{\alpha}{1-\alpha}\left[2\cdot||L||_F^2-\mathrm{Tr}(L^\dagger L\rho)\right].\end{aligned} \qquad (4.5.7)$$

From which follows the definition of the lower bounding function

$$f(x)=\frac{\alpha(2-\alpha)}{1-\alpha}||L||_F^2\cdot x+\frac{\alpha}{1-\alpha}\left[2\cdot||L||_F^2-\mathrm{Tr}(L^\dagger L\rho)\right]. \qquad (4.5.8)$$

We observe that in the previous equations the *quantum Tsallis entropy* of parameter $\alpha-1$ was found instead of the one of parameter α. Nonetheless, studying this quantity for $\alpha>2$ is the same as studying T_α for $\alpha\in(1,+\infty)$.

Contrary to what happened in the previous tests, if $\alpha>2$, the lower bounding function in Equation 4.5.8 is a monotonically increasing function of x. With a dual approach of Kobayashi's one, we could use these property to find an upper bound for *Tsallis entropy* in the long time limit $t\to\infty$. Thanks to the result in Equation 4.4.7 we can prove that $f(0)<0$ which would help the claim about the upper bound. Nevertheless, when computing such upper bound we obtain that it is bigger than the maximum value of *Tsallis entropy*. This means that even applying this inequalities no useful bound can be found.

Chapter 5

Entanglement Fidelity of common Quantum Channels

5.1 Introduction

In quantum Shannon theory, a source message can be described as a sequence of letters, where each letter is a pure state drawn from the ensemble $\{|\phi\rangle, q(x)\}$ [18].

Regardless of the choice of the ensemble, the sent letter will be represented by a density matrix

$$\rho = \begin{bmatrix} a & c \\ \overline{c} & b \end{bmatrix}, \tag{5.1.1}$$

where the elements $a, b \in \mathbb{R}$ and $c \in \mathbb{C}$.

In practice, the system representing the letter will interact with the environment. This interaction is represented by the introduction of a *Reference system* we cannot access. From this interaction, correlations between the two system generally arise in the form of *Quantum entanglement*. Once a letter is sent through a noisy quantum channel, its associated state is changed and this is reflected in a change of the correlations with the environment.

In many applications, it is in our interest to preserve the entanglement between the sub-system of interest and the reference system— or another system in general. M. A. Nielsen described how the maximization of this fidelity is fundamental for *Quantum error correction*[6], a corner-stone technique that allowed to build modern quantum devices. This concept can also be found in the study of *Quantum channels capacity* [8, 7], where different definitions of *quantum capacity* appear depending on whether we want to preserve entanglement or not. Datta et al. [5] revised the definition of the *Quantum rate distortion* problem where the distortion measure is based on *Entanglement fidelity*. Moreover, many quantum communication protocols, like *Quantum*

teleportation and *Superdense coding* involve sending one or more qubits through a channel while not acting on other correlated systems [11, 12, 25, 26].

To quantify how faithfully a quantum channel preserves this entanglement we use the *Entanglement fidelity*. Introduced by Schumacher in [4], he proved that the metric is a function of the density matrix ρ and the quantum channel $\mathcal{N}$ we apply to it— it can be computed without any relation to the reference system. For more details about the different type of fidelities and the meaning of *Entanglement fidelity* we invite the reader to refer to Chapter 7.

If we call A_μ a generic Kraus operator of the channel, the associated *Entanglement fidelity* can be computed as follows:

$$F_e(\rho, \mathcal{N}) = \sum_\mu \mathrm{Tr}(A_\mu \rho)\mathrm{Tr}(A_\mu^\dagger \rho). \tag{5.1.2}$$

The following inequalities hold

$$0 \leq F_e(\rho, \mathcal{N}) \leq 1. \tag{5.1.3}$$

If all the Kraus operators are hermitian, the equation above can be simplified to:

$$F_e(\rho, \mathcal{N}) = \sum_\mu \mathrm{Tr}^2(A_\mu \rho). \tag{5.1.4}$$

In the following sections we compute and analyze the value of *Entanglement fidelity* for different 1-qubit quantum channels, namely: *Random Pauli-X*, *Dephasing*, *Depolarizing*, *Werner-Holevo*, *Generalized Pauli* and *Amplitude damping* channels. The quantity is computed for the general density matrix defined in Equation 5.1.1.

All the examples will have the same structure: definition of the channel through Kraus operators, computation of $\mathrm{Tr}(A_\mu \rho)$ and $\mathrm{Tr}(A_\mu^\dagger \rho)$, computation of *Entanglement fidelity*.

5.2 F_e computation

In this section, the value of *Entanglement fidelity* for different channels is computed.
The channels are separated in two sections: the first three all depend on a parameter $u \in [0,1]$ and are based on Pauli matrices, the fourth one has no parameters, the fifth one depends on a matrix of parameters P and the last one depends on a parameter usually called γ.

5.2.1 Pauli channels

Random Pauli-X

The *Random Pauli-X channel* is defined as follows:

$$N_u(\rho) = u \cdot X\rho X^\dagger + (1-u)\rho \ ; \ u \in [0,1]. \tag{5.2.1}$$

It can be described with two Kraus operators:

$$\left\{\sqrt{u}X, \sqrt{1-u}I\right\}. \tag{5.2.2}$$

The *Entanglement fidelity* for this channel and the density matrix defined above is given by:

$$\begin{aligned} F_e(\rho, \mathcal{N}_u) &= \mathrm{Tr}^2(\sqrt{u} \cdot X\rho) + \mathrm{Tr}^2(\sqrt{1-u} \cdot I\rho) \\ &= u \cdot \mathrm{Tr}^2(X\rho) + (1-u). \end{aligned} \tag{5.2.3}$$

The las inequality holds because of the linearity of the trace and the fact that density matrices have unitary trace.
Now, we need to compute the matrix in the first factor of the sum:

$$X\rho = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & c \\ \overline{c} & b \end{bmatrix} = \begin{bmatrix} \overline{c} & b \\ a & c \end{bmatrix}. \tag{5.2.4}$$

Its trace is given by:

$$\mathrm{Tr}(X\rho) = 2 \cdot \Re(c). \tag{5.2.5}$$

We can now compute the *Entanglement fidelity*:

$$F_e(\rho, \mathcal{N}_u) = 1 + u\left[4 \cdot \Re^2(c) - 1\right], \tag{5.2.6}$$

where we recall from Section 2.1.2 that $\Re(c)$ represents the *real part* of c.

Dephasing channel

The *Dephasing channel* is defined as follows:

$$N_u(\rho) = u \cdot Z\rho Z^\dagger + (1-u)\rho \ ; \ u \in [0,1]. \tag{5.2.7}$$

It can be described with two Kraus operators:

$$\left\{\sqrt{u}Z, \sqrt{1-u}I\right\}. \tag{5.2.8}$$

The *Entanglement fidelity* for this channel and the density matrix defined above is given by:

$$\begin{aligned} F_e(\rho, \mathcal{N}_u) &= \mathrm{Tr}^2(\sqrt{u} \cdot Z\rho) + \mathrm{Tr}^2(\sqrt{1-u} \cdot I\rho) \\ &= u \cdot \mathrm{Tr}^2(Z\rho) + (1-u). \end{aligned} \tag{5.2.9}$$

We need to compute the matrix in the first factor of the sum:

$$Z\rho = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} a & c \\ \overline{c} & b \end{bmatrix} = \begin{bmatrix} a & c \\ -\overline{c} & -b \end{bmatrix}. \tag{5.2.10}$$

Its trace is given by:

$$\mathrm{Tr}(Z\rho) = a - b. \tag{5.2.11}$$

We can compute the *Entanglement fidelity*:

$$F_e(\rho, \mathcal{N}_u) = 1 + u\left[(a-b)^2 - 1\right]. \tag{5.2.12}$$

Depolarizing channel

The *Depolarizing channel* is defined as follows:

$$N_u(\rho) = (1-u)\rho + u \cdot \frac{I}{2} \; ; \; u \in [0,1]. \tag{5.2.13}$$

According to Nielsen and Chuang [17], we can write the channel with the following operator-sum representation:

$$\mathcal{N}_u(\rho) = \left(1 - \frac{3u}{4}\right)\rho + \frac{u}{4}\left(X\rho X + Y\rho Y + Z\rho Z\right). \tag{5.2.14}$$

It follows that the channel has the following Kraus operators:

$$\left\{\sqrt{1 - \frac{3u}{4}} \cdot I, \frac{\sqrt{u}}{2} \cdot X, \frac{\sqrt{u}}{2} \cdot Y, \frac{\sqrt{u}}{2} \cdot Z\right\}. \tag{5.2.15}$$

We already computed $\mathrm{Tr}(X\rho)$ and $\mathrm{Tr}(Z\rho)$ in the previous two sections.

$$Y\rho = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \cdot \begin{bmatrix} a & c \\ \overline{c} & b \end{bmatrix} = \begin{bmatrix} -i\overline{c} & -ib \\ ia & ic \end{bmatrix}. \tag{5.2.16}$$

So that the third trace is

$$\mathrm{Tr}(Y\rho) = -2 \cdot \Im(c), \tag{5.2.17}$$

where we recall from Section 2.1.2 that $\Im(c)$ represents the *imaginary part* of c.

It follows that the *Entanglement fidelity* for this channel is:

$$\begin{aligned} F_e(\rho, \mathcal{N}_u) &= \left(1 - \frac{3u}{4}\right) + \frac{u}{4}\mathrm{Tr}^2(X\rho) + \frac{u}{4}\mathrm{Tr}^2(Y\rho) + \frac{u}{4}\mathrm{Tr}^2(Z\rho) = \\ &= \left(1 - \frac{3u}{4}\right) + \frac{u}{4} \cdot 4 \cdot \Re^2(c) + \frac{u}{4} \cdot 4 \cdot \Im^2(c) + \frac{u}{4}(a-b)^2 = \\ &= 1 + u \cdot \left[\Re^2(c) + \Im^2(c) + \frac{(a-b)^2}{4} - \frac{3}{4}\right] \\ &= 1 + u \cdot \left[|c|^2 + \frac{(a-b)^2}{4} - \frac{3}{4}\right]. \end{aligned} \tag{5.2.18}$$

Werner-Holevo channel

Werner and Holevo ([27]) showed that for any dimension d, the map

$$\rho \to \Lambda_{WH}(\rho) = \frac{1}{d-1}(tr(\rho)I - \rho^T). \tag{5.2.19}$$

is a valid quantum channel.
When $d = 2$, the channel degenerates to the simple form

$$\Lambda_{WH}(\rho) = Y\rho Y. \tag{5.2.20}$$

In this case the only Kraus operator is Y and the *Entanglement fidelity* is given by

$$F_e(\rho, \Delta_{WH}) = \mathrm{Tr}^2(Y\rho) = 4 \cdot \Im^2(c). \tag{5.2.21}$$

Generalized Pauli channel

In their article *"Detecting positive quantum capacities of quantum channels"* [28], Singh and Datta studied the capacity of the *Generalized Pauli channel*. This channel is defined for every dimension d of the Hilbert space of a system. Moreover, its definition depends on a $d \times d$ matrix P such that $\sum_{i,j} p_{ij} = 1$. For a single qubit, its Kraus operators simplify to

$$\left\{\sqrt{p_{00}}I, \sqrt{p_{10}}X, \sqrt{p_{01}}Z, \sqrt{p_{11}}XZ\right\}$$

With an appropriate choice of P and letting $u = 1$, the channels presented in Sections 5.2.1, 5.2.1, 5.2.1 can be derived.

Exploiting the results of Section 5.2.1, we obtain the entanglement fidelity for this channel:

$$\begin{aligned} F_e(\rho,\mathcal{N}) &= p_{00}\mathrm{Tr}^2(I\rho) + p_{10}\mathrm{Tr}^2(X\rho) + p_{01}\mathrm{Tr}^2(Z\rho) + p_{11}\mathrm{Tr}^2(XZ\rho) = \\ &= p_{00} + 4\cdot p_{10}\cdot \Re^2(c) + p_{01}(a-b)^2 - 4\cdot p_{11}\cdot \Im^2(c), \end{aligned} \tag{5.2.22}$$

where $\mathrm{Tr}(XZ\rho)$ can be obtained observing that $XZ = -i\cdot Y$.

5.2.2 Amplitude damping channel

The *Amplitude damping channel* is defined through the following Kraus operators:

$$\left\{A_0 = \sqrt{\gamma}\cdot|0\rangle\langle 1|, A_1 = |0\rangle\langle 0| + \sqrt{1-\gamma}\cdot|1\rangle\langle 1|\right\}\ ;\ \gamma\in[0,1]. \tag{5.2.23}$$

The two associated matrices are:

$$A_0 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}\ ;\ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}. \tag{5.2.24}$$

The *Entanglement fidelity* for this channel and the density matrix defined above is given by:

$$F_e(\rho,\mathcal{N}) = \mathrm{Tr}(A_0\rho)\mathrm{Tr}(A_0^\dagger\rho) + \mathrm{Tr}^2(A_1\rho). \tag{5.2.25}$$

We compute the traces in the sum:

$$\begin{aligned} \mathrm{Tr}(A_0\rho) &= \mathrm{Tr}\left(\begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} a & c \\ \overline{c} & b \end{bmatrix}\right) \\ &= \mathrm{Tr}\left(\begin{bmatrix} \overline{c}\cdot\sqrt{\gamma} & b\cdot\sqrt{\gamma} \\ 0 & 0 \end{bmatrix}\right) \\ &= \overline{c}\cdot\sqrt{\gamma}, \end{aligned} \tag{5.2.26}$$

$$\begin{aligned} \mathrm{Tr}(A_0^\dagger\rho) &= \mathrm{Tr}\left(\begin{bmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{bmatrix}\cdot\begin{bmatrix} a & c \\ \overline{c} & b \end{bmatrix}\right) \\ &= \mathrm{Tr}\left(\begin{bmatrix} 0 & 0 \\ a\cdot\sqrt{\gamma} & c\cdot\sqrt{\gamma} \end{bmatrix}\right) \\ &= c\cdot\sqrt{\gamma}, \end{aligned} \tag{5.2.27}$$

$$\begin{aligned}\mathrm{Tr}(A_1\rho) &= \mathrm{Tr}\left(\begin{bmatrix}1 & 0\\ 0 & \sqrt{1-\gamma}\end{bmatrix}\cdot\begin{bmatrix}a & c\\ \overline{c} & b\end{bmatrix}\right)\\ &= \mathrm{Tr}\left(\begin{bmatrix}a & c\\ \overline{c}\cdot\sqrt{1-\gamma} & b\cdot\sqrt{1-\gamma}\end{bmatrix}\right)\\ &= a + b\cdot\sqrt{1-\gamma}.\end{aligned} \tag{5.2.28}$$

Now, we can compute the *Entanglement fidelity*:

$$F_e(\rho,\mathcal{N}) = \gamma\cdot|c|^2 + \left(a + b\cdot\sqrt{1-\gamma}\right)^2. \tag{5.2.29}$$

5.3 Summary

To better visualize the results obtained in this chapter we summarize them in the following table:

Channel $\mathcal{N}$	$F_e(\rho,\mathcal{N})$
Random Pauli-X	$1 + u\left[4\cdot\Re^2(c) - 1\right]$
Dephasing	$1 + u\left[(a-b)^2 - 1\right]$
Depolarizing	$1 + u\cdot\left[\Re^2(c) + \Im^2(c) + \frac{(a-b)^2}{4} - \frac{3}{4}\right]$
Generalized Pauli	$p_{00} + 4\cdot p_{10}\cdot\Re^2(c) + p_{01}(a-b)^2 - 4\cdot p_{11}\cdot\Im^2(c)$
Amplitude damping	$\gamma\cdot|c|^2 + (a + b\cdot\sqrt{1-\gamma})^2$
Werner-Holevo	$4\cdot\Im^2(c)$

Table 5.1: *Entanglement fidelity* for the studied quantum channels and ρ as defined in Equation 5.1.1.

Chapter 6

Entanglement Fidelity for a two-letter Quantum Message

6.1 Introduction

Now we apply the results of Chapter 5 to a scenario where we want to send a message of two possible quantum letters. The goal is to show how choosing appropriate pairs of letters can lead to an optimal preservation of entanglement between the letter and the reference system.

Let's consider the following pair of letters:

$$\begin{aligned} |\psi^+\rangle &= \sqrt{p}\,|0\rangle + \sqrt{1-p}\,|1\rangle \\ |\psi^-\rangle &= \sqrt{p}\,|0\rangle - \sqrt{1-p}\,|1\rangle\,. \end{aligned} \tag{6.1.1}$$

The following density matrix describe a system that has probability q of being in state $|\psi^+\rangle$ and $1-q$ of being in state $|\psi^-\rangle$:

$$\rho = q \cdot |\psi^+\rangle\langle\psi^+| + (1-q) \cdot |\psi^-\rangle\langle\psi^-|\,. \tag{6.1.2}$$

Here is the explicit form of the density matrix:

$$\rho = \begin{bmatrix} p & (2q-1)\cdot\sqrt{p(1-p)} \\ (2q-1)\cdot\sqrt{p(1-p)} & 1-p \end{bmatrix}. \tag{6.1.3}$$

Depending on the values of p and q we obtain different source messages. There are 5 extreme cases that are not relevant to real world applications and lead to special behavior:

- If $p = 0$ or $p = 1$, regardless of the value of q, we obtain a one letter alphabet which cannot convey any meaning.
- The cases $q = 0$ and $q = 1$ are analogous to the above ones.
- If $p = \frac{1}{2}$ the two letters $|\psi^+\rangle$ and $|\psi^-\rangle$ are orthogonal and so the message could as well be classical— as stated by Preskill in [18].

The remaining cases describe relevant scenarios of messages built with an alphabet of two non-orthogonal letters, each occurring with non-zero probability.

6.2 F_e computation

In the case of Equation 6.1.3, the parameters a, b, c defined in 5.1 are all real and have the following values:

$$\begin{aligned} a &= p \\ b &= 1 - p \\ c &= (2q - 1) \cdot \sqrt{p(1-p)}. \end{aligned} \tag{6.2.1}$$

Using the results obtained in Section 5.2, we can compute the value of the *Entanglement fidelity* for different channels when using a message composed of the letters in Equation 6.1.1. The results are summarized in Table 6.1.

Channel $\mathcal{N}$	$F_e(\rho, \mathcal{N})$
Random Pauli-X	$1 + u \cdot \left[4 \cdot (2q-1)^2 \cdot p(1-p) - 1\right]$
Dephasing	$1 + u\,[4 \cdot p(p-1)]$
Depolarizing	$1 + \frac{u}{4} \cdot \left\{\left[4 \cdot (2q-1)^2 \cdot p(1-p) - 1\right] + [4 \cdot p(p-1)] - 1\right\}$
Generalized Pauli	$p_{00} + p_{10} \cdot [4 \cdot (2q-1)^2 \cdot p(1-p) - 1] + p_{01} \cdot (2p-1)^2$
Amplitude damping	$p^2 + (1-\gamma)(1-p)^2 + p(1-p)\left[2\sqrt{1-\gamma} + \gamma(2q-1)^2\right]$
Werner-Holevo	0

Table 6.1: *Entanglement fidelity* for the studied quantum channels and ρ as defined in Equation 6.1.2.

6.3 F_e Analysis

In general, the *Entanglement fidelity* of the studied channels depends on three parameters: one that defines the channel— like u or γ — and two that define the letters, namely p and q.
In this section, some graphs that study the fidelities in terms of p and some values of q and u or γ are given. To get a better understanding on the relationship between F_e and the three parameters we added a 3D plot of the $Entanglement fidelity$ a the end of the sections related to the *Pauli channels* and the *Amplitude damping channel*.

In a real case scenario, we might not have the possibility to choose all of them. In fact, the parameter that defines the channel might depend on problem specification that we cannot control. The value of q depends on the specific messages the source produces and we may not be able to control it.

Luckily, the parameter p determines the particular states we use as letters and in principle we could prepare the states determined by every possible value. Even if in practice there might be some restrictions, this means that we have some freedom in the choice of the letters. With this tool, we could choose the letters that behave the best given the properties of the channel at hand.

6.3.1 Pauli channels

All the Pauli channels depend on a parameter u. When the parameter is zero, the channels act as an identity and they do not change the state. On the other hand, when $u = 1$, they fully affect the input state applying a Pauli matrix to it or combining more types of these applications. For this reasons, we expect the entanglement fidelity to be generally higher the smaller the u. In these cases more of the initial state is unchanged thanks to the identity.

Random Pauli-X channel

Recall from Table 6.1 that F_e for this channel is

$$F_e(\rho, \mathcal{N}_u) = 1 + u \cdot \big[4 \cdot (2q-1)^2 \cdot p(1-p) - 1\big].$$

For F_e is between 0 and 1 as stated in equation 5.1.3, the second operand in the sum must be negative. Moreover, as we were expecting, the smaller the value of u the more of the initial state is preserved. This is reflected in the fact that F_e might have a lower bound bigger then zero and the channel might not be able to completely destroy the entanglement.

Once the value of u and q is fixed, we are interested in studying which value of p would be the best to preserve entanglement as much as possible. From Figure 6.3.1 we can state that, no matter what the value of u and q are, the highest values of entanglement fidelity are achieved when p is close to 0.5.

As discussed in Section 6.1, when $p = 0.5$ the message degenerates to a classical one. Anyway, we can preserve the interesting properties of quantum messages by choosing p as close to 0.5 as possible given the constraints we might have.

The choice of p is influenced by the probability q of sending one of the two letters: the closer q gets to 0.5, the more values of p we could choose without affecting too much the value of the *Entanglement fidelity*.

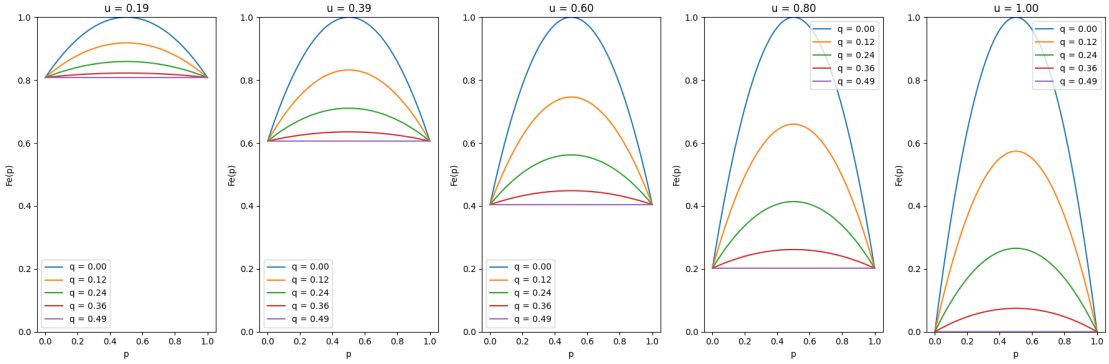

Figure 6.3.1: *Random Pauli-X channel*'s $F_e(p)$ for fixed values of u and q

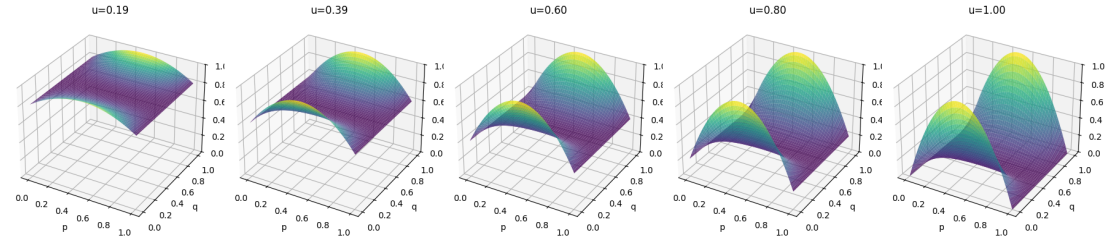

Figure 6.3.2: *Random Pauli-X channel*'s $F_e(p)$ for fixed values of u

Dephasing channel

The *Entanglement fidelity* for the *Dephasing channel* is given by Table 6.1:

$$F_e(\rho, \mathcal{N}) = 1 + u\,[4 \cdot p(p-1)]$$

In this case, it is a function of only u and p. Once u is fixed, for every value of q, F_e will be a convex quadratic function of p (Figure 6.3.3). This implies that the function has only one minimizer: $p = 0.5$.

Contrary to what happens for the Pauli-X channel (Section 6.3.1), the best values of F_e are obtained when p approaches one of the extremes of the interval. These extreme values represent

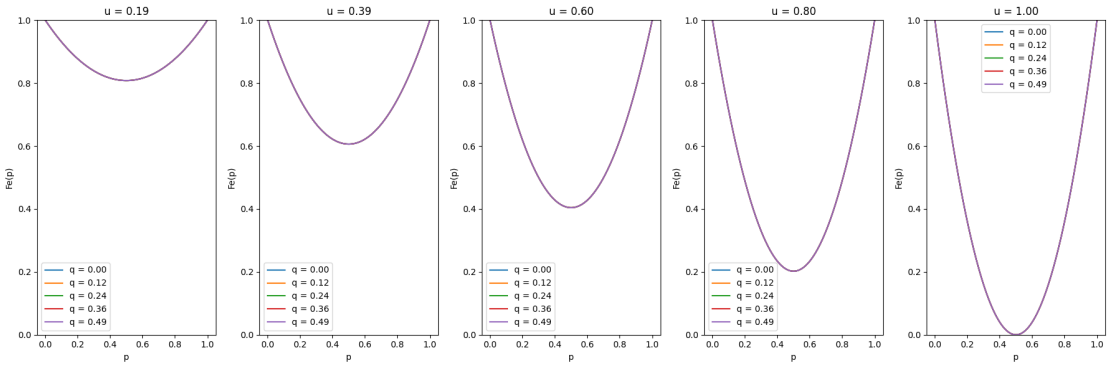

Figure 6.3.3: *Dephasing channel*'s $F_e(p)$ for fixed values of u and q

cases where the two letter states are the same and therefore are not useful. We can keep p as close as possible to these values in order to get the most entanglement preservation.

Even if it may not be feasible in practice, letting $p \to 0$ or $p \to 1$, we could theoretically achieve values of F_e arbitrarily close to 1.

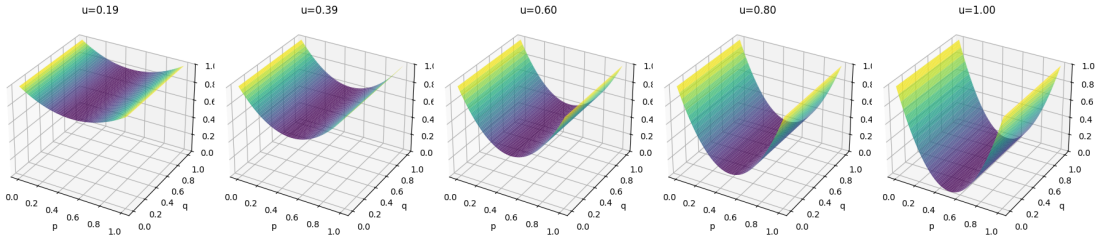

Figure 6.3.4: *Dephasing channel*'s $F_e(p)$ for fixed values of u

Depolarizing channel

The *Entanglement fidelity* for the *Depolarizing channel* is given by Table 6.1:

$$F_e(\rho, \mathcal{N}) = 1 + \tfrac{u}{4} \cdot \left[4 \cdot (2q-1)^2 \cdot p(1-p) + (2p-1)^2 - 3\right]$$

When the value of u and q is fixed, F_e is a convex quadratic function of p (Figure 6.3.5). As in the case of the *Dephasing channel* (Section 6.3.1), the function has one minimizer in $p = 0.5$ and the best values are achieved as p approaches the extremes of its range.

The big difference between the two channels lies in the maximum value of the *Entanglement fidelity*. When using a *Depolarizing channel*, it does not matter what value of p we choose, we cannot achieve $F_e = 1$ if $u \neq 0$.

On the other hand, differently from the Pauli-X channel case of Section 6.3.1, even if $u = 1$ the channel never completely destroys the entanglement— F_e has a non-zero minimum for every value of p and q.

As we could expect from the interpretation of the behavior of the channel, the bigger u is, the lower the maximum achievable *Entanglement fidelity* is— the closer u gets to 1 the bigger is the contribution of the maximally mixed state $\frac{I}{2}$ in the output state.

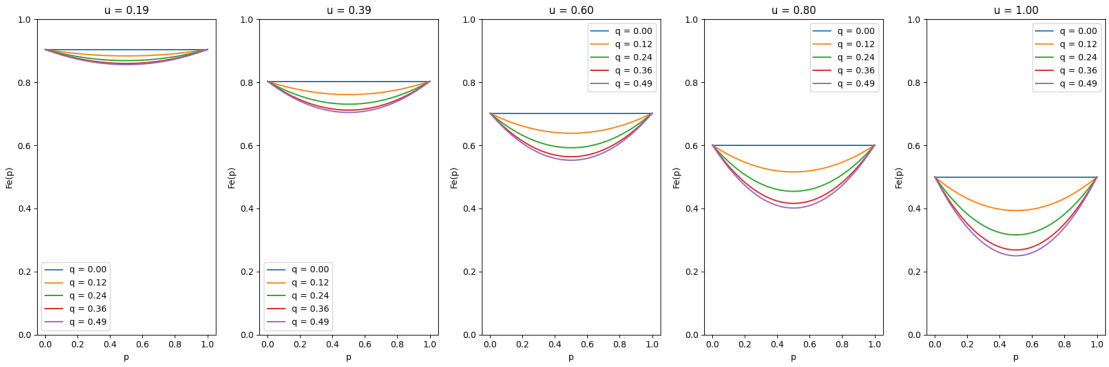

Figure 6.3.5: *Depolarizing channel*'s $F_e(p)$ for fixed values of u and q

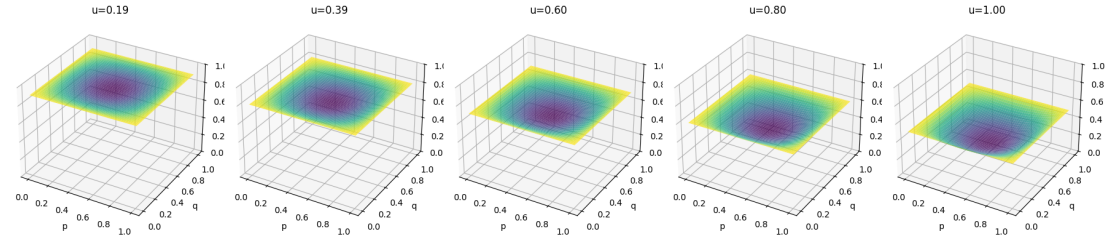

Figure 6.3.6: *Depolarizing channel*'s $F_e(p)$ for fixed values of u

Amplitude damping channel

The *Amplitude damping channel*'s fidelity (Table 6.1) has the form

$$F_e(\rho, \mathcal{N}) = p^2 + (1-\gamma)(1-p)^2 + p(1-p)\left[2\sqrt{1-\gamma} + \gamma(2q-1)^2\right]$$

Differently from the previous cases, the function shows an asymmetrical behavior with respect to $p = 0.5$ (see Figure 6.3.7). Moreover, its convexity changes for different values of q. Nonetheless, when p gets close to 1 the same happens to F_e.

The extreme scenario is not interesting in practical applications as discussed before (see Section 6.1). Anyway, we can conclude that when using an *Amplitude damping channel*, it is better to choose letters from Equation 6.1.1 where the main component in the superposition is $|0\rangle$.

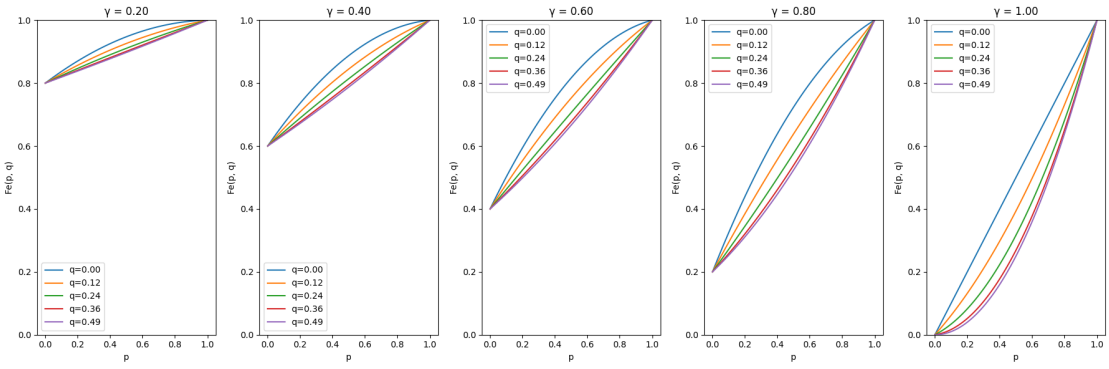

Figure 6.3.7: *Amplitude damping channel*'s $F_e(p)$ for fixed values of u and q

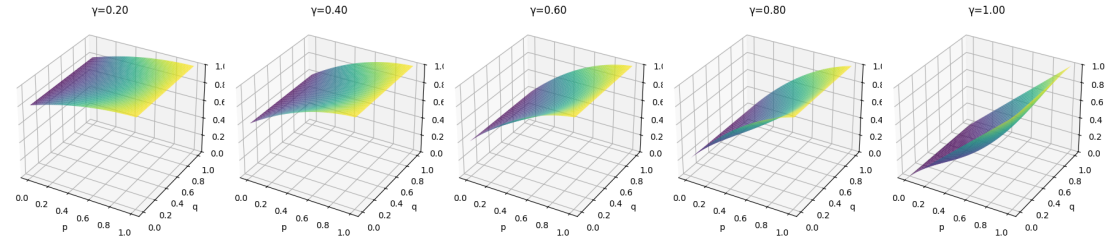

Figure 6.3.8: *Amplitude damping channel*'s $F_e(p)$ for fixed values of γ

6.4 Pauli channels ranking

The Pauli channels described in the previous sections have a similar structure that depends on the parameter u. Using one channel or the other introduces different types of noise in the state we send. In Section 6.3 we described how, once the channel is fixed, we can choose our letters to get the best *Entanglement fidelity* possible. In this section instead, we rank the three channels for a fixed value of q and for every possible value of p.
Recalling the results in Table 6.1 and fixing the value of u and q we define the following quantities:

- Pauli-X channel: $F_e^X(p) = 1 + u \cdot \left[4 \cdot (2q-1)^2 \cdot p(1-p) - 1\right]$
- Dephasing channel: $F_e^Z(p) = 1 + u\left[4 \cdot p(p-1)\right]$
- Depolarizing channel: $F_e^D(p) = 1 + \frac{u}{4} \cdot \left\{\left[4 \cdot (2q-1)^2 \cdot p(1-p) - 1\right] + \left[4 \cdot p(p-1)\right] - 1\right\}$

To simplify the notation in the following discussion we set

$$c = (2q-1)^2$$

Moreover, we consider $u \neq 0$ as no real application will include an identity channel.

By solving the inequalities, we can prove that the following results hold:

$$F_e^X(p) \geq F_e^Z(p) \quad \forall p \in [\frac{1}{2} - \frac{1}{2}\sqrt{\frac{c}{1+c}}, \frac{1}{2} + \frac{1}{2}\sqrt{\frac{c}{1+c}}], \tag{6.4.1}$$

$$F_e^D(p) \geq F_e^Z(p) \quad \forall p \in [\frac{1}{2} - \frac{1}{2}\sqrt{\frac{1+c}{3+c}}, \frac{1}{2} + \frac{1}{2}\sqrt{\frac{1+c}{3+c}}]. \tag{6.4.2}$$

When studying $F_e^D(p) \geq F_e^X(p)$, two cases must be distinguished. The subdivision is due to the nonexistence of real-valued solutions when $q \in \left(\frac{3-\sqrt{3}}{3+\sqrt{3}}, \frac{3+\sqrt{3}}{3+\sqrt{3}}\right)$.

Whenever $q \in \left(\frac{3-\sqrt{3}}{3+\sqrt{3}}, \frac{3+\sqrt{3}}{3+\sqrt{3}}\right)$:

$$F_e^D(p) \geq F_e^X(p) \quad \forall p \in [0,1]. \tag{6.4.3}$$

Otherwise:

$$F_e^D(p) \geq F_e^X(p) \quad \forall p \in [0, \frac{1}{2} - \frac{1}{2}\sqrt{\frac{3c-1}{1+3c}}] \cup [\frac{1}{2} + \frac{1}{2}\sqrt{\frac{3c-1}{1+3c}}, 1]. \tag{6.4.4}$$

p	**Ranking**
$\left[0, \frac{1}{2} - \frac{1}{2}\sqrt{\frac{1+c}{3+c}}\right)$	$F_e^Z(p) \geq F_e^D(p) \geq F_e^X(p)$
$\left[\frac{1}{2} - \frac{1}{2}\sqrt{\frac{1+c}{3+c}}, \frac{1}{2} - \frac{1}{2}\sqrt{\frac{c}{1+c}}\right)$	$F_e^D(p) \geq F_e^Z(p) \geq F_e^X(p)$
$\left[\frac{1}{2} - \frac{1}{2}\sqrt{\frac{c}{1+c}}, \frac{1}{2} - \frac{1}{2}\sqrt{\frac{3c-1}{1+3c}}\right)$	$F_e^D(p) \geq F_e^X(p) \geq F_e^Z(p)$
$\left[\frac{1}{2} - \frac{1}{2}\sqrt{\frac{3c-1}{1+3c}}, \frac{1}{2} + \frac{1}{2}\sqrt{\frac{3c-1}{1+3c}}\right)$	$F_e^X(p) \geq F_e^D(p) \geq F_e^Z(p)$
$\left[\frac{1}{2} + \frac{1}{2}\sqrt{\frac{3c-1}{1+3c}}, \frac{1}{2} + \frac{1}{2}\sqrt{\frac{c}{1+c}}\right)$	$F_e^D(p) \geq F_e^X(p) \geq F_e^Z(p)$
$\left[\frac{1}{2} + \frac{1}{2}\sqrt{\frac{c}{1+c}}, \frac{1}{2} + \frac{1}{2}\sqrt{\frac{1+c}{3+c}}\right)$	$F_e^D(p) \geq F_e^Z(p) \geq F_e^X(p)$
$\left[\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1+c}{3+c}}, 1\right]$	$F_e^Z(p) \geq F_e^D(p) \geq F_e^X(p)$

Table 6.2: Ranking of *Entanglement fidelity* for Pauli-X (F_e^X), Dephasing (F_e^Z) and Depolarizing channel (F_e^D); $q \in \left[0, \frac{3-\sqrt{3}}{3+\sqrt{3}}\right] \cup \left[\frac{3+\sqrt{3}}{3+\sqrt{3}}, 1\right]$

We can prove the following inequalities that are useful to define the relationships between the identified intervals of p:

$$\sqrt{\frac{3c-1}{1+3c}} \leq \sqrt{\frac{c}{1+c}} \leq \sqrt{\frac{1+c}{3+c}} \quad \forall c \in [0,1] \ (i.e.\ \forall q \in [0,1])\,. \tag{6.4.5}$$

6.4.1 Commonly used states

Instead of using two letter messages, many quantum communication protocols involve sending one state over a channel [25, 26, 12, 11]. The results presented so far, can be used to study this scenario by considering $q = 0$ or $q = 1$. The reason behind this is briefly described in Section 6.1.

With these values of q, the intervals in Tables 6.2 and 6.3 get simplified and the ranking of the channels can be found in Table 6.4.

Varying p and q, we can obtain the performance comparison for different states. We list in Table 6.5 the cases of some of the most common qubit states.
When dealing with composite systems, a set of ubiquitous states is the one of Bell states:

- $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
- $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$
- $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$
- $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

p	**Ranking**
$\left[0, \frac{1}{2} - \frac{1}{2}\sqrt{\frac{1+c}{3+c}}\right)$	$F_e^Z(p) \geq F_e^D(p) \geq F_e^X(p)$
$\left[\frac{1}{2} - \frac{1}{2}\sqrt{\frac{1+c}{3+c}}, \frac{1}{2} - \frac{1}{2}\sqrt{\frac{c}{1+c}}\right)$	$F_e^D(p) \geq F_e^Z(p) \geq F_e^X(p)$
$\left[\frac{1}{2} - \frac{1}{2}\sqrt{\frac{c}{1+c}}, \frac{1}{2} + \frac{1}{2}\sqrt{\frac{c}{1+c}}\right)$	$F_e^D(p) \geq F_e^X(p) \geq F_e^Z(p)$
$\left[\frac{1}{2} + \frac{1}{2}\sqrt{\frac{c}{1+c}}, \frac{1}{2} + \frac{1}{2}\sqrt{\frac{1+c}{3+c}}\right)$	$F_e^D(p) \geq F_e^Z(p) \geq F_e^X(p)$
$\left[\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1+c}{3+c}}, 1\right]$	$F_e^Z(p) \geq F_e^D(p) \geq F_e^X(p)$

Table 6.3: Ranking of *Entanglement fidelity* for Pauli-X (F_e^X), Dephasing (F_e^Z) and Depolarizing channel (F_e^D); $q \in \left(\frac{3-\sqrt{3}}{3+\sqrt{3}}, \frac{3+\sqrt{3}}{3+\sqrt{3}}\right)$

p	**Ranking**
$\left[0, \frac{1}{2} - \frac{1}{2\sqrt{2}}\right)$	$F_e^Z(p) \geq F_e^D(p) \geq F_e^X(p)$
$\left[\frac{1}{2} - \frac{1}{2\sqrt{(2)}}, \frac{1}{2} + \frac{1}{2\sqrt{2}}\right)$	$F_e^X(p) \geq F_e^D(p) \geq F_e^Z(p)$
$\left[\frac{1}{2} + \frac{1}{2\sqrt{2}}, 1\right]$	$F_e^Z(p) \geq F_e^D(p) \geq F_e^X(p)$

Table 6.4: Ranking of *Entanglement fidelity* for Pauli-X (F_e^X), Dephasing (F_e^Z) and Depolarizing channel (F_e^D); $q = 0 \vee q = 1$

q	**p**	**State**	**Ranking**
1	1	$\|0\rangle$	$F_e^Z(p) \geq F_e^D(p) \geq F_e^X(p)$
1	0	$\|1\rangle$	$F_e^Z(p) \geq F_e^D(p) \geq F_e^X(p)$
1	$\frac{1}{2}$	$\|+\rangle = \frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$	$F_e^X(p) \geq F_e^D(p) \geq F_e^Z(p)$
0	$\frac{1}{2}$	$\|-\rangle = \frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$	$F_e^X(p) \geq F_e^D(p) \geq F_e^Z(p)$

Table 6.5: Ranking of *Entanglement fidelity* for Pauli-X (F_e^X), Dephasing (F_e^Z) and Depolarizing channel (F_e^D) when sending common qubit states.

If the composite system given by the letter state and reference system is in one of these states, the entanglement between the two system is maximal. By computing the density matrices

$$|\Phi^+\rangle\langle\Phi^+|, |\Phi^-\rangle\langle\Phi^-|, |\Psi^+\rangle\langle\Psi^+|, |\Psi^-\rangle\langle\Psi^-|$$

and tracing out the reference system, we can show that each one of these states is a purification of the density matrix

$$\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}(|+\rangle\langle +| + |-\rangle\langle -|). \tag{6.4.6}$$

Additionally, it is trivial to prove that if we set $q = \frac{1}{2}$ and $p = \frac{1}{2}$, the density matrix in Equation 6.4.6 can be rewritten in terms of the letter states $|\psi^+\rangle$ and $|\psi-\rangle$:

$$\rho = \frac{1}{2}(|\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-|). \tag{6.4.7}$$

Notice that in this case a message of two possible letters is being sent. Because of these results and according to Table 6.3, if the initial state describing the composite system of letter and reference is a Bell states, the ranking between the Pauli channels applied to the letter state is:

$$F_e^D \geq F_e^X \geq F_e^Z.$$

Chapter 7

A brief discussion about Entanglement Preservation

In the evolution of *Quantum Shannon theory* many types of *fidelities* where defined. In Chapter 5 we introduced the *Entanglement fidelity* which is the main subject of the second part of this work. To dive deeper into the different *fidelities* definitions and their relationships we suggest the reading of the article "The entanglement fidelity and quantum error correction" from M.A. Nielsen [6].

Even though the meaning of the *Entanglement fidelity* is clearly stated by Schumacher in [4] and Nielsen and Chuang [17], the notion of "preserving entanglement" is not that straightforward. In this chapter, we explore some objections to this definition and give possible counter-arguments to them. In this discussion we will use the notation adopted by Datta, Hsieh and Wilde in [5] as this article is the reason we decided to study more about the topic.

Both Schumacher and Nielsen-Chuang descriptions refer to the same setting: a quantum system A is prepared in a state ρ that can be pure or mixed, the state is assumed to be entangled with a system R such that their joint state $|\psi^{\rho}_{RA}\rangle$ is a pure state, a channel $\mathcal{N}$ is applied to A while the reference system R is left unchanged. The evolution of the joint state is described by the channel $\mathrm{Id} \otimes \mathcal{N}$.

In this scenario, the authors introduce the *Entanglement fidelity* F_e:

$$F_e(\rho, \mathcal{N}) = \langle \psi^{\rho}_{RA}| \, \mathrm{Id} \otimes \mathcal{N}(|\psi^{\rho}_{RA}\rangle \langle \psi^{\rho}_{RA}|) \, |\psi^{\rho}_{RA}\rangle \, . \tag{7.0.1}$$

The state $|\psi^{\rho}_{RA}\rangle$ is pure by construction, if we use the notation $\rho' = \mathrm{Id} \otimes \mathcal{N}(|\psi^{\rho}_{RA}\rangle \langle \psi^{\rho}_{RA}|)$ to indicate the state of the joint system after the channel evolution, we can write F_e in terms of the fidelity between a density matrix and a pure state:

$$F_e = F(|\psi^{\rho}_{RA}\rangle \langle\psi^{\rho}_{RA}|, \rho')^2 = \langle\psi^{\rho}_{RA}| \rho' |\psi^{\rho}_{RA}\rangle . \tag{7.0.2}$$

We will refer to the quantity F as the *Static fidelity*.

This is how Nielsen and Chuang describe the meaning of F_e[17]:

> "The entanglement fidelity provides a measure of how well the entanglement between R and A is preserved by the process $\mathcal{N}$, with values close to 1 indicating that the entanglement has been well preserved, and values close to 0 indicating that most of the entanglement has been destroyed."

Both Schumacher and Nielsen and Chuang give this definition of *Entanglement fidelity* and then proceed to prove two facts about it: F_e does not depend on the purification of the system we choose and it is upper-bounded by the *Average fidelity* $\overline{F}$. If the initial state is described by $\rho = \sum_j p_j \rho_j$, the average fidelity for the channel $\mathcal{N}$ is:

$$\overline{F} = \sum_j pjF(\rho_j, \mathcal{N}(\rho_j))^2. \tag{7.0.3}$$

The authors describe $\overline{F}$ as a measure of how faithfully a channel conveys an ensemble of pure states. According to them, the relationship $F_e(\rho, \mathcal{N}) \leq \overline{F}$ shows how a channel that preserves the entanglement between the reference and the system is a channel that well preserves an ensemble source ρ.

The relationships we presented are proved in the mentioned resources and seem to be correct. Nevertheless, in our opinion, the description of the operational meaning of this quantities is ambiguous, especially for the concept of "entanglement preservation". Both authors just assign *Entanglement fidelity* the meaning of measuring entanglement preservation but do not describe what this preservation means.

7.1 Existing work on entanglement preservation meaning

The problem of describing the meaning of entanglement preservation has already been addressed by V.A. Mousolou in [10]. This article builds on top of an idea from Xiang [9]. In his work, Mousolou claims that *Entanglement fidelity* fails to measure entanglement preservation through quantum channels. If this was to be true, we could say that the distortion measure $d(\rho, \mathcal{N})$ used in Datta et al. [5] fails to take into account the preservation of entanglement in the encoding-decoding process.

Mousolou's argument is the following: if we start from a Bell state as a purification and apply a Pauli channel to the subsystem, the channel turns it into an orthogonal Bell state and the *Entanglement fidelity* will be equal to 0. As both states are maximally mixed, F_e fails to describe that the amount of entanglement has not changed.

7.1.1 Formalization of Mousolou's example

Let's consider a system in a maximally mixed state $\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$ and its purification $|\psi^{\rho}_{RA}\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The initial state of the composite system will be:

$$|\Phi^+\rangle\langle\Phi^+| = \tfrac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$$

Let's define the channel $\mathcal{N}(\rho) = X\rho X^\dagger$ as the evolution of system A. Because $X^\dagger X = I$, such channel is valid according to Choi-Kraus theorem. According to convention, we define the Bell state

$$|\Psi^+\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle).$$

We can show that $\mathrm{Id} \otimes \mathcal{N}(|\Phi^+\rangle\langle\Phi^+|) = |\Psi^+\rangle\langle\Psi^+|$. This means that the *Entanglement fidelity* for the evolution of A under $\mathcal{N}$ is given by:

$$F_e = \langle\Phi^+|\,|\Psi^+\rangle\langle\Psi^+|\,|\Phi^+\rangle = 0$$

We believe that this inconsistency is caused by the ambiguity in the definitions of "preserving entanglement" and "preserving a statistical ensemble".

7.2 Preserving a source ensemble

Nielsen and Chuang [17] define that a density matrix $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ represent a partially unknown state of a system that is in state $|\psi_i\rangle$ with probability p_i. When they talk about "preserving the state" they refer to two distinct quantities we already introduced:

- Static fidelity: $F(\rho, \mathcal{N}(\rho))$
- Average fidelity: $\overline{F} = \sum_j pjF(\rho_j, \mathcal{N}(\rho_j))^2$

The example in Section 7.1.1 shows how important it is to distinguish the meaning of this two measures. In the presented case, the initial system has $\frac{1}{2}$ probability of being in the state $|0\rangle$ and $\frac{1}{2}$ probability of being in $|1\rangle$. As the channel flips the value of the state, after the evolution, the density matrix representing the state is unchanged. This translates to the static fidelity being equal to 1—the state has been perfectly preserved.

Anyway, if we measure the fidelity for the individual pure states we will always get zero. This behavior follows from the definition of static fidelity for two orthogonal pure states. Such result leads to an average fidelity equal to zero.

It is not the same thing to have a channel that keeps the density matrix identical and to have a channel that preserve the single states from the ensemble that we are sending.
When it comes to reliable communication, we believe that what we want to refer to is the *Average fidelity*. It is true that the proposed channel does not change the density matrix but we have to take into account what would happen in a real system. Even if we have incomplete knowledge about the system's state, before the channel it will be in either $|0\rangle$ or $|1\rangle$. After the evolution, this state will be flipped and it will be maximally different to the sent one—situation that seems undesirable to us.

The definition based on *Average fidelity* is the one we think should be associated to the concept of "faithfully convey an ensemble of pure states".

7.3 Preserving entanglement

To describe what it means to "preserve entanglement" is a tougher problem than it is to describe the preservation of an ensemble of states.
Neither Schumacher [4] or Nielsen and Chuang [17] do not really state what it means. On the other hand, Mousolou [10] and Xiang [9] takes a clear position on the concept: a channel preserves entanglement if it turns an entangled state into one with an amount of entanglement close to the initial one. That is why Mousolou considers a channel to perfectly preserve entanglement when it turns a Bell state into an orthogonal one.

We think that *Entanglement fidelity* is not measuring the difference in the amount of entanglement.
Entanglement fidelity captures the preservation of the amount of entanglement taking into account the structure of the state. F_e is saying that the entanglement represented by $|\Phi^+\rangle$ is maximally different from the one described by $|\Psi^+\rangle$.
If we look at the two Bell states, we can see that the state of one system when we know the state of the other is opposite in the two cases.

Despite of the two Bell states encoding a very different information about the joint system, it is true that the amount of information they are related to is the same. We find it crucial to understand whether maintaining this amount of information in the encoding-decoding process is enough, or if it is also necessary to keep the state of the joint system close to the initial one.

Our first hypothesis is that both things have to be taken into account. Luckily, it seems that *Entanglement fidelity* is a good measure to capture them.

In any case, the definition of the concept of preservation given until now is very loose and it could be important to give a well rounded explanation.

There is another point where Xiang [9] is wrong in trying to confute the meaning of *Entanglement fidelity*. In his paper he states the following:

> "Schumacher has noted that the EF can be lowered by a local unitary operation but the entanglement cannot be so"

This statement is false: it is known that entanglement cannot be increased via *Local operation or classical communication* (LOCC) but it is not true that it cannot be preserved or decreased. The example of Section 7.1.1 shows a situation where a local operation preserves the amount of entanglement between two systems. Moreover, if we apply a local *Depolarizing channel* to the subsystem we obtain a state that is less entangled than the initial one.

7.3.1 Depolarizing channel example

As in Section 7.1.1 we take $\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$ and its maximally entangled purification $|\psi^{\rho}_{RA}\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The joint state density matrix is $|\Phi^+\rangle\langle\Phi^+|$.

We apply the local operation $\mathcal{N}(\sigma) = \frac{1}{2}\sigma + \frac{1}{4}\mathbb{I}_2$ to the subsystem A. The joint state density matrix is affected as follows:

$$Id \otimes \mathcal{N}(|\psi^{\rho}_{RA}\rangle\langle\psi^{\rho}_{RA}|) = \frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes \mathbb{I}_2. \tag{7.3.1}$$

The state in Equation 7.3.1 is not maximally entangled and so is less entangled then the initial one. This proves that entanglement between two systems can indeed be lowered by local unitary operations.

Chapter 8

Time Evolution of Entanglement Fidelity

8.1 Introduction

In Chapter 1 we introduced the concept of *Open quantum system* and in Chapter 5 the *Entanglement fidelity* measure. In Equation 5.1.2, it is explicit how F_e is dependent on the density matrix ρ associated to the sent letter. As a letter is a quantum system, all the concepts described in Chapter 1 apply. Especially, in a real application, the letter could interact with an environment and be subject to a time evolution. As the density matrix of a letter change in time before being sent, the *Entanglement fidelity* associated with it might change.

In this chapter, we compute the time derivative of *Entanglement fidelity* under different evolution models of the density matrix ρ.

8.2 Evolution under always-invertible maps

We introduced the quantum master equation associated to an always invertible dynamical map in Equation 3.2.5. We first compute the derivative of F_e assuming this general form of evolution.

Following from the linearity of the derivative and the rule for the derivative of a product of two functions, from Equation 5.1.2 we obtain

$$\begin{aligned}\frac{dF_e(\rho,\mathcal{N})}{dt} &= \sum_\mu \frac{d}{dt}\left[\mathrm{Tr}(A_\mu\rho)\mathrm{Tr}(A_\mu^\dagger\rho)\right] \\ &= \sum_\mu \left[\frac{d\mathrm{Tr}(A_\mu\rho)}{dt}\cdot\mathrm{Tr}(A_\mu^\dagger\rho) + \mathrm{Tr}(A_\mu\rho)\cdot\frac{d\mathrm{Tr}(A_\mu^\dagger\rho)}{dt}\right].\end{aligned} \tag{8.2.1}$$

Because of the linearity of the trace of a matrix and since A_μ does not depend on time, using Equation 3.2.5 the first derivative in the sum can be computed as

$$\begin{aligned}
\frac{d\mathrm{Tr}(A_\mu\rho)}{dt} &= \mathrm{Tr}\left(A_\mu \cdot \frac{d\rho}{dt}\right) \\
&= \mathrm{Tr}\left[A_\mu\left(-i[H_S,\rho] + \sum_j \gamma_j(t)\left[L_j(t)\rho L_j^\dagger(t) - \frac{1}{2}\{L_j^\dagger(t)L_j(t),\rho\}\right]\right)\right] \\
&= \Big\{ -i\cdot \mathrm{Tr}\left(A_\mu[H_S,\rho]\right) \\
&\quad + \sum_j \gamma_j(t)\left[\mathrm{Tr}\left(A_\mu L_j(t)\rho L_j^\dagger(t)\right) - \frac{1}{2}\mathrm{Tr}\left(A_\mu\{L_j^\dagger(t)L_j(t),\rho\}\right)\right]\Big\}.
\end{aligned} \tag{8.2.2}$$

The second derivative in the sum can be obtained by substituting A_μ with $A_\mu^\dagger$. We substitute the derivatives in Equation 8.2.1 and obtain our final result:

$$\begin{aligned}
\frac{dF_e(\rho,\mathcal{N})}{dt} =& \sum_\mu \Bigg\{ \mathrm{Tr}(A_\mu^\dagger\rho)\Big[-i\cdot \mathrm{Tr}\left(A_\mu[H_S,\rho]\right) \\
&\quad + \sum_j \gamma_j(t)\left(\mathrm{Tr}\left[A_\mu L_j(t)\rho L_j^\dagger(t)\right] - \frac{1}{2}\mathrm{Tr}\left[A_\mu\{L_j^\dagger(t)L_j(t),\rho\}\right]\right)\Big] \\
&\quad + \mathrm{Tr}(A_\mu\rho)\Big[-i\cdot \mathrm{Tr}\left(A_\mu^\dagger[H_S,\rho]\right) \\
&\quad + \sum_j \gamma_j(t)\left(\mathrm{Tr}\left[A_\mu^\dagger L_j(t)\rho L_j^\dagger(t)\right] - \frac{1}{2}\mathrm{Tr}\left[A_\mu^\dagger\{L_j^\dagger(t)L_j(t),\rho\}\right]\right)\Big]\Bigg\}.
\end{aligned} \tag{8.2.3}$$

If all the operators A_μ and $A_\mu^\dagger$ commute with ρ, Equation 8.2.3 simplifies to

$$\begin{aligned}\frac{dF_e(\rho,\mathcal{N})}{dt} =& \sum_\mu \Bigg\{ \mathrm{Tr}(A_\mu^\dagger \rho) \sum_j \gamma_j(t) \left[\mathrm{Tr}\left(A_\mu L_j(t) \rho L_j^\dagger(t) \right) - \mathrm{Tr}\left(A_\mu \rho L_j^\dagger(t) L_j(t) \right) \right] \\ &+ \mathrm{Tr}(A_\mu \rho) \sum_j \gamma_j(t) \left[\mathrm{Tr}\left(A_\mu^\dagger L_j(t) \rho L_j^\dagger(t) \right) - \mathrm{Tr}\left(A_\mu^\dagger \rho L_j^\dagger(t) L_j(t) \right) \right] \Bigg\}.\end{aligned} \tag{8.2.4}$$

since $\mathrm{Tr}\left(A_\mu [H_S, \rho]\right) = \mathrm{Tr}\left(A_\mu^\dagger [H_S, \rho]\right) = 0$, $\mathrm{Tr}(A_\mu \{L_j^\dagger(t) L_j(t), \rho\}) = 2 \cdot \mathrm{Tr}(A_\mu \rho L_j^\dagger(t) L_j(t))$ and similarly we can prove $\mathrm{Tr}(A_\mu^\dagger \{L_j^\dagger(t) L_j(t), \rho\}) = 2 \cdot \mathrm{Tr}(A_\mu^\dagger \rho L_j^\dagger(t) L_j(t))$. Recalling that if A, B are two operators: $[A, B] = AB - BA$ represents their *Commutator* and $\{A, B\} = AB + BA$ represents their *Anti-Commutator* [17], all the previous equalities can be proved by using the cyclic property of the trace.

If all the operators A_μ are Hermitian, i.e. $A_\mu = A_\mu^\dagger$, Equation 8.2.3 turns into

$$\begin{aligned}\frac{dF_e(\rho,\mathcal{N})}{dt} =& \sum_\mu \Bigg\{ 2 \cdot \mathrm{Tr}(A_\mu \rho) \Bigg[- i \cdot \mathrm{Tr}\left(A_\mu [H_S, \rho] \right) \\ &+ \sum_j \gamma_j(t) \left(\mathrm{Tr}\left[A_\mu L_j(t) \rho L_j^\dagger(t) \right] - \frac{1}{2} \mathrm{Tr}\left[A_\mu \{ L_j^\dagger(t) L_j(t), \rho \} \right] \right) \Bigg] \Bigg\}.\end{aligned} \tag{8.2.5}$$

The implication of both conditions happening at the same time is trivial.

8.3 Evolution under Lindblad master equation

Lindblad master equation defined in Equation 3.1.5 is a particular case of Equation 3.2.5 where only one decoherence operator L appears and is time independent. The time derivative of *Entanglement fidelity* under this type of Markovian evolution follows from the results in Section 8.2.

The derivative in Equation 8.2.2 simplifies to:

$$\begin{aligned}\frac{d\mathrm{Tr}(A_\mu \rho)}{dt} =& \Big\{ - i \cdot \mathrm{Tr}\left(A_\mu [H_S, \rho] \right) + \mathrm{Tr}(A_\mu L \rho L^\dagger) \\ &- \frac{1}{2} \left[\mathrm{Tr}(A_\mu L^\dagger L \rho) + \mathrm{Tr}(A_\mu \rho L^\dagger L) \right] \Big\}.\end{aligned} \tag{8.3.1}$$

So, the derivative in Equation 8.2.1 in this case is

$$\frac{dF_e(\rho,\mathcal{N})}{dt} = \sum_\mu \Bigg\{ \mathrm{Tr}(A_\mu^\dagger \rho) \Big[-i \cdot \mathrm{Tr}\left(A_\mu [H_S, \rho]\right) + \mathrm{Tr}(A_\mu L \rho L^\dagger) \\ - \frac{1}{2}\left[\mathrm{Tr}(A_\mu L^\dagger L \rho) + \mathrm{Tr}(A_\mu \rho L^\dagger L)\right] \Big] + \mathrm{Tr}(A_\mu \rho) \Big[-i \cdot \mathrm{Tr}\left(A_\mu^\dagger [H_S, \rho]\right) \\ + \mathrm{Tr}(A_\mu^\dagger L \rho L^\dagger) - \frac{1}{2}\left[\mathrm{Tr}(A_\mu^\dagger L^\dagger L \rho) + \mathrm{Tr}(A_\mu^\dagger \rho L^\dagger L)\right] \Big] \Bigg\}. \tag{8.3.2}$$

If all the operators A_μ and $A_\mu^\dagger$ commute with ρ we get

$$\frac{dF_e(\rho,\mathcal{N})}{dt} = \sum_\mu \Bigg\{ \mathrm{Tr}(A_\mu^\dagger \rho) \Big[\mathrm{Tr}(A_\mu L \rho L^\dagger) - \mathrm{Tr}(A_\mu \rho L^\dagger L) \Big] \\ + \mathrm{Tr}(A_\mu \rho) \Big[\mathrm{Tr}(A_\mu^\dagger L \rho L^\dagger) - \mathrm{Tr}(A_\mu^\dagger \rho L^\dagger L) \Big] \Bigg\}. \tag{8.3.3}$$

Instead, if all the operators A_μ are Hermitian

$$\frac{dF_e(\rho,\mathcal{N})}{dt} = \sum_\mu 2 \cdot \mathrm{Tr}(A_\mu \rho) \Big[-i \cdot \mathrm{Tr}\left(A_\mu [H_S, \rho]\right) + \mathrm{Tr}(A_\mu L \rho L^\dagger) \\ - \frac{1}{2}\left[\mathrm{Tr}(A_\mu L^\dagger L \rho) + \mathrm{Tr}(A_\mu \rho L^\dagger L)\right] \Big]. \tag{8.3.4}$$

8.4 Evolution under Pure decoherence model

The same procedure used in the previous two sections can be applied to non-Markovian evolution. The only requirement is to have a well defined time derivative of ρ. We compute here the time derivative of *Entanglement fidelity* assuming a time evolution of the state as in Equation 3.3.1. This time, Equation 8.2.2 becomes

$$\frac{d\mathrm{Tr}(A_\mu \rho)}{dt} = \mathrm{Tr}\left[A_\mu \left(\sigma_z \rho \sigma_z - \rho\right)\right]. \tag{8.4.1}$$

The the time derivative of F_e is given by:

$$\frac{dF_e(\rho,\mathcal{N})}{dt} = \sum_\mu \gamma(t) \Big\{ \mathrm{Tr}(A_\mu^\dagger \rho) \cdot \mathrm{Tr}\left[A_\mu \left(\sigma_z \rho \sigma_z - \rho\right)\right] \\ + \mathrm{Tr}(A_\mu \rho) \cdot \mathrm{Tr}\left[A_\mu^\dagger \left(\sigma_z \rho \sigma_z - \rho\right)\right] \Big\}. \tag{8.4.2}$$

In this case, having all the A_μ and $A_\mu^\dagger$ to commute with ρ does not change anything but if they are Hermitian we obtain

$$\frac{dF_e(\rho,\mathcal{N})}{dt} = \sum_\mu 2 \cdot \gamma(t) \mathrm{Tr}(A_\mu \rho) \cdot \mathrm{Tr}\left[A_\mu \left(\sigma_z \rho \sigma_z - \rho\right)\right]. \tag{8.4.3}$$

This last equality can be proved using properties of operators analogous to the ones discussed at the end of Section 8.2.

8.4.1 Evaluation for common quantum channels

Since the time derivative in Equation 8.4.2 depends on the Kraus representation of the channel and on the known matrix

$$\sigma_z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{8.4.4}$$

We can develop it further to obtain its value for the channels introduced in Chapter 5.

We consider again the general case of the density matrix in Equation 5.1.1 and proceed to evaluate the term

$$\begin{aligned} \sigma_z \rho \sigma_z - \rho &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & c \\ \overline{c} & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} a & c \\ \overline{c} & b \end{bmatrix} \\ &= \begin{bmatrix} a & c \\ -\overline{c} & -b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} a & c \\ \overline{c} & b \end{bmatrix} \\ &= \begin{bmatrix} a & -c \\ -\overline{c} & b \end{bmatrix} - \begin{bmatrix} a & c \\ \overline{c} & b \end{bmatrix} \\ &= -2 \begin{bmatrix} 0 & c \\ \overline{c} & 0 \end{bmatrix}. \end{aligned} \tag{8.4.5}$$

To simplify the following computations we define

$$C = \begin{bmatrix} 0 & c \\ \overline{c} & 0 \end{bmatrix} \tag{8.4.6}$$

and observe that C is a particular case of ρ where $a = b = 0$. Thanks to this and the results obtained in Chapter 5, we can prove the following useful results:

$$\begin{aligned}
&\mathrm{Tr}(C) = 0 \\
&\mathrm{Tr}(XC) = \mathrm{Tr}(X\rho) = 2 \cdot \Re(c) \\
&\mathrm{Tr}(ZC) = 0 \\
&\mathrm{Tr}(YC) = \mathrm{Tr}(Y\rho) = -2 \cdot \Im(c) \\
&\mathrm{Tr}(XZC) = \mathrm{Tr}(XZ\rho) = 2 \cdot i \cdot \Im(c).
\end{aligned} \tag{8.4.7}$$

Given these results, we can rewrite Equation 8.4.2 as

$$\frac{dF_e}{dt} = -2 \cdot \gamma(t) \sum_{\mu} \left[\mathrm{Tr}(A_\mu^\dagger \rho)\mathrm{Tr}(A_\mu C) + \mathrm{Tr}(A_\mu \rho)\mathrm{Tr}(A_\mu^\dagger C) \right] \tag{8.4.8}$$

and whenever all the operators A_μ are Hermitian we get

$$\frac{dF_e}{dt} = -4 \cdot \gamma(t) \sum_{\mu} \mathrm{Tr}(A_\mu \rho)\mathrm{Tr}(A_\mu C). \tag{8.4.9}$$

By substituting the Kraus operators for the different channels and using the results presented above we can compute the derivative of F_e for the different quantum channels.

Random Pauli-X channel

The Kraus operators for this channel are

$$\left\{ \sqrt{u}X, \sqrt{1-u}I \right\}$$

and are both Hermitian. This implies that the time derivative of *Entanglement fidelity* can be obtained from Equation 8.4.3 and is given by

$$\begin{aligned}
\frac{dF_e}{dt} &= -4 \cdot \gamma(t) \left[u \cdot \mathrm{Tr}(X\rho)\mathrm{Tr}(XC) + (1-u)\mathrm{Tr}(\rho)\mathrm{Tr}(C) \right] \\
&= -4 \cdot u \cdot \gamma(t)\mathrm{Tr}(X\rho)\mathrm{Tr}(XC) \\
&= -16 \cdot u \cdot \gamma(t) \cdot \Re^2(c).
\end{aligned} \tag{8.4.10}$$

Dephasing channel

The Kraus operators for this channel are

$$\left\{ \sqrt{u}Z, \sqrt{1-u}I \right\}$$

and are both Hermitian. This implies that the time derivative of *Entanglement fidelity* can be obtained from Equation 8.4.3 and is given by

$$\begin{aligned}\frac{dF_e}{dt} &= -4 \cdot \gamma(t) \left[u \cdot \mathrm{Tr}(Z\rho)\mathrm{Tr}(ZC) + (1-u)\mathrm{Tr}(\rho)\mathrm{Tr}(C)\right] \\ &= -4 \cdot u \cdot \gamma(t)\mathrm{Tr}(Z\rho)\mathrm{Tr}(ZC) \\ &= 0.\end{aligned} \tag{8.4.11}$$

Then, we can state that if the letter is sent through a *Dephasing channel*, the associated *Entanglement fidelity* is constant in time. This does not imply that the letter is preserved during the decoherence process but that the capacity of the channel to preserve the entanglement between letter and reference system is not affected.

Depolarizing channel

The Kraus operators for this channel are

$$\left\{\sqrt{1-\tfrac{3}{4}u}I, \tfrac{\sqrt{u}}{2}X, \tfrac{\sqrt{u}}{2}Y, \tfrac{\sqrt{u}}{2}Z\right\}$$

The time derivative of *Entanglement fidelity* can be obtained from Equation 8.4.3 and is given by

$$\begin{aligned}\frac{dF_e}{dt} &= -4 \cdot \gamma(t)\Bigg[\frac{u}{4}\mathrm{Tr}(X\rho)\mathrm{Tr}(XC) + \frac{u}{4}\mathrm{Tr}(Y\rho)\mathrm{Tr}(YC) \\ &\quad + \frac{u}{4}\mathrm{Tr}(Z\rho)\mathrm{Tr}(ZC) + \left(1-\frac{3}{4}u\right)\mathrm{Tr}(\rho)\mathrm{Tr}(C)\Bigg] \\ &= -4 \cdot u \cdot \gamma(t)\left[\Re^2(c) + \Im^2(c)\right] \\ &= -4 \cdot u \cdot |c|^2.\end{aligned} \tag{8.4.12}$$

Werner-Holevo channel

This channel has only one Kraus operator given by the *Pauli Y* matrix that is Hermitian. Again, the time derivative of *Entanglement fidelity* can be obtained from Equation 8.4.3 and is given by

$$\begin{aligned}\frac{dF_e}{dt} &= -4 \cdot \gamma(t) \cdot \mathrm{Tr}(Y\rho)\mathrm{Tr}(YC) \\ &= -16 \cdot \gamma(t) \cdot \Im^2(c).\end{aligned} \tag{8.4.13}$$

Generalized Pauli channel

The Kraus operators for this channel are

$$\left\{\sqrt{p_{00}}I, \sqrt{p_{10}}X, \sqrt{p_{01}}Y, \sqrt{p_{11}}XZ\right\}.$$

The time derivative of *Entanglement fidelity* can be obtained from Equation 8.4.3 and is given by

$$\begin{aligned}\frac{dF_e}{dt} &= -4\cdot\gamma(t)\Big[p_{10}\mathrm{Tr}(X\rho)\mathrm{Tr}(XC) + p_{01}\mathrm{Tr}(Z\rho)\mathrm{Tr}(ZC) \\ &\quad + p_{11}\mathrm{Tr}(XZ\rho)\mathrm{Tr}(XZC) + p_{01}\mathrm{Tr}(\rho)\mathrm{Tr}(C)\Big] \\ &= -16\cdot\gamma(t)\left[p_{10}\cdot\Re^2(c) - p_{11}\cdot\Im^2(c)\right].\end{aligned} \tag{8.4.14}$$

Amplitude Damping channel

To avoid confusion with the *decay rate* function let's define the Kraus operator of the channel in terms of the parameter u:

$$\left\{A_0 = \sqrt{u}\cdot|0\rangle\langle 1|, A_1 = |0\rangle\langle 0| + \sqrt{1-u}\cdot|1\rangle\langle 1|\right\}$$

where the matrices will take the form

$$A_0 = \begin{bmatrix}0 & \sqrt{u}\\ 0 & 0\end{bmatrix} \;;\; A_1 = \begin{bmatrix}1 & 0\\ 0 & \sqrt{1-u}\end{bmatrix}. \tag{8.4.15}$$

We recall from Section 5.2.2 that the following results hold:

$$\begin{aligned}\mathrm{Tr}(A_0\rho) &= \overline{c}\cdot\sqrt{u}\\ \mathrm{Tr}(A_0^\dagger\rho) &= c\cdot\sqrt{u}\\ \mathrm{Tr}(A_1\rho) &= a + b\sqrt{1-u}.\end{aligned} \tag{8.4.16}$$

Since the C can be seen as a particular case of ρ with $a = b = 0$ we obtain also

$$\begin{aligned}\mathrm{Tr}(A_0C) &= \overline{c}\cdot\sqrt{u}\\ \mathrm{Tr}(A_0^\dagger C) &= c\cdot\sqrt{u}\\ \mathrm{Tr}(A_1C) &= 0.\end{aligned} \tag{8.4.17}$$

We can now compute the time derivative of F_e using Equation 8.4.8

$$\begin{aligned}\frac{dF_e}{dt} &= -2\cdot\gamma(t)\Big[\mathrm{Tr}(A_0^\dagger\rho)\mathrm{Tr}(A_0C)+\mathrm{Tr}(A_0\rho)\mathrm{Tr}(A_0^\dagger C)+2\mathrm{Tr}(A_1\rho)\mathrm{Tr}(A_1C)\Big]\\ &= -4\cdot u\cdot\gamma(t)\cdot|c|^2.\end{aligned} \tag{8.4.18}$$

8.5 Evolution for two-level system in a dissipative environment

Similarly to what we did in Section 8.4 we proceed to compute the time derivative of F_e assuming a non-Markovian evolution given by Equation 3.3.7.
The derivative of $\mathrm{Tr}(A_\mu\rho)$ is:

$$\begin{aligned}\frac{d\mathrm{Tr}(A_\mu\rho)}{dt} =&\Big\{-\frac{i}{4}\cdot S(t)\cdot\mathrm{Tr}\left(A_\mu[\sigma_z,\rho]\right)+\gamma(t)\Big[\mathrm{Tr}\left(A_\mu\sigma_-\rho\sigma_+\right)\\ &-\frac{1}{2}\mathrm{Tr}\left(A_\mu\left\{\sigma_+\sigma_-,\rho\right\}\right)\Big]\Big\}.\end{aligned} \tag{8.5.1}$$

By substituting this and the time derivative of $\mathrm{Tr}(A_\mu^\dagger\rho)$ in Equation 8.2.1 we get:

$$\begin{aligned}\frac{dF_e(\rho,\mathcal{N})}{dt} =&\sum_\mu\Bigg\{\mathrm{Tr}(A_\mu^\dagger\rho)\Big[-\frac{i}{4}\cdot S(t)\cdot\mathrm{Tr}\left(A_\mu[\sigma_z,\rho]\right)+\gamma(t)\Big[\mathrm{Tr}\left(A_\mu\sigma_-\rho\sigma_+\right)\\ &-\frac{1}{2}\mathrm{Tr}\left(A_\mu\left\{\sigma_+\sigma_-,\rho\right\}\right)\Big]\Big]+\mathrm{Tr}(A_\mu\rho)\Big[-\frac{i}{4}\cdot S(t)\cdot\mathrm{Tr}\left(A_\mu^\dagger[\sigma_z,\rho]\right)\\ &+\gamma(t)\Big[\mathrm{Tr}\left(A_\mu^\dagger\sigma_-\rho\sigma_+\right)-\frac{1}{2}\mathrm{Tr}\left(A_\mu^\dagger\left\{\sigma_+\sigma_-,\rho\right\}\right)\Big]\Big]\Bigg\}.\end{aligned} \tag{8.5.2}$$

This time, if all the A_μ and $A_\mu^\dagger$ commute with ρ the derivative get simplified to

$$\begin{aligned}\frac{dF_e(\rho,\mathcal{N})}{dt} =&\sum_\mu\Bigg\{\mathrm{Tr}(A_\mu^\dagger\rho)\cdot\gamma(t)\Big[\mathrm{Tr}\left(A_\mu\sigma_-\rho\sigma_+\right)-\mathrm{Tr}\left(A_\mu\rho\sigma_+\sigma_-\right)\Big]\\ &+\mathrm{Tr}(A_\mu\rho)\cdot\gamma(t)\Big[\mathrm{Tr}\left(A_\mu^\dagger\sigma_-\rho\sigma_+\right)-\mathrm{Tr}\left(A_\mu^\dagger\rho\sigma_+\sigma_-\right)\Big]\Bigg\}.\end{aligned} \tag{8.5.3}$$

If A_μ is Hermitian, Equation 8.5.2 simplifies to

$$\begin{aligned}\frac{dF_e(\rho,\mathcal{N})}{dt} = \sum_\mu 2 \cdot \mathrm{Tr}(A_\mu \rho)\Big[- \frac{i}{4} \cdot S(t) \cdot \mathrm{Tr}\left(A_\mu[\sigma_z, \rho]\right) \\ + \gamma(t)\Big(\mathrm{Tr}\left[A_\mu \sigma_- \rho \sigma_+\right] - \frac{1}{2}\mathrm{Tr}\left[A_\mu \left\{\sigma_+\sigma_-, \rho\right\}\right]\Big)\Big].\end{aligned} \tag{8.5.4}$$

Again, these results follow from properties analogous from those described at the end of Section 8.2.

Chapter 9

Conclusions and Future work

In this work, we investigated various aspects of two of the main information measures used in *Quantum Shannon theory*: *von Neumann Entropy* and *Entanglement fidelity*. We analyzed the time evolution of both when measured on the state of an *open quantum system* undergoing Markovian and non-Markovian decoherence processes. Additionally, we briefly discussed a possible extension of already existing results on *VNE* to its generalizations: *quantum Rényi* and *Tsallis* entropies.

We tried to apply the procedure previously described by K. Kobayashi [2] to search a bound on *von Neumann entropy* in the case of non-Markovian evolution. We chose to use two non-Markovian evolution models: one representing a case of *pure decoherence* and one representing a two-level system in a dissipative environment [3]. Our work shows how the removal of the Markovian assumption, present in Kobayashi's work, does not allow to draw the same conclusion by replicating its procedure. In the models we studied, this seems to be strongly associated to the presence of a sign-varying *decay rate* in the evolution of ρ associated to non-Markovianity.

In the following chapter, we studied the same idea but applying it to *Rényi* and *Tsallis* entropies— this time keeping the Markovian evolution assumption. We computed the time derivatives of these two quantities using the *Lindblad master equation* and described the most relevant tests we made to find a possible lower bound. Unfortunately, the more complicated structure of the derivatives complicates the usage of Kobayashi's idea and we have not found similar results for the two generalizations.

In the second part of the thesis we studied the *Entanglement fidelity* of some of the most common quantum channels. We computed its values in two different scenarios: the case of a density matrix in a general form and the one of a probabilistic mixture of two parametrized states $|\psi_+\rangle$ and $|\psi_-\rangle$. Defining a two-letters alphabet parametrized on p allowed us to compare how the choice of this parameters— and so of the letters— affects the capacity of a noisy

quantum channel to preserve the entanglement between the sent letter and an inaccessible reference system. We then ranked the F_e of *Random Pauli-X*, *Dephasing* and *Depolarizing* channel depending on the relationship between the parameter p and the probability q of having a letter in the state $|\psi_+\rangle$.

To complement this analysis of the *Entanglement fidelity*, we presented some objections to the meaning given to it by Schumacher in his definition [4]. We gave our perspective on the meaning of "entanglement preservation" and presented some counter-arguments to the objections.

To conclude, we put together the two main topics of our work by computing the time derivative of *Entanglement fidelity* under different types of evolution, describing how this metric is affected by decoherence if the letter is considered to interact with a reference system. We worked on four different models: a general form of *Lindblad master equation* with an arbitrary number of time-dependent *decoherence operators*, the simplified *Lindblad equation* with one time-independent *operator* and the two non-Markovian ones presented before. For the *Pure decoherence* model we also computed the time derivative for all the quantum channels studied in the previous chapters.

Our work suggests many possibilities for additional research. Given the results of applying Kobayashi's ideas to non-Markovian evolution and to *quantum Rényi* and *Tsallis* entropies it is clear that more work could be done in order to verify the existence, or not, of lower bounds on such quantities in the long time limit. The same work we did for the presented quantum channels can be extended to more complex channels like *Gaussian Bosonic channels* that present the difficulty of including sums of an infinite number of terms in the definition of their Kraus operators. The idea of having a parametrized letters can be further explored to find optimal alphabets in practical applications. This can be motivated by the fact that using quantum states as letters gives us more freedom than the classical counterpart where mostly we just consider 0 and 1. Moreover, the study of the time derivative of *Entanglement fidelity* could suggest to study the evolution of other *quantum information measures* under the assumption of different *decoherence processes*— both Markovian or not.

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