Politecnico di Torino

Master's Degree in Physics of Complex Systems

Entanglement and Non-Locality in Continuous-Variable Quantum Systems

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1 Introduction

At the end of the 19th century, it seemed that the theoretical framework of physics was already complete. Until then, Newtonian mechanics, in conjunction with Maxwell's theory of electromagnetism, appeared sufficient to explain all physical phenomena, from the motion of planets to the properties of electric charges. With the exception of a few anomalies, classical physics seemed sufficient to explain all known phenomena. Most physicists believed these anomalies would eventually be resolved within a classical framework. However, the experimental and theoretical investigation of these unresolved issues finally brought down classical physics as a general theory and reduced it to a great but bounded model that only worked within the particular regimes. Classical physics had developed for describing the macroscopic phenomena within the human experience: the motion of falling objects, projectiles, orbital paths of planets, and the sluggish flow of electric charges in electric circuits. It was formulated based on large, slow-moving bodies, and in this domain, it remains an outstanding scientific framework. Yet, at the same time, there was no warranty that the classical physics might give an equally reliable description of reality under extreme conditions-like systems that involve speeds comparable to the speed of light or dimensions as small as atoms and elementary particles. One of the prominent fields that originated the failure of classical physics was the investigation on the propagation of light and their emission. In the Michelson-Morley experiment, performed in 1881 [1], showed that the velocity of light does not depend upon the observer frame. This was a conclusion contradicted by the Galilean relativity or, correspondingly, by classical principle of a velocity composition. Besides, Maxwell's equations seemed not to be consistent with the Galilean transformations since their form was changing under the transformations between different inertial frames. These contradictions impelled Albert Einstein, in 1905, to return to the principle of relativity again and to sacrifice on its behalf the classical concepts about space and time; thus, special relativity was born.

Meanwhile, the studies of blackbody radiation presented another serious problem to classical physics: in 1900, Max Planck suggested that atoms emit not continuous energy, but in discrete quanta, thereby laying the foundations for quantum mechanics [2]. The hypothesis put forward by Planck was a serious digression from the concepts of classical physics, and in 1905 Einstein used this very quantum hypothesis in his explanation of the photoelectric effect; another phenomenon which classical theories could not explain.

Further acceleration of the rise of quantum mechanics was due to discoveries about the structure of the atom. In 1897, with the discovery of the electron, J.J. Thomson proved that the atom was not an indivisible fundamental unit but was actually composed of subatomic particles. The further development of atomic and nuclear models then showed that on this scale, classical physics was quite insufficient and required a radically new approach.

The crisis of classical physics marked the beginning of a period of intense scientific development. The emergence of new interpretative models allowed for the prediction of previously unknown phenomena and paved the way for overcoming the wave-particle duality. In classical physics, particle-like phenomena and wave-like phenomena are

treated as distinct, with a clear separation between them. A given physical phenomenon could be described either as corpuscular or wave-like, but never both simultaneously. However, in the microscopic world, this strict distinction no longer holds: radiation can exhibit particle-like behavior, while particles can display wave-like properties. The behavior of matter waves required a new theoretical framework, leading to the development of quantum mechanics (also known as wave mechanics) in 1925, independently formulated by Erwin Schrödinger and Werner Heisenberg.

This was the origin of fundamental conceptual problems associated with wave-particle duality: A wave is a field quantity oscillating in space and time, whereas until then, one had attributed a particle to be a localized object following a well-defined trajectory. In 1926 Erwin Schrödinger formulated a differential equation (now known as the Schrödinger equation) describing the time evolution of a wave function Ψ for material particles. In the same year, the German physicist Max Born proposed interpreting the wave function Ψ (which is a function of a complex variable) as a *probability wave*. In quantum mechanics, the wave associated with a particle should be understood as a *probability wave function* oscillating in time and space, where a greater amplitude corresponds to a higher probability of finding the particle. The probability p that a matter particle is located within a spatial region of volume dV during the time interval dt is proportional to the square of the wave function.

Faced with this new theoretical framework, physicists have provided various explanations that, broadly speaking, can be grouped into two schools of thought. The first, to which the majority of physicists adhere, is associated with the Copenhagen interpretation. According to this school of thought, the goal of a physical theory is not to describe an objective reality independent of the observer but rather to predict the outcomes of experiments. The second school of thought, on the other hand, prioritizes the descriptive aspect of a theory over its predictive power. In other words, physicists belonging to this school argue that a theory should not only provide predictions about the behavior of physical systems but also offer a clear picture of what actually happens. In particular, supporters of this viewpoint long believed that quantum indeterminism was a sign of the incompleteness of quantum mechanics, rejecting the idea that reality was inherently non-deterministic. They argued that there must exist a deeper level of reality, beyond what quantum mechanics describes, where the world would return to being fully deterministic. According to this perspective, quantum indeterminism arises because quantum mechanics does not account for additional degrees of freedom, known as hidden variables. If these hidden variables were known, the description of physical phenomena would become completely deterministic.

It is in this context that the famous 1935 paper by Einstein, Podolsky, and Rosen (EPR) was formulated [3]. Einstein and his collaborators can undoubtedly be placed within this second school of thought. In their work, they demonstrated that if two fundamental principles, referred to as the principle of reality and the principle of locality, are assumed to be valid, then the formalism of quantum mechanics leads to a contradiction, unless one admits the existence of hidden variables. To Einstein and his colleagues, these two principles appeared so fundamental and self-evident that they considered them indisputable. However, subsequent debates over the following decades suggested that these principles might not be as natural as they initially seemed.

It was about thirty years after the EPR paper that, in 1964, John Bell published what

was really a seminal work, still today the object of an intense activity of both studies and experiments [4]. Bell demonstrated that if exists a fundamental theory that is both realistic (meaning physical properties exist independently of measurement) and deterministic, while also satisfying locality assumption (no information can travel faster than light) then certain specific constraints must apply to the correlations between spatially separated microscopic systems. These constraints, known as Bell inequalities, have set bounds on how strongly two separated systems can be correlated under the assumption of local hidden variable theories. However, Bell also showed that quantum mechanics violates such limits. That is, any theory that tries to replace quantum mechanics with a local hidden variable model will fail because it cannot reproduce predictions given by quantum mechanics. Bell's theorem thus showed a remarkable boundary between the determinism in the classical case and quantum physics, making quantum physics incompatible with any theory supporting the local realistic hypothesis. But these implications of Bell's theorem, were, for the time being, merely theoretical. An essential question still had to be answered: Can these predictions be tested experimentally? Through the following decades, physicists searched for an empirical determination of whether nature indeed violated Bell inequalities and therefore could not support any local hidden variable theory.

The first experimental tests took place in the 1970s and 1980s, most notably by Alain Aspect and his collaborators. The experiments by Aspect provided the first strong evidence that indeed quantum mechanics violates Bell inequalities and, therefore, that quantum entanglement displays nonlocal correlations which no classical hidden variable theory is able to explain. Later results were refined through even more precise experiments that ruled out any conceivable loophole that might have allowed for alternative explanations of the violations. Today, the work of Bell is at the root of quantum information science; it stimulates intensive works in the field of quantum cryptography, quantum computing, and quantum teleportation. Experimental confirmation of Bell's inequalities has finally fixed not only the nonlocal character of quantum mechanics but also opened the way for the technological revolution promoted by quantum mechanics in the 21st century.

Although entanglement and non-locality are closely connected, it is very important to treat them as different concepts. Entanglement is a quantum correlation of states of particles, which does not depend on the distance between them. The property that guarantees that upon the instant measurement of a state of one particle, the state of the other is immediately determined, even when separated by light-years. However, entanglement by itself does not necessarily contradict local realism and can exist without violating any classical or hidden variable theories. Non-locality, on the other hand, specifically refers to the violation of Bell's inequalities, which demonstrates that no local hidden variable theory can fully account for the behavior of quantum particles. Unlike entanglement, non-locality directly challenges classical notions of causality and space-time, revealing that quantum mechanics allows for correlations that defy any explanation based on local interactions. While all non-local systems are entangled, the reverse is not always true; there exist entangled states that do not exhibit non-locality.

And this is precisely where this work comes in. The aim of thesis is to provide an analysis of the profound connection that exists between entanglement and non-locality, with a focus on continuous variable systems.

While discrete-variable quantum mechanics (e.g., qubits) has been widely studied in quantum computing and cryptography [5, 6], continuous-variable systems offer alternative advantages. Gaussian states, for example, provide a powerful and experimentally accessible framework for quantum communication and quantum metrology [7, 8, 9, 10]. Understanding how entanglement and non-locality manifest in continuous-variable systems is essential for advancing these fields.

In Chapter 2, we begin by introducing the quantum theory of light, which provides a foundation for understanding quantum states relevant to quantum optics. The chapter covers the quantization of the electromagnetic field, leading to a discussion of different types of quantum states such as Fock states, coherent states, and squeezed states. These states play a crucial role in quantum information processing, metrology, and communication. The treatment of squeezed states, in particular, highlights their importance in reducing quantum noise beyond the standard quantum limit, making them essential for high-precision measurements.

The Chapter 3 introduces the key concepts of quantum information theory, starting with fundamental elements such as qubits and qudits, the basic units of quantum information. The discussion then expands to quantum measurements and their impact on quantum states. The chapter also introduces entanglement in discrete systems, discussing how quantum correlations emerge and how they can be quantified through Negativity measure. The Chapter 4 is the focused on continuous variable quantum systems, including the phase-space formalism and the mathematical description of Gaussian states. We describe in more details the entanglement in continuous-variable systems, with a focus on two-mode Gaussian states. We explore the structure of entanglement measures, including methods for characterizing maximal entanglement at fixed local and global purities, as well as the role of generalized entropy in defining entanglement properties. These results are crucial for understanding the different ways in which quantum correlations can manifest in Gaussian states.

In Chapter 5, we shift our attention to non-locality, starting with a discussion of the Einstein-Podolsky-Rosen (EPR) argument and the Bell inequalities, which provide a fundamental limit for local theories, defining the maximum strength of correlations that can be explained within a local hidden variable framework. The chapter presents a mathematical characterization of non-local correlations, distinguishing no-signaling correlations, quantum correlations, and the local polytope of Bell inequalities. The discussion also covers Gisin's theorem, which establishes the link between entanglement and non-locality in pure states, and extends to mixed states where this relationship becomes more complex. The chapter also includes a brief overview of multipartite non-locality and its interplay with entanglement. The chapter investigates a specific measure of multipartite entanglement, the tangle, and considers scenarios where entanglement does not necessarily imply non-locality. In Chapter 6 we focus our attention on non-locality in continuous-variable systems, where traditional Bell inequalities may not directly apply. The chapter explores new forms of Bell-type inequalities that can be tested using parity measurements, particularly in the context of the EPR state. The discussion also includes an analysis of phase-space representations and the role of Wigner functions in detecting non-local correlations.

2 Quantum theory of light

In this section we will introduce a series of quantum states which are optimal for the characterization of optical fields. The first step is to carry out the process of quantizing the electromagnetic field, confining it within a closed volume. We will use an expansion of the vector potential for the electromagnetic field in terms of cavity modes. This approach reduces the problem to the quantization of the harmonic oscillator corresponding to each individual cavity mode, as has been done in [11]. The first type of state we will introduce during this process is the so-called Fock's state (or number state), which represents the number of excitations for each mode of the cavity even if a more typical optical field involves the superposition of number states. One such field is the coherent state, which has the minimum uncertainty in amplitude and phase allowed by the uncertainty principle, making it the closest quantum mechanical state possible to a classical field. A more exotic set of states of the electromagnetic field are the squeezed states. These are also minimum-uncertainty states, but, unlike the coherent states, the noise is not uniformly distributed in phase. Squeezed states may have applications in measurements below the vacuum-noise limit. Consequently, the noise in one quadrature component is reduced at the expense of increased noise in the orthogonal component. We introduce the basic properties of squeezed states and discuss their properties

2.1 Field quantization

We approach to the derivation of quantized electromagnetic field starting from the free field that obeys the source-free Maxwell equations:

$$\nabla \cdot \mathbf{B} = 0, \tag{2.1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.2}$$

$$\nabla \cdot \mathbf{E} = 0, \tag{2.3}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \tag{2.4}$$

where $\mathbf{B} = \mu_0 \mathbf{H}$, $\mathbf{D} = \varepsilon_0 \mathbf{E}$, with μ_0 and ε_0 being the magnetic permeability and electric permittivity of free space, and $\mathbf{c} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$. Maxwell's equations are gauge-invariant when no sources are present. A convenient choice of gauge for problems in quantum optics is the Coulomb gauge. In the Coulomb gauge, both $\mathbf{B}$ and $\mathbf{E}$ can be determined from a vector potential $\mathbf{A}(\mathbf{r},t)$ as follows:

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{2.5}$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \tag{2.6}$$

with the Coulomb gauge condition

$$\nabla \cdot \mathbf{A} = 0. \tag{2.7}$$

Substituting (2.5) and (2.6) into (2.4), we find that $\mathbf{A}(\mathbf{r},t)$ satisfies the wave equation

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0. \tag{2.8}$$

We separate the vector potential into two complex terms:

$$\mathbf{A}(\mathbf{r},t) = \mathbf{A}^{(+)}(\mathbf{r},t) + \mathbf{A}^{(-)}(\mathbf{r},t), \tag{2.9}$$

where $\mathbf{A}^{(+)}(\mathbf{r},t)$ contains all amplitudes that vary as $e^{-i\omega t}$ for $\omega > 0$ and $\mathbf{A}^{(-)}(\mathbf{r},t)$ contains all amplitudes that vary as $e^{i\omega t}$ and $\mathbf{A}^{(-)} = \left(\mathbf{A}^{(+)}\right)^*$.
It is more convenient to deal with a discrete set of variables rather than the whole continuum. We shall therefore describe the field restricted to a certain volume of space and expand the vector potential in terms of a discrete set of orthogonal mode functions:

$$\mathbf{A}^{+}(\mathbf{r},t) = \sum_k c_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t} \tag{2.10}$$

where the Fourier coefficients c_k are constant for a free field. The set of vector mode functions $\mathbf{u}_k(\mathbf{r})$ which correspond to the frequency ω_k will satisfy the wave equation

$$\left(\nabla^2 + \frac{\omega_k^2}{c^2}\right)\mathbf{u}_k(\mathbf{r}) = 0, \tag{2.11}$$

provided the volume contains no refracting material. The mode functions are also required to satisfy the transversality condition,

$$\nabla \cdot \mathbf{u}_k(\mathbf{r}) = 0. \tag{2.12}$$

The mode functions form a complete orthonormal set

$$\int \mathbf{u}_k(\mathbf{r}) \cdot \mathbf{u}_{k'}(\mathbf{r}) d^3 r = \delta_{kk'}. \tag{2.13}$$

And they depend on the boundary conditions of the physical volume under consideration, e.g., periodic boundary conditions corresponding to traveling waves or conditions appropriate for reflecting walls which lead to standing waves. For example, the plane wave mode functions appropriate to a cubical volume of side L may be written as

$$\mathbf{u}_k(\mathbf{r}) = L^{-3/2}\hat{\mathbf{e}}^{(\lambda)} e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{2.14}$$

where $\hat{\mathbf{e}}^{(\lambda)}$ is the unit polarization vector. The mode index $\mathbf{k}$ describes several discrete variables, the polarization index $\lambda = (1,2)$, and the three Cartesian components of the propagation vector $\mathbf{k}$. Each component of the wave vector $\mathbf{k}$ takes the values

$$k_x = \frac{2\pi n_x}{L}, \quad k_y = \frac{2\pi n_y}{L}, \quad k_z = \frac{2\pi n_z}{L}, \quad n_x, n_y, n_z = 0, \pm 1, \pm 2, \ldots. \tag{2.15}$$

The polarization vector $\hat{\mathbf{e}}^{(\lambda)}$ is required to be orthogonal to $\mathbf{k}$ by the transversality condition (2.12). The vector potential may now be written in the form

$$\mathbf{A}(\mathbf{r},t) = \sum_k \left(\frac{\hbar}{2\varepsilon_0 \omega_k}\right)^{1/2} \left[\mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t} \hat{a}_k + \mathbf{u}_k^*(\mathbf{r}) e^{i\omega_k t} \hat{a}_k^\dagger\right]. \tag{2.16}$$

The corresponding form for the electric field is

$$\mathbf{E}(\mathbf{r},t) = \mathbf{i} \sum_k \left(\frac{\hbar\omega_k}{2\varepsilon_0}\right)^{1/2} \left[\mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t} \hat{a}_k - \mathbf{u}_k^*(\mathbf{r}) e^{i\omega_k t} \hat{a}_k^\dagger\right]. \tag{2.17}$$

The normalization factors have been chosen such that the amplitudes $\hat{a}_k$ and $\hat{a}_k^\dagger$ are dimensionless.
In classical electromagnet theory, these Fourier amplitudes are complex numbers. Quantization of the electromagnetic field is accomplished by changing $\hat{a}_k$ and $\hat{a}_k^\dagger$ to be mutually adjoint operators. Since photons are bosons the appropriate commutation relations to choose for the operators $\hat{a}_k$ and $\hat{a}_k^\dagger$ are the boson commutation relations:

$$[\hat{a}_k, \hat{a}_{k'}] = 0, \quad [\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}. \tag{2.18}$$

The dynamical behavior of the electric-field amplitudes may then be described by an ensemble of independent harmonic oscillators obeying the above commutation relations. The quantum states of each mode may now be discussed independently of one another. The state in each mode may be described by a state vector $|\Psi\rangle_k$ of the Hilbert space appropriate to that mode. The states of the entire field are then defined in the tensor product space of the Hilbert spaces for all of the modes.
The Hamiltonian for the electromagnetic field is given by

$$H = \frac{1}{2} \int_V \left(\varepsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0} \mathbf{B}^2\right) dV. \tag{2.19}$$

Substituting (2.17) for $\mathbf{E}$ and the equivalent expression for $\mathbf{H}$ and making use of the conditions (2.12) and (2.13), the Hamiltonian may be reduced to the form

$$H = \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right). \tag{2.20}$$

This represents the sum of the number of photons in each mode multiplied by the energy $\hbar\omega_k$ of each photon. Hence, $\hbar\omega_k/2$ representing the energy of the vacuum state. We shall now consider these properties for three different quantum states of light.

2.1.1 Fock states

The Hamiltonian (2.20) has the eigenvalues $\hbar\omega_k n_k$, where n_k is an integer ($n_k = 0, 1, 2, \ldots, \infty$). The eigenstates are written as $|n_k\rangle$ and are known as number or Fock states. They are eigenstates of the number operator $\hat{N}_k = \hat{a}_k^\dagger \hat{a}_k$,

$$\hat{a}_k^\dagger \hat{a}_k |n_k\rangle = n_k |n_k\rangle, \tag{2.21}$$

moreover

$$\langle n_k | \hat{a}_k^\dagger \hat{a}_k | n_k \rangle = \| \hat{a}_k | n_k \rangle \|^2 \geq 0 \rightarrow n_k \geq 0. \tag{2.22}$$

Furthermore, the action of $\hat{a}_k^\dagger$ (raising operator) and $\hat{a}_k$ (lowering operator) is easily deduced by assuming that $[\hat{a}_k, \hat{a}_k^\dagger] = \mathrm{I}$. We can start noticing that $\hat{a}_k^\dagger |n_k\rangle$ and $\hat{a}_k |n_k\rangle$ are eigenvectors of the number operator:

$$\begin{aligned} &\hat{n}_k \hat{a}_k |n_k\rangle = \hat{a}_k^\dagger \hat{a}_k \hat{a}_k |n_k\rangle = (\hat{a}_k \hat{a}_k^\dagger - 1)\hat{a}_k |n_k\rangle = \hat{a}_k \hat{n}_k |n_k\rangle - \hat{a}_k |n_k\rangle = (n_k - 1)\hat{a}_k |n_k\rangle , \\ &\hat{n}_k \hat{a}_k^\dagger |n_k\rangle = \hat{a}_k^\dagger \hat{a}_k \hat{a}_k^\dagger |n_k\rangle = \hat{a}_k^\dagger (\hat{a}_k^\dagger \hat{a}_k + 1) |n_k\rangle = \hat{a}_k^\dagger (\hat{n}_k + 1) |n_k\rangle = (n_k + 1)\hat{a}_k^\dagger |n_k\rangle . \end{aligned} \tag{2.23}$$

State $\hat{a}_k |n_k\rangle$ thus diagonalizes $\hat{n}_k$ with eigenvalue $(n_k - 1)$ showing how necessarily $\hat{a}_k |n_k\rangle = C_k |n_k - 1\rangle$ and similarly have to be $\hat{a}_k^\dagger |n_k\rangle = D_k |n_k + 1\rangle$. To determine C_k and D_k consider the two scalar products

$$\langle n_k | \hat{a}_k^\dagger \hat{a}_k | n_k \rangle = \langle n_k - 1 | C_k^* C_k | n_k - 1 \rangle \rightarrow n_k = |C_k|^2 \rightarrow C_k = e^{i\alpha_k} \sqrt{n_k} \tag{2.24}$$

and

$$\begin{aligned} &(\langle n_k | \hat{a}_k)\hat{a}_k^\dagger |n_k\rangle = \langle n_k + 1 | D_k^* D_k | n_k + 1 \rangle \rightarrow \langle n_k | (\hat{n}_k + 1) | n_k \rangle = |D_k|^2 \\ &\rightarrow \langle n_k | \hat{n}_k | n_k \rangle + 1 = |D_k|^2 \rightarrow D_k = e^{i\beta_k} \sqrt{n_k + 1}. \end{aligned} \tag{2.25}$$

If we now assume $\alpha_k, \beta_k = 0$ it becomes evident action of $\hat{a}_k$ and $\hat{a}_k^\dagger$ on Fock state:

$$\begin{aligned} \hat{a}_k |n_k\rangle &= \sqrt{n_k} |n_k - 1\rangle , \\ \hat{a}_k^\dagger |n_k\rangle &= \sqrt{n_k + 1} |n_k + 1\rangle . \end{aligned} \tag{2.26}$$

Therefore $\hat{a}_k^\dagger, \hat{a}_k$ represent the creation and the annihilation of a photon with wavevector $\vec{k}$ and polarization $\hat{e}_k$. Since (2.26) the ground state of the oscillator (or vacuum state of the field mode) is defined by

$$\hat{a}_k|0\rangle = 0. \tag{2.27}$$

The state vectors for the higher excited states may be obtained from the vacuum by successive application of the creation operator.

$$|n_k\rangle = \frac{(\hat{a}_k^\dagger)^{n_k}}{(n_k!)^{1/2}}|0\rangle, \quad n_k = 0,1,2,\ldots \tag{2.28}$$

The number states are orthogonal

$$\langle n_k|m_k\rangle = \delta_{mn}, \tag{2.29}$$

and complete

$$\sum_{n_k=0}^{\infty} |n_k\rangle\langle n_k| = 1. \tag{2.30}$$

Since the norm of these eigenvectors is finite, they form a complete set of basis vectors for a Hilbert space.

2.1.2 Coherent states

Coherent states are a fundamental concept in quantum mechanics, introduced to bridge the gap between classical and quantum descriptions of physical systems. They were first formulated by Erwin Schrödinger [12] in the early days of quantum mechanics and have since been extensively studied and generalized for various applications in quantum physics and beyond. Coherent states were initially introduced in response to a critique that the wave functions defined by Schrödinger did not exhibit classical motion. He developed solutions that were Gaussian in nature, with the expectation values of position and momentum oscillating in time similarly to classical harmonic oscillators [13]. Furthermore, the product of the uncertainty in position and momentum for a coherent state is the minimum allowed by the uncertainty principle, and this is the reason why they are the closest quantum mechanical states to a classical description of the field. The introduction of coherent states, later developed extensively by Roy Glauber in the 1960s [14, 15], was motivated by the necessity of factorizing to all orders the electromagnetic field correlation function. According to Glauber [15], field coherent states can be constructed using any of the following three mathematical definitions.

Definition 1: The coherent states $|\alpha\rangle$ are quantum states that satisfy the minimum-uncertainty relationship

$$(\Delta p)^2(\Delta q)^2 = \frac{1}{4}, \tag{2.31}$$

with the position and momentum operators defined as

$$\hat{q} = \frac{1}{\sqrt{2}}(a + a^\dagger), \quad \hat{p} = \frac{1}{i\sqrt{2}}(a - a^\dagger), \tag{2.32}$$

and the generic variance and expectation value given by

$$\begin{aligned} (\Delta A)^2 &\equiv \langle\alpha|\,(\hat{A} - \langle\hat{A}\rangle)^2\,|\alpha\rangle \\ \langle\hat{A}\rangle &= \langle\alpha|\hat{A}\,|\alpha\rangle\,. \end{aligned} \tag{2.33}$$

It is worth noticing that coherent states are not the only one that satisfy the condition (2.31) as we will see later.

Definition 2: The coherent states $|\alpha\rangle$ are eigenstates of the harmonic-oscillator annihilation operator a,

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle\,, \tag{2.34}$$

where α is a complex number.

Definition 3: The coherent states $|\alpha\rangle$ can be obtained by applying a displacement operator $D(\alpha)$ on the vacuum state of the harmonic oscillator,

$$|\alpha\rangle = D(\alpha)\,|0\rangle\,, \tag{2.35}$$

where the displacement operator $D(\alpha)$ is defined as

$$D(\alpha) = \exp\big(\alpha a^\dagger - \alpha^* a\big). \tag{2.36}$$

where α is an arbitrary complex number. The following discussion was developed on the basis of the work of V.Penna [16].

2.1.3.1 Minimum Uncertainty Coherent States

In quantum optics the simplest example from which the field coherent states are uniquely defined is precisely the harmonic oscillator. In this respect its Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{q}^2}{2}. \tag{2.37}$$

As is well known, the Heisenberg uncertainty principle (UP) states that

$$(\Delta q)^2(\Delta p)^2 \geq \frac{\hbar^2}{4}, \tag{2.38}$$

where $(\Delta q)^2 = \langle q^2\rangle - \langle q\rangle^2$ and $(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2$ represent the variance of q and p, respectively. Our initial goal is to demonstrate that MUCS are the states that optimize the Heisenberg UP, where inequality (2.38) reduces to

$$(\Delta q)^2(\Delta p)^2 = \frac{\hbar^2}{4}. \tag{2.39}$$

This can be shown by considering the general case of two operators $\hat{A}$ and $\hat{B}$ such that $[\hat{A},\hat{B}] = i\hat{G}$, where the operator $\hat{G}$ is not necessarily the identity $c\mathbb{I}$ (up to a constant factor $c \in \mathbb{R}$), as in the case of canonical coordinates $\hat{q}$ and $\hat{p}$.

Notation: We consider a pair of non-commutating Hermitian operators, $\hat{A}$ and $\hat{B}$:

$$[\hat{A},\hat{B}] = i\hat{G}, \tag{2.40}$$

where $\hat{G}$ is generally an operator. Note that since $\hat{A} = \hat{A}^\dagger$ and $\hat{B} = \hat{B}^\dagger$, the operator $\hat{G}$ is also Hermitian ($\hat{G}^\dagger = (-i[\hat{A},\hat{B}])^\dagger = i(\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger = i(\hat{B}\hat{A} - \hat{A}\hat{B}) = -i[\hat{A},\hat{B}] = \hat{G}$). Denote by

$$\langle F\rangle = \langle\psi|F|\psi\rangle \tag{2.41}$$

the expectation value of a generic operator F on some physical state $|\psi\rangle$ of the system, and by

$$(\Delta F)^2 = \langle F^2\rangle - \langle F\rangle^2 \tag{2.42}$$

its uncertainty (or variance) on the same state. The general form of the Heisenberg UP is thus

$$(\Delta A)^2(\Delta B)^2 \geq \frac{|\langle G\rangle|^2}{4}. \tag{2.43}$$

Taking now into account the non-hermitian operator defined as $\hat{A} + i\sigma\hat{B}$, where $\sigma = \langle\hat{G}\rangle/2\Delta_B^2$ (for simplicity, we will denote $\Delta B := \Delta_B$), we are going to prove that Eq.(2.38) is a strict equality:

$$(\hat{A} + i\sigma\hat{B})|\psi\rangle = \lambda|\psi\rangle \rightarrow \Delta_A\Delta_B = \frac{\langle\hat{G}\rangle}{2} \tag{2.44}$$

where $|\psi\rangle$ is an eigenstate of the non-Hermitian operator. This is immediately obtained

noticing that the eigenvalue equation in (2.44) yields $\lambda = \langle\hat{A}\rangle + i\sigma\langle\hat{B}\rangle$. Therefore we can rewrite the eigenvalue equation as

$$(\hat{A} - \langle\hat{A}\rangle)|\psi\rangle = -i\sigma(\hat{B} - \langle\hat{B}\rangle)|\psi\rangle \tag{2.45}$$

and then if the term $(\hat{A} - \langle\hat{A}\rangle)$ acts on both sides, we find

$$(\hat{A} - \langle\hat{A}\rangle)^2|\psi\rangle = -i\sigma(\hat{A} - \langle\hat{A}\rangle)(\hat{B} - \langle\hat{B}\rangle)|\psi\rangle \tag{2.46}$$

with the product $(\hat{A} - \langle\hat{A}\rangle)$ that can be manipulated as

$$\begin{aligned}(\hat{A} - \langle\hat{A}\rangle)(\hat{B} - \langle\hat{B}\rangle) &= [\hat{A} - \langle\hat{A}\rangle, \hat{B} - \langle\hat{B}\rangle] + (\hat{B} - \langle\hat{B}\rangle)(\hat{A} - \langle\hat{A}\rangle) \\ &= i\hat{G} + (\hat{B} - \langle\hat{B}\rangle)(\hat{A} - \langle\hat{A}\rangle)\end{aligned} \tag{2.47}$$

where we have used $[\hat{A} - \langle\hat{A}\rangle, \hat{B} - \langle\hat{B}\rangle] = [\hat{A}, \hat{B}] = iG$. Our eigenvalue equation thus becomes

$$(\hat{A} - \langle\hat{A}\rangle)^2|\psi\rangle = \sigma\hat{G}|\psi\rangle + (i\sigma)^2(\hat{B} - \langle\hat{B}\rangle)^2|\psi\rangle, \tag{2.48}$$

and we just need to project $|\psi\rangle$ on the latter equation to determine the variance

$$\begin{aligned}\Delta_A^2 = \langle\psi|(\hat{A} - \langle\hat{A}\rangle)^2|\psi\rangle &= -i\sigma\langle\psi|iG + (\hat{B} - \langle\hat{B}\rangle)(\hat{A} - \langle\hat{A}\rangle)|\psi\rangle \\ &= \sigma\langle\psi|G|\psi\rangle - \sigma^2\langle\psi|(\hat{B} - \langle\hat{B}\rangle)^2|\psi\rangle \\ &= \frac{\langle\hat{G}\rangle^2}{\Delta_B^2}\end{aligned} \tag{2.49}$$

from which the direct equality is now proven

$$\Delta_A^2\Delta_B^2 = \frac{\langle G\rangle^2}{4} \quad \blacksquare. \tag{2.50}$$

2.1.3.2 Annihilation operator coherent states

In this section we want to construct coherent states as the eigenstates of the annihilation operator $\hat{a}$. We can start again from the definition of the Hamiltonian of harmonic oscillator (2.37). The natural description of the HO Hamiltonian is in terms of *Schrödinger (canonical) algebra*. Although in this text we use the name of Schrödinger to define an algebra that is generated by position and momentum, this is not unique in literature, since there are numerous texts among which [17] that identify this algebra with the name of *Weyl-Heisenberg algebra*. Regardless the proper name we present its own generators:

$$\mathscr{A}_S = [\mathbb{I}, \hat{q}, \hat{p}], \quad [\hat{q}, \hat{p}] = i\hbar. \tag{2.51}$$

The Schrödinger algebra is referred to be the Hamiltonian generating algebra for the

harmonic oscillator (HO) because it can be expressed as a quadratic function of the algebra generators $\hat{q}$ and $\hat{p}$. In this sense, $\mathscr{A}_S$ is the Hamiltonian generating algebra for any one-dimensional potential problem described by $H = p^2/2m + V(x)$ (where $V(x)$ is the potential energy) or, more generally, for any system whose Hamiltonian is such that $H = F(x,p)$. The introduction of the annihilation operator

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} - \frac{i\hat{p}}{m\omega}\right) \tag{2.52}$$

allows one to write

$$\hat{q} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \tag{2.53}$$

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a} - \hat{a}^\dagger), \tag{2.54}$$

and

$$H = \hbar\omega\left(\hat{n} + \frac{1}{2}\right), \tag{2.55}$$

where $\hat{n} = \hat{a}^\dagger\hat{a}$ is the so-called *number operator*, and we use the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = \mathbb{I}. \tag{2.56}$$

It is now clear why there is no agreement in defining the appropriate name for the algebra associated with the HO Hamiltonian. The form (2.55) is obtained in term of $\hat{a}, \hat{a}^\dagger$, that are the generators of the previously mentioned *Weyl-Heisenberg algebra*:

$$A_{WH} = \{\mathbb{I}, \hat{a}, \hat{a}^\dagger\}, \tag{2.57}$$

that is equivalent to the Schrödinger one since $\hat{a}, \hat{a}^\dagger$ are a linear combination of position and momentum. It is also an Hamiltonian generating algebra for the HO, in that $\hat{H}$ can be expressed in terms of quantities $(\mathbb{I}, \hat{a}, \hat{a}^\dagger)$ with well-defined commutators. On the other hand, by observing that

$$[\hat{a}, \hat{n}] = [\hat{a}, \hat{a}^\dagger]\hat{a} = \hat{a}, \tag{2.58}$$

$$[\hat{a}^\dagger, \hat{n}] = \hat{a}^\dagger[\hat{a}^\dagger, \hat{a}] = \hat{a}^\dagger, \tag{2.59}$$

one recognizes that H can be written within a more general algebraic structure. If an algebra is such that some Hamiltonian can be written as a linear combination of its generators, then such an algebra is referred to as the *dynamical algebra* for the

Hamiltonian under examination. Hence, since $\hat{q}^2, \hat{p}^2 \notin A_S$ and $\hat{n} \notin A_{WH}$, neither the Schrödinger algebra nor the Weyl-Heisenberg algebra is a dynamical algebra for the HO. The so-called *Weyl algebra*, generated by

$$A_W = \{\mathbb{I}, \hat{a}, \hat{a}^\dagger, \hat{n}\}, \quad [\hat{a}, \hat{a}^\dagger] = \mathbb{I}, \quad [\hat{a}, \hat{n}] = \hat{a}, \quad [\hat{a}^\dagger, \hat{n}] = \hat{a}^\dagger. \tag{2.60}$$

Therefore, we can now take expression (2.52) and examine the eigenstates of the annihilation operator:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \tag{2.61}$$

expanding these states in the eigenfunction basis to get:

$$\hat{a}|\alpha\rangle = \hat{a}\sum_{n=0}^{\infty} f_n|n\rangle = \sum_{n=1}^{\infty}\sqrt{n}f_n|n-1\rangle = \sum_{n=0}^{\infty}\sqrt{n+1}f_{n+1}|n\rangle = \alpha\sum_{n=0}^{\infty} f_n|n\rangle, \tag{2.62}$$

where the coefficients f_n of the series are related by the recursive relation $\sqrt{n}f_n = \alpha f_{n-1}$. In this way, we end up with:

$$f_n = \frac{\alpha^n}{\sqrt{n!}}f_0, \tag{2.63}$$

which leads to:

$$|\alpha\rangle = f_0\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{2.64}$$

with f_0 determined due to the normalization condition:

$$1 = \langle\alpha|\alpha\rangle = |f_0|^2\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!} = |f_0|^2 e^{|\alpha|^2}, \tag{2.65}$$

from which:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{2.66}$$

The last step now consists in evaluating the uncertainty relation of q and p over the eigenstates (2.61):

$$\langle q\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\alpha + \alpha^*), \tag{2.67}$$

$$\langle p\rangle = i\sqrt{\frac{\hbar m\omega}{2}}(\alpha - \alpha^*), \tag{2.68}$$

$$\langle q^2\rangle = \frac{\hbar}{2m\omega}[(\alpha+\alpha^*)^2+1], \tag{2.69}$$

$$\langle p^2\rangle = \frac{\hbar}{2m\omega}[1-(\alpha-\alpha^*)^2], \tag{2.70}$$

from which finally:

$$\Delta_q^2 = \langle q^2\rangle - \langle q\rangle^2 = \frac{\hbar}{2m\omega}, \tag{2.71}$$

$$\Delta_p^2 = \langle p^2\rangle - \langle p\rangle^2 = \frac{\hbar m\omega}{2}, \tag{2.72}$$

$$\Delta_q^2\Delta_p^2 = \frac{\hbar^2}{4} \tag{2.73}$$

Equation (2.73) explicitly shows that the eigenstates $|\alpha\rangle$ of the annihilation operator are minimum uncertainty states, i.e., they are annihilation operator coherent states (AOCS). This means that AOCS $\rightarrow$ MUCS. Another relevant feature is that coherent states contain an indefinite number of photons. This may be made apparent by considering an expansion of the coherent states in the number states basis. Taking the scalar product of both sides of (2.61) with $\langle n|$ we find the recursion relation

$$\langle n+1|\alpha\rangle = \frac{\alpha}{\sqrt{n+1}}\langle n|\alpha\rangle. \tag{2.74}$$

It follows that

$$\langle n|\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}\langle 0|\alpha\rangle. \tag{2.75}$$

We may expand $|\alpha\rangle$ in terms of the number states $|n\rangle$ with expansion coefficients $\langle n|\alpha\rangle$ as follows:

$$|\alpha\rangle = \sum_n |n\rangle\langle n|\alpha\rangle = \langle 0|\alpha\rangle\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{2.76}$$

The squared length of the vector $|\alpha\rangle$ is thus

$$\langle\alpha|\alpha\rangle = |\langle 0|\alpha\rangle|^2\sum_n \frac{|\alpha|^{2n}}{n!}. \tag{2.77}$$

It is easily seen that

$$\langle 0|\alpha\rangle = \langle 0|D(\alpha)|0\rangle = e^{-|\alpha|^2/2}. \tag{2.78}$$

Thus $|\alpha\rangle$ and the coherent states are normalized. The coherent state may then be expanded in terms of the number states as

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{2.79}$$

We note that the probability distribution of photons in a coherent state is a Poisson distribution

$$P(n) = |\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!}, \tag{2.80}$$

where $|\alpha|^2$ is the mean number of photons $\langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2$. The product rule of coherent states is

$$|\langle\beta|\alpha\rangle|^2 = e^{-|\alpha-\beta|^2}. \tag{2.81}$$

Thus the coherent states are not orthogonal although two states $|\alpha\rangle$ and $|\beta\rangle$ become approximately orthogonal in the limit $|\alpha-\beta| \to \infty$. The coherent states form a two-dimensional continuum of states and are, in fact, overcomplete. The completeness relation

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = 1, \tag{2.82}$$

may be proved as follows. We use the expansion (43) to give

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle\alpha| = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|n\rangle \langle m|}{\pi\sqrt{n!m!}} \int e^{-|\alpha|^2} \alpha_m^* \alpha_n d^2\alpha \tag{2.83}$$

Changing to polar coordinates this becomes

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle\alpha| = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|n\rangle \langle m|}{\pi\sqrt{n!m!}} \int_0^{\infty} r dr e^{-r^2} r^{n+m} \int_0^{2\pi} d\theta e^{i(n-m)\theta} \tag{2.84}$$

Using

$$\int_0^{2\pi} d\theta e^{i(n-m)\theta} = 2\pi\delta_{m,n} \tag{2.85}$$

we have

$$\int \langle\alpha|\hat{A}|\alpha\rangle \frac{d^2\alpha}{\pi} = \sum_{n,m} \frac{\delta_{mn}}{n!} \langle n|\hat{A}|m\rangle = \sum_n \langle n|\hat{A}|n\rangle. \tag{2.86}$$

Thus we have

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle\alpha|) = \sum_{n=0}^{\infty} \frac{|n\rangle \langle n|}{n!} \int_0^{\infty} d\varepsilon e^{-\varepsilon} \varepsilon^n \tag{2.87}$$

where $\varepsilon = r^2$. Since the integral is equal to $n!$ we obtain

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle\alpha|) = \sum_{n=0}^{\infty} |n\rangle \langle n| = 1 \tag{2.88}$$

Following from the completeness relation for the number states.

2.1.3.3 Displaced Operator representation

The last step consists in proving the *Definition 3*. To prove the equivalence of this last definition we need to consider a generic coherent state as a linear combination of Fock states:

$$|\alpha\rangle = \sum_{n=1}^{\infty} c_n |n\rangle, \quad \text{where} \quad c_n = \langle n|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}. \tag{2.89}$$

The terms are weighted by coefficients that are set by means the normalization condition $\langle\alpha|\alpha\rangle = 1$. We have already seen that is possible to define a Fock state as the result of the iterative action of the creation operator on the vacuum state (2.28). So we get:

$$\begin{aligned} |\alpha\rangle &= e^{-\frac{|\alpha|^2}{2}} \sum_{n=1}^{\infty} \frac{\alpha^n \hat{a}^{\dagger n}}{\sqrt{n!}} |0\rangle, \\ &= e^{-\frac{|\alpha|^2}{2}} e^{\alpha \hat{a}^{\dagger}} |0\rangle, \\ &= e^{-\frac{|\alpha|^2}{2}} e^{\alpha \hat{a}^{\dagger}} e^{-\alpha^* \hat{a}} |0\rangle. \end{aligned} \tag{2.90}$$

To go from the second line to the third we used $e^{-\alpha^* \hat{a}} |0\rangle = |0\rangle$. In order to derive the displacement operator we need of the operator theorem [18]:

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}, \tag{2.91}$$

which holds when $[A,[A,B]] = [B,[A,B]] = 0$. We can write $D(\alpha)$ (2.36) as

$$D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha a^{\dagger}} e^{-\alpha^* a} \quad \blacksquare. \tag{2.92}$$

The displacement operator $D(\alpha)$ has the following properties:

$$D^{\dagger}(\alpha) = D^{-1}(\alpha) = D(-\alpha), \tag{2.93}$$

$$D^{\dagger}(\alpha) a D(\alpha) = a + \alpha, \tag{2.94}$$

$$D^{\dagger}(\alpha) a^{\dagger} D(\alpha) = a^{\dagger} + \alpha^*. \tag{2.95}$$

The coherent state $|\alpha\rangle$ is generated by operating with $D(\alpha)$ on the vacuum state:

$$|\alpha\rangle = D(\alpha)|0\rangle. \tag{2.96}$$

The coherent states are eigenstates of the annihilation operator $\hat{a}$. This may be proved as follows:

$$D^{\dagger}(\alpha)aD(\alpha)|0\rangle = (a+\alpha)|0\rangle = \alpha|0\rangle. \tag{2.97}$$

Multiplying both sides by $D(\alpha)$ we arrive at the eigenvalue equation

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{2.98}$$

Since a is a non-Hermitian operator, its eigenvalues α are complex.

2.1.3 Squeezed states

Coherent states, previously identified as states of minimum uncertainty, are not the only quantum states that satisfy this condition. Another important class of states that meet the minimum uncertainty requirement is known as *squeezed states*. These states were first introduced in 1926 by Schrödinger [19] in the context of classical states of quantum harmonic oscillators, although the term "squeezed states" was later coined by Hollenhorst in 1979 [20] . Unlike coherent states, squeezed states exhibit reduced noise in one quadrature at the expense of increased noise in the conjugate quadrature, in accordance with the Heisenberg uncertainty principle. Coherent states represent a special case within this broader family of minimum-uncertainty states, characterized by equal noise in both quadratures. We proceed to the description of these states by what is derived from Milburn and Walls [11]. We shall begin our discussion by defining a family of minimum-uncertainty states. Let us calculate the variances for the position and momentum operators for the harmonic oscillator:

$$\hat{q} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^{\dagger}), \quad \hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}-\hat{a}^{\dagger}). \tag{2.99}$$

from which derive:

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{q}+i\hat{p}), \quad \hat{a}^{\dagger} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{q}-i\hat{p}). \tag{2.100}$$

In Eq. (2.100), we see that up to normalization factors the position and the momentum are the real and imaginary parts of the annihilation operator. Let us now define the dimensionless pair of conjugate variables:

$$\hat{X}_1 = \sqrt{\frac{2m\omega}{\hbar}}\hat{q}, \quad \hat{X}_2 = \sqrt{\frac{2}{\hbar m\omega}}\hat{p}. \tag{2.101}$$

These operators represent the *quadratures* of a single mode, in classical terms corresponding to the real and imaginary parts of the oscillator's complex amplitude. We may write the annihilation operator as a combination of two Hermitian operators:

$$\hat{a} = \frac{\hat{X}_1 + i\hat{X}_2}{2}. \tag{2.102}$$

$\hat{X}_1$ and $\hat{X}_2$, the real and imaginary parts of the complex amplitude, give dimensionless amplitudes for the two quadrature phases of the modes. They obey the following commutation relation

$$[\hat{X}_1, \hat{X}_2] = 2i. \tag{2.103}$$

The corresponding uncertainty principle is

$$(\Delta\hat{X}_1)^2(\Delta\hat{X}_2)^2 \geq 1. \tag{2.104}$$

This relation with the equals sign defines a family of minimum-uncertainty states. The coherent states are a particular minimum-uncertainty state with

$$(\Delta X_1)^2 = (\Delta X_2)^2 = 1. \tag{2.105}$$

The coherent state $|\alpha\rangle$ has the mean complex amplitude α and it is a minimum-uncertainty state for $\hat{X}_1$ and $\hat{X}_2$, with equal uncertainties in the two quadrature phases. A coherent state may be represented by an "error circle" in a complex amplitude plane whose axes are X_1 and X_2 (Figure 1). The center of the error circle lies at $\frac{1}{2}\langle X_1 + iX_2\rangle = \alpha$ and the radius $\Delta X_1 = \Delta X_2 = 1$ accounts for the uncertainties in X_1 and X_2. There is obviously a whole family of minimum-uncertainty states defined by (2.105). If we plot ΔX_1 against ΔX_2, the minimum-uncertainty states lie on a hyperbola (2.1.3). Only points lying to the right of this hyperbola correspond to physical states. The coherent state with $\Delta X_1 = \Delta X_2$ is a special case of a more general class of states which may have reduced uncertainty in one quadrature at the expense of increased uncertainty in the other ($\Delta X_1 < 1, \Delta X_2 > 1$). These states correspond to the shaded region in Figure 1. Such states we shall call squeezed states [21]. They may be generated by using the unitary squeeze operator [22].

$$S(\varepsilon) = \exp\left(\frac{1}{2}\varepsilon^* a^2 - \frac{1}{2}\varepsilon a^{\dagger 2}\right), \tag{2.106}$$

where $\varepsilon = re^{i\theta}$. Note the squeeze operator obeys the relations

$$S^\dagger(\varepsilon) = S^{-1}(\varepsilon) = S(-\varepsilon) \tag{2.107}$$

and has the following useful transformation properties

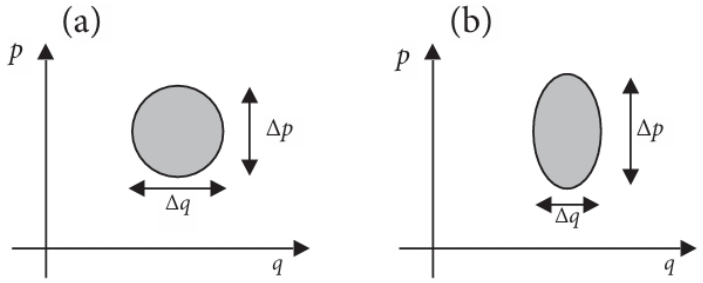

Figure 1: Phase space representation showing contours of constant uncertainty for (a) coherent state and (b) squeezed state. [11]

$$S^\dagger(\varepsilon)aS(\varepsilon) = a\cosh r - a^\dagger e^{i\theta}\sinh r \tag{2.108}$$
$$S^\dagger(\varepsilon)a^\dagger S(\varepsilon) = a^\dagger\cosh r - ae^{-i\theta}\sinh r \tag{2.109}$$
$$S^\dagger(\varepsilon)(Y_1+iY_2)S(\varepsilon) = Y_1e^{-r} + iY_2e^{r}. \tag{2.110}$$

where

$$Y_1 + iY_2 = (X_1 + iX_2)e^{-i\theta}. \tag{2.111}$$

Is a rotated complex amplitude. The squeezing operator attenuates one component of the (rotated) complex amplitude, and it amplifies the other component. The degree of attenuation and amplification is determined by $r = |\varepsilon|$, that is called the *squeezing factor*. The squeezed state $|\alpha,\varepsilon\rangle$ is given by first applying the squeezing operator on the vacuum state and then displacing it

$$|\alpha,\varepsilon\rangle = D(\alpha)S(\varepsilon)\,|0\rangle \tag{2.112}$$

A squeezed state has the following expectation values and variances

$$\begin{aligned}
&\langle X_1 + iX_2\rangle = (Y_1+iY_2)e^{-i\phi} = 2\alpha,\\
&\Delta Y_1 = e^{-r}, \quad \Delta Y_2 = e^{r},\\
&\langle N\rangle = |\alpha|^2 + \sinh^2 r,\\
&(\Delta N)^2 = |\alpha\cosh r - \alpha^* e^{i2\theta}\sinh r|^2 + 2\cosh^2 r\sinh^2 r.
\end{aligned} \tag{2.113}$$

3 Elements of Quantum information Theory

In this chapter we introduce the fundamental elements of quantum information theory, providing the basis for understanding key concepts and tools required in the broader

discussion of quantum mechanics. The chapter begins with the definition of *qubits*, the fundamental units of quantum information, and proceeds to explore their generalization to *qudits*, providing insight into higher-dimensional quantum systems. Next, the concept of quantum measurement is introduced, emphasizing the differences between pure and mixed states, and the probabilistic nature of quantum systems. The discussion then goes through the introduction of *bipartite entanglement* in discrete systems, presenting its formal definition and methods of quantification, focusing our attention on a specific one; The *Negativity*.

3.1 Qubit

In classical information theory, information is encoded in a binary variable that can take the value 0 or 1. Modern computers typically encode this information in the form of electrical voltage or current pulses that can assume two distinct values. Similarly to the classical bit, a qubit in quantum information is defined as a system that resides in a two-dimensional Hilbert space, denoted as $\mathscr{H}^{(2)}$. Let $|0\rangle$ and $|1\rangle$ represent a basis for this Hilbert space. Unlike a classical bit, a qubit can exist in a superposition of states $|0\rangle$ and $|1\rangle$. Generally, the state of a qubit is expressed as:

$$|\psi\rangle = a|0\rangle + b|1\rangle, \tag{3.1}$$

where a and b must satisfy the normalization condition $|a|^2 + |b|^2 = 1$. The ability to prepare qubits in arbitrary superpositions of two states leads to various phenomena. When measuring qubits, or more generally quantum states, the outcome of measurements is inherently random. This randomness might seem problematic and could make quantum computation appear intractable; however, this is not the case. The ability to prepare qubits in specific states deterministically makes quantum computation at least as powerful as classical computation. Moreover, leveraging the capability to prepare qubits in superpositions, a quantum computer can outperform a classical computer in certain computational tasks. Qubits can be encoded in the polarization state of a photon (e.g., horizontal and vertical polarization) or in two distinct energy levels of an atom, ion, or molecule.

3.2 Qudit

The concept of qubits can be extended to higher-dimensional systems known as qudits. A qudit is a system that resides in a d-dimensional Hilbert space $\mathscr{H}^{(d)}$. Let $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$ denote a basis for this Hilbert space. Similarly to a qubit, a qudit can be in a superposition of all basis states:

$$|\psi\rangle = \sum_{i=0}^{d-1} a_i |i\rangle, \tag{3.2}$$

where the coefficients a_i must satisfy the normalization condition $\sum_{i=0}^{d-1} |a_i|^2 = 1$.

3.3 Entanglement in discrete systems

Consider a state formed by two qubits, the first belonging to space $\mathscr{H}_A$ with basis $\{|0\rangle_A, |1\rangle_A\}$, and the second to the space $\mathscr{H}_B$ with basis $\{|0\rangle_B, |1\rangle_B\}$. We can write, for example, $|00\rangle = |0\rangle_A \otimes |0\rangle_B$, with $\otimes$ being the tensor product, or briefly $|00\rangle = |0\rangle_A |0\rangle_B$. The possible values that the qubits can take are:

$$|00\rangle = |0\rangle_A |0\rangle_B, \quad |01\rangle = |0\rangle_A |1\rangle_B, \quad |10\rangle = |1\rangle_A |0\rangle_B, \quad |11\rangle = |1\rangle_A |1\rangle_B. \tag{3.3}$$

Now consider a generic state of $\mathscr{H}_A$ in normalized form:

$$|\phi\rangle_A = \lambda_A |0\rangle_A + \mu_A |1\rangle_A, \quad |\lambda_A|^2 + |\mu_A|^2 = 1 \tag{3.4}$$

and a generic state of $\mathscr{H}_B$ also in normalized form:

$$|\phi\rangle_B = \lambda_B |0\rangle_B + \mu_B |1\rangle_B, \quad |\lambda_B|^2 + |\mu_B|^2 = 1. \tag{3.5}$$

The state obtained from the tensor product of the two is:

$$\begin{aligned} |\phi\rangle_A \otimes |\phi\rangle_B &= \lambda_A\lambda_B |0_A 0_B\rangle + \lambda_A\mu_B |0_A 1_B\rangle + \mu_A\lambda_B |1_A 0_B\rangle + \mu_A\mu_B |1_A 1_B\rangle \\ &= \lambda_A\lambda_B |00\rangle + \lambda_A\mu_B |01\rangle + \mu_A\lambda_B |10\rangle + \mu_A\mu_B |11\rangle . \end{aligned} \tag{3.6}$$

It is noted that the states $|\phi\rangle_A \otimes |\phi\rangle_B$ represent only a small subset of the states in the space $\mathscr{H}_A \otimes \mathscr{H}_B$. A generic state of this space has the following form:

$$\begin{aligned} |\psi\rangle &= \alpha_{00} |0_A 0_B\rangle + \alpha_{01} |0_A 1_B\rangle + \alpha_{10} |1_A 0_B\rangle + \alpha_{11} |1_A 1_B\rangle \\ &= \alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{10} |10\rangle + \alpha_{11} |11\rangle . \end{aligned} \tag{3.7}$$

For the state $|\psi\rangle$ to be of the form $|\phi\rangle_A \otimes |\phi\rangle_B$, it must be $\alpha_{00}\alpha_{11} = \alpha_{10}\alpha_{01}$.
Now consider the state:

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle). \tag{3.8}$$

The state $|\phi\rangle$ belongs to the space $\mathscr{H}_A \otimes \mathscr{H}_B$ but is not in the form $|\phi\rangle_A \otimes |\phi\rangle_B$, since:

$$\alpha_{00} = \alpha_{11} = 0, \quad \alpha_{01} = \alpha_{10} = \frac{1}{\sqrt{2}} \rightarrow \alpha_{00}\alpha_{11} \neq \alpha_{10}\alpha_{01}. \tag{3.9}$$

A state like this, which cannot be written in the form $|\phi\rangle_A \otimes |\phi\rangle_B$, is called an entangled state. An entangled state is one that cannot be separated, meaning it is not possible to define a pair of states $|\phi\rangle_A, |\phi\rangle_B$ such that their composition $|\phi\rangle_A \otimes |\phi\rangle_B = |\phi\rangle$. We have thus seen, in the simplest case possible, how it is possible to recognize an entangled state by introducing the concept of *inseparability* of the state. It is now natural to wonder how this concept could be applied to more complex pure states, i.e. with higher dimensions Hilbert spaces, and mixed states.

3.3.1 Pure States

In the previous section we have introduced the non-separability of a state as necessary and sufficient condition to identify entanglement. We have already seen how easy is to deal with a bipartite system composed by quibits. We want to extend the treatment to bipartite systems whose individual components belong to Hilbert spaces of greater dimension, or when the two arguments x_1 and x_2 of the bipartite wave function $\Psi(x_1,x_2)$ can take only finite numbers of discrete values $x_1 = x_{1,1}, x_{1,2}, \ldots, x_{1,d}$ and $x_2 = x_{2,1}, x_{2,2}, \ldots, x_{2,\tilde{d}}$, where d and $\tilde{d}$ are the dimensionalities of the single-particle Hilbert spaces for the particles 1 and 2 (in a general case d and $\tilde{d}$ can differ from each other). A general bipartite wave function with discrete variables can be expanded in a double sum of direct products of columns $|j\rangle$ and $|k\rangle$:

$$|\psi(1,2)\rangle = \sum_{j,k} C_{j,k} |j\rangle \otimes |k\rangle , \tag{3.10}$$

where the arguments $1,2$ of the bipartite wave function and the labels (1) and (2) refer to particles 1 and 2 (often denoted as A and B). Clearly, the bipartite wave function $|\psi(1,2)\rangle$ is a column with $d \times \tilde{d}$ lines. In this section we will consider just the case in which $d = \tilde{d}$. To derive the Schmidt decomposition is necessary to introduce the *Singular-Value-Decomposition or SVD* of the matrix C that contains all the coefficients of the expansion. Specifically, the singular value decomposition of C is equal to:

$$C = UDV. \tag{3.11}$$

Here D is a diagonal matrix with non-negative real numbers on the diagonal, while U, V are complex unitary matrix. Thus we can write:

$$|\psi(1,2)\rangle = \sum_{i,j,k} u_{ji} d_{ii} v_{ik} |j\rangle \otimes |k\rangle . \tag{3.12}$$

If we now define $|i_1\rangle \equiv \sum_j u_{ji} |j\rangle$, $|i_2\rangle \equiv \sum_k v_{ik} |k\rangle$ and $\lambda_i \equiv d_{ii}$, we get:

$$|\psi(1,2)\rangle = \sum_i \lambda_i |i_1\rangle \otimes |i_2\rangle . \tag{3.13}$$

This result is very useful. As a taste of its power, consider the following consequence: let $|\psi\rangle$ be a pure state of a composite system, 1,2. Then by the Schmidt decomposition:

$$\rho_1 = \sum_i \lambda_i^2 |i_1\rangle\langle i_1| \quad \text{and} \quad \rho_2 = \sum_i \lambda_i^2 |i_2\rangle\langle i_2|, \tag{3.14}$$

so the eigenvalues of ρ_1 and ρ_2 are identical, namely λ_i^2 for both density operators. Many important properties of quantum systems are completely determined by the eigenvalues of the reduced density operator of the system, so for a pure state of a composite system such properties will be the same for both systems. A general definition of entanglement for pure bipartite states is related to the assumption about non-factorization of the corresponding bipartite wave function: the state is entangled if

$$|\psi(1,2)\rangle \neq |\phi_1\rangle \otimes |\phi_2\rangle \tag{3.15}$$

The Schmidt decomposition (3.13) perfectly agree with this definition, and it provide both a deeper understanding of conditions under which entanglement occurs and an evaluation of its degree. Indeed, in terms of the Schmidt decomposition, factorization occurs only if one of the parameters λ_i equals unity with all other λ_i equal to zero, e.g., $\lambda_i = \delta_{i,1}$. In all other cases, with several non-zero values of λ_i, the Schmidt decompositions contain more than one term, the wave function (3.13) is unfactorable, and the state is entangled. Actually, in these cases, the Schmidt decompositions show directly how many products of Schmidt modes have to be summed to reproduce an unfactorable bipartite wave function. The number of terms in the Schmidt decompositions characterizes the amount of uncertainty in the localization of particles 1 and 2 in the pairs of adjoint Schmidt modes and, hence, it characterizes the degree of entanglement.

3.3.2 Mixed States

A mixed quantum state $\hat{\rho}$ can be written as a convex combination of pure states:

$$\hat{\rho} = \sum_k p_k |\psi_k\rangle\langle\psi_k|. \tag{3.16}$$

This expression reveals how a state described by the density matrix ρ is constructed. Specifically, the pure state $|\psi_1\rangle$ is prepared with probability p_1, $|\psi_2\rangle$ with probability p_2, and so on. In the work of Adesso and Illuminati [23] is pointed out how the key challenge here lies in the fact that such a decomposition is not unique. Beyond the pure states, there are infinitely many ways to decompose a given ρ, meaning the mixed state can be prepared using countless different methods. This non-uniqueness has significant implications when it comes to analyzing the entanglement properties of mixed states. Imagine that we have prepared a system, consisting of two subsystems A and B, which is divided between Alice and Bob. The two observers are located in distant laboratories and have access to a set of possible measurements that can be made on their own system. It is important to point out that the two parties make measurements at the same time, so as to avoid any kind of mutual dependence. The latter condition implies that the operation is local ('LO'). However, there is no reason to make the operations of separated labs totally independent. Classical communication ('CC') can essentially be performed perfectly using standard technologies, and so we may also use such communication to coordinate the quantum actions of the different labs. The sequence of these two procedures is called *Local Operations and Classical Communication* (LOCC). Allowing classical communication in the set of LOCC operations means that they are not completely local, this means that the results of coordinate measurements possibly revealing some correlations between them. Due to the ambiguity in how the state is prepared, it is impossible to determine upfront whether these correlations arise from quantum interactions between the subsystems (indicating entanglement) or whether they are due to classical processes such as local operations and classical communication (LOCC), suggesting classical correlations instead. As a result, a mixed state is

called *separable* (or classically correlated) if it can be prepared through at least one method that uses only LOCC. On the other hand, a mixed state is deemed *entangled* (or quantumly correlated) if none of the infinite possible decompositions involve only LOCC [24]. However, determining whether a mixed state is separable requires analyzing all the possible decompositions to see if there is one that is a convex combination of product states, which would imply the state is not entangled. This process, however, is clearly impractical due to the infinite number of possible decompositions. We therefore generalize the definition of pure-state entanglement via the non-factorizability of the total state vector to mixed states through non-separability or inseparability of the total density operator. A general quantum state of two-party system is separable if its total density operator is a mixture (a convex sum) of product states

$$\rho_{AB} = \sum_{\lambda} p_{\lambda} \rho_A^{\lambda} \otimes \rho_B^{\lambda} \tag{3.17}$$

Otherwise is inseparable. In general, it is a nontrivial question whether a given density operator is separable or inseparable. Nonetheless, a very convenient method of testing for inseparability is Peres's [25] partial-transpose criterion (*PPT*). The derivation of this separability condition is best done by writing the density matrix elements explicitly, with all their indices

$$\rho_{m\mu,n\nu} = \sum_{\lambda} p_{\lambda} \left(\rho_A^{\lambda}\right)_{mn} \otimes \left(\rho_B^{\lambda}\right)_{\mu\nu} \tag{3.18}$$

Latin indices refer to the first subsystem, Greek indices to the second one (the subsystems may have different dimensions). We aim for quantum density matrices to possess eigenvalues that are non-negative rather than focusing solely on ensuring non-negative elements. Achieving the former condition is notably more challenging. Let's proceed to introduce a new matrix definition.

$$\sigma_{n\mu,m\nu} = \rho_{m\mu,n\nu} \tag{3.19}$$

The Latin indices of ρ have been transposed, but not the Greek ones. This is not a unitary transformation but, nevertheless, the s matrix is Hermitian. When Eq. (3.17) is valid, we have

$$\sigma_{AB} = \sum_{\lambda} p_{\lambda} \left(\rho_A^{\lambda}\right)^T \otimes \rho_B^{\lambda} \tag{3.20}$$

Since the transposed matrices $\left(\rho_A^{\lambda}\right)^T = \left(\rho_A^{\lambda}\right)^*$ are nonnegative matrices with unit trace, they can also serve as valid density matrices. Consequently, none of the eigenvalues of σ_{AB} are negative, a prerequisite for Equation (3.17) to remain valid. Notably, the eigenvalues of σ_{AB} remain invariant under separate unitary transformations U_A and U_B of the bases employed by the two observers. In such instances, ρ undergoes transformation as follows:

$$\rho_{AB} \longrightarrow (U_A \otimes U_B)\rho_{AB}(U_A \otimes U_B)^{\dagger} \tag{3.21}$$

and we then have

$$\sigma_{AB} \longrightarrow ((U_A)^T \otimes U_B)\sigma_{AB}((U_A)^T \otimes U_B)^\dagger \tag{3.22}$$

which also is unitary transformation, leaving the eigenvalues of σ invariant. Peres' criterion states that if a state ρ is separable, then its partial transpose ρ_1^T (with respect, for example, to subsystem S_1) is a valid density matrix, specifically positive semidefinite, $\rho^T \geq 0$. Naturally, the same holds for ρ_2^T with respect to subsystem S_2. Positivity of the partial transpose (PPT) is therefore a necessary condition for separability [25]. The converse (i.e., $\rho^T \geq 0$ implies separability) is generally false, but it has been proven true for low-dimensional systems, specifically bipartite systems with Hilbert state space of dimensionality 2×2 and 2×3. In these cases, the PPT property is equivalent to separability [26].

3.4 Entanglement quantification

Entanglement quantification involves measuring the degree of entanglement present in a quantum system. Different entanglement measures have been developed to capture various aspects of entanglement, each suited to specific types of quantum states and operational paradigms [27, 28]. For instance, measures like negativity, concurrence, entanglement of formation, entropy of entanglement and robustness of entanglement are widely used to quantify entanglement in different contexts [29, 30]. However, all those different measures must fulfill the following properties to be considered a valid one:

- **Monotonicity under Local Operations and Classical Communication (LOCC)**: An entanglement measure $E(\rho)$ must not increase under LOCC operations. This reflects the idea that entanglement cannot be created at a distance using only local operations and classical communication.

  $$E(\rho) \geq E(\Lambda_{\text{LOCC}}(\rho)), \tag{3.23}$$

  where Λ_{LOCC} represents a LOCC operation.

- **Invariance under Local Unitary Transformations**: Entanglement must remain unchanged if local unitary operations are applied to the subsystems.

  $$E(\rho) = E(U_A \otimes U_B \rho U_A^\dagger \otimes U_B^\dagger), \tag{3.24}$$

  where U_A and U_B are unitary operators acting locally on the subsystems.

- **Non-negativity**: A valid entanglement measure must be non-negative for all quantum states, and it should be zero for separable (non-entangled) states.

  $$E(\rho) \geq 0, \quad E(\rho) = 0 \quad \text{if} \quad \rho \text{ is separable.} \tag{3.25}$$

- **Convexity (or Strong Monotonicity)**: An entanglement measure should be convex, meaning that mixing entangled states cannot increase the overall entanglement.
$$E\left(\sum_i p_i \rho_i\right) \leq \sum_i p_i E(\rho_i). \tag{3.26}$$

- **Asymptotic Behavior**: For a large number of copies of a state ρ, the measure of entanglement should be consistent with the von Neumann entropy asymptotically.
$$E(\rho^{\otimes n}) \sim nE(\rho) \quad \text{as} \quad n \to \infty. \tag{3.27}$$

- **Normalization**: The measure should be normalized such that maximally entangled states (e.g., Bell states for two qubits) have a maximum fixed value. For example, it is often normalized so that $E(\text{Bell state}) = 1$.

- **Continuity**: The measure should be continuous with respect to the quantum state ρ, meaning that small changes in the state should not lead to large variations in the entanglement measure.

- **Optional: Asymmetry with Respect to Subsystems**: For multipartite systems, the measure might distinguish between different subsystems, though not all entanglement measures require this property.

It is relevant, in pure bipartite quantum states, to consider the so-called *von Neumann entropy* [31]. Given a pure state ρ_{AB} of two subsystems A and B, we define the states $\rho_A = \text{Tr}_B[\rho_{AB}]$ and $\rho_B = \text{Tr}_A[\rho_{AB}]$, where the partial trace has been taken over one subsystem, either A or B. Then, the von Neumann entropy of the reduced density operators is given by
$$S(\rho_A) = -\text{Tr}(\rho_A \ln \rho_A) = -\text{Tr}(\rho_B \ln \rho_B). \tag{3.28}$$

In the case of a separable pure joint state, $S(\rho_A)$ is zero, and for maximally entangled states, it gives $\ln 2$. With regard to mixed states, we analyze a very effective measure of entanglement in its quantification, which will also be useful in the domain of continuous variables and it is the *Negativity* [32].

3.4.1 Negativity

Previously we have defined , by means the positivity of the partial transpose density matrix (*PPT* criterion), a way to distinguish inseparable from separable states. *Negativity* essentially measures the degree to which ρ^{T_A} fails to be positive, and therefore it can be regarded as a quantitative version of Peres' criterion for separability [25]. In this section we shell follow the review of G. Vidal and R.F. Werner [32] who from the trace norm of ρ^{T_A}, denoted by $\|\rho^{T_A}\|_1$, they have constructed two useful quantities: *Negativity* and *Logarithmic negativity*. Regardless the existence of negative eigenvalues ρ^{T_A}, its trace sum to unit, while the trace norm of any Hermitian operator $\hat{A}$ is $\|A\|_1 \equiv \text{tr}\sqrt{A^\dagger A}$, which is equal to the sum of the absolute values of the eigenvalues of $\hat{A}$. From these conditions we can easily derive the negativity measure, since:

$$
\begin{aligned}
Tr(\rho^{T_A}) = 1 = \sum_{i:\lambda_i>0} \lambda_i - \sum_{i:\lambda_i<0} |\lambda_i|, \\
||\rho^{T_A}|| = \sum_{i:\lambda_i>0} \lambda_i + \sum_{i:\lambda_i<0} |\lambda_i|.
\end{aligned}
\tag{3.29}
$$

The final result is:

$$
N(\rho) \equiv \frac{\|\rho^{T_A}\|_1 - 1}{2}, \tag{3.30}
$$

which corresponds to the absolute value of the sum of negative eigenvalues of ρ^{T_A} [33]. Notice that for any separable state ρ_s [24],

$$
\rho_s = \sum_k p_k |e_k, f_k\rangle\langle e_k, f_k|; \quad p_k \geq 0, \quad \sum_k p_k = 1, \tag{3.31}
$$

its partial transposition is also a separable state [25],

$$
\rho_s^{T_A} = \sum_k p_k |e_k^*, f_k\rangle\langle e_k^*, f_k| \geq 0, \tag{3.32}
$$

and therefore $\|\rho_s^{T_A}\|_1 = 1$ and $N(\rho_s) = 0$. As they proved, $N(\rho)$ does not increase under LOCC, i.e., it is an entanglement monotone [34], and as such it can be used to quantify the degree of entanglement in composite systems. We will also consider the *logarithmic negativity*:

$$
E_N(\rho) \equiv \log_2 \|\rho^{T_A}\|_1, \tag{3.33}
$$

which again exhibits some form of monotonicity under LOCC (it does not increase during deterministic distillation protocols) and is, remarkably, an additive quantity. We have seen in previous section what are the requirements that a proper entanglement measure must satisfy. In this prospective it is necessary to prove *convexity*, *monotonicity* [32, 28].

Proposition 1: N is a convex function, i.e.,

$$
N\left(\sum_i p_i \rho_i\right) \leq \sum_i p_i N(\rho_i), \tag{3.34}
$$

whenever the ρ_i are Hermitian, and $p_i \geq 0$ with $\sum_i p_i = 1$. The *convexity* in this case is given for free, since $N(\rho) = \|(\rho^{T_A}\|_1 - 1)/2$ includes $\|\cdot\|_1$, that satisfies like any norm, the triangle inequality and is homogeneous of degree 1 for positive factors, hence convex. However, What is not immediately clear is that the $\|\rho\|_1$, when it is defined in terms of the eigenvalues. This is shown best by rewriting it as a variational expression. The main reason for recalling this standard observation from the theory of the trace norm is that the same variational expression will be crucial for showing monotonicity

under LOCC operations. The variational expression is simply the representation of a general Hermitian matrix A as a difference of positive operators: Since we are in finite dimension we can always write

$$A = a^+\rho^+ - a^-\rho^-, \tag{3.35}$$

where $\rho^\pm \geq 0$ are density matrices ($\mathrm{tr}[\rho^\pm] = 1$) and $a^\pm \geq 0$ are positive numbers. Note that by taking the trace of this equation we simply have $\mathrm{tr}[A] = a^+ - a^-$.

Lemma 1: *For any Hermitian matrix A there is a decomposition of the form (3.35) for which $a^+ + a^-$ is minimal. For this decomposition, $\|A\|_1 = a^+ + a^-$, and a^- is the absolute sum of the negative eigenvalues of A.*

Proof: Let P^- be the projector onto the negative eigenvalued subspace of A, and $N = -\mathrm{tr}[AP^-]$ the absolute sum of the negative eigenvalues. We can reverse the decomposition (3.35) to obtain that $A + a^-\rho^-$ is positive semidefinite. This implies that

$$0 \leq \mathrm{tr}[(A + a^-\rho^-)P^-] = -N + a^-\mathrm{tr}[\rho^-P^-]. \tag{3.36}$$

But $\mathrm{tr}[\rho^-P^-] \leq 1$, that is $a^- \geq N$. This bound can be saturated with the choice $a^-\rho^- \equiv -P^-AP^-$ (corresponding to the Jordan decomposition of A, where ρ^- and ρ^+ have disjoint support) ■.

For the negativity we therefore get the formula

$$N(A) = \inf\{a^- \mid A = a^+\rho^+ - a^-\rho^-\}, \tag{3.37}$$

where the infimum is over all density matrices $\rho^\pm$ and $a^\pm \geq 0$. Another remarkable property of $N(\rho)$ is the easy way in which $N(\rho_1 \otimes \rho_2)$ relates to the negativity of ρ_1 and that of ρ_2. For the entanglement measure treated, they get additivity for free [32]. As they showed, starting from the identity

$$\|\rho_1 \otimes \rho_2\|_1 = \|\rho_1\|_1\|\rho_2\|_1, \tag{3.38}$$

which is best shown by using the definition of the trace norm via eigenvalues, and observe that partial transposition commutes with taking tensor products. After taking logarithms, we find for the logarithmic negativity:

$$E_N(\rho_1 \otimes \rho_2) = E_N(\rho_1) + E_N(\rho_2). \tag{3.39}$$

It might seem from this that E_N is a candidate for the much sought for canonical measure of entanglement. However, it has other drawbacks. For instance, it is not convex, as is already suggested by the combination of a convex functional (the trace norm) with

the concave log function, which implies that it increases under some LOCC.

The last step is to prove that Negativity is an entanglement monotone. By definition, a LOCC operation consists of a series of steps where each party performs a local measurement and communicates the outcome to the others. The choice of local measurement can depend on the results of previous steps, but for simplicity we will consider a single-step procedure. For an initial state ρ, the measurement outcome is represented by an index i, which occurs with probability p_i, and the resulting state is denoted as ρ_i'. A valid entanglement monotone $E(\rho)$ [34] must satisfy:

$$E(\rho) \geq \sum_i p_i E(\rho_i'). \tag{3.40}$$

This property can be proved by analyzing a single round of an LOCC protocol, involving only one local operation. Since the measure N does not distinguish between Alice and Bob, it suffices to consider a local operation performed by Bob. The most general local measurements can be represented by a set of completely positive linear maps M_i, which act as:

$$M_i(\rho) = p_i \rho_i', \tag{3.41}$$

subject to the normalization condition:

$$\sum_i \mathrm{Tr}[M_i(\rho)] = \mathrm{Tr}(\rho). \tag{3.42}$$

For a local measurement performed by Bob, we can write:

$$M_i(\rho) = (I_A \otimes M_i)\rho(I_A \otimes M_i^\dagger), \tag{3.43}$$

where the operators M_i are known as Kraus operators. They arise naturally in the mathematical framework describing the evolution of open quantum systems and general quantum operations. In quantum mechanics, unitary evolution fully describes an *isolated system* via the transformation:

$$\rho' = U\rho U^\dagger. \tag{3.44}$$

However, when a quantum system interacts with an external environment, the evolution is no longer unitary, requiring a more general description. To model such processes, consider a bipartite system composed of a system A and an environment B, initially in a product state:

$$\rho_{AB} = \rho_A \otimes |0\rangle_B\langle 0|. \tag{3.45}$$

The total system evolves under a global unitary transformation U_{AB}, leading to the final state:

$$\rho'_{AB} = U_{AB}(\rho_A \otimes |0\rangle_B\langle 0|)U_{AB}^\dagger. \tag{3.46}$$

Since the environment is not accessible, we obtain the reduced state of system A by taking the partial trace:

$$\rho'_A = \mathrm{Tr}_B\left[U_{AB}(\rho_A \otimes |0\rangle_B\langle 0|)U_{AB}^\dagger\right]. \tag{3.47}$$

Expanding in an orthonormal basis $\{|j\rangle_B\}$ for system B, we obtain the *operator-sum representation*, also known as the *Kraus representation*:

$$\rho'_A = \sum_i M_i \rho_A M_i^\dagger, \tag{3.48}$$

where the Kraus operators are given by:

$$M_i = \langle i|U_{AB}|0\rangle_B. \tag{3.49}$$

These operators satisfy the *completeness relation*:

$$\sum_i M_i^\dagger M_i = I, \tag{3.50}$$

ensuring that the transformation is trace-preserving ■.
Now we can compute the right-hand side of the inequality (3.40), we observe that:

$$M_i(\rho)^{T_A} = M_i(\rho^{T_A}), \tag{3.51}$$

which follows directly from (3.43) expressing ρ as a sum of (not necessarily positive) tensor products. For Alice's local operations, a similar formula holds, with the Kraus operators replaced by their complex conjugates. The partial transpose of ρ can be decomposed as:

$$\rho^{T_A} = (1+N)\rho_+ - N\rho_-, \tag{3.52}$$

where $\rho_\pm$ are density operators and $N = N(\rho)$ is the negativity of the state. The same decomposition applies to the partially transposed output states:

$$p_i(\rho'_i)^{T_A} = M_i(\rho)^{T_A} = M_i(\rho^{T_A}) = (1+N)M_i(\rho_+) - NM_i(\rho_-). \tag{3.53}$$

Dividing by p_i, we obtain a decomposition consistent with the definition of $N(\rho'_i)$. The coefficient a_- is given by N/p_i, which must be greater than or equal to the infimum:

$$N(\rho'_i) \leq \frac{N}{p_i}. \tag{3.54}$$

Multiplying by p_i and summing over all i, we arrive at the desired inequality:

$$\sum_i p_i N(\rho_i') \leq N(\rho) \quad \blacksquare. \tag{3.55}$$

4 Continuous Variables

Expanding beyond the discrete framework, the chapter explores *continuous-variable (CV) systems*, which play a fundamental role in the description of quantum optical fields [5, 35]. It addresses key topics such as the quantum phase-space formalism, and the mathematical characterization of Gaussian states, providing a foundation for the analysis of entanglement and separability in continuous-variable quantum systems. The chapter also introduces Simon's separability criterion for Gaussian states, including the its symplectic representation and the Duan bound, along with key measures like negativity, to distinguish entangled states from separable ones. These tools will be essential for exploring the intricate relationship between entanglement and non-locality in quantum systems.

A natural extension of qudits can be obtained by taking the limit $d \to \infty$, resulting in an infinite-dimensional Hilbert space $\mathscr{H}$. The orthonormal basis of this space is denoted by $\{|n\rangle, n \in \mathbb{N}\}$, and a generic state can be expressed as:

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \tag{4.1}$$

where the coefficients c_n satisfy the normalization condition $\sum_{n=0}^{\infty} |c_n|^2 = 1$. This Hilbert space, known as the Fock space, is used to describe systems with a variable or indefinite number of identical particles and provides a convenient basis for systems like harmonic oscillators, including the electromagnetic field. In the case of a harmonic oscillator, the state of the system can be described in the Fock basis. While this basis is highly useful, it is sometimes more convenient to represent the system's state in terms of position or momentum or, in the case of an electromagnetic field, in terms of its quadratures. For this purpose, we consider a continuous basis composed of eigenstates of the position operator $\hat{q}$ and the momentum operator $\hat{p}$, defined by the relations:

$$\hat{q}|q\rangle = q|q\rangle, \tag{4.2}$$

$$\hat{p}|p\rangle = p|p\rangle. \tag{4.3}$$

These are not strictly eigenstates in the sense that they are non-normalizable and do not belong to the Hilbert space $L^2(\mathbb{R})$. However, they are orthogonal and satisfy the normalization conditions:

$$\langle q|q'\rangle = \delta(q - q'), \tag{4.4}$$

$$\langle p|p'\rangle = \delta(p-p'), \tag{4.5}$$

where δ represents the Dirac delta function. The states $|q\rangle$ and $|p\rangle$ are idealized states corresponding to precise position or momentum measurements. Although physically unrealistic, as they would require infinite energy to create, they are mathematically useful and form a complete basis:

$$\int_{-\infty}^{\infty} dq\,|q\rangle\langle q| = \int_{-\infty}^{\infty} dp\,|p\rangle\langle p| = 1. \tag{4.6}$$

Using this completeness relation, any state can be expressed in the position or momentum basis:

$$|\psi\rangle = \int_{-\infty}^{\infty} \psi(q)|q\rangle\,dq = \int_{-\infty}^{\infty} \phi(p)|p\rangle\,dp, \tag{4.7}$$

where the wavefunctions $\psi(q)$ and $\phi(p)$ are square-integrable and have norm one. The position operator $\hat{q}$ and momentum operator $\hat{p}$ do not commute and satisfy the commutation relation $[\hat{q},\hat{p}] = i\hbar$, indicating that the descriptions in the position and momentum basis are not independent. Using the commutation relation, $\psi(q)$ and $\phi(p)$ can be shown to be Fourier transforms of each other.

Proof: To prove this property we can express:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dpdq\,|q\rangle\langle q|\hat{p}\,|p\rangle\langle p| = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dpdq\,p|q\rangle\langle q|\,|p\rangle\langle p|. \tag{4.8}$$

Here, we have multiplied the momentum operator $\hat{p}$ on both sides by the completeness relations; on the left, expressed in the position basis, and on the right, expressed in the momentum basis. Since the action of the momentum operator onto position basis is:

$$\hat{p}\,|q\rangle = -i\hbar\frac{\partial}{\partial q}\,|q\rangle\,, \tag{4.9}$$

we can rewrite (4.8) in terms of $\psi_p(q) = \langle q|p\rangle$:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dpdq\,|q\rangle\left(-i\hbar\frac{\partial}{\partial q}\psi_p(q)\right)\langle p| = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dpdq\,|q\rangle\,(p\psi_p(q))\,\langle p|. \tag{4.10}$$

The two integrals are equivalent when their integrands are identical, therefore, we can obtain a first order differential equation for $\psi_p(q)$:

$$-i\hbar\frac{\partial}{\partial q}\psi_p(q) = p\psi_p(q), \tag{4.11}$$

whose solution is easily derived:

$$\psi_p(q) = Ae^{i\frac{pq}{\hbar}}. \tag{4.12}$$

We know that the momentum eiganstates are not normalizable, but we can use the normalization condition of delta function to define the constant A. We can start from its definition:

$$\int_{-\infty}^{+\infty} \psi^*_{p'}(q)\psi_{p''}(q)dq = \delta(p' - p''). \tag{4.13}$$

The integral becomes:

$$|A|^2 \int_{-\infty}^{+\infty} e^{-\frac{i}{\hbar}(p'-p'')q} dq = 2\pi\hbar\delta(p' - p'') \rightarrow A = \frac{1}{\sqrt{2\pi\hbar}}. \tag{4.14}$$

We can finally use this result to highlight the link between the two wavefunctions:

$$\psi(q) = \langle q|\psi\rangle = \langle q| \int_{-\infty}^{+\infty} \phi(p)\,|p\rangle\, dp = \int_{-\infty}^{+\infty} \phi(p)\,\langle q|p\rangle\, dp = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \phi(p) e^{\frac{ipq}{\hbar}} dp. \tag{4.15}$$

Taking the inverse Fourier transform we get

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(q) e^{\frac{-ipq}{\hbar}} dq \quad \blacksquare. \tag{4.16}$$

This implies that it is impossible to know both position and momentum with arbitrary precision, as expressed by the Heisenberg uncertainty principle. Mathematically, this principle bounds the product of the root mean square deviations of position and momentum:

$$\sigma_q \sigma_p \geq \frac{1}{2}. \tag{4.17}$$

An example of a continuous state can be given by considering $\psi(q)$ as a Gaussian function centered at the origin with a width σ:

$$\psi(q) = \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-q^2/(4\sigma^2)}, \tag{4.18}$$

$$\phi(p) = \frac{(2\sigma^2)^{1/4}}{\pi^{1/4}} e^{-p^2\sigma^2}. \tag{4.19}$$

The mean square deviations for this state yield $\sigma_q = \sigma$ and $\sigma_p = 1/(2\sigma)$, showing

that it saturates the Heisenberg uncertainty relation. As σ decreases, reducing uncertainty in position, the uncertainty in momentum increases. In the limit $\sigma \to 0$, $\psi(q)$ converges to a Dirac delta distribution, indicating complete certainty in position and infinite uncertainty in momentum. It is important to note that the Fock basis approach and the position or momentum eigenstate approach are entirely equivalent. Indeed, it is possible to construct a discrete basis for the Hilbert space $L^2(\mathbb{R})$, as in the case of the eigenvectors of the harmonic oscillator Hamiltonian, which are proportional to Hermite polynomials. This is because the Hilbert space $L^2(\mathbb{R})$ is separable.

4.1 Multimodes systems

The systems that will be discussed in this section are systems with N canonical degrees of freedom. These N modes could represent the modes of an Harmonic oscillator (2.55) as we have already seen in Sec. (2.1). Each of them is identified by the number operator $\hat{n}_k$, whose state $|n\rangle_k$ belongs to the single mode Fock space $\mathscr{H}_k$. The global continuous-variable (CV) system [36],[37] of N canonical bosonic modes is then described by a Hilbert space $\mathscr{H} = \bigotimes_{k=1}^{N} \mathscr{H}_k$, resulting from the tensor product structure of infinite-dimensional Fock spaces $\mathscr{H}_k$. Although the description in terms of operator number is the most natural, it is not the most convenient. We have seen that because of their similarity to classical states, coherent states are a much more powerful tool for analysis of quantum systems. The coherent state for single mode is the eigenvator of the annihilation operator $\hat{a}_k$, i.e, $\hat{a}_k|\alpha\rangle_k = \alpha|\alpha\rangle_k$. This operator can be expressed in terms of quadratures as expressed by (2.102), whose commutation relation is given by (2.103).
Coherent states result from applying the single-mode Weyl displacement operator $\hat{D}_k$ to the vacuum $|0\rangle_k$,

$$|\alpha\rangle_k = \hat{D}_k(\alpha)|0\rangle_k. \tag{4.20}$$

Since we are dealing with a system of independent bosonic modes, the overall coherent state is obtained by applying the N-mode Weyl operators $\hat{D}_\xi$ to the global vacuum $|0\rangle = \otimes|0\rangle_k$. In order to give an explicit expression of $\hat{D}_\xi$ is necessary to group together, in a single vector $\hat{R} \in \mathbb{R}^{2N}$, all the modes:

$$\hat{\mathbf{R}} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_N, \hat{p}_N)^{\mathrm{T}}, \tag{4.21}$$

which allows us to write the bosonic commutation relations between the quadrature phase operators in a compact form:

$$[\hat{R}_k, \hat{R}_l] = 2i\Omega_{kl}, \tag{4.22}$$

where Ω is the symplectic form given by:

$$\Omega = \bigoplus_{k=1}^{N} \omega, \quad \text{with} \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.23}$$

Finally we get:

$$\hat{D}_\xi = e^{i\hat{R}^T \Omega \xi}, \tag{4.24}$$

with $\xi \in \mathbb{R}^{2N}$. One then has

$$|\xi\rangle = \hat{D}_\xi |0\rangle. \tag{4.25}$$

4.2 Quantum phase-space formalism

In quantum mechanics, the density matrix formalism is widely used to describe the quantum state of a system, particularly when dealing with mixed states or open systems interacting with the environment. The density matrix encodes the full information of the quantum state, including probabilities and coherences, and it is essential for evaluating expectation values of observables. However, when dealing with continuous variable systems, such as quantum states of light, it is often convenient to describe quantum states in phase space, similar to classical mechanics. In the quantum phase space representation, quantum states are mapped to quasi-probability distributions that describe the state in terms of position and momentum (or other conjugate variables, such as the quadratures of the electromagnetic field). These quasi-probability distributions provide a more intuitive picture of quantum phenomena and allow the use of classical-like methods to analyze quantum states. Three primary functions are commonly used to represent quantum states in phase space: *the Glauber-Sudarshan P function [38, 39], the Husimi Q function [19], and the Wigner function [40]*. Each of these functions offers a different perspective on the quantum state and has its unique applications. It is particularly useful for visualizing and analyzing quantum states in continuous variable systems because it connects the quantum description of the system with the classical phase space, offering an intuitive picture of quantum properties like superposition and entanglement. We want to give now a mathematical characterization of the mentioned quasi-probabilities, following the work of C.Gerry and P.Knight [41]. Starting from the most general definition for a mixture of quantum states

$$\hat{\rho} = p_i \sum_i |\Psi_i\rangle \langle\Psi_i| \tag{4.26}$$

we can always write the density matrix in terms of entries of the number state basis:

$$\hat{\rho} = \mathbb{I}\hat{\rho}\mathbb{I} = \sum_m |m\rangle \langle m| \hat{\rho} \sum_n |n\rangle \langle n| = \sum_n \sum_m |m\rangle \rho_{m,n} \langle n| . \tag{4.27}$$

On the other hand, resolving unity with coherent states on both sides of ρ, results in:

$$\iint \frac{d^2\alpha' d^2\alpha''}{\pi} \left\langle \alpha' \middle| \hat{\rho} \middle| \alpha'' \right\rangle \left| \alpha' \right\rangle \left\langle \alpha'' \right| \tag{4.28}$$

But there is yet another way to represent $\hat{\rho}$ in terms of coherent states, namely

$$\hat{\rho} = \int P(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha, \tag{4.29}$$

where $P(\alpha)$ is a weight function often referred to as the *Glauber–Sudarshan P function* [38]. This expression represents $\hat{\rho}$ in terms of coherent states. The right-hand side is the "diagonal" form of the density operator, and $P(\alpha)$ serves a role similar to phase-space distributions in statistical mechanics. In this context, the real and imaginary parts of α are the phase space variables. Since $\hat{\rho}$ is a Hermitian operator, $P(\alpha)$ must be a real function and correctly normalized. But for some quantum states of the field, $P(\alpha)$ can have properties quite unlike those of any true probability distribution where one would expect to have $P(\alpha) \geq 0$. Although $P(\alpha)$ can be interpreted as a classical distribution, this is not always true for every quantum system. In certain cases, the $P(\alpha)$ function may take on negative values in specific regions of phase space, thereby reflecting non-classical properties of physical reality. Before facing the main advantages of this distribution, we need to give an explicit description of it, using the results of Metha [42]:

$$P(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2} \int e^{|u|^2} \langle -u|\rho|u\rangle e^{u^*\alpha - u\alpha^*}\, d^2u. \tag{4.30}$$

where $|-u\rangle$ and $|u\rangle$ are coherent vectors. At this point, we introduce the optical equivalence theorem of Sudarshan [39]. Suppose we have a "normally ordered" function of the operators $\hat{a}$ and $\hat{a}^\dagger$, $G^{(N)}(\hat{a},\hat{a}^\dagger)$, where the annihilation operators stand to the right of the creation operators:

$$\hat{G}^{(N)}(\hat{a},\hat{a}^\dagger) = \sum_n \sum_m C_{nm} (\hat{a}^\dagger)^n \hat{a}^m. \tag{4.31}$$

The average of this function is:

$$\begin{aligned}
\langle G^{(N)}(\hat{a},\hat{a}^\dagger)\rangle &= \mathrm{Tr}\left(\hat{G}^{(N)}(\hat{a},\hat{a}^\dagger)\hat{\rho}\right) \\
&= Tr \int P(\alpha) \sum_{n,m} C_{n,m} (\hat{a}^\dagger)^n \hat{a}^m |\alpha\rangle \langle\alpha| d^2\alpha \\
&= \int P(\alpha) \sum_{n,m} C_{n,m} \langle\alpha| (\hat{a}^\dagger)^n \hat{a}^m |\alpha\rangle d^2\alpha \\
&= \int P(\alpha) \sum_{n,m} C_{n,m} (\alpha^*)^n \alpha^m d^2\alpha \\
&= \int P(\alpha) G^{(N)}(\alpha,\alpha^*) d^2\alpha.
\end{aligned} \tag{4.32}$$

This is *the optical equivalence theorem*: the expectation value of a normally ordered operator is simply the P function weighted average of the function obtained from the operator by replacing $\hat{a} \to \alpha$ and $\hat{a}^\dagger \to \alpha^*$.
As mentioned above, the P-representation is not unique. Therefore, we can connect the

density operator $\hat{\rho}$ to another quasi-distribution function called *the Q, or Husimi function* [19]. It is possible to derive this new formulation representing a generic operator $\hat{B}$ in the "diagonal" coherent state form, sometimes referred to as the P-representation. For an operator $\hat{B}$, the P-representation is given by:

$$\hat{B} = \int B_p(\alpha, \alpha^*)|\alpha\rangle\langle\alpha| d^2\alpha. \tag{4.33}$$

The average of $\hat{B}$ is:

$$\langle\hat{B}\rangle = \mathrm{Tr}(\hat{B}\hat{\rho}) = \sum_n \langle n| \int B_p(\alpha, \alpha^*)|\alpha\rangle\langle\alpha|\hat{\rho}|n\rangle d^2\alpha, \tag{4.34}$$

which simplifies to:

$$\int B_p(\alpha, \alpha^*)\langle\alpha|\hat{\rho}|\alpha\rangle d^2\alpha. \tag{4.35}$$

Evidently, the expectation value of the density operator with respect to the coherent state also plays the role of a phase-space probability distribution. This is usually called *the Q, or Husimi function* [19]:

$$Q(\alpha) = \frac{\langle\alpha|\hat{\rho}|\alpha\rangle}{\pi}. \tag{4.36}$$

Unlike the P function, the Q function is positive for all quantum states and it is devoted to the evaluation of *antinormally ordered* functions. The last, and the most usefull quasi-distribution, is the so called *Wigner function*. It provides a way to describe the quantum state in terms of position and momentum, similar to classical phase-space distributions. For an arbitrary density operator $\hat{\rho}$, the Wigner function is defined as:

$$W(q,p) \equiv \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \left\langle q + \frac{x}{2}\right| \hat{\rho} \left| q - \frac{x}{2}\right\rangle e^{ipx/\hbar} dx, \tag{4.37}$$

where $|q \pm \frac{x}{2}\rangle$ are the position eigenstates. This expression represents how the quantum state is mapped into phase space in terms of the position q and momentum p.
For a *pure state*, where $\hat{\rho} = |\psi\rangle\langle\psi|$, the Wigner function can be written as:

$$W(q,p) \equiv \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \psi^*\left(q - \frac{x}{2}\right)\psi\left(q + \frac{x}{2}\right) e^{ipx/\hbar} dx, \tag{4.38}$$

where $\psi(q \pm \frac{x}{2})$ are the wavefunctions in position space. When the Wigner function is integrated over the momentum p, the result is the *probability density in position space*:

$$\int_{-\infty}^{\infty} W(q,p) dp = |\psi(q)|^2, \tag{4.39}$$

which corresponds to the usual probability density for finding the particle at position

q. Similarly, integrating over the position q gives the *probability density in momentum space*:

$$\int_{-\infty}^{\infty} W(q,p)dq = |\varphi(p)|^2, \tag{4.40}$$

where $\varphi(p)$ is the wavefunction in momentum space, related to the position wavefunction $\psi(q)$ through a Fourier transform. Although the Wigner function resembles a probability distribution, it is not a true probability function because it can take negative values for certain non-classical quantum states. However, it is extremely useful for calculating averages, provided that the operators involved are expressed in Weyl-ordered (symmetrically ordered) form in terms of the position and momentum operators $\hat{q}$ and $\hat{p}$. All these quasi-distributions can be grouped in a compact way by means the *s-parameterized function* of Cahill and Glauber [43]:

$$\chi(\xi,s) = \mathrm{Tr}\left[\hat{\rho}\exp\left(\xi\hat{a}^\dagger - \xi^*\hat{a} + s|\xi|^2/2\right)\right] = \mathrm{Tr}\left[\hat{\rho}\hat{D}_\xi\right] e^{s|\xi|^2/2} \tag{4.41}$$

such that $\chi(\xi,0) = \chi_W(\xi)$, $\chi(\xi,1) = \chi_P(\xi)$, $\chi(\xi,-1) = \chi_Q(\xi)$. The family of characteristic functions is related, via complex *Fourier transform*, to the quasi-probability distributions W_s, which provide another complete description of the quantum states:

$$W_s(\alpha) \equiv \frac{1}{\pi^2}\int \exp(\xi^*\alpha - \xi\alpha^*)\chi(\xi,s)\,d^2\xi. \tag{4.42}$$

Finally, we have the ingredients to generalize to an N-modes bosonic system, described by the Hamiltonian (2.55). As shown in [23], in this case the s-ordered characteristic functions:

$$\chi(\xi,s) = \mathrm{Tr}[\rho\hat{D}_\xi]e^{s\|\xi\|^2/2}, \tag{4.43}$$

with $\xi \in \mathbb{R}^{2N}$, and $\|\cdot\|$ denoting the Euclidean norm in $\mathbb{R}^{2N}$. The vector ξ belongs to the real $2N$-dimensional space $\Gamma = (\mathbb{R}^{2N}, \Omega)$, known as phase space, analogous to classical Hamiltonian dynamics. From the definition of characteristic functions, we observe that in the phase space picture, the tensor product structure is replaced by a direct sum structure, such that the N-mode phase space $\Gamma = \bigoplus_k \Gamma_k$, where $\Gamma_k = (\mathbb{R}^2, \omega)$ is the local phase space associated with mode k. At the end, the global quasi-distributions are given by:

$$W_s(\xi) = \frac{1}{\pi^2}\int_{\mathbb{R}^{2N}} \chi_s(\kappa)e^{i\kappa^T\Omega\xi}\,d^{2N}\kappa. \tag{4.44}$$

The quasi-probability distributions of integer orders W_{-1}, W_0, and W_1 are associated with the antinormally ordered, symmetrically ordered, and normally ordered expressions of operators, respectively. Specifically, if the operator $\hat{O}$ can be expressed as $\hat{O} = f(\hat{a}_k, \hat{a}_k^\dagger)$ for $k = 1,\ldots,N$, where f is a symmetrically ordered function of the field operators, then [43, 44]:

$$\mathrm{Tr}[\rho\hat{O}] = \int_{\mathbb{R}^{2N}} W_0(\kappa)\bar{f}(\kappa)\, d^{2N}\kappa, \tag{4.45}$$

where $\bar{f}(\kappa)$ is the Weyl transform of the operator f. The function $\overline{f}(\kappa)$ is defined as $f(\kappa_k + i\kappa_{k+1}, \kappa_k - i\kappa_{k+1})$, and f takes the same form as the operatorial function previously introduced. The same relationship holds between W_{-1} and the antinormally ordered expressions of the operators, and between W_1 and the normal ordering. We also recall that the normally ordered function of a given operator is provided by its Wigner representation. This entails the following equalities for the trace:

$$1 = \mathrm{Tr}(\rho) = \int_{\mathbb{R}^{2N}} W(\kappa)\, d^{2N}\kappa = \chi(0), \tag{4.46}$$

and for the purity μ [45] of a state ρ:

$$\mu = \mathrm{Tr}(\rho^2) = \int_{\mathbb{R}^{2N}} W^2(\kappa)\, d^{2N}\kappa = \int_{\mathbb{R}^{2N}} |\chi(\xi)|^2\, d^{2N}\xi. \tag{4.47}$$

These expressions will be useful in the following.

4.3 Mathematical description of Gaussian states

In the light of what has been seen in the previous section, a Gaussian state is defined when the characteristic function and the quasi-probability distributions are Gaussian functions. Thus, a Gaussian state in quantum mechanics is uniquely determined by its first and second statistical moments of the quadrature field operators [23]. These quadrature operators describe the continuous variables, such as position and momentum in quantum optics. Specifically, the first moments and the covariance matrix (CM) contain all the relevant information required to fully describe the state. The first moments are defined by the vector

$$\overline{R} = (\langle\hat{R}_1\rangle, \langle\hat{R}_2\rangle, \ldots, \langle\hat{R}_N\rangle), \tag{4.48}$$

where $\langle\hat{R}_i\rangle$ represents the expectation value of the i-th quadrature operator $\hat{R}_i$. The second moments are captured by the covariance matrix σ, which is defined as:

$$\sigma_{ij} = \frac{1}{2}\langle\hat{R}_i\hat{R}_j + \hat{R}_j\hat{R}_i\rangle - \langle\hat{R}_i\rangle\langle\hat{R}_j\rangle. \tag{4.49}$$

The covariance matrix contains essential information about the quantum state, such as correlations between the quadrature operators. Importantly, it determines quantities like entanglement, purity, and squeezing. While the first moments can provide information about the displacement of the quantum state in phase space, they can be arbitrarily adjusted by applying local unitary operations like displacement operators. Specifically, single-mode displacements, often implemented via the Weyl operator, shift the state in phase space without altering its fundamental properties, such as entropy or entanglement. As such, these first moments do not carry significant informational content,

and for most analytical purposes, they are often set to zero. This assumption greatly simplifies the analysis of Gaussian states.
When the first moments are set to zero, the Wigner function, which provides a quasi-probability distribution in phase space, takes a particularly simple form. For an N-mode Gaussian state, the Wigner function is given by:

$$W(R) = \frac{e^{-\frac{1}{2}R\sigma^{-1}R^T}}{\pi^N\sqrt{\det\sigma}}, \tag{4.50}$$

where R is the phase-space vector $(q_1, p_1, \ldots, q_N, p_N) \in \mathbb{R}^{2N}$, representing the real quadrature variables. The covariance matrix σ characterizes the shape and orientation of the Gaussian function in phase space. The Wigner function provides a complete phase-space representation of the Gaussian state, capturing its quantum properties, including possible non-classical behavior. Despite the infinite-dimensional nature of the Hilbert space for continuous variable systems, the full description of a Gaussian state (up to local unitary operations) is encoded in the $2N \times 2N$ covariance matrix σ. In practice, we often use σ interchangeably to refer to both the Gaussian state and the matrix of second moments. This matrix reflects correlations between the canonical continuous variables and is central to analyzing the state's quantum features. From the perspective of statistical mechanics, the elements of the covariance matrix σ correspond to two-point truncated correlation functions between the quadrature operators. Each entry can be thought of as encoding information about the correlations between specific quadrature pairs. These elements can also be related to energies by multiplying them by the mode's energy level spacing $\hbar\omega_k$, where ω_k is the frequency of the k-th mode. In this way, $\mathrm{Tr}\,\sigma$ gives a quantity proportional to the mean energy of the Gaussian state, corresponding to the expectation value of the non-interacting Hamiltonian. In conclusion, the covariance matrix contains all the information required to describe a system composed of N independent modes. For this description to be fully equivalent, it is necessary to include the Heisenberg uncertainty principle and a constraint that ensures the positive semi-definiteness of the density matrix ρ. Those two requirements impose a condition on σ known as the Robertson-Schrödinger uncertainty relation [46]. This condition ensures that the quantum uncertainty relations are respected and is mathematically expressed as:

$$\sigma + i\Omega \geq 0, \tag{4.51}$$

The inequality $\sigma + i\Omega \geq 0$ is both necessary and sufficient for σ to represent a valid physical state [47, 48]. It is a general condition applicable to both Gaussian and non-Gaussian states and guarantees that the quantum state satisfies the uncertainty principle. In this preliminary overview, an important example of a two-mode Gaussian state is mentioned: the two-mode squeezed state $|\psi_{\mathrm{sq}}\rangle_{i,j} = \hat{U}_{i,j}(r)(|0\rangle_i \otimes |0\rangle_j)$ with squeezing factor $r \in \mathbb{R}$. The (phase-free) two-mode squeezing operator is given by

$$\hat{U}_{i,j}(r) = \exp\left[-\frac{r}{2}(\hat{a}_i^\dagger\hat{a}_j^\dagger - \hat{a}_i\hat{a}_j)\right]. \tag{4.52}$$

The significance of a two-mode squeezed state lies in its nature as an entangled state for any non-zero value of the squeezing parameter ($r > 0$). This means that as the squeezing parameter r increases, the noise in one quadrature grows while it is simultaneously reduced in the complementary one. In the asymptotic regime, where $r \to \infty$, this state can simulate, with an arbitrarily high degree of accuracy, the ideal Einstein-Podolsky-Rosen (EPR) state, which is represented by a non-normalizable and non-physical wavefunction. Therefore, two-mode squeezed states are crucial as entangled resources for practical implementations of continuous-variable (CV) quantum information protocols [36]. They play a central role in the study of the entanglement properties of Gaussian states. Now we derive explicitly the covariance matrix for this relevant case. First of all, we need to define the action of the squeezing operator on the quadratures:

$$\begin{aligned}
\hat{U}_{i,j}^{\dagger}(r)\hat{q}_i\hat{U}_{i,j}(r) &= \hat{U}_{i,j}^{\dagger}(r)(\hat{a}_i + \hat{a}_i^{\dagger})\hat{U}_{i,j}(r) \\
&= \left((\hat{a}_i\cosh(r) + \hat{a}_j{}^{\dagger}\sinh(r)) + (\hat{a}_i{}^{\dagger}\cosh(r) + \hat{a}_j\sinh(r))\right) \\
&= \left(\hat{a}_i\cosh(r) + \hat{a}_i^{\dagger}\cosh(r) + \hat{a}_j^{\dagger}\sinh(r) + \hat{a}_j\sinh(r)\right) \\
&= \cosh(r)(\hat{a}_i + \hat{a}_i^{\dagger}) + \sinh(r)(\hat{a}_j^{\dagger} + \hat{a}_j) \\
&= \hat{q}_i\cosh(r) + \hat{q}_j\sinh(r).
\end{aligned} \tag{4.53}$$

and

$$\begin{aligned}
\hat{U}_{i,j}^{\dagger}(r)\hat{p}_i\hat{U}_{i,j}(r) &= \hat{U}_{i,j}^{\dagger}(r)\frac{1}{i}(\hat{a}_i - \hat{a}_i^{\dagger})\hat{U}_{i,j}(r) \\
&= \frac{1}{i}\left((\hat{a}_i\cosh(r) + \hat{a}_j{}^{\dagger}\sinh(r)) - (\hat{a}_i{}^{\dagger}\cosh(r) + \hat{a}_j\sinh(r))\right) \\
&= \frac{1}{i}\left(\hat{a}_i\cosh(r) - \hat{a}_i^{\dagger}\cosh(r) + \hat{a}_j^{\dagger}\sinh(r) - \hat{a}_j\sinh(r)\right) \\
&= \frac{i}{i}\cosh(r)(\hat{a}_i - \hat{a}_i^{\dagger}) - \frac{1}{i}\sinh(r)(\hat{a}_j - \hat{a}_j^{\dagger}) \\
&= \hat{p}_i\cosh(r) - \hat{p}_j\sinh(r).
\end{aligned} \tag{4.54}$$

From these results we can construct the covariance matrix, starting from its general definition:

$$\sigma_{sq}^{i,j}(r) = \begin{pmatrix} \langle\hat{q}_i^2\rangle & \langle\hat{p}_i\hat{q}_i\rangle & \langle\hat{q}_j\hat{q}_i\rangle & \langle\hat{p}_j\hat{q}_i\rangle \\ \langle\hat{q}_i\hat{p}_i\rangle & \langle\hat{p}_i^2\rangle & \langle\hat{q}_j\hat{p}_i\rangle & \langle\hat{p}_j\hat{p}_i\rangle \\ \langle\hat{q}_i\hat{q}_j\rangle & \langle\hat{p}_i\hat{q}_j\rangle & \langle\hat{q}_j^2\rangle & \langle\hat{p}_j\hat{q}_j\rangle \\ \langle\hat{q}_i\hat{p}_j\rangle & \langle\hat{p}_i\hat{p}_j\rangle & \langle\hat{q}_j\hat{p}_j\rangle & \langle\hat{p}_j^2\rangle \end{pmatrix} \tag{4.55}$$

Where the average is on the squeezed vacuum state:

$$
\begin{aligned}
\langle \hat{q}_i^2 \rangle_{sq} &= \langle 0,0| \hat{U}_{i,j}^{\dagger}(r) \hat{q}_i^2 \hat{U}_{i,j}(r) |0,0\rangle \\
&= \langle 0,0| \hat{U}_{i,j}^{\dagger}(r) \hat{q}_i \hat{U}_{i,j}(r) \hat{U}_{i,j}^{\dagger}(r) \hat{q}_i \hat{U}_{i,j}(r) |0,0\rangle \\
&= \langle 0,0| \left(\hat{U}_{i,j}^{\dagger}(r) \hat{q}_i \hat{U}_{i,j}(r) \right)^2 |0,0\rangle \\
&= \langle 0,0| \left(\hat{q}_i \cosh(r) + \hat{q}_j \sinh(r) \right)^2 |0,0\rangle \\
&= \cosh^2(r) \langle 0,0| \hat{q}_i^2 |0,0\rangle + \sinh^2(r) \langle 0,0| \hat{q}_j^2 |0,0\rangle \\
&= \cosh^2(r) \langle 0,0| (\hat{a}_i^2 + \hat{a}_i^{\dagger 2} + 2\hat{a}_i^{\dagger}\hat{a}_i + 1) |0,0\rangle + \\
&+ \sinh^2(r) \langle 0,0| (\hat{a}_j^2 + \hat{a}_j^{\dagger 2} + 2\hat{a}_j^{\dagger}\hat{a}_j + 1 |0,0\rangle \\
&= \cosh^2(r) + \sinh^2(r) = \cosh(2r)
\end{aligned}
\tag{4.56}
$$

In the second line we have employed the unitary property of the squeezing operator, i.e, $\hat{U}^{\dagger}\hat{U} = \mathbb{I}$. Following the same procedure we find all the entries of the matrix:

$$\langle \hat{q}_i^2 \rangle_{sq} = \cosh^2(r) \langle \hat{q}_i^2 \rangle_0 + \sinh^2(r) \langle \hat{q}_j^2 \rangle_0 = \cosh(2r), \tag{4.57}$$

$$\langle \hat{p}_i^2 \rangle_{sq} = \cosh^2(r) \langle \hat{p}_i^2 \rangle_0 + \sinh^2(r) \langle \hat{p}_j^2 \rangle_0 = \cosh(2r), \tag{4.58}$$

$$\langle \hat{q}_i \hat{q}_j \rangle_{sq} = 2\sinh(r)\cosh(r)(\langle \hat{q}_i^2 \rangle_0 + \langle \hat{q}_j^2 \rangle_0) = \sinh(2r), \tag{4.59}$$

$$\langle \hat{p}_i \hat{p}_j \rangle_{sq} = 2\sinh(r)\cosh(r)(\langle \hat{p}_i^2 \rangle_0 + \langle \hat{p}_j^2 \rangle_0) = -\sinh(2r). \tag{4.60}$$

The two-mode squeezed state covariance matrix is finally derived:

$$\sigma_{\mathrm{sq}}^{i,j}(r) = \begin{pmatrix} \cosh(2r) & 0 & \sinh(2r) & 0 \\ 0 & \cosh(2r) & 0 & -\sinh(2r) \\ \sinh(2r) & 0 & \cosh(2r) & 0 \\ 0 & -\sinh(2r) & 0 & \cosh(2r) \end{pmatrix} \quad \blacksquare. \tag{4.61}$$

4.3.1 Symplectic operations

When we describe the state of a system by its density matrix, we know that there are unitary operations, associated with at most quadratic Hamiltonian in the modes, that maintain unchanged the trace and positivity of the density operator, i.e. they leave the state unchanged. This type of transformation results, in the space of phases, in the existence of operations, so-called *symplectic operations*, i.e. linear transformation S which preserves the symplectic form Ω:

$$S^T \Omega S = \Omega. \tag{4.62}$$

Symplectic transformations on a $2N$-dimensional phase space form the (real) symplectic group $Sp(2N, \mathbb{R})$. Such transformations act linearly on first moments and by

congruence on covariance matrices, $\sigma \to S\sigma S^T$. Equation (4.62) implies $\mathrm{Det}(S) = 1$, $\forall S \in Sp_{(2N,\mathbb{R})}$. For instance, the two-mode squeezing operator equation corresponds to the symplectic transformation

$$S_{i,j}(r) = \begin{pmatrix} \cosh r & 0 & \sinh r & 0 \\ 0 & \cosh r & 0 & -\sinh r \\ \sinh r & 0 & \cosh r & 0 \\ 0 & -\sinh r & 0 & \cosh r \end{pmatrix}, \tag{4.63}$$

where the matrix is understood to act on the pair of modes i and j. In this way, the two-mode squeezed state, equation (4.61), can be obtained as

$$\sigma_{i,j}^{sq}(r) = S_{i,j}(r)\mathbb{I}_4 S_{i,j}^T(r), \tag{4.64}$$

exploiting the fact that the CM of the two-mode vacuum state is the 4×4 identity matrix. We note that local symplectic operations belong to the group $Sp(2,\mathbb{R})^{\oplus N}$. They correspond, at the Hilbert space level, to tensor products of unitary transformations, each acting on the state space of a single mode. It is useful to note that the determinants of each 2×2 submatrix of an N-mode CM, are all invariants under local symplectic operations $S \in Sp(2,\mathbb{R})^{\oplus N}$. This mathematical property reflects the physical requirement that marginal informational properties and correlations between individual subsystems cannot be altered by local operations alone.

4.3.2 Symplectic eigenvalues and invariants

If symplectic operations reflect the action of unitary transformations in phase space, it is natural to ask which quantities, derived from σ, remain invariant under the action of such operations. To define the first invariant, it is necessary to introduce the symplectic transformation that diagonalizes a Gaussian state in the basis of normal modes. Through this decomposition, thanks to Williamson's theorem [49], the CM of an N-mode Gaussian state can always be written in the so-called Williamson normal or diagonal form

$$\sigma = S^T \nu S, \tag{4.65}$$

where $S \in Sp(2N,\mathbb{R})$ and ν is the CM

$$\nu = \bigoplus_{k=1}^{N} \begin{pmatrix} \nu_k & 0 \\ 0 & \nu_k \end{pmatrix}. \tag{4.66}$$

The physical interpretation of these symplectic eigenvalues becomes evident when comparing the diagonalized Williamson form with the covariance matrix of a thermal state. We can focus our attention on a generic single-mode system k (in the following discussion, we will omit the subscript k, which will be implicitly understood), whose

covariance matrix is a 2×2 matrix and the first moment vector is a 2-dimensional vector:

$$\sigma = \begin{pmatrix} 2\langle\hat{q}^2\rangle_{\hat{\rho}} - 2\langle\hat{q}\rangle_{\hat{\rho}}^2 & \langle\hat{q}\hat{p}+\hat{p}\hat{q}\rangle_{\hat{\rho}} - 2\langle\hat{q}\rangle_{\hat{\rho}}\langle\hat{p}\rangle_{\hat{\rho}} \\ \langle\hat{p}\hat{q}+\hat{q}\hat{p}\rangle_{\hat{\rho}} - 2\langle\hat{p}\rangle_{\hat{\rho}}\langle\hat{q}\rangle_{\hat{\rho}} & 2\langle\hat{p}^2\rangle_{\hat{\rho}} - 2\langle\hat{p}\rangle_{\hat{\rho}}^2 \end{pmatrix}, \tag{4.67}$$

For a single harmonic oscillator, the thermal state covariance matrix is easy to compute. Let us consider the thermal state at temperature T:

$$\hat{\rho} = \frac{1}{Z}e^{-\beta\hat{H}} = \frac{1}{Z}e^{-\frac{\hbar\omega}{k_BT}\hat{a}^\dagger\hat{a}}, \tag{4.68}$$

where

$$Z = \mathrm{Tr}\left(e^{-\frac{\hbar\omega}{k_BT}\hat{a}^\dagger\hat{a}}\right) = \sum_{n=0}^{\infty}\langle n|e^{-\frac{\hbar\omega}{k_BT}\hat{a}^\dagger\hat{a}}|n\rangle = \sum_{n=0}^{\infty}e^{-\frac{\hbar\omega}{k_BT}n} = \frac{e^{\frac{\hbar\omega}{k_BT}}}{e^{\frac{\hbar\omega}{k_BT}}-1} = \frac{1}{1-e^{-\frac{\hbar\omega}{k_BT}}}. \tag{4.69}$$

The expectation values of $\hat{q}$ and $\hat{p}$ in a thermal state are zero, which can be quickly verified by considering:

$$\mathrm{Tr}\left[e^{-C\hat{a}^\dagger\hat{a}}(\hat{a}\pm\hat{a}^\dagger)\right] = \sum_n\langle n|e^{-C\hat{a}^\dagger\hat{a}}(\hat{a}\pm\hat{a}^\dagger)|n\rangle = \sum_n e^{-Cn}\langle n|(\hat{a}\pm\hat{a}^\dagger)|n\rangle = 0. \tag{4.70}$$

There are four covariances to compute. First, the diagonal ones:

$$\begin{aligned}
\langle\hat{q}^2\rangle_{\hat{\rho}} &= \frac{1}{Z}\mathrm{Tr}\left(e^{-\frac{\hbar\omega}{k_BT}\hat{a}^\dagger\hat{a}}(\hat{a}^\dagger+\hat{a})^2\right) \\
&= \frac{1}{Z}\sum_{n=0}^{\infty}\langle n|e^{-\frac{\hbar\omega}{k_BT}\hat{a}^\dagger\hat{a}}(\hat{a}^\dagger+\hat{a})^2|n\rangle \\
&= \frac{1}{Z}\sum_{n=0}^{\infty}e^{-\frac{\hbar\omega}{k_BT}n}\langle n|\left(\hat{a}^\dagger\hat{a}^\dagger+\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger\right)|n\rangle \\
&= \frac{1}{Z}\sum_{n=0}^{\infty}e^{-\frac{\hbar\omega}{k_BT}n}(2n+1) \\
&= \frac{1}{Z}\frac{e^{\frac{\hbar\omega}{k_BT}}(1+e^{\frac{\hbar\omega}{k_BT}})}{(e^{\frac{\hbar\omega}{k_BT}}-1)^2} \\
&= \coth\left(\frac{\hbar\omega}{2k_BT}\right),
\end{aligned} \tag{4.71}$$

and

$$
\begin{aligned}
\langle \hat{p}^2 \rangle_{\hat{\rho}} &= -\frac{1}{Z}\mathrm{Tr}\left(e^{-\frac{\hbar\omega}{k_B T}\hat{a}^\dagger \hat{a}} (\hat{a}^\dagger - \hat{a})^2 \right) \\
&= -\frac{1}{Z}\sum_{n=0}^{\infty} \langle n| e^{-\frac{\hbar\omega}{k_B T}\hat{a}^\dagger \hat{a}} (\hat{a}^\dagger - \hat{a})^2 |n\rangle \\
&= -\frac{1}{Z}\sum_{n=0}^{\infty} e^{-\frac{\hbar\omega}{k_B T} n} \langle n| \left(\hat{a}^\dagger \hat{a}^\dagger + \hat{a}\hat{a} - \hat{a}^\dagger \hat{a} - \hat{a}\hat{a}^\dagger \right) |n\rangle \\
&= \frac{1}{Z}\sum_{n=0}^{\infty} e^{-\frac{\hbar\omega}{k_B T} n} (2n+1) \\
&= \frac{1}{Z}\frac{e^{\frac{\hbar\omega}{k_B T}}(1+e^{\frac{\hbar\omega}{k_B T}})}{(e^{\frac{\hbar\omega}{k_B T}}-1)^2} \\
&= \coth\left(\frac{\hbar\omega}{2k_B T} \right).
\end{aligned}
\tag{4.72}
$$

The non-diagonal components can be easily proven to be zero since $(\hat{a}^\dagger \pm \hat{a})(\hat{a} \mp \hat{a}^\dagger) = \hat{a}\hat{a} + \hat{a}^\dagger \hat{a}^\dagger$, which have zero trace. Therefore, the covariance matrix for the thermal state is:

$$
\sigma = \begin{pmatrix} \nu & 0 \\ 0 & \nu \end{pmatrix}, \quad \nu = \coth\left(\frac{\hbar\omega}{2k_B T} \right) > 1, \tag{4.73}
$$

where $\nu \to 1$ in the limit $T \to 0$ ■.
The conclusion is that the Williamson form of the covariance matrix corresponds to a tensor product state with a diagonal density matrix $\rho^{\otimes}$ equal to:

$$
\rho^{\otimes} = \bigotimes_k \frac{2}{\nu_k + 1} \sum_{n=0}^{\infty} \left(\frac{\nu_k - 1}{\nu_k + 1} \right)^n |n\rangle_{kk}\langle n|, \tag{4.74}
$$

where $|n\rangle_k$ denotes the number state of order n in the Fock space $\mathscr{H}_k$. In the Williamson form, each mode with frequency ω_k is a Gaussian state in thermal equilibrium at a temperature T_k, characterized by a Bose–Einstein statistical distribution of the thermal photons n_k, with average

$$
\bar{n}_k = \frac{\nu_k - 1}{2} = \frac{1}{\exp\left(\frac{\hbar\omega_k}{k_B T_k} \right) - 1}. \tag{4.75}
$$

The N quantities ν_k form the symplectic spectrum of the CM σ and are invariant under the action of global symplectic transformations on the matrix σ. The symplectic eigenvalues can be computed as the orthogonal eigenvalues of the matrix $|i\Omega\sigma|$ [50] and are thus determined by N invariants of the characteristic polynomial of such a matrix [51]. Since these symplectic operations are the counterpart in phase-space of unitary transformations in the Hilbert space, we expect them to maintain the global properties of the system, encoded in CM matrix, unchanged. One of this global symplectic invariant

is simply the determinant of CM, that follows from the condition $Det(S) = 1$. in fact we have that:

$$Det\,\sigma = Det\left(S^T \nu S\right) = Det\,\nu = \prod_{k=1}^{N} \nu_k^2. \tag{4.76}$$

The second global invariant is called *seralian* [52] $\Delta(\sigma)$, that is defined as:

$$\Delta(\sigma) = \sum_k \nu_k^2. \tag{4.77}$$

The proof of this result is derived in Serafini et all paper [52], in which the two-mode Gaussian state is treated, but a further genralization was given by [53]. This specific case is useful to underling, in the next chapter, some features of entanglement as a consequence of marginal and global purity. We can now proceed to the analysis by introducing the block form of the covariance matrix:

$$\sigma = \begin{pmatrix} \alpha & \gamma \\ \gamma^T & \beta \end{pmatrix} \tag{4.78}$$

In which each block is a $2x2$ matrix. This CM matrix can be always recast into another, different from the fully diagonal one (4.65), named *standard form* [54] by means of a proper symplectic transformation. In two-mode gaussian state it's easy to decompose these operations in terms of local transformations and local one [51]:

$$S \in Sp_{(4,\mathbb{R})} = S_{loc}(r_1, r_2) R(\xi) S_{tm}(r) R(\eta) S_l. \tag{4.79}$$

Transformation S is made up by a local operation S_l, two rotations $R(\phi)$, with

$$R(\phi) = \begin{pmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & \cos\phi & 0 & -\sin\phi \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & \sin\phi & 0 & \cos\phi \end{pmatrix}, \tag{4.80}$$

a global squeezing $S_{tm}(r) = \mathrm{diag}(e^r, e^{-r}, e^{-r}, e^r)$ and a local squeezing $S_{loc}(r_1, r_2) = S_{sm}(r_1, 0) \oplus S_{sm}(0, r_2)$, resulting from the direct product of two single-mode squeezing operators with null phase. It is very important noticing that even if $S_{tm}(r)$ acts on both modes at the same time it does not introduce mixing of them, in other words it does not introduce non-local effect. The consequence of this is that $S_{tm}(r) = S_{loc}(r, -r)$, so that the only global (non-local) operations in the decomposition of Eq. (4.79) are the two rotations. We note that an equivalent decomposition has been recently demonstrated for generic multimode pure Gaussian states [55]. For any covariance matrix σ there exists a local canonical operation $S_{sf} = S_1 \oplus S_2$ which brings σ to the "standard form" σ_{sf} [54]:

$$S_{sf}^T \sigma S_{sf} = \sigma_{sf} \equiv \begin{pmatrix} a & 0 & c_+ & 0 \\ 0 & a & 0 & c_- \\ c_+ & 0 & b & 0 \\ 0 & c_- & 0 & b \end{pmatrix}, \tag{4.81}$$

where a, b, c_+, c_- are determined by the four local symplectic invariants:

$$\det\sigma = (ab - c_+^2)(ab - c_-^2), \quad \det\alpha = a^2, \quad \det\beta = b^2, \quad \text{and} \quad \det\gamma = c_+ c_-. \tag{4.82}$$

while the *seralian* is equal to:

$$\Delta(\sigma) = det\,\alpha + det\,\beta + 2det\,\gamma. \tag{4.83}$$

The invariance of $\Delta(\sigma)$ is easy to prove. it is clearly invariant under local transformation, i.e, $S_{loc}(r_1, r_2), S_{tm}(r)$ and S_l. As for the non-local rotations which enter in the definition of S, let us notice that they act on covariance matrices of the following form:

$$\tilde{\sigma} = \begin{pmatrix} u & 0 & j & 0 \\ 0 & v & 0 & k \\ j & 0 & w & 0 \\ 0 & k & 0 & z \end{pmatrix}, \tag{4.84}$$

and if we define two others submatrices of σ:

$$\delta = \begin{pmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \sigma_{22} & \sigma_{24} \\ \sigma_{42} & \sigma_{44} \end{pmatrix}. \tag{4.85}$$

for which one has $\Delta(\tilde{\sigma}) = \mathrm{Tr}(\delta\varepsilon)$. Such an expression is manifestly invariant under the action of identical rotations $R(\phi)$ on the submatrices δ and γ, proving the statement. We can now use these invariant forms to derive the eigenvalues in the case of *two-mode Gaussian states*, since we know:

$$\begin{aligned} Det\,\sigma &= \nu_-^2 \nu_+^2, \\ \Delta(\sigma) &= \nu_-^2 + \nu_+^2 \end{aligned} \tag{4.86}$$

and inverting the relations we end up with the definition of eigenvalues in term of the invariants:

$$\nu_\pm^2 = \frac{\Delta(\sigma) \pm \sqrt{\Delta(\sigma)^2 - 4Det(\sigma)}}{2}. \tag{4.87}$$

Now is evident how the uncertainty principle (4.51) can be recast, in the two-mode Gaussian states, in terms of its invariant. For the standard form of the covariance matrix (4.81) the condition (4.51) implies that the determinant of the entire matrix is greater or

equal to zero. If we impose this requirement we discover a key relation between $\Delta(\sigma)$ and $Det(\sigma)$:

$$\Delta(\sigma) \leq 1 + Det(\sigma). \tag{4.88}$$

In the Chapter 6 will be clear the physical meaning of these two quantities.

4.3.3 Symplectic representation of uncertainty principle

We have seen that is always possible to bring a generic covariance matrix to its Williamson form, i.e, a fully diagonal matrix. In this picture, the uncertainty principle (4.51) can be recast in a simplified form. If we apply the symplectic operation to (4.51):

$$S(\sigma + i\Omega)S^T \geq 0 \rightarrow S\sigma S^T + i\Omega \geq 0. \tag{4.89}$$

The first term of the inequality is the Williamson form of the matrix σ. The statement means that the sum of the two matrices has to be a semi positive definite matrix, i.e, its eigenvalues must be greater or equal than zero. Since we are not interested in the actual values it is sufficient, to satisfy the uncertainty principle, that the determinant must be greater or equal than zero. This condition impose a lower bound on the eigenvalues ν_k:

$$\nu_k \geq 1. \tag{4.90}$$

Without loss of generality, one can rearrange the modes of an N-mode state such that the corresponding symplectic eigenvalues are sorted in ascending order:

$$\nu_- \equiv \nu_1 \leq \nu_2 \leq \cdots \leq \nu_{N-1} \leq \nu_N \equiv \nu_+. \tag{4.91}$$

With this notation, the uncertainty relation reduces to

$$\nu_1 \geq 1. \tag{4.92}$$

We remark that the full saturation of the uncertainty principle can only be achieved by pure N-mode Gaussian states, for which $\nu_i = 1\ \forall i = 1,\ldots,N$, meaning that the Williamson normal form of any pure Gaussian state is the vacuum $|0\rangle$ of the N-mode Hilbert space $\mathscr{H}$.

4.3.4 Degree of information in Gaussian states

When talking about the degree of information contained in a system it is natural to refer to the results obtained within the information theory from Shannon, in a classical framework. He wondered whether it was possible to define a measure for quantifying the degree of information of a discrete source, or equivalently the degree of ignorance of the system itself, understood as the capability to predict the output of a test prior the

measurement. A quantitative measure for the average amount of information that we expect to gain in this kind of test can be defined as follows. Let $p_1, ..., p_N$ be the known probabilities of the various outcomes of the test that we intend to perform. Namely, if we imagine the same test applied to n identically prepared systems, and if n is a large number, we expect about $n_j = np_j$ outcomes of type j. In a test of a quantum system it is only possible to give a prediction of its possible results, based on $\{p_1, ...p_N\}$, therefore we cannot say with certainty in which order the results will be found. The number of different possibilities to arrange these n outcomes is equal to $n!/n_1!...n_N!$ and if $n \to \infty$ we can express this ratio by means the Stirling's approximation:

$$log\left(\frac{n}{n_1!...n_N!}\right) \approx nlog(n) - n - \sum_j n_j log(n_j) - n_j = -n\sum_j p_j log(p_j). \tag{4.93}$$

The expression

$$S := -\sum_j p_j log(p_j) \tag{4.94}$$

is called *Shannon entropy*. It is a measure of the disorder in the system and the predictability of its results. To fully understand the meaning of this measure, we can imagine to deal with a quantum test, knowing the probabilities of each possible outcome. It is convenient to discuss two two extreme cases; that are the pure state and the completely mixed state. In the first case, the ideal one, the system will be described by a density matrix given by:

$$\hat{\rho} = |\psi\rangle \langle\psi| . \tag{4.95}$$

Therefore, if you perform the measurement associated with the eigenstate that describes the system, you get a distribution that focuses on a single value, there will be a coefficient $p_k = 1$ and all others equal to zero. The Shannon entropy in this case will be zero, which indicates that we will have full predictability of results before making the measurement itself. However, in an experimental setup it is not realistically possible to construct a completely pure state, since there will always be some degree of correlation between the system and the environment (measuring instruments etc.), therefore when making the partial trace, by removing the degrees of freedom from the environment, we obtain a density matrix that describes a mixed state, then a diagonal matrix. If for every test we have maximal ignorance, namely $p_k = 1/N$, the preparation is called a random mixture, and it is represented by the density matrix of the following form:

$$\hat{\rho} = \mathbb{I}/N. \tag{4.96}$$

The coefficients that define the mixture vary between 0 and 1. The case of maximum disorder is such that each state is equally likely and consequently all coefficients shall

be uniformly distributed. This is the case when the Shannon entropy is at its maximum, so we have complete uncertainty about the predictability of the results before making the measurement. We have seen how the Shannon entropy applies to classical variables with known probability distributions. Assuming we have a quantum system, identified by its density matrix, we are interested in quantifying the degree of information contained in it. To do this we can introduce the quantum equivalent of Shannon's entropy, so-called *Von Neumann's entropy*:

$$S(\rho_A) = -\operatorname{Tr}(\rho_A \ln \rho_A) = -\operatorname{Tr}(\rho_B \ln \rho_B). \tag{4.97}$$

When we diagonalize ρ, its eigenvalues represent the probabilities $\{p_i\}$ associated with the pure states in the mixed-state decomposition. The von Neumann entropy then reduces to the Shannon entropy calculated over the eigenvalues:

$$S(\rho) = -\sum_i p_i \log p_i. \tag{4.98}$$

In the first part we have seen how important it is to know the *purity* of the system under consideration since this coincides with the degree of information that the state carries with it [56]. For a quantum state represented by the density matrix ρ, the purity $\mu(\rho)$ is given by:

$$\mu(\rho) = \operatorname{Tr}(\rho^2). \tag{4.99}$$

This measure of purity, a key concept in quantum information, indicates the extent to which a state is mixed or pure. For states within a Hilbert space $\mathscr{H}$ of dimension N, purity ranges from $\frac{1}{N} \leq \mu \leq 1$, where the lower bound corresponds to a completely mixed state, or random ensemble, and the upper bound is reached only by pure states. In continuous-variable (CV) systems where $N \to \infty$, the minimum purity approaches zero asymptotically, reflecting a state of maximal mixedness or randomness. The complement of purity, often referred to as *impurity* or the degree of mixedness, represents the extent of our uncertainty regarding the quantum state prior to measurement. Impurity can be effectively captured by the *linear entropy* $S_L(\rho)$, which is defined by:

$$S_L(\rho) = \frac{N}{N-1}(1-\mu) = \frac{N}{N-1}\left(1 - \operatorname{Tr}(\rho^2)\right), \tag{4.100}$$

where S_L ranges between 0 for pure states and 1 for maximally mixed states. This metric is widely used in quantum information theory for its direct relationship with purity and computational ease. Additionally, the linear entropy serves as a straightforward measure of a state's mixedness, offering insights into the "noise" or uncertainty inherent in the quantum state. More generally, the mixedness of a quantum state can be fully described by evaluating its *Schatten p-norms* [57]:

$$\|\rho\|_p = \left(\operatorname{Tr}|\rho|^p\right)^{1/p} = \left(\operatorname{Tr}(\rho^p)\right)^{1/p}, \quad p \geq 1. \tag{4.101}$$

These norms offer a family of metrics that measure various aspects of a state's mixedness. For $p = 2$, the Schatten norm is directly tied to purity μ, connecting it to the linear entropy. The p-norms are particularly useful as they are multiplicative under tensor products, making them essential in constructing *generalized entropies* [58, 59]. The *generalized entropy* $S_p(\rho)$ for a given p is defined as:

$$S_p(\rho) = \frac{1 - \mathrm{Tr}(\rho^p)}{p-1}, \quad p > 1. \tag{4.102}$$

This measure scales from 0, representing pure states, up to $\frac{1}{p-1}$ for fully mixed states with completely degenerate spectra. Generalized entropies provide an adaptable framework for studying mixedness across different quantum systems, as they highlight how much a state deviates from purity and help characterize the information-theoretic properties of quantum states with complex eigenstructures. This generalized entropy has a very powerful definition in terms of symplectic eigenvalues as it has been proved in [60]. We have already seen how each covariance matrix can be related to a diagonal form (4.65) by means symplectic operations, corresponding to a tensor product of thermal states is given by (4.74). where ν_k are the symplectic eigenvalues, and $|n\rangle_k$ is the occupation number states of the k-th mode. The single mode density matrix is given by:

$$\rho^{\otimes}_{\tilde{\nu}_k} = \frac{2}{\nu_k + 1} \sum_{n=0}^{\infty} \left(\frac{\nu_k - 1}{\nu_k + 1} \right)^n |n\rangle\langle n|, \tag{4.103}$$

and consequently ρ^p will be equal to:

$$(\rho^{\otimes}_{\tilde{\nu}_k})^p = \left(\frac{2}{\nu_k + 1} \right)^p \sum_{n=0}^{\infty} \left(\frac{\nu_k - 1}{\nu_k + 1} \right)^{np} |n\rangle\langle n|. \tag{4.104}$$

The eingenvalues ν_k are greater or equal than one, as imposed by uncertainty condition, so we are dealing with a geometric sum. Accordingly to this observation it is now easy to derive the trace of single mode density matrix:

$$\lambda_k^p = \left(\frac{2}{\nu_k + 1} \right)^p \frac{1}{1 - \left(\frac{\nu_k - 1}{\nu_k + 1} \right)^p} = \frac{2^p}{(\nu_k + 1)^p - (\nu_k - 1)^p}. \tag{4.105}$$

Generalizing to the N-mode systems we get:

$$\mathrm{Tr}(\rho^p) = \prod_{k=1}^{N} g_p(\nu_k), \quad g_p(x) = \frac{2^p}{(x+1)^p - (x-1)^p} \quad \blacksquare. \tag{4.106}$$

From this definition it is possible to derive the purity in terms of simplectic eigenvalues, since for $p = 2$:

$$Tr(\rho^2) = \mu = \frac{1}{\prod_{i=1}^{N} \nu_i} = \frac{1}{\sqrt{Det(\sigma)}}. \tag{4.107}$$

We have now all the ingredients for evaluating the extreme values of the *generalized entropy* at fixed purity. Unlike the single mode case, in which all the entropies S_p's (and Von Newmann entropy as well) are just increasing functions of $Det(\sigma)$, the multi-mode states show a very different behavior. In particular we can distinguish two opposite trend depending on which kind of exponent we are considering. Indeed, without going deeply into the mathematics, it is worth mentioning which are the extreme values of the entropy function when changing p. In particular when $p < 2$ the generalized entropy is a concave function for each s_i:

$$(p-1)S_p = 1 - \left(\prod_{i=1}^{n-1} g_p(s_i)\right) g_p\left(\frac{1}{\mu \prod_{i=1}^{n-1} s_i}\right), \quad 1 \leq s_i \leq \frac{1}{\mu \prod_{i\neq j} s_j}, \tag{4.108}$$

and it is minimum when, at given purity, the symplectic spectrum is partially degenerate, with $\nu_1 = \cdots = \nu_{n-1} = 1, \nu_n = \frac{1}{\mu}$. For $p < 2$, the mixedness of the states with minimal generalized entropies at given purity is therefore concentrated in one quadrature:

$$S_p^{\min}(\mu) = \frac{1 - g_p\left(\frac{1}{\mu}\right)}{p-1}. \tag{4.109}$$

The maximum value $S_p^{max}(\mu)$ is achieved by states with a completely degenerate symplectic spectrum: $\nu_1 = \cdots = \nu_n = \mu^{-1/n}$, yielding:

$$S_p^{\max}(\mu) = \frac{1 - g_p\left(\mu^{-1/n}\right)^n}{p-1}. \tag{4.110}$$

The instance $p >$ can be treated in the same way, with the major difference that the function Sp of Eq.(4.108) is convex with respect to any si for any value of the s_i's. This leads to a complete reversal of the results that were previously found; For $p > 2$, the states with minimal $S_p^{\min}(\mu)$ at given purity μ are those with a fully distributed symplectic spectrum, with:

$$S_p^{\min}(\mu) = \frac{1 - g_p\left(\mu^{-1/n}\right)^n}{p-1}. \tag{4.111}$$

On the other hand, the states with maximal $S_p^{\max}$ at given purity μ are those with a spectrum of the kind $\nu_1 = \cdots = \nu_{n-1} = 1, \nu_n = \frac{1}{\mu}$. Therefore:

$$S_p^{\max}(\mu) = \frac{1 - g_p\left(\frac{1}{\mu}\right)}{p-1}. \tag{4.112}$$

As final remark we notice that the distance $|S_p^{\max} - S_p^{\min}|$ decreases with increasing p. This is due to the fact that the quantity S_p carries less information with increasing p, and the knowledge of μ provides a more precise bound on the value of S_p.

4.3.5 Pure states: phase-space Schmidt decomposition

We address the decomposition of a multimode pure Gaussian state with respect to a bipartite division of the modes, following the derivation of A. Botero and B.Reznik [55], who proved that for any such division the state can always be expressed as a product state involving entangled two-mode squeezed states and single-mode local states at each side. The character of entanglement of the state can therefore be understood modewise; that is, a given mode on one side is entangled with only one corresponding mode of the other, and therefore the total bipartite entanglement is the sum of the modewise entanglement. To begin with, suppose a collection of N canonical systems or "modes" is partitioned into two sets, i.e., Alice's $A = \{A_1, \ldots, A_m\}$ and Bob's $B = \{B_1, \ldots, B_n\}$, of sizes m and n, respectively. If the quantum state of the modes is a pure Gaussian state $|\psi\rangle_{AB}$, the following theorem characterizes the entanglement between Alice and Bob.

Theorem 1. *A Gaussian pure state $|\psi\rangle_{AB}$ for $m+n$ modes A and B may always be written as*

$$|\psi\rangle_{AB} = |\tilde{\psi}_1\rangle_{\tilde{A}_1\tilde{B}_1} \otimes |\tilde{\psi}_2\rangle_{\tilde{A}_2\tilde{B}_2} \otimes \cdots \otimes |\tilde{\psi}_s\rangle_{\tilde{A}_s\tilde{B}_s} \otimes |0\rangle_{A_F} \otimes |0\rangle_{B_F} \tag{4.113}$$

for some $s \leq \min(m,n)$, where $\tilde{A} = \{\tilde{A}_1, \ldots, \tilde{A}_m\}$ and $\tilde{B} = \{\tilde{B}_1, \ldots, \tilde{B}_n\}$ are new sets of modes obtained from A and B, respectively, through local linear canonical transformations. The states $|\tilde{\psi}_k\rangle$ are two-mode squeezed states [61] of the form

$$|\tilde{\psi}_k\rangle_{\tilde{A}_k\tilde{B}_k} = \frac{1}{\sqrt{Z_k}} \sum_n e^{-\beta_k n/2} |n\rangle_{\tilde{A}_k} |n\rangle_{\tilde{B}_k}, \tag{4.114}$$

entangling the modes $\tilde{A}_k$ and $\tilde{B}_k$ for $k \leq s$, and $|0\rangle_{A_F}$ and $|0\rangle_{B_F}$ are products of oscillator ground states for the remaining modes in $\tilde{A}$ and $\tilde{B}$, respectively. Each pairwise entangled state of the form (4.114) is expressed in terms of an appropriate Fock state basis for the mode pair subspace and yields a two-mode density matrix:

$$\hat{\rho}_{\tilde{A}_k\tilde{B}_k} = \frac{1}{Z_k} \sum_m \sum_n e^{-\frac{\beta_k}{2}(n-m)} |n\rangle_{\tilde{A}_k} |n\rangle_{\tilde{B}_k} \langle m|_{\tilde{B}_k} \langle m|_{\tilde{A}_k}. \tag{4.115}$$

From this we can derive the reduced density matrix:

$$\hat{\rho}_{\tilde{A}_k} = Tr_{\tilde{B}_k}(\hat{\rho}_{\tilde{A}_k\tilde{B}_k}) = \frac{e^{-\beta_k \tilde{n}_k}}{\mathrm{Tr}(e^{-\beta_k \tilde{n}_k})} = \frac{1}{Z_k} \sum_n e^{-\beta_k n} |n\rangle_{kk}\langle n|, \tag{4.116}$$

where $\rho_{\tilde{A}_k}$ is an oscillator thermal state for the k-th mode. We notice that this formulation is completely equivalent to (4.74), by assuming that:

$$\beta_k = \ln\left(\frac{\nu_k + 1}{\nu_k - 1}\right), \quad \langle \tilde{n_k} \rangle = \frac{\nu_k - 1}{2}. \tag{4.117}$$

Note that as a consequence of the uncertainty principle, admissible Gaussian states

satisfy the condition $\forall k, \nu_k \geq 1$, with pure Gaussian states when $\forall k, \nu_k = 1$. For $\nu_k = 1$, $\rho_k = |0\rangle_k\langle 0|$ is obtained as the limit of Eq. (4.116) as $b_i \to \infty$. Finally we have that reduced density matrix for either set takes the form of tensor product:

$$\hat{\rho}_A = \bigotimes_{k=1}^{s} \hat{\rho}_{\tilde{A}_k}. \tag{4.118}$$

We now proceed with the proof of *Theorem 1*:

Proof: When we perform the Schmidt decomposition (3.13) automatically we get the diagonal form of the partial density matrices for A and B:

$$\rho_A = \sum_a \lambda_a |\phi_a\rangle\langle\phi_a|, \quad \rho_B = \sum_a \lambda_a |\chi_a\rangle\langle\chi_a|, \tag{4.119}$$

which are, by construction, of equal rank and spectrum, thus showing that the λ_a's are unique. The basis states $|\phi_a\rangle_A$ and $|\chi_a\rangle_B$ are also unique (up to phase factors) for nondegenerate λ_a, and otherwise may be chosen to be elements of any orthonormal basis spanning the degenerate subspace. Now, if $|\psi\rangle_{AB}$ is Gaussian, then the reduced density matrices, ρ_A and ρ_B, are also Gaussian. We have seen that a Gaussian state is fully characterized by its own covariance matrix. This matrix has an equivalent completely diagonal representation, known as the Williamson form (4.65), which is obtained through the action of appropriate symplectic operations. The density matrix associated with this diagonal form is expressed as the tensor product of single-mode density matrices (4.74). Thus, ρ_A and ρ_B can be recast into the form (4.74). Suppose that there are s modes in A and t modes in B with symplectic eigenvalue $\nu = 1$. Since the remaining modes factor out from the respective density matrices as projection operators onto their ground states, we may factor $|\psi\rangle_{AB}$ as

$$|\psi\rangle_{AB} = |\tilde{\psi}\rangle_{AB} \otimes |0\rangle_{\tilde{A}_F} \otimes |0\rangle_{\tilde{B}_F}, \tag{4.120}$$

where $|0\rangle_{\tilde{A}_F}$ and $|0\rangle_{\tilde{B}_F}$ are collective ground states of the modes with $\nu = 1$ and $|\tilde{\psi}\rangle_{AB}$ is the generally entangled state for the remaining modes $\tilde{A}_1, \ldots, \tilde{A}_s$ and $\tilde{B}_1, \ldots, \tilde{B}_t$. Concentrate then on $|\tilde{\psi}\rangle_{AB}$, the partial density matrices of which may be written as

$$\tilde{\rho}_A = \sum_{\vec{n}_A} \frac{e^{-\vec{\beta}_A \cdot \vec{n}_A}}{Z^{(A)}} |\vec{n}_A\rangle\langle\vec{n}_A|, \quad \tilde{\rho}_B = \sum_{\vec{n}_B} \frac{e^{-\vec{\beta}_B \cdot \vec{n}_B}}{Z^{(B)}} |\vec{n}_B\rangle\langle\vec{n}_B|, \tag{4.121}$$

where $\vec{n}_A = \{n_{A_1}, \ldots, n_{A_s}\}^T$ and $\vec{n}_B = \{n_{B_1}, \ldots, n_{B_t}\}^T$ are s- and t-dimensional vectors representing occupation number distributions on each side and $\vec{\beta}_A = \{\beta_{\tilde{A}_1}, \ldots, \beta_{\tilde{A}_s}\}^T$ and $\vec{\beta}_B = \{\beta_{\tilde{B}_1}, \ldots, \beta_{\tilde{B}_t}\}^T$ represent the distributions of thermal parameters on each side. Now, from our previous discussion, both density matrices have the same rank and the same eigenvalues. This means that there must exist a one-to-one pairing between the occupation number distributions $\vec{n}_A$ and $\vec{n}_B$, such that

$$\vec{\beta}_A \cdot \vec{n}_A = \vec{\beta}_B \cdot \vec{n}_B \tag{4.122}$$

We now observe that the pairing $\vec{n}_A \Leftrightarrow \vec{n}_B$ is a *homogeneous linear map*, since $\vec{n}_A = 0$ and $\vec{n}_B = 0$ are paired (all β's $\neq 0$) and $(\vec{n}_A + \vec{n}'_A, \vec{n}_B + \vec{n}'_B)$ satisfies Eq. (4.122) if $(\vec{n}_A, \vec{n}_B)$ and $(\vec{n}'_A, \vec{n}'_B)$ satisfy Eq. (4.122). However, if a linear map is one-to-one then the domain and range have the same dimensions. Thus we see that $s = t$; in other words, the number of modes in A and B with symplectic eigenvalues different from $1/2$ is the same. Now, label the modes on each side in ascending order of β, so that $0 < \beta_{\tilde{A}_1} \leq \beta_{\tilde{A}_2} \leq \cdots \leq \beta_{\tilde{A}_s}$ and $0 < \beta_{\tilde{B}_1} \leq \beta_{\tilde{B}_2} \leq \cdots \leq \beta_{\tilde{B}_s}$. Consider first the case $\vec{n}_A = \{1, 0, \ldots, 0\}^T$, yielding the smallest nonzero value of $\vec{\beta}_A \cdot \vec{n}_A$. By construction, this distribution must be paired with the smallest nonzero value of $\vec{\beta}_B \cdot \vec{n}_B$, which is (or can be taken to be in the case of degenerate $\beta_{\tilde{B}_1}$) $\vec{n}_B = \{1, 0, \ldots, 0\}^T$. We thus find that Eq. (4.122) has a solution provided that $\beta_{\tilde{A}_1} = \beta_{\tilde{B}_1}$ (hence $v_{A_1} = v_{B_1}$), and by the linearity property we find for any $\vec{n}_A$ the map $n_{\tilde{A}_1} \to n_{\tilde{B}_1} = n_{\tilde{A}_1}$. At this point, we can repeat the procedure but applied to the subspace of the remaining modes, in other words, solve for a map between $\vec{n}'_A = \{0, n_{A_2}, \ldots, n_{A_s}\}$ and $\vec{n}'_B = \{0, n_{B_2}, \ldots, n_{B_s}\}$ such that $\vec{\beta}_A \cdot \vec{n}'_A = \vec{\beta}_B \cdot \vec{n}'_B$. By a similar argument, we find that $\beta_{\tilde{A}_2} = \beta_{\tilde{B}_2}$ and $n_{\tilde{A}_2} = n_{\tilde{B}_2}$. Iterating the procedure until all the components are exhausted, we find that the admissible solutions to Eq. (4.122) are $\vec{n}_A = \vec{n}_B$ (with a freedom of reordering the labels of degenerate modes), provided that $\vec{\beta}_A = \vec{\beta}_B$.
Reconstructing the Schmidt decomposition of $|\tilde{\psi}\rangle_{AB}$ from ρ_A and ρ_B, we see that

$$|\tilde{\psi}\rangle_{AB} = \frac{1}{\sqrt{Z}} \sum_{\vec{n}} e^{-\vec{\beta}\cdot\vec{n}/2} |\vec{n}\rangle |\vec{n}\rangle = \bigotimes_{i=1}^{s} \left(\sum_n \frac{e^{-\beta_i n/2}}{\sqrt{Z_i}} |n\rangle_{\tilde{A}_i} |n\rangle_{\tilde{B}_i} \right). \quad \blacksquare \tag{4.123}$$

Thus, $|\psi\rangle_{AB} = |\tilde{\psi}\rangle_{AB} \otimes |0\rangle_{A_F} \otimes |0\rangle_{B_F}$ is of the form (4.113). As can be expected, the most complete understanding of the behaviour of Gaussian states under Gaussian operations has been reached for pure states. Indeed, necessary and sufficient conditions for the possible state transformations under local Gaussian operations (GLOCC) can be given [37]. Local means here that the applied Gaussian quantum operations are applied locally, accompanied by classical communication. This result crucially depends on a normal form that can be obtained for any bipartite system of continuous variables [62, 63, 55]. For any pure Gaussian state with covariance matrix σ of an $n \times n$-mode system, there exist local symplectic transformations S_A, S_B such that

$$(S_A \oplus S_B)\sigma^{sq}(S_A \oplus S_B)^T = \bigoplus_{k=1}^{s} \begin{bmatrix} \cosh(2r_k) & 0 & \sinh(2r_k) & 0 \\ 0 & \cosh(2r_k) & 0 & -\sinh(2r_k) \\ \sinh(2r_k) & 0 & \cosh(2r_k) & 0 \\ 0 & -\sinh(2r_k) & 0 & \cosh(2r_k) \end{bmatrix} \tag{4.124}$$

with $r_k \in [0, \infty)$. In other words, by means of local unitary Gaussian operations the pure

Gaussian state described by σ^{sq} can be transformed into a tensor product of two-mode squeezed states, characterized by squeezing parameter r_k (see Figure 2).

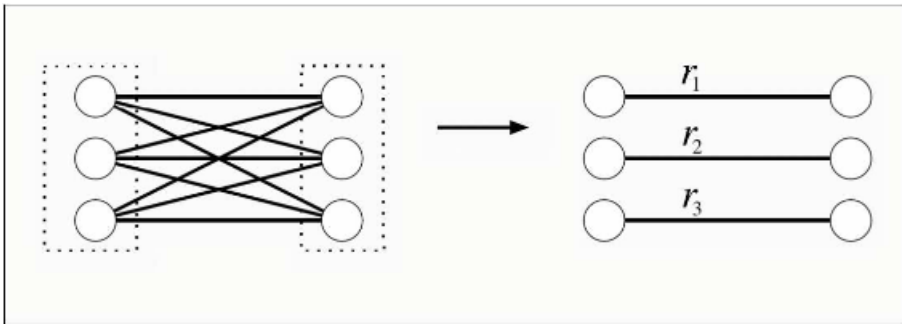

Figure 2: Schmidt decomposition of pure bi-partite Gaussian states.

We will now show that for (generally mixed) Gaussian states with some local symmetry constraints, a similar phase-space reduction is available, such that multimode properties (like entanglement) can be unitarily reduced to two-mode ones.

4.3.6 Symmetric and bisymmetric states

As mentioned in the previous section, we will analyze a class of extremely interesting states, namely symmetric and bisymmetric states, which have unique characteristics in the study of entanglement. In fact, following the work of Adesso, Serafini and Illuminati [64], we discover that the entanglement between the m-mode and the n-mode blocks can then be completely concentrated on a single pair of modes by means of local unitary operations alone, emphasizing the importance of studying entanglement between two modes. To identify these operations we will start from the description of the two separate sets, identified by the corresponding covariance matrices, thus finding their eigenvalues. This first step is essential because the joint system has the same spectrum of the two covariance matrices. Identifying the eigenvalues is a necessary step, because through the associated eigenvectors we can obtain the $\boldsymbol{T}$ matrix that diagonalizes the covariance matrix. Finally we will find what is the relation between $\boldsymbol{T}$ and the symplectic operation that brings the covariance matrix in Williamson form [49]. We shall say that a multimode Gaussian state is fully symmetric if it is invariant under the exchange of any two modes. In the following, we will consider the fully symmetric m-mode and n-mode Gaussian states ρ_{α^m} and ρ_{β^n}, with CMs σ_{α^m} and σ_{β^n}. Due to symmetry, we have that

$$\boldsymbol{\sigma}_{\alpha^m} = \begin{pmatrix} \alpha & \varepsilon & \cdots & \varepsilon \\ \varepsilon & \alpha & \cdots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \cdots & \alpha \end{pmatrix}, \quad \boldsymbol{\sigma}_{\beta^n} = \begin{pmatrix} \beta & \zeta & \cdots & \zeta \\ \zeta & \beta & \cdots & \zeta \\ \vdots & \vdots & \ddots & \vdots \\ \zeta & \zeta & \cdots & \beta \end{pmatrix}, \tag{4.125}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\beta}$, and $\boldsymbol{\zeta}$ are 2×2 real symmetric submatrices (the symmetry of $\boldsymbol{\varepsilon}$ and

$\boldsymbol{\zeta}$ stems again from the symmetry under the exchange of any two modes). All the properties related to correlations and entropic measures of multimode Gaussian states are invariant under local, single-mode symplectic operations. A first preliminary fact, analogous to the standard form reduction of two-mode states, is that is always possible, by means single-mode symplectic operations, to bring the submatrices into a diagonal form $\beta = diag(b,b)$ and $\zeta = diag(z_1,z_2)$ as is proven in [64]. The same results hold for α and ε. A second aspect that will be useful later on is related to the spectrum of the two covariance matrices, which is represented by two eigenvalues ν_{β}^{-} and $\nu_{\beta^n}^{+}$:

$$\nu_{\beta}^{-} = \sqrt{(b-z_1)(b-z_2)}, \quad \nu_{\beta^n}^{+} = \sqrt{[b+(n-1)z_1][b+(n-1)z_2]}, \tag{4.126}$$

where ν_{β}^{-} is the $(n-1)$-times degenerate eigenvalue and they are related to $(n-1)$ eigenvetors constructed in the following way:

$$\{\mathbf{v}_i\}, \quad \text{for } i = 1,\ldots,n-1, \tag{4.127}$$

such that

$$\mathbf{v}_i = (0,\cdots,0,\underbrace{\mathbf{v}^T}_{\text{mode i}},\underbrace{-\mathbf{v}^T}_{\text{mode i+1}},0,\cdots,0) \tag{4.128}$$

where, for convenience, we have introduced the two-dimensional vector

$$\mathbf{v} = \left(i\frac{b-z_2}{\nu_{\beta}^{-}},1\right)^T. \tag{4.129}$$

The $\mathbf{v}_i$ are $(n-1)$ linearly independent vectors that satisfy the relation:

$$i\boldsymbol{\Omega}\boldsymbol{\sigma}_{\beta^n}\mathbf{v}_i = \nu_{\beta}^{-}\mathbf{v}_i, \tag{4.130}$$

thus proving that the symplectic eigenvalue ν_{β}^{-} of $\boldsymbol{\sigma}_{\beta^n}$ is $(n-1)$-times degenerate ■. The entire derivation is left to [64]. The remaining linear independent eigenvector of $i\boldsymbol{\Omega}\boldsymbol{\sigma}_{\beta^n}$ is the vector.

$$\mathbf{w}^T = (w^T,\cdots,w^T)^T \tag{4.131}$$

with

$$w^T = (\sqrt{i(b+(n-1)z_1)},\sqrt{i(b+(n-1)z_2)})^T. \tag{4.132}$$

It is immediate to verify that such a vector is associated with the eigenvalue $\nu_{\beta^n}^{+}$. The analysis on the individual symmetric state contains relevant information related to the

joint system which results from a correlated combination of the fully symmetric blocks $\boldsymbol{\sigma}_{\alpha^m}$ and $\boldsymbol{\sigma}_{\beta^n}$,

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\sigma}_{\alpha^m} & \boldsymbol{\Gamma} \\ \boldsymbol{\Gamma}^T & \boldsymbol{\sigma}_{\beta^n} \end{pmatrix}, \tag{4.133}$$

where $\boldsymbol{\Gamma}$ is a $2m \times 2n$ real matrix formed by identical 2×2 blocks $\boldsymbol{\gamma}$. Clearly, $\boldsymbol{\Gamma}$ is responsible for the correlations existing between the m-mode and the n-mode parties. Once again, the identity of the submatrices $\boldsymbol{\gamma}$ is a consequence of the local invariance under mode exchange, internal to the m-mode and n-mode parties. States of the form of Eq. (4.133) will be henceforth referred to as *bisymmetric*. We recall that our goal is to derive the symplectic operation that focus all the correlation on a single pair of modes. To do that we have to go trough the analysis of the spectrum of $\boldsymbol{\sigma}$. The main result, proven by A. Serafini, G. Adesso, and F. Illuminati [64], is that the spectrum of the bysimmetric matrix includes two degenerate eigenvalues. The first one is ν_α^- of the reduced matrix $\boldsymbol{\sigma}_{\alpha^m}$,with multiplicity $(m-1)$, and the second one is ν_β^- of the reduced matrix $\boldsymbol{\sigma}_{\beta^n}$, with multiplicity $(n-1)$.

Proof: Let us consider the standard forms of the blocks $\boldsymbol{\sigma}_{\alpha^m}$ and $\boldsymbol{\sigma}_{\beta^n}$, while keeping the 2×2 submatrices $\boldsymbol{\gamma}$ in arbitrary, generally nonsymmetric, form. Let us next focus on the block $\boldsymbol{\sigma}_{\beta^n}$ and define the vectors $\overline{\mathbf{v}}_i$ by

$$\overline{\mathbf{v}}_i = (0, \ldots, 0, \mathbf{v}_i^T)^T. \tag{4.134}$$

They are the vectors obtained from the vectors $\mathbf{v}_i$'s of Eq. (4.128) by appending to them $2m$ null entries on the left. Because of the identity of the blocks $\boldsymbol{\gamma}$, their contributions to the secular equation cancel out and it is straightforward to verify that the vectors $\overline{\mathbf{v}}_i$'s are $n-1$ eigenvectors of $i\boldsymbol{\Omega}\boldsymbol{\sigma}$ with eigenvalue $\mathbf{v}_\beta^-$. The same argument holds considering the submatrix $\boldsymbol{\sigma}_{\alpha^m}$, thus completing the proof ■.

With these results, it is now possible to assess the bipartite entanglement of bisymmetric multimode Gaussian states and demonstrate that it can consistently be localized or concentrated through unitary operations.

Proof: The vectors $\overline{\mathbf{v}}_i$ of Eq. (4.134), with the first $2m$ entries equal to 0, are, by construction, simultaneous eigenvectors of $i\boldsymbol{\Omega}\boldsymbol{\sigma}_{\beta^n}$ and $i\boldsymbol{\Omega}\boldsymbol{\sigma}$, with the same (degenerated) eigenvalue. This fact suggests that the phase-space modes corresponding to such eigenvectors are the same for $\boldsymbol{\sigma}$ and for $\boldsymbol{\sigma}_{\beta^n}$. Then, bringing by means of a local symplectic operation the CM $\boldsymbol{\sigma}_{\beta^n}$ in Williamson form, any $(2n-2) \times (2n-2)$ submatrix of $\boldsymbol{\sigma}$ will be diagonalized because the normal modes are common to the global and local CMs. In other words, no correlations between the m-mode party with reduced CM $\boldsymbol{\sigma}_{\alpha^m}$ and such modes will be left: all the correlations between the m-mode and n-mode parties will be concentrated in the two conjugate quadratures of a single mode of the n-mode block. Going through the same argument for the m-mode block with CM $\boldsymbol{\sigma}_{\alpha^m}$ would prove the proposition and show that the whole entanglement between the two

multimode blocks can always be concentrated in only two modes, one for each of the two multimode parties ■.

The last part of the statement consist in finding a suitable unitary operation. We need to find a link between the symplectic operation that bring $\boldsymbol{\sigma}$ in Williamson form $\boldsymbol{\nu}$ and the trasformation that diagonalize $i\boldsymbol{\Omega}\boldsymbol{\sigma}$. This connection is not obvious since the normal form associated with $\boldsymbol{\sigma}$ is invariant under local rotations (this local freedom is always present in the selection of normal modes) and, due to degeneracy, also under global symplectic rotations of the modes associated with the degenerate eigenvalue ν_β^-. Thus there is an ambiguity in selecting the eigenvectors of $i\boldsymbol{\Omega}\boldsymbol{\sigma}$ and therefore in determining the transformation that diagonalizes it. Before proceeding with the mathematical details it is necessary to compare the spectrum of these two matrices. Regarding $i\boldsymbol{\Omega}\boldsymbol{\sigma}$ we have that the spectrum is given by the set $\{\pm\nu_k\}$ where $|\nu_k|$ are the symplectic eigenvalues of $\boldsymbol{\sigma}$ and at the same time of $\boldsymbol{\nu}$. We can now assume the complete knowledge of the $2(m+n)$ column-vectors normalized eigenvectors of $i\boldsymbol{\Omega}\boldsymbol{\sigma}$, given by $\{\mathbf{w}_i\}$, and it os possible to construct with them the matrix $\boldsymbol{T}$ that diagonalize it:

$$\mathbf{T} = (\xi_1\mathbf{w}_1, \ldots, \xi_k\mathbf{w}_k). \rightarrow \mathbf{T}^{-1}(i\boldsymbol{\Omega}\boldsymbol{\sigma})\mathbf{T} = \mathbf{D}, \tag{4.135}$$

where $\boldsymbol{D}$ is a diagonal matrix of the form:

$$\boldsymbol{D} = diag(\nu_\alpha^+, -\nu_\alpha^+, \nu_\alpha^-, -\nu_\alpha^- \ldots, \nu_\beta^+, -\nu_\beta^+, \nu_\beta^-, -\nu_\beta^- \ldots). \tag{4.136}$$

At this point it is sufficient to find a unitary operation that brings $i\boldsymbol{\Omega}\boldsymbol{\nu}$ into $\boldsymbol{D}$ and compare the two results. We can proceed by observing that the 2×2 matrix $i\boldsymbol{\omega}$ is diagonalized by the unitary transformation $\overline{\mathbf{U}}$, with

$$\overline{\mathbf{U}} = \frac{1}{\sqrt{2}}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}, \tag{4.137}$$

so that $\overline{\mathbf{U}}^\dagger ia\boldsymbol{\omega}\overline{\mathbf{U}} = \mathrm{diag}(a,-a)$ (where a is any complex number). We can then define the matrix $\mathbf{U} = \overline{\mathbf{U}}^{\oplus(m+n)}$, which is local in the sense that it is block diagonal and acts on each mode separately, such that for any normal form $\boldsymbol{\nu}$,

$$\mathbf{U}^{-1}i\boldsymbol{\Omega}\boldsymbol{\nu}\mathbf{U} = \mathbf{D}. \tag{4.138}$$

Let us next denote by $\mathbf{S}$ one of the symplectic transformations that bring $\boldsymbol{\sigma}$ in normal form: $\mathbf{S}^{\mathrm{T}}\boldsymbol{\sigma}\mathbf{S} = \boldsymbol{\nu}$. It is then easy to see that

$$\mathbf{D} = \mathbf{T}^{-1}(i\boldsymbol{\Omega}\boldsymbol{\sigma})\mathbf{T} = \mathbf{U}^{-1}(i\boldsymbol{\Omega}\boldsymbol{\nu})\mathbf{U} = \mathbf{U}^{-1}(i\boldsymbol{\Omega}\mathbf{S}^{\mathrm{T}}\boldsymbol{\sigma}\mathbf{S})\mathbf{U} = \mathbf{U}^{-1}\mathbf{S}^{-1}(i\boldsymbol{\Omega}\boldsymbol{\sigma})\mathbf{S}\mathbf{U}, \tag{4.139}$$

and therefore

$$\mathbf{S} = \mathbf{T}\mathbf{U}^{-1} = \mathbf{T}\mathbf{U}^{\dagger}, \tag{4.140}$$

where in the last equation (4.139) we have exploited the fundamental property of symplectic transformations: $\mathbf{S}^{-1T}\boldsymbol{\Omega}\mathbf{S}^{-1} = \boldsymbol{\Omega}$. Equation (4.140) shows that there must exist some symplectic transformation that diagonalizes $i\boldsymbol{\Omega}\boldsymbol{\sigma}$ and satisfies the further condition given by Eq. (4.140). In fact, it is obvious that not every $\mathbf{T}$ diagonalizing $i\boldsymbol{\Omega}\boldsymbol{\sigma}$ is a symplectic transformation when multiplied on the right by $\mathbf{U}^{\dagger}$. Vice versa, if this last condition holds, the symplectic operation that brings $\boldsymbol{\sigma}$ in normal form is given by Eq. (4.140). The modes that diagonalize the quadratic form $\boldsymbol{\sigma}$ in phase space can be reconstructed in terms of $\mathbf{S}$: since they are linear combinations of the original modes and $\mathbf{S}^{\mathrm{T}}\boldsymbol{\sigma}\mathbf{S}$ is diagonal, they can be expressed by real column vectors identified by the columns of $\mathbf{S}$.
We can now go back to our original problem: leaving aside the involved task of exactly determining which choice of the eigenvectors of $i\boldsymbol{\Omega}\boldsymbol{\sigma}$ leads to a symplectic transformation of the form Eq. (4.140), we are anyway assured that in the subspace associated with the eigenvalues $\pm\nu_{\beta}^{-}$, such eigenvectors must be linear combinations of the $\bar{\mathbf{v}}_i$'s defined in Eq. (4.134) (with their first $2m$ entries, related to the m-mode party, set equal to 0). Therefore, the transformation $\mathbf{T}$ reads, in general,

$$\mathbf{T} = \begin{pmatrix} \mathbf{T}_{1,1} & \cdots & \mathbf{T}_{1,m} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{T}_{m,1} & \cdots & \mathbf{T}_{m,m} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{T}_{m+1,1} & \cdots & \mathbf{T}_{m+1,m} & \mathbf{T}_{m+1,m+1} & \cdots & \mathbf{T}_{m+1,m+n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{T}_{m+n,1} & \cdots & \mathbf{T}_{m+n,m} & \mathbf{T}_{m+n,m+1} & \cdots & \mathbf{T}_{m+n,m+n} \end{pmatrix}, \tag{4.141}$$

where $\mathbf{0}$ stands for 2×2 null matrices and $\mathbf{T}_{i,j}$ are 2×2 blocks, whose exact form is unessential to our aims. Exploiting Eq. (4.140), for the last $2(n-1)$ columns of $\mathbf{S}$ we obtain, in terms of 2×2 matrices,

$$(\underbrace{\mathbf{0},\cdots,\mathbf{0}}_{\text{first } m \text{ modes}},\mathbf{U}^{*}\boldsymbol{T}_{2,i}^{T},\cdots,\mathbf{U}^{*}\boldsymbol{T}_{n,i}^{T})^{T} \tag{4.142}$$

Due to the presence of the first m null entries, the $n-1$ modes determined by Eq. (4.142) are normal modes of both the global CM $\boldsymbol{\sigma}$ and the local CM $\boldsymbol{\sigma}_{\beta^n}$. An analogous proof, going along the same lines of reasoning, holds for the reduced CM $\boldsymbol{\sigma}_{\alpha^m}$: it can be reduced to a local normal form that shares $m-1$ normal modes with the global CM $\boldsymbol{\sigma}$. *These results imply that the form in which all the correlations between the two parties are shared only by a single mode of the n-mode party and by a single mode of the m-mode party can be obtained by means of local symplectic (unitary) operations, namely by the symplectic operations bringing the block $\boldsymbol{\sigma}_{\beta^n}$ and the block $\boldsymbol{\sigma}_{\alpha^m}$ in Williamson form* [49].

4.3.7 Separability for Gaussian states

In this section, we aim to define the concept of *separability* in the context of continuous variables, following the approach outlined by Simon [65], which generalizes the Peres-Horodecki criterion presented earlier (3.19). Simon observed that, in the continuous-variable case, the partial transpose operation can be interpreted geometrically as a *mirror reflection in phase space*. A key insight of his work is the realization that this geometric interpretation extends naturally to the Wigner phase space, offering a visually intuitive framework. In this context, separability imposes a stricter constraint on the second moments (uncertainties) than the traditional uncertainty principle; even commuting variables are subject to this additional condition. This restriction is central to proving that the Peres-Horodecki criterion serves as a necessary and sufficient condition for separability in all bipartite Gaussian states. To illustrate this, we consider a bipartite system comprising two modes, one associated with Alice and the other with Bob. To analyze the partial transpose operation in the Wigner representation, it is helpful to arrange the phase-space variables and Hermitian canonical operators into four-dimensional column vectors:

$$\xi = (q_1, p_1, q_2, p_2)^T \quad \hat{\xi} = (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2)^T \tag{4.143}$$

It follows from this definition that the partial transpose operation on the bipartite density operator transcribes faithfully into the following transformation on the Wigner distribution, since the relations between the two (4.37):

$$\mathrm{PT} : W(q_1, p_1, q_2, p_2) \to W(q_1, p_1, q_2, -p_2). \tag{4.144}$$

This corresponds to a mirror reflection or "local time reversal" which inverts only the p_2 coordinate:

$$\mathrm{PT} : \xi \to \Lambda\xi, \quad \Lambda = \mathrm{diag}(1, 1, 1, -1). \tag{4.145}$$

The Peres-Horodecki separability criterion reads as follows: *if $\hat{\rho}$ is separable, then its Wigner distribution necessarily transforms into another Wigner distribution under the phase space mirror reflection* Λ. The distribution $W(\Lambda\xi)$, like $W(\xi)$, should possess the "Wigner quality" for any separable bipartite state. Roughly speaking, local time reversal, defined by Λ as above, is a symmetry in the subspace of separable states. The Peres-Horodecki criterion has important implications for the uncertainties or second moments. We recall the compact statement for the *uncertainty principle*:

$$\boldsymbol{\sigma} + i\Omega \geq 0 \tag{4.146}$$

The uncertainty principle, as given in equation (4.146), is a direct consequence of the commutation relation (4.22) and the non-negativity of $\hat{\rho}$. In order to manipulate (4.146) it is convenient to write it in equivalent way, introducing an operator $\hat{Q} = \hat{h}\hat{h}^\dagger$, with

$\hat{h} = c_1\hat{\xi}_1 + c_2\hat{\xi}_2 + c_3\hat{\xi}_3 + c_4\hat{\xi}_4 = \mathbf{c}^T\hat{\xi}$. The uncertainty principle is equivalent to the condition $\langle\hat{Q}\rangle = \mathrm{tr}(\hat{Q}\hat{\rho}) \geq 0$.

Proof: Starting from the definition of the operator $\hat{Q}$ we notice that its average value is given by:

$$\langle\hat{Q}\rangle = \langle\hat{h}\hat{h}^\dagger\rangle_\rho = \langle(\mathbf{c}^T\hat{\xi})(\mathbf{c}^T\hat{\xi})^\dagger\rangle_\rho = \mathbf{c}^T\langle\hat{\xi}\hat{\xi}^T\rangle_\rho\mathbf{c}^* = \mathbf{c}^\dagger\langle\hat{\xi}\hat{\xi}^T\rangle_\rho\mathbf{c}, \tag{4.147}$$

where we have used the property $\langle\hat{Q}\rangle = \langle\hat{Q}\rangle^*$. We can now exploit the connection with the covariance matrix whose entries are defined in the following way:

$$\sigma_{ij} = \frac{1}{2}\langle\{\xi_i\xi_j\}\rangle_\rho - \langle\xi_i\rangle\langle\xi_j\rangle \rightarrow \langle\xi_i\xi_j\rangle = \sigma_{ij} + \langle\xi_i\rangle\langle\xi_j\rangle. \tag{4.148}$$

Arbitrarily imposing the first moment equal to zero we have that $\langle\hat{Q}\rangle = \mathbf{c}^\dagger\boldsymbol{\sigma}\mathbf{c} \geq 0$, since $\boldsymbol{\sigma}$ is semi-positive definite, ending the proof ■.

Viewed somewhat differently, it is equivalent to the statement that for every pair of real four-vectors $\mathbf{d}$ and $\mathbf{d}'$, the Hermitian operators $\hat{X}(\mathbf{d}) = \mathbf{d}^T\hat{\boldsymbol{\xi}} = d_1\hat{q}_1 + d_2\hat{p}_1 + d_3\hat{q}_2 + d_4\hat{p}_2$ and $\hat{X}(\mathbf{d}') = \mathbf{d}'^T\hat{\boldsymbol{\xi}} = d'_1\hat{q}_1 + d'_2\hat{p}_1 + d'_3\hat{q}_2 + d'_4\hat{p}_2$ obey

$$\langle(\Delta\hat{X}(\mathbf{d}))^2\rangle + \langle(\Delta\hat{X}(\mathbf{d}'))^2\rangle \geq |\mathbf{d}'^T\Omega\mathbf{d}| = \left|(d_1d'_2 - d_2d'_1 + d_3d'_4 - d_4d'_3)\right| \tag{4.149}$$

where $\Omega = \begin{pmatrix}\boldsymbol{\omega} & 0\\ 0 & \boldsymbol{\omega}\end{pmatrix}$ and $\boldsymbol{\omega} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$.

Proof. Starting from the *uncertainty principle* we remember that, for two non-commuting Hermitian operators, it holds:

$$[\hat{A},\hat{B}] = i\hat{G} \rightarrow (\Delta A)^2(\Delta B)^2 \geq \frac{|\langle G\rangle|^2}{4}. \tag{4.150}$$

In our case

$$[\hat{X}(\mathbf{d}),\hat{X}(\mathbf{d}')] = i(d_1d'_2 - d_2d'_1 + d_3d'_4 - d_4d'_3) \tag{4.151}$$

Now we can derive the (4.149) from:

$$\begin{aligned}\langle\left(\Delta\hat{X}(\mathbf{d})-\Delta\hat{X}(\mathbf{d}')\right)^2\rangle &= \langle(\Delta\hat{X}(\mathbf{d}))^2+(\Delta\hat{X}(\mathbf{d}'))^2-2\Delta\hat{X}(\mathbf{d})\Delta\hat{X}(\mathbf{d}')\rangle \geq \\ &\geq \langle(\Delta\hat{X}(\mathbf{d}))^2+(\Delta\hat{X}(\mathbf{d}'))^2 \\ &\quad -\left|(d_1d_2'-d_2d_1'+d_3d_4'-d_4d_3')\right|\rangle \geq 0 \\ \text{so} \\ &\langle(\Delta\hat{X}(\mathbf{d}))^2\rangle+\langle(\Delta\hat{X}(\mathbf{d}))^2\rangle \geq \\ &\geq \left|(d_1d_2'-d_2d_1'+d_3d_4'-d_4d_3')\right| \quad \blacksquare.\end{aligned} \tag{4.152}$$

Under the Peres-Horodecki partial transpose, the Wigner distribution undergoes mirror reflection, and it follows from equation (4.149) that the variances change to $\tilde{\boldsymbol{\sigma}}=\Lambda\boldsymbol{\sigma}\Lambda$. Since $W(\Lambda\xi)$ must be a Wigner distribution if the state under consideration is separable, we have

$$\tilde{\boldsymbol{\sigma}}+i\Omega\geq 0 \tag{4.153}$$

as a necessary condition for separability. We may write it also in the equivalent form:

$$\boldsymbol{\sigma}+i\tilde{\Omega}\geq 0,\quad \tilde{\Omega}=\Lambda\Omega\Lambda=\begin{pmatrix}\boldsymbol{\omega} & 0\\ 0 & -\boldsymbol{\omega}\end{pmatrix}, \tag{4.154}$$

so that the separability of $\hat{\rho}$ implies an additional restriction that has the same form as equation (4.149), with $\mathbf{d}'^T\boldsymbol{\Omega}\mathbf{d}$ on the right-hand side replaced by $\mathbf{d}'^T\tilde{\Omega}\mathbf{d}$. Combined with equation (4.149), this restriction reads:

$$\langle\Delta\hat{X}(\mathbf{d})^2\rangle+\langle\Delta\hat{X}(\mathbf{d}')^2\rangle\geq|d_1d_2'-d_2d_1'|+|d_3d_4'-d_4d_3'|, \tag{4.155}$$

for all $\mathbf{d},\mathbf{d}'$.

Proof. We begin from the *uncertainty principle* (4.149) and the partial reflection form of it (4.154)

$$\begin{aligned}\langle(\Delta\hat{X}(\mathbf{d}))^2\rangle+\langle(\Delta\hat{X}(\mathbf{d}'))^2\rangle &\geq |\mathbf{d}'^T\Omega\mathbf{d}|=A \\ \langle(\Delta\hat{X}(\mathbf{d}))^2\rangle+\langle(\Delta\hat{X}(\mathbf{d}'))^2\rangle &\geq |\mathbf{d}'^T\tilde{\Omega}\mathbf{d}|=B.\end{aligned} \tag{4.156}$$

Where $A=|C+D|$ and $B=|C-D|$, with $D=d_1d_2'-d_2d_1'$ and $C=d_3d_4'-d_4d_3'$. It follows that

$$\begin{aligned}\langle(\Delta\hat{X}(\mathbf{d}))^2\rangle+\langle(\Delta\hat{X}(\mathbf{d}'))^2\rangle &\geq \max(A,B)= \\ &= \max(|C+D|,|C-D|)=|C|+|D| \quad \blacksquare.\end{aligned} \tag{4.157}$$

This restriction, to be obeyed by all separable states, is generically stronger than the

usual uncertainty principle (4.149). For instance, let $\hat{X}(\mathbf{d})$ commute with $\hat{X}(\mathbf{d}')$; i.e., let $\mathbf{d}'^T\,\Omega\mathbf{d} = 0$. If the state is separable, then $\hat{X}(\mathbf{d})$ and $\hat{X}(\mathbf{d}')$ cannot both have arbitrarily small uncertainties unless $\mathbf{d}'^T\,\tilde{\Omega}\mathbf{d} = 0$ as well, i.e., unless $d_1 d_2' - d_2 d_1' = 0$ and $d_3 d_4' - d_4 d_3' = 0$. As an example, $\hat{X} = \hat{x}_1 + \hat{p}_1 + \hat{x}_2 + \hat{p}_2$ and $\hat{Y} = \hat{x}_1 - \hat{p}_1 - \hat{x}_2 + \hat{p}_2$ commute, but the sum of their uncertainties in any separable state is ≥ 4. The Peres-Horodecki condition (4.155) can be simplified. Real linear canonical transformations of a two-mode system constitute the ten-parameter real symplectic group $\mathrm{Sp}(4,\mathbb{R})$. For every real 4×4 matrix $S \in \mathrm{Sp}(4,\mathbb{R})$, the irreducible canonical Hermitian operators $\hat{\boldsymbol{\xi}}$ transform among themselves, leaving the fundamental commutation relation (4.146) invariant:

$$S \in \mathrm{Sp}(4,\mathbb{R}): \quad S\boldsymbol{\Omega}S^T = \boldsymbol{\Omega}, \quad \hat{\boldsymbol{\xi}} \to \hat{\boldsymbol{\xi}}' = S\hat{\boldsymbol{\xi}}, \quad [\hat{\xi}_a', \hat{\xi}_b'] = i\Omega_{ab}. \tag{4.158}$$

The symplectic group acts unitarily and irreducibly on the two-mode Hilbert space. Let $U(S)$ represent the (infinite-dimensional) unitary operator corresponding to $S \in \mathrm{Sp}(4,\mathbb{R})$. It transforms the bipartite state vector $|\psi\rangle$ to $|\psi'\rangle = U(S)|\psi\rangle$, and hence the density operator $\hat{\rho}$ to $\hat{\rho}' = U(S)\hat{\rho}U(S)^\dagger$. This transformation takes a strikingly simple form in the Wigner description, and this is one reason for the effectiveness of the Wigner picture in handling canonical transformations:

$$S: \hat{\rho} \to U(S)\hat{\rho}U(S)^\dagger \quad \Leftrightarrow \quad W(\boldsymbol{\xi}) \to W(S^{-1}\boldsymbol{\xi}) \tag{4.159}$$

The bipartite Wigner distribution simply transforms as a scalar field under $\mathrm{Sp}(4,\mathbb{R})$. The variance matrix transforms in the following manner:

$$S \in \mathrm{Sp}(4,\mathbb{R}): \quad \boldsymbol{\sigma} \to \boldsymbol{\sigma}' = S\boldsymbol{\sigma}S^T. \tag{4.160}$$

The uncertainty relation (4.146) has an $\mathrm{Sp}(4,\mathbb{R})$-invariant form (recall $S\Omega S^T = \Omega$). But separable states have to respect not just (4.146), but also the restriction (4.154), and this requirement is preserved only under the six-parameter $\mathrm{Sp}(2,\mathbb{R}) \times \mathrm{Sp}(2,\mathbb{R})$ subgroup of $\mathrm{Sp}(4,\mathbb{R})$ corresponding to independent local linear canonical transformations on the subsystems of Alice and Bob:

$$S_{\mathrm{local}} \in \mathrm{Sp}(2,\mathbb{R}) \times \mathrm{Sp}(2,\mathbb{R}): \quad S_{\mathrm{local}} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \quad S_1\boldsymbol{\omega}S_1^T = \boldsymbol{\omega}, \quad S_2\boldsymbol{\omega}S_2^T = \boldsymbol{\omega}. \tag{4.161}$$

It is desirable to cast the Peres-Horodecki condition (4.155) in an $\mathrm{Sp}(2,\mathbb{R}) \times \mathrm{Sp}(2,\mathbb{R})$-invariant form. To this end, let us write the variance matrix $\boldsymbol{\sigma}$ in the block form:

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^T & \boldsymbol{\beta} \end{pmatrix}. \tag{4.162}$$

The physical condition (4.146) implies $\boldsymbol{\alpha} \geq \mathbb{I}$ and $\boldsymbol{\beta} \geq \mathbb{I}$. As can be seen from (4.160), the local group changes the blocks of $\boldsymbol{\sigma}$ in the following manner:

$$\boldsymbol{\alpha} \to S_1 \boldsymbol{\alpha} S_1^T, \quad \boldsymbol{\beta} \to S_2 \boldsymbol{\beta} S_2^T, \quad \boldsymbol{\gamma} \to S_1 \boldsymbol{\gamma} S_2^T. \tag{4.163}$$

Thus, the $\mathrm{Sp}(2,\mathbb{R}) \times \mathrm{Sp}(2,\mathbb{R})$ invariants associated with $\boldsymbol{\sigma}$ are:

$$I_1 = \det \boldsymbol{\alpha}, \quad I_2 = \det \boldsymbol{\beta}, \quad I_3 = \det \boldsymbol{\gamma}, \quad I_4 = \mathrm{tr}(\boldsymbol{\alpha\omega\gamma\omega\beta\omega\gamma}^T \boldsymbol{\omega}), \tag{4.164}$$

and the determinant of $\boldsymbol{\sigma}$ is an obvious invariant, but it is a function of the I_k's, namely,

$$\det \boldsymbol{\sigma} = I_1 I_2 + I_3^2 - I_4. \tag{4.165}$$

We claim that the uncertainty principle (4.146) is equivalent to the $\mathrm{Sp}(2,\mathbb{R}) \times \mathrm{Sp}(2,\mathbb{R})$-invariant statement:

$$\det \boldsymbol{\alpha} \det \boldsymbol{\beta} + (1 - \det \boldsymbol{\gamma})^2 - \mathrm{tr}(\boldsymbol{\alpha\omega\gamma\omega\beta\omega\gamma}^T \boldsymbol{\omega}) \geq (\det \boldsymbol{\alpha} + \det \boldsymbol{\beta}). \tag{4.166}$$

Under the Peres-Horodecki partial transpose or mirror reflection, the variance matrix $\boldsymbol{\sigma}$ transforms to $\tilde{\boldsymbol{\sigma}} = \Lambda \boldsymbol{\sigma} \Lambda$. Specifically, $\boldsymbol{\gamma}$ transforms to $\boldsymbol{\gamma}\sigma_3$ and $\boldsymbol{\beta}$ transforms to $\sigma_3 \boldsymbol{\beta} \sigma_3$, while $\boldsymbol{\alpha}$ remains unchanged (σ_3 is the diagonal Pauli matrix: $\sigma_3 = \mathrm{diag}(1,-1)$). As a result, $I_3 = \det \boldsymbol{\gamma}$ changes sign, while I_1, I_2, and I_4 remain unchanged. Consequently, condition (4.154) for $\tilde{\boldsymbol{\sigma}}$ takes a form identical to equation (4.166) with only the sign in front of $\det \boldsymbol{\gamma}$ in the second term on the left-hand side reversed. Thus, the requirement that the variance matrix of a separable state has to obey (4.154), in addition to the fundamental uncertainty principle (4.146), takes the form:

$$\det \boldsymbol{\alpha} \det \boldsymbol{\beta} + (1 - |\det \boldsymbol{\gamma}|)^2 - \mathrm{tr}(\boldsymbol{\alpha\omega\gamma\omega\beta\omega\gamma}^T \boldsymbol{\omega}) \geq (\det \boldsymbol{\alpha} + \det \boldsymbol{\beta}). \tag{4.167}$$

This is the final form of our necessary condition on the variance matrix of a separable bipartite state. This condition is invariant not only under $Sp(2,\mathbb{R}) \times Sp(2,\mathbb{R})$ but also under mirror reflection, as it should be. It constitutes a complete description of the implication the Peres-Horodecki criterion has for the second moments. The Peres-Horodecki criterion (4.167) is a necessary and sufficient condition for separability, for all bipartite Gaussian states, but the proof of this statement is left to [65].

4.3.7.1 Symplectic form of separability criterion

The PPT (Positive Partial Transpose) criterion has an elegant symplectic representation. The partially transposed matrix $\tilde{\sigma}$ of any N-mode Gaussian covariance matrix (CM) is

still a positive and symmetric matrix. As such, it admits a Williamson normal-mode decomposition [49], given by

$$\tilde{\sigma} = S^T \tilde{\nu} S, \tag{4.168}$$

where $S \in \mathrm{Sp}(2N, \mathbb{R})$ and $\tilde{\nu}$ is the CM

$$\tilde{\nu} = \bigoplus_{k=1}^{N} \begin{pmatrix} \tilde{\nu}_k & 0 \\ 0 & \tilde{\nu}_k \end{pmatrix}. \tag{4.169}$$

The N quantities $\tilde{\nu}_k$ are the symplectic eigenvalues of the partially transposed CM $\tilde{\sigma}$. The symplectic spectrum $\{\nu_k\}$ of σ encodes the structural and informational properties of a Gaussian state. The partially transposed spectrum $\{\tilde{\nu}_k\}$ encodes the qualitative characterization of entanglement in the state. The PPT condition (4.153), i.e., the uncertainty relation for $\tilde{\sigma}$, can be equivalently recast in terms of the parameters $\tilde{\nu}_k$ as

$$\tilde{\nu}_k \geq 1. \tag{4.170}$$

We can, without loss of generality, rearrange the modes of an N-mode state such that the corresponding symplectic eigenvalues of the partial transpose $\tilde{\sigma}$ are sorted in ascending order,

$$\tilde{\nu}_- \equiv \tilde{\nu}_1 \leq \tilde{\nu}_2 \leq \cdots \leq \tilde{\nu}_{N-1} \leq \tilde{\nu}_N \equiv \tilde{\nu}_+, \tag{4.171}$$

in analogy to what is the previous section for the spectrum of σ. With this notation, the PPT criterion across an arbitrary bipartition reduces to

$$\tilde{\nu}_1 \geq 1 \quad \text{for all separable Gaussian states.} \tag{4.172}$$

If $\tilde{\nu}_1 < 1$, the corresponding Gaussian state σ is entangled.

4.3.7.2 Another criterion for separability: Duan Bound

Another inseparability criterion, applicable to two mode continuous-variable systems and expressed in terms of an inequality for certain variances involving position and momentum operators, was derived by Duan, et al. [54] using a strategy independent of the partial transpose. Interestingly, they find that for any separable state, there exists a lower bound to the total variance. To be more general, they consider the following type of EPR-like operators:

$$\begin{aligned} \hat{u} &= |a|\hat{x_1} + \frac{1}{a}\hat{x_2} \\ \hat{v} &= |a|\hat{p_1} - \frac{1}{a}\hat{p_2} \end{aligned} \tag{4.173}$$

where a is a real (non zero) coefficient, and $\hat{x}_i$ and $\hat{p}_i$ are position and momentum operators for the respective subsystems. For any separable state, the total variance of any pair of EPR-like operators in the form of Eqs. (4.173) should satisfy a lower bound indicated by the following theorem:

Theorem 2: Sufficient criterion for inseparability: For any separable quantum state $\hat{\rho}$, the total variance of a pair of EPR-like operators defined by Eqs. (4.173) with the commutators $[\hat{x}_j, \hat{p}_{j'}] = i\delta_{jj'} \quad (j, j' = 1,2)$ satisfies the inequality

$$\langle (\Delta\hat{u})^2 \rangle_\rho + \langle (\Delta\hat{v})^2 \rangle_\rho \geq a^2 + \frac{1}{a^2} \tag{4.174}$$

where $(\Delta\hat{u})^2$ and $(\Delta\hat{v})^2$ denote the variances of the EPR-like operators $\hat{u}$ and $\hat{v}$, respectively. This result is easily proved by considering that

$$\begin{aligned} Tr\{\rho\,(\Delta\hat{u})^2\} =& Tr\{\left(\sum_i c_i \rho_{i1} \otimes \rho_{i2}\right)(\Delta\hat{u})^2\} \\ & \sum_i c_i Tr\{(\rho_{i1} \otimes \rho_{i2})\,(\Delta\hat{u})^2\} = \sum_i c_i \langle (\Delta\hat{u})^2 \rangle_i \end{aligned} \tag{4.175}$$

Same result for $(\Delta\hat{u})^2$. At this point we can explicitly proceed through the calculation:

$$\begin{aligned} \langle (\Delta\hat{u})^2 \rangle_\rho + \langle (\Delta\hat{v})^2 \rangle_\rho =& \sum_i c_i \left(\langle \hat{u}^2 \rangle_i + \langle \hat{v}^2 \rangle_i \right) - \langle \hat{u} \rangle_\rho^2 - \langle \hat{v} \rangle_\rho^2 \\ =& \sum_i c_i \left(a^2 \langle \hat{x_1}^2 \rangle_i + \frac{1}{a^2} \langle \hat{x_2}^2 \rangle_i + a^2 \langle \hat{p_1}^2 \rangle_i + \frac{1}{a^2} \langle \hat{p_2}^2 \rangle_i \right) \\ & + 2\frac{a}{|a|} \left(\sum_i c_i \langle \hat{x_1} \rangle_i \langle \hat{x_2} \rangle_i - \sum_i c_i \langle \hat{p_1} \rangle \langle \hat{p_2} \rangle_i \right) - \langle \hat{u} \rangle_p^2 - \langle \hat{v} \rangle_p^2 \\ =& \sum_i c_i \left(a^2 \langle \left(\Delta\hat{x_1}^2\right) \rangle_i + \frac{1}{a^2} \langle \left(\Delta\hat{x_2}^2\right) \rangle_i + a^2 \langle \left(\Delta\hat{p_1}^2\right) \rangle_i + \frac{1}{a^2} \langle \left(\Delta\hat{p_2}^2\right) \rangle_i \right) \\ & + \sum_i c_i \langle \hat{u} \rangle_i^2 + \left(\sum_i c_i \langle \hat{u} \rangle_i \right)^2 + \sum_i c_i \langle \hat{v} \rangle_i^2 + \left(\sum_i c_i \langle \hat{v} \rangle_i \right)^2 . \end{aligned} \tag{4.176}$$

In Eq. (4.176), the symbol $\langle \cdots \rangle_i$ denotes the average over the product density operator $\rho_{i1} \otimes \rho_{i2}$. It follows from the uncertainty relation that

$$\langle (\Delta\hat{x}_j)^2 \rangle_i + \langle (\Delta\hat{p}_j)^2 \rangle_i \geq |[\hat{x}_j, \hat{p}_j]| = 1 \quad \text{for} \quad j = 1,2, \tag{4.177}$$

and, moreover, by applying the Cauchy-Schwarz inequality

$$\left(\sum_i p_i \langle p_i \hat{u} \rangle_i\right)^2 \leq \left(\sum_i p_i\right)\left(\sum_i p_i \langle \hat{u} \rangle_i^2\right), \tag{4.178}$$

we know that the last line of Eq. (4.176) is bounded from below by zero. Hence, the total variance of the two EPR-like operators $\hat{u}$ and $\hat{v}$ is bounded from below by $a^2 + \frac{1}{a^2}$ for any separable state. This completes the proof of the theorem. Note that this theorem in fact gives a set of inequalities for separable states. The operators $\hat{x}_j, \hat{p}_j\,(j = 1,2)$ in the definition (4.173) can be any local operators satisfying the commutators $[\hat{x}_j, \hat{p}_{j'}] = i\delta_{jj'}$. In particular, if we apply an arbitrary local unitary operation $U_1 \otimes U_2$ to the operators $\hat{u}$ and $\hat{v}$, the inequality (4.174) remains unchanged. Note also that without loss of generality we have taken the operators x_j and p_j dimensionless. For inseparable states, the total variance of the $\hat{u}$ and $\hat{v}$ operators is required by the uncertainty relation to be larger than or equal to $\left|a^2 - \frac{1}{a^2}\right|$, which reduces to zero for $a = 1$. For separable states, the much stronger bound given by Eq. (4.174) must be satisfied.
As final remark is worth mentioning the link between the Simon and Duan criteria, and to do that we can consider again the Hermitian operators $\hat{X}(\mathbf{d}) = \mathbf{d}^T \hat{\boldsymbol{\xi}} = d_1\hat{q}_1 + d_2\hat{p}_1 + d_3\hat{q}_2 + d_4\hat{p}_2$ and $\hat{X}(\mathbf{d}') = \mathbf{d}'^T \hat{\boldsymbol{\xi}} = d'_1\hat{q}_1 + d'_2\hat{p}_1 + d'_3\hat{q}_2 + d'_4\hat{p}_2$. The vectors $\mathbf{d}$ and $\mathbf{d}'$ are required to be real without any other constraint. Since this freedom of choice we can reconstruct the two EPR-like operators $\hat{u}$ and $\hat{v}$ (when $a = 1$). Therefore, if $\mathbf{d} = (1,0,1,0)$ and $\mathbf{d}' = (0,1,0,-1)$ we have that:

$$\begin{aligned} \hat{X}(\mathbf{d}) &= \hat{q}_1 + \hat{q}_2, \\ \hat{X}(\mathbf{d}') &= \hat{p}_1 - \hat{p}_2. \end{aligned} \tag{4.179}$$

and the Simon separability condition (4.155) coincides with the Duan bound:

$$\langle (\Delta\hat{X}(\mathbf{d}))^2 \rangle + \langle (\Delta\hat{X}(\mathbf{d}))^2 \rangle \geq 2 \quad \blacksquare. \tag{4.180}$$

4.3.7.3 Negativity in symplectic representation

Previously we have introduced Negativity as an entanglement measure that has the clear interpretation as the sum of absolute values of negative eigenvalues of the partial transpose density matrix. We want to apply the general definition of sec.(3.2.2.1) to Gaussian states, highlighting the link between this entanglement measure and symplectic eigenvalues. The first step consists in representing the norm of density matrix in terms of those eigenvalues. It is useful to recall the single-mode density matrix, as the N-mode generalization follows directly from this analysis, given that the global density matrix is expressed as the tensor product of the single-mode ones:

$$\rho_{\tilde{\nu}_k}^{\otimes} = \frac{2}{\tilde{\nu}_k + 1} \sum_{n=0}^{\infty} \left(\frac{\tilde{\nu}_k - 1}{\tilde{\nu}_k + 1}\right)^n |n\rangle\langle n|, \tag{4.181}$$

where $\tilde{\nu}_k$ are the eigenvalues of $\tilde{\sigma}$, i.e, the covariance matrix related to the partial transpose of the density matrix ρ. If we consider the norm of this expression, we obtain different results depending on the value of each $\tilde{\nu}_k$. The norm is:

$$||\rho_{\tilde{\nu}_k}^{\otimes}||_1 = \frac{2}{\tilde{\nu}_k+1}\sum_{n=0}^{\infty}\left|\left(\frac{\tilde{\nu}_k-1}{\tilde{\nu}_k+1}\right)\right|^n. \tag{4.182}$$

When $\tilde{\nu}_k \geq 1$, so the state is separable, the sum converges to one and the negativity is zero as expected. Otherwise, when $\tilde{\nu}_k < 1$ the norm converges to:

$$||\rho_{\tilde{\nu}_k}^{\otimes}||_1 = \frac{2}{\tilde{\nu}_k+1}\frac{1}{1-\left|\frac{\tilde{\nu}_k-1}{\tilde{\nu}_k+1}\right|} = \frac{2}{\tilde{\nu}_k+1}\frac{1}{1-\frac{1-\tilde{\nu}_k}{\tilde{\nu}_k+1}} = \frac{1}{\tilde{\nu}_k}. \tag{4.183}$$

The negativity is now given by :

$$N(\rho_{\tilde{\nu}_k}^{\otimes}) = \frac{1}{2}\left(\frac{1}{\tilde{\nu}_k}-1\right) \quad \blacksquare. \tag{4.184}$$

We can now generalize which have been studied for a single mode to N-mode system by following [50, 60]. The negativity of a Gaussian state with CM σ is given by

$$N(\sigma) = \begin{cases} \frac{1}{2}\left(\prod_k \tilde{\nu}_k^{-1}-1\right), & \text{for } k:\tilde{\nu}_k<1, \\ 0 & \text{if } \tilde{\nu}_i \geq 1 \forall i, \end{cases} \tag{4.185}$$

where the set $\{\tilde{\nu}_k\}$ consists of the symplectic eigenvalues of the partially transposed CM $\tilde{\sigma}$. Accordingly, the logarithmic negativity reads

$$E_N(\sigma) = \begin{cases} -\sum_k \log \tilde{\nu}_k, & \text{for } k:\tilde{\nu}_k<1, \\ 0 & \text{if } \tilde{\nu}_i \geq 1 \forall i. \end{cases} \tag{4.186}$$

For $\tilde{\nu}_- \geq 1$, the state is separable, otherwise it is entangled; the smaller the $\tilde{\nu}_-$, the more entangled is the corresponding Gaussian state.

4.3.8 Characterizing Two-Mode Entanglement by Information Measures

We have seen that entanglement or more precisely bipartite quantum correlations in pure states are defined in terms of the amount of local/marginal entropy . This identification disappears when it comes to mixed states, due to the information loss on a global scale about their state since it's impossible to distinguish between quantum correlation from classical one. In this chapter we review the results of G. Adesso, A. Serafini, and F. Illuminati [66] who demonstrate that the combined information of both, global and marginal degrees provides a perfect measure for entanglement in terms of purities or more generally speaking by generalized p-entropy (3.1.3.4). It has been proved [66] the existence of strict upper and lower bounds on the entanglement and the existence of extremally (maximally and minimally) entangled states at fixed global and marginal degrees of information. The study of a two-mode system is more efficient by means another structure of the CM, the so called *standard form* [54].

$$S^T \sigma S = \sigma_{\rm sf} \equiv \begin{pmatrix} a & 0 & c_+ & 0 \\ 0 & a & 0 & c_- \\ c_+ & 0 & b & 0 \\ 0 & c_- & 0 & b \end{pmatrix}. \tag{4.187}$$

States whose standard form fulfills $a = b$ are said to be symmetric. Let us recall that any pure state is symmetric and fulfills $c_+ = -c_- = \sqrt{a^2 - 1}$. The standard form covariances a, b, c_+, and c_- can be determined in terms of the two local symplectic invariants

$$\mu_1 = (\det \alpha)^{-1/2} = \frac{1}{a}, \quad \mu_2 = (\det \beta)^{-1/2} = \frac{1}{b}, \tag{4.188}$$

which are the marginal purities of the reduced single-mode states, and of the two global symplectic invariants

$$\mu = (\det \sigma)^{-1/2} = \left[(ab - c_+^2)(ab - c_-^2)\right]^{-1/2}, \quad \Delta = a^2 + b^2 + 2c_+c_-, \tag{4.189}$$

which are the global purity and the seralian, respectively. Eqs. (4.188), (4.189) can be inverted to provide the following physical parametrization of two-mode states in terms of the four independent parameters μ_1, μ_2, μ, and Δ [66]:

$$a = \frac{1}{\mu_1}, \quad b = \frac{1}{\mu_2}, \quad c_\pm = \frac{\sqrt{\mu_1 \mu_2}}{4}(\varepsilon_+ \pm \varepsilon_-), \tag{4.190}$$

with

$$\varepsilon_\mp \equiv \sqrt{\left[\Delta - \frac{(\mu_1 \mp \mu_2)^2}{\mu_1^2 \mu_2^2}\right]^2 - \frac{4}{\mu^2}}. \tag{4.191}$$

The uncertainty principle and the existence of the radicals appearing in (4.191) impose the following constraints on the four invariants in order to describe a physical state:

$$0 \leq \mu_{1,2} \leq 1, \tag{4.192}$$

$$\mu_1 \mu_2 \leq \mu \leq \frac{\mu_1 \mu_2}{\mu_1 \mu_2 + |\mu_1 - \mu_2|}, \tag{4.193}$$

$$\frac{2}{\mu} + \frac{(\mu_1 - \mu_2)^2}{\mu_1^2 \mu_2^2} \leq \Delta \leq 1 + \frac{1}{\mu^2}. \tag{4.194}$$

We can now provide the separability condition for the two-mode state, recalling that the PPT criterion, in the continuous variables environment, coincides with a time reflection in the phase space [65], translates in a change of sign of $\det(\gamma)$. So we end up with

$\tilde{\Delta} = \det\alpha + \det\beta - 2\det\gamma = \Delta - 4\det\gamma$. The symplectic eigenvalues of the CM $\boldsymbol{\sigma}$ and of its partial transpose $\tilde{\boldsymbol{\sigma}}$ are promptly determined in terms of symplectic invariants:

$$2\nu_{\mp}^2 = \Delta \mp \sqrt{\Delta^2 - \frac{4}{\mu^2}}, \quad 2\tilde{\nu}_{\mp}^2 = \tilde{\Delta} \mp \sqrt{\tilde{\Delta}^2 - \frac{4}{\mu^2}}. \tag{4.195}$$

The PPT criterion yields a state σ separable if and only if $\tilde{\nu}_- \geq 1$. At the end, Negativity provides an efficient measure of entanglement:

$$E_N = \max(0, -\log\tilde{\nu}_-). \tag{4.196}$$

4.3.8.1 Maximal entanglement at fixed local purities

As mentioned earlier, the purpose of this discussion is to understand how entanglement is defined for mixed Gaussian states. Once the key variables have been identified, it is only necessary to understand how they are related to entanglement. A first relevant analysis consists in identifying the relationship between entanglement and global purity when μ_1, μ_2 and the seralian Δ are fixed. We have seen see that the inseparability of a state lies completely within the sign of the smallest eigenvalue of the covariance matrix, as determined by Simon [65]. This eigenvalue is a monotonic function of global purity (4.195), so when it increases it will also increase $\tilde{\nu}_-$ and consequently the logarithmic negativity increases (4.196). Clearly, when μ_1, μ_2 and Δ are fixed, the global purity cannot vary freely but must satisfy very precise bounds which depend on these constant quantities (4.193). It becomes interesting to understand the effects on the entanglement when we are at the extremes of this domain. By imposing the saturation of the upper bound in (4.195), $\mu = \mu_{\max}(\mu_1, \mu_2) \equiv \frac{\mu_1\mu_2}{\mu_1\mu_2 + |\mu_1 - \mu_2|}$, we can identify the states with the highest purity for given marginals. Furthermore, selecting $\mu = \mu_{\max}(\mu_1, \mu_2)$ ensures that the upper and lower bounds of Δ in (4.194) coincide, making Δ uniquely defined in terms of μ_1 and μ_2. This indicates that the two-mode states with maximal purity for fixed marginals are indeed the *Gaussian maximally entangled states* for fixed marginal mixednesses (GMEMMS).

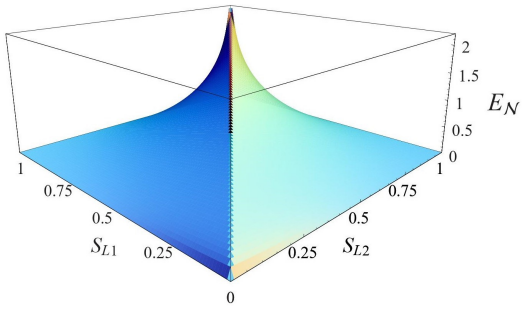

Figure 3: Plot of the maximal entanglement achievable by quantum systems with given marginal linear entropies: logarithmic negativity of continuous variable GMEMMS [66]

In Fig. 3, the logarithmic negativity of GMEMMS is depicted as a function of the marginal linear entropies, $S_{L1,2} \equiv 1 - \mu_{1,2}$. This description shows, for mixed Gaussian states, that the maximum entangled state coincides with the maximum degree of marginal disorder. This is a behavior absolutely analogous to that which happens for pure states. For the latter, in fact, the more the global system is entangled the greater the entropy of the individual components. This is a clear effect of the inherent correlations between the two subsystems, which distribute information in pairs and not in individual components. As an example we can think about the Bell state that is a pure maximal entangled state ($\mu = 1$), with marginal purities $\mu_{1,2} = 1/2$. Conversely, the entanglement decreases when the difference between the marginal mixednesses increases. This result aligns with the intuition that subsystems sharing quantum correlations should exhibit comparable amounts of quantum information. Additionally, it is worth highlighting that the "minimally" entangled states for fixed marginals, which reach the lower bound in (4.194) (i.e., $\mu = \mu_1\mu_2$), correspond to tensor product states. These are states where no correlations (quantum or classical) exist between the subsystems.

4.3.8.2 Extremal entanglement at fixed global and local purities

What has been demonstrated so far, using simple analytical bounds, reveals a general trend: entanglement increases with higher global purity, and with lower marginal purities and smaller differences between them. The next step of the work [66] is now to employ the joint information about global and marginal purities to provide a meaningful characterization of entanglement, both qualitatively and quantitatively. It is possible to give a physical meaning of the seralian Δ in characterizing the properties of Gaussian states. In fact, to this end, they analyze the dependence of the symplectic eigenvalue $\tilde{\nu}_-$ on Δ, for fixed μ_1, μ_2, and μ:

$$\frac{\partial \tilde{\nu}_-^2}{\partial \Delta}\Big|_{\mu_1,\mu_2,\mu} = \frac{1}{2}\left(\frac{\tilde{\Delta}}{\sqrt{\tilde{\Delta}^2 - \frac{1}{4\mu^2}}} - 1\right) > 0. \tag{4.197}$$

The smallest symplectic eigenvalue of the partially transposed state is strictly monotonic in Δ. Consequently, the entanglement of a generic Gaussian state σ with a given global purity μ and marginal purities μ_1, μ_2 increases strictly as Δ decreases. The seralian Δ thus acquires a clear physical interpretation: for given global and marginal purities, it determines the amount of entanglement in the state. The Δ bounds (4.194) elucidates the link between global and marginal purities, since they define the upper and lower bounds to the only free parameter Δ. The focus of this analysis is on characterizing Gaussian states with extreme entanglement properties, specifically those that are maximally and minimally entangled for a given set of global and marginal purities. Attention is first directed toward states that reach the lower bound presented in equation (4.194), which corresponds to maximally entangled configurations. These states define a particular class known as Gaussian Most Entangled Mixed States (GMEMS) for specified global and local purities. Notably, this class is shown to include asymmetric two-mode squeezed thermal states as described by equation (aggiungere citazione su 2 mode squeezed state). For these states, the squeezing parameter and symplectic spectrum are defined as follows:

$$\tanh(2r) = \frac{2}{\mu_1+\mu_2}\sqrt{\mu_1\mu_2 - \frac{\mu_1^2\mu_2^2}{\mu}}, \tag{4.198}$$

$$\nu_\pm^2 = \frac{1}{\mu} + \frac{(\mu_1-\mu_2)^2}{2\mu_1^2\mu_2^2} \pm \frac{|\mu_1-\mu_2|}{2\mu_1\mu_2}\sqrt{\frac{(\mu_1-\mu_2)^2}{\mu_1^2\mu_2^2} + \frac{4}{\mu}}. \tag{4.199}$$

It is observed that nonsymmetric two-mode squeezed thermal states become separable when the condition

$$\mu \leq \frac{\mu_1\mu_2}{\mu_1+\mu_2-\mu_1\mu_2} \tag{4.200}$$

is satisfied. This result implies that Gaussian states with purities in the separable region, as defined by this inequality, do not exhibit entanglement. The authors next consider the states that saturate the upper bound in equation (4.194). These states determine the class of Gaussian Least Entangled Mixed States (GLEMS) for given global and local purities. Outside the separable region (where every Gaussian state can be considered a GLEMS with zero entanglement), they satisfy the relation

$$\Delta = 1 + \frac{1}{\mu^2}. \tag{4.201}$$

This condition implies that the symplectic spectrum of these states takes the form $\nu_- = 1$, $\nu_+ = 1/\mu$. Therefore, GLEMS are identified as mixed Gaussian states with partial minimum uncertainty, aligning with their property of minimal entanglement, making them, in some sense, the most classical Gaussian states. According to the PPT criterion outlined in the study, GLEMS are separable only if

$$\mu \leq \frac{\mu_1 \mu_2}{\sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2 \mu_2^2}}. \tag{4.202}$$

In the range

$$\frac{\mu_1 \mu_2}{\mu_1 + \mu_2 - \mu_1 \mu_2} < \mu \leq \frac{\mu_1 \mu_2}{\sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2 \mu_2^2}}, \tag{4.203}$$

both separable and entangled states are found. Conversely, the region

$$\mu > \frac{\mu_1 \mu_2}{\sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2 \mu_2^2}} \tag{4.204}$$

only accommodates entangled states. The narrow region defined by inequality (4.203) is therefore identified as the only region where both entangled and separable Gaussian mixed states coexist (*coexistence region*). The study emphasizes that knowledge of μ_1, μ_2, and μ provides a precise characterization of Gaussian state entanglement. Quantitative knowledge of the local and global purities enables reliable entanglement quantification as well. Outside the separable region, GMEMS achieve maximum logarithmic negativity $E_N^{\max}(\mu_1, \mu_2, \mu)$, while in the entangled region, GLEMS attain minimum logarithmic negativity $E_N^{\min}(\mu_1, \mu_2, \mu)$, where

$$E_N^{\max} = -\frac{1}{2} \log \left[-\frac{1}{\mu} + \left(\frac{\mu_1 + \mu_2}{2\mu_1^2 \mu_2^2} \right) \left(\mu_1 + \mu_2 - \sqrt{(\mu_1 + \mu_2)^2 - \frac{4\mu_1^2 \mu_2^2}{\mu}} \right) \right], \tag{4.205}$$

$$E_N^{\min} = -\frac{1}{2} \log \left[\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2} - \frac{1}{2\mu^2} - \frac{1}{2} - \sqrt{\left(\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2} - \frac{1}{2\mu^2} - \frac{1}{2} \right)^2 - \frac{1}{\mu^2}} \right]. \tag{4.206}$$

The entanglement of Gaussian states with given global and marginal purities is quantified by the "average logarithmic negativity" [60]

$$\overline{E_N}(\mu_1, \mu_2, \mu) \equiv \frac{E_N^{\max}(\mu_1, \mu_2, \mu) + E_N^{\min}(\mu_1, \mu_2, \mu)}{2}. \tag{4.207}$$

It is important to note that the entanglement measure, for fixed global and marginal purities, is a 'privileged' measure of mixedness in continuous variable systems, because it is possible to give an explicit expression to quantify extramally entangled states (4.206),(4.207). Specifically, S_L can be measured using feasible direct methods, provided some prior knowledge about the state, such as its Gaussian nature. These

methods, including single-photon detection schemes [67] and emerging quantum network architectures [68], eliminate the need for complete homodyne reconstruction of the density matrix. Recently, a significant and promising advancement in this area has been achieved: an experimental demonstration of direct photon detection for measuring the squeezing and purity of a single-mode squeezed vacuum. This setup required only a tunable beam splitter and a single-photon detector [69].

4.3.8.3 Extremal entanglement at fixed global and local generalized entropies

The linear entropy S_L is not the only possible measure of mixedness of a state. The aim of this section is to characterize entanglement of two-mode (mixed) Gaussian state, reviewing [66], by exploiting the explicit behavior of the global invariant Δ at fixed global and marginal entropies (for $p \neq 2$), and its relation with the logarithmic negativity E_N. In the paragraph (3.1.3.4) we have pointed out that the generalized entropy is bounded by the lower and upper limits which identify, as in the previous case, two extremal classes of states still identified as the nonsymmetric squeezed thermal states (GMEMS) and the mixed states with partial minimum uncertainty (GLEMS). However, it is important to note that the seralian Δ no longer maintains a monotonic relationship with the entanglement of the state when the generalized entropies are fixed. In order to compare the results we will obtain with those of the previous section, where the protagonists were global and marginal purities, we proceed with a description that follows the same steps but for a general p, distinguishing cases in which $p < 2$ and $p > 2$.
First of all we need of another parametrization of the covariance matrix σ in terms of local purities $\mu_{1,2}$ (or any other marginal $S_{p1,p2}$ because all the local, single mode entropies are equivalent for any value of the integer p), global entropy S_p, and the global symplectic invariant Δ. From Eqs.(4.102),(4.106),(4.87) we notice that $S_p \equiv S_p(\Delta, R)$, where $R = 2/\mu$. Since our analysis focuses on the study of the entanglement properties for fixed p-entropy, it is useful to highlight the relationship between Δ and μ. Our interest lies in how these two realities change so that the entropy remains constant. To this end we analyze:

$$\left.\frac{\partial \mu}{\partial \Delta}\right|_{S_p} = -\frac{2}{R^2}\left.\frac{\partial R}{\partial \Delta}\right|_{S_p}. \tag{4.208}$$

We can exploit the invariance of the entropy to rewrite this derivative, since:

$$dS_p = 0 = \left.\frac{\partial S_p}{\partial \Delta}\right|_R d\Delta + \left.\frac{\partial S_p}{\partial R}\right|_R dR, \rightarrow \left.\frac{\partial R}{\partial \Delta}\right|_{S_p} = -\frac{\partial S_p/\partial \Delta|_R}{\partial S_p/\partial R|_\Delta}, \tag{4.209}$$

obtaining

$$\left.\frac{\partial \mu}{\partial \Delta}\right|_{S_p} = \frac{2}{R^2}\frac{\partial S_p/\partial \Delta|_R}{\partial S_p/\partial R|_\Delta} = \frac{2}{R^2}\frac{N_p(\Delta, R)}{D_p(\Delta, R)}. \tag{4.210}$$

At the end, all the information is encoded inside the ratio $N_p(\Delta, R)/D_p(\Delta, R)$. This is an increasing function of p, with a zero when $p = 2$. In particular for $p < 2$ is negative with absolute minimum at (-1), in the asymptotic regime $(\Delta \to 2, R \to 2, p \to 1)$.

This means that in this regime Δ and μ are inversely proportional, in fact, when Δ is minimum (maximum), the purity μ is maximum (minimum). For $p > 2$ the ratio is positive; therefore, Δ and μ are in this case proportional. The analysis of this trend allows us to easily identify which states correspond to the extreme values of Δ, since for fixed values of marginal and global entropy, the negativity depends exclusively on Δ, so that to extreme values of Δ correspond extreme values of entanglement. We therefore conclude that all Gaussian states with maximal Δ for any fixed triple of values of global and marginal entropies are GLEMS. Vice versa, one can show that all Gaussian states with minimal Δ for any fixed triple of values of global and marginal entropies are Gaussian maximally entangled mixed states (GMEMS) . In summary, we have demonstrated that GMEMS and GLEMS consistently always represent the extremally entangled Gaussian states, regardless of the specific choice of generalized global and marginal entropic measures used. This indicates that maximally and minimally entangled states in continuous variable (CV) systems are remarkably resilient to variations in the chosen measures of mixedness. This behavior contrasts sharply with discrete variable systems, where selecting different measures of mixedness leads to distinct classes of maximally entangled states [70].

Thus, this dependence between Δ and μ from the parameter p, or the type of entropy measure considered, is clearly reflected on the dependence which has the smallest eigenvalue from the serialism Δ. For a given p, the smallest symplectic eigenvalue of the partially transposed covariance matrix, used to quantify entanglement, is not always a monotonic function of the symplectic invariant Δ when global and marginal p-entropies are fixed. This leads to unexpected outcomes in the relationship between extremal Δ values and the entanglement of states. For $p < 2$: The surfaces representing GMEMS (Gaussian states with minimal Δ) and GLEMS (Gaussian states with maximal Δ become more distinct as p decreases. For $p > 2$: The distinction between GMEMS and GLEMS diminishes as p increases. In a specific range of global and marginal entropies, these two classes of states swap their roles. GMEMS (typically associated with minimal Δ) now represent minimally entangled states, while GLEMS (typically associated with maximal Δ) now represent maximally entangled states. This inversion of roles is a general phenomenon that always occurs for all $p > 2$, reflecting the nontrivial connection between Δ, entanglement, and the generalized p-entropies. In the following we will provide the general idea to understand how this inversion comes out. To explore this intriguing behavior, we examine how the symplectic eigenvalue $\tilde{\nu}_-$ depends on the global invariant Δ, while keeping the marginals and the p-entropy S_p fixed for a generic p. Using Maxwell's relations, the derivative can be expressed as:

$$\kappa_p \equiv \left.\frac{\partial(2\tilde{\nu}_-^2)}{\partial\Delta}\right|_{S_p} = \left.\frac{\partial(2\tilde{\nu}_-^2)}{\partial\Delta}\right|_{R} - \left.\frac{\partial(2\tilde{\nu}_-^2)}{\partial R}\right|_{\Delta} \cdot \frac{\left.\frac{\partial S_p}{\partial\Delta}\right|_R}{\left.\frac{\partial S_p}{\partial R}\right|_\Delta}. \tag{4.211}$$

When $\kappa_p > 0$, GMEMS and GLEMS maintain their conventional roles, with GLEMS representing minimally entangled states and GMEMS representing maximally entangled states. Conversely, for $\kappa_p < 0$, their roles are reversed. At the critical node where $\kappa_p = 0$, GMEMS and GLEMS exhibit the same level of entanglement; this implies that the entanglement of all Gaussian states is completely determined by local purities and

S_p, being independent of Δ. Without going into mathematical details, described in [60], we show the key idea to determine this nodal surface in $p > 2$ regime. To do this, we look for a non-trivial solution of the equation $\kappa_p = 0$, for any Δ value (being the entanglement independent from it on the nodal surface). If we choose $\Delta = 1 + R^2/4$ (that saturates the Heisenberg uncertainty relation and is satisfied by GLEMS) the nodal surface has the following form:

$$S_p = S_p^{\kappa}(\mu_1, \mu_2) \equiv \frac{1 - g_p \left[\left(\mu_p^{\kappa}(\mu_1, \mu_2) \right)^{-1} \right]}{p - 1}. \tag{4.212}$$

The entanglement of all Gaussian states whose entropies lie on the surface $S_p^{\kappa}(\mu_1, \mu_2)$ is completely determined by the knowledge of μ_1, μ_2, and S_p. The explicit expression of the function $\mu_p^{\kappa}(\mu_1, \mu_2)$ depends on p, but, being the global purity of physical states, is constrained by the inequality:

$$\mu_1 \mu_2 \leq \mu_p^{\kappa}(\mu_1, \mu_2) \leq \frac{\mu_1 \mu_2}{\mu_1 \mu_2 + |\mu_1 - \mu_2|}. \tag{4.213}$$

After verifying the existence of extremally entangled Gaussian states, it is necessary to provide only a quantitative measure of these. We can formally define the maximal entanglement $E_N^{\max}(S_{p1,2}, S_p)$ as the logarithmic negativity attained by GMEMS (or GLEMS, below the inversion nodal surface for $p > 2$). In a similar way, in the entangled region, GLEMS (or GMEMS, below the inversion nodal surface for $p > 2$) achieve the minimal logarithmic negativity $E_N^{\min}(S_{p1,2}, S_p)$. The explicit analytical expressions of these quantities are unavailable for any $p \neq 2$ due to the transcendence of the conditions relating S_p to the symplectic eigenvalues.
In summary, we have identified the presence of both maximally and minimally entangled two-mode Gaussian states at fixed local and global generalized p-entropies. The analytical properties of these states have been thoroughly examined for all values of p. Notably, for $p \leq 2$, the minimally entangled states are minimum uncertainty states that saturate the Heisenberg principle, whereas the maximally entangled states are non-symmetric two-mode squeezed thermal states. Interestingly, for $p > 2$ and within specific ranges of the entropic measures, the roles of these states are reversed. In this case, two-mode squeezed thermal states, often considered the continuous variable analogs of maximally entangled states, instead become minimally entangled.

5 Non-Locality

In classical physics, it is assumed that the physical properties of an object exist independently of observation. For example, the position and velocity of a particle are considered real attributes that are merely *revealed* by measurements. This view aligns with everyday intuition: objects possess inherent properties regardless of whether they are being observed. However, the development of quantum mechanics in the 1920s and 1930s introduced a revolutionary shift. Quantum mechanics suggests that an unobserved particle does not possess definite physical properties. Instead, such proper-

ties arise *as a result of measurements* performed on the system. For instance, a qubit does not simultaneously possess well-defined values for 'spin in the z-direction (σ_z)' and 'spin in the x-direction (σ_x).' Quantum mechanics provides probabilistic rules, based on the state vector $|\psi\rangle$, for predicting the possible outcomes of these measurements. The counterintuitive nature of quantum mechanics led to strong objections from many physicists, including Albert Einstein. In the famous 'EPR paper' (1935)[3], co-authored with Nathan Rosen and Boris Podolsky, Einstein argued that quantum mechanics is *incomplete*. The EPR paradox centered on the notion of 'elements of reality.' Einstein and his colleagues proposed that if one could predict the value of a property with certainty, without disturbing the system, then this property must correspond to an element of physical reality. The EPR *thought experiment* highlighted the peculiar phenomenon of *entanglement*, where measurements on one particle instantaneously affect another, regardless of the distance between them. Einstein referred to this phenomenon as "spooky action at a distance". They proved that it is possible to simultaneously determine, with completely certainty, both the relative position and momentum of a pair of spinless particles, in contrast to what is predicted by quantum mechanics, thus concluding that quantum mechanics was an incomplete theory. Thanks to this result the EPR paper aimed to force a return to a classical worldview where properties exist independently of measurement. Nearly three decades later, physicist John Bell [4] formulated a theoretical framework to test the validity of the EPR argument. Bell's argument include *locality* (the measurement on a subsystem does not influence the observation of the other one) and *realism* (the physical properties of a system exist independently from the observation or measurement) as the keys hypothesis to derive the famous inequality. It is important to stress that Bell's inequality *is not* a result of quantum mechanics, and it consists in a upper limit for all local physical theories. The violation of Bell's inequality is equivalent to say that there is not any local theory that properly describe the system, suggesting non-local features as expected from quantum mechanics. This does not say anything about the completeness of quantum mechanics, therefore there is no contradiction between the results of Bell and what said by Einstein, but it validates the hypothesis that the reality is non-local. Experimental tests of Bell's inequality, beginning with those conducted by Alain Aspect in the 1980s [71], conclusively demonstrated that Nature does not conform to the classical worldview envisioned by Einstein and the EPR paper. Instead, the experimental results align with the predictions of quantum mechanics, confirming the non-local correlations inherent in entangled states. In the following we will describe in more details the EPR idea and the consecutive Bell response.

5.1 EPR argument

In a 1935 paper [3], Einstein,Podolsky and Rosen introduced a thought experiment to argue that quantum mechanics was not a complete physical theory. Known today as the "EPR paradox," the thought experiment was meant to demonstrate the innate conceptual difficulties of quantum theory. The key aspects on which the discussion is developed are two: (1) *"every element of the physical reality must have a counter-part in the physical theory*; (2) *"If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity,*

the there exist an element of physical reality corresponding to this physical quantity. To prove incomplitness of quantum mechanics we can introduce the following thought experiment. Let's suppose we have two systems, I and II, that exist in an entangled state, such that :

$$|\Psi\rangle = \sum_n p_n |a_n\rangle_1 \otimes |b_n\rangle_2 \tag{5.1}$$

That is, the tensor product between the state of particle I and II, weighed by the coefficients p_n where $|a_n\rangle_1$ and $|b_n\rangle_2$ are the eigenvectors of an operator $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ respectively. Suppose we perform a measure, for system I, the physical quantity associated with operator $\hat{\mathbf{A}}$. Due to the collapse of the wave function, we know the state of particle II instantly and with absolute certainty. This implies, that the quantity related to the operator $\hat{\mathbf{B}}$ has an element of physical reality. It is important to stress that the EPR argument considered measurement as a local process. In their reasoning, they assumed that the act of measurement on one part of an entangled system could not instantaneously influence the other part, in accordance with the principle of local realism. Suppose, however, that instead of measuring B on particle II we measured some other observable, call it $\hat{\mathbf{D}}$, with eigenstates $|d_n\rangle_2$, for which it holds that $\left[\hat{\mathbf{B}}, \hat{\mathbf{D}}\right] \neq 0$. We then have to rewrite the state as

$$|\Psi\rangle = \sum_n f_n |c_n\rangle_1 \otimes |d_n\rangle_2 \tag{5.2}$$

where now $|c_n\rangle_1$ is an eigenstate of some other operator $\hat{\mathbf{C}}$ for particle I. With the same procedure we can predict with probability one the value of the quantity $\hat{\mathbf{D}}$ for the second system, after measuring the first. It results that also $\hat{\mathbf{D}}$ has an element of physical reality. Assuming that the wave function does contain a complete description of the two-particle system it would seem that the argument of EPR establishes that it is possible to assign two different states ($|b_n\rangle_2$ and $|c_n\rangle_2$) to the same reality. Nevertheless, two physical quantities represented by operators which do not commute cannot have simultaneous reality. The EPR paradox concluded that the quantum mechanical description of physical reality given by the wave function is not complete.

5.2 Bell' inequalities : An experimental approach

In 1964, Bell proved that the predictions of quantum theory are incompatible with those of any physical theory satisfying a natural notion of locality [4]. He showed that in a *Gedankenexperiment of Bohm'* (a variant of that of EPR) no local hidden-variable theory can reproduce all of the statistical predictions of quantum mechanics. In order to carry out Bell's idea we introduce the main aspects of Alain Aspect work [71]. In the following a picture of the experimental setup :

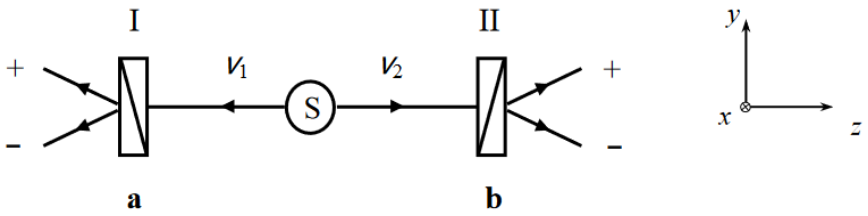

Figure 4: Einstein-Podolsky-Rosen-Bohm Gedankenexperiment with photons [71].

In which we have a source (S) emitting pairs of photons with frequencies ν_1 and ν_2, counterpropagating along Oz. We perform linear polarization measurements on the two photons, with analysers I and II. The analyser I, in orientation **a**, is followed by two detectors, giving results $+$ or $-$, corresponding to a linear polarization found parallel or perpendicular to **a**. The analyser II, in orientation **b**, acts similarly. Suppose that the polarization part of the state vector describing the pair is:

$$|\Psi(\nu_1,\nu_2)\rangle = \frac{1}{\sqrt{2}}(|x\rangle\otimes|x\rangle + |y\rangle\otimes|y\rangle) \tag{5.3}$$

It is crucial to note that the joint state is not factorizable, therefore this can only be thought of globally, as an *entangled state*. It is easy to derive the quantum mechanical predictions for these measurements of polarization. The probability to measure the polarization of a single photon, given the orientation of the detectors **a**, **b**, is equal to:

$$P_+(a) = P_-(a) = \frac{1}{2} \tag{5.4}$$

$$P_+(b) = P_-(b) = \frac{1}{2}, \tag{5.5}$$

while the joint probabilities are:

$$P_{++}(a,b) = P_{--}(a,b) = \frac{1}{2}\cos^2(a,b) \tag{5.6}$$

$$P_{+-}(a,b) = P_{-+}(a,b) = \frac{1}{2}\sin^2(a,b) \tag{5.7}$$

A convenient way to measure the amount of correlations between random quantities is to calculate the correlation coefficient. For the polarization measurements considered above, it is equal to :

$$E_{QM}(a,b) = P_{++}(a,b) + P_{--}(a,b) - P_{+-}(a,b) - P_{-+}(a,b) = \cos 2(a,b) \tag{5.8}$$

We note that in case of parallel polarizers $(a,b) = 0$ we have perfect correlations, i.e.

$E_{QM} = 1$, so, even if the individual measurement gives random results, these random results are correlated, as expressed by equation (5.8). For parallel (or perpendicular) orientations of the polarizers, the correlation is total ($|E_{QM}| = 1$).

5.2.1 A counterintuitive example

From the previous paragraphs, we have seen that quantum correlations show properties that challenge intuition. Further evidence of this unusual behavior is illustrated by a two-step experiment. First of all, we assume that we have measured only the state of the first photon along the direction **a**, obtaining as a result + with probability $\frac{1}{2}$. Of course, for the collapse of the wave function, we know that the first particle is in the state $|\mathbf{a}\rangle_1$, while the second one is still undefined. We have to project the initial state on a new base $\{|\mathbf{a},x\rangle, |\mathbf{a},y\rangle\}$. We get:

$$\left|\Psi'(\nu_1,\nu_2)\right\rangle = \frac{1}{C}(\langle\mathbf{a}|_1 \otimes \mathbb{I}_2)\,|\Psi(\nu_1,\nu_2)\rangle \tag{5.9}$$

where $\mathbf{a} = \alpha\,|x\rangle_1 + \beta\,|y\rangle_1$ and $\mathbb{I}$ is the identity matrix for system II. Writing explicitly we have:

$$\begin{aligned}\frac{1}{C}(\langle\mathbf{a}|_1 \otimes \mathbb{I}_2)\,|\Psi(\nu_1,\nu_2)\rangle &= \frac{1}{\sqrt{2}C}(\langle\mathbf{a}|_1 \otimes \mathbb{I}_2)(|x_1,x_2\rangle + |y_1,y_2\rangle) = \\ &= \frac{1}{\sqrt{2}C}(\alpha\,\langle x|_1\,|x\rangle_1\,|x\rangle_2 + \beta\,\langle y|_1\,|y\rangle_1\,|y\rangle_2) \\ &= \frac{1}{\sqrt{2}C}(\alpha\,|x\rangle_2 + \beta\,|y\rangle_2)\end{aligned} \tag{5.10}$$

By means normalization condition we obtain $C = \sqrt{\frac{(\alpha^2+\beta^2)}{2}}$ with $\alpha^2 + \beta^2 = 1$. We notice that the coefficients are the same for both the particles states. So finally:

$$\left|\Psi'(\nu_1,\nu_2)\right\rangle = |\mathbf{a}\rangle_1 \otimes |\mathbf{a}\rangle_2 \tag{5.11}$$

Immediately after the first measurement, photon ν_1 takes on the polarization **a**. This is straightforward, as the photon has passed through a polarizer aligned along **a**, and the outcome + was observed. What is more surprising, however, is that the distant photon ν_2, which has not interacted with any polarizer yet, is projected into the same polarization state **a**, having a polarization parallel to that of photon ν_1. While this conclusion seems counterintuitive, it leads to the correct final result. Using Malus' law, a subsequent measurement performed on the photon ν_2 along a direction **b** yields the probability $P_{++}(\mathbf{a},\mathbf{b})$, which is the probability of finding the photon ν_2 with polarization + along **b**, since the photon ν_1 was measured with polarization + along **a**. This probability is given by:

$$P_{++}(\mathbf{a},\mathbf{b}) = \frac{1}{2}\cos^2(\theta), \tag{5.12}$$

where θ is the angle between the polarization directions **a** and **b**. The calculation in two steps yields the same result as a direct computation. Additionally, this process suggests a physical interpretation of the two-step measurement: the first measurement on photon ν_1 immediately influences the polarization state of photon ν_2, even though photon ν_2 has not yet encountered a polarizer. This is a hallmark of quantum entanglement, where the measurement of one particle instantaneously determines the state of the other, regardless of the distance separating them.

5.2.2 Supplemetary parameters

The scenario presented above seems in contradiction with relativity. According to Einstein, what happens in a given region of space-time cannot be influenced by an event happening in a region of space-time that is separated by a space like interval. It therefore not unreasonable to try to find more acceptable pictures for understanding the EPR correlations. Following the intuition of Einstein-Podolski-Rosen these correlations could be explained by a local theory, assuming that the pairs emitted by the source share hidden properties that predetermine the measurement result. It is then sufficient to admit that half the pairs are emitted with the property $++$, and half with the property $--$, to reproduce all the results of measurement in this configuration. Note however that such properties, differing from one pair to another one, are not taken into account by the quantum mechanical state vector $|\Psi(\nu_1,\nu_2)\rangle$ which is the same for all pairs. Therefore, we don't know the actual state of the source, that in principle could contain supplementary information that allows us to determine with certainty the result of a measurement. This is why Einstein concludes that quantum mechanics is not complete. And this is why such additional properties are referred to as *supplementary parameters or hidden variables*. It can be hoped to recover the statistical quantum mechanical predictions when averaging over the supplementary parameters. We formalized the idea of a local theory more precisely. The assumption of locality implies that we should be able to identify a set of past factors, described by some variables λ, having a joint causal influence on both outcomes, and which fully account for the dependence between **a** and **b**. Once all such factors have been taken into account, the residual indeterminacies about the outcomes must now be decoupled; that is, the probabilities for **a** and **b** should factorize:

$$P(A,B|ab,\lambda) = P(A|a,\lambda)P(B|b,\lambda) \tag{5.13}$$

Where $A(\lambda,\mathbf{a}), B(\lambda,\mathbf{b}) \in (+1,-1)$ are the possible outcomes at the two polarizers. This factorability condition simply expresses the fact that we have found an explanation according to which the probability for **a** depends only on the past variables λ and on the local measurement $A(\lambda,\mathbf{a})$, but not on the distant measurement and outcome, and analogously for the probability to obtain $B(\lambda,\mathbf{b})$. The variable λ will not necessarily be constant for all runs of the experiment, even if the procedure which prepares the particles to be measured is held fixed, because λ may involve physical quantities that are not fully controllable. The different values of λ across the runs should thus be char-

acterized by a probability distribution $\rho(\lambda)$. Combined with the above factorizability condition, we can thus write

$$P(A,B|ab) = \int d\lambda \rho(\lambda) P(A|a,\lambda) P(B|b,\lambda). \tag{5.14}$$

A particular Supplementary Parameter Theory is completely defined by the explicit form of the function $\rho(\lambda)$, $A(\lambda,\mathbf{a})$ and $B(\lambda,\mathbf{b})$. It is then easy to express the probabilities of the various results of measurements. For instance, noting that the function $\frac{1}{2}(A(\lambda,\mathbf{a})+1)$ assumes the value $+1$ for the $+$ result, and 0 otherwise (and similarly $\frac{1}{2}(1-B(\lambda,\mathbf{b}))$ assumes the value +1 for the $-$ result, and 0 otherwise, we can write

$$P(+,-|ab) = \int d\lambda \rho(\lambda) \frac{1}{2}(A(\lambda,\mathbf{a})+1) \frac{1}{2}(1-B(\lambda,\mathbf{b}))$$

Similarly, the correlation function assumes the simple form

$$E(a,b) = \int \rho(\lambda) A(\lambda,\mathbf{a}) B(\lambda,b)$$

Now let's suppose we have two possible orientation for each polarizer and we choose randomly one of those. In this situation we have four possible configurations: $(\mathbf{a},\mathbf{b})$,$(\mathbf{a},\mathbf{b'})$,$(\mathbf{a'},\mathbf{b})$,$(\mathbf{a'},\mathbf{b'})$. To derive the Bell'inequality it is necessary to introduce a linear combination of the correlation coefficients, one for each possible configuration:

$$S(\mathbf{a},\mathbf{a'};\mathbf{b},\mathbf{b'}) = E(\mathbf{a},\mathbf{b}) - E(\mathbf{a},\mathbf{b'}) + E(\mathbf{a'},\mathbf{b}) + E(\mathbf{a'},\mathbf{b'}) \tag{5.15}$$

introducing the explicit form of the correlations we get

$$S(\mathbf{a},\mathbf{a'};\mathbf{b},\mathbf{b'}) = \int d\lambda \rho(\lambda) s(\mathbf{a},\mathbf{a'};\mathbf{b},\mathbf{b'};\lambda)$$

where

$$|s(\mathbf{a},\mathbf{a'};\mathbf{b},\mathbf{b'};\lambda)| = |A(\lambda,\mathbf{a})(B(\lambda,b) - B(\lambda,b')) + A(\lambda,\mathbf{a'})(B(\lambda,b) + B(\lambda,b'))| \leq 2$$

and finally *CHSH* formulation is derived

$$|E(\mathbf{a},\mathbf{b}) - E(\mathbf{a},\mathbf{b'}) + E(\mathbf{a'},\mathbf{b}) + E(\mathbf{a'},\mathbf{b'})| \leq 2 \tag{5.16}$$

In conclusion we can assert that, if a local theory exist, it must fullfil this inequality. Note that no assumptions of determinism or of a "classical behavior" are being involved in Eq.(5.16) : we assumed that **A** (and similarly **B**) is only probabilistically determined by the orientation of the polaryzer **a** and the variable λ, with no restrictions on the physical laws governing this causal relation.

5.2.3 Comparison between Classical and Quantum predictions

We now want to compare quantum predictions with newly derived inequalities and for this purpose we recall:

$$S_{QM} = \cos 2(a,b) - \cos 2(a,b') + \cos 2(a',b) + \cos 2(a',b') \tag{5.17}$$

If we assume that the angle ($\theta = 2|a-b|$) between the two polarizers is the only relevante quantity, and $(a,b') = (a,b) + (a',b) + (a',b')$, then the maximum will accour for:

$$(a,b) = (a',b) = (a',b') = \theta \tag{5.18}$$

and

$$\sin\theta = \sin 3\theta \tag{5.19}$$

than we get the maximum violation of Bell'inequalities, since:

$$S_{QM} = 2\sqrt{2} \quad for \quad \theta = \pm\frac{\pi}{8} \tag{5.20}$$

$$S_{QM} = -2\sqrt{2} \quad for \quad \theta = \pm\frac{3\pi}{8} \tag{5.21}$$

The cornerstone of Bell's arguments, as emphasized in all his works, is the assumption of locality in the formalism. Specifically, it is assumed that the result $A(\lambda,a)$, representing the outcome of the measurement at polarizer I, is independent of the orientation b of the distant polarizer II, and vice versa. Similarly, the probability distribution $\rho_\lambda(\lambda)$, which describes how particle pairs are emitted, is also presumed to be independent of the orientations a and b. This locality assumption is critical because Bell's inequalities would not hold without it. If the measurement outcomes or the probability distribution were allowed to depend on both polarizer settings, $A(\lambda,a,b)$ or $\rho_\lambda(\lambda,a,b)$, the derivation of Bell's inequalities would fail. To summarize, two key hypotheses are necessary for Bell's inequalities to arise, and consequently, for the conflict between quantum and classical mechanics to emerge:

1. *Distant correlations can be explained by introducing supplementary parameters, or hidden variables, carried by the particles, consistent with Einstein's notion that spatially separated objects possess distinct physical realities.*

2. *The quantities $A(\lambda)$, $B(\lambda)$, and $\rho(\lambda)$ adhere to the locality condition, meaning they are unaffected by the orientations of distant polarizers.*

This underlines the common assertion that quantum mechanics challenges Local Realism.

5.3 Mathematical characterization of non local correlations

The previous paragraph served as an introduction to the Bell inequalities, introduced by a simple experiment carried out by two distant observers who had two possible configurations of the experimental setup and two possible outcomes for each of them. The aim of this section is to generalize what has been seen above, introducing a wider range of possible configurations and results. We are always dealing with two remote observers, Alice and Bob, who conduct measurements on a common physical system, such as a pair of entangled particles, therefore each observer can select from a set of **m** distinct measurements to perform on their respective system. These measurements can each result in Δ possible outcomes. To abstractly depict this scenario, we refer to Alice and Bob's setup as a "black box". Here, each observer independently selects an input (a measurement setting), and the black box generates an output (a measurement outcome) accordingly. This scenario is commonly known as a *Bell scenario*. Let $p(ab|xy)$ denote the joint probability of obtaining output pair (a,b) given input pair (x,y). In the context of a Bell scenario, these joint probabilities, totaling $\Delta^2 m^2$, offer a comprehensive characterization of the scenario.We label the set $\mathbf{p} = \{p(ab|xy)\}$ as a *behavior*. Casually, we dub them the correlations dictating the operation of the shared black box between Alice and Bob. Conceptually, a behavior maps to a point p within the probability space $P \subset \mathbb{R}^{\Delta^2 m^2}$, subject to positivity constraints $p(ab|xy) \geq 0$ and normalization constraints $\sum_{a,b} p(ab|xy) = 1$. Due to these constraints, P forms a subspace of $\mathbb{R}^{\Delta^2 m^2}$ with a dimension of $dim\ P = (\Delta^2 - 1)m^2$, but the presence of a specific physical model underlying the observed correlations in a Bell scenario imposes further constraints on the behaviors **p**, allowing us to identify three primary types of correlation. To address this issue, we will rely on the review of Brunner et all [72], which contains an extensive analysis of the concept of non-locality. One of those constrain has been already presented in the previous paragraph, where we have employed the locality condition, which is equivalent to factorizability of the joint probability as it is shown in (5.14). All the elements of P that satisfy this condition belong to the set of *local behaviors* $\mathscr{L}$. Clearly, this is not the only existing constraint, and below we will define two additional sets and how they relate to each other.

5.3.1 No-signaling correlations

The more general constraints on the behaviour **p** that can be considered are the *no signaling* constrains. [73, 74]. They are formalized in the following way:

$$\sum_{b=1}^{\Delta} p(ab|xy) = \sum_{b=1}^{\Delta} p(ab|xy'), \qquad for\ all \qquad a,x,y,y' \tag{5.22}$$

$$\sum_{a=1}^{\Delta} p(ab|xy) = \sum_{a=1}^{\Delta} p(ab|x'y), \qquad for\ all \qquad b,y,x,x' \tag{5.23}$$

The physical interpretation of these constraints is that Alice's local marginal probabilities are independent of Bob's measurement setting y, so Bob cannot communicate with

her based on his input (and vice versa). Let $\mathscr{NS}$ denote the set of behaviors satisfying the no-signaling constraints (5.22). It ha been proved in [75] that $\mathscr{NS}$ is an affine subspace of $\mathbb{R}^{\Delta^2 m^2}$ of dimension:

$$dim\mathscr{NS} = 2(\Delta-1)m + (\Delta-1)^2 m^2 =: t. \tag{5.24}$$

While every local behavior adheres to the no-signaling constraint, the opposite isn't always true. Some no-signaling behaviors exist that don't meet the criteria for locality. Consequently, the collection of local correlations is definitively less extensive than that of no-signaling correlations; that is, $\mathscr{L} \subset \mathscr{NS}$. Correlations that cannot be written in the form (5.14) are said to be *non local*.

5.3.2 Quantum correlations

The last type of correlations to be defined is that of quantum correlations, in the set $\mathscr{Q}$, which corresponds to a subset of P that can be written as

$$p(ab|xy) = tr(\rho_{AB} M_{a|x} \otimes M_{b|y}) \tag{5.25}$$

Where ρ_{AB} is the density matrix related to the joint quantum state in $\mathscr{H}_{\mathscr{A}} \otimes \mathscr{H}_{\mathscr{B}}$ of aribitrary dimension, while $M_{a|x}$ and $M_{b|y}$ are the measurment operators on Alice and Bob setup respectively. It is important to note that, without loss of generality, we can always assume the quantum state to be pure and the measurement operators to be orthogonal projectors. If necessary, this can be achieved by extending the dimension of the Hilbert space. Thus, a quantum behavior can equivalently be written as

$$p(ab|xy) = \langle \psi | M_{a|x} \otimes M_{b|y} | \psi \rangle, \tag{5.26}$$

where $M_{a|x} M_{a'|x} = \delta_{aa'} M_{a|x}$ and $\sum_a M_{a|x} = \mathbb{I}_A$, with a similar definition for the operators $M_{b|y}$. Another way of defining quantum behaviors is by relaxing the requirement of a tensor product structure between Alice's and Bob's systems. Instead, we simply require that their local operators commute [76]. The corresponding set is denoted by Q_0, meaning that a behavior $\mathbf{p}$ belongs to Q_0 if

$$p(ab|xy) = \langle \psi | M_{a|x} M_{b|y} | \psi \rangle, \tag{5.27}$$

where $|\psi\rangle$ is a state in a Hilbert space $\mathscr{H}$, and $M_{a|x}$ and $M_{b|y}$ are orthogonal projectors on $\mathscr{H}$ that define valid measurements and satisfy the commutation relation $[M_{a|x}, M_{b|y}] = 0$. In the following sections, we discuss the properties of $\mathscr{L}$, $\mathscr{Q}$, and $\mathscr{NS}$ in more detail. In particular, we see how it is possible to decide if a given behavior belongs or not to one of these sets. We show how each set can be characterized in terms of Bell-type inequalities and discuss how to compute bounds for Bell-type expression for behaviors in $\mathscr{L}$, $\mathscr{Q}$, and $\mathscr{NS}$.

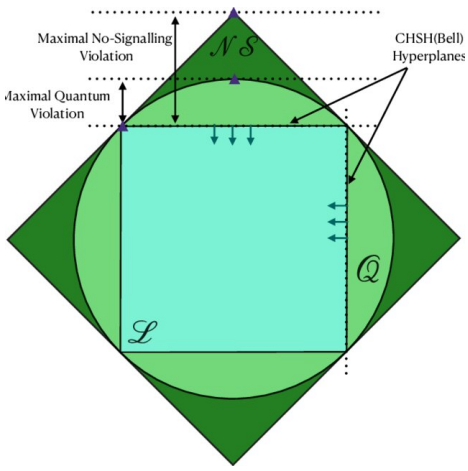

Figure 5: Depiction of the no-signaling ($\mathcal{NS}$), quantum ($\mathcal{Q}$), and local ($\mathcal{L}$) sets. It's noteworthy that there are strict inclusions $\mathcal{L} \subset \mathcal{Q} \subset \mathcal{NS}$. Furthermore, $\mathcal{NS}$ and $\mathcal{L}$ are polytopes, meaning they can be defined as the convex combination of a finite number of extremal points. On the other hand, the set $\mathcal{Q}$ is convex but not a polytope. The hyperplanes defining the set $\mathcal{L}$ correspond to Bell inequalities [72].

5.3.3 The local polytope and Bell inequalities

The next step is to investigate how Bell inequalities, i.e., the hyperplanes characterizing the set $\mathcal{L}$, can be found. For this goal we rely on the review of V.Scarani [77] in which an example is introduced explaining the mechanisms underlying classical correlations. Suppose that because of an extraordinary committee, employees of an investment firm start *simultaneously* to sell and buy shares. This simultaneous operation is not surprising, since we expect them to act according to precise directives or even to have organized themselves in advance before any operation. This example illustrates *the only two classical mechanisms that explain correlations between distant parties: communication (a.k.a. signaling) and pre-established agreement*. We focus now on the second aspect avoiding *signaling* for the moment. The default agreements require the parties to plan their behavior for each input pair in advance, so that each party can produce the result alone, rather than from the other side. This means that exist some parameter λ, such that:

$$p(\lambda|x,y) = p(\lambda). \tag{5.28}$$

A condition defined as *measurement independence*, which allow to factorize the probability as in (5.13). Integrating over λ, we get (5.14). In this example, we naturally assumed that the agents *know exactly* what to do once they learn the result of the committee. However, no such restriction was imposed on the mathematics: the probabilities $p(a|x,\lambda)$ and $p(b|y,\lambda)$ are only required to be valid probability distributions.

Deterministic local variables (DLVs) are a special case where, for any given input, the outcome is fully determined by λ. In this case, we have:

$$p(a|x,\lambda) = \begin{cases} 1 & \text{if exists } \lambda \text{ that determines } a \text{ given } x, \\ 0 & \text{otherwise.} \end{cases} \tag{5.29}$$

and

$$p(b|y,\lambda) = \begin{cases} 1 & \text{if exists } \lambda \text{ that determines } b \text{ given } y, \\ 0 & \text{otherwise.} \end{cases} \tag{5.30}$$

Another way to describe deterministic local variables is by listing the outcomes for all possible inputs:

$$\lambda_{\text{DL}} \equiv \{a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_m\} \in A^{|X|} \times B^{|Y|}, \tag{5.31}$$

From this, it's clear that the number of deterministic local points is $\Delta^m \times \Delta^m$.
The significance of deterministic local variables is demonstrated by the following result, first proved by Fine in 1982 [78]:

Proposition: *A family of probability distributions $p_{X,Y}$ can be explained by pre-established agreement if and only if it can be explained by deterministic local variables.*

Proof. The "if" direction is straightforward. The inverse is not so obvious because we need to prove that, for each fixed λ, a deterministic model has the same statistics of the initial stochastic model. To simplify things, let's label all the possible outcomes for Alice (the same holds for Bob) and group together in a set : $A = \{1, 2, \ldots, m\}$. From this we can introduce the cumulative distribution $\Sigma(a) = \sum_{\alpha \leq a} p(\alpha|x,\lambda)$, where $a \in A$. It can clearly be computed in the local variable (LV) model. Now, we introduce a new local parameter μ_A, belonging to uniform distribution, and then output a according to the following deterministic rule:

$$p_d(a|x,\lambda,\mu) = \begin{cases} 1 & \text{if } \Sigma(a-1) \leq \mu_A < \Sigma(a), \\ 0 & \text{otherwise.} \end{cases} \tag{5.32}$$

If μ_A is drawn from a uniform distribution, the original stochastic distribution is recovered:

$$\int_0^1 d\mu\, p_d(a|x,\lambda,\mu_A) = \int_{\Sigma(a-1)}^{\Sigma(a)} d\mu_A = p(a|x,\lambda). \tag{5.33}$$

Thus, equation (5.14) can be rewritten as:

$$p_{\text{LV}}(a,b|x,y) = \int d\lambda\, \rho(\lambda) \int_0^1 d\mu_A \int_0^1 d\mu_B\, p_d(a|x,\lambda,\mu_A) p_d(b|y,\lambda,\mu_B), \tag{5.34}$$

which is the desired convex sum of deterministic local variables (LVs) for the extended variable $\lambda' \equiv (\lambda, \mu_A, \mu_B)$ with distribution $\rho'(\lambda')d\lambda' = \rho(\lambda)d\lambda d\mu_A d\mu_B$. It immediately follows that the finite set of λ's, defined by the Δ^{2m} deterministic local points, is sufficient to describe any LV statistics. Each $p_d(a|x,\lambda,\mu)$, given by equation (5.32), is one of the Δ^m deterministic points for A, and similarly for B ■.
What has been just tested is valid for any pair a and b and any pair of outcomes, therefore it is possible to define the behavior $\mathbf{p}$ in terms of:

$$d_\lambda(a,b|x,y) = \begin{cases} 1 & \text{if exist } \lambda \text{ that determines } a,b \\ 0 & \text{otherwise} \end{cases} \tag{5.35}$$

Since $\mathbf{d}_\lambda \in \mathscr{L}$ we finally have:

$$\mathbf{p} = \sum_\lambda \rho_\lambda \mathbf{d}_\lambda \quad \blacksquare. \tag{5.36}$$

Bell inequalities are a consequence of properties of the set $\mathscr{L}$ as shown in [72]. The latter is a compact, closed and convex set (the same holds for $\mathscr{Q}$ and $\mathscr{NS}$ [79]). That is, if $\mathbf{p}_1$ and $\mathbf{p}_2$ belong to the local set, the the mixture $\mu\mathbf{p}_1 + (1-\mu)\mathbf{p}_2$ is also in the set. According to the hyperplane separation theorem, for each behavior $\hat{\mathbf{p}} \in \mathbb{R}^t$ (where $t = dim\mathscr{L}$) that does not belong to the set, there exists a hyperplane that separates $\hat{\mathbf{p}}$ from the corresponding set. In other words, if $\hat{\mathbf{p}} \notin \mathscr{L}$, then there is an inequality of the form:

$$\mathbf{s}\cdot\mathbf{p} = \sum_{abxy} s_{ab}^{xy} p(ab|xy) \leq S_L. \tag{5.37}$$

The coefficients s_{ab}^{xy} depend on the specific Bell expression and they can be $+1$ or -1. The inequality is satisfied by all $\mathbf{p} \in \mathscr{L}$, but violated by $\hat{\mathbf{p}}$ because $\mathbf{s}\cdot\hat{\mathbf{p}} > S_L$. In the case of the local set $\mathscr{L}$, these inequalities are simply the Bell inequalities. Since the set $\mathscr{L}$ is the convex hull of a finite number of points, it forms a *polytope*. The local deterministic behaviors $\mathbf{d}_\lambda$ represent the vertices, or extreme points, of this polytope. According to Minkowski's theorem [80], a fundamental result in polyhedral theory, a polytope can be described in two ways: either as the convex hull of its vertices or as the intersection of a finite number of half-spaces. Therefore, we can express:

$$\mathbf{p} \in \mathscr{L} \quad \text{if and only if} \quad \mathbf{s}_i \cdot \mathbf{p} \leq S_i^L \quad \forall i \in I, \tag{5.38}$$

where I represents a finite set of linear inequalities. Conversely, if $\mathbf{p}$ is non-local, it will violate at least one of these inequalities in Eq. (5.38). Therefore, the local set $\mathscr{L}$ can be fully characterized by a finite set of Bell inequalities.

5.3.4 Quantum bound: The Tsirelson bound

Before delving into the detailed computation of the quantum bound S_q associated with a Bell expression, we provide a brief overview of the structure of the quantum set $\mathscr{Q}$

[72]. Recall that a behavior p is considered quantum if, as defined in Eq. (5.25), it can be expressed as $p(ab|xy) = \langle \psi | M_{a|x} | M_{b|y} | \psi \rangle$, where $|\psi\rangle$ is a state in a Hilbert space $\mathscr{H}$, and $M_{a|x}$ and $M_{b|y}$ are orthogonal projectors on $\mathscr{H}$ representing proper measurements, satisfying $\left[M_{a|x}, M_{b|y}\right] = 0$. (For the characterization of the quantum set, it is convenient to assume commutation relations rather than a tensor product structure, and we adopt this approach throughout the remainder of this section.) As previously mentioned, the local set $\mathscr{L}$ is strictly contained within the quantum set $\mathscr{Q}$, implying the existence of quantum behaviors that exhibit nonlocality; hence, in general, $S_q > S_l$. Any quantum behavior must satisfy two fundamental requirements to be nonlocal: firstly, the measurements performed by Alice and Bob must be non-commuting [78], and secondly, the state ρ must be entangled. Let's now narrow our attention to the specific task of calculating the quantum bound for a Bell expression. Remember that the quantum set $\mathscr{Q}$, like any convex and compact set, can be characterized by an infinite system of linear inequalities, known as quantum Bell inequalities. For any arbitrary Bell expression $\mathbf{s}$, its corresponding quantum bound is defined as follows:

$$S_q = \max_{p \in \mathscr{Q}} \mathbf{s} \cdot \mathbf{p} = \max_{S} \|\hat{S}\|, \tag{5.39}$$

where

$$\hat{S} = \sum_{abxy} s_{ab}^{xy} M_{a|x} M_{b|y} \tag{5.40}$$

The Bell operator associated with $\mathbf{s}$, denoted as $\|\hat{S}\|$, represents the spectral norm (largest eigenvalue) of $\hat{S}$, and the optimization described above is performed over all possible Bell operators S associated with s. This optimization encompasses all conceivable measurements $M_{a|x}$ and $M_{b|y}$, where the coefficients s_{ab}^{xy} are determined by the choice of $\mathbf{s}$. In the specific case of the *CHSH* expression, the Bell operator is defined as $\hat{S} = \hat{A}_1 \otimes \hat{B}_1 + \hat{A}_1 \otimes \hat{B}_2 + \hat{A}_2 \otimes \hat{B}_1 - \hat{A}_2 \otimes \hat{B}_2$, where $\hat{A}_x$ and $\hat{B}_y$ represent arbitrary ± 1-eigenvalued observables. In particular, a quantum state ρ violates the CHSH inequality if and only if there exist measurements such that $\mathrm{Tr}(\rho \hat{S}) > 2$. Before introducing specific states that violate the CHSH inequality, let us present a generic result, known as *Tsirelson's bound* [81] and derived in V.Scarani work [77]:

Theorem 3.1 : *Measurements on quantum systems can violate the CHSH inequality at most up to* $S \leq 2\sqrt{2}$.

Proof : To prove this result, we need to find an upper bound for the largest eigenvalue of $\hat{S}$, denoted as $||\hat{S}||_\infty$. By construction, we have $||\hat{A}_x||_\infty = ||\hat{B}_y||_\infty = \mathbb{I}$, $\hat{A}_x^2 = \mathbb{I}_{d_A}$, and $\hat{B}_y^2 = \mathbb{I}_{d_B}$, where d_A and d_B represent the dimensions of Hilbert spaces, which can be left unspecified and may even be infinite. To obtain the bound, we work with the square of the CHSH operator:

$$\hat{S}^2 = 4\,\mathbb{I} \otimes \mathbb{I} - [\hat{A}_1, \hat{A}_2] \otimes [\hat{B}_1, \hat{B}_2] \tag{5.41}$$

Indeed, $||[\hat{A}_1,\hat{A}_2]||_\infty = ||\hat{A}_1\hat{A}_2 - \hat{A}_1\hat{A}_2||_\infty \leq ||\hat{A}_1\hat{A}_2||_\infty + ||\hat{A}_2\hat{A}_1||_\infty \leq 2||\hat{A}_1||_\infty||\hat{A}_2||_\infty = 2$, where the last inequality follows from the general property $|xy| \leq |x||y|$.
Similarly, we find $||[\hat{B}_0,\hat{B}_1]||_\infty \leq 2$. Therefore, we have:

$$||\hat{S}^2||_\infty \leq 8 \quad \blacksquare. \tag{5.42}$$

which proves the claim.

3.4.5.1 Entanglement and non-locality for two quibit state

In Section (5.2), we derived Bell's inequalities starting from a maximally entangled state, observing that, with appropriate orientations of the polarizers, it is possible to achieve the maximum violation of the inequalities, consistent with the Tsirelson's bound (5.42). It is therefore important to explore the relationship between entanglement and non-locality in a generic two-qubit state:

$$|\Psi(\theta)\rangle = \cos\theta|0\rangle\otimes|0\rangle + \sin\theta|1\rangle\otimes|1\rangle, \tag{5.43}$$

characterized by the following density matrix :

$$\rho = \frac{1}{4}\left(\mathbb{I}\otimes\mathbb{I} + \mathbf{r}_\rho\cdot\boldsymbol{\sigma}\otimes\mathbb{I} + \mathbb{I}\otimes\mathbf{s}_\rho\cdot\boldsymbol{\sigma} + \sum_{i,j=x,y,z} T_\rho^{ij}\sigma_i\otimes\sigma_j\right). \tag{5.44}$$

The first term $\mathbb{I}\otimes\mathbb{I}$ represents the identity operator for both qubits, indicating the "unpolarized" or maximally mixed part of the density matrix. The second $\mathbf{r}_\rho\cdot\boldsymbol{\sigma}\otimes\mathbb{I}$: Here, $\mathbf{r}_\rho$ is the Bloch vector representing the local polarization of the first qubit, and $\boldsymbol{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$ are the Pauli matrices. This term describes the effect of the first qubit's polarization while leaving the second qubit unchanged. $\mathbb{I}\otimes\mathbf{s}_\rho\cdot\boldsymbol{\sigma}$: Similarly, $\mathbf{s}_\rho$ is the Bloch vector for the second qubit. This term describes the second qubit's local polarization, while the first qubit remains unchanged.
$\sum_{i,j=x,y,z} T_\rho^{ij}\sigma_i\otimes\sigma_j$: This term accounts for the correlations between the two qubits. The elements T_ρ^{ij} form a 3×3 matrix that describes how measurements on the first qubit along the axis i (where $i = x,y,z$) are correlated with measurements on the second qubit along the axis j.

Theorem 3.2: *The maximum value of the CHSH inequality, achievable using von Neumann measurements on any two-qubit state ρ, is given by*:

$$\langle\hat{S}\rangle = 2\sqrt{\lambda_1+\lambda_2} \tag{5.45}$$

where λ_1 and λ_2 are the two largest eigenvalues of the symmetric matrix $T_\rho^t T_\rho$, and T_ρ^t is the transpose of T_ρ. The proof also outlines how to choose measurement settings that reach this maximum [26].

Proof: To find the maximum value of $\langle \hat{S} \rangle = \mathrm{Tr}(\rho \hat{S})$, we begin with the following expression:

$$\langle \hat{S} \rangle = \mathbf{a}_1 \cdot \big(T_\rho(\mathbf{b}_1 + \mathbf{b}_2)\big) + \mathbf{a}_2 \cdot \big(T_\rho(\mathbf{b}_1 - \mathbf{b}_2)\big) \tag{5.46}$$

Here, we take advantage of the fact that the sum and difference of two vectors are always orthogonal. The vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are chosen such that $(\mathbf{b}_1 \cdot \mathbf{b}_2) = \cos 2\chi$, and $\mathbf{a}_1$ and $\mathbf{a}_2$ are unit vectors.We can use this information for simplify the previous average value, since:

$$|\mathbf{b}_1 + \mathbf{b}_2|^2 = |\mathbf{b}_1|^2 + |\mathbf{b}_2|^2 + 2\mathbf{b}_1 \cdot \mathbf{b}_2 \tag{5.47}$$

we recall that $\mathbf{b}_1$ and $\mathbf{b}_2$ are unit vectors so we get

$$|\mathbf{b}_1 + \mathbf{b}_2|^2 = 2 + 2\cos(2\chi) \tag{5.48}$$

And the magnitude of the sum is:

$$|\mathbf{b}_1 + \mathbf{b}_2| = \sqrt{2 + 2\cos(2\chi)} \tag{5.49}$$

We can simplify $\cos(2\chi)$ using the trigonometric identity $\cos(2\chi) = 2\cos^2(\chi) - 1$. Substituting this into the previous expression, we obtain:

$$|\mathbf{b}_1 + \mathbf{b}_2| = \sqrt{2 + 2(2\cos^2(\chi) - 1)} = \sqrt{4\cos^2(\chi)} = 2\cos(\chi). \tag{5.50}$$

Since the magnitude of $\mathbf{b}_1 + \mathbf{b}_2$ is $2\cos(\theta)$ and the vector $\mathbf{b}_1 + \mathbf{b}_2$ are orthogonal to $\mathbf{b}_1 - \mathbf{b}_2$, it must be parallel to a unit vector $\mathbf{c}$, while $\mathbf{b}_1 - \mathbf{b}_2$ must be parallel to $\mathbf{c}^\perp$. Therefore, we can write:

$$\begin{aligned} \mathbf{b}_1 + \mathbf{b}_2 &= 2\cos(\chi)\mathbf{c} \\ \mathbf{b}_1 - \mathbf{b}_2 &= 2\cos(\chi)\mathbf{c}^\perp \end{aligned} \tag{5.51}$$

Looking for the maximum is equivalent to find the four directions such that:

$$\max_{\mathbf{a}_1,\mathbf{a}_2,\mathbf{b}_1,\mathbf{b}_2} \langle \hat{S} \rangle = \max_{\mathbf{b}_1,\mathbf{b}_2} 2\left(\cos\chi ||T_\rho \mathbf{c}|| + \sin\chi ||T_\rho \mathbf{c}^\perp||\right) \tag{5.52}$$

Next, using the well-known optimization formula $\max_\chi x\cos\chi + y\sin\chi = \sqrt{x^2+y^2}$, achievable when $\cos(\chi) = x/\sqrt{x^2+y^2}$. The expression becomes:

$$2\max_{\mathbf{c},\mathbf{c}^\perp} \sqrt{||T_\rho \mathbf{c}||^2 + ||T_\rho \mathbf{c}^\perp||^2} \tag{5.53}$$

Finally, the norm $||T_\rho \mathbf{c}||^2$ is given by the expression $(\mathbf{c} \cdot T_\rho^t T_\rho \mathbf{c})$, where $T_\rho^t T_\rho$ is a symmetric and positive matrix. This ensures that the optimization is achieved by choosing

$\mathbf{c}$ and $\mathbf{c}^{\perp}$ as the two eigenvectors corresponding to the two largest eigenvalues of $T_{\rho}^{t}T_{\rho}$. Thus, the maximum violation of the CHSH inequality is achieved with these specific measurement settings, proving the bound $\langle\hat{S}\rangle_{max} = 2\sqrt{\lambda_1 + \lambda_2}$, where λ_1 and λ_2 are the two largest eigenvalues of $T_{\rho}^{t}T_{\rho}$ ■.
This proof also provides the method to reconstruct the optimal measurement settings that lead to the maximal quantum violation of the CHSH inequality. The correlation matrix $T_{\Psi(\theta)}$ matrix related to (5.43) is given by:

$$T_{\Psi(\theta)} = \begin{pmatrix} \sin 2\theta & 0 & 0 \\ 0 & -\sin 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.54}$$

The maximum value of the CHSH expression for this state can be calculated as:

$$\langle\hat{S}\rangle_{max} = \max_{\mathbf{a}_1,\mathbf{a}_2,\mathbf{b}_1,\mathbf{b}_2} \langle\Psi(\theta)|\hat{S}|\Psi(\theta)\rangle = 2\sqrt{1+\sin^2 2\theta} \tag{5.55}$$

This value is always greater than 2 unless $\sin 2\theta = 0$, which corresponds to the unentangled (product) states. Thus, *all pure entangled states of two qubits violate the CHSH inequality for appropriately chosen measurement settings*. Furthermore, only maximally entangled states can reach the maximal violation $S = 2\sqrt{2}$.
To determine the corresponding measurement settings, consider that the eigenvector associated with the largest eigenvalue of $T_{\rho}^{t}T_{\rho}$ is $\mathbf{c} = \hat{z}$. The orthogonal subspace, being degenerate, allows for the choice of any vector; for instance, we can take $\mathbf{c}^{\perp} = \hat{x}$. With this choice, we have:

$$\mathbf{b}_1 = \cos(\chi)\hat{z} + \sin\chi\hat{x}, \quad \mathbf{b}_2 = \cos\chi\hat{z} - \sin\chi\hat{x} \tag{5.56}$$

where $\cos\chi = \frac{1}{\sqrt{1+\sin^2 2\theta}}$. Moreover, $\mathbf{a}_1$ must be the unit vector aligned with $T_{\rho}\mathbf{c}$, and $\mathbf{a}_2$ aligned with $T_{\rho}\mathbf{c}^{\perp}$, which gives:

$$\mathbf{a}_1 = \hat{z}, \quad \mathbf{a}_2 = \hat{x} \tag{5.57}$$

With this, we now have a complete understanding of how the CHSH inequality is violated by two-qubit states. In the next section, we will explore how the CHSH inequality can be adapted to demonstrate that all pure entangled states violate a Bell inequality.

5.3.5 Gisin's Theorem

The fact that all pure entangled states violate a Bell inequality is usually referred to as *Gisin's theorem*, since Nicolas Gisin was the first to ask the question and to answer it for bipartite states [82]. Popescu and Rohrlich extended the proof to the general case shortly after [83]. Here, I follow this procedure.

Lemma 3.1: *Any bipartite entangled pure state (i.e., of any dimensionality) violates a Bell inequality.*

Proof: We start with the Schmidt decomposition of a pure bipartite state $|\Psi\rangle$:

$$|\Psi\rangle = \sum_{k=0}^{d-1} c_k |k\rangle \otimes |k\rangle \tag{5.58}$$

where we can define the bases such that $c_0 \geq c_1 \geq ... \geq c_{d-1} \geq 0$. The state is entangled if and only if $c_1 \neq 0$.
We can now rewrite this state by separating out the two largest Schmidt coefficients, c_0 and c_1, as follows:

$$|\Psi\rangle = \sqrt{c_0^2 + c_1^2} \left(\frac{c_0}{\sqrt{c_0^2 + c_1^2}} |0\rangle \otimes |0\rangle + \frac{c_1}{\sqrt{c_0^2 + c_1^2}} |1\rangle \otimes |1\rangle \right) + \sqrt{1 - c_0^2 - c_1^2} |\Psi'\rangle \tag{5.59}$$

where $|\Psi'\rangle$ is the normalized projection of $|\Psi\rangle$ onto the subspace orthogonal to the span of $|0\rangle \otimes |0\rangle$ and $|1\rangle \otimes |1\rangle$. The state $|\Psi'\rangle$ is given by:

$$|\Psi'\rangle = \frac{1}{\sqrt{1 - c_0^2 - c_1^2}} \sum_{k=2}^{d-1} c_k |k\rangle \otimes |k\rangle \tag{5.60}$$

Thus, we can express the full state as:

$$|\Psi\rangle = \sqrt{c_0^2 + c_1^2} (\cos\theta |0\rangle \otimes |0\rangle + \sin\theta |1\rangle \otimes |1\rangle) + \sqrt{1 - c_0^2 - c_1^2} |\Psi'\rangle \tag{5.61}$$

where

$$\cos\theta = \frac{c_0}{\sqrt{c_0^2 + c_1^2}}, \quad \sin\theta = \frac{c_1}{\sqrt{c_0^2 + c_1^2}} \tag{5.62}$$

Now consider the operators

$$A_x = \mathbf{a}_x \cdot \boldsymbol{\sigma} \oplus \mathbb{I}', \quad B_y = \mathbf{b}_y \cdot \boldsymbol{\sigma} \oplus \mathbb{I}' \tag{5.63}$$

where Pauli matrices $\boldsymbol{\sigma}$ act in the subspace Span$(|0\rangle, |1\rangle)$, and

$$\mathbb{I}' = |2\rangle\langle 2| + \cdots + |d-1\rangle\langle d-1| \tag{5.64}$$

is the identity on the orthogonal complement of that subspace. By choosing the measurement vectors that lead to the optimal violation (5.56),(5.57), one can reach the value of the CHSH parameter S given by:

$$S = (c_0^2 + c_1^2)2\sqrt{1+\sin^2 2\theta} + (1 - c_0^2 - c_1^2)2 \tag{5.65}$$

which is larger than 2 as soon as $c_1 > 0$, as claimed.

5.4 New features testing Non-locality

Although Bell's theorem is conceptually clear, the experimental verification of Bell inequalities has historically encountered a variety of challenges, both technical and conceptual in nature. This section examines the obstacles that have arisen in the execution of these tests and outlines the historical development of such experiments. Bell's introduction of his inequalities was initially perceived as a theoretical instrument for investigating the principles of local realism. However, despite the clarity of the mathematical framework, the transition to experimental implementation presented significant difficulties. When Bell introduced his inequalities in 1964, no experimental design had yet been conceived that could practically test them. The challenge was twofold: first, a conceptual gap in translating Bell's theoretical framework into a testable experimental setup; and second, the experimental obstacles of the time, such as the difficulty in reliably generating and measuring entangled particles. Additionally, ensuring the spatial separation of measurement stations, which is essential to confirm that no signal could influence both measurements in accordance with relativity, required highly precise timing and detection capabilities that were beyond the technological capabilities of the era. These challenges were gradually overcome, and with the reformulation of Bell's inequality into the more experimentally accessible Clauser-Horne-Shimony-Holt (CHSH) inequality in 1969 [84], the first significant test was conducted in 1972. John Clauser and Stuart Freedman were among the first to conduct tests using photon pairs generated from atomic cascades [85] in which they provided evidence of a violation of Bell's inequality, suggesting that quantum mechanics could indeed predict correlations that local hidden variable theories could not elucidate. However, several critical limitations diminished the conclusiveness of their results, since a significant issue was the efficiency of the detectors utilized in the experiment, which could only detect a small fraction of the emitted photons, leaving open the possibility that the undetected photons might conform to local hidden variable theories. This concern, known as the *detection loophole*, remained a pivotal limitation in Bell tests for many years. Additionally, the experiment did not involve rapid changes in measurement settings, which raised the possibility that a hidden signal could affect both measurement stations. This introduced the *locality loophole*, which questioned whether the observed violations were genuinely indicative of quantum nonlocality or merely the result of some communication between the particles. The early 1980s marked a pivotal moment in the testing of Bell's inequalities, largely attributed to the experiments conducted by Alain Aspect and his team in France [71]. These investigations rectified several deficiencies found in prior tests and introduced critical innovations that significantly advanced the discipline. A key aspect of Aspect's work was the use of rapid-switching measurement settings. In these experiments, the settings of the measurement devices were modified while the entangled photons were already in motion, ensuring that the decision regarding mea-

surement settings was made after the photons had been emitted. This approach was vital in addressing the locality loophole, as it made it highly unlikely for any hidden signal to influence both measurements. Additionally, Aspect's experiments increased the distance between the detectors, further minimizing the potential for local communication between the particles. These innovations provided robust evidence supporting quantum mechanics, although the detection loophole continued to be a concern due to the limited efficiency of the photon detectors utilized at that time. The detection loophole has emerged as a particularly persistent challenge in the context of Bell tests. When detectors fail to capture all emitted particles from a source, it creates an opportunity for local hidden variable theories to account for the observed correlations by utilizing the undetected particles. To definitively eliminate the possibility of local hidden variable theories, it became essential to enhance the efficiency of the detectors to a point where the majority of emitted particles were successfully detected. This requirement presented a considerable obstacle, particularly in experiments involving photons. A first attempt for build up a free loopholes experiment was given by Philippe H. Eberhard in 1992 [86], by means an alternative inequality, equivalent to CHSH one. This new inequality does not require the fair sampling assumption (it counts for undetected events) and can be tested by detectors with lower efficiency ($\eta \approx 66.7\%$). A significant breakthrough that effectively tackled these issues emerged in 2013 , when various independent teams conducted loophole-free Bell tests. These experiments successfully closed both the detection and locality loopholes. For instance, Ronald Hanson's research group at Delft University employed entangled electrons in diamond, which were separated by distances greater than one kilometer, for their study [87]. Simultaneously, Anton Zeilinger's team carried out similar experiments utilizing entangled photons [88]. These investigations represented a critical milestone in the field, as they provided the first definitive tests of Bell's inequalities.

5.4.1 Escaping detection loophole

Gisin's theorem for pure states clarified the relationship between entangled pure states and non-locality. The maximum violation of Bell inequalities is observed for maximally entangled states. However, in experimental tests, we are often constrained by instrumentation with less-than-perfect efficiency. As we will see, this highlights an unexpected relationship between entanglement and non-locality. In this section we exploit the results of Méthot and Scarani [89]. They analyze the same physical setup as described in the previous section, where two binary measurements are performed on each qubit of an entangled pair. Ideally, each measurement yields one of two possible outcomes: $+$ or $-$. However, in a real experiment, a third possible outcome, denoted $\perp$, can occur. This corresponds to the case where the detector fails to register the particle, i.e., it does not fire. Physicists often assume *fair sampling*, meaning that the particles detected are a representative sample of the total particle set. In other words, the likelihood of a detector not firing is assumed to be completely uncorrelated with the two-qubit state being measured. To investigate the *detection loophole*, the CHSH inequality is rewritten in a form first derived by Clauser and Horne [90]. The inequality takes the following form:

$$Pr_{A_1B_1}[++] + Pr_{A_1B_2}[++] + Pr_{A_2B_1}[++] - Pr_{A_2B_2}[++] - Pr_{A_1}[+] - Pr_{B_1}[+] \leq 0. \tag{5.66}$$

We want to evaluate those probabilities related to a generic pure entangled state $|\psi(\theta)\rangle$ given by

$$|\psi(\theta)\rangle = \cos(\theta)\,|0,0\rangle + \sin(\theta)\,|1,1\rangle \tag{5.67}$$

where $\theta \in [0, \frac{\pi}{4}]$. When describing qubits, we follow the convention that $|0\rangle$ and $|1\rangle$ are the eigenstates of the Pauli matrix σ_z, corresponding to the eigenvalues $+1$ and -1, respectively. Any projective measurement on a qubit can be represented by a projection onto the eigenstates of a Pauli matrix $\mathbf{n}\cdot\boldsymbol{\sigma}$, where $\mathbf{n}$ is a normalized unit vector. Quantum mechanics indicates that when we perform measurements on the state $|\psi(\theta)\rangle$ with detectors of efficiency η, the measurement outcomes are influenced by the detector's efficiency. Specifically, the probability of detecting a particle is directly proportional to η, and thus the measurement statistics will reflect this detector efficiency. The results can be described by the following relations:

$$\begin{aligned} Pr_{A_i}[+] &= \eta\,\langle\psi(\theta)|\left(\frac{1}{2}(I+\mathbf{a}\cdot\boldsymbol{\sigma})\right)\otimes\mathbb{I}_2\,|\psi(\theta)\rangle = \frac{\eta}{2}\left(1+a_z^i\cos(2\theta)\right) \\ Pr_{B_i}[+] &= \eta\,\langle\psi(\theta)|\left(\frac{1}{2}(I+\mathbf{b}\cdot\boldsymbol{\sigma})\right)\otimes\mathbb{I}_2\,|\psi(\theta)\rangle = \frac{\eta}{2}\left(1+b_z^i\cos(2\theta)\right) \end{aligned} \tag{5.68}$$

for single counting. While for joint probabilities:

$$\begin{aligned} Pr_{A_i,B_j}[++] &= \langle\psi(\theta)|\left(\frac{\eta}{2}(I+\mathbf{a}\cdot\boldsymbol{\sigma})\right)\otimes\left(\frac{\eta}{2}(I+\mathbf{b}\cdot\boldsymbol{\sigma})\right)|\psi(\theta)\rangle \\ &= \eta^2\left(\frac{1}{4}+\frac{1}{4}b_z^j\cos(2\theta)+\frac{1}{4}a_z^i\cos(2\theta)+E(\mathbf{a}_i,\mathbf{b}_j)\right) \end{aligned} \tag{5.69}$$

In which we used the correlation function

$$\begin{aligned} E(\mathbf{a}_i,\mathbf{b}_j) &= \langle\psi(\theta)|\,(\mathbf{a}_i\cdot\boldsymbol{\sigma})\otimes(\mathbf{b}_j\cdot\boldsymbol{\sigma})\,|\psi(\theta)\rangle \\ &= a_z^i b_z^j + \sin(2\theta)\left(a_x^i b_x^j - a_y^i b_y^j\right). \end{aligned} \tag{5.70}$$

Inserting those relations inside (5.66) one finds that the inequality can be violated if and only if the efficiency of the detector is high enough, namely

$$\eta > \eta_c(\theta) = \min_{\mathbf{a}_i,\mathbf{b}_j}\left[\frac{4+2\cos(2\theta)(a_z^1+b_z^1)}{2+2\cos(2\theta)(a_z^1+b_z^1)+\langle\hat{S}\rangle_{max}}\right], \tag{5.71}$$

If we now recall the settings that maximized the expectation value of the Bell' operator from the Sec. (5.3.4) , we have:

$$\langle \hat{S} \rangle_{max} = 2\sqrt{1+\sin^2(2\theta)}. \tag{5.72}$$

The criterion (5.71) can be seen as a measure of non-locality, since (intuitively) the more non-local a state is, the easier its non-locality is to be revealed in an imperfect measurement. It is then easy to verify that, as θ increases, $\eta_c(\theta)$ decreases, with the minimum obtained at $\theta = \frac{\pi}{4}$. It thus seems that everything is as it should be. However, maximizing the function $\langle \hat{S} \rangle_{\max}$ is not equivalent to directly minimizing $\eta_c(\theta)$. In fact, if we assume to consider non-maximally entangled states ($\theta < \frac{\pi}{4}$), $\eta_c(\theta)$ is minimized by settings which are not those that maximize the violation of the CHSH inequality (even though they still give a violation), and moreover, with the optimized settings, $\eta_c(\theta)$ decreases as θ decreases. In particular, one has

$$\eta_c(\theta \to 0) \to \frac{2}{3}. \tag{5.73}$$

It could be argued that this peculiar behavior arises not from an intrinsic anomaly in non-locality or entanglement, but rather from the specific choice of the inequality under consideration. However, it has been proved in [91] that for a maximally entangled state and detectors with an efficiency $\eta \leq \frac{3}{4}$, there exists an explicit local model which recovers the quantum predictions. This means that for $\frac{2}{3} < \eta \leq \frac{3}{4}$, the maximally entangled state can in no way close the detection loophole, while some non-maximally entangled states can.

5.5 Werner states

It's now evident that the relationship between non-locality and entanglement is not so obvious. We know that entanglement is a necessary condition for the violation of Bell'inequalities, and we wonder if it is also sufficient. This section is meant to be a proof that this condition does not hold in general, so there could exist entangled states that can be described using an LHV model. Thanks to [77] we introduce this relevant result:

Theorem 3.3: *There exist mixed states that are entangled, but nevertheless cannot violate any Bell inequality.*

The states of interest are called *Werner states* and are mixed states that combine a pure maximal entangled state with a fully mixed one [24]. The general expression is the following:

$$\rho_w = p\left|\Psi^-\right\rangle\left\langle\Psi^-\right| + (1-p)\frac{\mathbb{I}}{4}, \quad p \in [0,1] \tag{5.74}$$

The joint probability of obtaining results a and b when Alice measures along **a** and Bob measures along **b** is given by a convex combination of two contributions, weighted by the parameter p:

$$P_w(a,b \mid \mathbf{a},\mathbf{b}) = pP_{\Psi^-}(a,b \mid \mathbf{a},\mathbf{b}) + (1-p)P_{\text{mixed}}(a,b) \tag{5.75}$$

where

$$\begin{aligned} P_{\Psi^-}(a,b \mid \mathbf{a},\mathbf{b}) &= \frac{1}{4}\left\langle \Psi^- \right| (\mathbb{I} + a(\mathbf{a}\cdot\sigma)) \otimes ((\mathbb{I} + b(\mathbf{b}\cdot\sigma)) \left| \Psi^- \right\rangle \\ &= \frac{1}{4}\left(1 - ab(\mathbf{a}\cdot\mathbf{b})\right), \\ P_{\text{mixed}}(a,b) &= \frac{1}{4}. \end{aligned} \tag{5.76}$$

Using the criterion of the negative partial transposition, it can be proved that Werner states are separable for $p \leq 1/3$ and entangled otherwise. The statistics of von Neumann measurements on Werner states are given by:

$$P_w(p) = \left\{ P(a,b \mid \mathbf{a},\mathbf{b}) = \frac{1}{4}\left(1 - ab(\mathbf{a}\cdot\mathbf{b})p\right), \quad a,b \in \{-1,+1\}, \quad \mathbf{a},\mathbf{b} \in \mathbb{S}^2 \right\} \tag{5.77}$$

Lemma 3.2: *The set $P_w(p)$ can be reproduced by a local hidden variable (LHV) model if $p \leq \frac{1}{2}$.*

Proof: It is sufficient to prove the case for $p = \frac{1}{2}$, as any state ρ_w with $p < \frac{1}{2}$ can be generated by mixing $\rho_{1/2}$ with white noise. In each run of the experiment, the pre-shared local variable is a vector λ, which is uniformly drawn from the unit sphere $\mathbb{S}^2$ with the distribution $\rho(\lambda)d\lambda = \frac{1}{4\pi}\sin\theta d\theta d\phi$ in spherical coordinates. Alice's box simulates the measurement of a single spin prepared in the direction λ, giving the probability:

$$P_A^{\lambda}(a \mid \mathbf{a}) = \frac{1}{2}\left(1 + a\mathbf{a}\cdot\lambda\right) \tag{5.78}$$

Bob's box outputs $b = -\text{sign}(\mathbf{b}\cdot\lambda)$, meaning $b = +1$ if $\mathbf{b}\cdot\lambda \leq 0$ and $b = -1$ if $\mathbf{b}\cdot\lambda > 0$. Therefore, the joint probability is:

$$P(a,+1 \mid \mathbf{a},\mathbf{b}) = \int_{S^2} d\lambda\, \rho(\lambda) P_A^{\lambda}(a \mid \mathbf{a}) \delta_{\mathbf{b}\cdot\lambda \leq 0} = \frac{1}{4} + \frac{1}{2}a \int_{\mathbf{b}\cdot\lambda \leq 0} d\lambda\, \rho(\lambda)\mathbf{a}\cdot\lambda \tag{5.79}$$

To compute the integral, we use spherical coordinates such that $\mathbf{b}$ is aligned with $\hat{z}$ (i.e., $\theta = 0$). The integral becomes:

$$\int_{\mathbf{b}\cdot\lambda \leq 0} d\lambda\, \rho(\lambda)\mathbf{a}\cdot\lambda = \frac{1}{4\pi}\int_{\pi/2}^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \left[(a_x\cos\phi + a_y\sin\phi)\sin\theta + a_z\cos\theta\right] \tag{5.80}$$

Evaluating the integral:

$$= \frac{1}{2} a_z \int_{\pi/2}^{\pi} d\theta \sin\theta \cos\theta = -\frac{1}{4} a_z \tag{5.81}$$

Substituting this result into the expression for $P(a,+1 \mid \mathbf{a},\mathbf{b})$ and recalling that $a_z = \mathbf{a}\cdot\mathbf{b}$, we recover the desired result for $b=+1$. The calculation for $b=-1$ changes only in the bounds of the last integral and yields the correct result as well. This concludes the proof of the lemma. Since all Werner states with $\frac{1}{3} < p \leq \frac{1}{2}$ are entangled, this also proves Theorem 3.3 ■.

5.6 Non-Locality in Multipartite Quantum Systems: An Overview

The study of bipartite systems has provided significant insights into the intricate relationship between entanglement and non-locality. However, this profound connection becomes even more complex as the number of parties sharing quantum information increases. The first distinction becomes evident when the inseparability of a state is analyzed. Unlike the bipartite case, where a state can be either separable or entangled, the multipartite scenario presents a variety of possibilities, as shown in [92]. In the following, we outline the five relevant classes of interest. Let us consider three qubits A, B, and C. We classify their possible states according to whether they are separable or not with respect to the different qubits. In particular, according to whether they can be written in one or more of the following forms:

$$\rho = \sum_i |a_i\rangle_A\langle a_i| \otimes |b_i\rangle_B\langle b_i| \otimes |c_i\rangle_C\langle c_i|, \tag{5.82}$$

$$\rho = \sum_i |a_i\rangle_A\langle a_i| \otimes |\phi_i\rangle_{BC}\langle \phi_i|, \tag{5.83}$$

$$\rho = \sum_i |b_i\rangle_B\langle b_i| \otimes |\phi_i\rangle_{AC}\langle \phi_i|, \tag{5.84}$$

$$\rho = \sum_i |c_i\rangle_C\langle c_i| \otimes |\phi_i\rangle_{AB}\langle \phi_i|. \tag{5.85}$$

Here, $|a_i\rangle$, $|b_i\rangle$, and $|c_i\rangle$ are (unnormalized) states of systems A, B, and C, respectively, and $|\phi_i\rangle$ are states of two systems. From these definitions, we can construct five disjoint classes of states:

Class 1: Fully inseparable states
These are states that cannot be written in any of the forms (5.82),(5.83),(5.84),(5.85). An example of a fully inseparable state is the GHZ state:

$$|GHZ\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}, \tag{5.86}$$

which is a maximally entangled state of the three qubits [93].

Class 2: 1-qubit biseparable states
Biseparable states with respect to qubit A are states that are separable with respect to the first qubit but non-separable with respect to the other two qubits. These states can be written in the form (5.83) but not in the forms (5.84) or (5.85). A trivial example of such a state is:

$$|0\rangle_A \otimes |\Phi^+\rangle_{BC}, \tag{5.87}$$

where $|\Phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ is a maximally entangled state of two qubits.

Class 3: 2-qubit biseparable states
Biseparable states with respect to qubits A and B are states that are separable with respect to the first and second qubits but non-separable with respect to the third qubit. These states can be written in the forms (5.83) and (5.84) but not in the form (5.85).

Class 4: 3-qubit biseparable states
These are states that can be written in the forms (5.83), (5.84), and (5.85), but not in the form (5.82).

Class 5: Fully separable states
These are states that can be written in the form (1a). A trivial example of such a state is:

$$|0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C. \tag{5.88}$$

In summary, for a generic N-multimode system, we see that if $\rho_N^{T_{A_k}} \geq 0$, then it can be written in the form:

$$\rho_N = \sum_i |a_i\rangle_{A_k}\langle a_i| \otimes |\phi_i\rangle_{\text{rest}}\langle\phi_i|, \tag{5.89}$$

and therefore ρ_N is separable with respect to particle A_k. On the other hand, if considering all possible partitions of the qubits into two sets it turns out that, for each partition, the partial transpose with respect to one of the sets is positive, then ρ_N is fully separable.

5.6.1 multimode entanglement measure: Tangle

The *tangle* is a fundamental measure of entanglement, particularly for bipartite qubit systems [94]. To define it, consider two qubits, A and B, whose joint state is described by the density matrix ρ_{AB}, which may be either pure or mixed. The *spin-flipped* density matrix is introduced as:

$$\tilde{\rho}_{AB} = (\sigma_y \otimes \sigma_y)\rho_{AB}^*(\sigma_y \otimes \sigma_y), \tag{5.90}$$

where the asterisk denotes complex conjugation in the standard computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The matrix σ_y, expressed in this basis, is given by:

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{5.91}$$

This spin-flip operation plays a crucial role in capturing entanglement properties because it effectively acts as a *time-reversal transformation* on the spin degrees of freedom. It ensures that the mathematical structure of entanglement correlations is correctly preserved, which is essential for defining the *Concurrence* [95], a key intermediate quantity in the calculation of the tangle. Both ρ_{AB} and $\tilde{\rho}_{AB}$ are positive operators, implying that their product $\rho_{AB}\tilde{\rho}_{AB}$, although generally non-Hermitian, has only *real and non-negative eigenvalues*. Denoting the square roots of these eigenvalues, arranged in descending order, as $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, we define the *concurrence* of ρ_{AB} as:

$$C_{AB} = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}. \tag{5.92}$$

The *tangle* is then given by the squared concurrence:

$$\tau_{AB} = C_{AB}^2 = \left[\max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}\right]^2. \tag{5.93}$$

Thus, the *spin-flipped density matrix is a necessary tool* for computing the concurrence, which quantifies the degree of quantum correlations between the two qubits. A value of $\tau_{AB} = 0$ corresponds to an unentangled state, while $\tau_{AB} = 1$ represents a maximally entangled state. The physical significance of this measure arises from the inherently quantum nature of correlations. Unlike classical correlations, quantum entanglement cannot be freely distributed among multiple subsystems.
To illustrate this concept, consider two spin-$\frac{1}{2}$ particles, A and B, prepared in a singlet state, which is a maximally entangled state. One might ask whether A (or B) can simultaneously be entangled with a third particle, C. However, this is not possible because if A were entangled with C, then the pair AB would also be entangled with C, leading to a mixed-state density matrix. This contradicts the fact that the singlet state is a pure state. More generally, a less extreme version of this restriction must hold: if A is partially entangled with B, then A can only be partially entangled with C as well. In summary, the goal is to quantify how the entanglement between two subsystems limits their capability to be entangled with other subsystems. Mathematically, this is captured by the *monogamy of entanglement* inequality:

$$\tau_{AB} + \tau_{AC} \leq \tau_{A(BC)}. \tag{5.94}$$

This inequality expresses that a qubit A has a fixed amount of entanglement with the bipartite system BC. This total entanglement serves as an upper bound for A's individual entanglement with qubits B and C. Furthermore, the portion of entanglement allocated to qubit B (as measured by the tangle) is not available for entanglement with

qubit C. This fundamental constraint on quantum correlations reflects the intrinsic limits on how quantum entanglement can be distributed within a multipartite system.
As the final step in this description, we introduce the concept of *residual tangle*, which plays a crucial role in characterizing the distribution of entanglement in multipartite systems. The residual tangle is defined as the portion of entanglement that cannot be attributed to bipartite correlations alone, effectively capturing the genuinely multipartite nature of entanglement:

$$\tau_{\text{res},A} = \tau_{A(BC)} - \tau_{AB} - \tau_{AC} \tag{5.95}$$

If the residual tangle is nonzero, this indicates that some entanglement is genuinely shared among all three qubits and is not reducible to simple bipartite correlations. This concept will be essential in the subsequent discussion, where we analyze the connection between entanglement and nonlocality in multipartite systems. The study of residual tangle allows us to explore whether systems exhibiting strong multipartite entanglement necessarily display stronger violations of Bell-type inequalities, thereby shedding light on the fundamental relationship between these two key quantum properties.

5.6.2 Multipartite non locality

After defining the criteria to distinguish the various classes of entangled states, we can explore the relationship between entanglement and the concept of multimode nonlocality. As expected, understanding the connection between entanglement and nonlocality becomes even more challenging in this context. To begin, we extend the concepts already discussed for the bipartite case by introducing an additional observer, Charlie (C), to the configuration previously referred to as the Bell scenario, where Alice and Bob are two independent operators communicating via classical channels. The experiment is thus characterized by the joint probability distribution:

$$p(abc|xyz) = \int d\lambda\, q(\lambda)\, p_\lambda(a|x)\, p_\lambda(b|y)\, p_\lambda(c|z), \tag{5.96}$$

where λ is a shared local random variable and $\int d\lambda\, q(\lambda) = 1$. Correlations are said to be local if they can be written in this form and non-local otherwise. In the multipartite case, the notion of non-locality can be refined in several ways. For example, consider a joint distribution of the form:

$$p(abc|xyz) = p(ab|xy) \cdot p(c|z), \tag{5.97}$$

where Charles is uncorrelated with Alice and Bob. These correlations can violate the locality condition (5.96) if $p(ab|xy)$ is non-local, even though there is no non-locality between Alice, Bob, and Charles as a whole. In other words, such correlations exhibit only bipartite non-locality. In contrast, one can consider a scenario where all three parties are nonlocally correlated. This is referred to as *genuine multipartite nonlocality*, which represents the strongest form of multipartite nonlocality.

5.6.3 Genuine multipartite non locality

The concept of genuine multipartite nonlocality was first introduced by Svetlichny in 1987 [96]. This notion extends the idea of nonlocality to include correlations that cannot be decomposed into any form of bipartite nonlocality combined with local correlations. To illustrate, consider a joint probability distribution $p(abc|xyz)$. Genuine multipartite nonlocality implies that $p(abc|xyz)$ cannot be expressed in the following form:

$$p(abc|xyz) = \int d\lambda\, q(\lambda)\, p_\lambda(ab|xy)\, p_\lambda(c|z) + \int d\mu\, q(\mu)\, p_\mu(bc|yz)\, p_\mu(a|x) + \int d\nu\, q(\nu)\, p_\nu(ac|xz)\, p_\nu(b|y), \tag{5.98}$$

where λ, μ, and ν are hidden variables, and the terms $p_\lambda(ab|xy)$, $p_\mu(bc|yz)$, and $p_\nu(ac|xz)$ represent bipartite nonlocal correlations, while $p_\lambda(c|z)$, $p_\mu(a|x)$, and $p_\nu(b|y)$ represent how the remaining observer is locally correlated to the others. This decomposition accounts for all possible bipartite splits of the system, and if $p(abc|xyz)$ cannot be written in this form, the system exhibits genuine multipartite nonlocality.

5.6.2.1 Alternative Definitions

The main issue with the definition of genuine multipartite nonlocality proposed by Svetlichny is that it allows correlations that violate the no-signaling condition. According to this definition, the correlations among three parties, Alice, Bob, and Charles, can be expressed as a convex combination of three terms, each representing a scenario where two of the three parties share a nonlocal resource while the third party remains locally correlated with them. However, Svetlichny does not impose any restriction on the bipartite correlations within this decomposition, allowing them to violate the *no-signaling condition* (5.22). This means that, in certain configurations, Alice could *instantly* influence Bob's outcome, or vice versa, merely by choosing her measurement. This creates a logical and causal inconsistency, as the choice of measurement by one observer could have an immediate impact on the other result, regardless of the spatial separation. In a setup where the order of measurements is not predetermined, this leads to a causal paradox similar to the classical *grandfather paradox*. The grandfather paradox, well known in the literature on time travel, arises when an event in the present influences the past in such a way that it prevents its own existence. Applied to the context of multipartite nonlocality, this paradox occurs when two observers can mutually influence each other through physically meaningful correlations. If Alice and Bob can communicate instantaneously and the measurement protocol does not follow a predetermined order, then two contradictory scenarios emerge: if Alice measures before Bob, her result can instantly influence Bob's, but if Bob measures first, then his result can instantly influence Alice's. If the order of measurements is not predefined, there could be a step in which Alice influences Bob, followed by another step where Bob influences Alice. This implies that the two observers influence each other in a

circular way, making it impossible to determine who decided their outcome first. This is equivalent to saying that the future influences the past, just like in the grandfather paradox, where a future decision (Bob's result) can influence a past choice (Alice's result) and vice versa.
To solve this issue, alternatives to Svetlichny's definition have been proposed. One of the most immediate solutions is to impose the no-signaling constraint on all bipartite correlations appearing in the decomposition. This approach has been explored by Almeida,et al. [97] and Barrett, et al. [98], who introduced the set of correlations denoted as $S_{ns}^{2|1}$, which includes only those decompositions that respect the no-signaling condition. In this way, it is ensured that two parties can share nonlocal correlations without being able to communicate instantaneously, thus preserving the causal consistency of the theory. The last definition is based on *time-ordering* requirement of all bipartite correlations. We denote $p_A^{T_{AB}}(ab|xy)$ as the probability distribution that is time-order dependent: when Alice influences Bob we have $p_A^{T_{AB}}(ab|xy) = p_A^{A<B}(ab|xy)$, and vice-versa $p_A^{T_{AB}}(ab|xy) = p_A^{B<A}(ab|xy)$ when Bob influences Alice. It is not allowed a mutual influence to avoid the problem discussed above. We define a new set $S_{to}^{2|1}$ of two-way time ordered correlations that contains all distributions that can be written in the form:

$$p(abc|xyz) = \int d\lambda\, q(\lambda)\, p_\lambda(ab|xy)^{T_{AB}}\, p_\lambda(c|z) + \int d\mu\, q(\mu)\, p_\mu^{T_{AC}}(bc|yz)\, p_\mu(a|x) + \int d\nu\, q(\nu)\, p_\nu^{T_{BC}}(ac|xz)\, p_\nu(b|y). \tag{5.99}$$

It is important to emphasize that the three definitions of multipartite nonlocality are not equivalent. Moreover, the relationship between the different sets is as follows:

$$\mathscr{L} \subset S_{ns}^{2|1} \subset S_{to}^{2|1} \subset S_{Svet}^{2|1}. \tag{5.100}$$

5.6.4 Entanglement vs Non locality in multipartite systems

The aim of this section is to give an overview of the non trivial connection between entanglement and non-locality in multipartite systems. Despite the additional complexities, significant progress has been made over time. One of the first relevant results was demonstrated by Popescu [83], who showed that all pure N-partite states exhibit nonlocality, generalizing the result previously established by Gisin in the bipartite case. To provide a generalized inequality we can start from a tripartite system of quibits: A,B and C. In the following we include the same assumptions on which Bell' inequality has been derived: *locality* and *realism*. The first condition ensures the factorizability of the joint probability distribution:

$$P(a_i, b_j, c_k, \lambda) = P(a_i, \lambda)P(b_j, \lambda)P(c_k, \lambda) \tag{5.101}$$

where a_i, b_j and c_k are possible outcomes for the three observer A,B and C. Now as-

sume that C measures its system and the output is give by c_k, and this result is classically communicated to A and B. We want to evaluate the correlation between the two, conditional the result of C:

$$E(A,B,c_k) = \frac{\sum_{i,j} a_i b_j \int d\lambda P(a_i,b_j,c_k,\lambda) p(\lambda)}{\sum_{i,j} \int d\lambda P(a_i,b_j,c_k,\lambda) p(\lambda)} \tag{5.102}$$

Since the denominator is normalized we get:

$$\int d\lambda E(A,\lambda) E(B,\lambda) P(c_k) p(\lambda), \quad E(X,\lambda) \equiv \sum_i x_i P(x_i,\lambda). \tag{5.103}$$

Now

$$-2 \leq E(A,\lambda)\left[E(B,\lambda)+E(B',\lambda)\right] + E(A',\lambda)\left[E(B,\lambda)-E(B',\lambda)\right] \leq 2. \tag{5.104}$$

The sum is linear in each expectation value and each of takes one of the two extreme values ± 1, therefore, we can multiply for $p(\lambda)P(c_k,\lambda)/P(c_k)$ and integrate, we obtain a CHSH inequality for bipartite system A, B, conditional on the measurement of C:

$$-2 \leq E(A,B,c_k) + E(A,B',c_k) + E(A',B,c_k) - E(A,B,c_k) \leq 2. \tag{5.105}$$

Ultimately, the problem of identifying non-local aspects reduces to the bipartite case, which has already been analyzed. Consequently, the results obtained by Gisin apply, implying that if the bipartite state is entangled, it will necessarily violate Bell inequalities. The generalization to an N-parties system is straightforward, because it is sufficient to repeat the same derivation, taking into account the $N-2$ outcomes, reducing the problem to a CHSH conditional on all the $N-2$ measurements. We can now test quantum predictions against this inequality to determine the non-local nature of the correlations. The correlation terms are obtained by taking the expectation values over a state derived from the original one through projection onto the $N-2$ known states. It remains to check that the projection of $N-mode$ system $|\Phi\rangle$ onto product of states of the other $N-2$, leaves the remaining two systems in an entangled state.

Proof: The proof proceeds by assuming the opposite, assuming that the entangled state $|\Phi\rangle$, projected onto the state associated with the $N-2$ systems, is factorized:

$$\left\langle e^{i_3}\right|_3 \left\langle e^{i_4}\right|_4 \cdots \left\langle e^{i_N}\right|_N |\Phi\rangle = |\psi\rangle_1 \left|\psi'\right\rangle_2, \tag{5.106}$$

where $\left|e^{i_j}\right\rangle_j$ is an element of a given basis of the jth system, while $|\psi\rangle_1$ and $|\psi'\rangle_2$ are the states of the two remaining systems. Regardless the choice of indices $i_3,\cdots,i_N$ they are, by assumption, in a product states. Despite this, they could in principle depend on the indices:

$$
\begin{aligned}
|\psi\rangle_1 &= |\psi(i_3,\cdots,i_N)\rangle_1 \\
\left|\psi'\right\rangle_2 &= \left|\psi'(i_3,\cdots,i_N)\right\rangle_2 .
\end{aligned}
\tag{5.107}
$$

If we take a different element belonging to the same basis of jth system, $\left|e^{i'_j}\right\rangle_j$ either $|\psi\rangle_1$ or $|\psi'\rangle_2$ must remain unchanged (up to a phase factor). Let suppose that they both change. If we know replace $\left|e^{i_j}\right\rangle_j$ with a linear combination of $\left|e^{i'_j}\right\rangle_j$ and $\left|e^{i_j}\right\rangle_j$ in direct product basis (5.106), we introduce correlations between $|\psi\rangle_1$ and $|\psi'\rangle_2$, contrary to the original assumption. We conclude that the two single states can not have the dependence on the same index. We can write, without loss of generality:

$$
\begin{aligned}
|\psi\rangle_1 &= |\psi(i_3,\cdots,i_k)\rangle_1 \\
\left|\psi'\right\rangle_2 &= \left|\psi'(i_{k+1},\cdots,i_N)\right\rangle_2 .
\end{aligned}
\tag{5.108}
$$

Now the identity operator can be resolved into a complete sum over orthogonal projections:

$$
\begin{aligned}
|\Phi\rangle &= \sum_{i_3,\cdots i_N} \left|e^{i_N}\right\rangle_N \cdots \left|e^{i_3}\right\rangle_3 \left\langle e^{i_3}\right|_3 \cdots \left\langle e^{i_N}\right|_N |\Phi\rangle = \\
&= \sum_{i_3\cdots i_k} |\psi(i_3,\cdots,i_k)\rangle_1 \left|e^{i_3}\right\rangle_3 \cdots \left|e^{i_k}\right\rangle_k \sum_{i_{k+1}\cdots i_N} \left|\psi'(i_{k+1},\cdots,i_N)\right\rangle_1 \left|e^{i_{k+1}}\right\rangle_{k+1} \cdots \left|e^{i_N}\right\rangle_N ,
\end{aligned}
\tag{5.109}
$$

showing that $|\Phi\rangle$ factorize into a product, contradicting the assumption that it was an entangled state ■.
It is important to note that Popescu's solution is limited to systems that preserve entanglement between pairs. Consequently, it cannot be applied to states belonging to the GHZ equivalence class, as the partial trace over $N-2$ subsystems defines a separable bipartite state. As a result, no violation of Popescu inequality is ever observed for genuine multipartite entanglement. We can easily verify this statement for a system with $N=3$, by computing the partial trace over a single qubit. The global state is:

$$
|GHZ\rangle = \frac{1}{\sqrt{2}} \left(|000\rangle + |111\rangle\right). \tag{5.110}
$$

The density matrix will be:

$$
\rho_{GHZ} = \frac{1}{2} \left(|000\rangle\langle 000| + |000\rangle\langle 111| + |111\rangle\langle 000| + |111\rangle\langle 111| \right). \tag{5.111}
$$

We can trace out one of the three quibits, the first for example, and study the separability of the bipartite density matrix:

$$
\rho_{23}^{GHZ} = Tr_1(\rho_{GHZ}) = \frac{1}{2} \left(|00\rangle\langle 00| + |11\rangle\langle 11|\right), \tag{5.112}
$$

Showing that it is separable, so, no quantum behaviors can be observed just considering a pair of system. This consideration highlights the need to define inequalities that are specifically suited for testing genuinely entangled states.
The first criterion for identifying genuine multipartite nonlocality was formulated by Svetlichny in 1987 [96]. Specifically addressing a three-party system, he derived a Bell-type inequality that remains valid for any probability distribution satisfying Eq. (5.99). Consequently, any violation of this inequality serves as conclusive evidence for the existence of genuine tripartite nonlocality. Furthermore, such a violation inherently implies the presence of genuine tripartite entanglement.
We now focus on the scenario where each party j performs one of two possible measurements, denoted x_j and x'_j. All measurements are dichotomic, meaning their outcomes are given by $a_j = \pm 1$ and $a'_j = \pm 1$. Svetlichny demonstrated that the following inequality:

$$S_3 = a_1a_2a'_3 + a_1a'_2a_3 + a'_1a_2a_3 - a'_1a'_2a'_3 + a'_1a'_2a_3 + a'_1a_2a'_3 + a_1a'_2a'_3 - a_1a_2a_3 \leq 4 \tag{5.113}$$

holds for any probability distribution satisfying Eq. (5.99). Here, the terms should be interpreted as expectation values, where, for instance, $a_1a_2a'_3$ represents the expectation value of the product of measurement outcomes when the measurements performed are x_1, x_2, and x'_3. To gain further insight into Svetlichny's inequality and to establish that its violation implies genuine multipartite nonlocality, we adopt the approach of Bancal, Brunner et al. [99]. We first rewrite the inequality as:

$$S_3 = Sa'_3 + S'a_3 \leq 4, \tag{5.114}$$

where $S = a_1a_2 + a_1a'_2 + a'_1a_2 - a'_1a'_2$ is the CHSH expression, and $S' = a'_1a'_2 + a'_1a_2 + a_1a'_2 - a_1a_2$ is an equivalent form obtained by permuting primed and non-primed measurements. Observing the structure of the inequality, we note that Charlie's input setting determines which version of the CHSH game Alice and Bob are playing. When Charlie receives input x'_3, Alice and Bob play the standard CHSH game; when Charlie receives x_3, they play its symmetric counterpart. Thus, for any bipartite model of Eq. (5.99), the bound $S_3 \leq 4$ holds. Considering the bipartition $A|BC$, Bob is aware of the CHSH game version to be played with Alice, as he is in the same partition as Charlie. However, since the CHSH game is inherently nonlocal, Alice and Bob cannot overcome the local bound $S = 2$ or $S' = 2$, given that they are spatially separated. Consequently, the inequality $S_3 \leq 4$ remains valid for the bipartition $A|BC$. The same argument applies to the bipartition $B|AC$. Since the polynomial is symmetric under the permutation of parties, it follows that $S_3 \leq 4$ for all bipartitions. The inequality (5.113) serves as a powerful tool for detecting genuine multipartite nonlocality in significant classes of quantum states, including the GHZ state. Svetlichny's inequality has been extended to accommodate an arbitrary number of parties, n, as demonstrated by Collins [100] and Seevinck and Svetlichny [101]. Following a similar approach used to derive Svetlichny's inequality from CHSH , it is possible to obtain the generalized form:

$$S_n = S_{n-1}a'_n + S'_{n-1}a_n \leq 2^{n-1}, \tag{5.115}$$

where S'_{n-1} is derived from S_{n-1} by exchanging a_1 with a'_1 and vice versa [99].
At this stage, it is natural to ask whether entanglement and nonlocality in multipartite systems are equivalent concepts. In the case of pure bipartite states, Gisin's theorem provides an affirmative answer to this question, a result that is also emphasized in Popescu's work. In these cases, entanglement and nonlocality are indeed synonymous. However, this equivalence no longer holds for genuinely multipartite entangled states, as demonstrated in [102]. In the cited work, the violation of Svetlichny inequalities is analyzed for two specific states belonging to the *GHZ class*: the *generalized GHZ (GGHZ) states* and the *maximal slice (MS) states*, defined as follow:

$$|\psi_{\text{GGHZ}}\rangle = \cos\theta_1 |000\rangle + \sin\theta_1 |111\rangle, \tag{5.116}$$

$$|\psi_{\text{MS}}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |11\rangle (\cos\theta_3 |0\rangle + \sin\theta_3 |1\rangle)). \tag{5.117}$$

For instance, the well-known GHZ state, which belongs to both subsets, corresponds to the specific values $\theta_1 = \pi/4$ and $\theta_3 = \pi/2$.
Like other Bell-type inequalities, the Svetlichny inequality is expressed in terms of the expectation value of a Bell-type operator $\hat{S}$, which satisfies the bound:

$$|\langle \hat{S} \rangle| \leq 4. \tag{5.118}$$

However, it has been shown that the maximum expectation values of $\hat{S}$ for the GGHZ and MS states are given by:

$$\hat{S}_{\max}(\psi_{\text{GGHZ}}) = \begin{cases} 4\sqrt{1-\tau(\psi_{\text{GGHZ}})} & \text{if } \tau(\psi_{\text{GGHZ}}) \leq \frac{1}{3}, \\ 4\sqrt{2\tau(\psi_{\text{GGHZ}})} & \text{if } \tau(\psi_{\text{GGHZ}}) \geq \frac{1}{3}, \end{cases} \tag{5.119}$$

$$\hat{S}_{\max}(\psi_{\text{MS}}) = 4\sqrt{1+\tau(\psi_{\text{MS}})}. \tag{5.120}$$

Here $\tau(\psi)$ is the residual tangle (5.95) and quantifies tripartite entanglement, with:

$$\tau(\psi_{\text{GGHZ}}) = \sin^2 2\theta_1, \quad \tau(\psi_{\text{MS}}) = \sin^2\theta_3. \tag{5.121}$$

These results indicate that the relationship between entanglement and nonlocality is non-trivial in the multipartite case. While nonlocality can be inferred from entanglement in bipartite pure states, multipartite entanglement does not necessarily imply the violation of Bell-type inequalities such as the Svetlichny bound. Although both states belong to the same class, they exhibit completely different behaviors. The most intriguing case is the GGHZ state, as the violation of Svetlichny inequalities is not always

guaranteed. Specifically, for values of three-tangle $\tau(\psi_{\text{GGHZ}}) \leq \frac{1}{3}$, the expectation value of the Bell operator $\hat{S}$ is a monotonically decreasing function of τ. In the limiting case where $\tau \to 0$, the expectation value approaches to:

$$\langle \hat{S} \rangle_{max} \to 4, \tag{5.122}$$

which implies that no violation is possible in this regime. Conversely, for $\tau \geq \frac{1}{3}$, the behavior of $\langle \hat{S} \rangle_{max}$ is reversed, as the maximum violation is now proportional to τ. To ensure that the Svetlichny inequality is violated, we impose the condition that the argument of the square root in $\hat{S}_{\text{max}}$ must be greater than 1. From this, we obtain the necessary and sufficient condition for the violation of Svetlichny inequalities:

$$\tau > \frac{1}{2}. \tag{5.123}$$

This result highlights that, despite being a multipartite entangled state, the GGHZ state does not always exhibit nonlocal correlations, reinforcing the fundamental distinction between entanglement and nonlocality in multipartite quantum systems. On the other side all Maximal Slice (MS) states consistently violate the Svetlichny inequality, indicating that these states always exhibit some form of genuine multipartite nonlocality. Equation (5.120) is directly analogous to the well-known result for two-qubit systems, where bipartite entanglement determines the maximum violation of the CHSH inequality (5.55).

As anticipated in the previous paragraph, however, the definition of multimode nonlocality is not unique. This lack of a unified definition leads to inconsistencies in determining the relationship between entanglement and nonlocality. It has been shown by Almeida, et al.[97] that all connected graph states are fully genuinely nonlocal within the no-signaling approach discussed in Sec.(5.6.2.1), while using the definition based on time ordering (5.99), numerical evidence suggests that all pure genuine tripartite entangled qubit states are genuine tripartite nonlocal [98]. Below, we present a table summarizing the results obtained for both the bipartite and multipartite cases.

State Type	Bipartite Entanglement	Multipartite Entanglement
Pure States	A pure bipartite state is given by: $\|\psi\rangle = \sum_i c_i \|i\rangle_1 \|i\rangle_2.$ If at least two coefficients $c_i \neq 0$, the state is entangled, leading to the violation of Bell inequalities → Gisin' theorem. entanglement ⇔ nonlocality	**Genuinely Entangled:** Not decomposable as biseparable mixtures: $\|GHZ\rangle = \frac{1}{\sqrt{2}}(\|000\rangle + \|111\rangle).$ Nonlocality depends on the specific state. For $\|\psi_{\text{GGHZ}}\rangle$, entanglement ⇔ nonlocality if and only if $\tau(\psi_{GGHZ}) > \frac{1}{2}$. For $\|\psi_{MS}\rangle$, entanglement ⇔ nonlocality $\forall\tau(\psi_{MS})$.
		Biseparable: Separable in some bipartition but entangled in others. $\|\Phi\rangle = \otimes_{i=3}^{N} \|\phi\rangle_i \otimes \|\psi\rangle_{1,2}$ where $\|\psi\rangle_{1,2}$ is an entangled bipartite state. In this case, entanglement ⇔ nonlocality
Mixed States	if: $\rho_{AB} \neq \sum_i c_i \rho_i^A \otimes \rho_i^B,$ then the state is entangled. However, in this case: entanglement ⇏ nonlocality, but entanglement ⇐ nonlocality Example: Werner states $\rho_w = p\left\|\Psi^-\right\rangle\left\langle\Psi^-\right\| + (1-p)\frac{\mathbb{I}}{4},$ $\frac{1}{3} < p \leq \frac{1}{2}.$	Can be a mixture of separable, biseparable, or genuinely entangled states.

Table 1: Comparison of Bipartite and Multipartite Non-locality in Pure and Mixed States.

6 Non-Locality in continuous variable systems

We have seen, for the discrete case, how the non-local behavior of the correlations between two particles is revealed by the violation of Bell's inequalities. In the context

of continuous variable systems (position and momentum), the description takes place in phase space by means quasi-distributions. The most relevant of these is the Wigner function, which has the unique property of being negative in certain regions of phase space. When this occurs, it implies non-local properties of the described system. This naturally raises the question of whether the sign of the Wigner function is a necessary and sufficient condition to predict non-local aspects. One of the first to attempt to answer this question was Bell, who showed that an EPR state, whose non-local properties were experimentally proved by Bhom, can be described in phase space by a always positive Wigner function [103]. This implies that the EPR state allows a description by means of a local model. We briefly report the simple derivation that Bell proposed. The idea is to measure the positions of two spinless particles. In the EPR' paper [3] they measured also the momenta, but the easiest way to measure the momenta of free particles is just to wait a long time and measure their positions. We can imagine to measure at two different times t_1, t_2 the positions of the two particles and evaluating:

$$\hat{q}_1 + t_1 \frac{\hat{p}_1}{m_1}; \quad \hat{q}_2 + t_2 \frac{\hat{p}_2}{m_2}. \tag{6.1}$$

The time t_1 and t_2 plays the role of the two polarizer settings in the Bohm experimental setup. The QM probability of finding, at times t_1 and t_2 respectively, the particles at positions q_1 and q_2 respectively, is

$$\rho(q_1, q_2, t_1, t_2) \tag{6.2}$$

with

$$\rho = |\psi(q_1, q_2, t_1, t_2)|^2. \tag{6.3}$$

The two-time wave function ψ satisfies the two Schrödinger equations

$$i\hbar \frac{\partial \psi}{\partial t_1} = H_1 \psi = \left(\frac{\hat{p}_1^2}{2m_1} \right) \psi \tag{6.4}$$

$$i\hbar \frac{\partial \psi}{\partial t_2} = H_2 \psi = \left(\frac{\hat{p}_2^2}{2m_2} \right) \psi \tag{6.5}$$

with

$$i\hat{p}_1 = \hbar \frac{\partial}{\partial q_1}, \quad i\hat{p}_2 = \hbar \frac{\partial}{\partial q_2}. \tag{6.6}$$

For simplicity, we will consider the case of equal masses, and take units such that

$$m_1 = m_2 = \hbar = 1. \tag{6.7}$$

The same ρ, (6.2), can be obtained from the corresponding two-time Wigner distribution:

$$\rho = \iint \frac{dp_1 dp_2}{2\pi} W(q_1, q_2, p_1, p_2, t_1, t_2) \tag{6.8}$$

where

$$W = \iint dy_1 dy_2 \, e^{-i(p_1 y_1 + p_2 y_2)/\hbar} \psi\left(q_1 + \frac{y_1}{2}, q_2 + \frac{y_2}{2}, t_1, t_2\right) \psi^*\left(q_1 - \frac{y_1}{2}, q_2 - \frac{y_2}{2}, t_1, t_2\right) \tag{6.9}$$

From (6.4),

$$\left(\frac{\partial}{\partial t_1} + p_1 \frac{\partial}{\partial q_1}\right) W = \left(\frac{\partial}{\partial t_2} + p_2 \frac{\partial}{\partial q_2}\right) W = 0. \tag{6.10}$$

That is, W evolves exactly as does a probability distribution for a pair of freely-moving classical particles:

$$W(q_1, q_2, p_1, p_2, t_1, t_2) = W(q_1 - p_1 t_1, q_2 - p_2 t_2, p_1, p_2, t_1, t_2) \tag{6.11}$$

When W happens to be initially nowhere negative, the classical evolution (6.11) preserves the non-negativity. The original EPR wave function

$$\delta\left((q_1 + \frac{1}{2} q_0) - (q_2 - \frac{1}{2} q_0)\right), \tag{6.12}$$

assumed to hold at $t_1 = t_2 = 0$, gives

$$W(q_1, q_2, p_1, p_2, 0, 0) = \delta(q_1 - q_2 + q_0) 2\pi\hbar \delta(p_1 + p_2). \tag{6.13}$$

Since the initial Wigner function is non-negative, the time evolution will preserve this condition. Thus in this case thr EPR correlations are precisely those between two classical particles in indipendent free motion. The question posed then by Bell was whether the negativity of the Wigner function was a *sufficient* condition for violating local realism. To answer to this question we have to wait until 1998, when Konrad Banaszek and Krzysztof Wòdkiewicz [104, 105] made an essential contribution in understanding the connections between non-locality and quasi-distributions in phase space. Below we present the main results of their research.

6.1 A Bell inequality from parity measurement

In the systems we are considering, states live in infinite-dimensional Hilbert spaces, so it is complicated to define a measure equivalent to the discrete case of non-locality. One would have to use a continuous Bell inequality for two parties. However, to the best of our knowledge, there is no continuous variable Bell inequality for setups with two parties with quantum violations. Another approach involves directly measuring a binary quantity to test the CHSH inequality. As suggested in [106] a natural choice for such a binary measurement is the parity operator, defined as

$$\hat{P} = e^{i\pi\hat{n}}, \tag{6.14}$$

where $\hat{n} = \hat{a}^\dagger\hat{a}$ is the number operator. The parity operator has two eigenvalues, $+1$ and -1 (like the two possible outputs of the detectors in Sec.(5.2)), corresponding to the subspaces spanned by Fock states with an even and odd number of photons, respectively. To test the CHSH inequality, we must define different measurement settings for Alice and Bob. This is done using the displaced parity operator:

$$\hat{\Pi}(\alpha) = \hat{D}(\alpha)\hat{P}\hat{D}^\dagger(\alpha), \tag{6.15}$$

where $\hat{D}(\alpha) = \hat{D}(\nu,\mu) = \exp\big(\alpha a^\dagger - \alpha^* a\big)$ is the phase-space displacement operator, with $\alpha = (\nu + i\mu)/\sqrt{2}$. Since the displacement operator $\hat{D}(\alpha)$ is unitary, it does not alter the spectrum of the parity operator, meaning $\hat{\Pi}(\alpha)$ retains its binary nature. It follows that any local realistic theory should satisfy the following bound on the CHSH inequality

$$B = \langle\hat{\Pi}(\alpha)\otimes\hat{\Pi}(\beta)\rangle + \langle\hat{\Pi}(\alpha)\otimes\hat{\Pi}(\beta')\rangle + \langle\hat{\Pi}(\alpha')\otimes\hat{\Pi}(\beta)\rangle - \langle\hat{\Pi}(\alpha')\otimes\hat{\Pi}(\beta')\rangle \leq 2. \tag{6.16}$$

Interestingly, it has been demonstrated that using displaced parity measurements, one can reveal the non-locality of quantum states even when their Wigner function is strictly positive [104, 105]. It is also noteworthy that the expectation value of the displaced parity operator is directly related to the Wigner function $W(\alpha)$ of the state as follows:

$$\langle\hat{\Pi}(\alpha)\rangle = \frac{\pi}{2}W(\alpha). \tag{6.17}$$

Similarly, the expectation value of the tensor product of two displaced parity operators corresponds to the two-mode Wigner function of the state. This will play an essential role in the experimental setup discussed, where the Wigner function can be directly measured.

6.1.1 Non-locality of the Einstein-Podolsky-Rosen state

As seen earlier, Bell's attempt to reveal non-local properties of the EPR state highlighted the absence of a clear link between the negativity of Wigner function and

non-locality. The relation between the EPR correlations and the Wigner distribution function has been treated in several works [107, 108, 109, 110, 111, 112, 113], however, none of them presented results that contradicted those obtained by Bell. The only aspect that was improved concerned the problem of normalization of the EPR state, which was addressed through the proposal of a normalizable state that simulates the EPR correlations. This problem has been solved by a "smoothing" procedure of the original wave function (6.12). An example of this procedure, with a clear application to quantum optics, was the use of a two-mode squeezed vacuum state produced in a process of non-degenerate optical parametric amplification (NOPA). As mentioned in the previous paragraph, it was not until the works of Banaszek and Wódkiewicz that the relationship between the Wigner function and non-locality was clarified [104, 105]. In fact, they demonstrated that, despite the Wigner function of the EPR state being positive definite, it still provides direct evidence of the nonlocal correlations exhibited by this state. They showed that the positivity or the negativity of the quasi-distribution has a rather weak relation to the non-locality of quantum correlations. In fact, the NOPA wave function violates the Bell inequality and that the original EPR wave function (6.12) exhibits strong non-locality, but one should be careful with the singular limit of strong squeezing (in this limit, the NOPA state reduces to the EPR state). The NOPA phase space will be parametrized by two complex coherent state amplitudes α and β corresponding, respectively, to (x_1, p_1) and (x_2, p_2). The starting point of their proof is an observation that the two-mode Wigner function $W(\alpha;\beta)$ can be expressed as

$$W(\alpha;\beta) = \frac{4}{\pi^2}\langle\hat{\Pi}(\alpha;\beta)\rangle, \tag{6.18}$$

where $\Pi(\alpha,\beta)$ is a quantum expectation value of a product of displaced parity operators:

$$\hat{\Pi}(\alpha;\beta) = \hat{D}_1(\alpha)(-1)^{\hat{n}_1}\hat{D}_1^{\dagger}(\alpha)\otimes\hat{D}_2(\beta)(-1)^{\hat{n}_2}\hat{D}_2^{\dagger}(b). \tag{6.19}$$

As the measurement of the parity operator yields only one of two values, $+1$ or -1, there exists an apparent analogy between the measurement of the parity operator and of the spin-$\frac{1}{2}$ projectors. The solid angle defining the direction of the spin measurement is now replaced by the coherent displacement describing the shift in phase. As Eq. (6.19) clearly demonstrates, the correlation functions measured in such experiments are given, up to a multiplicative constant, by the joint Wigner function of the system. Consequently, we have the fundamental relation

$$E(\alpha;\beta) \equiv \Pi(\alpha;\beta). \tag{6.20}$$

The original EPR state is an unnormalizable delta function. In order to avoid problems arising from this singularity, we will consider a normalizable state that can be generated in a NOPA. Such a state is characterized by the dimensionless effective interaction time r (the squeezing parameter). The Wigner function of this NOPA state is well known and is given by

$$\Pi(\alpha;\beta) = \exp\left[-2\cosh(2r)(|\alpha|^2+|\beta|^2)+2\sinh(2r)(\alpha\beta+\alpha^*\beta^*)\right]. \qquad (6.21)$$

The Wigner function of the original EPR state is obtained in the limit $r \to \infty$.
The correlation function is measured for any of four combinations of $\alpha = 0, \sqrt{J}$ and $\beta = 0, -\sqrt{J}$, where J is a positive constant characterizing the magnitude of the displacement. From these quantities, we construct the combination

$$B = \Pi(0;0)+\Pi(\sqrt{J};0)+\Pi(0;-\sqrt{J})-\Pi(\sqrt{J};-\sqrt{J}), \qquad (6.22)$$

which evaluates to

$$B = 1+2\exp(-2J\cosh(2r))-\exp\left(-4Je^{-2r}\right). \qquad (6.23)$$

For local theories, B satisfies the inequality $-2 \le B \le 2$. One of the components of the above combination describes perfect correlations: $\Pi(0,0) = 1$, obtained for a direct measurement of the parity operator with no displacements applied. This is a manifestation of the fact that in the parametric process, photons are always generated in pairs. With increased r, the violation of Bell's inequality is observed for smaller J. In the asymptotic regime ($r \to \infty$ and $J \ll 0$) we may approximate $\cosh 2r$, appearing in the argument of the first exponent in Eq. (6.23), by $\frac{e^{2r}}{2}$. Then a straightforward calculation shows that the maximum value of B (for this particular selection of coherent displacements) is obtained for

$$Je^{2r} = \frac{1}{3}\ln 2, \qquad (6.24)$$

and equals

$$B = 11-3\cdot 2^{4/3} \approx 2.19. \qquad (6.25)$$

Thus, in the limit $r \to \infty$, when the original EPR state is recovered, a significant violation of Bell's inequality takes place. The current discussion shows that although the original EPR wave function (6.12) display strong non-locality. It class of states violate the Bell inequality for a positive Wigner function. This example refutes various conjectures that linked the positivity or negativity of the Wigner function to the violation of local realism. The authors have shown that in quantum mechanics, the correlations can indeed be expressed by the Wigner function itself. This is because the Wigner function can be directly associated with the parity operator, which can be measured in a photon-photon coincidence experiment. The Wigner representations of quantum observables cannot be in general interpreted as phase-space distributions of possible experimental outcomes. In particular, the Wigner representation of the parity operator is not a bounded reality corresponding to the dichotomic result of the measurement. This enables violation of Bell's inequalities even for quantum states described by positive-definite Wigner functions.

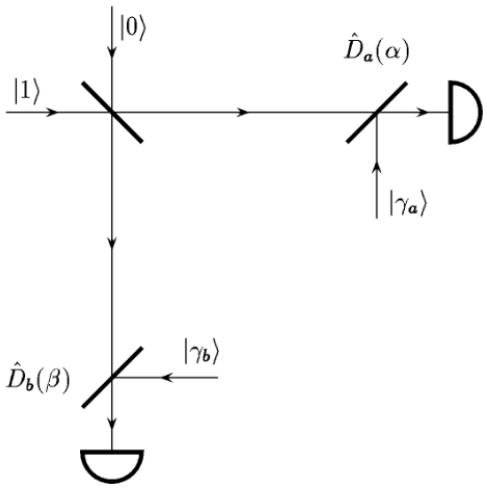

Figure 6: The optical setup proposed to demonstrate quantum nonlocality in phase space. The exemplary source of nonclassical correlated radiation is a single photon incident on a 50:50 beam splitter, which generates a quantum singletlike state. The measuring devices are photon counting detectors preceded by beam splitters. The beam splitters have the transmission coefficient close to one and strong coherent states injected into the auxiliary ports. In this limit, they effectively perform coherent displacements $\hat{D}(\alpha)$ and $\hat{D}(\beta)$ on the two modes of the input field. [105].

6.1.2 Testing Quantum Non-locality in Phase Space

The other important result obtained by Banaszek and Wòdkiewicz [105] it is the proof that the Wigner function and Q function, which are phase-space quasi-distribution functions, can directly represent non-local correlations in quantum mechanics. Specifically, the authors propose an experimental setup to test quantum non-locality using photon counting experiments that can be linked to these quasi-distributions. The setup to demonstrate quantum non-locality in the phase space is presented in Fig.6. For concreteness, we will assume that the source of the correlated state of light is a single photon impinging on a 50:50 beam splitter. The outgoing modes are labeled a and b. From the following discussion, it will be clear that the same scheme can be employed to test the non-local character of any correlated state of modes a and b, and that the corresponding Wigner and Q functions will serve as indicators of non-local correlations. The quantum state of our exemplary source, written in terms of the outgoing modes, takes a form analogous to the singlet state of two spin-$\frac{1}{2}$ particles:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle_a|0\rangle_b - |0\rangle_a|1\rangle_b). \tag{6.26}$$

They have demonstrated how the non-locality of this state is revealed by the Wigner and Q functions. Each measuring apparatus in the setup consists of a photon counting

detector, preceded by a beam splitter with power transmission T. The second input port of the beam splitter is fed with a highly excited coherent state $|\gamma\rangle$. As is well-known [114, 115, 116], in the limit $T \to 1$ and $\gamma \to \infty$, the effect of the beam splitter is described by the displacement operator $\hat{D}(\sqrt{1-T}\gamma)$, with the parameter equal to the amplitude of the reflected part of the coherent state. In contrast to the standard approach, They were interested in events when no photons were registered. Let us assign 1 to *no-count event* and 0 otherwise. This establishes a strict analogy with two-particle coincidence experiments, where each of the spatially separated analyzers provides a binary outcome. The role of adjustable parameters of the analyzers is now played by coherent displacements a and b. Consequently, all Bell inequalities derived for a measurement of local realities bounded by 0 and 1 can be applied to test the nonlocal character of correlations obtained in our setup. This type of measurement is described by a pair of two orthogonal projection operators depending on the coherent displacement $\alpha = \sqrt{1-T}\gamma$:

$$\hat{Q}(\alpha) = \hat{D}(\alpha)|0\rangle\langle 0|\hat{D}^{\dagger}(\alpha), \tag{6.27}$$

and

$$\hat{P}(\alpha) = \hat{D}(\alpha)\sum_{n=1}^{\infty}|n\rangle\langle n|\hat{D}^{\dagger}(\alpha), \tag{6.28}$$

which satisfy the completeness relation:

$$\hat{Q}(\alpha) + \hat{P}(\alpha) = \mathbb{I}. \tag{6.29}$$

The joint probability for no photon detections at both detectors is expressed as:

$$Q_{ab}(\alpha,\beta) = \langle\Psi|\hat{Q}_a(\alpha)\otimes\hat{Q}_b(\beta)|\Psi\rangle = \frac{1}{2}|\alpha-\beta|^2 e^{-|\alpha|^2-|\beta|^2}. \tag{6.30}$$

Meanwhile, the probability of no detection at one detector, say detector a, is given by:

$$Q_a(\alpha) = \langle\Psi|\hat{Q}_a(\alpha)\otimes\mathbb{I}_b|\Psi\rangle = \frac{1}{2}\left(|\alpha|^2+1\right)e^{-|\alpha|^2}, \tag{6.31}$$

and similarly, for detector b, we have:

$$Q_b(\beta) = \langle\Psi|\mathbb{I}_a\otimes\hat{Q}_b(\beta)|\Psi\rangle = \frac{1}{2}\left(|\beta|^2+1\right)e^{-|\beta|^2}. \tag{6.32}$$

To test nonlocality, measurements are carried out for two settings of coherent displacements: one with zero displacement and another with either α for mode a or β for mode b. Using these four measurement outcomes, we can form the Clauser-Horne combination [90] as:

$$CH = Q_{ab}(0,0) + Q_{ab}(\alpha,0) + Q_{ab}(0,\beta) - Q_{ab}(\alpha,\beta) - Q_a(0) - Q_b(0), \qquad (6.33)$$

which must satisfy the inequality $-1 \leq CH \leq 0$ in any local hidden variable theory. Next, assume the coherent displacements α and β have the same magnitude, i.e., $|\alpha|^2 = |\beta|^2 = J$, and differ only by a phase factor $\beta = e^{2i\omega}\alpha$. Substituting these values, the Clauser-Horne expression simplifies to:

$$CH = -1 + Je^{-J} - 2Je^{-2J}\sin^2\omega. \qquad (6.34)$$

The largest violation occurs when the phase difference ω is such that the term involving $\sin^2\omega$ is minimized, which happens when the coherent displacements have opposite phases, i.e., $\beta = -\alpha$. Finally the great result is now evident, since with a single-type measurement, they obtain the violation of CH inequalities of correlations described by $Q_ab(\alpha,\beta)$. This one is equal, up to a multiplicative constant $1/\pi^2$, to Q function of the initial state $|\Psi\rangle$. This definition has an obvious operational meaning, as we have discussed an experiment in which the non-local character of the Q function can be tested. To provide an operational interpretation of the Wigner function, they examined next the case where the detectors are able to distinguish the number of photons absorbed. We assign the outcome $+1$ or -1 to each event, depending on whether an even or odd number of photons is detected. This type of measurement can be described using two projection operators:

$$\hat{\Pi}_{+1}(\alpha) = \hat{D}(\alpha)\sum_{k=0}^{\infty}|2k\rangle\langle 2k|\hat{D}^{\dagger}(\alpha), \qquad (6.35)$$

and

$$\hat{\Pi}_{-1}(\alpha) = \hat{D}(\alpha)\sum_{k=0}^{\infty}|2k+1\rangle\langle 2k+1|\hat{D}^{\dagger}(\alpha). \qquad (6.36)$$

The correlation function measured in our setup is determined by the expectation value of the following operator:

$$\hat{\Pi}_{ab}(\alpha,\beta) = \left(\hat{\Pi}_{+1}(\alpha) - \hat{\Pi}_{-1}(\alpha)\right) \otimes \left(\hat{\Pi}_{+1}(\beta) - \hat{\Pi}_{-1}(\beta)\right). \qquad (6.37)$$

The link with the Wigner function ,of the state $|\Psi\rangle$, is immediately clear if we rewrite this operator in terms of the parity operator $-1^{\hat{n}_a+\hat{n}_b}$:

$$\hat{\Pi}_{ab}(\alpha,\beta) = \hat{D}(\alpha)\hat{D}(\beta)(-1)^{\hat{n}_a+\hat{n}_b}\hat{D}^{\dagger}(\alpha)\hat{D}^{\dagger}(\beta), \qquad (6.38)$$

which is one of equivalent definitions of the Wigner function [117, 118]. An easy calculation provides the expectation value of the operator $\hat{\Pi}_{ab}(\alpha,\beta)$ for the state $|\Psi\rangle$:

$$\Pi_{ab}(\alpha,\beta) = \langle\Psi|\hat{\Pi}_{ab}(\alpha,\beta)|\Psi\rangle = (2|\alpha-\beta|^2-1)e^{-2|\alpha|^2-2|\beta|^2}. \tag{6.39}$$

We now introduce the average of the Bell operator:

$$B = \Pi_{ab}(0,0)+\Pi_{ab}(\alpha,0)+\Pi_{ab}(0,\beta)-\Pi_{ab}(\alpha,\beta), \tag{6.40}$$

where local hidden variable theories impose the constraint $-2 \leq B \leq 2$. For simplicity, we assume the magnitudes of the coherent displacements to be equal, i.e., $|\alpha|^2 = |\beta|^2 = J$, with a certain phase difference between them, $\beta = e^{2i\omega}\alpha$. Under these assumptions, the combination B takes the form

$$B = -1+(4J-2)e^{-2J}-8J\sin^2(\omega)e^{-4J}. \tag{6.41}$$

This result violates the lower bound imposed by local hidden variable theories. The strongest violation occurs when $\omega = \frac{\pi}{2}$, that is, when the coherent displacements have opposite phases. Therefore, its equivalence to the correlation function that violates the CHSH inequality was also derived for the Wigner function. It is interesting at this point to understand whether the non-locality just demonstrated is in any way related to the non-positivity of the Wigner function. The Wigner function of the state $|\Psi\rangle$, which contains only a single photon, is not positive definite and demonstrates the non-local nature of quantum correlations. This non-local characteristic of the phase-space Wigner function can be directly observed in an experiment involving a detector capable of resolving the number of absorbed photons. However, it is important to note that if one performs the same measurement for an incoherent mixture of the two components that make up the state $|\Psi\rangle$, the joint correlation is given by

$$(2|\alpha|^2+2|\beta|^2-1)e^{-2|\alpha|^2-2|\beta|^2}. \tag{6.42}$$

This joint correlation corresponds to the Wigner function of the incoherent mixture. While this function is also not positive definite, it does not display quantum interference effects. As a result, in this case, the Bell inequality is not violated. This highlights that the non-positivity of the Wigner function alone does not guarantee the violation of local realism. In conclusion, we have shown that phase-space quasi-distribution functions, specifically the Wigner function and the Q function, encode explicit information regarding the non-locality of entangled quantum states. This stems from the fact that these two quasi-probability distributions are directly linked to non-local correlation functions, which can be experimentally measured in a class of photon-counting experiments involving the application of coherent displacements.

7 Conclusions

As anticipated in the introduction, the central objective of this work has been to provide a detailed analysis of entanglement and its non-trivial relationship with non-locality,

with a particular emphasis on continuous-variable systems. In the first part of this thesis, we focused on the quantization of the electromagnetic field, which serves as the foundation for defining the quantum states used in quantum optics. We introduced key quantum states such as Fock states, coherent states, and squeezed states, which are particularly relevant for quantum communication. Among these, Gaussian states play a fundamental role due to their analytical tractability and direct applicability in quantum information processing [119, 7, 10, 9].

The subsequent section was dedicated to introducing fundamental concepts in quantum information theory, including the definition of qubits and qutrits, as well as the characterization of entanglement in discrete-variable systems. Once the basic concepts were established, we aimed to develop a systematic methodology to distinguish entangled states from separable ones. This led us to introduce the Positive Partial Transpose (PPT) criterion, which provides a clear entanglement measure: Negativity.

The study of discrete systems was later extended to cases where the Hilbert space is infinite-dimensional, specifically to continuous-variable (CV) systems described by position and momentum operators. These systems are often analyzed using semi-classical representations of the density matrix, employing quasi-probability distributions. While these distributions mimic classical behavior, they also retain purely quantum features that cannot be described classically. One of the most important tools in this context is the Wigner function, which allows for a phase-space representation of quantum states. A particular class of states examined within this framework is the Gaussian state class, characterized by a Gaussian Wigner function. This property implies that the entire information defining the state, including its entanglement properties, is fully contained in the covariance matrix σ. In fact, the subsequent analysis exploits this covariance matrix, along with its physically equivalent representations, which can be obtained through appropriate symplectic transformations.

The next step was to precisely define entanglement in continuous-variable systems, and one of the most elegant approaches to this was provided by Simon's criterion. Simon reformulated the PPT criterion in terms of mirror reflection in phase space, offering a fundamental tool for determining the separability of Gaussian states in continuous-variable systems. Then we have extended the Negativity measure to CV systems by exploiting it as a function of the eigenvalues of the covariance matrix.

The next section focuses on the analysis of entanglement for mixed Gaussian states, examining its dependence on both marginal and global mixedness, quantified through the generalized p-entropy. The discussion highlights the existence of two classes of extremal entangled states: Gaussian Least Entangled Mixed States (GLEMS) and Gaussian Maximally Entangled Mixed States (GMEMS). The GLEMS can be regarded as semi-classical states, as their symplectic spectrum partially saturates the Heisenberg minimum uncertainty condition ($\nu_- = 1, \nu_+ > 1$). Conversely, GMEMS belong to the class of asymmetric two-mode squeezed thermal states, which exhibit stronger entanglement properties. A key result of this analysis is that, although the existence of extremal states is robust across different entropic measures, their role is reversed for entropy measures with $p > 2$ for specific values of global and marginal purities. In this regime, GLEMS become the Gaussian maximally entangled states, while GMEMS transform into the Gaussian least entangled states.

The second half of this work focuses on defining the concept of non-locality and its

relationship with entanglement. This concept is introduced through the study of Bell's work and the inequalities he formulated. As discussed, these inequalities can be violated by entangled states, demonstrating the impossibility of simulating quantum correlations using a local hidden variable model. The Gisin theorem is presented, showing that all entangled pure states violate Bell's inequalities, thereby exhibiting non-locality. However, when extending this result to mixed states, this double implication no longer holds. In fact, there exist entangled states that do not violate Bell's inequalities. A paradigmatic example of such mixed states is given by Werner states.
The complexity of the entanglement-nonlocality relationship increases significantly in multipartite systems, where multiple parties share quantum correlations. Unlike the bipartite case, multipartite non-locality manifests in different forms, including:

- **Genuine Multipartite Non-Locality**: Systems where all subsystems share *non-local correlations* that cannot be reduced to bipartite interactions.
- **Bipartite Non-Locality in Multipartite Systems**: Cases where *non-local correlations exist only between specific pairs of subsystems*.

Gisin's results have been generalized to the multipartite case by Popescu, who demonstrated that *multipartite states preserving pairwise entanglement always violate Bell inequalities*. Consequently, in this scenario, entanglement and non-locality become equivalent concepts. For what concern the Genuine multipartite non-locality a fundamental result is *Svetlichny's theorem*, which extends Bell inequalities to three or more parties. If violated, it implies that the observed correlations cannot be decomposed into a combination of bipartite non-locality and local correlations. However, unlike in bipartite pure states where entanglement always implies non-locality multipartite entanglement does not always lead to multipartite non-locality. This distinction underscores the increasing complexity of non-local correlations in multi-party quantum systems.
In the case of continuous-variable (CV) systems, the concept of non-locality extends beyond the traditional Bell inequalities formulated for discrete systems. Given that all the information about a quantum state is contained in its Wigner function, it is natural to ask whether non-local properties can also be inferred from it. A common assumption in early studies of non-locality was that a negative Wigner function was a necessary condition for the violation of Bell inequalities. However, several studies, including Bell's own work, have demonstrated that this is not necessarily the case. A striking example is provided by the Einstein-Podolsky-Rosen (EPR) state, which, despite being highly non-local, is described by a positive Wigner function. This result refuses the conjecture that Wigner negativity is a strict prerequisite for non-locality. Instead, it has been shown that non-locality can be directly analyzed through the expectation values of displaced parity operators, that is inherently non-Gaussian operator, which are closely related to the Wigner function. The key advantage of the parity operator is that, despite acting in an infinite-dimensional Hilbert space, its two eigenvalues (+1 or -1) allow it to simulate the binary nature of spin measurements in traditional Bell tests. This property enables the derivation of a CHSH-type Bell inequality in phase space, which can be used to test non-locality in continuous-variable systems without requiring spin-like observables. This leads to the remarkable conclusion that an EPR state can violate

Bell inequalities based on parity measurements, despite its Wigner function remaining entirely positive. These results have significant implications for the development of continuous-variable quantum technologies, including quantum cryptography, quantum networks, and high-precision quantum sensing. The distinction between entanglement and non-locality in CV systems suggests that Gaussian quantum information protocols may require additional non-Gaussian resources to fully exploit the advantages of non-local correlations.

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