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Generic construction of multipartite entanglement measures

Supervisors

Prof. Davide GIROLAMI

Prof. Vlatko VEDRAL

Candidate

Michele MINERVINI

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"The important thing is not to stop questioning. Curiosity has its own reason for existing. One cannot help but be in awe when he contemplates the mysteries of eternity, of life, of the marvellous structure of reality. It is enough if one tries merely to comprehend a little of this mystery every day."

—Albert Einstein

Summary

Quantum physics is widely recognized as the most successful but counterintuitive physical theory. It has enabled significant technological advancements across diverse fields by providing incredibly accurate predictions about the behaviour of microscopic particles. However, despite its success, many quantum phenomena are still beyond our classical intuition. Since the 1980s, the scientific community has undergone a paradigm shift in the way they study quantum phenomena. Instead of treating them as inexplicable conundrums, they are now viewed as useful resources. This shift led to the emergence of quantum information science, which explores the advantages that quantum theory can offer in processing and transferring information. One of the most useful and still fascinating features of quantum physics is entanglement. It occurs when a quantum state of two or more particles becomes correlated in such a way that the states of the individual particles cannot be described independently; instead, the overall system must be described as a whole. Entanglement is a key resource in quantum information theory, enabling tasks beyond classical resources. It can be manipulated, broadcast, controlled, and distributed. For this reason, it is important to somehow measure the amount of entanglement present in a certain system. However, until now only entanglement for bi-partite systems has been extensively studied and understood.
Understanding and quantifying entanglement for multipartite states remain an ongoing and crucial pursuit in advancing our comprehension of quantum phenomena and leveraging quantum information science for practical applications. Its complexity stems from the myriad ways entanglement can manifest among multiple particles. In the thesis, I start exploring the partitionability of a system of N elements. I point out its connection with entanglement, which showcases the multifaceted nature of multipartite quantum systems. The variety of possible partitions within an N-partite system highlights the diverse nature of multipartite entanglement, leading to numerous distinct types and degrees of entanglement. My first attempt in this thesis is to generalize entanglement of formation -a measure of bipartite entanglement- to the multipartite case. I term it k-NonSep entanglement since it quantifies entanglement in k-non-separable states. This formulation extends the concept of entanglement from bipartite scenarios by measuring the mixedness

of subsystems within N-partite systems. Proofs of its well-defined properties in accordance with three axioms ($\mathbb{A}_{1-3}$) of a valid entanglement measure demonstrate its consistency as a fruitful entanglement measure. However, this measure manifests limitations and downsides. Remarkably, it is not able to take into account the deeper properties of entanglement, such as whether a state belongs to a certain entanglement class under LOCCs. I highlight its inability to distinguish between the classes of the states $|GHZ\rangle$ and $|W\rangle$.
This limitation prompts the introduction of an alternative approach. I cope with the genuine k–partite entanglement of a quantum state, and I first investigate its physical interpretation. It represents how this state differs from the one that can be constructed through LOCCs by the N laboratories when these are divided into subsets of at most k laboratories: it is within these subsets that the state is formed beyond the limitations imposed by LOCC. I propose a novel framework for quantifying genuine multipartite entanglement. It is a generic construction of a measure based on the idea of partitioning a multipartite system into groups of at most k qubits and, for each group, evaluating the sum of bipartite entanglement between each particle. This refined approach successfully rectified the prior limitations. Moreover, due to its construction, the measure satisfies the desiderata properties of an entanglement measure. Further, the evaluation of this entanglement measure for key states corroborates its efficacy in delineating genuine multipartite entanglement, reaffirming its applicability and relevance in characterizing complex quantum states. This methodological framework adds to the toolkit of quantum information theory, aiding in the systematic exploration and understanding of multipartite entanglement in quantum systems. Further research can explore its evaluation for more general multipartite quantum states and its connections to other entanglement measures. Furthermore, it would be intriguing to explore potential applications in quantum computing tasks and idealize experiments to measure entanglement for multipartite states, which could confirm or refute the predicted value with this measure.

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Chapter 1

Introduction

"The inner mysteries of quantum mechanics require a willingness to extend one's mental processes into a strange world of phantom possibilities, endlessly branching into more and more abstruse chains of coupled logical networks, endlessly extending themselves forward and even backwards in time."

—J. C. Ward [1]

Quantum theory emerged in the early 20th century and evolved into a fully-fledged physical theory during the 1920s and 1930s. Its impact was felt not only in science but also on a philosophical level. The elementary units that constitute our world adhere to quantum mechanical laws that contradict what human beings naturally experience. Although it has been incredibly successful in practical applications, our understanding of quantum mechanics is still incomplete and remains an important objective of physics research.

As technology evolves, we will increasingly depend on the quantum world [2]. Therefore, it is essential to develop a solid understanding of the boundary between quantum and classical. Quantum computation and quantum information offer new types of resources that are fundamentally different from traditional resources in classical information theory. One of the best-understood of these resources is quantum entanglement, which is a quantum correlation that is stronger than classical correlation and is believed by many to embody the essence of quantum mechanics.

We say 'best understood', but that is not saying a whole lot! Although we have not yet developed a general theory of quantum entanglement, there has been promising progress towards it. This progress has revealed an interesting structure of the entangled states and some remarkable connections between the properties of noisy

quantum channels and various types of entanglement transformation. However, developing a general theory of entanglement in multi-partite systems is still a challenge. In this thesis, I aim to contribute to the understanding of quantum systems made up of many subsystems by presenting a measure that quantifies its entanglement.

An early account of such a composite quantum system and the possibility of the paradoxical behaviour of this system was pointed out by Albert Einstein, Boris Podolsky, and Nathan Rosen in 1935 [3]. The phenomenon described in the EPR paper was named entanglement by Schrodinger in the same year [4] and qualitatively analyzed by John Bell in 1964 [5]. First experiments to test the nonlocal nature of quantum mechanics were proposed and performed by John F. Clauser and coworkers [6, 7] and Alain Aspect and coworkers [8]. In the 1980s and 1990s, entanglement gained renewed attention and became a crucial component of quantum information theory, the field of research that it belongs to [9]. It is now widely recognized that entanglement is not only a topic for philosophical debates, but it has also emerged as a new quantum resource. Entanglement can be manipulated, controlled, broadcasted, and distributed. It is noteworthy that, while entanglement itself does not carry information, it can be used to accomplish a variety of tasks. Thus, it is important to establish a means of quantifying the amount of entanglement present in a given system.

Initially, entanglement measures were developed based on the entanglement-separability paradigm. However, it has been recently discovered that there are other kinds of quantum states that also exhibit nonclassical features aside from entangled states (for more details, refer to [10]). While entanglement is the foundation of many fundamental quantum tasks and is often considered synonymous with quantum correlations in early studies, it is now recognized that the notion of quantum correlations has a much broader scope. Entanglement, in particular, is the most important type of quantum correlations since it can be identified as nonlocal quantum correlations.

The fields of quantum computation and quantum information theory are at the forefront of a remarkable scientific endeavour. This endeavour seeks to bridge the gap between quantum mechanics and their respective classical counterparts and explore the physics of information in the quantum realm. It holds great promise for developing groundbreaking technological applications. Central to these pursuits is the phenomenon of entanglement, which lies at the heart of quantum mechanics and plays a pivotal role in the advancement of quantum information science. The aim of this thesis is to find a suitable measure to quantify entanglement in multipartite systems. This measure must satisfy certain properties to be valid. To accomplish this objective, an introductory chapter will lay out the fundamental theoretical concepts and mathematical tools necessary to understand the subsequent chapters.

Chapter 2

Introduction to entanglement

"In any field, find the strangest thing and then explore it."
—J.A. Wheeler

2.1 Quantum information theory

Quantum mechanics is considered the most precise and comprehensive explanation of the universe we know. It also forms the foundation for understanding the principles of quantum information theory, a field that explores the fundamental aspects that govern the transmission, processing, and storage of information using quantum systems. Unlike classical information theory which deals with bits and classical logic gates, quantum information theory leverages the unique properties of quantum mechanics, such as superposition and entanglement, to offer new computational paradigms and secure communication protocols.
At the core of quantum information theory are *quantum states*, which describe the properties of a quantum system (a detailed description of them is given in section 2.2). Unlike classical bits, which can only exist in one of two states (0 or 1), quantum systems can exist in superpositions of multiple states simultaneously. This superposition enables quantum systems to process information in parallel, potentially leading to computational speed-ups in certain tasks. Operations on quantum systems are done through *quantum gates*, the quantum analogs of classical logic gates, allowing for the manipulation and transformation of quantum information.
The hallmark of quantum mechanics is its non-deterministic feature. *Measurements* of quantum systems are probabilistic and affect the state of the system. Observables, such as position, momentum, or spin, correspond to the properties of a system

that can be measured. The outcomes of measurements are obtained as eigenvalues of the corresponding observables, and the associated probabilities are given by the squared absolute values of the coefficients in the superposition, on the basis of the eigenvectors of the observable being measured. The measurement process "collapses" the system's state to one of the possible measurement outcomes.
Quantum transformations which are plagued by errors are called *quantum channels*. These channels can be noisy, introducing errors and disturbances during transmission. The study of quantum error correction codes and fault-tolerant quantum computation aims to mitigate the effects of noise and ensure reliable quantum information processing.

Quantum information theory also involves the development of measures to quantify various properties of quantum states, channels, and processes. These measures provide insights into the amount of information contained in quantum systems, the degree of entanglement, the efficiency of quantum algorithms, and the capacities of quantum channels. Quantum complexity theory explores the computational resources needed to solve problems on quantum computers. It includes investigating classes of problems that can be solved efficiently with quantum algorithms.
Moreover, quantum information theory has significant implications for secure communication. Quantum cryptography protocols utilize quantum mechanics principles, such as the no-cloning theorem and quantum entanglement, to achieve information-theoretic security.

We have said that one of the properties of quantum mechanics that empowers quantum information theory is *entanglement*. It is a unique property of quantum systems and plays a central role in quantum information theory. It occurs when the quantum states of two or more particles become correlated in such a way that the states of the individual particles cannot be described independently. Instead, the overall system must be described as a whole (more details can be found in section 2.4). Entangled states possess non-classical correlations, and they can be exploited for various quantum information tasks, including quantum communication, teleportation, and quantum cryptography.
The next sections of this chapter provide the necessary background knowledge of quantum mechanics and information theory needed for a thorough grasp of the phenomenon of entanglement.

2.2 Quantum states

At a first stage, quantum mechanics can be formulated using the language of state vectors $|\Psi\rangle$. Indeed, associated with any isolated physical system is a complex vector space with inner product, which is a Hilbert space $\mathcal{H}$, known as the *state space* of the system, and the system is completely described by its *state vector*, which is a unit vector in the system's state space.
The simplest quantum mechanical system, and the system which quantum computing is most concerned with, is the *qubit*. A qubit has a two-dimensional state space. Suppose $|0\rangle$ and $|1\rangle$ form an orthonormal basis for that state space. Then an arbitrary state vector in the state space can be written as

$$|\Psi\rangle = a\,|0\rangle + b\,|1\rangle \tag{2.1}$$

where a and b are complex numbers. The condition that $|\Psi\rangle$ be a unit vector, $\langle\Psi|\Psi\rangle = 1$, is therefore equivalent to $|a|^2 + |b|^2 = 1$. This condition is called *normalization condition* for state vectors.
Quantum mechanics allows also the construction of multiparticle systems through *tensor product*, which is a way of putting vector spaces together to form larger vector spaces. Suppose V and W are Hilbert spaces of dimension m and n respectively. Then $V \otimes W$ is an mn dimensional Hilbert space. The elements of $V \otimes W$ are linear combinations of tensor products $|v\rangle \otimes |w\rangle$ of elements $|v\rangle$ of V and $|w\rangle$ of W. We often use the abbreviated notations $|v\rangle\,|w\rangle$, $|v, w\rangle$ or even $|vw\rangle$ for the tensor product $|v\rangle \otimes |w\rangle$. This can be extended to the general case of a composite system made up of k constituents, for which the Hilbert space is the tensor product of the Hilbert spaces of the component systems $\mathcal{H} = \otimes_{i=1}^{k} \mathcal{H}_i$[1], and the joint state vector of the total system has the form $|\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_k\rangle$ and its linear combinations, where $|\phi_i\rangle$ is the state for system i.

An alternate formulation is possible using a tool known as the density matrix operator $\hat{\rho}$, which is mathematically equivalent to the state vector approach, but it provides a much more convenient language for thinking about some commonly encountered scenarios in quantum mechanics. Indeed, state vector formulation makes sense only for pure states, but in quantum mechanics, there can be also mixed states, which cannot be represented by state vectors. Let us analyze their analogies and differences.
Suppose we know precisely that a quantum system is a state $|\Psi\rangle$. We shall call this situation a **pure state**. The *density matrix operator* related to $|\Psi\rangle$ is defined

[1] throughout this thesis, we will assume that all Hilbert spaces are finite-dimensional unless stated otherwise.

as its outer product of the state vector with itself

$$\hat{\rho} = |\Psi\rangle \langle\Psi|^{2}. \tag{2.2}$$

Instead, suppose now that we do not know exactly in which quantum state the system is. We call this situation a **mixed state**. We could think of M virtual copies of the system, where each copy is in a different state $|\Psi_m\rangle$. Our ignorance about the actual state of the system is described by a density matrix operator characterized by a *statistical distribution* of pure state density matrices

$$\hat{\rho} = \sum_m p_m \ \hat{\rho}_m = \sum_m p_m \, |\Psi_m\rangle \langle\Psi_m| \tag{2.3}$$

where $|\Psi_m\rangle \langle\Psi_m|$ is the density matrix operator related to the m-th possible state, p_m are a set of real non-negative numbers ($0 \leq p_m \leq 1$ and $\sum_m p_m = 1$) describing the probabilities that the system is in the state $|\Psi_m\rangle$.
Both pure and mixed density matrix operators fulfill some fundamental properties:

1. **Hermiticity**

$$\boldsymbol{\hat{\rho} = \hat{\rho}^{\dagger}}$$

   - for pure states:

$$(|\Psi\rangle \langle\Psi|)^{\dagger} = |\Psi\rangle \langle\Psi| \iff \hat{\rho}^{\dagger} = \hat{\rho}$$

   - for mixed states:
     each single pure state $\hat{\rho}_m = |\Psi_m\rangle \langle\Psi_m|$ is Hermitean ($\hat{\rho}_m = \hat{\rho}_m^{\dagger}$) and the weights p_m are real numbers, then

$$\hat{\rho} = \sum_m p_m \hat{\rho}_m \qquad \Rightarrow \qquad \hat{\rho} = \hat{\rho}^{\dagger}$$

2. **Trace equals 1**

$$\mathbf{Tr}\, \boldsymbol{\hat{\rho} = 1}$$

   - for pure states:
     using any basis Φ_α of the Hilbert space such that $|\Psi\rangle = \sum_\alpha c_\alpha |\Phi_\alpha\rangle$,

$$\begin{aligned} \mathrm{Tr}\, \hat{\rho} = \mathrm{Tr}(|\Psi\rangle \langle\Psi|) = \mathrm{Tr}\left(\sum_\alpha c_\alpha |\Phi_\alpha\rangle \, c_\alpha^* \langle\Phi_\alpha| \right) = \\ = \sum_\alpha c_\alpha c_\alpha^* \, \mathrm{Tr}(|\Phi_\alpha\rangle \langle\Phi_\alpha|) = \sum_\alpha c_\alpha c_\alpha^* = \sum_\alpha |c_\alpha|^2 = 1 \end{aligned}$$

[2] In this section I stress the notation for which operators in quantum mechanics are marked with the hat symbol $\hat{}$. In the next sections, I will neglect this notation, giving it as implicit when we are in the presence of operators.

- for mixed states:

$$\mathrm{Tr}\,\hat{\rho} = \mathrm{Tr}\left(\sum_m p_m \hat{\rho}_m\right) = \sum_m p_m \underbrace{\mathrm{Tr}\,\hat{\rho}_m}_{=1} = \sum_m p_m = 1$$

3. **Positivity**

$$\boldsymbol{\langle \Phi |\, \hat{\rho}\, | \Phi \rangle \geq 0 \qquad \forall \text{ state } \Phi}$$

- for pure states:

$$\langle \Phi |\, \hat{\rho}\, | \Phi \rangle = \langle \Phi | \Psi \rangle \langle \Psi | \Phi \rangle = |\, \langle \Psi | \Phi \rangle\, |^2 \geq 0$$

- for mixed states:

$$\langle \Phi |\, \hat{\rho}\, | \Phi \rangle = \langle \Phi | \sum_m p_m \hat{\rho}_m\, | \Phi \rangle = \sum_m \underbrace{p_m}_{\geq 0} \underbrace{\langle \Phi |\, \hat{\rho}_m\, | \Phi \rangle}_{\geq 0} \geq 0$$

The density matrix operator is, therefore, a *positive semi-definite, Hermitian matrix with unit trace* which provides a comprehensive representation of the quantum states, both pure or mixed. But, actually, there is a criterion to distinguish a pure state from a mixed one. For a pure state, the density matrix is equal to its square (and to any of its integer powers)[3]:

$$\hat{\rho}^2 = |\Psi\rangle \underbrace{\langle \Psi | \Psi \rangle}_{=1} \langle \Psi | = |\Psi\rangle \langle \Psi | = \hat{\rho}$$

$$\boldsymbol{\Rightarrow \hat{\rho}^2 = \hat{\rho}}$$

While this is not true for mixed states:

$$\begin{aligned} \hat{\rho}^2 &= \left(\sum_m p_m |\Psi_m\rangle \langle \Psi_m|\right)\left(\sum_n p_n |\Psi_n\rangle \langle \Psi_n|\right) = \\ &= \sum_{m,n} p_m p_n |\Psi_m\rangle \underbrace{\langle \Psi_m | \Psi_n \rangle}_{\delta_{m,n}} \langle \Psi_n| = \\ &= \sum_m p_m^2 |\Psi_m\rangle \langle \Psi_m| \neq \underbrace{\sum_m p_m |\Psi_m\rangle \langle \Psi_m|}_{=\hat{\rho}} \end{aligned}$$

[3]This is precisely the property of projector operators, $\hat{P}^n = \hat{P}$ (idempotence). Indeed, the pure state density matrix operator $\hat{\rho}$ related to $|\Psi\rangle$ is a *projection operator*, in the sense that its action on a generic state $|\Phi\rangle$ of the system is to return the state obtained by projecting $|\Phi\rangle$ along the direction of $|\Psi\rangle$,

$$\hat{\rho}\,|\Phi\rangle = |\Psi\rangle \underbrace{\langle \Psi | \Phi \rangle}_{\text{coeff}}$$

and the result is thus a state "directed along the direction of $|\Psi\rangle$".

$$\Rightarrow \hat{\boldsymbol{\rho}}^{\mathbf{2}} \neq \hat{\boldsymbol{\rho}}$$

From the idempotence and property 2, for pure states we have $\mathrm{Tr}\,\hat{\rho}^2 = 1$. Instead, for mixed states we have

$$\begin{aligned}\mathrm{Tr}\,\hat{\rho}^2 &= \mathrm{Tr}\left(\sum_m p_m^2 \left|\Psi_m\right\rangle \left\langle\Psi_m\right|\right) = \\ &= \sum_m p_m^2 \underbrace{\mathrm{Tr}(\left|\Psi_m\right\rangle \left\langle\Psi_m\right|)}_{=1} = \\ &= \sum_m p_m^2 < 1\end{aligned}$$

Thus, a useful criterion to distinguish pure states from mixed ones is given by evaluating the trace of the square of the density matrix:

$$\begin{aligned}\boxed{\mathrm{Tr}\,\hat{\rho}^2 = 1} \quad &\Longleftrightarrow \quad \text{pure state} \\ \boxed{\mathrm{Tr}\,\hat{\rho}^2 < 1} \quad &\Longleftrightarrow \quad \text{mixed state}\end{aligned} \tag{2.4}$$

We have seen that we have a pure state when we know precisely that the quantum system is in a state $|\Psi\rangle$. However, when we measure an observable A in an experiment, still the result is not deterministic. Indeed if we repeat the experiment many times, by preparing every time the system in the *same* state $|\Psi\rangle$, we obtain in general different outcomes. The possible outcomes are the set $\{a_\alpha\}$ of eigenvalues of the operator $\hat{A}$ associated with the observable A we are considering. This lack of determinism is an intrinsic feature of Quantum Mechanics[4] and is unavoidable, even for pure states.
Instead, in the case of mixed states, we have a twofold uncertainty: a *Statistical uncertainty*, related to the fact that we do not know in which state the system is (this is similar to the case of a classical system); a *Quantum uncertainty*, related to the fact that, even if we knew in what state the system is, the outcomes of an experiment would, in general, be not deterministic.
Through the density matrix formalism, we can predict the statistics of experimental outcomes, though. Indeed, both for pure and mixed states, the expectation value of an observable A is given by

$$\langle\hat{A}\rangle = \mathrm{Tr}\big(\hat{\rho}\hat{A}\big) = \mathrm{Tr}\big(\hat{A}\hat{\rho}\big). \tag{2.5}$$

This relation shows that, once the density matrix operator ρ is known, we can compute the expectation value of any observable A as a trace.

[4]With his famous statement "Gott würfelt nicht", Einstein expressed his doubts about the completeness of such a non-deterministic theory.

Another advantage of dealing with density matrix representation is that, as far as pure states are concerned, using the wavefunction Ψ or the related density matrix operator $\hat{\rho} = |\Psi\rangle \langle\Psi|$ is exactly equivalent, with the additional advantage of the *lack of phase factor ambiguity*. Indeed, for any operator $\hat{A}$, the expectation value is given by

$$\langle \hat{A} \rangle = \int d\mathbf{R} \Psi^*(\mathbf{R}) \hat{A} \Psi(\mathbf{R}) = \mathrm{Tr}\Big(\hat{\rho}\hat{A}\Big);$$

this means that the density matrix operator $\hat{\rho} = |\Psi\rangle \langle\Psi|$ provides the same physical information as the wavefunction $\Psi(\mathbf{R})$. Furthermore, while the wavefunction $\Psi(\mathbf{R})$ is always determined up to an overall phase factor $e^{i\theta}$, the density matrix operator does not have such ambiguity

$$\left|e^{i\theta}\Psi\right\rangle \left\langle e^{-i\theta}\Psi\right| = |\Psi\rangle \langle\Psi| .$$

It is useful to introduce here a specific mixed state: the *completely mixed density operator*. It is denoted as ρ_{mixed} and represents a maximally mixed state where all pure states are equally probable. In other words, it corresponds to a state with no distinguishable features or coherence. Each diagonal element of the density operator is equal for a maximally mixed state in a d-dimensional space, indicating an equal probability for each pure state. Let us denote this probability as p, which is the same for all pure states. Since the trace of the density operator should be equal to 1, we have $\mathrm{Tr}(\rho_{mixed}) = d * p = 1$; solving for p, we get $p = 1/d$. The diagonal elements of the density operator ρ_{mixed} can be written as $\rho_{mixed}(i, i) = p = 1/d$ for all i, where i ranges from 1 to d. Therefore, the completely mixed density operator in d-dimensional space is given by

$$\rho_{mixed} = \frac{1}{d} \mathbb{I} \tag{2.6}$$

where $\mathbb{I}$ is the $d \times d$ identity matrix. We will see in section 2.3 that the completely mixed state is the state with maximal entropy in a d-dimensional Hilbert space.

We have already seen that quantum mechanics allows also the construction of multiparticle systems through tensor product and we have shown the joint state vector of a system composed of n subsystem has the form $|\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_n\rangle$ or all its linear combinations. This can be reformulated also in terms of density matrix operators: the joint state of the total system is $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n$.
Namely, *n-qubit states* can be written as a linear combination of tensor products of Pauli matrices:

$$\rho = \frac{1}{2^n} \sum_{i_1,\ldots,i_n=0}^{n} t_{i_1 \ldots i_n} \; \sigma_{i_1}^1 \otimes \cdots \otimes \sigma_{i_n}^n \tag{2.7}$$

where σ_0 is the identity operator in the Hilbert space of qubit k, and the $\sigma_{i_k}^k$ correspond to the Pauli operators for three orthogonal directions $i_k = 1, 2, 3$.

Perhaps the deepest application of the density operator is as a descriptive tool for *subsystems* of a composite system. Such a description is provided by the *reduced density operator*. Suppose we have physical systems A and B, whose state is described by a density operator ρ^{AB}. The reduced density operator for system A is defined by

$$\rho^A = \mathrm{Tr}_B(\rho^{AB}) \tag{2.8}$$

where Tr_B is a map of operator known as the *partial trace* over system B. The partial trace is defined by

$$\mathrm{Tr}_B(|a_1\rangle \langle a_2| \otimes |b_1\rangle \langle b_2|) = |a_1\rangle \langle a_2| \, \mathrm{Tr}(|b_1\rangle \langle b_2|) \tag{2.9}$$

where $|a_1\rangle$ and $|a_2\rangle$ are any two vectors in the sate space of A, and $|b_1\rangle$ and $|b_2\rangle$ are any two vectors in the state space of B. This can be generalized for a general n-composite quantum system.

To sum up, we have observed that expressing quantum mechanics in terms of the density operator is mathematically equivalent to using the state vector. However, the density operator approach has two significant advantages when it comes to understanding quantum mechanics: firstly, it is useful for describing quantum systems whose state is uncertain, which can be done through mixed state density matrices; secondly, it is helpful for describing subsystems of a composite quantum system using reduced-density matrices.

2.3 Entropy and information

A breakthrough in quantum information theory is treating quantum states as information and asking information-theoretic questions about them.
We first start by presenting the *classical information theory*. Its main measure is the **Shannon entropy**. It is used to quantify the information in a source X that produces messages x_i with probabilities p_i:

$$H(p) = -\sum_i p_i \log(p_i)^5. \tag{2.10}$$

That is, the Shannon entropy measures the uncertainty associated with a classical probability distribution.
The **joint entropy**, given a pair of random variables X and Y, is given by

$$H(X,Y) = -\sum_{x,y} p(x,y) \log p(x,y) \tag{2.11}$$

and may be extended in an obvious way to any vector of random variables.
The **relative entropy** is a useful measure (see [11]) of the closeness of two probability distributions $\{p_i\}$ and $\{q_i\}$ from the same source X:

$$H(p||q) = \sum_i p_i \log\left(\frac{p_i}{q_i}\right). \tag{2.12}$$

It can be proved that the relative entropy is non-negative, $H(p||q) \geq 0$.
Correlations between two different random variables X and Y are measured by the **Shannon mutual information**:

$$H(X:Y) = H(X) + H(Y) - H(X,Y) = H(X) - H(X|Y) \tag{2.13}$$

where $H(X,Y) = -\sum_{ij} p_{ij} \log(p_{ij})$ is the joint entropy and p_{ij} is the probability of outcomes x_i and y_j both occurring. The mutual information measures how much information X and Y have in common, so it is intuitively clear that this is a good measure of correlations, since it shows how far a joint distribution is from the product one in which all the correlations have been destroyed, or alternatively, how distinguishable a correlated state is from a completely uncorrelated one. Indeed, it may also be defined as a special case of the relative entropy, since it is a measure

[5]Given two observables A and B with corresponding probabilities $p(a_i)$ and $p(b_j)$ of observable A of being a_i and of B being b_j, then $S(A) = -\sum_i p(a_i) \log p(a_i) = -\sum_{ij} p(a_i, b_j) \log \sum_j p(a_i, b_j)$, same for $S(B) = -\sum_j p(b_j) \log p(b_j) = -\sum_{ij} p(a_i, b_j) \log \sum_i p(a_i, b_j)$, where we have used the fact that $\sum_j p(a_i, b_j) = p(a_i)$ and $\sum_i p(a_i, b_j) = p(b_j)$.

of how distinguishable a joint probability distribution p_{ij} is from the completely uncorrelated pair of distributions $p_i p_j$,

$$H(p_{ij}||p_i p_j) = H(p_i) + H(p_j) - H(p_{ij}). \tag{2.14}$$

The **conditional entropy** is given by

$$H(X|Y) = H(X,Y) - H(Y) = H(X) - H(X:Y) \tag{2.15}$$

and it measures how uncertain we are, on average, about the value of X given that we know the value of Y.

In physics, the main contrast between quantum and classical systems can be observed in the superposition principle. While classical systems can only exist in a single state, quantum systems can exist in states that involve combinations of different elements of a basis. This unique characteristic of quantum systems is what makes information theory based on quantum mechanics fundamentally different from the classical information theory that has been described so far. [12]. Thus, the Shannon entropy is replaced with the **Von Neumann entropy**:

$$S(\rho) = -Tr(\rho \log \rho). \tag{2.16}$$

The von Neumann entropy is the quantum equivalent of the Shannon entropy, representing the number of qubits required to transmit quantum states from a statistical source. Formally, $S(\rho) = H(\lambda) = -\sum_i \lambda_i \log \lambda_i$, where $\lambda = \{\lambda_i\}$ are the eigenvalues of the state ρ.
The **quantum relative entropy** of a state ρ with respect to another state σ is defined as

$$S(\rho||\sigma) = -S(\rho) - Tr(\rho \log \sigma). \tag{2.17}$$

It can be shown that the quantum relative entropy, as the classical one, is non-negative, $S(\rho||\sigma) \geq 0$.
In a d-dimensional Hilbert space the entropy is at most $\log d$. The entropy is equal to $\log d$ if and only if the system is in the completely mixed state $\mathbb{I}/d$ shown in eq. (2.6). This result follows from the non-negativity of the relative entropy, $0 \leq S(\rho||\mathbb{I}/d) = -S(\rho) + \log d$.
Suppose a composite system AB is in a pure state, then S(A) = S(B). This can be shown in the following way. From the Schmidt decomposition, we know that the eigenvalues of the density operators of systems A and B are the same[6]. Since the

[6]For the Schmidt decomposition, if $|\psi\rangle$ is a pure state of a composite system, AB, then there exist orthonormal states $|i_A\rangle$ for system A, and orthonormal states $|i_B\rangle$ of system B such that $|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$.

entropy is determined completely by the eigenvalues, we have $S(A) = S(B)$. This property of the von Neuman entropy will be used in section 2.4.3 for the definition of an entanglement measure, the entropy of entanglement.
The **joint entropy** $S(\rho_{AB})$ for a composite system ρ_{AB} with two subsystems A and B is given by

$$S(\rho_{AB}) = -Tr(\rho_{AB} \log \rho_{AB}). \tag{2.18}$$

When one system learns information about another, their states become correlated. The degree of correlation can be quantified by the **Von Neumann mutual information** between the two subsystems:

$$I(\rho_{A:B}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}). \tag{2.19}$$

This is usually used to measure the total correlations between the two subsystems of a bipartite quantum system. As in the classical case (2.14), the mutual information is the relative entropy between ρ_{AB} and $\rho_A \otimes \rho_B$.
Quantum entropy fulfills the following properties:

1. *additivity*: $S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$
2. *concavity*: $S(\sum_i \lambda_i \rho_i) \geq \sum_i \lambda_i S(\rho_i)$
3. *subadditivity*: $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$
4. *strong subadditivity*: $S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})$.

The first property is the same as in classical information theory, namely the entropies of independent systems add up. The concavity simply reflects the fact that "mixing increases uncertainty". It is also worth mentioning that the consequence of the strong subadditivity is the so called weak subadditivity described by the *Araki Lieb inequality* [13]:

$$S(\rho_A) + S(\rho_B) \geq S(\rho_{AB}) \geq |S(\rho_A) - S(\rho_B)|. \tag{2.20}$$

Physically, the left-hand side -subadditivity- implies that we have more information (less uncertainty) in an entangled state than if the two states are treated separately. This arises naturally since by treating the subsystems separately we have neglected the correlations (entanglement). We note that if the composite system is in a pure state, then $S(\rho_{AB}) = 0$, and from the right-hand side it follows that $S(\rho_A) = S(\rho_B)$.

2.4 Entanglement

> *"I would not call [entanglement] one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought."*
>
> *—Erwin Schrödinger [14]*

Quantum mechanics violates daily intuition not only because the measured outcome can only be predicted probabilistically but also because of a quantum-specific correlation called **entanglement**, which is usually said to be the characteristic trait of quantum mechanics. All started with the recognition by Einstein, Podolsky, and Rosen (in the original paper [3], the authors didn't consider qubits but position and momentum operators, here I present the reformulation awarded to David Bohm [15]) that two-qubit states such as the superposition $|\psi\rangle = |00\rangle + |11\rangle$, where $|0\rangle$ and $|1\rangle$ are the eigenstates of σ_z, have some kind of non-local "action at a distance" since a measure of the first qubit somehow "changes" the state of the second qubit, no matter how far away it is: if I measure one qubit and obtain $|0\rangle$ ($|1\rangle$), I know immediately that a measurement on the other, on the same basis, will also return the state $|0\rangle$ ($|1\rangle$).
Inspired by the EPR paper, Schrodinger analyzed the physical consequences of quantum formalism. He noticed that the two-particle EPR state does not allow for individual states of the subsystems. This can be shown for any of the Bell states

$$\left|\Psi^{\pm}\right\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}} \quad , \quad \left|\Phi^{\pm}\right\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}} \tag{2.21}$$

which are considered to be the unit of entanglement (also called e-bit) for bipartite systems. For example, let's consider $|\Psi^+\rangle = |00\rangle + |11\rangle$ (where we are neglecting a normalization factor $\frac{1}{\sqrt{2}}$) and two generic single qubit states $|a\rangle = a_0 |0\rangle + a_1 |1\rangle$ and $|b\rangle = b_0 |0\rangle + b_1 |1\rangle$, then we have

$$|a\rangle |b\rangle = a_0 b_0 |00\rangle + a_0 b_1 |01\rangle + a_1 b_0 |10\rangle + a_1 b_1 |11\rangle ;$$

if $|\Psi^+\rangle = |a\rangle |b\rangle$, then we must have $a_0 b_0 = 1,\ a_0 b_1 = 0,\ a_1 b_0 = 0,\ a_1 b_1 = 1$ since $\{|ij\rangle\}$ is an orthonormal basis. If $a_0 b_1 = 0$, then $a_0 = 0$ or $b_1 = 0$; but $a_0 = 0$ is contradiction to $a_0 b_0 = 1$ and $b_1 = 0$ is contradiction to $a_1 b_1 = 1$. Thus Bell states cannot be written as a product state of two single-qubit states

$$\left|\Psi^+\right\rangle \neq |a\rangle |b\rangle .$$

Then Schrödinger concluded: "Thus one disposes provisionally (until the entanglement is resolved by actual observation) of only a common description of the

two in that space of higher dimension. This is the reason that knowledge of the individual systems can decline to the scantiest, even to zero, while that of the combined system remains continually maximal. The best possible knowledge of a whole does not include the best possible knowledge of its parts — and this is what keeps coming back to haunt us" [4]. A pictorial example of this can be given considering again the Bell states $|\Psi^{\pm}\rangle$ and $|\Phi^{\pm}\rangle$. These are pure states of joint systems AB. Their density matrices are given by

$$\rho_{AB}^{\Psi} = \left|\Psi^{\pm}\right\rangle\left\langle\Psi^{\pm}\right| = \frac{1}{2}(|00\rangle\langle 00| \pm |00\rangle\langle 11| \pm |11\rangle\langle 00| + |11\rangle\langle 11|),$$

$$\rho_{AB}^{\Phi} = \left|\Phi^{\pm}\right\rangle\left\langle\Phi^{\pm}\right| = \frac{1}{2}(|01\rangle\langle 01| \pm |01\rangle\langle 10| \pm |10\rangle\langle 01| + |10\rangle\langle 10|).$$

The reduced density matrices are given by

$$\rho_{A}^{\Psi} = \mathrm{Tr}_B(\left|\Psi^{\pm}\right\rangle\left\langle\Psi^{\pm}\right|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{\mathbb{I}}{2},$$

$$\rho_{A}^{\Phi} = \mathrm{Tr}_B(\left|\Phi^{\pm}\right\rangle\left\langle\Phi^{\pm}\right|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{\mathbb{I}}{2}.$$

The latter are mixed states, being $\mathrm{Tr}\left((\frac{\mathbb{I}}{2})^2\right) = 1/2 \leq 1$ (we used the criterion (2.4)). So, for all the Bell states, while the state of the composite system ρ_{AB} is pure and so we have the complete possible knowledge of it, the states of the subsystems are completely mixed and so highly uncertain. This is another hallmark of quantum entanglement: *the whole is more definite than the parts*, and it turns out to be a criterion to identify entanglement for pure states (we will define this criterion in section 2.4.1): entangled states provide information about the whole system and subsystems through a profoundly nonclassical relationship.
Schrödinger's observation that an entangled state gives us more information about the total system than about subsystems can be quantified simply by using von Neumann entropy. Let us consider a pure state of a composite system $|AB\rangle$. We have that $|A\rangle$ is a pure state if and only if there is no entanglement.[7] Hence, $S(A) \neq 0$ if and only if $|AB\rangle$ is entangled. The conditional entropy is given by $S(B|A) = S(A,B) - S(A)$ and, as $S(A,B) = 0$ since $|AB\rangle$ is a pure state, we have

[7]Let show this property. ($\Leftarrow$) Suppose there is no entanglement, then we can write the state as a product state $\rho_{AB} = |a\rangle_A\langle a|_A \otimes |b\rangle_B\langle b|_B$; the reduced density matrix is $\rho_A = \mathrm{Tr}_B\,\rho_{AB} = |a\rangle\langle a|\,\mathrm{Tr}(|b\rangle\langle b|) = |a\rangle\langle a|$, so we have $\mathrm{Tr}\left(\rho_A^2\right) = 1$ and thus ρ_A is a pure state. ($\Rightarrow$) Suppose now $|A\rangle$ is a pure state; because $|AB\rangle$ is a pure state, we can use the Schmidt decomposition and write $|AB\rangle = \sum_i \lambda_i |i_A\rangle|i_B\rangle$; then $\rho_A = \mathrm{Tr}_B(|AB\rangle\langle AB|) = \sum_{i,j}\lambda_i\lambda_j |i_A\rangle\langle j_A|\,\mathrm{Tr}(|i_B\rangle\langle j_B|) = \sum_{i,j}\lambda_i\lambda_j |i_A\rangle\langle j_A|\,\delta_{i,j} = \sum_j \lambda_j^2 |j_A\rangle\langle j_A|$; then, because ρ_A is a pure state, we have $\lambda_j = 1$ and otherwise 0 for others $i \neq j$; it follows that $|\psi\rangle = |j_A\rangle|j_B\rangle$, thus $|\psi\rangle$ is a product state and there is no entanglement. □

therefore $S(B|A) = -S(A)$. Thus we obtain that $S(B|A) < 0$ if and only if $|AB\rangle$ is entangled, that is $S(A) \geq 0$. So, when the conditional entropy is negative, it implies that the joint system AB has less entropy than what would be expected if the subsystems A and B were independent. As we said, this distinct property of entangled states can be used as a criterion to distinguish them from separable states. However, this is a sufficient condition only for pure bipartite states. There are many other criteria to detect entanglement. I will mention some of them later in this thesis (see Chapter (2.4.1)).

Among the criteria used to detect entanglement is the violation of Bell's inequalities, which have an interesting historical origin. After the EPR paper showed the strange property of entanglement (that Einstein called 'spooky action at a distance'), among physicists there was a general discomfort stemming from the apparent conflict between quantum mechanics and the reigning opinion that a complete physical theory should be local and realistic. According to such a theory, (i) a measurement made at one point in space should not be influenced by anything outside its past light cone; (ii) physical properties should have a well-defined value whether they are measured or not (it is believed Einstein once said "I like to think that the moon is there even if I am not looking at it"). The idea that observations made on one part of an entangled pair could instantaneously project the other into a well-defined quantum state contradicts both these axioms, suggesting (i) that faster-than-light communication is possible, and (ii) that physical quantities lack objective reality. Quantum mechanics was therefore deemed incomplete, and local 'hidden variable' (LHV) models were proposed to explain the manifestation of this 'spooky action at a distance'. After many years of heated debate, British physicist John Bell concluded that empirical evidence alone would resolve the controversy and in 1966 published details of an experiment designed to test the deterministic worldview [16]. The key result of this publication was a measurable inequality, derived independently of any type of mathematical formalism, which should be respected by any LHV. A more general version of Bell's inequality was measured for the first time in the 1980s in a series of experiments by Aspect et al. [8], whose results conclusively precluded the existence of local hidden variables, therefore ascribing to quantum mechanics the characteristics of a non-local theory. Since its inception, most of Bell's experiments have yielded similar conclusions: the crucial factor in the Bell theorem is the disparity between the quantum and classical descriptions of correlations, which becomes increasingly apparent. On a fundamental level, nature presents us with a new type of statistical correlations that do not carry any meaningful information, and are encoded in the quantum states of compound systems through entanglement. They are "non-local" in the sense, that

(i) they cannot be described by a LHVM;

(ii) they are non-signalling as the local measurements performed on spatially

separated systems cannot be used to transmit messages.

Thus It has become evident that entanglement is not only a topic of philosophical discussions but also a novel quantum resource that cannot be replicated through classical resources. Remarkably, entanglement is a resource that, though it does not carry information itself, can help in such tasks. One of the applications that shows the operational capabilities of entangled states is *quantum teleportation*. It is a technique for moving quantum states around, even in the absence of a quantum communications channel linking the sender of the quantum state to the recipient, but only via sharing a maximally entangled state and a classical channel between the two parties.

Unfortunately, quantum entanglement has three disagreeable but fascinating features: it has a complex structure, is fragile to the environment, and cannot be increased on average when systems are spatially separated. In recent years considerable effort has been expended on the characterization, manipulation, and quantification of entanglement, attempting to give answers to fundamental questions as: i) how to detect entanglement theoretically and in laboratory; ii) how to prevent an inevitable process of degradation of entanglement; iii) how to characterize, control, and quantify entanglement. In this dissertation, I will stress more on quantification of entanglement, since I will focus on finding a suitable measure for entanglement for multipartite systems.

Until now, though, we mentioned only the case of bipartite systems. Traditionally, starting with Bell, the example which has been most studied was that of non-local correlations between two remote quantum particles. However, it is now clear that the correlations among more than two remote particles present novel and highly nontrivial aspects compared to two-particle entanglement (see for example the correlations displayed by GHZ state [17]). *Multipartite states* refer to quantum states that involve more than two subsystems or particles; for example, a tripartite state would involve three subsystems, so a multipartite state can involve an arbitrary number of subsystems. These states describe the collective quantum properties of the entire system, including entanglement between the subsystems. When considering multipartite scenarios, the phenomenology becomes much more complex compared to the bipartite case. There are super-exponentially many ways to partition the constituents of an N-partite quantum system into non-overlapping groups, with each such partitioning giving rise to a legitimate locality constraint. As a result of this complexity, the theory of multi-partite entanglement is much less canonical than the bipartite version. By this, we mean that in the two-party case questions like “What is a natural unit of entanglement?” or “When is there a maximally entangled state ” tend to have unambiguous natural answers, while this is always never true for more than two subsystems. In addition, there is a basically unique way of quantifying bipartite pure state entanglement, whereas the “right” multi-partite measure strongly depends on the intended use of the entangled states,

as shown in [18]. Another fundamental property of multipartite entanglement is monogamy, which restricts entanglement distribution among subsystems. In simple terms, it means that if two subsystems are maximally entangled with each other, they cannot be maximally entangled with any other subsystems simultaneously. This property is often referred to as "**entanglement monogamy**". It reflects the fact that *entanglement is a limited resource in quantum systems*. When distributing entanglement between subsystems, the amount available for sharing with other subsystems is reduced.
We will deepen these features of multipartite systems in chapter 3, while a detailed definition of multipartite entangled state is already given in the next section 2.4.1, where criteria to detect entanglement will be presented, followed by a description of how entanglement can be manipulated and, mainly, quantified, even if still for the simplistic case of bipartite states.

2.4.1 Entanglement characterization

From the passage from classical to quantum, the "effect" of the replacement of the classical concept of phase space by abstract Hilbert space creates a gap in describing composite systems. Indeed, consider a multipartite system comprising N subsystems. According to classical description, the total state space of the system is the Cartesian product of the N subsystem spaces implying that the state of the entire system can always be expressed as a product of the states of each of the N individual systems. On the opposite, according to the principles of quantum mechanics, the total Hilbert space, denoted as $\mathcal{H}$, is represented as a tensor product of the subsystem spaces

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N, \tag{2.22}$$

where $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_N$ are separable complex Hilbert spaces which are associated with the quantum systems $1, 2, \ldots, N$, respectively.

For a *bipartite* system, determining if it is entangled should not, at least ideally, be difficult. Indeed, in the case of *pure* bipartite state $|\Psi_{AB}\rangle \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, we have an entangled state if and only if it can not be written as a product of two vectors corresponding to Hilbert spaces of subsystems:

$$|\Psi_{AB}\rangle \neq |\phi_A\rangle\,|\psi_B\rangle\,. \tag{2.23}$$

In the opposite case the state is called *product state*. In general, if the vector $|\Psi_{AB}\rangle$

is written in any orthonormal product basis $\{|e_A^i\rangle \otimes \left|e_B^j\right\rangle\}$ as follows[8]

$$|\Psi_{AB}\rangle = \sum_{i=1}^{d_A}\sum_{j=1}^{d_B} A_{ij}^{\Psi} \left|e_A^i\right\rangle \otimes \left|e_B^j\right\rangle,$$

then it is product if and only if the matrix of coefficients $A^{\Psi} = \{A_{ij}^{\Psi}\}$ is of rank one.
In the case of *mixed* bipartite state, we have an entangled state if and only if it can not be written as a mixture of product states:

$$\rho_{AB} \neq \sum_{i=1}^{k} p_i \rho_A^i \otimes \rho_B^i \tag{2.24}$$

where the Caratheodory bound[9] is kept: $k \leq (dim\mathcal{H}_{AB})^2$. In the opposite case the state is called *separable*.

Let's now focus on the case of systems made of more than two subsystems. The superposition principle allows us to write the total state of the system in the following form

$$|\psi\rangle = \sum_{\mathbf{i}_N} c_{\mathbf{i}_N} |\mathbf{i}_N\rangle, \tag{2.25}$$

where $\mathbf{i}_N = i_1, i_2, \ldots, i_N$ is the multi-index and $|\mathbf{i}_N\rangle = |i_1\rangle \otimes |i_2\rangle \cdots \otimes |i_N\rangle$, which cannot be, in general, described as a product of states of individual subsystems[10] $|\psi\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_N\rangle$. This means that it is generally impossible to assign a single state vector to any of the N subsystems. This formally expresses the phenomenon of entanglement which, in contrast to classical superposition, allows the construction of exponentially large superpositions with only a linear amount of physical resources. This property is often referred to as "quantum parallelism" and it is just what allows us to perform nonclassical tasks.
In practice we mostly encounter mixed states rather than pure. When dealing with mixed states, entanglement is no longer just equated to non-product states as it is with pure states. Rather, a mixed state of N systems is considered entangled if it cannot be expressed as a combination of product states. This is referred to as a convex combination

$$\rho \neq \sum_i p_i \ \rho_1^i \otimes \cdots \otimes \rho_N^i. \tag{2.26}$$

---

[8]here the orthonormal basis $\{\left|e_X^i\right\rangle\}$ spans subspace $\mathcal{H}_X$, $X = A, B$.

[9]The Caratheodory theorem states that the number l in the convex combination can be bounded by the square of the dimension of the global Hilbert space (see [19]).

[10]Sometimes instead of notation $|\psi\rangle \otimes |\phi\rangle$ we use simply $|\psi\rangle\,|\phi\rangle$ and for $|i\rangle \otimes |j\rangle$ even the shorter $|ij\rangle$.

The states that are not entangled according to the above definition are referred to as *separable*.
However, compared to the bipartite case, the qualitative definition of separability and entanglement is wide-ranging in the multipartite case. There are many types of separability, including full separability which generalizes bipartite separability. Indeed, there are multiple ways of partitioning a many-body system. In particular, the number of ways of partitioning an N-partite system is given by the Bell number and it scales exponentially as N grows (in Appendix A a code for evaluating the total number of partitioning a set into disjoint subsets and the Bell number is shown). Let's see what a partition consists of. Given a N-partite quantum system $\mathcal{X}_N = \left\{\mathcal{X}_{[1]}, \mathcal{X}_{[2]}, \ldots, \mathcal{X}_{[N]}\right\}$[11], a ***k*-partition** (with $k = 2, 3, \ldots, N$) of a quantum system $\mathcal{H}$ given in (2.22) is a set $\left\{\mathcal{X}_1, \ldots, \mathcal{X}_k\right\}$ of nonempty subsets of $\{1, \ldots, N\}$ which satisfies:

a) $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset$ for all $i \neq j$ (with $i, j \in \{1, \ldots, k\}$) and $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \cdots \cup \mathcal{X}_k = \{1, \ldots, N\}$;

b) $\mathcal{H}^{\mathcal{X}_1} \otimes \mathcal{H}^{\mathcal{X}_2} \otimes \cdots \otimes \mathcal{H}^{\mathcal{X}_k} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, where $\mathcal{H}^{\mathcal{X}_i} = \bigotimes_{j=1}^{k_i} \mathcal{H}_{n_{ij}}$ if $\mathcal{X}_i = \{n_{i_1}, n_{i_2}, \ldots, n_{i_{k_i}}\}$ with $n_{i_1} < n_{i_2} < \cdots < n_{i_{k_i}}$.

Throughout this thesis, a k-partition is denoted by $\mathcal{X}_1|\mathcal{X}_2|\ldots|\mathcal{X}_k$. For the sake of clarity and to become familiar with the notation adopted, we provide an example for the case $N = 4$. The quantum system is given by $\mathcal{X}_4 = \left\{\mathcal{X}_{[1]}, \mathcal{X}_{[2]}, \mathcal{X}_{[3]}, \mathcal{X}_{[4]}\right\}$. The 1-partition is simply given by $\mathcal{X}_{1234}$. The 2-partitions are: $\mathcal{X}_{12}|\mathcal{X}_{34}$, $\mathcal{X}_{13}|\mathcal{X}_{24}$, $\mathcal{X}_{14}|\mathcal{X}_{23}$. The 3-paritions are: $\mathcal{X}_1|\mathcal{X}_{23}|\mathcal{X}_4$, $\mathcal{X}_1|\mathcal{X}_{24}|\mathcal{X}_3$, $\mathcal{X}_1|\mathcal{X}_{34}|\mathcal{X}_2$, $\mathcal{X}_2|\mathcal{X}_{14}|\mathcal{X}_3$, $\mathcal{X}_2|\mathcal{X}_{13}|\mathcal{X}_4$, $\mathcal{X}_3|\mathcal{X}_{12}|\mathcal{X}_4|$. The 4-parition is given by $\mathcal{X}_1|\mathcal{X}_2|\mathcal{X}_3|\mathcal{X}_4$.
A *pure* state $|\Psi\rangle$ is called ***k*-separable** if there exists a k-partition $\mathcal{X}_1|\mathcal{X}_2|\ldots|\mathcal{X}_k$ of $\mathcal{H}$ such that $|\Psi\rangle$ can be written as the product of at most $k \leq N$ states $\left|\psi_i^{\mathcal{X}_i}\right\rangle$ belonging to a set of non overlapping partitions $i = 1, \ldots, k$:

$$|\Psi\rangle = \left|\Psi_1^{\mathcal{X}_1}\right\rangle \otimes \left|\Psi_2^{\mathcal{X}_2}\right\rangle \otimes \cdots \otimes \left|\Psi_k^{\mathcal{X}_k}\right\rangle \tag{2.27}$$

where $\left|\Psi_i^{\mathcal{X}_i}\right\rangle \in \mathcal{H}^{\mathcal{X}_i}$ is a pure state of $\mathcal{H}^{\mathcal{X}_i}$. Instead, an m-partite *mixed* state ρ is called ***k*-separable** if it can be written as a convex combination of k-separable pure states:

$$\rho = \sum_i p_i \left|\Psi_i\right\rangle \left\langle\Psi_i\right|, \tag{2.28}$$

[11]For the sake of clarity, let us point out the notation adopted: $\mathcal{X}_{[i]}$ indicates a single subsystem i; $\mathcal{X}_i$ indicates a subset of subsystems, that is $\mathcal{X}_i = \{\mathcal{X}^i_{[1]}, \mathcal{X}^i_{[2]}, \ldots, \mathcal{X}^i_{[m_i]}\}$; then $\mathcal{X}_N$ indicates the full set of subsystems.

where $|\Psi_i\rangle$ might be k-separable under different k-partitions, and $p_i \geq 0$, $\sum_i p_i = 1$. In particular, for an N-partite quantum state ρ, if it is N-separable, it is called **fully separable**

$$\rho = \sum_{i=1}^{l} p_i \ \rho_1^i \otimes \cdots \otimes \rho_N^i \tag{2.29}$$

where the Caratheodory bound is kept: $l \leq (dim\mathcal{H}_{\mathcal{X}_1 \ldots \mathcal{X}_N})^2$. Instead, if it is not k-separable for any $k > 1$ (or, equivalently, not 2-separable), we say it is a **genuinely mulipartite entangled state**.

In this way, we obtain a *hierarchy for entanglement*, where k-separable classes are considered to be more entangled than the l-separable ones for $k < l$. Indeed, there exist different classes of entangled states depending on how many parties are entangled across certain partitions. For example a pure state of three parties A, B, C of the form $|\Psi\rangle_{ABC} = |\phi\rangle_{AB} \otimes |\chi\rangle_C$ is in general entangled, as long as $|\phi\rangle_{AB}$ is entangled, however party C is not correlated with A and B and in this sense state $|\Psi\rangle_{ABC}$ is not fully or genuinely multipartite entangled. In other words, every subsystem in a genuinely multipartite entangled state is entangled with the others. Besides the level of multipartite entanglement, a further complication to the complete characterization of the entanglement of a state may be which particular parties are entangled. Indeed, consider a tripartite state σ_{ABC} and suppose it is biseparable. It remains then to know which particular decomposition it admits, for instance, can it be expressed as $\sigma = p_1\sigma_A \otimes \sigma_{BC} + p_2\sigma_{AB} \otimes \sigma_C$? Then, we will call a state *decomposable with respect to a set of partitions* of the parties if it can be written as a convex combination of states which are separable with respect to any of these (and only these) partitions.

Therefore, it is now plain to see that, in practice, it is hard to decide if a given state is separable or entangled based on the definition itself. Indeed, one of the fundamental problems concerning entanglement is the so-called **separability problem**. Some operational criteria have been defined to identify separable states, even if their straightforward application is for the bipartite case, while their generalization for the multipartite case is hard and not obvious. Below some *separability criteria* for the *bipartite* case are reported:

1. *Entropy inequalities*
   We have already seen that one of the aspects of entanglement involves a profoundly nonclassical relationship between the information entangled states provide us about the entire system and the information they provide us about subsystems. Indeed, we have that for separable states these α-entropy inequalities hold:

$$S_\alpha(\rho_{AB}) \geq S_\alpha(\rho_A), \quad S_\alpha(\rho_{AB}) \geq S_\alpha(\rho_B) \tag{2.30}$$

where $S_\alpha(\rho) = (1-\alpha)^{-1} \log \mathrm{Tr}\, \rho^\alpha$ is the α-Renyi entropy for $\alpha \geq 0$ [20], here $\rho_A = \mathrm{Tr}_B(\rho_{AB})$ and similarly for ρ_B. If α tends to 1 decreasing, one obtains the von Neumann entropy $S_1(\rho) = S(\rho)$ as a limiting case. Thus, the entropy of a subsystem can be greater than the entropy of the total system only when the state is entangled.

2. *Positive partial transpose (PPT) criterion*
It was proposed in [21] and states that if ρ_{AB} is separable then the new matrix $\rho_{AB}^{T_B}$, where the operation T_B is called partial transpose, with matrix elements defined in some fixed product basis as

$$\langle m| \langle \mu| \rho_{AB}^{T_B} |n\rangle |\nu\rangle = \langle m| \langle \nu| \rho_{AB} |n\rangle |\mu\rangle ,$$

is a density operator (i.e. has nonnegative spectrum), which means automatically that $\rho_{AB}^{T_B}$ is also a quantum state (it also guarantees positivity of $\rho_{AB}^{T_A}$ defined in an analogous way). A fundamental fact is that PPT condition is a necessary and sufficient condition for separability of $2 \otimes 2$ and $2 \otimes 3$ cases. Thus it gives a complete characterization of separability in those cases.

3. *Separability via positive, but not completely positive maps*
It has been recognized that a state ρ_{AB} is separable if and only if this condition

$$[\mathbb{I} \otimes \Lambda_B](\rho_{AB}) \geq 0 \tag{2.31}$$

is satisfied for all P but not CP[12] maps Λ: $\mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$ where $\mathcal{H}_A$, $\mathcal{H}_B$ describe the left and right subsystems of the system AB.

4. *Separability via entanglement witnesses*
Entanglement witnesses are observables that completely characterize separable states and have been found very important in experimental detection of entanglement. Their origin stems from geometry: the convex sets can be described by hyperplanes (see Fig. 2.1). This translates into the statement that the state ρ_{AB} belongs to the set of separable states if it has nonnegative mean value $\mathrm{Tr}(W\rho_{AB}) \geq 0$ for all observables W that (i) have at least one negative eigenvalue and (ii) have nonnegative mean value on product states or, equivalently, satisfy the nonnegativity condition $\langle \psi_A| \langle \phi_B| W |\psi_A\rangle |\phi_B\rangle \geq 0$ for all pure product states $|\psi_A\rangle |\phi_B\rangle$.

[12]The map Λ is completely positive if and only if $\mathbb{I} \otimes \Lambda$ is positive for identity map $\mathbb{I}$ on any finite-dimensional system.

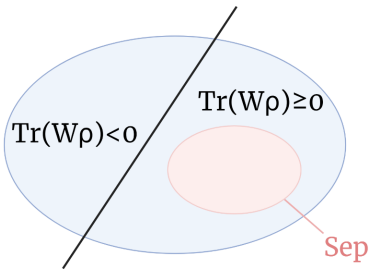

Figure 2.1: The big blue circle represents all possible states, and the smaller red one represents separable states. The line corresponds to the hyperplane representing the entanglement witness W. All states that are located to the right of this hyperplane or belong to it, including all separable states, provide a non-negative mean value of the witness. This can be expressed as $\mathrm{Tr}(W\rho_{sep}) \geq 0$. On the other hand, states that are located to the left of the hyperplane are considered entangled states and can be detected by the witness.

There are many other criteria to detect separable states. A detailed list of them can be found in [19].
In the multipartite case, as we have seen, the qualitative definition of separability and entanglement is much richer than in the bipartite case. The characterization of separability in terms of positive but not completely positive maps and witnesses generalizes naturally, but the extension of separability criteria to the multipartite case is not straightforward and, in general, more difficult.

2.4.2 Entanglement manipulation

Entanglement is by no means a comprehensively understood phenomenon. However, despite its more esoteric characteristics, it is now well-established as a real physical resource. Unfortunately, entanglement is often somewhat fragile and can be diminished or destroyed if the entangled systems experience contact with a noisy environment. By the way, entanglement can be manipulated under various sets of operations, even if a complete understanding of what is possible and impossible has not been reached yet.
It was realized (see [22]) that a natural way to manipulate entanglement is through *local operations and classical communication* (LOCC), which cannot create entanglement for free. Using LOCC operations to study resource transformation is best illustrated in the quantum teleportation process: here, all the distant parties (Alice and Bob) are allowed to perform arbitrary local quantum operations and send classical information: It is not allowed to transfer quantum systems between labs. This is because classical bits are unable to carry quantum information or

create entanglement. Therefore, entanglement processing is a natural class for manipulating entanglement as a resource.
So LOCCs are a special subset of all physically realizable operations on the global system (a more detailed analysis can be found in [23]). Those are implemented in such a way that a multipartite quantum system is restricted to acting locally on its respective subsystems by performing measurements and more general quantum operations; however in order to enhance their measurement strategies, the parties are free to communicate any classical data, which includes the sharing of randomness and previous measurement results (see Fig. 2.2). One then distinguishes various subclasses of operations in general, as described in [19].

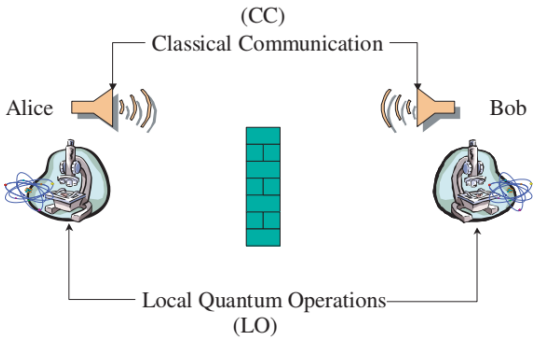

Figure 2.2: In a standard quantum communication scenario, Alice and Bob can perform localized generalized measurements in their respective labs and communicate classically. The brick wall signifies the impossibility of coherent exchange of quantum particles between Alice and Bob. This set of operations is generally referred to as LOCC. (reference [24])

The most general quantum operation that transforms one quantum state into the other is a *probabilistic* or *stochastic* physical operation of the type

$$\rho \longrightarrow \frac{\Lambda(\rho)}{\mathrm{Tr}(\Lambda(\rho))} \tag{2.32}$$

with trace nonincreasing CP map, i.e. a map satisfying $\mathrm{Tr}(\Lambda(\rho)) \leq 1$ for any state ρ, which can be expressed in the form

$$\Lambda(\rho) = \sum_i V_i \rho V_i^\dagger \tag{2.33}$$

with $\sum_i V_i^\dagger V_i \leq \mathbb{I}$. The operation above takes place with the probability $\mathrm{Tr}(\Lambda(\rho))$ which depends on the argument ρ. The probability is equal to one if and only if the CP map Λ is trace-preserving (which corresponds to $\sum_i V_i^\dagger V_i = \mathbb{I}$ in (2.33)); in such a case Λ is called *deterministic* or a *quantum channel*.

(a) From two partially entangled states to one completely entangled state

(b) From a completely entangled state to two partially entangled states

Figure 2.3: Entanglement Distillation (a) and Dilution (b)

So established *distant lab* (or LOCC) paradigm plays a fundamental role in entanglement theory. Important results have been achieved within this paradigm, including a framework for manipulating pure states. Then, since in the laboratory one usually meets mixed states representing *noisy entanglement*, not much useful for quantum information processing, it has been demonstrated that pure entanglement can be distilled from noisy entanglement in the asymptotic regime, a process called **entanglement distillation**, which converts a partially entangled state to a maximally entangled state (Fig. 2.3a). The other is **entanglement dilution**, which converts a maximally entangled state to a given partially entangled state (Fig. 2.3b). These two processes will then be used in the next section 2.4.3 to define operational measures of entanglement.

2.4.3 Entanglement measures

Measures of entanglement play a crucial role in quantifying and characterizing the degree of entanglement present in quantum systems. The question remains open about how much entanglement a certain state contains. However, this question is not entirely well defined unless we state what physical circumstances characterize the amount of entanglement. Indeed entanglement is a multifaceted and complex phenomenon, and there is no single measure that captures all its aspects perfectly. Thus researchers have developed various entanglement measures based on different mathematical frameworks, such as *entropy-based measures*, *geometric measures*, and *operational measures*. Moreover, entanglement theory is an active area of research, and new measures continue to be proposed and explored. This ongoing development is driven by the need to deepen our understanding of entanglement and uncover new properties.
An entanglement measure is a mathematical quantity that should capture the essential features that we associate with entanglement and ideally should be related to some operational procedure. To be more precise, an entanglement measure $E(\rho)$

is a mapping from density matrices into positive real numbers:

$$\rho \to E(\rho) \in \mathbb{R}^+.$$

There are some fundamental axioms that every entanglement measure has to satisfy:

($\mathbb{A}_1$) $E(\rho) = 0$ if and only if ρ is separable;

($\mathbb{A}_2$) local unitary operations leave $E(\rho)$ invariant;

($\mathbb{A}_3$) $E(\rho)$ does not increase over LOCC.

Any function E satisfying these conditions is called *entanglement monotone*.
The origin of condition ($\mathbb{A}_1$) is that separable states are known to contain no entanglement (as shown in section 2.4.1). The reason for condition ($\mathbb{A}_2$) is that local unitary transformations represent a local change of basis only and leave quantum correlations unchanged, so this ensures that a local change of basis does not affect the amount of entanglement. Formally, for a N-partite system we have $E(\rho) = E(U_1 \otimes U_2 \otimes \cdots \otimes U_N \ \rho \ U_1^\dagger \otimes U_2^\dagger \otimes \cdots \otimes U_N^\dagger)$. The reason for condition ($\mathbb{A}_3$) is that any increase in correlations achieved by LOCC should be classical (see section 2.4.2) and therefore entanglement should not be increased.
For reasons of mathematical convenience, there can be some additional requirements for entanglement measures, such as:

- *Convexity*: for a mixed state $\rho = \sum_i p_i \rho_i$ where the p_i are the weights of the different ρ_i components, an entanglement measure is convex if

$$E\left(\sum_i p_i \rho_i\right) \leq \sum_i p_i E(\rho_i). \tag{2.34}$$

  Requiring this mathematical useful property is sometimes justified as capturing the loss of information, i.e. describing the process of going from a selection of identifiable states ρ_i that appear with rates p_i to a mixture of these states of the form $\rho = \sum_i p_i \rho_i$.

- *Additivity*: given a state ρ, we say an entanglement measure is additive if

$$E(\rho^{\otimes n}) = nE(\rho) \tag{2.35}$$

  is satisfied for all integers n. Some significant entanglement measures do not satisfy this condition, and for this reason, additivity is not included as a basic axiom.
  A much stronger requirement could be to demand *full additivity*, by which we mean that for any pair of states σ and ρ we have $E(\sigma \otimes \rho) = E(\sigma) + E(\rho)$. This requirement may not be satisfied by all quantities that meet the basic

properties. Indeed, even such basic measures as distillable entanglement (it will be treated later) may not satisfy this property. For this reason, we have not included the full additivity in our set of properties.

Some entanglement measures have an operational definition. Others are constructed on the basis of the above axioms, and not all have a physical interpretation.The following summary of entanglement measures is intended as a qualitative overview, rather than a mathematically detailed discussion. I have focused on measures relevant to the present work - entanglement of formation - or have an intuitive physical interpretation - distance-based measure -, and I have dealt only with the bipartite case, being this case the simplest one and being the generalization to the multipartite case the objective of the next chapter. I refer the reader to [25, 24, 26] and references therein for a more comprehensive and fully mathematical treatment.
So here is a brief list of the most commonly used entanglement measures in the bipartite case:

- **Distillable entanglement**
  Entanglement distillation is an entanglement purification protocol whereby Alice and Bob share n copies of an entangled state ρ, perform LOCC and obtain $k < n$ copies of a Bell state. The distillable entanglement is defined as the optimal ratio k/n yielded by this process in the limit of large n. It is interesting to note that all entangled bipartite states can be distilled, however this may require the help of an activator state. Activator states, on the other hand, cannot be distilled; their entanglement is inaccessible, hence the term 'bound entanglement' by which we refer to these states.

- **Entanglement cost**
  It represents the number of singlet states N one has to share to distil an arbitrary state, such that errors become infinitesimal in the limit of large N. It is therefore defined in relation to a process that is the opposite of distillation.

- **Distance-based measure**
  The entanglement of a bipartite quantum state ρ_{AB} may be quantified by how distinguishable it is from the 'nearest' separable state,

$$E(\rho) = \min_{\sigma \in \mathcal{S}} D(\rho||\sigma) \tag{2.36}$$

  where D is any measure of distance between the two density matrices ρ and σ such that $E(\rho)$ satisfies the axioms reported previously, and $\mathcal{S}$ is the set of separable states. To satisfy condition $(\mathbb{A}_1)$ it is sufficient to demand that $D(\rho||\sigma) = 0$ if and only if $\rho = \sigma$. Due to the invariance of D under local unitary transformations, condition $(\mathbb{A}_2)$ is automatically satisfied: $D(\rho||\sigma) =$

$D(U\rho U^\dagger||U\sigma U^\dagger)$. For condition ($\mathbb{A}_3$) to be satisfied it is sufficient to demand that $D(\rho||\sigma)$ has the property that it is non-increasing under every completely positive trace-preserving map Θ, i.e. $D(\Theta\rho||\Theta\sigma) \leq D(\rho||\sigma)$[13]. This can easily be seen in the following. If σ^* is a separable density operator that realizes the minimum of eq. (2.36), then because $\Theta\mathcal{S} \subset \mathcal{S}$ we find $E(\rho) := D(\rho||\sigma^*) \geq D(\Theta\rho||\Theta\sigma^*) \geq \min_{\sigma\in\mathcal{S}} D(\Theta\rho||\sigma) = E(\Theta\rho)$.
The amount of entanglement given by eq. (2.36) can be interpreted as finding a state σ^* in $\mathcal{S}$ that is closest to ρ under the measure D (see Fig. 2.4 for pictorial representation).

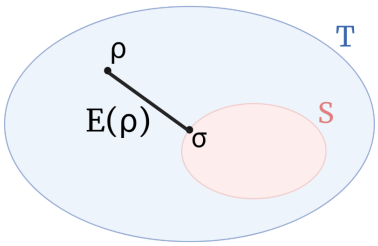

Figure 2.4: The set of all density matrices, $\mathcal{T}$, is represented by the outer circle. Its subset, a set of separable states $\mathcal{S}$ is represented by the inner circle. A state ρ belongs to the entangled states, and σ^* is the separable state that minimizes the distance $D(\rho||\sigma)$, thus representing the amount of entanglement in ρ.

Entanglement can be measured using relative entropy as a distance measure [27]. The relative entropy is a non-negative quantity (see section 2.3) and, due to this property, it often appears in the context of distance measure though technically it is not a distance, e.g. it is not symmetric. **Relative entropy of entanglement**, defined as

$$E_{RE}(\rho_{AB}) = \min_{\sigma_{AB}\in\mathcal{S}} S(\rho_{AB}||\sigma_{AB}) \tag{2.37}$$

has been shown to be a useful measure of entanglement and that fulfils the three fundamental properties ($\mathbb{A}_{1-3}$).

- **Entropy of entanglement**
  This measure is applied to *pure* bipartite state, and says that entanglement E is defined as the von Neumann entropy of either of the two subsystems A and

[13]Indeed, CPTP maps are more general than unitary operations. Thus if condition ($\mathbb{A}_3$) is satisfied for CPTP maps, it is satisfied also for unitary operations.

B:

$$\begin{aligned} E(|\psi\rangle) &= S(\rho_A) = -\operatorname{Tr}(\rho_A \log_2 \rho_A) \\ &= S(\rho_B) = -\operatorname{Tr}(\rho_B \log_2 \rho_B) \end{aligned} \tag{2.38}$$

where ρ_A is the partial trace of $|\psi\rangle\langle\psi|$ over subsystem B, and ρ_B is defined similarly. We have already shown in 2.3 why for pure bipartite states we have $S(\rho_A) = S(\rho_B)$.
Let's now show that the entropy of entanglement fulfills the three axioms and so is a suitable measure:

($\mathbb{A}_1$) We have already shown in section 2.4 that if a pure bipartite state has no entanglement, then its subsystems are pure states; thus if $|\psi\rangle$ is a pure product state, then $E(|\psi\rangle) = S(\rho_A) = S(\rho_B) = 0$.

($\mathbb{A}_2$) Since the eigenvalues of the reduced density matrix of subsystems are invariant under local unitary transformation, so its von Neumann entropy is also invariant under local unitary operation, so $E(|\psi\rangle)$ is invariant under local unitary operation: $E(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger) = E(\rho_{AB})$, where $\rho_{AB} = |\psi\rangle\langle\psi|$.

($\mathbb{A}_3$) Let us prove monotonicity under LOCC by exploiting the fact that quantum operations never increase mutual information: suppose AB is a composite quantum system and $\mathcal{E}$ is a trace-preserving quantum operation on system B; let $I(A:B)$ denote the mutual information between system A and B -as defined in (2.19)- before $\mathcal{E}$ is applied to system B, and $I(A':B')$ the mutual information after $\mathcal{E}$ is applied to system B. Then $I(A':B') \leq I(A:B)$.

Proof. The action of $\mathcal{E}$ on B may be simulated by introducing a third system C, initially in a pure state $|0\rangle$, and a unitary interaction U between C and B (see [12]). The action of $\mathcal{E}$ on B is equivalent to the action of U followed by discarding system C. Letting primes denote the state of the systems after U has acted we have $I(A:B) = I(A:B,C)$ because C starts out in a product state with AB, and clearly $I(A:B,C) = I(A':B',C')$. Discarding systems cannot increase mutual information, so $I(A':B') \leq I(A':B',C')$. Putting it all together gives $I(A':B') \leq I(A:B)$, as required. □

Since in our case AB is a pure state, we have $S(\rho_{AB}) = 0$ and $S(\rho_A) = S(\rho_B)$. Thus from $I(A':B') \leq I(A:B)$ we obtain $E(|\psi'\rangle) \leq E(|\psi\rangle)$.

- **Entanglement of formation**
  This measure was introduced in [28] and its principal objective is of quantifying the entanglement of mixed states, i.e. states which are a mixture of

entangled pure states. Indeed, we have already shown that entanglement of pure bipartite states can be evaluated via the entropy of entanglement (2.38), thus entanglement of formation for a pure state $|\psi\rangle$ is simply given by

$$E_F(|\psi\rangle) = S(\rho_A) = S(\rho_B) \tag{2.39}$$

where $\rho_A = \mathrm{Tr}_B |\psi\rangle \langle\psi|$ and $\rho_B = \mathrm{Tr}_A |\psi\rangle \langle\psi|$. Once imposed a measure on pure states, the entanglement of formation (EoF) for mixed states is built by convex roof and is then defined as the minimal average entanglement over all pure state decompositions:

$$E_F(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \; E_F(|\psi_i\rangle) \tag{2.40}$$

where the minimum is taken over all ensembles $\{p_i, |\psi_i\rangle\}$, that is over all the possible realizations of the state $\rho_{AB} = \sum_i p_i |\psi_i\rangle \langle\psi_i|$, $E_F(|\psi_i\rangle)$ is given by (3.3), and p_i represents the probability associated with each pure state.
It can be acknowledged that EoF satisfies the essential axioms ($\mathbb{A}_{1-3}$) for being an entanglement measure since the entropy of entanglement -so the entanglement of formation for pure states- satisfies them.

As was said, the above quantities only focus on measuring bipartite entanglement, though multipartite generalizations can sometimes be derived. In particular, in the next chapter, I will generalize the entanglement of formation in the multipartite case.

Chapter 3

Multipartite entanglement

"The enormous usefulness of mathematics in natural sciences is something bordering on the mysterious, and there is no rational explanation for it. It is not at all natural that "laws of nature" exist, much less that man is able to discover them. The miracle, of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve."

—Eugene P. Wigner

Multipartite entanglement, in addition to being a captivating phenomenon, is a crucial aspect of quantum computation because it permits a speed-up over classical computation, as pointed out by Jozsa and Linden [29]. However, identifying entanglement in multipartite states is challenging due to exponentially increasing degrees of freedom. On the other hand, though the N-number of quantum systems can have various kinds of entanglement, much effort has been devoted to detecting its strongest form, *genuine multipartite entanglement* (GME), since it has multiple applications in quantum information and computational tasks (see [19]). Two widely studied multipartite entangled states are (i) Greenberger-Horne-Zeilinger (GHZ) [17] and (ii) W states [30]. These two states are inequivalent, i.e. they cannot be transformed into each other by LOCCs. Maximally entangled states have found applications in diverse fields, such as quantum teleportation [31], quantum secret sharing [32], superdense coding [33], and enhancing the computational power [34]. The stronger non-locality displayed by maximally multipartite entangled states also leads to many theoretical and experimental interests in quantum physics. However, the occurrence of genuine multipartite entanglement is not straightforward and easily detectable. A complete characterization and quantification of multipartite

entanglement is an ongoing goal in quantum information theory. With this thesis, I aim to contribute to these scientific endeavours.

We already introduced in section 2.4.1 the characterization of entanglement through separability. That was sufficient to characterize entanglement in bipartite states. However, in the multipartite case, this is no longer sufficient to provide a complete description of entanglement. Indeed, while an N-partite (mixed) Genuine Multipartite Entangled state is necessarily N-partite entangled, if for example it is found to be bi-separable it remains to be resolved what kind of k-partite entanglement exists in this state: being bi-separable just implies that k ranges from $\lceil N/2 \rceil$ to $N-1$. A N-partite state which is found to be k-separable can exhibit anything from $\lceil N/k \rceil$ to $(N-k+1)$-partite entanglement. Thus, we will introduce some mathematical concepts from partitioning theory to have a more comprehensive and useful characterization of a multipartite quantum system.

We will see that there are two different ways to characterize the entanglement of N-partite quantum systems [35], one is according to the question "How many partitions are separable?", the other is "How many particles are entangled". The former is described by k-separability (as we have already seen in chapter 2), and the latter leads to k-partite entanglement. k-separability provides fine graduation of entanglement of states according to their degrees of separability, so we will introduce a measure to quantify the entanglement of such states. We will start by introducing a measure for k–non-separable states. Then we will manage to extend that definition in the case of k–producible states. However, we will see that such an extension does not bring any useful description of entanglement. Thus we will present another general construction of a measure suitable for quantifying genuine multipartite enetanglement. These tasks will be accomplished through the exploitation of the concepts of partionability and producibility [36].
In this way, we will analyze two different manifestations of multipartite entanglement: *k–non-separable entanglement* and *k–partite entanglement*. These are two different concepts of multipartite entanglement, although 2–non-separable entanglement is equivalent to genuine N–partite entanglement and N–non-separable entanglement is equivalent to 2–partite entanglement. So, both k–non-separable and k–partite entanglement can be used to characterize multipartite entanglement. In the upcoming section, we will illustrate that these two occurrences of entanglement stem from two different ways of partitioning an N–partite system $\mathcal{X}_N$: a partition $\mathcal{X}_1|\mathcal{X}_2|\mathcal{X}_3|\ldots$ (where a part $\mathcal{X}_i$ is a subsystem, possibly consisting of several elementary subsystems, e.g. particles) is *k–partionable* if the number of subsystems is *at least* k ($\mathcal{X}_1|\mathcal{X}_2|\ldots|\mathcal{X}_k$), while it is *$k$–producible* if all the subsystems contain *at most* k elementary subsystems ($\mathcal{X}_i = \left\{\mathcal{X}_{[1]}, \mathcal{X}_{[2]}, \ldots, \mathcal{X}_{[l]}\right\}$ with $l \leq k$ for each i).

3.1 Partitioning a N-partite quantum system

A finite dimensional N-partite quantum system, $\mathcal{X}_N = \left\{\mathcal{X}_{[1]}, \mathcal{X}_{[2]}, \ldots, \mathcal{X}_{[N]}\right\}$[1], is described by a density matrix ρ_N, being $\rho_{[i]}$ the states of the subsystems $\mathcal{X}_{[i]}$, $i = 1,2,\ldots,N$. Thus, the state of the total system ρ_N belongs to the Hilbert space $\mathcal{H}$ that decomposes into a direct product of N subspaces $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, where the dimension of the Hilbert space $\mathcal{H}_i$ is assumed to be finite.
Given a N-partite quantum system $\mathcal{X}_N$, we may distinguish between *k-partitionability* and *k-producibility* of entanglement - the latter also called entanglement depth. Namely, **k–partitionability** is about the number of subsystems separable from one another, and not sensitive to the size of entangled subsystems. In the previous chapter, it was understood that when we talked about k–partition in section 2.4.1 we meant k–partitionable partition. The notion of **k–producibility**, instead, is designed to be sensitive to the question 'how many particles are entangled?', so it is about the size of the largest entangled subsystem, and not sensitive to the number of subsystems separable from one another (for more details, see [37, 38]). Thus, the degree of two different aspects of multipartite entanglement can be described. They arise from two different ways of partitioning an ensemble of N subsystems: those that cannot be factorized in at least k parts, and those that cannot be factorized in parts of size at most k, respectively.
The k–producibly separable states will be simply called *k–producible states*, while k–partitionibly separable states will be called *k–separable states*, as we got used to. However, as we can see from their definitions and in the examples shown in Figure 3.1, it is not clear how partitionability and producibility are related to each other in entanglement theory. To better understand them, let us point out their properties.

3.1.1 k-partitionability

Let us start by analyzing the *properties of partitionability*. The concept of partitionability is linked to the separability problem already covered in section 2.4.1. A N–partite pure state $|\psi\rangle \in \mathcal{H}$ is called k–separable (or, extensively, k–partitionably separable) if there is a k–partitionable partition $\mathcal{X}_1|\mathcal{X}_2|\ldots|\mathcal{X}_k$ of $\{1,2,\ldots,N\}$ into k pairwise disjoint subsets: $\mathcal{X}_N = \cup_{i=1}^{k}\mathcal{X}_i$ with $\mathcal{X}_i = \left\{\mathcal{X}^i_{[1]}, \mathcal{X}^i_{[2]}, \ldots, \mathcal{X}^i_{[m_i]}\right\}$ such that

$$|\psi_{k-sep}\rangle = |\psi\rangle_{\mathcal{X}_1} |\psi\rangle_{\mathcal{X}_2} \ldots |\psi\rangle_{\mathcal{X}_k} \tag{3.1}$$

[1]For the sake of clarity, let us point out the notation adopted: $\mathcal{X}_{[i]}$ indicates a single subsystem i; $\mathcal{X}_i$ indicates a subset of subsystems, that is $\mathcal{X}_i = \{\mathcal{X}^i_{[1]}, \mathcal{X}^i_{[2]}, \ldots, \mathcal{X}^i_{[m_i]}\}$; then $\mathcal{X}_N$ indicates the full set of subsystems.

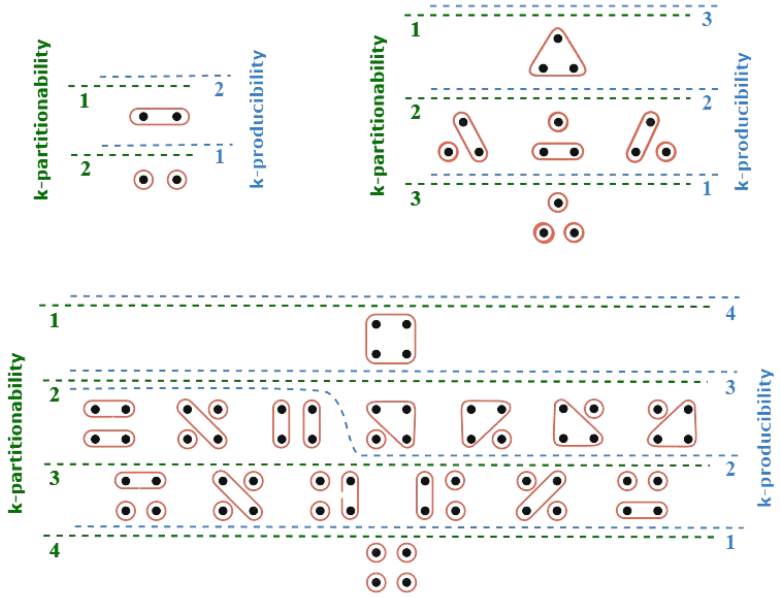

Figure 3.1: The down-sets corresponding to k–partitionability and k–producibility, illustrated for $N = 2, 3$ and 4. The two kinds of down-sets contain the elements below the specific green and blue dashed lines.

where $|\psi\rangle_{\mathcal{X}_i}$ is the state of subsystems $\mathcal{X}^i_{[1]}, \mathcal{X}^i_{[2]}, \ldots, \mathcal{X}^i_{[m_i]}$ in $\mathcal{H}_{\mathcal{X}_i} = \bigotimes_{j=1}^{m_i} \mathcal{H}_j$. That is, an N–partite pure state is k–separable, if and only if it can be written as a product of k substates. Then, an N–partite mixed state ρ is called k–separable if it can be written as a convex combination of at most k–separable pure states

$$\rho_{k-sep} = \sum_i p_i \left|\psi^i_{k-sep}\right\rangle \left\langle\psi^i_{k-sep}\right| \tag{3.2}$$

where $\left|\psi^i_{k-sep}\right\rangle$ might be k–separable under different partitions. That is, an N-partite mixed state ρ is k–separable if and only if it has a decomposition into k-separable pure states. In particular, an N–partite state is referred to as fully separable, if and only if it is N-separable. While it is called *genuinely N–partite entangled* if and only if it is not bi–separable (2–non-separable).
So, each k–separable state with $k < N$ has some degree of entanglement. For this reason, we will identify k–partitionability with ***k*–non-separable entanglement**. Indeed, if any k–partitionable subset of a state has some degree of entanglement, then it means that that state is not k–separable.
In section 2.4.3 we have presented entanglement measures for bi-partite states ($k = 2$). Bi–partite entanglement is well understood and, now, accessible from an experimental viewpoint too. Instead, multi–partite entanglement ($k > 2$) characterization and quantization is still not very understood and it is currently one

of the ever-expanding horizons of quantum information. This is also because, in an N–partite state, there are many ways to share entanglement. For example, there are $N(N-1)/2$ different pairs of particles that can display bipartite entanglement. In general, there are

$$\binom{N}{k} = \frac{N!}{k!(N-k)!}$$

different kinds of k–partitionable entanglement in an N–partite state, where $N \geq k$, given by the different ways of realizing a k–partitonable partition.
In general, k–separable mixed states are not separable within any subset, which makes k-separability - and so k–partitionable entanglement - even more difficult to detect. As an example, we see bi-separable states that are entangled with respect to a fixed bipartition. It has been observed that certain states, which are a combination of other states that are separable with respect to some bipartition, still exhibit entanglement. This means that such states cannot be expressed as a combination of separable states with respect to a fixed bipartition, nor can they be expressed as a combination of fully separable states. For example, three-qubit states $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{23} |0\rangle_1$, $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{13} |0\rangle_2$, and $|\psi_3\rangle| = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{12} |0\rangle_3$ are 2-separable under 2–partitionable partitions $1|23$, $13|2$, and $12|3$, respectively. Their convex combinations $\rho = p_1 |\psi_1\rangle \langle\psi_1| + p_2 |\psi_2\rangle \langle\psi_2| + p_3 |\psi_3\rangle \langle\psi_3|$ (with $p_i > 0, \sum_{i=1}^{3} p_i = 1$) are mixtures of bi–separable states with respect to different partitions, and therefore bi–separable. However, as can be easily checked, ρ is entangled with respect to each fixed bipartition, that is, it can not be written as a convex combination of bi-separable states with respect to a fixed bipartition.

To summarize, the study of partitionability in quantum systems offers insights into the intricate nature of entanglement. Partitionability delineates the ability to decompose multipartite quantum states into smaller subsystems and characterizes the entanglement present within these partitions. A state is considered k-separable if it can be factored into k substates, each pertaining to a distinct partition, thus revealing a form of entanglement within each subset. This task is quite demanding since the number of possible partitions of a set of N elements grows exponentially with N. Detecting k-separability in mixed states becomes even more arduous, since such states might not exhibit separability within any individual subset. Even states that are k-separable under certain partitions can be entangled when considered in different partitions. For instance, mixtures of bipartite-separable states across various partitions can still retain entanglement, making their characterization complex and challenging.
In the next chapter we will present a measure capable of quantifying this manifestation of entanglement: it will be a generalization of the entanglement of formation for multipartite states and it will be suitable for evaluating the degree of entanglement in k–non-separable quantum states.

3.1.2 k-producibility

Let us now analyze the *properties of producibility*. We will adhere to the formulation Girolami, Tuffarelli, and Susa provided in [39]. As we have seen before, a finite dimensional N-partite quantum system $\mathcal{X}_N = \{\mathcal{X}_{[1]}, \mathcal{X}_{[2]}, \ldots, \mathcal{X}_{[N]}\}$ is described by a density matrix ρ_N, being $\rho_{[i]}$ the states of the subsystems $\mathcal{X}_{[i]}$, $i = 1,2,\ldots,N$. Let us consider clusters, described by density matrices ρ_{k_j}, forming a coarse grained partition $\{\mathcal{X}_{k_1}, \mathcal{X}_{k_2}, \ldots, \mathcal{X}_{k_m}\}$, with $\sum_{j=1}^{m} k_j = N$, $k_j \leq k$, where each cluster $\mathcal{X}_{k_j}$ includes up to k subsystems, e.g. $\mathcal{X}_{k_j} = \{\mathcal{X}_{[1]}, \mathcal{X}_{[2]}, \ldots, \mathcal{X}_{[k_j]}\}$. Now we consider the set of all k-producible partitions of a state ρ_N:

$$\mathcal{P}_k = \left\{ \rho_N = \bigotimes_{j=1}^{m} \rho_{k_j},\ \sum_{j=1}^{m} k_j = N,\ k = \max\{k_j\} \right\}. \tag{3.3}$$

The complete chain for producibility hierarchy reads $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_{N-1} \subset \mathcal{P}_N$; an illustrative representation is given in Fig. 3.2.

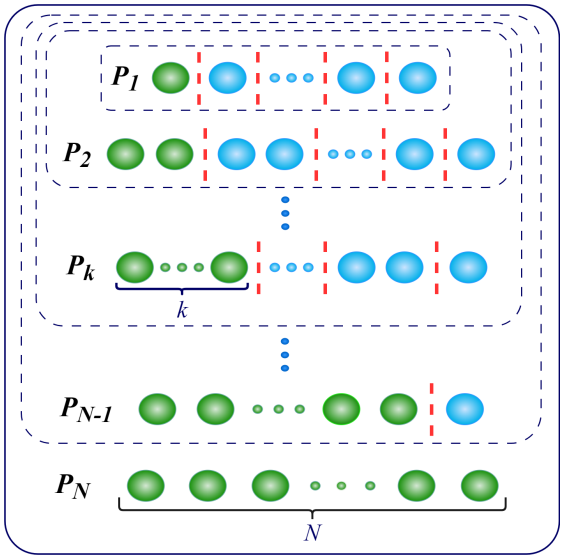

Figure 3.2: Multipartite producibility hierarchy. Given a system of N particles (blue spheres), the sets P_k, $k = 1, 2, \ldots, N$, consist of states displaying up to k-partite entaglement. The green k spheres identify the largest subset of a coarse-grained partition (the dashed red lines separate each cluster).

We say that a *pure* N–partite state $|\psi_N\rangle$ is **k–producible**, or producible by k–partite entanglement, if it belongs in $\mathcal{P}_k$ and it can be expressed as

$$|\psi_{k-\text{prod}}\rangle = |\psi\rangle_{\mathcal{X}_{k_1}} \otimes |\psi\rangle_{\mathcal{X}_{k_2}} \otimes \cdots \otimes |\psi\rangle_{\mathcal{X}_{k_m}}, \tag{3.4}$$

where each $|\psi\rangle_{\mathcal{X}_{k_i}}$ is a pure state corresponding to the cluster $\mathcal{X}_{k_i}$ and they are states of maximally k parts. So, in this definition $m \geq N/k$ has to hold. To

the sake of clarity, this has to be verified in order to $|\psi\rangle$ being k–producible: $\exists \mathcal{X}_{k_1}|\mathcal{X}_{k_2}|\ldots|\mathcal{X}_{k_m} \ni |\psi\rangle_{\mathcal{X}_{k_i}}$ is the state of subsystems $\mathcal{X}^{k_i}_{[1]}, \mathcal{X}^{k_i}_{[2]}, \ldots, \mathcal{X}^{k_i}_{[p]}$ in $\mathcal{H}_{\mathcal{X}_{k_i}} = \bigotimes_{j=1}^{p} \mathcal{H}_j$ and $p \leq k$.
Then, we then say that a *mixed* state ρ is **k–producible** if and only if it can be expressed as a convex combination of k–producible pure states like in (3.4):

$$\rho = \sum_i \lambda_i \left|\psi^i_{k-\text{prod}}\right\rangle \left\langle\psi^i_{k-\text{prod}}\right|, \tag{3.5}$$

where $\sum_i \lambda_i = 1,\ \lambda_i \geq 0$. That is, a mixed state which is k–producible requires only the generation of k–partite entangled states and mixing for its production. Consequently, a mixed state ρ contains k–partite entanglement if and only if it cannot be decomposed into a convex sum of products of density matrices with all density matrices involving less than k particles: at least one of the terms is a k–partite entangled density matrix.
With this formalism, a state is **genuine k–partite entangled** if it is k–producible but not $k+1$–producible.
An extension of producibility is the concept of **depth of entanglement** (see [37]): it is the minimal k for which a given multipartite state ρ admits a decomposition such as (3.5). It characterizes the minimal number of particles that are entangled. More precisely, for a pure k–product state such as in (3.1):

$$|\psi_{k-sep}\rangle = |\psi\rangle_{\mathcal{X}_1} |\psi\rangle_{\mathcal{X}_2} \ldots |\psi\rangle_{\mathcal{X}_k}$$

with $\bigcup \mathcal{X}_i = \left\{\mathcal{X}_{[1]}, \mathcal{X}_{[2]}, \ldots, \mathcal{X}_{[N]}\right\}$ and $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset$ for $i \neq j$, the entanglement depth of $|\psi_{k-sep}\rangle$, denoted by $\mathbf{D}(\boldsymbol{\psi}_{\mathbf{k-sep}})$, is defined as the largest cardinality of $\mathcal{X}_i$. And an N-particle state ρ_N has entanglement depth D if, for all decomposition of $\rho_N = \sum_i p_i |\psi^i_N\rangle \langle\psi^i_N|$, there exists one pure state $|\psi^i_N\rangle$ that has at least D–particle entanglement. More precisely, the entanglement depth of ρ_N is defined as follows

$$\mathbf{D}(\boldsymbol{\rho}_{\mathbf{N}}) = \min_{\rho_N = \sum p_i |\psi_i\rangle\langle\psi_i|} \max_i D(\psi^i_N) \tag{3.6}$$

where each $|\psi^i_N\rangle$ is an N–particle pure state and $D(\psi^i_N)$ is the entanglement depth of $|\psi^i_N\rangle \langle\psi^i_N|$. That is, ρ_N has *genuine D–partite entanglement*. Obviously in general, all $1 \leq D(\rho_N) \leq N$ are possible for ρ_N. An N-partite density matrix ρ_N has entanglement depth at least $\lceil N/k \rceil$ if it is not k–separable. The entanglement depth $D(\rho_N) = 1$ corresponds to a fully separable state (N-separable), and the entanglement depth $D(\rho_N) = N$ corresponds to a genuine N–partite entangled state (i.e. not biseparable).In this sense, the entanglement depth characterizes the minimum number of particles that cannot be separated within the system.
In systems lacking spatial ordering, entanglement depth is a crucial variable for characterizing entanglement properties. However, in systems with spatial order such

as spin chains, or in the presence of gradient fields, the dynamics of a system may also depend on whether entanglement exists only between neighbouring particles or also between distant particles.The concept of entanglement width will be used to describe the property of entanglement (see [40] for more details). The width of entanglement w of a pure state $|\psi\rangle = \bigotimes_j |\psi_j\rangle$ is defined as the maximal distance w of two entangled particles within the states $|\psi_j\rangle$. A completely separable state exhibits an entanglement width of $w = 1$. The entanglement width of a mixed state is defined by the minimum with w overall decomposition $\rho = \sum_j p_j |\psi_j\rangle \langle\psi_j|$, that is

$$w(\rho) = \min_{\sum_j p_j |\psi_j\rangle\langle\psi_j|} [\max_j \{w(\psi_j)\}]. \tag{3.7}$$

By definition, the entanglement depth is a lower bound of the entanglement width (even if the entanglement width does not make any statement about the entanglement depth). In Fig. 3.3 an exemplary representation of entanglement depth and width is shown.

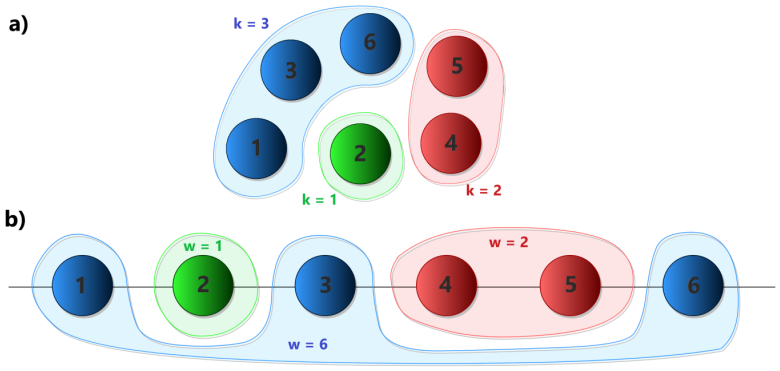

Figure 3.3: Comparison of entanglement depth (a) and entanglement width (b) of the state $|\psi\rangle = |\psi_{3,6}\rangle \otimes |\psi_{4,5}\rangle \otimes |\psi_2\rangle$. While entanglement depth disregards spatial ordering, entanglement width requires it, e.g. in a spin chain. The entanglement depth of the state $|\psi\rangle$ in (a) is given by $D(\psi) = 3$, since maximally three particles are entangled. This is a lower bound on the entanglement width in (b), which equals $w = 6$, since entanglement occurs between particles that are six units apart in the chain.

To summarize, the presented analysis delves into the properties of producibility within multipartite quantum systems. The concept explores the hierarchical nature of multipartite entanglement through the lens of k-producible partitions. These partitions, forming a coarse-grained description of the system, highlight the emergence of entanglement among subsystems. Pure states are deemed k-producible if they can be expressed as products of pure states within clusters of at most k elements, while mixed states are k-producible when they can be expressed as convex

combinations of such pure k-producible states. Genuine k-partite entanglement is defined for states that are k-producible but not $(k+1)$-producible.

After having introduced the notions of partitionability and producibility along with their properties, in the next sections we will exploit these concepts to define measures of entanglement suitable to describe the two manifestations of multipartite entanglement: the k–non-separable entanglement and the k–partite one. For the latter, we will start with an already proposed measure for k–non-separable entanglement: we will generalize it and show that it is a valid measure of entanglement, even if not able to give satisfying insights into entanglement of multipartite states. Then, we will build on a generic construction for a genuine multipartite entanglement measure.

3.2 Entanglement measure for k–non-separable states

In [41] a k–entanglement measure E_k based on the use of the von Neumann entropy was proposed, with k the size of a split of a N–partite system. Here, I will propose a variation of that formula, which will be a generalization of the entanglement of formation in the multipartite case able to quantify the entanglement in k–non-separable states, and then I will prove that it is a well-defined entanglement measure. This approach is based on the fact that, since a bipartite state is entangled if and only if its subsystems are mixed (as shown in chapter 2), then it is at least plausible to generalize this approach for the multipartite scenario and to think that "*the more mixed the marginals, the more entangled is the state*". Thus a measure of mixedness (entropies) of the subsystems should lead to a motivated measure of entanglement of the whole N-partite system. One possibility is that if one considers the entanglement with respect to a k–separable partition, then one simply sums up the measures of mixedness of the subsystems with respect to that partition.
So, let $|\psi\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ be a N–partite *pure* state, the entanglement for multipartite k–non-separable state $\rho_N = |\psi\rangle\langle\psi|$ (simply denoted as k–NonSep entanglement, $E^{(k)}$) is defined as follows:

$$E^{(k)}(\rho_N) = \min_{\{\mathcal{X}_1|\mathcal{X}_2|\ldots|\mathcal{X}_k\}} \frac{1}{k}\sum_{i=1}^{k} S(\rho_{\mathcal{X}_i}), \tag{3.8}$$

where the minimum is taken over the set of all k–partitionable partitions of ρ_N, $\{\mathcal{X}_1|\mathcal{X}_2|\ldots|\mathcal{X}_k\}$, $S(\rho_{\mathcal{X}_i}) = -\operatorname{Tr}(\rho_{\mathcal{X}_i}\log\rho_{\mathcal{X}_i})$ is the von Neumann entropy of $\rho_{\mathcal{X}_i} = \operatorname{Tr}_{\bar{\mathcal{X}}_i}(|\psi\rangle\langle\psi|)$, and $\bar{\mathcal{X}}_i$ is the complementary set of $\mathcal{X}_i$ in $\{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_k\}$. The factor $1/k$ takes into account the fact that, for higher orders of k, so for higher orders of partitionable partitions, the entanglement of the state is expected to be

lower. So we can interpret this measure as given by the k–partition of ρ_N that minimizes the average of the entropy of all the subsystems $\mathcal{X}_i$, $i = 1, \ldots, k$, that compose that partition.
Then, thanks to the convex roof construction, for a *mixed* state ρ_N the k–NonSep entagnlement is defined as:

$$E^{(k)}(\rho_N) = \inf_{\{p_j, |\psi_j\rangle\}} \sum_j p_j E^{(k)}(|\psi_j\rangle \langle\psi_j|), \tag{3.9}$$

where the infimum is taken over all ensemble decompositions of ρ_N.

Now we have to prove that k–NonSep entanglement is a well-defined entanglement measure. Thus, we have to prove that it fulfils the three axioms ($\mathbb{A}_{1-3}$) presented in section 2.4.3. Before these, we will prove an additional property ($\mathbb{A}_0$), convexity, for which we have that mixing of states does not increase entanglement of formation.

($\mathbb{A}_0$) Consider two N–partite quantum systems $\rho, \sigma \in \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ and $t \in [0,1]$, then $E^{(k)}$ is convex with respect to input states:

$$E^{(k)}\Big(t\rho + (1-t)\sigma\Big) \leq tE^{(k)}(\rho) + (1-t)E^{(k)}(\sigma). \tag{3.10}$$

Proof.
Let us start by considering ρ and σ as *pure* states. In this case, the proof is obvious because of the convex roof construction by which the k–NonSep entanglement (3.9) is defined.
Now let us consider the case in which ρ and σ are *mixed* states. Let us denote by $Q_1 = \{p_i, |\psi_i\rangle\}$ the set of pure states decomposition of ρ, and $Q_2 = \{q_j, |\phi_j\rangle\}$ the set of pure states decomposition of σ, such that $\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$, $\sigma = \sum_j q_j |\phi_j\rangle \langle\phi_j|$. Consider $\gamma = t\rho + (1-t)\sigma$. By construction, one pure state decomposition of γ is given by the pure state decomposition of ρ plus the one of σ: $t\sum_i p_i |\psi_i\rangle \langle\psi_i| + (1-t)\sum_j q_j |\phi_j\rangle \langle\phi_j|$. However, γ may have other pure states decomposition $\gamma = \sum_l r_l |\eta_l\rangle \langle\eta_l|$, and among them there can be some that provide the infimum of the k–NonSep entanglement of γ. This can be expressed more formally by noting that, if we denote as $Q = \{r_l, |\eta_l\rangle\}$ the set of pure states decomposition of γ, we have $Q_1 \cup Q_2 \subseteq Q$, and so

$$\begin{aligned} E^{(k)}(t\rho + (1-t)\sigma) &= \inf_Q \sum_l r_l E^{(k)}(|\eta_l\rangle \langle\eta_l|) \\ &\leq \inf_{Q_1 \cup Q_2} \Big[t \sum_i p_i E^{(k)}(|\psi_i\rangle \langle\psi_i|) + (1-t) \sum_j q_j E^{(k)}(|\phi_j\rangle \langle\phi_j|) \Big] \\ &= tE^{(k)}(\rho) + (1-t)E^{(k)}(\sigma). \end{aligned} \tag{3.11}$$

□

($\mathbb{A}_1$) Given an N–partite quantum state $\rho \in \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$, then it is k–separable if and only if $E^{(k)}(\rho) = 0$.

Proof.
Let us first demonstrate this property for a pure state $\rho = |\psi\rangle\langle\psi|$.

($\Rightarrow$) If $\rho = |\psi\rangle\langle\psi|$ is a k–separable pure state relative to a k–partition $\mathcal{X}_1|\mathcal{X}_2|\ldots|\mathcal{X}_k$, it is clear that $S(\rho_{\mathcal{X}_i}) = 0$ for all $i = 1,2,\ldots,k$. So $E^{(k)}(|\psi\rangle\langle\psi|) = \min_{\{\mathcal{X}_1|\mathcal{X}_2|\ldots|\mathcal{X}_k\}} \frac{1}{k}\sum_{i=1}^{k} S(\rho_{\mathcal{X}_i}) = 0$.

($\Leftarrow$) If $E^{(k)}(|\psi\rangle\langle\psi|) = 0$, then there exist a k–partition $\mathcal{X}_1|\mathcal{X}_2|\ldots|\mathcal{X}_k$ such that $S(\rho_{\mathcal{X}_i}) = 0$ for all i, which holds if and only if each $\rho_{\mathcal{X}_i}$ is pure, $i = 1,2,\ldots,k$. Let us prove that this implies that $\rho = |\psi\rangle\langle\psi|$ is k–separable. To do so we will use an iterative procedure, showing first that ρ is bi–separable, then 3–separable, until showing that it is k–separable.
Let us split the N–partite quantum system into a bipartition according to $\mathcal{X}_1$, such that $\mathcal{H} = \mathcal{H}^{\mathcal{X}_1} \otimes \mathcal{H}^{\bar{\mathcal{X}}_1}$. Let

$$|\psi\rangle = \sum_i \lambda_i^{(1)} \left|a_i^{\mathcal{X}_1}\right\rangle \left|b_i^{\bar{\mathcal{X}}_1}\right\rangle$$

be the Schmidt decomposition of $|\psi\rangle$ with respect to the 2–partition $\mathcal{X}_1|\bar{\mathcal{X}}_1$. Then we have

$$\rho = \sum_{i,j} \lambda_i^{(1)}\lambda_j^{(1)} \left|a_i^{\mathcal{X}_1}\right\rangle\left\langle a_j^{\mathcal{X}_1}\right| \otimes \left|b_i^{\bar{\mathcal{X}}_1}\right\rangle\left\langle b_j^{\bar{\mathcal{X}}_1}\right|$$

$$\rho_{\mathcal{X}_1} = \mathrm{Tr}_{\bar{\mathcal{X}}_1}(\rho) = \sum_i (\lambda_i^{(1)})^2 \left|a_i^{\mathcal{X}_1}\right\rangle\left\langle a_i^{\mathcal{X}_1}\right|$$

$$\rho_{\bar{\mathcal{X}}_1} = \mathrm{Tr}_{\mathcal{X}_1}(\rho) = \sum_i (\lambda_i^{(1)})^2 \left|b_i^{\bar{\mathcal{X}}_1}\right\rangle\left\langle b_i^{\bar{\mathcal{X}}_1}\right|.$$

Since $\rho_{\mathcal{X}_1}$ is a pure state, there exist i such that $\lambda_i^{(1)} = 1$ and $\lambda_j^{(1)} = 0$ for all $j \neq i$. Consequently, we have $\rho_{\mathcal{X}_1} = \left|a_i^{\mathcal{X}_1}\right\rangle\left\langle a_i^{\mathcal{X}_1}\right|$, $\rho_{\bar{\mathcal{X}}_1} = \left|b_i^{\bar{\mathcal{X}}_1}\right\rangle\left\langle b_i^{\bar{\mathcal{X}}_1}\right|$. Then we have

$$\rho = \left|a_i^{\mathcal{X}_1}\right\rangle\left\langle a_i^{\mathcal{X}_1}\right| \otimes \left|b_i^{\bar{\mathcal{X}}_1}\right\rangle\left\langle b_i^{\bar{\mathcal{X}}_1}\right| = \rho_{\mathcal{X}_1} \otimes \rho_{\bar{\mathcal{X}}_1},$$

so ρ is 2–separable with respect to the 2–partition $\mathcal{X}_1|\bar{\mathcal{X}}_1$.
Next we split the subsystem $\bar{\mathcal{X}}_1$ into a bipartition according to $\mathcal{X}_2$, such that $\mathcal{H}^{\bar{\mathcal{X}}_1} = \mathcal{H}^{\mathcal{X}_2} \otimes \mathcal{H}^{\bar{\mathcal{X}}_1 \setminus \mathcal{X}_2}$, where $\bar{\mathcal{X}}_1 \setminus \mathcal{X}_2$ is the complement of $\mathcal{X}_2$ in $\bar{\mathcal{X}}_1 = \left\{\mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_k\right\}$. Repeating the same argument as before, we consider the

Schmidt decomposition of $\left|\psi_{\bar{\mathcal{X}}_1}\right\rangle$ with respect to the 2–partition $\mathcal{X}_2|\{\bar{\mathcal{X}}_1 \setminus \mathcal{X}_2\}$ of $\bar{\mathcal{X}}_1$

$$\left|\psi_{\bar{\mathcal{X}}_1}\right\rangle = \sum_i \lambda_i^{(2)} \left|c_i^{\mathcal{X}_2}\right\rangle \left|d_i^{\bar{\mathcal{X}}_1 \setminus \mathcal{X}_2}\right\rangle$$

and

$$\rho_{\bar{\mathcal{X}}_1} = \sum_{i,j} \lambda_i^{(2)} \lambda_j^{(2)} \left|c_i^{\mathcal{X}_2}\right\rangle \left\langle c_j^{\mathcal{X}_2}\right| \otimes \left|d_i^{\bar{\mathcal{X}}_1 \setminus \mathcal{X}_2}\right\rangle \left\langle d_j^{\bar{\mathcal{X}}_1 \setminus \mathcal{X}_2}\right|$$

Since $\rho_{\mathcal{X}_2}$ is a pure state, we have $\mathrm{Tr}\Big[\mathrm{Tr}_{\mathcal{X}_2}(\rho_{\bar{\mathcal{X}}_1})\Big]^2 = \mathrm{Tr}\Big[\mathrm{Tr}_{\bar{\mathcal{X}}_1 \setminus \mathcal{X}_2}(\rho_{\bar{\mathcal{X}}_1})\Big]^2 = 1$, and then

$$\rho_{\bar{\mathcal{X}}_1} = \rho_{\mathcal{X}_2} \otimes \rho_{\bar{\mathcal{X}}_1 \setminus \mathcal{X}_2};$$

so ρ is 3–separable:

$$\rho = \rho_{\mathcal{X}_1} \otimes \rho_{\mathcal{X}_2} \otimes \rho_{\bar{\mathcal{X}}_1 \setminus \mathcal{X}_2}.$$

Continuing this procedure, in the end we obtain that $\rho = |\psi\rangle\langle\psi|$ is k–separable:

$$\rho = \rho_{\mathcal{X}_1} \otimes \rho_{\mathcal{X}_2} \otimes \cdots \otimes \rho_{\mathcal{X}_k}.$$

Let us now demonstrate this property for a mixed state $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$.

($\Rightarrow$) If ρ is k–separable, then each $|\psi_i\rangle$ is k–separable, with $E^{(k)}(|\psi_i\rangle\langle\psi_i|) = 0$, and by the convex roof construction shown in (3.9) we have $E^{(k)}(\rho) = 0$.

($\Leftarrow$) If $E^{(k)}(\rho) = 0$, then, due to the definition (3.9), there exists an optimal ensemble decomposition $\{p_i, |\psi_i\rangle\}$, such that $E^{(k)}(\rho_N) = \inf_{\{p_i,|\psi_i\rangle\}} \sum_i p_i E^{(k)}(|\psi_i\rangle\langle\psi_i|) = 0$, that is $\sum_i p_i E^{(k)}(|\psi_i\rangle\langle\psi_i|) = 0$. This implies that, for every i, we have $E^{(k)}(|\psi_i\rangle\langle\psi_i|) = 0$, that is $|\psi_i\rangle$ is k–separable for every i, so $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ is k–separable.

□

The axiom ($\mathbb{A}_1$) provide that if $E^{(k)}(\rho) > 0$ then ρ is not k–separable, from which the name "entanglement of formation for multipartite k–nonseparable state" derives.

($\mathbb{A}_2$) Given an N–partite quantum state $\rho \in \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$, if $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_N$ are unitary operators acting on $\mathcal{H}_1, \mathcal{H}_2 \ldots, \mathcal{H}_N$ respectively, then $E^{(k)}(\rho)$ is invariant under local unitary transformation:

$$E^{(k)}\Big(\mathcal{U}_1 \otimes \mathcal{U}_2 \otimes \cdots \otimes \mathcal{U}_N \; \rho \, \mathcal{U}_1^\dagger \otimes \mathcal{U}_2^\dagger \otimes \cdots \otimes \mathcal{U}_N^\dagger\Big) = E^{(k)}(\rho). \tag{3.12}$$

Proof.
For a pure state $|\psi\rangle$, since the eigenvalues of the reduced density matrices of the subsystems $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_k$ (the subsystems of the k–partition that minimizes $\frac{1}{k}\sum_{i=1}^{k} S(\rho_{\mathcal{X}_i})$) do not change under local unitary operations, so their von Neuman entropy are also invariant and as a consequence even $E^{(k)}(|\psi\rangle\langle\psi|)$ is invariant under local unitary transformations.
For a mixed state $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, this property stems directly from the pure states case, due to the convex roof construction shown in (3.9). □

($\mathbb{A}_3$) Given an N–partite quantum state $\rho \in \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$, if $\mathcal{E}$ is a local operation and classical communication (LOCC) procedure, then $E^{(k)}(\rho)$ does not increase over $\mathcal{E}$:

$$E^{(k)}\Big(\mathcal{E}(\rho)\Big) \leq E^{(k)}(\rho). \tag{3.13}$$

Proof.
We have already proven in section 2.4.3, in the case of a bipartite state, the validity of this property for the entropy of entanglement and, consequently, for the entanglement of formation. We will exploit that result for the following proof.
Let us start from the pure state case $\rho = |\psi\rangle\langle\psi|$. Let $\mathcal{X}_1|\mathcal{X}_2|\ldots|\mathcal{X}_k$ be a k–partition of $\{\mathcal{X}_{[1]}, \mathcal{X}_{[2]}, \ldots, \mathcal{X}_{[N]}\}$. Now we consider all the k bipartitions $\mathcal{X}_i|\bar{\mathcal{X}}_i$ for $i = 1, \ldots, k$, and we assume that $E_i(|\psi\rangle\langle\psi|) = S(\rho_{\mathcal{X}_i})$ is the entanglement of formation (as defined in (3.3)) of $\rho = |\psi\rangle\langle\psi|$ with respect to the 2–partition $\mathcal{X}_i|\bar{\mathcal{X}}_i$.[2] For what we have already proven in the bipartite case, we have $E_i\Big(\mathcal{E}(|\psi\rangle\langle\psi|)\Big) \leq E_i(|\psi\rangle\langle\psi|)$. So

$$\begin{aligned} E^{(k)}\Big(\mathcal{E}(|\psi\rangle\langle\psi|)\Big) &:= \min_{\{\mathcal{X}_1|\mathcal{X}_2|\ldots|\mathcal{X}_k\}} \frac{1}{k}\sum_{i=1}^{k} E_i\Big(\mathcal{E}(|\psi\rangle\langle\psi|)\Big) \\ &\leq \min_{\{\mathcal{X}_1|\mathcal{X}_2|\ldots|\mathcal{X}_k\}} \frac{1}{k}\sum_{i=1}^{k} E_i(|\psi\rangle\langle\psi|) := E^{(k)}(|\psi\rangle\langle\psi|). \end{aligned}$$

Now let us consider the mixed state case $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$. Here we will use

[2]We remember that for a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, the entanglement of formation of a pure state $\rho_{AB} = |\psi\rangle\langle\psi|$ is simply given by $E_F(\rho_{AB}) = S(\rho_A) = S(\rho_B)$.

the previous result for pure states and convexity property ($\mathbb{A}_0$):

$$\begin{aligned} E^{(k)}\Big(\mathcal{E}(\rho)\Big) &= E^{(k)}\Big(\sum_j p_j \mathcal{E}(|\psi_j\rangle\langle\psi_j|)\Big) \xrightarrow{(\mathbb{A}_0)} \\ &\leq \sum_j p_j E^{(k)}\Big(\mathcal{E}(|\psi_j\rangle\langle\psi_j|)\Big) \xrightarrow{\text{pure states}} \\ &\leq \sum_j p_j E^{(k)}(|\psi_j\rangle\langle\psi_j|). \end{aligned}$$

So we have $E^{(k)}\Big(\mathcal{E}(\rho)\Big) \leq \inf_{\{p_j,|\psi_j\rangle\}} \sum_j p_j E^{(k)}(|\psi_j\rangle\langle\psi_j|) := E^{(k)}(\rho)$. □

Now we present two examples to show this entanglement measure in action. We will deal with the $|GHZ\rangle$ and the $|W\rangle$ states, which are the three qubits states with maximum entanglement [42], so we expect that the k–NonSep entanglement is consistent with that. Let us start with the $|GHZ\rangle$ state

$$|GHZ\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}. \tag{3.14}$$

Its density matrix is given by

$$\overset{GHZ}{\rho}_{ABC} = \frac{1}{2}\Big(|000\rangle\langle 000| + |000\rangle\langle 111| + |111\rangle\langle 000| + |111\rangle\langle 111|\Big). \tag{3.15}$$

In order to address the partitions useful for the evaluation of the k–NonSep entanglement of $|GHZ\rangle$, we compute its reduced density matrices by taking the partial trace over the subsystems A, B and C:

$$\begin{aligned} \overset{GHZ}{\rho}_{AB} &= \mathrm{Tr}_C(\overset{GHZ}{\rho}_{ABC}) = \frac{1}{2}\Big(|00\rangle\langle 00| + |11\rangle\langle 11|\Big) \\ &= \overset{GHZ}{\rho}_{AC} = \overset{GHZ}{\rho}_{BC} \end{aligned} \tag{3.16}$$

$$\begin{aligned} \overset{GHZ}{\rho}_{A} &= \mathrm{Tr}_{BC}(\overset{GHZ}{\rho}_{ABC}) = \frac{1}{2}\Big(|0\rangle\langle 0| + |1\rangle\langle 1|\Big) \\ &= \overset{GHZ}{\rho}_{B} = \overset{GHZ}{\rho}_{C} \end{aligned} \tag{3.17}$$

The von Neumann entropies are:

$$\begin{aligned} S(\overset{GHZ}{\rho}_{AB}) = S(\overset{GHZ}{\rho}_{AC}) = S(\overset{GHZ}{\rho}_{BC}) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = 1 \\ S(\overset{GHZ}{\rho}_{A}) = S(\overset{GHZ}{\rho}_{B}) = S(\overset{GHZ}{\rho}_{C}) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = 1 \end{aligned} \tag{3.18}$$

Thus, the measures of k–NonSep entanglement of $|GHZ\rangle$ are:

$$E^{(3)}(|GHZ\rangle\langle GHZ|) = \min_{\mathcal{S}_3} \frac{1}{3}\sum_{i=1}^{3} S(\rho_i) = 1$$

$$E^{(2)}(|GHZ\rangle\langle GHZ|) = \min_{\mathcal{S}_2} \frac{1}{2}\sum_{i=1}^{2} S(\rho_i) = 1$$

where with $\mathcal{S}_k$ we indicated the set of the k–partitions of the system ABC ($\mathcal{S}_2 = \{AB|C, AC|B, BC|A\}$, $\mathcal{S}_3 = \{A|B|C\}$). So we obtained that $|GHZ\rangle$ is not fully separable ($E^{(3)} \neq 0$), and it is not 2–separable ($E^{(2)} \neq 0$), thus $|GHZ\rangle$ displays genuinely three-partite entanglement, as we expected.
Let us now consider the $|W\rangle$ state

$$|W\rangle = \frac{|001\rangle + |010\rangle + |100\rangle}{\sqrt{3}} \tag{3.19}$$

and let us repeat the same procedure of $|GHZ\rangle$. We expect that also $|W\rangle$ is genuinely entangled. The density matrix is given by

$$\begin{aligned}\rho^{W}_{ABC} = \frac{1}{3}\Big(&|100\rangle\langle 100| + |100\rangle\langle 010| + |100\rangle\langle 001| + \\ &|010\rangle\langle 100| + |010\rangle\langle 010| + |010\rangle\langle 001| + \\ &|001\rangle\langle 100| + |001\rangle\langle 010| + |001\rangle\langle 001|\Big).\end{aligned} \tag{3.20}$$

The reduced density matrixes are:

$$\begin{aligned}\rho^{W}_{AB} = \rho^{W}_{AC} = \rho^{W}_{BC} = \frac{1}{3}\Big(&|00\rangle\langle 00| + |10\rangle\langle 10| + \\ &|10\rangle\langle 01| + |01\rangle\langle 10| + |01\rangle\langle 01|\Big)\end{aligned} \tag{3.21}$$

$$\rho^{W}_{A} = \rho^{W}_{B} = \rho^{W}_{C} = \frac{1}{3}\Big(2\,|0\rangle\langle 0| + |1\rangle\langle 1|\Big) \tag{3.22}$$

The von Neumann entropies are (this quantities are evaluated also in Appendix B):

$$\begin{aligned}S(\rho^{W}_{AB}) = S(\rho^{W}_{AC}) = S(\rho^{W}_{BC}) = -\frac{1}{3}\log\frac{1}{3} - \frac{2}{3}\log\frac{2}{3} = 0.918 \\ S(\rho^{W}_{A}) = S(\rho^{W}_{B}) = S(\rho^{W}_{C}) = -\frac{1}{3}\log\frac{1}{3} - \frac{2}{3}\log\frac{2}{3} = 0.918\end{aligned} \tag{3.23}$$

Thus, the measures of k–NonSep entanglement of $|W\rangle$ are:

$$E^{(3)}(|W\rangle\langle W|) = \min_{\mathcal{S}_3} \frac{1}{3}\sum_{i=1}^{3} S(\rho_i) = 0.918$$

$$E^{(2)}(|W\rangle\langle W|) = \min_{\mathcal{S}_2} \frac{1}{2}\sum_{i=1}^{2} S(\rho_i) = 0.918.$$

$\boldsymbol{\rho_N}$	$\boldsymbol{E^{(2)}}$	$\boldsymbol{E^{(3)}}$
$\|GHZ\rangle$	1	1
$\|W\rangle$	0.9	0.9

Table 3.1: k–non-separable entanglement values for GHZ and W state.

In Table 3.1 the values computed for the two canonical states GHZ and W are reported. So we obtained that $|W\rangle$ is not fully separable ($E^{(3)} \neq 0$), and it is not 2–separable ($E^{(2)} \neq 0$), thus $|W\rangle$ is genuinely entangled, as we expected. As we can see, we have got reasonable results. We have obtained that these states are genuinely three partite entangled.

In summary, we have introduced a multipartite entanglement measure termed as k-NonSep Entanglement, a generalization of entanglement of formation for multipartite systems that quantifies entanglement in k-non-separable states. The formulation extends the concept of entanglement from bipartite scenarios by measuring the mixedness of subsystems within N-partite systems. Proofs of its well-defined properties in accordance with three axioms ($\mathbb{A}_{1-3}$) demonstrate its consistency as a bona fide entanglement measure. Further, exemplification with key states such as the $|GHZ\rangle$ and $|W\rangle$ states corroborates its efficacy in delineating genuine multipartite entanglement, reaffirming its applicability and relevance in characterizing complex quantum states. This methodological framework adds to the toolkit of quantum information theory, aiding in the systematic exploration and understanding of multipartite entanglement in quantum systems.
However, it is important to note that this measure has its limitations and downsides. One of the main limitations is that it does not take into account the deeper properties of entanglement, such as whether a state belongs to a certain entanglement class under LOCC (see [30]). For example, using solely this measure we would not be able to distinguish between $|GHZ\rangle$ and $|W\rangle$. Indeed, we know that they belong to two different LOCC classes: $|W\rangle$ from $|GHZ\rangle$ are distinguished by the robustness over particle loss, in the sense that $|W\rangle$ retains bipartite entanglement when one if its particle is traced out, on the contrary of $|GHZ\rangle$. This behaviour is not reflected in any way by the k–NonSep measure. In addition, calculating the k-NonSep measure is a complex task for states with more than three particles. This is because the measure involves adding the entropies of all the possible separable portions of the substate, which grows exponentially as the number of particles increases. Therefore, calculating the k-NonSep measure becomes extremely challenging for states with more than three particles.

In the next section, we address the problem of defining a more suitable and useful measure of multipartite entanglement. We will do this for the k–partite entanglement. We will use a framework similar to the one in [39], which identifies

distance-based measures to describe genuine multipartite correlations in classical and quantum systems. We instead will use solely a measure of bipartite entanglement to describe genuine multipartite entanglement.

3.3 Generic construction of genuine multipartite entanglement measure

Given the measure of entanglement for k–non-separable states, it is at least legitimate to generalize that measure to the case in which we have k–producible states, thus managing to measure genuine multipartite entanglement of quantum systems. To do so, we will adopt the formulation used in [39], where a framework to describe genuine multipartite correlations in classical and quantum systems was proposed through the identification of distance-based measures.
Let us start with the *pure* state $\rho_N = |\psi\rangle\langle\psi|$. Similarly to equation (3.8), we define the measure of genuine entanglement of order higher than k as follows:

$$E^{k\to N}(\rho_N) = \min_{\mathcal{P}_k} \frac{1}{m}\sum_{i=1}^{m} S(\rho_i) \tag{3.24}$$

where the minimum is taken over all the k–producible partitions $\mathcal{P}_k = \left\{\bigotimes_{i=1}^{m}\rho_{k_i}, \sum_{i=1}^{m} k_i = N, k = \max\{k_i\}\right\}$, $m \leq \lceil \frac{N}{k} \rceil$ is the number of marginals $\rho_{k_i} = \mathrm{Tr}_{N-k_i}(\rho_N)$ belonging in the k–producible partition that fulfil the minimization, and the introduction of the factor $1/m$ has the same meaning of the introduction of $1/k$ for the k–non-separable entanglement (3.8), thus getting the meaning of an arithmetic mean of the entropy of the subsystems.
Then, thanks to the convex roof construction, for a *mixed* state ρ_N the genuine entanglement of order higher than k–is defined as:

$$E^{k\to N}(\rho_N) = \inf_{\{p_j, |\psi_j\rangle\}} \sum_j p_j E^{k\to N}(|\psi_j\rangle\langle\psi_j|) \tag{3.25}$$

where the infimum is taken over all ensemble decompositions of ρ_N.
The fulfilment of properties $\mathbb{A}_{1-3}$ is straightforward since this measure is a generalization of the already proved measure for k–non-separable states.

Now, as an illustrative example, we evaluate this measure of entanglement for the $|GHZ\rangle$ and $|W\rangle$ states, and we verify that this measure is consistent with what we expect. We make use of the calculations performed in the previous section 3.2. So, for the $|GHZ\rangle$ state, we have

$$E^{2\to 3}(|GHZ\rangle\langle GHZ|) = \min_{\mathcal{P}_2} \frac{1}{m}\sum_{i=1}^{m} S(\rho_i) = \frac{1}{2}(1+1) = 1$$

$\boldsymbol{\rho_N}$	$\boldsymbol{E^2}$	$\boldsymbol{E^3}$
$\|GHZ\rangle$	0	1
$\|W\rangle$	0	0.9

Table 3.2: k–partite entanglement values for GHZ and W state using measure (3.24).

$$E^{1\to 3}(|GHZ\rangle\langle GHZ|) = \min_{\mathcal{P}_1} \frac{1}{m}\sum_{i=1}^{m} S(\rho_i) = \frac{1}{3}(1+1+1) = 1,$$

while for the $|W\rangle$ state we have

$$E^{2\to 3}(|W\rangle\langle W|) = \min_{\mathcal{P}_2} \frac{1}{m}\sum_{i=1}^{m} S(\rho_i) = \frac{1}{2}(0.918+0.918) = 0.918$$

$$E^{1\to 3}(|W\rangle\langle W|) = \min_{\mathcal{P}_1} \frac{1}{m}\sum_{i=1}^{m} S(\rho_i) = \frac{1}{3}(0.918+0.918+0.918) = 0.918.$$

On the other hand, it is more interesting and useful to measure the amount of genuine k–partite entanglement rather than the genuine entanglement of order higher than k. To do so, we could use the same formulation pointed out in [39] and define the genuine k–partite entanglement of formation as follows:

$$E^k(\rho_N) = E^{k-1\to N}(\rho) - E^{k\to N}(\rho). \tag{3.26}$$

However, we can already see that this is not consistent with what we know about the $|GHZ\rangle$ and $|W\rangle$ states. Indeed, we expect to have only tripartite entanglement for the $|GHZ\rangle$ state, while both bipartite and tripartite entanglement for the $|W\rangle$ state (see [42]), i.e. the entanglement of $|W\rangle$ is more robust under particle loss than the one of $|GHZ\rangle$. But, while this is true for the $|GHZ\rangle$ state

$$E^3(\rho^{GHZ}) = E^{2\to 3}(\rho^{GHZ}) = 1$$

$$E_2(\rho^{GHZ}) = E^{1\to 3}(\rho^{GHZ}) - E^{2\to 3}(\rho^{GHZ}) = 0,$$

instead for the $|W\rangle$ we do not obtain what we expect:

$$E^3(\rho^{W}) = E^{2\to 3}(\rho^{W}) = 0.918$$

$$E^2(\rho^{W}) = E^{1\to 3}(\rho^{W}) - E^{2\to 3}(\rho^{W}) = 0.$$

These values are also reported in Table 3.2 which gives a more direct glimpse of the erroneous similar behaviour of GHZ state and W one.
Thus, using the definition of genuine entanglement of formation of order higher

than k (3.24) and formula (3.26) we were not able to describe the presence of bipartite entanglement in the $|W\rangle$ state. The origin of this setback can be found in the fact that for both the $|GHZ\rangle$ and the $|W\rangle$ state the reduced density matrices ρ_{AB} and ρ_A have the same entropy (see calculations in (3.18),(3.23)), thus there is no effective way to distinguish between the entanglement of formation defined in (3.24) for 1–producible partition and 2–producible one. This implies that formula (3.26) gives zero genuine bipartite entanglement for both $|GHZ\rangle$ and $|W\rangle$.

Hence, a different formulation for a measure of genuine multipartite entanglement is needed. We will propose a generic construction of genuine k–partite entanglement measures. It exploits the concept of producibility and is solely based on a generic measure of bipartite entanglement. Thus, for construction, it will fulfil the known properties for a valid entanglement measure. Then, we will present some examples that validate the measure for some canonical quantum states.

In the first chapter, we have shown that entanglement in a bipartite system represents how far a state is from the one created through LOOC between two laboratories. We first address the problem of extending this interpretation to the case of a multipartite system, and of defining a construction to measure its genuine multipartite entanglement. For genuine k–partite entangled state we intend a multipartite state for which there are groups of at most k entangled particles. So it comes in handy to use the concept of producibility. We will use a framework similar to the one in [39], which identifies distance-based measures to describe genuine multipartite correlations in classical and quantum systems. We instead will use solely a measure of bipartite entanglement to describe genuine multipartite entanglement.

Let us start with the interpretation of k-partite entanglement. Let us consider a multipartite quantum system $\mathcal{X}_N$ of N qubits distributed among remote parties (laboratories) that can only interact via LOCC. In a similar way to the bipartite entanglement, k–partite entanglement of a state represents how this state differs from one that can be constructed through LOCC by the N laboratories when these are divided into subsets of at most k laboratories. In other words, it is within these subsets that our state is formed beyond the limitations imposed by LOCC.
Now we describe the construction of a measure of genuine k–partite entanglement. First, we partition the system into groups of at most k qubits, thus k laboratories, by considering the set of k–producible partitions $\mathcal{P}_k$ (3.3). To take into account the entanglement present in each group, so in each marginal state ρ_{k_i}, we sum the bipartite entanglement between each individual qubit and the other qubits. But, to avoid redundancy while evaluating the entanglement between one qubit and the other qubits, we trace out the qubits for which the bipartite entanglement has been already taken in. In other words, by considering the bipartite entanglement between each qubit and the remaining ones, this approach takes into account

how the marginal state of k qubits ρ_k differs overall from the state that can be created through LOCC between each individual laboratory and the remaining $k-1$ laboratories, so if only LOCC were allowed between the k laboratories. This procedure is then repeated for all the marginal states ρ_{k_i} composing the k–producible partition of the system $\mathcal{X}_N$. A pictorial representation of this construction in the case of a system with four qubits is illustrated in Figure 3.4. In this way, we obtain

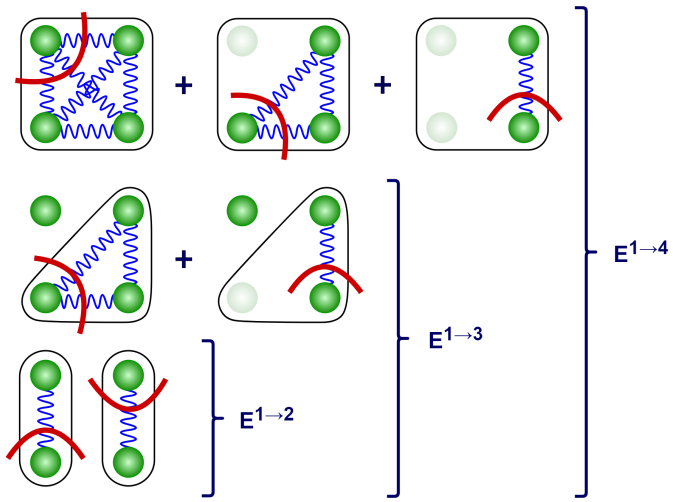

Figure 3.4: Construction of genuine multipartite entanglement of order lower or equal than k, $E^{1\to k}$, in the case of a system with four qubits (green spheres). Each row is related to a different way of partitioning the system of four qubits: four, three, and two producible partitions. Subsets of the partitions are encased in black lines. Blue zigzag lines indicate the possible entanglement present between two qubits. Red arches represent the bipartition for which the entanglement between one qubit and the remaining ones is evaluated. Light green spheres indicate the qubits traced out once their bipartite entanglement has been considered. So, genuine multipartite entanglement of order equal or lower than k, $E^{1\to k}$, is assessed by evaluating the maximum of the sum of all bipartite entanglements over the different $k, k-1, \ldots, 2$ producible partitions.

the genuine multipartite entanglement of order equal or lower than k, which is defined as:

$$E^{1\to k}(\rho_N) = \max_{\mathcal{P}_k} \sum_{i=1}^{m} E(\rho_{k_i}), \tag{3.27}$$

where the maximum is taken over all the k-producible partitions of ρ_N and $E(\rho_{k_i})$ is given by the sum of the bipartite entanglement of formation of ρ_{k_i} and of all its

$\boldsymbol{\rho_N}$	$\boldsymbol{E^2}$	$\boldsymbol{E^3}$	$\boldsymbol{E^4}$
$\|GHZ_3\rangle$	0	1	
$\|W_3\rangle$	0.5	0.9	
$\|\psi^{\pm}/\phi^{\pm}\rangle \otimes \|\psi^{\pm}/\phi^{\pm}\rangle$	2	0	0
$\|GHZ_4\rangle$	0	0	1

Table 3.3: Genuine k–partite entanglement values for GHZ state of three qubits, W_3 state, a state given by the tensor product of two Bell states $|\psi^{\pm}\rangle$ / $|\phi^{\pm}\rangle$, GHZ of four qubits.

$k_i - 2$ subsystems obtained tracing out one subsystem at a time:

$$E(\rho_{k_i}) = \max\Big\{E_{\mathcal{X}_{[1]}}(\rho_{k_i}) + \sum_{j=2}^{k_i-1} E_{\mathcal{X}_{[j]}}\Big(\mathrm{Tr}_{\mathcal{X}_{[1]},\ldots,\mathcal{X}_{[j-1]}}(\rho_{k_i})\Big)\Big\}, \tag{3.28}$$

where the maximum is taken over all the possible permutations of the subsystems $\mathcal{X}_{[1]}, \ldots, \mathcal{X}_{[k_i]}$ composing ρ_{k_i}, obviously $E(\rho_{[l]}) = 0$ that is the entanglement of a single subsystem is zero, and $E_{\mathcal{X}_{[l]}}$ is the bipartite entanglement with respect to subsystem $\mathcal{X}_{[l]}$: if ρ_{k_i} describes the state of the quantum systems $\mathcal{X}_{[1]}, \ldots, \mathcal{X}_{[k_i]}$, then its bipartite entanglement with respect to $\mathcal{X}_{[l]}$ is:

$$E_{\mathcal{X}_{[l]}}(\rho_{k_i}) = E_2(\mathcal{X}_{[l]} : \mathcal{X}_{[1]} \ldots \mathcal{X}_{[l-1]}\mathcal{X}_{[l+1]} \ldots \mathcal{X}_{[k_i]}). \tag{3.29}$$

Any of the measures fulfilling properties $\mathbb{A}_{1-3}$ can be used to measure bipartite entanglement E_2 in (3.29).
Then, the genuine k-partite entanglement is given by the difference between multipartite entanglement of order equal or lower than k and the one equal or lower than $k-1$:

$$E^k = E^{1\to k} - E^{1\to k-1}. \tag{3.30}$$

It is straightforward to see that measure (3.27), and consequently also (3.30), satisfy the desiderata properties $\mathbb{A}_{1-3}$ for an entanglement measure. Indeed, by construction, (3.27) is given by the sum of bipartite entanglements. Thus, assuming that the used measure of bipartite entanglement fulfills the properties $\mathbb{A}_{1-3}$, then those properties are also accomplished by (3.27). In particular, the measure of genuine k–partite entanglement E^k is always bigger or equal to zero, since we always have $E^{1\to k} \geq E^{1\to k-1}$.

To check the validity of this measure, now we evaluate the genuine k-partite entanglement for some canonical examples. As a bipartite entanglement measure, we use the entanglement of formation:

$$E_2(|\psi\rangle) = S(\rho_A) = S(\rho_B)$$

for pure states $\rho_{AB} = |\psi\rangle\langle\psi|$ and

$$E_2(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \; E_2(|\psi_i\rangle)$$

for mixed states $\rho_{AB} = \sum_j p_j |\psi_j\rangle\langle\psi_j|$. Here are the quantum states considered:

- the GHZ state, for which we obtain $E^{1\to3}\left(\rho_3^{GHZ}\right) = 1$, $E^{1\to2}\left(\rho_3^{GHZ}\right) = 0$;
- similarly, for the GHZ state of N qubits, $E^{1\to N}\left(\rho_N^{GHZ}\right) = 1$ and $E^{N}\left(\rho_N^{GHZ}\right) = 1$;
- the W state of three qubits, for which $E^{1\to3}\left(\rho_3^{W}\right) = 1.4$, $E^{1\to2}\left(\rho_3^{W}\right) = 0.5$;
- we evaluate also the genuine tripartite and bipartite entanglement values for the states given by the linear combinations of $|GHZ\rangle$ and $|W\rangle$. Their values for different combinations are shown in Fig. 3.5, which exhibits some peculiar behaviors;
- another sanity check is verifying that a state given by the tensor product of N Bell states, so with $2N$ qubits, has bipartite entanglement equal to N and k–partite entanglement equal to 0 for $k \geq 2$. In the case $N = 2$, we have $E^{1\to2} = 2$, $E^{1\to3} = 2$, $E^{1\to4} = 2$.

The genuine k–partite entanglement values for all these cases are reported in Table 3.3. For all these cases we get values in agreement with what we expected.

In this comprehensive discussion on measuring genuine multipartite entanglement, a novel approach is proposed to construct a measure for genuine k–partite entanglement in quantum systems. Initially, an attempt was made to extend an existing measure, as described in equation (3.26), to assess genuine k–partite entanglement. However, this measure exhibited shortcomings when applied to canonical states like the $|GHZ\rangle$ and $|W\rangle$ states. Specifically, it failed to delineate bipartite entanglement accurately within these states, a pivotal distinction. This limitation prompted the introduction of an alternative approach. This method leverages the concept of producibility and relies solely on a generic measure of bipartite entanglement, addressing the limitations posed by LOCC. The developed measure, described by equations (3.27) to (3.30), successfully satisfies the desired properties for a valid entanglement measure. Notably, the proposed measure is corroborated through rigorous evaluations of various canonical quantum states, such as GHZ and W states, and even linear combinations of these states. The results obtained are in accordance with the expected behaviour, validating the effectiveness of the devised measure.

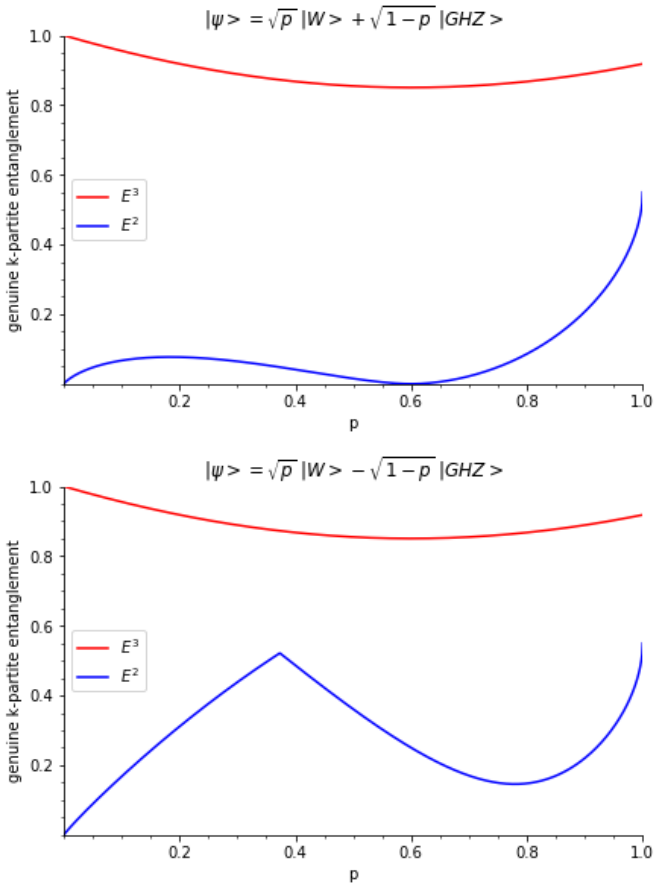

Figure 3.5: Genuine tripartite and bipartite entanglement values for the states given by the linear combinations of $|GHZ\rangle$ and $|W\rangle$.

Chapter 4

Conclusions and future developments

Entanglement is a unique property of quantum systems and plays a central role in quantum information theory. While bi-partite entanglement has been extensively studied and experimentally observed, multi-partite entanglement presents a challenging frontier in quantum information theory. Understanding and quantifying entanglement for multipartite states remains an ongoing and crucial pursuit in advancing our comprehension of quantum phenomena and leveraging quantum information science for practical applications. Its complexity stems from the myriad ways entanglement can manifest among multiple particles. The exploration of partitionability and its connection to entanglement showcases the multifaceted nature of those quantum systems. The variety of possible partitions within an N-partite system highlights the diverse nature of multipartite entanglement, leading to numerous distinct types and degrees of entanglement.

In this thesis, we have first introduced a generalization to the multipartite case of the entanglement of formation measure. We termed it k-NonSep entanglement and pointed out how it quantifies entanglement in k-non-separable states. This formulation extends the concept of entanglement from bipartite scenarios by measuring the mixedness of subsystems within N-partite systems. Proofs of its well-defined properties in accordance with three axioms ($\mathbb{A}_{1-3}$) of a valid entanglement measure have demonstrated its consistency as a fruitful entanglement measure. However, this measure manifested limitations and downsides. Remarkably, it was not able to take into account the deeper properties of entanglement, such as whether a state belongs to a certain entanglement class under LOCC. For instance, we remarked on the inability of this measure to distinguish between the states $|GHZ\rangle$ and $|W\rangle$. This limitation prompted the introduction of an alternative approach. First, we investigated the physical interpretation of genuine k–partite entanglement of a

quantum state. It represents how this state differs from the one that can be constructed through LOCC by the N laboratories when these are divided into subsets of at most k laboratories: it is within these subsets that the state is formed beyond the limitations imposed by LOCC. Then, we presented a generic construction for a genuine multipartite entanglement measure. This construction is based on the idea of partitioning a multipartite system into groups of at most k qubits and, for each group, evaluating the sum of bipartite entanglement between each particle. Due to this construction, the measure satisfies the desiderata properties of an entanglement measure. Further, the evaluation of this entanglement measure for key states corroborated its efficacy in delineating genuine multipartite entanglement, reaffirming its applicability and relevance in characterizing complex quantum states. This methodological framework adds to the toolkit of quantum information theory, aiding in the systematic exploration and understanding of multipartite entanglement in quantum systems.

In summary, we have proposed a novel framework for quantifying genuine multipartite entanglement leveraging the concept of producibility and relying solely on a measure of bipartite entanglement. This refined approach successfully rectified the prior limitations, exhibiting consistency and accuracy when evaluated across a spectrum of quantum states. Particularly, it was possible to distinguish between different entanglement classes in states like $|W\rangle$ and $|GHZ\rangle$. Moreover, the formalism satisfies the properties required to be fulfilled by any valid entanglement measure.

Our method offers a promising avenue to explore and quantify multipartite entanglement, providing a robust tool applicable to various quantum states, further advancing our understanding of multipartite correlations in quantum mechanics. Further research can explore its evaluation for more general multipartite quantum states and its connections to other entanglement measures. Furthermore, it would be intriguing to explore potential applications in quantum computing and idealize experiments to measure entanglement for multipartite states, which could confirm or refute the predicted value of our measure.

Appendix A

Code for counting partitions

First, we show the Python code for counting the number of ways to partition a set into k disjoint subsets.
For example, let us consider the case with $N = 3$ and $k = 2$. The set is given by {1,2,3}. We can partition it into 2 subsets in following ways:
$\{\{1,2\},\{3\}\}, \{\{1\},\{2,3\}\}, \{\{1,3\},\{2\}\}$. So we have in total 3 different ways of partitioning a set of 3 elements into 2 disjoint subsets.
The following code is based on a recursive approach which is explained here. There are two cases:

1. The previous $N-1$ elements are divided into k partitions, i.e $S(N-1,k)$ ways. Put this nth element into one of the previous k partitions. So, count $= k * S(N-1,k)$

2. The previous $N-1$ elements are divided into $k-1$ partitions, i.e $S(N-1,k-1)$ ways. Put the N-th element into a new partition (single element partition). So, count $= S(N-1,k-1)$

3. Total count $= k * S(N-1,k) + S(N-1,k-1)$.

```
% # A Python3 program to count the number of partitions of a set with
    N elements into k subsets

# Returns count of different partitions of n elements in k subsets

def countP(N, k):

    # Base cases
    if (N == 0 or k == 0 or k > n):
        return 0
```

```
    if (k == 1 or k == n):
        return 1

    # S(N+1, k) = k*S(N, k) + S(N, k-1)
    return (k * countP(N-1, k) +
                countP(N-1, k-1))
```

Since two recursive functions are called for every value of N, then the time complexity of the above code is exponential. More specifically, the time complexity is $O(2^N)$.

More efficient solutions can be devised by exploiting dynamic programming. Indeed, the solution can be optimized by reducing the overlapping subproblems. This can be done by avoiding the recomputation of the same subproblems by constructing a temporary array dp[ ][ ], as it is shown in the code below.

```
# A Dynamic Programming based Python3 program to count the number of
    partitions of a set with N elements into k subsets

# Returns count of different partitions of N elements in k subsets
def countP(N, k):

    # Table to store results of subproblems
    dp = [[0 for i in range(k + 1)]
             for j in range(N + 1)]

    # Base cases
    for i in range(N + 1):
        dp[i][0] = 0

    for i in range(k + 1):
        dp[0][k] = 0

    # Fill rest of the entries in  dp[][] in bottom up manner
    for i in range(1, n + 1):
        for j in range(1, k + 1):
            if (j == 1 or i == j):
                dp[i][j] = 1
            else:
                dp[i][j] = (j * dp[i-1][j] + dp[i-1][j-1])

    return dp[N][k]
```

The time complexity of the dynamic programming code is $O(N \times k)$.

Now we show the Python code for evaluating the Bell number, that is the total number of ways to partition a set of N elements into k subsets. The value of N-th

Bell Number is the sum of $S(N,k)$ for $k = 1$ to N: $Bell(N) = \sum_{k=1}^{N} S(N,k)$.
For example, let us consider the case with $N = 3$. The set is given by {1,2,3}. We can partition it in the following ways:
$\{\{1\},\{2\},\{3\}\}, \{\{1,2\},\{3\}\}, \{\{1\},\{2,3\}\}, \{\{1,3\},\{2\}\}, \{\{1,2,3\}\}$. So we have in total 5 different ways of partitioning a set of 3 elements.

```
# python program to find the number of ways of partitioning it
N = 5
s = [[0 for _ in range(N+1)] for _ in range(N+1)]
for i in range(N+1):
    for j in range(N+1):
        if j > i:
            continue
        elif(i==j):
            s[i][j] = 1
        elif(i==0 or j==0):
            s[i][j]=0
        else:
            s[i][j] = j*s[i-1][j] + s[i-1][j-1]
ans = 0
for i in range(0,N+1):
    ans+=s[N][i]
print(ans)
```

Appendix B

Qiskit code

The following Qiskit code is useful for the evaluation of the reduced density matrices and the entropies for the W state.

```
# importing lbraries
from qiskit import QuantumCircuit, Aer, execute, quantum_info,
    QuantumRegister, ClassicalRegister
from qiskit.quantum_info import DensityMatrix, Statevector
import numpy as np

# creating the W state
q = QuantumRegister(3)
rot = 2* np.arcsin(1/np.sqrt(3))
circuit = QuantumCircuit(q)
circuit.ry(rot, q[0])
circuit.x(q[0])
circuit.ch(q[0], q[1])
circuit.x(q[1])
circuit.ccx(q[0], q[1], q[2])
circuit.x(q[1])
circuit.x(q[0])

backend = Aer.get_backend('statevector_simulator')
result = execute(circuit, backend).result()
statevector = result.get_statevector()
density_matrix = DensityMatrix(statevector)

# evaluating the reduced density matrix tracing out 1 qubit
qubits_to_trace_out_1 = [0]
reduced_density_matrix_1 = quantum_info.partial_trace(density_matrix,
    qubits_to_trace_out_1)
entropy_1 = quantum_info.entropy(reduced_density_matrix_1)

```

```
# evaluating the reduced density matrix tracing out 2 qubits
qubits_to_trace_out_2 = [1,2]
reduced_density_matrix_2 = quantum_info.partial_trace(density_matrix,
    qubits_to_trace_out_2)
entropy_2 = quantum_info.entropy(reduced_density_matrix_2)

# evaluating the entanglement of formation of \rho_AB
E_AB = quantum_info.entanglement_of_formation(
    reduced_density_matrix_1)
```

Bibliography

[1] JC Ward. *Memoirs of a theoretical physicist*. 2004 (cit. on p. 1).

[2] Jonathan P. Dowling and Gerard J. Milburn. *Quantum Technology: The Second Quantum Revolution*. 2002. arXiv: quant-ph/0206091 [quant-ph] (cit. on p. 1).

[3] A. Einstein, B. Podolsky, and N. Rosen. «Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?» In: *Phys. Rev.* 47 (10 May 1935), pp. 777–780. DOI: 10.1103/PhysRev.47.777. URL: https://link.aps.org/doi/10.1103/PhysRev.47.777 (cit. on pp. 2, 14).

[4] Erwin Schrödinger. «Die gegenwärtige Situation in der Quantenmechanik». In: *Naturwissenschaften* 23 (1935), pp. 807–812 (cit. on pp. 2, 15).

[5] J. S. Bell. «On the Einstein Podolsky Rosen paradox». In: *Physics Physique Fizika* 1 (3 Nov. 1964), pp. 195–200. DOI: 10.1103/PhysicsPhysiqueFizika.1.195. URL: https://link.aps.org/doi/10.1103/PhysicsPhysiqueFizika.1.195 (cit. on p. 2).

[6] John F. Clauser, Michael A. Horne, Abner Shimony, and Richard A. Holt. «Proposed Experiment to Test Local Hidden-Variable Theories». In: *Phys. Rev. Lett.* 23 (15 Oct. 1969), pp. 880–884. DOI: 10.1103/PhysRevLett.23.880. URL: https://link.aps.org/doi/10.1103/PhysRevLett.23.880 (cit. on p. 2).

[7] John F. Clauser and Abner Shimony. «Bell's theorem. Experimental tests and implications». In: *Reports on Progress in Physics* 41 (1978), pp. 1881–1927 (cit. on p. 2).

[8] Alain Aspect, Philippe Grangier, and Gérard Roger. «Experimental Realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A New Violation of Bell's Inequalities». In: *Phys. Rev. Lett.* 49 (2 July 1982), pp. 91–94. DOI: 10.1103/PhysRevLett.49.91. URL: https://link.aps.org/doi/10.1103/PhysRevLett.49.91 (cit. on pp. 2, 16).

[9] Richard P Feynman. «Simulating physics with computers». In: *International journal of theoretical physics* 21.6/7 (1982), pp. 467–488 (cit. on p. 2).

[10] Gerardo Adesso, Marco Cianciaruso, and Thomas R Bromley. «An introduction to quantum discord and non-classical correlations beyond entanglement». In: *arXiv preprint arXiv:1611.01959* (2016) (cit. on p. 2).

[11] V. Vedral. «The role of relative entropy in quantum information theory». In: *Reviews of Modern Physics* 74.1 (Mar. 2002), pp. 197–234. DOI: 10.1103/revmodphys.74.197. URL: https://doi.org/10.1103%2Frevmodphys.74.197 (cit. on p. 11).

[12] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, 2011. ISBN: 9781107002173. URL: https://www.amazon.com/Quantum-Computation-Information-10th-Anniversary/dp/1107002176?SubscriptionId=AKIAIOBINVZYXZQZ2U3A&tag=chimbori05-20&linkCode=xm2&camp=2025&creative=165953&creativeASIN=1107002176 (cit. on pp. 12, 29).

[13] H. Araki and E. H. Lieb. «Entropy inequalities». In: *Commun. Math. Phys.* 18 (1970), pp. 160–170. DOI: 10.1007/BF01646092 (cit. on p. 13).

[14] E. Schrödinger. «Discussion of Probability Relations between Separated Systems». In: *Mathematical Proceedings of the Cambridge Philosophical Society* 31.4 (1935), pp. 555–563. DOI: 10.1017/S0305004100013554 (cit. on p. 14).

[15] David Bohm. *Quantum theory*. Prentice-Hall, New York, 1951 (cit. on p. 14).

[16] John S. Bell. «On the Problem of Hidden Variables in Quantum Mechanics». In: *Rev. Mod. Phys.* 38 (3 July 1966), pp. 447–452. DOI: 10.1103/RevModPhys.38.447. URL: https://link.aps.org/doi/10.1103/RevModPhys.38.447 (cit. on p. 16).

[17] Daniel M. Greenberger, Michael A. Horne, and Anton Zeilinger. «Multiparticle Interferometry and the Superposition Principle». In: *Physics Today* 46.8 (Aug. 1993), pp. 22–29. ISSN: 0031-9228. DOI: 10.1063/1.881360. eprint: https://pubs.aip.org/physicstoday/article-pdf/46/8/22/8306195/22_1_online.pdf. URL: https://doi.org/10.1063/1.881360 (cit. on pp. 17, 31).

[18] Michael Walter, David Gross, and Jens Eisert. *Multi-partite entanglement*. 2017. arXiv: 1612.02437 [quant-ph] (cit. on p. 18).

[19] Ryszard Horodecki, Paweł Horodecki, Michał Horodecki, and Karol Horodecki. «Quantum entanglement». In: *Reviews of Modern Physics* 81.2 (June 2009), pp. 865–942. DOI: 10.1103/revmodphys.81.865. URL: https://doi.org/10.1103%2Frevmodphys.81.865 (cit. on pp. 19, 23, 24, 31).

[20] Martin Müller-Lennert, Frédéric Dupuis, Oleg Szehr, Serge Fehr, and Marco Tomamichel. «On quantum Rényi entropies: A new generalization and some properties». In: *Journal of Mathematical Physics* 54.12 (Dec. 2013). ISSN: 1089-7658. DOI: 10.1063/1.4838856. URL: http://dx.doi.org/10.1063/1.4838856 (cit. on p. 22).

[21] Asher Peres. «Separability Criterion for Density Matrices». In: *Physical Review Letters* 77.8 (Aug. 1996), pp. 1413–1415. DOI: 10.1103/physrevlett.77.1413. URL: https://doi.org/10.1103%2Fphysrevlett.77.1413 (cit. on p. 22).

[22] Charles H. Bennett, Gilles Brassard, Sandu Popescu, Benjamin Schumacher, John A. Smolin, and William K. Wootters. «Purification of Noisy Entanglement and Faithful Teleportation via Noisy Channels». In: *Physical Review Letters* 76.5 (Jan. 1996), pp. 722–725. DOI: 10.1103/physrevlett.76.722. URL: https://doi.org/10.1103%2Fphysrevlett.76.722 (cit. on p. 23).

[23] Eric Chitambar, Debbie Leung, Laura Mančinska, Maris Ozols, and Andreas Winter. «Everything You Always Wanted to Know About LOCC (But Were Afraid to Ask)». In: *Communications in Mathematical Physics* 328.1 (Mar. 2014), pp. 303–326. DOI: 10.1007/s00220-014-1953-9. URL: https://doi.org/10.1007%2Fs00220-014-1953-9 (cit. on p. 24).

[24] Martin B. Plenio and S. Virmani. *An introduction to entanglement measures*. 2006. arXiv: quant-ph/0504163 [quant-ph] (cit. on pp. 24, 27).

[25] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight. «Quantifying Entanglement». In: *Physical Review Letters* 78.12 (Mar. 1997), pp. 2275–2279. DOI: 10.1103/physrevlett.78.2275. URL: https://doi.org/10.1103%2Fphysrevlett.78.2275 (cit. on p. 27).

[26] Luigi Amico, Rosario Fazio, Andreas Osterloh, and Vlatko Vedral. «Entanglement in many-body systems». In: *Rev. Mod. Phys.* 80 (2008), pp. 517–576. DOI: 10.1103/RevModPhys.80.517. arXiv: quant-ph/0703044 (cit. on p. 27).

[27] V. Vedral and M. B. Plenio. «Entanglement measures and purification procedures». In: *Physical Review A* 57.3 (Mar. 1998), pp. 1619–1633. ISSN: 1094-1622. DOI: 10.1103/physreva.57.1619. URL: http://dx.doi.org/10.1103/PhysRevA.57.1619 (cit. on p. 28).

[28] Charles H. Bennett, David P. DiVincenzo, John A. Smolin, and William K. Wootters. «Mixed-state entanglement and quantum error correction». In: *Physical Review A* 54.5 (Nov. 1996), pp. 3824–3851. DOI: 10.1103/physreva.54.3824. URL: https://doi.org/10.1103%2Fphysreva.54.3824 (cit. on p. 29).

[29] Richard Jozsa and Noah Linden. «On the role of entanglement in quantum-computational speed-up». In: *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences* 459.2036 (Aug. 2003), pp. 2011–2032. DOI: 10.1098/rspa.2002.1097. URL: https://doi.org/10.1098%2Frspa.2002.1097 (cit. on p. 31).

[30] W. Dür, G. Vidal, and J. I. Cirac. «Three qubits can be entangled in two inequivalent ways». In: *Phys. Rev. A* 62 (6 Nov. 2000), p. 062314. DOI: 10.1103/PhysRevA.62.062314. URL: https://link.aps.org/doi/10.1103/PhysRevA.62.062314 (cit. on pp. 31, 46).

[31] Anders Karlsson and Mohamed Bourennane. «Quantum teleportation using three-particle entanglement». In: *Phys. Rev. A* 58 (6 Dec. 1998), pp. 4394–4400. DOI: 10.1103/PhysRevA.58.4394. URL: https://link.aps.org/doi/10.1103/PhysRevA.58.4394 (cit. on p. 31).

[32] Mark Hillery, Vladimir Bužek, and Andre Berthiaume. «Quantum secret sharing». In: *Physical Review A* 59.3 (Mar. 1999), pp. 1829–1834. DOI: 10.1103/physreva.59.1829. URL: https://doi.org/10.1103%2Fphysreva.59.1829 (cit. on p. 31).

[33] Pankaj Agrawal and Arun Pati. «Perfect teleportation and superdense coding with W states». In: *Phys. Rev. A* 74 (6 Dec. 2006), p. 062320. DOI: 10.1103/PhysRevA.74.062320. URL: https://link.aps.org/doi/10.1103/PhysRevA.74.062320 (cit. on p. 31).

[34] Ellie D'Hondt and Prakash Panangaden. «The Computational Power of the W and GHZ states». In: (2006). arXiv: quant-ph/0412177 [quant-ph] (cit. on p. 31).

[35] Otfried Gühne and Géza Tóth. «Entanglement detection». In: *Physics Reports* 474.1-6 (Apr. 2009), pp. 1–75. DOI: 10.1016/j.physrep.2009.02.004. URL: https://doi.org/10.1016%2Fj.physrep.2009.02.004 (cit. on p. 32).

[36] G. Andrews. *The Theory of Partitions*. Cambdridge University Press, 1976 (cit. on p. 32).

[37] Otfried Gühne, Géza Tóth, and Hans J Briegel. «Multipartite entanglement in spin chains». In: *New Journal of Physics* 7.1 (Nov. 2005), p. 229. DOI: 10.1088/1367-2630/7/1/229. URL: https://dx.doi.org/10.1088/1367-2630/7/1/229 (cit. on pp. 33, 37).

[38] Marcus Huber, Marti Perarnau-Llobet, and Julio I. de Vicente. «Entropy vector formalism and the structure of multidimensional entanglement in multipartite systems». In: *Physical Review A* 88.4 (Oct. 2013). DOI: 10.1103/physreva.88.042328. URL: https://doi.org/10.1103%2Fphysreva.88.042328 (cit. on p. 33).

[39] Davide Girolami, Tommaso Tufarelli, and Cristian E. Susa. «Quantifying Genuine Multipartite Correlations and their Pattern Complexity». In: *Physical Review Letters* 119.14 (Oct. 2017). DOI: 10.1103/physrevlett.119.140505. URL: https://doi.org/10.1103%2Fphysrevlett.119.140505 (cit. on pp. 36, 46–49).

[40] Sabine Wölk and Otfried Gühne. «Characterizing the width of entanglement». In: *New Journal of Physics* 18.12 (Dec. 2016), p. 123024. DOI: 10.1088/1367-2630/aa5015. URL: https://doi.org/10.1088%2F1367-2630%2Faa5015 (cit. on p. 38).

[41] Szilard Szalay. «Multipartite entanglement measures». In: *Physical Review A* 92.4 (Oct. 2015). DOI: 10.1103/physreva.92.042329. URL: https://doi.org/10.1103%2Fphysreva.92.042329 (cit. on p. 39).

[42] W. Dür, G. Vidal, and J. I. Cirac. «Three qubits can be entangled in two inequivalent ways». In: *Physical Review A* 62.6 (Nov. 2000). DOI: 10.1103/physreva.62.062314. URL: https://doi.org/10.1103%2Fphysreva.62.062314 (cit. on pp. 44, 48).

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Acronyms

LOCC
Local operations and classical communication

SLOCC
Stochastic local operations and classical communication

PPT
Positive partial transpose

CP
Completely positive

CPTP
Completely positive trace preserving

GME
Genuine multipartite entanglement

GHZ
Greenberger-Horne-Zeilinger

EPR
Einstein-Podolski-Rosen

LHV
Local hidden variable

POVM
Positive operator-valued measure