Optimal growth in an uncertain environment

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Abstract

Bet-hedging in biological phenomena is a well recognized process that can manifest itself as phenotypic switching: an individual has to choose the best strategy to adapt to a stochastic environment in order to maximize the growth rate of its own population. Fluctuations can lead to the extinction of the population itself, no matter how high the growth rate is. The same model can be applied to Kelly' s horse races studies, where the gambler has to choose the best strategy to maximize his payoff. We propose to add a definition of "risk" to Kelly's horse races model in order to map the phenotypic switching model into Kelly's one, taking into account the "bankruptcy" probability, that we want to minimize. We end up considering the volatility as a measure of risk, as well as the probability of extinction itself and comparing the different definitions. Results show a similarity between the two notions. Furthermore, we analyse the model with a game-theoretic approach, to see if there is any possible reduction, under some particular conditions, of the set of strategies into a smaller one, called the "essential part of the game". We found a general way to map an optimization problem to a game-theoretic one, showing how reduction leads to the same results.

1 Phenotypic switching in a stochastic environment

In an environment characterized by unpredictable fluctuations, populations may find it advantageous to prioritize long-term risk reduction over short-term reproductive success. This strategy serves to safeguard individuals from the potential stochastic and severe environmental variations that could occur. This adaptive mechanism, known as 'bet-hedging,' becomes evident in a straightforward example involving a population of bacteria inhabiting an environment where antibiotics may sporadically appear.

As studied in [8], [16], [5] let us consider a biological population of individuals which exhibit only two phenotypes A and B. The rate of switching, for an individual, from A to B and from B to A will be represented, respectively, by π_1 and π_2. In this simple framework, let us also assume that the environment can present itself only in two different states, namely S_1 and S_2. Stochastic environmental transitions between these two states will be described by fixed rates κ_1 (from S_1 to S_2) and κ_2 (from S_2 to S_1). The population vector describing the number of individuals in each phenotype at a time t will be denoted by $\mathbf{N}(t) = (N_A(t), N_B(t))^{\mathbf{T}}$. The subpopulation of individuals with phenotype A grows when placed in the environment i with the growth rate k_{A_i}, while the other subpopulation with phenotype B grows with rate k_{B_i} . Since the environmental switching rates are fixed, our aim will be to understand what is the best strategy of switching from phenotype A to B, namely the rate π_1, or from B to A, π_2. The reactions can be summarized in the vector equation below [8].

$$\frac{d}{dt}\mathbf{N}(t) = M_{S(t)}\mathbf{N}(t) \tag{1}$$

with matrices

$$M_{S_1} = \begin{pmatrix} k_{A_1} - \pi_1 & \pi_2 \\ \pi_1 & k_{B_1} - \pi_2 \end{pmatrix}$$

$$\mathrm{M}_{S_2} = \begin{pmatrix} k_{A_2} - \pi_1 & \pi_2 \\ \pi_1 & k_{B_2} - \pi_2 \end{pmatrix} \tag{2}$$

From these, one can define the first quantity of interest, the growth rate of the exponential solution:

$$\Lambda_t = \frac{1}{t}\log\frac{N(t)}{N(0)} \tag{3}$$

with $N(t) = N_A(t) + N_B(t)$ and we will consider $\Lambda = \lim_{t\to\infty}\Lambda_t$. From [8], following the Hufton–Lin–Galla approach, the variance of the growth rate can be defined for this system of stochastic switching, so we have an expression for $\mathrm{Var}(\Lambda_t)$. We now analyze the relation between the average growth rate Λ and the asymptotic behavior of the finite time growth rate variance, which we denoted as $\mathrm{Var}(\Lambda)$. As stated before defining bet-hedging, higher growth rate can lead to higher fluctuations (or risk) and therefore a suitable trade-off between average growth rate and variance may be preferable. The vector (π_1, π_2) represents the strategy of the individuals once the environmental parameters (κ_1 and κ_2) are fixed. We introduced α as a suitable risk-aversion parameter to give different weights to the trade-off, so that an α of 0.5 means that we are giving the same importance on maximizing the growth rate of the population as minimizing the fluctuations around its average value. Thus, we define the objective function, that has to be maximized to find the optimal strategies (π_1^*, π_2^*):

$$J(\pi_1, \pi_2) = \alpha\Lambda - (1-\alpha)\sqrt{Var(\Lambda)} \tag{4}$$

One can easily see that, for what concerns the phenotypic switching problem, the standard deviation of the growth rate is used as definition of risk, i.e. the quantity we want to minimize. A numerical maximization of this objective function leads to the following plot in Figure 1

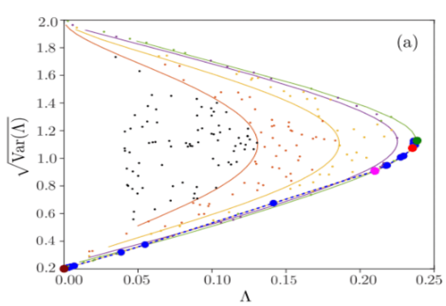

Figure 1: Mean-variance trade-off for the long term growth rate. The filled (blue or other colors) circles represent points in the Pareto front computed by minimizing the objective function. Figure from [8].

The numerical optimization shows that whatever random strategy our population adopts, will always be limited by a trade-off branch of the parabola, which is the one below in the figure. Kelly' s strategy, in financial terms, is the green point in figure 1, where $\alpha = 1$, i.e. we' re just maximizing the long-term growth rate, without any constraint on the volatility. Similar results about the plots and the trade-off have been achieved concerning gambling models ([7], [9]) and Markovitz portfolio studies ([10]).

2 Optimal strategies for the gambling problem

2.1 Diagonal case introduction

To show the mapping between the phenotypic and the gambling problem, let us introduce a simple gambling problem inspired by Kelly's horse races model [7], in which we consider M horses. The gambler, each run, can bet a quantity b_x on each horse $x = 1, 2, ..., M$ such that $\sum_{x=1}^{M} b_x = 1$, meaning that the gambler is investing all of its own capital, each run. For all x's, $b_x > 0$. This specifies that the gambler bets on all horses but only makes money from the horse x that wins (diagonal case). o_x represents the odds paid by the bookmaker when the horse x wins, and the probability for x to win is given by p_x . We can define $r_x = \frac{1}{o_x}$ for a matter of notation.

An essential characteristic of the model is its repetitive nature where the capital gained in one race is reinvested in the subsequent one. Therefore, the gambler's capital, denoted as C_{N+1} after $N+1$ races, is related to their capital after N races, represented as C_N, through the equation

$$C_{N+1} = o_{x_{N+1}} b_{x_{N+1}} C_N \text{ with probability } p_x \tag{5}$$

where $x_{N+1} = 1, ..., M$ is the horse that won the N+1-th race. Again, one can define the long-term growth rate as in the phenotypic switching framework. Since the process is multiplicative, if $N >> 1$, thanks to Central Limit Theorem

$$\lim_{N\to\infty} \frac{1}{N} \log C_N = \sum_x p_x \log o_x b_x := \langle W \rangle \tag{6}$$

We immediately get the similarity (even though the time here is discrete and measured in number of runs) with the biological case as soon as we define the quantity W_x (equivalent of Λ_t) as $W_x = \log(o_x b_x)$. Its average with respect to the ensemble will be denoted as $\langle W \rangle$, and named the average $log - capital$.

Kelly' s strategy, as stated before, is obtained by maximizing the average growth rate over the betting strategy with the constraint on normalization, not taking into account fluctuations in the optimization. An easy computation leads to $b_x^* = p_x \ \forall x$, i.e. a proportional strategy which overlooks risks.

In practice gamblers and investors know that optimal Kelly can be "too risky", that' s why in our optimization problem, one should take into account a definition of risk.

Then, starting from a certain capital C_0 (i.e. an initial $log - capital$ $\log C_0$) we have seen that the multiplicative process leads to some easy relations of the type:

$$C_{n+1} = o_{x_{n+1}} b_{x_{n+1}} C_n$$

which means that after N races, the capital will be

$$C_N = \prod_{n=1}^{N} o_{x_n} b_{x_n} C_0$$

hence

$$\lim_{N\to\infty} \frac{1}{N} \log \frac{C_N}{C_0} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \log\left(o_{x_n} b_{x_n}\right)$$

$$\xrightarrow{CLT} \sum_x p_x \log\left(o_x b_x\right)$$

which, considering that $\log C_0$ becomes negligible in the limit $N \gg 1$, makes us obtain the canonical relation expressed in 6. See the Appendix (A.1) for a slightly more detailed calculation. With this relation we define the log-capital after N races as $log - capital(N) = \sum_{n=1}^{N} \log\left(o_{x_n} b_{x_n}\right)$, since it follows from the Central Limit Theorem (every race outcome is independent from the one before and equally distributed, and so are the logarithms of these random variables), but we still haven't mentioned the fluctuations of this random variable, i.e. the volatility. We will refer to the $log - capital$ also as W, growth rate or wealth, throughout the article. Cases for non independent races and the role of inference to learn from previous outcomes have been studied in [1], [12].

2.2 Volatility as a measure of risk

There have been many attempts to generalize Kelly's horse races model, some ideas came from the possibility to implement adaptive control ([1], [3]), some others trying to implement different definitions of risk ([6], [4]). We initially suggest to implement a risk definition for the gambler using volatility.

We can proceed exactly as in the case for the phenotypic switching: we construct a utility function as the one in [9]

$$J(\alpha, \vec{b}) = \alpha\langle W\rangle - (1-\alpha)\sigma_W \tag{7}$$

where σ_W is nothing but the standard deviation of the random variable W_x, which obviously depends on the betting strategy, and it's called **volatility**. It has to be said that in the following we will solve the optimization problem with respect to the betting strategies only when the problem is concave, since the optimization will actually end up in a maximum. For the same reasons, we will analyse only the trade-off branch.

Two horses numerical optimization The numerical optimization for the two horses case is a simple one. The probabilities for the two horses, in the example made, are described by the vector $p = (0.1, 0.9)$, while if one defines $r_x = \frac{1}{o_x}$ $\forall x$ then the bookmaker sets $r_x = (0.7, 0.3)$. We showed that in the case for two horses, in the trade-off branch, the optimization of the objective function J, for different values of α, gives a so called "Pareto front" along which the non-optimized strategies lie as well. In Figure 2 the blue dots correspond to 100 random strategies that led to certain values of $\langle W\rangle$ and σ_W, they lie exactly in the optimized strategy line (red one).

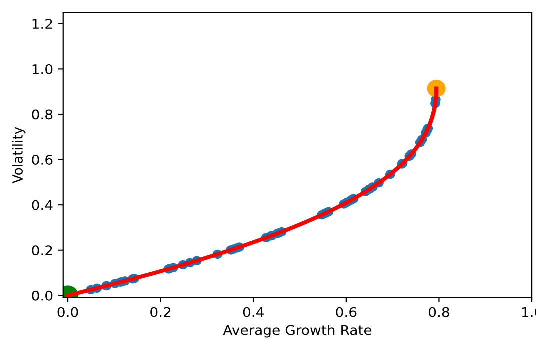

Figure 2: Mean-variance trade-off for 2 horses case with $p = (0.1, 0.9)$ and $r_x = (0.7, 0.3)$. The blue points correspond to random selected strategies, whereas the red line comes from the maximization of the objective function, hence is the realization of the trade-off. The green point correspond to the *null strategy* which has no risk, but at the same time no payoff, while the yellow point corresponds to the $Kelly's\ strategy$, leading to the highest growth rate but also to high fluctuations.

Three horses numerical optimization For the case of three horses the situation is different: random strategies will not anymore correspond to the solution of the optimization problem, but they will always lie above that line, thus, for a given average growth rate, giving an higher risk. In the following numerical simulation we' ve used $p = (0.1, 0.2, 0.7)$ and $r = (0.7, 0.1, 0.2)$ as values for the horses winning probability and for the bookmaker odds. The results are shown in Figure 3.

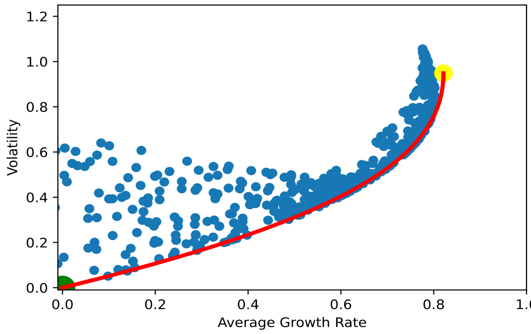

Figure 3: Mean-variance trade-off for 3 horses case with $p = (0.1, 0.2, 0.7)$ and $r = (0.7, 0.1, 0.2)$. The blue points correspond to random selected strategies and now they lie always above the optimal, whereas the red line comes from the maximization of the objective function. Again, the green point corresponds to the *null strategy*, while the yellow point corresponds to the $Kelly'$ *s strategy*.

The main question we would like to ask in the following section is: is volatility the best measure of risk we can provide? Can we define extinction/bankruptcy in a different way?

3 Extinction probability as new definition of risk

3.1 Probability of extinction

We want now to analyse how to deal with the case in which our log-capital (growth rate) reaches a certain boundary level. The point of this is indeed to study with what frequency, hence probability, our log-capital during time is going to reach the "extinction level" which in the context of gambling stands for going bankrupt, whereas meaning the actual extinction of the population in the case of phenotypic switching. For simplicity, we will still consider a diagonal case, meaning that whenever an horse wins, the actual gain will be proportional only to the bet put on that specific horse. Mathematically speaking this means that the odds matrix $\mathbf{O}$ is diagonal, exactly as considered up to now, so that o_x has just a single index standing for the horse that won.

Let's call x the log-capital after N races, which is nothing but a sum of i.i.d. random variables:

$$x = \sum_{n=1}^{N} \log(o_{x_n} b_{x_n}) = S_N \tag{8}$$

If $N \gg 1$ for the CLT one gets:

$$S_N \sim \mathcal{N}(N\langle W\rangle, N\sigma_W^2) \tag{9}$$

with for each random variable:

$$\langle W\rangle = \sum_x p_x \log(o_x b_x)$$

$$\sigma_W^2 = \sum_x p_x \log^2(o_x b_x) - \langle W\rangle^2$$

Relation 9 can be checked by a simple example for 2 horses. Let's consider values $p = (0.2, 0.8)$, $r = (0.6, 0.4)$, $b = (0.4, 0.6)$. One can compute the analytical average growth rate and variance, obtaining $\langle W\rangle = 0.243$ and $\sigma_W^2 = 0.106$. Running then many simulations of $N = 100$ runs, we may plot an histogram showing the distribution of S_{100} from the outcomes we got (Figure 4).

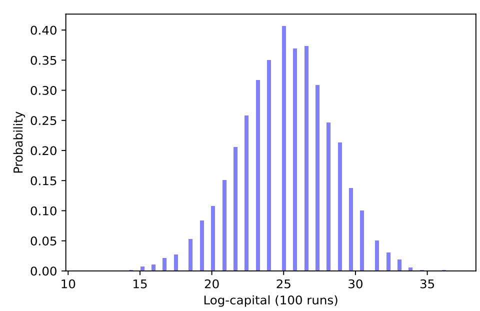

Figure 4: Histogram showing the distribution of 5000 different outcomes for the final log-capital measured in 100 runs. Parameters are $p = (0.2, 0.8)$, $r = (0.6, 0.4)$, $b = (0.4, 0.6)$

As we can see, the log-capital, since $N \gg 1$ follows a gaussian distribution with $N\langle W\rangle$ (from the plot we see it to be around 24.3) while computing the variance this results in $\sigma_{S_{100}}^2 = 10.6 = N\sigma_W^2$ as expected.

Hence, if the number of runs is high enough, the log-capital is gaussian distributed. Thus, calling B the value of the boundary leading us to extinction, we can compute analytically the probability of reaching and being below B **at the "time"** $t = N$:

$$P(x \leq B) = \int_{-\infty}^{B} \frac{e^{-\frac{(x - N\langle W\rangle)^2}{2N\sigma_W^2}}}{\sqrt{2\pi\sigma_W^2 N}} dx \tag{10}$$

which leads to the solution

$$P(x \leq B) = \frac{1}{2}\left[1 + erf\left(\frac{B - N\langle W\rangle}{\sqrt{2N\sigma_W^2}}\right)\right] \quad (11)$$

with $erf(x)$ being the error function. The problem with this definition of risk is the fact that it's computing the probability to be extincted only at the specific "time" $\mathbf{t = N}$, without considering that perhaps, bankruptcy, has been reached already at earlier times.

3.2 Geometric brownian motion mapping

From mathematical considerations, we see that x is nothing but a random variable whose dynamics is described by a drift-diffusion model, given by the fact that, when N is large enough, the probability distribution describing the value of the wealth (growth rate) at a certain time N, is a gaussian distribution whose mean value and variance increase linearly with time (number of runs). If one discretizes the time as the number of runs and assumes to start from an initial wealth x_0, one could write the distribution $\phi_{x_0}(x,t)$, i.e. the probability of having a $log - capital$ x at time t starting from x_0, from 9 as:

$$\phi_{x_0}(x,t) = \frac{e^{-\frac{(x-\langle W\rangle t - x_0)^2}{2\sigma_W^2 t}}}{\sqrt{2\pi\sigma_W^2 t}} \quad (12)$$

which is nothing but the solution of the Fokker-Planck equation for a random variable following a geometric brownian motion with $\langle W\rangle$ as mean and σ_W as standard deviation. x, in this sense, becomes a random variable following its respective Langevin equation with the just specified drift and noise. Further considerations about the geometric brownian motion model can be found in [11].
Since now we want to be sure that the $log-capital$ doesn't go below the threshold B with $B < x_0$ **from whatever time up to time** $\mathbf{t = N}$, we use the **image method**: first, we start taking a linear combination of two solutions, one that started in x_0 and an other one that started in m

$$P(x,t) = \phi_{x_0}(x,t) - e^{-\eta}\phi_m(x,t) \quad (13)$$

This is still going to be a solution and m and and η are parameters still to be determined. The first condition is to impose that, **at time t=0**, $P(x = B, 0) = 0$ thus giving

$$P(x = B, 0) = 0 \implies e^{\frac{(B-x_0)^2}{2\sigma_W^2 t}} = e^{-\eta + \frac{(B-m)^2}{2\sigma_W^2 t}}$$

hence

$$(B - x_0)^2 = (B - m)^2 \iff$$

$$m = 2B - x0$$

the second is instead to impose the same **at whatever other time** $\mathbf{t > 0}$, thus giving

$$P(x = B, t) = 0 \;\forall t > 0 \implies$$

$$(B - \langle W\rangle t - x_0)^2 = 2\sigma_W^2 \eta t + (B - \langle W\rangle t - 2B + x_0)^2$$

hence

$$-2B\langle W\rangle t + 2x_0\langle W\rangle t = 2\sigma_W^2\eta t + 2B\langle W\rangle t - 2x_0\langle W\rangle t$$

implying

$$\eta = \frac{2(x_0 - B)\langle W\rangle}{\sigma_W^2}$$

Thus, the probability to have a $log - capital$ x at time t, constrained on having zero probability for $x = B$ $\forall t$ reads:

$$P(x,t) = \frac{1}{\sqrt{2\pi\sigma_W^2 t}}\left[e^{-\frac{(x-\langle W\rangle t - x_0)^2}{2\sigma_W^2 t}} + \right.$$
$$\left. -e^{-\frac{2\langle W\rangle(x_0-B)}{\sigma_W^2}} e^{-\frac{(x-\langle W\rangle t - 2B + x_0)^2}{2\sigma_W^2 t}}\right]$$

Now, to compute the probability to be above the barrier at any time up to t, i.e. the survival probability $S(t)$ (detailed calculations in Appendix A.2) one computes, defining $erfc(x) = 1 - erf(x)$

$$S(t) = \int_B^{+\infty} P(x,t)\, dx =$$

$$= \frac{1}{2}\text{erfc}\left(\frac{B - x_0 - \langle W\rangle t}{\sigma_W\sqrt{2t}}\right) +$$
$$-\frac{1}{2}e^{-\frac{2\langle W\rangle(x_0-B)}{\sigma_W^2}}\text{erfc}\left(\frac{-B + x_0 - \langle W\rangle t}{\sigma_W\sqrt{2t}}\right)$$

when $t \to 0$ we get:

$$S(t) \xrightarrow{t\to 0}$$

$$\frac{1}{2}\left[\text{erfc}(-\infty) - e^{-\frac{2\langle W\rangle(x_0-B)}{\sigma_W^2}}\text{erfc}(+\infty)\right] = 1$$

whereas for $t \to \infty$ we have:

$$S(t) \xrightarrow{t\to\infty} 1 - e^{-\frac{2\langle W\rangle(x_0-B)}{\sigma_W^2}}$$

hence, to be sure to be extincted at large times one needs obviously $\langle W\rangle \ll \sigma_W^2$ or, alternatively, extinction is reached for

$$\langle W \rangle \sim \frac{\sigma_W^2}{2(B - x_0)}$$

meaning that for $B < x_0$ one would need a negative growth rate (or a positive one but $B > x_0$) to be sure to reach bankruptcy at large times, otherwise the probability of extinction reaches an asymptotic value.

Once we have the survival probability up to time t, we can compute the extinction probability as the complementary probability distribution $P_{ext}(t) = 1 - S(t)$.

The plot of the survival probability with respect to time is shown in Figure 5, whereas the same quantity with respect to the boundary level is shown in Figure 6.

From these plots, and from the limits considerations, it's easy to see that, as time passes, when the drift is positive, the survival probability doesn't saturate to 0 at large times, but reaches the asymptotic value computed before. It's easier to go broke at earlier times since I start close to the threshold, whereas as time passes, the $log-capital$ increases linearly with time while we know, from CLT, that the standard deviation grows slowlier, even if it starts from an higher value. For what concerns the survival probability with respect to the boundary values, at fixed time, it's easy to see that the probability of going broke increases as B increases, reaching a probability of extinction of 1 as $B = x_0 = 10$, since the survival one reaches 0.

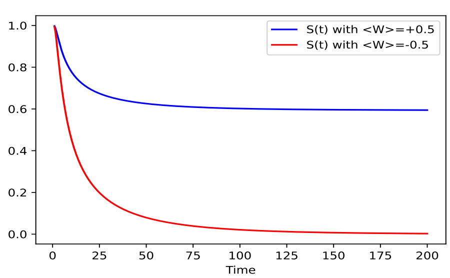

Figure 5: Survival probability with respect to time from the analytical computation with $x_0 = 10$, $\langle W \rangle = 1$, $\sigma_W^2 = 10$, $B = 1$.

To understand at what times is more likely to get extincted, one could compute the **first passage time probability distribution** as

$$FPT(t) = -\frac{dS(t)}{dt} =$$

$$= \frac{\langle W \rangle}{2\sqrt{2\pi\sigma_W^2 t}} \left(e^{-\frac{(x_0 - B - \langle W \rangle t)^2}{2\sigma_W^2 t}} e^{-\frac{2\langle W \rangle (x_0 - B)}{\sigma_W^2}} + \right.$$

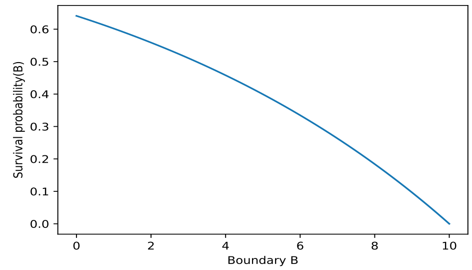

Figure 6: Survival probability with respect to the position of B from the analytical computation with $x_0 = 10$, $\langle W \rangle = 1$, $\sigma_W^2 = 10$, $t = 100$.

$$\left. e^{-\frac{(B - x_0 - \langle W \rangle t)^2}{2\sigma_W^2 t}} \right)$$

Thus, plotting the function as shown in figure 7, we see how the extinction is more likely at short times, while it becomes unlikely at large times. This is given by the fact that the drift (growth rate) is increasing with time faster (linearly $\sim t$) than the fluctuations (volatility) ($\sim \sqrt{t}$), leading us to have always less probability to go broke at later times.

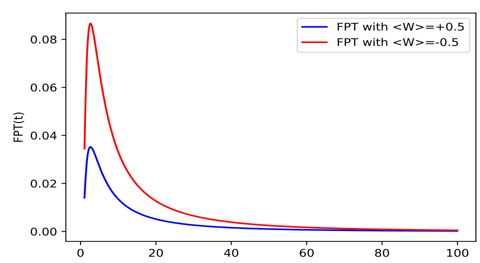

Figure 7: First passage time (FPT) probability distribution with respect to time with $x_0 = 10$, $\langle W \rangle = 1$, $\sigma_W^2 = 10$, $B = 1$.

3.3 Kelly' s implementation

The idea now is to check how the analytical prediction of the mapping to a geometric brownian motion coincides with Kelly's gambling problem. To do this, we implemented a Python code that counts an average on how many times, given the value of the boundary, we get extincted, directly simulating a $time - series$ of 100 runs, and comparing it to the analytical calculation of the geometric brownian motion mapping in which first we get the statistics from Kelly's model, then we compute it for given boundaries.

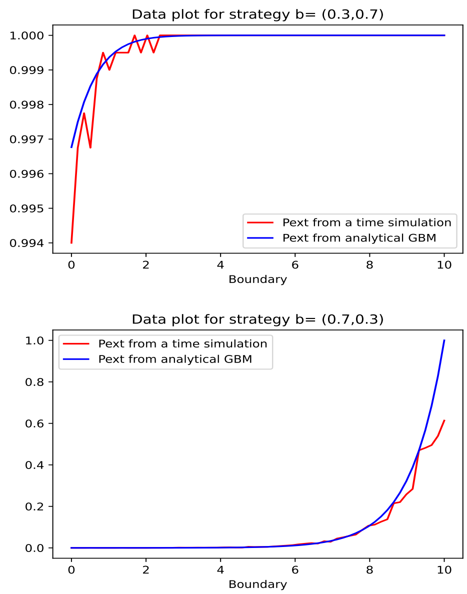

Figure 8: Analytical (GBM) and simulated (time-series) extinction probability comparison for $p = (0.7, 0.3)$, $r = (0.4, 0.6)$, with an initial capital of 10, a number of races of 100. The plots are for strategies $b = (0.3, 0.7)$, $b = (0.7, 0.3)$

Two horses case For this case, we set the probability of the horses to win to be $p = (0.7, 0.3)$ and the bookmaker odds to be $r = (0.4, 0.6)$, we set a number of runs of 100, an initial $log - capital$ $x_0 = 10$, and 4000 simulations to average on for each B we are changing. We do this for many different possible strategies . Let's clarify this again: $\langle W \rangle$ and σ_W^2 will be computed by the Kelly's relations we obtained before given r, p, b, the geometric brownian motion extinction probability will take this as input, and will be plotted for different boundaries having a fixed number of runs of 100 (from Figure 5 we've seen this time is enough for P_{ext} to settle). At the same time, we compare this to a direct simulation of a 100 races in which the player is gambling, and we measure the frequency of going broke over a 4000 different simulations for each boundary B. The results of this are shown in figure 8 for different values of $\vec{b}$.

Plot (a) in figure 8 is done for a strategy $b = (0.3, 0.7)$ which is exactly the opposite as Kelly's strategy: we are betting in an opposite way with respect to the probability of the horses to win. Firstly, the results are qualitatively matching since the GBM mapping seems to predict correctly the time series simulation, considering that the excursion in the y axis is small. Secondly, we notice that we are betting very similar to the bookmaker odds ($r = (0.4, 0.6)$) and totally differently from the real probability of winning. Thus, the probability of extinction is really high since our $log - capital$ is not going to increase much and the fluctuations can easily bring us to bankruptcy. The case of $b = r = (0.4, 0.6)$ is not shown since it corresponds to the $null - strategy$, meaning that everything I win, is going to be kept by the bookmaker, having no risk, but also no growth. So betting like that and starting from $x_0 > B$ will always lead me to a $P_{ext} = 0$, but without any capital gained.
Plot (b) is showing Kelly's strategy, so that for a fixed B, the extinction probability is lower than its opposite strategy, since we are copying the environment. From these considerations we can already see how the extinction probability is strongly related to the volatility and it may be used as definition of risk on its own. The GBM model seems to well predict the real simulation, in both cases obviously leading to a $P_{ext} = 1$ when $B = x_0 = 10$, but it's easy to see how the real discrete simulation, based on a time-series, is giving sort of "steps" following only on average the theoretical calculations.

Since $\langle W \rangle$ and σ_W are very small compared to the range of boundaries we analysed, we could actually try to simulate many runs, in smaller range for B's to see a possible actual behaviour of the extinction probability on a more specific interval, as has been done in Figure 9.

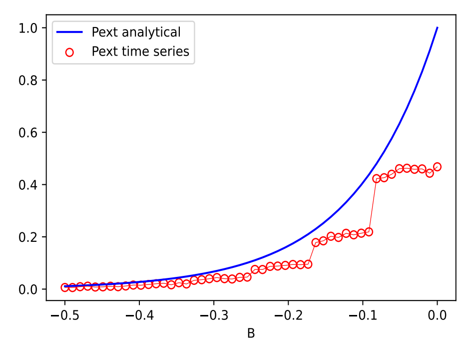

Figure 9: Analytical (GBM) and simulated (time-series) extinction probability comparison for $p = (0.4, 0.6)$, $r = (0.4347, 0.5)$ (unfair odds), with an initial capital of 0, a number of races of 100, a number of simulations for each boundary equal to 4000. The simulation is run for Kelly' s strategy.

We notice that, for the time simulation, there is the appearence of constant steps of probability, that still follow the analytical behaviour on average. This represents the main difference between the continuous GBM model we mapped into and the real discrete behaviour of Kelly's model. Indeed, we notice that, when B is low enough, no particular steps appear, since the trajectories that

lead to extinction must be more or less of the same kind: they must go down a lot, to reach the boundary.
When B starts increasing instead, trajectories are more free to be different, as long as they end up crossing the boundary, to give that frequency of extinction. As B changes continuously, the discrete trajectories aren't able to catch the small changement, whereas the GBM model does. This means that there are values of B for which a sort of "phase transition" appears: a lot of the many trajectories that were able to cross the boundary with certain configurations, will not be able to do it anymore, since a lot of the possible configurations are now cut out with respect to before.

For this reason, it' s interesting to show the behaviour of these trajectories for different values of B, as shown in Figure 10.

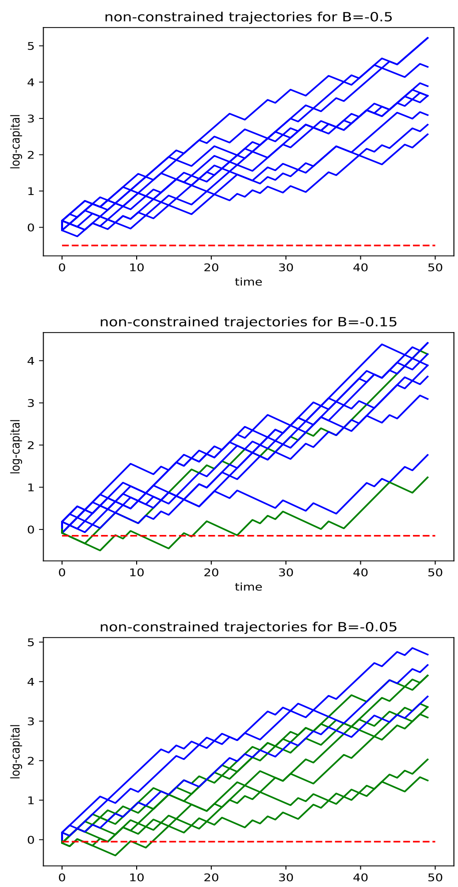

Figure 10: Surviving trajectories (blue) and extincting ones (green) for $p = (0.4, 0.6)$ and $r = (0.4347, 0.5)$, initial wealth of 0 and different boundaries.

Three horses case Considering now three horses, we analyse only the case in which $b = p$, hence when we use Kelly' s strategy, since we know that for each boundary B, this should lead to the highest value of extinction probability, being the strategy that brings to the highest volatility, i.e. fluctuations. The results are not that different from the ones with two horses. One of the strategy is plot in Figure 11.

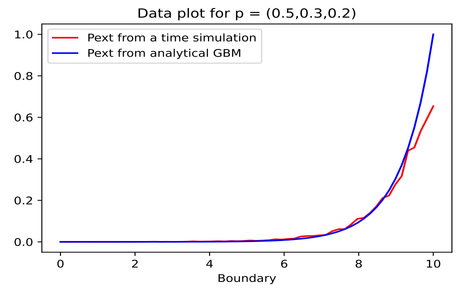

Figure 11: Analytical (GBM) and simulated (time-series) extinction probability comparison for three horses with $r = (0.4, 0.2, 0.4)$, using Kelly's strategy. Initial capital of 10, 100 races used.

3.4 On the minimization of a functional: Karush-Kuhn-Tucker conditions

For the sake of the minimization/maximization of an objective function, useful here and in the next sections, we analyse a mathematical useful tool known as **Karush-Kuhn-Tucker conditions (KKT)**. This method is useful whenever one has to optimize a function with respect to certain variables, with equalities and **inequalities** constraints, thus needing functional theory.

Formal statement of the problem Given functions f, g_1,...,g_m and h_1,...,h_l defined on some domain $\Omega \subset \mathbf{R}^n$ the optimization problem has the form

$$\min_{\mathbf{x}\in\Omega} f(\mathbf{x})$$

$$\text{subject to} \;\; g_i(\mathbf{x}) \leq 0 \; \forall i \in [1, \; m]$$

$$\text{and} \;\; h_j(\mathbf{x}) = 0 \; \forall j \in [1, \; n]$$

To perform the optimization, one should define the Lagrangian functional $\mathcal{L}$ as

$$\mathcal{L}(\mathbf{x}, \mu, \lambda) = f(\mathbf{x}) + \mu^T \mathbf{h}(\mathbf{x}) + \lambda^T \mathbf{g}(\mathbf{x}) \quad (14)$$

Let's focus on the case in which we have only inequalities as constraints, i.e.

$$\mathcal{L}(\mathbf{x}, \mu, \lambda) = f(\mathbf{x}) + \lambda^T \mathbf{g}(\mathbf{x})$$

such that

$$g_i(\mathbf{x}) \leq 0 \ \forall i \in [1,\ m]$$

Since our problem is to find a minimum (we will assume our problem to be convex, such that it'll be necessary only to study the first derivative), respecting the inequality constraints, there can happen only two cases:

- **Case 1: the constraint is not active**
  The minimum of $f(\mathbf{x})$ is already inside the feasibile area for some of the constraints (suppose for $k \in \mathbf{K} \subset [1,\ m]$), since the solution without those constraints is already fulfilling $g_k(\mathbf{x}^*) < 0$ for $k \in \mathbf{K}$. In this case, the problem reduces to minimize only $f(\mathbf{x})$ with respect to $\mathbf{x}$ i.e. you can assume $\lambda_k = 0$ for those $k \in \mathbf{K}$, providing that you check that the solution of the optimization $\mathbf{x}^*$, fulfills $g_k(\mathbf{x}^*) < 0 \ \forall k \in \mathbf{K}$.

- **Case 2: the constraint is active**
  The minimum of $f(\mathbf{x})$ is not, by itself, inside the feasible area for some of the constraints ($l \in \mathbf{L} = [1,\ m] \setminus \mathbf{K}$). This means that, to go as close as possible to the real minimum, but still respecting the constraints, these have to become equalities, since the best that the function can do to try to go to its minimum, it's localizing the optimal at the border of the feasible zone given by $g_l(\mathbf{x}) \leq 0$ for $l \in \mathbf{L}$, as close as possible to the real minimum. Then the optimization problem becomes:

  $$-\nabla_{\mathbf{x}} f(\mathbf{x}) = \lambda_l \nabla_{\mathbf{x}} g_l(\mathbf{x})$$

  for all the $l \in \mathbf{L}$ constraints that fulfill this second case. The assumption on the equality can be satisfied if and only if we check that $\lambda_l > 0 \ \forall l \in \mathbf{L}$, since the constraints must be active ($\lambda_l \neq 0$). The sign $\lambda_l > 0$ means that also the gradient of the function must point in the same direction as the perpendicular to the feasible contour, to ensure that $\mathbf{x}^*$ will be as close as possible to the real minimum of $f(\mathbf{x})$.

The method can also be adapted to find a minimum when the constraints are of the form $g_l \geq 0$, just checking that instead $\lambda_l < 0$. In the same way, we can also generalize the problem when we talk about a maximization

$$\max_{\mathbf{x}\in\Omega} f(\mathbf{x})$$

$$\text{subject to } \ g_i(\mathbf{x}) \leq 0 \ \forall i \in [1,\ m]$$

$$\text{and } \ h_j(\mathbf{x}) = 0 \ \forall j \in [1,\ n]$$

This problem is equivalent to state that we want the $\min_{\mathbf{x}\in\Omega}(-f(\mathbf{x}))$. This means that the minimization problem has now to be done for

$$\mathcal{L}(\mathbf{x}, \mu, \lambda) = -f(\mathbf{x}) + \mu^T \mathbf{h}(\mathbf{x}) + \lambda^T \mathbf{g}(\mathbf{x}) \qquad (15)$$

So when we differentiate this, the second case is still respected provided that now we check $\lambda_l < 0 \ \forall l \in \mathbf{L}$, whereas the problem

$$\max_{\mathbf{x}\in\Omega} f(\mathbf{x})$$

$$\text{subject to } \ g_i(\mathbf{x}) \geq 0 \ \forall i \in [1,\ m]$$

$$\text{and } \ h_j(\mathbf{x}) = 0 \ \forall j \in [1,\ n]$$

will need to check the conditions $\lambda_l > 0 \ \forall l \in \mathbf{L}$, for the constraints that belong to the second case.

Further analysis of KKT conditions and their applications to gambling problems are discussed in [15]. KKT maximization will be used in the following sections for the optimization of different objective functions.

3.5 Extinction probability as definition of risk

We now want to consider a different trade-off to obtain optimal strategies in gambling. The aim is to use the definition of extinction/bankruptcy probability to constrain the problem, in order to maximize the growth rate without letting the probability of getting extinct too high. Suppose we want to obtain the best optimal way to bet when the environment and the bookmaker odds are set, always in the case of diagonal odds, starting from an initial $log - capital$ of $x_0 = 1$, while asking for the probability of extinction to be $P_{ext} = P(W_{min} < B) < \beta$, where $W_{min} = \min_t[log - capital(t)]$. The problem rephrases as:

$$\max_{\vec{b}} \langle W \rangle = \max_{\vec{b}} \ \mathbf{E}(\log(\tfrac{b_x}{r_x}))$$

subject to

$$\sum_x b_x = 1$$
$$\mathbf{P}(W_{min} < B) < \beta$$

with variables $\vec{b}$, where $B,\ \beta \in (0, 1)$ are given parameters. The last constraint limits the probability of a drop in total wealth ($log - capital$) below value B to be no more than β. For example, we might take $B = 0.7$ and $\beta = 0.1$, meaning that we require the probability of a drawdown of more than 30% to be less than 10%. Unfortunately this problem is, as far as we know, a difficult optimization one in general. Luckily, as stated and proven in [6] (see Appendix A.3), we can solve a related and easier problem. Indeed, it has been shown that:

$$\mathbf{E}\left(\left(\frac{b_x}{r_x}\right)^{-\lambda}\right) \leq 1 \implies \mathbf{P}(W_{min} < B) < \beta$$

where λ is defined as

$$\lambda = \frac{\log \beta}{\log B} \in \ (+\infty, 0] \text{ as } \beta \in (0,1)$$

This means that, varying the maximum extinction probability allowed, hence varying λ, we can constrain our optimization to be more or less open to factoring in risk, while betting. The optimization problem thus becomes

$$\max_{\vec{b}} \langle W \rangle = \max_{\vec{b}} \ \mathbf{E}\left(\log\left(\tfrac{b_x}{r_x}\right)\right)$$

subject to

$$\sum_x b_x = 1$$
$$\mathbf{E}\left(\left(\tfrac{b_x}{r_x}\right)^{-\lambda}\right) \leq 1$$

This problem can now be treated, with Python maximization libraries or exploiting KKT conditions analytically to actually see what case and conditions we are satisfying, depending on the value of β or, once the extinction threshold B is fixed, of λ.

KKT analysis Exploiting the KKT maximization problem definition, since we are dealing with the problem of maximizing a function ($\langle W \rangle$) with an equality constraint and an inequality one, we can write the *Lagrangian* of the problem as

$$\mathcal{L}(\mathbf{b}, \kappa, \mu) = \sum_x p_x \log\left(\frac{b_x}{r_x}\right) +$$
$$-\kappa\left[\mathbf{E}\left(\left(\tfrac{b_x}{r_x}\right)^{-\lambda}\right) - 1\right] + \mu\left(\sum_x b_x - 1\right) \qquad (16)$$

Equation 3.5 is working in the same framework of KKT equations, where we have just two lagrange multipliers, one for the equality (μ) and one for the inequality (κ). The minus in front of the inequality constraint is put for notation, thus leading us to check $\kappa > 0$ and not $\kappa < 0$. As we can see, if we are in the case of $\kappa = 0$, then the constraint will not be active, but we' ll have to check that the strict inequality is satisfied, if it is not the case, the solution will be located on the contour of the feasible region and we will have to check then that $\kappa > 0$. Let's remember that the problem is concave (see Appendix A.4), so the study of the first derivative is enough to find the value of maximum.

$$\frac{\partial \mathcal{L}}{\partial b_x} = \frac{p_x}{b_x}\left[1 + \kappa\lambda\left(\frac{r_x}{b_x}\right)^{\lambda}\right] + \mu = 0 \qquad (17)$$

Let' s analyse the different cases for the inequality, keeping in mind that the normalization constraint instead must be always satisfied.

- **The constraint is not active:** Here we assume that the maximum of the growth rate is already in the feasible zone, without the need of the constraint. Hence, we **assume** $\kappa = 0$, and then we **check** $\mathrm{E}\left(\left(\frac{b_x^*}{r_x}\right)^{-\lambda}\right) < 1$. Equation 17 reads:

$$\frac{\partial \mathcal{L}}{\partial b_x} = \frac{p_x}{b_x} + \mu = 0$$

and with the constraint on normalization, the solution will be

$$b_x^* = p_x \quad \forall x$$

Hence, for this case, we recover Kelly' s strategy.

- **The constraint is active:** Here we assume that the maximum of the growth rate, on its own, is not in the feasible zone for the constraint, so we it needs to be on the contour, as stated before. Hence, we **assume** $\mathrm{E}\left(\left(\frac{b_x^*}{r_x}\right)^{-\lambda}\right) = 1$, and then we **check** $\kappa > 0$. The equation for this case is obviously exactly equation 17, but it's not possible to find an explicit analytical expression for b_x^*.

Numerical optimization We run a simulation to maximize the constrained problem, since analytically the computation is tough. We consider the case for three horses: fixing the value of $B = 0.78$ [6] and starting from an initial $log - capital$ $x_0 = 1$, we use $p = (0.1, 0.2, 0.7), r = (0.7, 0.1, 0.2)$. With these values, we analyse the maximization that leads to the optimal strategy $\mathbf{b}^*$ for different values of $\beta \in (0,1)$. Once the optimal strategy is obtained for a fixed β, we can compute the value of $\langle W \rangle$, σ_W, $P_{ext} = \beta$ for that particular strategy obtained by this maximization. Hence, we can obtain a Pareto front (a plot ($\langle W \rangle - \sigma_W$)), a plot ($\langle W \rangle - P_{ext}$), and one ($\sigma_W - P_{ext}$) for this new case where P_{ext} is used as definition of risk, following the problem we just optimized.

Thus, we can compare these three plots with what obtained using the usual definition of risk, the volatility, where the optimal strategies have been obtained, for different values of B from the maximization of the objective function in relation (7). In that case, keep in mind that P_{ext} has instead to be computed analytically from the GBM assumption once we obtained the value for $\langle W \rangle$ and σ_W. The comparison is shown in figure 12

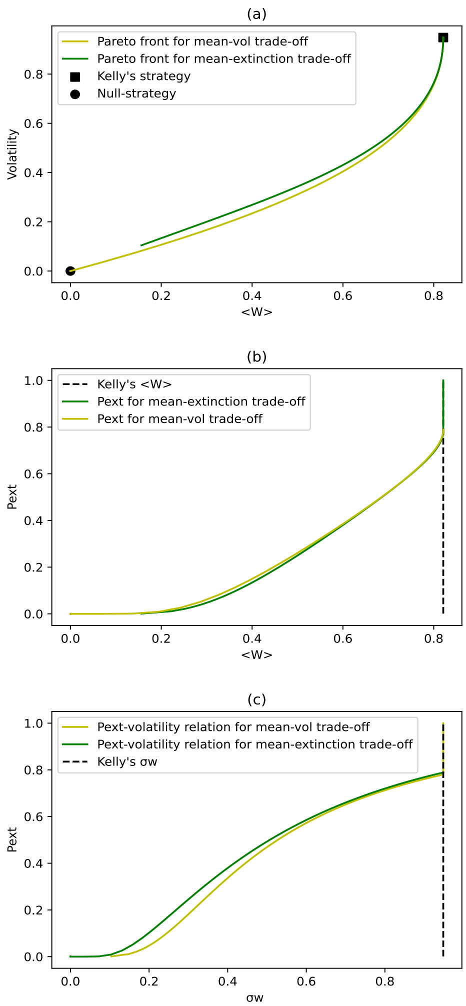

Figure 12: Plot (a): comparison for the Pareto front for the two definitions of risk: the yellow one uses volatility as risk, the green one uses the extinction probability as risk. Plot (b) and (c) shows the comparison respectively for the Pext-$\langle W \rangle$ and Pext-σ_W relation: the red one uses volatility as risk, the blue one uses the extinction probability instead. For both risk definitions, the initial $log - capital$ has been set to 1, $p = (0.1, 0.2, 0.7)$ and $r = (0.7, 0.1, 0.2)$. For the first optimization we used parameters $t = 100$, $B_1 = 0.87$, for the second $B_2 = 0.78$.

As we can see, the Pareto fronts (a) coincide almost perfectly no matter the boundary parameters chosen. On the other hand, the Extinction Probability plots (b) coincide, for the different definitions of risk, only when is set a boundary for the first optimization of $B_1 = 0.87$ while $B_2 = 0.78$ for the second. We suspect an explanation for this 0.09 discrepancy, but in any case, results show that the definition of risk as the *volatility* has almost the same Pareto front as when risk is defined as the *extinction probability*, meaning they are associated to similar notions of risk.
Moreover, it's relevant to notice the similarity of the results even though we worked in different specific frameworks for the two definitions: when volatility is used as risk, we're asking to maximize an objective function J that takes into account the trade-off through the risk-aversion parameter α, so a part from normalization, the problem has no constraints. When the extinction is used as risk, instead, the problem is asking to **always** maximize the average wealth, constraining the problem to fix a boundary for the extinction probability. In this case β becomes our risk-aversion parameter, since its value fixes the importance we're giving into taking into account risk.
For this reason, also the way P_{ext} is computed between the two optimizations is not the same at all: in the first, it comes from a continuous GBM mapping, in the second, it's just the bound itself. This similarity being discovered from these comparisons then, proved an important point to solve the optimization trade-off for a gambling problem, showing connections between different notions of risk in bet-hedging frameworks.

Important considerations It' s interesting to notice one thing: after a certain value of $\beta = \beta^*$ a phase transition appears in plot (b) of Figure 12: Kelly' s strategy is always the optimal one when the probability of extinction becomes high enough, as shown by the vertical line appearing in Figure 12. This obviously makes sense, intuitively, but thanks to the mathematical tools given by the KKT analysis, we can understand why this happens, for specific values of p and r. Indeed, to have the solution of the maximization problem to be Kelly' s strategy, we've seen that we should end up in the first case of KKT analysis. But to be in this case, we need to be sure that, in the region $\beta \in (\beta^*, 1)$, $\mathbf{E}\left(\left(\frac{b_x^*}{r_x}\right)^{-\lambda}\right) < 1$. Since Kelly' s strategy is chosen, this condition becomes

$$p_0^{1-\lambda} r_0^{\lambda} + p_1^{1-\lambda} r_1^{\lambda} + p_2^{1-\lambda} r_2^{\lambda} < 1 \qquad (18)$$

One can check that this condition is always satisfied for these values of p and r, in the range $\beta = (\beta^*, 1)$ as proven in figure 13. This explains why Kelly's strategy is always chosen as optimal for this interval of β values: extinction is allowed to happen more frequently, so that Kelly's strategy is the best option to maximize the average wealth, even though its extinction frequency is high.

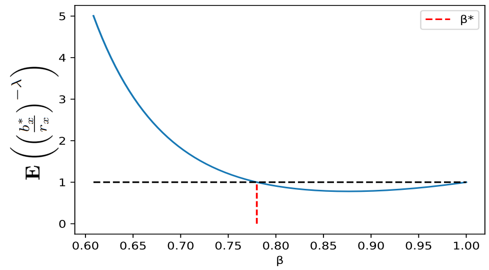

Figure 13: $\mathbf{E}\left(\left(\frac{b_x^*}{r_x}\right)^{-\lambda}\right)$ value when $\mathbf{b}^* = \mathbf{p}$, to check that for high values of β this always satisfies inequality 18

.

Condition 18 is never satisfied if we choose $r_i > p_i \ \forall i$, i.e. a particular case of unfair odds, indeed, the phase transition disappears choosing precisely this condition on p and r as shown in Figure 14. This means that the bookmaker is betting "even better" than the environment, so Kelly's strategy would never be a good choice, since the bookmaker would keep all of our gain.

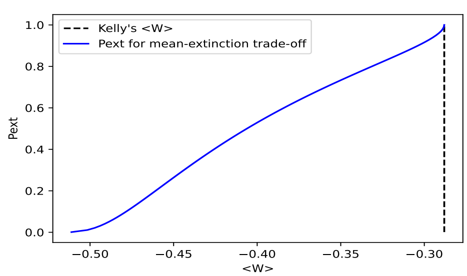

Figure 14: P$_{ext}$$-\langle W \rangle$ relation using the extinction as definition of risk with $p = (0.1, 0.2, 0.7)$ and $r = (0.7, 0.2, 0.8)$

On the other hand, condition 18 is always satisfied, hence Kelly is always optimal, for $r_i < p_i \ \forall i$, a particular case of superfair odds, as shown in Figure 15. This indeed means that the bookmaker isn't following the environment properly, not keeping enough when we "copy" it: Kelly's strategy is always optimal.

For what concerns the $null - strategy$, it's interesting to analyse it from KKT perspective. The expression of the constraint, since $\beta \to 0$ in this case ($\lambda \to +\infty$) becomes

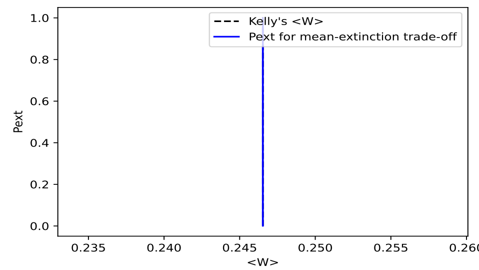

Figure 15: P$_{ext}$$-\langle W \rangle$ relation using the extinction as definition of risk with $p = (0.1, 0.2, 0.7)$ and $r = (0.1, 0.1, 0.6)$

$$p_0 \left(\frac{r_0}{b_0}\right)^{\lambda} + p_1 \left(\frac{r_1}{b_1}\right)^{\lambda} + p_2 \left(\frac{r_2}{b_2}\right)^{\lambda} < 1$$

which implies $r_i \le b_i$

- If $r_i = b_i \ \forall i$, the solution is determined by the second case of the KKT ones, hence following the $null - strategy$.
- If $r_i < b_i \ \forall i$, the solution is determined by the first case of the KKT ones, hence following Kelly. Then $b_i = p_i$ and the condition becomes $r_i < p_i \ \forall i$, which was exactly analysed more in general, and not only for $\beta \to 0$, already before (Figure 15).
- If $r_i < b_i$ for just some i's and $r_j = b_j$ for some $j \neq i$, the inequality is still satisfied, and the solution will still be following Kelly.

Moreover, as can be seen from plot (a) in Figure 12, the new framework is not able to "reach" the $null - strategy$ from its optimization, no matter the value of β: this is due to the fact that now we're **always** asking to maximize the average $log - capital$, so that the $null - strategy$ could never be optimal, in contrast with the optimization using the objective function where choosing $\alpha = 0$ made $\langle W \rangle$ to lose its importance in the maximization. Having shown again another big difference between the frameworks, the similarity between the definitions look even more astonishing. In conclusion for this section, we show some trajectories under the constraint of being limited (upper bound) in their maximum extinction probability. The plots can be seen in Figure 16

4 Game-Theoretic approach to Kelly's problem

We now analyse a different framework, using Game Theory to find the optimal strategy for a

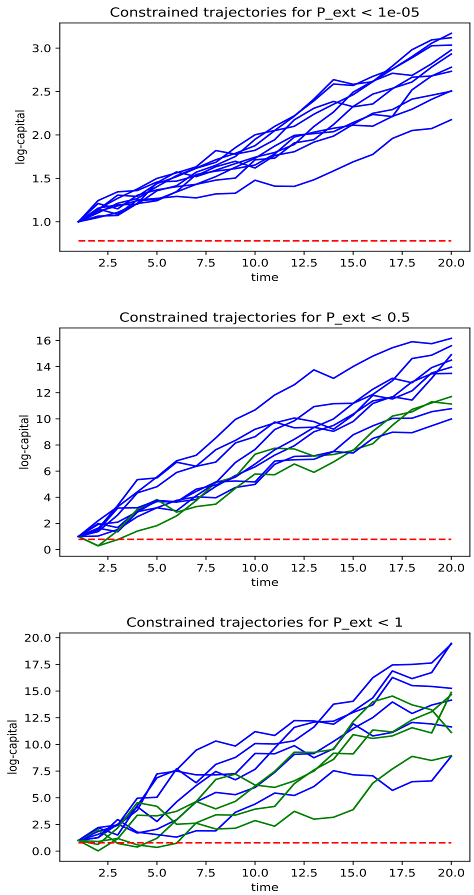

Figure 16: Constrained simulated surviving trajectories (blue) and extincting ones (green) with different values for β in an interval of $t = 20$ with $B = 0.78$, $p = (0.1, 0.2, 0.7)$ and $r = (0.7, 0.1, 0.2)$ (fair odds).

gambler in Kelly's problem instead of using any trade-off objective function. The main idea of the approach will be to consider a zero-sum game between the gambler and the environment, representing respectively player one and player two. The optimization problem will be based on the fact that player one will try to maximize its own payoff, while player two will try to minimize it.

4.1 Non-diagonal case

Up to now we based our model on a simple assumption: whenever an horse wins, the gambler gains a payoff based on what he had bet and the odds of that particular horse x. Let us now consider the matrix of the odds $\mathbf{O}$ to be non-diagonal. This means that the payoff for each run will be a linear combination of the bets the gambler put on each horse y, given that x has won that particular run. At time (run) t, the gain of the gambler will be:

$$\log\left(\sum_y o_{x(t)y} b_y\right) = \log\left(\sum_y \frac{b_y}{r_{x(t)y}}\right)$$

Using CLT as before the average growth rate is

$$\langle W(\mathbf{p}, \mathbf{b})\rangle = \sum_x p_x \log\left(\sum_y o_{xy} b_y\right)$$

4.2 Introduction to Game Theory

In game theory, when considering a zero-sum game (a game in which what is won by a player is totally lost by the other), we can consider that each player has a set of different possible strategies [2]. The convention on the gain is defined with respect to the gain of player one, meaning that if the game is **non-mixing** when we construct the **Game matrix A**, each element a_{ij} of the matrix will represent what player one will gain if he (the gambler) chooses to bet on i and player two (the environment) chooses j.
If the game is **mixing**, meaning that each player has to bet on more than one horse i, then we can just talk about an expected payoff, since each player is trying to diversificate between many horses, and not betting everything on just one of them.
If the game is **fully mixing**, it means that each player has to put a bet strictly higher than 0, on each of the horses. In our framework, considering for example the case for three horses, player one being the gambler, player two being the environment, $\mathbf{A} = \mathbf{O}^{\mathbf{T}} \in \mathbf{R}^{3x3}$ (transposing it is necessary to make the gambler to be player one in this framework), having a fully-mixing game will mean that $p_j \neq 0 \; \forall j = 0, 1, 2$ (every horse can win with a certain non-zero probability) and also $b_i \neq 0 \; \forall i = 0, 1, 2$ (the gambler has to bet something higher than 0 on all the horses). The expected payoff, mapping the game theoretical approach to our model, will be then

$$\mathbf{E}(\mathbf{b}, \mathbf{p}) = \sum_{x=0}^{2} p_x \log\left(\sum_{y=0}^{2} o_{xy} b_y\right)$$

If the environment doesn't play optimally, it will increase the gambler's payoff, whereas if the gambler doesn't play optimally he will gain less. Thus, calling b^* and p^* the optimal strategies for player one and player two, it's true that

$$\mathbf{E}(\mathbf{b}, \mathbf{p}^*) \leq \mathbf{E}(\mathbf{b}^*, \mathbf{p}^*) \leq \mathbf{E}(\mathbf{b}^*, \mathbf{p})$$

One can also define the **min-max solution** [2]

$$v^+ = \min_p \; \max_b \; \mathbf{E}(\mathbf{b}, \mathbf{p})$$

And the **max-min one**

$$v^+ = \max_b \min_p \mathbf{E}(\mathbf{b}, \mathbf{p})$$

When the game is fully-mixing for both the players, defining v the value of the game, so the expected payoff for player one, then

$$v^- = v^+ = v = \mathbf{E}(i, P) = \mathbf{E}(B, j) \; \forall i, j$$

where $\mathbf{E}(i, P)$ represents the average gain **for the gambler** conditioned on the fact that the gambler has chosen to invest everything on horse i, whereas $\mathbf{E}(B, j)$ represents the average gain **always for the gambler** but conditioned on the fact that the environment is allowing only horse j to win.

In particular, it's true that

$$p_j > 0 \; \forall j \implies \mathbf{E}(B, 0) = \mathbf{E}(B, 1) = \mathbf{E}(B, 2) = v$$

and using these two equalities and the normalization condition, remembering matrix $\mathbf{R} = \mathbf{O^{-1}}$ one finds the optimal strategy for the gambler to be ([13], [2])

$$b_x^* = \sum_y \frac{\sum_x r_{xy}}{\sum_l r_{ly}} p_y \tag{19}$$

For the same reason, it's true that

$$b_i > 0 \; \forall i \implies \mathbf{E}(0, P) = \mathbf{E}(1, P) = \mathbf{E}(2, P) = v$$

and using these two equalities and the normalization condition, the optimal strategy for the environment will be

$$p_x^* = \frac{\sum_l r_{lx}}{\sum_{xy} r_{xy}} \tag{20}$$

Considering that both players are playing optimally, the optimal solutions will be

$$b_x^* = \frac{\sum_l r_{xl}}{\sum_{xy} r_{xy}} \quad p_x^* = \frac{\sum_l r_{lx}}{\sum_{xy} r_{xy}} \tag{21}$$

Let's keep in mind that these game-theoretic results have been obtained under the assumption of both $b_i > 0 \; \forall i$ and $p_j > 0 \; \forall j$, so the game has to be **fully-mixing** [2]. To obtain relations 21 one needs the matrix of the odds $\mathbf{O}$ to be invertible and simplex preserving. Thus, these results, are considered true under the assumption of a fully-mixing game and an invertible odds matrix. It's also important to underline that this game-theoretic approach is not taking into account **any definition of risk**, since its only aim is to maximize player's one expected payoff, hence the average growth rate (average $log - capital$) we've always considered.

A three horses example Let's analyze how the game-theoretic optimization works for a simple case of three horses in which we allow only the gambler to play optimally, since in our model the environment is actually not really playing, its strategy is fixed and he's not trying to minimize the gambler's gain. We will specify both a diagonal case and a non-diagonal one, the respective odds matrices will be

$$\mathbf{O}_d = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \mathbf{O}_{nd} = \begin{pmatrix} 2 & \frac{2}{3} & 1 \\ \frac{5}{6} & \frac{5}{3} & \frac{5}{6} \\ 1 & \frac{2}{3} & 2 \end{pmatrix} \tag{22}$$

We are giving an input environment strategy of $p_{input} = (p[0], p[1], p[2]) = (0.2, 0.5, 0.3)$ even though we will analyse the optimal strategies b_0^*, b_1^*, b_2^* as a function of $p_0 \in (0, 1)$ and the other values used will be obtained by $p_1 = \frac{p[1]}{p[1]+p[2]}(1 - p_0)$ and $p_2 = \frac{p[2]}{p[1]+p[2]}(1 - p_0)$. This said, we can use equation 19, remembering $r_{xy} = (o^{-1})_{xy}$, to optimize such a problem for both the matrices considered. Before showing the results obtained by these equations that should work only in the fully-mixing framework, we could solve the problem with KKT conditions, that work in a more general one, without any assumption needed. This relies on the general problem

$$\max_{\vec{b}} \langle W \rangle = \max_{\vec{b}} \mathbf{E}\left(\log\left(\sum_y \frac{b_x}{r_{xy}}\right)\right)$$

subject to

$$\begin{gathered} \sum_x b_x = 1 \\ b_x \geq 0 \; \forall x \end{gathered}$$

With this statement, the problem is now allowing also non fully-mixing strategies, thus solving it should make us obtain a more general solution for the optimization we wanted.

4.3 KKT approach for the optimization

This problem can, again, be solved and analysed thanks to the KKT conditions. The problem rephrases in the maximization of the functional

$$\begin{gathered} \mathcal{L}(\mathbf{b}, \lambda, \mu) = \\ \mathbf{E}\left(\log\left(\sum_y \frac{b_x}{r_{xy}}\right)\right) + \sum_x \lambda_x b_x + \mu\left(\sum_x b_x - 1\right) \end{gathered}$$

Since the problem is concave for $\mathbf{b}$ (see proof in Appendix A.5), we will just need to set the first derivative to 0 to obtain the point of maximum, i.e. the optimal strategy $\mathbf{b}^*$.

$$\frac{\partial \mathcal{L}}{\partial b_x} = \sum_k p_k \frac{o_{kx}}{\sum_y o_{ky} b_y} + \lambda_x + \mu = 0$$

Let's write down all the mutually exclusive cases:

- **One of the bets $\mathbf{b_i = 0}$, the other twos are $\mathbf{b_j > 0}$:** In this case one has to solve the system of equations

  $$\begin{cases} \frac{\partial \mathcal{L}}{\partial b_i} = \sum_k p_k \frac{o_{ki}}{\sum_{y \neq i} o_{ky} b_y} + \lambda_i + \mu = 0 \\ \frac{\partial \mathcal{L}}{\partial b_j} = \sum_k p_k \frac{o_{kj}}{\sum_{y \neq i} o_{ky} b_y} + \mu = 0 \quad for\ j \neq i \\ \sum_{x \neq i} b_x = 1 \end{cases}$$

  for the variables b_j's, λ_i and μ. This case can be the solution if and only if, in the end

  $$\begin{cases} b_j > 0 \quad j \neq i \\ \lambda_i > 0 \end{cases}$$

  This case can happen for all $i \in 0, 1, 2$, so three different subcases of these have to be analysed.

- **Two bets $\mathbf{b_i = 0}$, $\mathbf{b_j = 0}$ whereas the third $\mathbf{b_l = 1}$:** In this case it's easy to show that $b_l = 1$ (because of normalization) when $l \neq i, j$ and $\mu = -1$. The system becomes

  $$\begin{cases} \sum_k p_k \frac{o_{ki}}{o_{kl}} + \lambda_i - 1 = 0 \\ \sum_k p_k \frac{o_{kj}}{o_{kl}} + \lambda_j - 1 = 0 \end{cases}$$

  for the variables λ_i and λ_j, since $b_l = 1$ and $\mu = -1$ have already been found. This case can be the solution if and only if, in the end

  $$\begin{cases} \lambda_i > 0 \\ \lambda_j > 0 \end{cases}$$

  Again, this case can happen for all $l \in 0, 1, 2$, so three different subcases.

- **None of the bets is zero, hence $\mathbf{b_x > 0}$ $\forall$x:** The equations for this case should be exactly the same as equations 19, since the conditions are the same

  $$\begin{cases} \sum_k p_k \frac{o_{kx}}{\sum_y o_{ky} b_y} + \mu = 0 \ for\ x = 0, 1, 2 \\ \sum_x b_x = 1 \end{cases}$$

  and has to be solved for b_0, b_1, b_2, μ. This is the solving case if and only if, in the end

  $$\begin{cases} b_0 > 0 \\ b_1 > 0 \\ b_2 > 0 \end{cases}$$

For each different value of $p = (p_0, p_1, p_2)$ one will have to analyse all these cases to see to which of these the solution belongs. This is obviously complicated analytically, but we can solve it numerically.

Numerical solution of the KKT approach for the three horses example Solving these equations numerically, for the same example stated before, we can compare the KKT maximization, that doesn't require any assumption on the fully-mixing condition of the game, with the game-theoretical solution obtained by solving equation 19, which instead required the game to be fully-mixing. This comparison is shown in Figure 17.

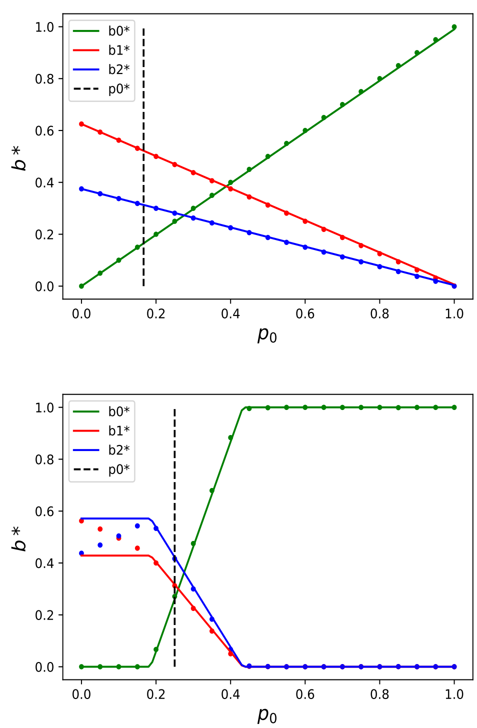

Figure 17: KKT maximization comparison with the game-theoretical solution of equation 19 for gambler's strategy for a diagonal case (plot 1) and for a non-diagonal one (plot 2). KKT maximization is shown by the full dots, solution to equation 19 is instead represented by the continuous lines.

These plots have been already obtained using simulated annealing instead of KKT conditions, but the latter has produced better results that actually match more the solutions of the game-theoretic approach, when in a fully mixing framework.

It's interesting to notice that, since we're just allowing the gambler to use an optimal strategy for different strategies of the environment, only in some cases (some values of p_0) the game can become non fully-mixing. Indeed, there are intervals within which one or more $b_i = 0$, so the game becomes non fully-mixing for the specific choice of the environment strategy.

As expected, the KKT maximization shows that the game-theoretic approach is far from reality for small p_0's, where indeed we have a case of non fully-mixing game. It's also important, at this point, to see how the intensity of the average growth rate ($log - capital$) behaves with different strategies with respect to a change in p_0, and hence, to a change in p. Results of this are shown in Figure 18.

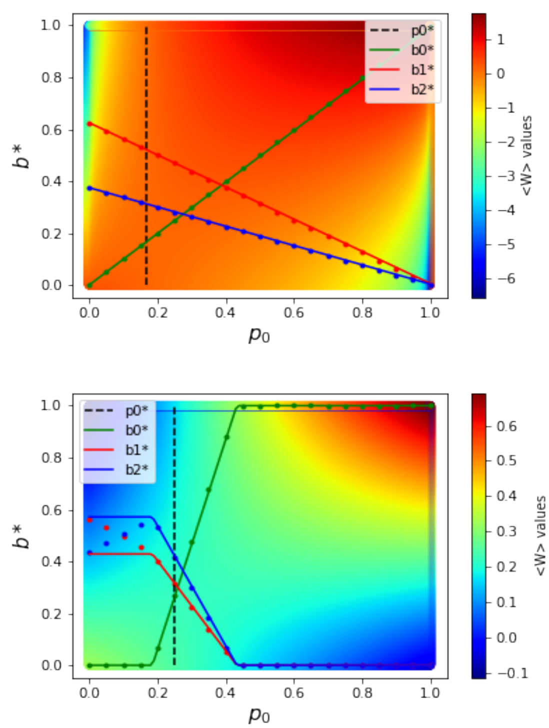

Figure 18: KKT maximization comparison with the game-theoretical solution of equation 19 for gambler's strategy for a diagonal case (plot 1) and for a non-diagonal one (plot 2). Colored plot represents the intensity of the average growth rate. KKT maximization is shown by the full dots, solution to equation 19 is instead represented by the continuous lines.

Figure 18 shows the same results as [14] for growth rate intensity, for diagonal and non-diagonal case. As intuitively guessed, one can see that whenever b_0 is following p_0, hence Kelly's strategy, the growth rate is the highest, so across the green line one has the highest of the $\langle W \rangle$ for a fixed p_0 and it becomes the highest possible when we move further from p_0^* optimal, since we are not only following Kelly' s strategy, but the environment itself is playing not optimally for him, making simpler for the gambler to copy him and gaining more.
Moreover, in the non-diagonal case, we can see the appearence of a saddle point for the average growth rate, due to the fact that moving on the green line always represents the best (Kelly's) strategy **for the gambler**, when plotted with respect to p_0. On the contrary, environment's best choice will be the one that minimizes this growth rate. The identifiable saddle point then is nothing but the intersection of the two hyperparabolas (here visible just in one dimension) describing the gambler and the environment strategies' resulting expected growth rate, one trying to maximize it, the other to minimize it.
The question now relies on trying to understand if there is a way to use the game-theoretic approach for a non fully-mixing game, in order to check that this gives the correct result, as obtained by the KKT maximization.

4.4 Non fully-mixing game: The essential part of the game

As stated before, to apply the relations obtained in equations 21, one needs to work under the equivalent assumptions of

- Fully mixing game
- $\mathbf{O^T}$ being invertible and simplex preserving

Now on, we're going to work with a different example for what concerns p, the strategy of the environment. Indeed, the assumption on the fully-mixing game stated before is defined for a game **before** one of the player has chosen a specific strategy. On the contrary, it's important to notice that **we are choosing the strategy** p, being optimal or not, varying the value of it depending on p_{input}. When our game starts, in principle, as a fully mixing one, the optimal strategies (b^*, p^*) will show no zeros. Nevertheless, since we are not optimizing p in reality, but analysing instead b^* for different values of it, it can happen that the game gets to be non fully-mixing only for some values of p_0, i.e. of p. The fixed "strategy" of the environment allows the game to become non fully-mixing even though, if both the player would have played, it would have been fully-mixing instead. That's what happened in the example before and what is proven in Figure 17. That's why, from now on, we are going to work with a $p_{input} = (0.2, 0.8, 0)$, so that the strategy chosen for the environment, will always have $p_2 = 0$, and, as a result, $b_2 = 0$, allowing the game to be non fully-mixing, for whatever value of p_0, since p_2, b_2 are fixed and not depending on it.

The essential part of the game When the game is not fully-mixing, the game-theoretical predictions are not following the correct maximization. Having defined the odds matrix $\mathbf{O^T}$ (so player one is the gambler), if one has

$$\left(\mathbf{O^T}\right)_{ik} \geq \left(\mathbf{O^T}\right)_{jk} \ \forall k$$

the gambler would always gain more betting on row i instead of row j. This means that $b_j^* = 0$ and the matrix can be reduced having its row j removed ([2]). The same happens for player two, who wants to minimize gambler's payoff, so whatever column has each of its element greater than the same elements of another one, can be removed

$$\left(\mathbf{O^T}\right)_{ki} \leq \left(\mathbf{O^T}\right)_{kj} \ \forall k$$

thus, column j can be removed and $p_j^* = 0$.

Hence, if one would want to go from a 3x3 problem being non fully-mixing to a 2x2 fully-mixing, one would need to remove a row l and a column m, then fixing

$$b_l = 0, \ p_m = 0$$

The reduction from a non fully-mixing to a fully-mixing problem allows us to isolate only what is called **the essential part of the game** [2], and it can be done in two different ways depending on the framework:

- **Both players are trying to play optimally**
  In this case the game is uniquely defined by the odds matrix, which is not invertible, thus there's no possibility to compute the optimal strategies (for both players) from equations in 21. If the odds matrix has a row and a column which can be removed, the problem can be reduced to a fully mixing one, the matrix becomes invertible and the equations 21 give the correct results. This is not our case since we are analyzing different strategies for the environment, varying p_0.
- **Just the gambler wants to play optimally**
  In this case the "game" is defined by the odds matrix **and** the strategy p we' re assuming the environment to use in this specific case. To have a reduction we need at least one $p_j = 0$ for $j \in 0, 1, 2$, even if the odds matrix is invertible. Indeed, since we are already defining one of the strategy by ourselves the definition of the matrix is not important. In this case the game theoretic formulas are computable, since $\mathbf{O}$ may be invertible, but they will not give the correct result until we perform the reduction. The reduction, as explained before, will be done removing column j and row j (if $b_j^* = 0$) and again the problem will reduced to a 2x2 fully-mixing one, and the equations 21 will give the correct results.

Reduction example As stated before, we are working in the second framework described, since we choose an input strategy for the environment to be $p_{input} = (0.2, 0.8, 0)$, so that while p_0 varies, $p_2 = 0$ always, and to have a reduction we need $b_2^* = 0 \ \forall p_0$, which appears from calculations. This means that, no matter whether $\mathbf{O}$ is invertible or not, the game will always be non fully-mixing for the input strategy chosen. Let's consider the case in which $\mathbf{O}$ is non diagonal and **not invertible**, as

$$\mathbf{O}_{ninv} = \begin{pmatrix} 2 & \frac{2}{3} & 1 \\ \frac{5}{6} & \frac{\frac{5}{3}}{\frac{3}{3}} & \frac{5}{6} \\ 1 & \frac{2}{3} & 1 \end{pmatrix} \tag{23}$$

It' s easy to see that the game is not fully-mixing **a priori from the choice of p** because we can remove both the last row and the last column, but also **because of p**, since the strategy we set led to $p_2 = 0, \ b_2^* = 0$. Indeed, no results can be computed by the game theoretic fully-mixing equations, since $\mathbf{O}$ is not invertible.

Let's consider instead the case in which $\mathbf{O}$ is non diagonal but **invertible** as

$$\mathbf{O}_{inv} = \begin{pmatrix} 2 & \frac{2}{3} & 1 \\ \frac{5}{6} & \frac{\frac{5}{3}}{\frac{3}{3}} & \frac{5}{6} \\ 1 & \frac{2}{3} & 2 \end{pmatrix} \tag{24}$$

The game is fully-mixing **a priori**, since now no rows or columns can be removed. If one would have not chosen a strategy for the environment, then both the players would have tried to play optimally, with a solution correctly predicted by equations 21.

$$b^* = (0.3, 0.4, 0.3) \qquad p^* = (0.25, 0.5, 0.25)$$

Nevertheless, **because of the choice of the strategy** $p \neq p^*$ that leads to $p_2 = 0, \ b_2 = 0$, the game becomes non fully-mixing and evidently, we are not letting the environment to play optimally. Indeed, trying to compare KKT with game-theory, one ends up with Figure 19

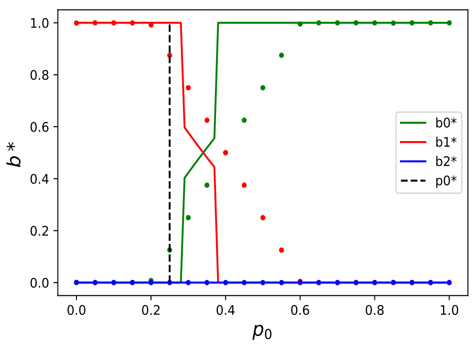

Figure 19: KKT maximization comparison with the game-theoretical solution of equation 19 for gambler's strategy with new parameters and invertible odds matrix. KKT maximization is shown by the full dots, solution to equation 19 is instead represented by the continuous lines.

Again as expected, the results can now be computed, since $\mathbf{O}$ is invertible, but they are wrong, since the the framework is not fully-mixing.

Now, what we want to prove is that, with $p_2 = 0$, $b_2 = 0$ by hypothesis, the matrix can be reduced to a 2x2 one, taking advantage only on the **essential part of the game**, removing the last row and the last column:

$$\mathbf{O}_{red} = \begin{pmatrix} 2 & \frac{2}{3} \\ \frac{5}{6} & \frac{5}{3} \end{pmatrix} \quad (25)$$

The game is now fully-mixing, since $\mathbf{O}$ is reduced and what is associated to 0's in the environment and gambler's strategy has been removed from it. Applying the game-theoretical approach to this reduced matrix, so solving equations 21 on a problem which is now fulfilling the assumptions, one gets the results shown in Figure 20.

As expected, the reduction led to a new 2x2 fully-mixing game, in which we are neglecting the horse which is never going to be chosen, and we focus on applying the game-theoretic approach on the essential part of the game. Since the essential part now fulfills the assumptions under which relations 21 have been derived, the theory works again, showing that it coincides with the actual KKT Kelly's maximization solution.

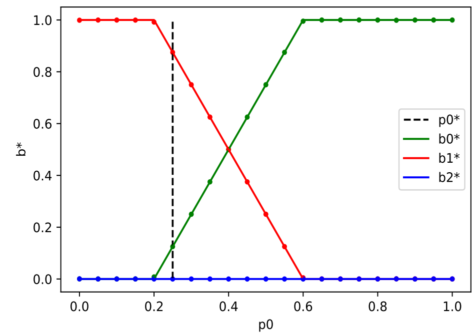

Figure 20: KKT maximization comparison with the game-theoretical optimization for gambler's strategy with new parameters and reduced odds matrix: solution for the essential part of the game. KKT maximization is shown by the full dots, solution to equation 19 is instead represented by the continuous lines.

5 Conclusions

Throughout the article, we analysed a simple gambling problem inspired by Kelly's horse races model. We discussed the characteristics of the model for the diagonal and non-diagonal case. For the first case, we tried to extend the work of Kelly ([7] and [9]) introducing at first the volatility as a measure of risk, and then implementing a geometric brownian motion approach [11] to model the behavior of the $log - capital$ and to define an **extinction probability**, later used as new definition of risk.
Since the GBM continuous mapping showed to be in agreement with the races simulation, we compared the results of the two definitions and obtained that the Pareto fronts and the relations between extinction probability and the statistical quantities are very similar. There seems to be a discrepancy between the definition of the *boundary B* for what concerns the analytical expression of the extinction probability from GBM and the risk-constrained extinction threshold defined in ([6]). We think that this discrepancy may be caused by the mapping of the problem from continuous to discrete since it's just quantitative. Nevertheless, we suspect a better explanation not yet found for this behaviour.
In any case, we proved the **usefulness of the geometric brownian motion** approach to map the behaviour of the $log - capital$/wealth during time, as shown by Figures 7, 8, 10, whereas we' ve shown the **similarities and discrepancies between the two definitions of risk** in Figure 12 for the main quantities of interest. A new def-

inition of risk appliable in bet-hedging scenarios related to econophysics, biophysics, neuroscience, social sciences, would be useful to construct new and more specific optimization problems in which the fluctuations of the quantity to optimize (in our case $\langle W \rangle$) are not the main source of real risk (i.e. in some frameworks fluctuations can be useful to grow faster, whereas the extinction/bankruptcy probability is always a quantity one wants to minimize). The generality of the approach we studied just for a simple gambling problem, can be easily specified and mapped into different bet-hedging realizations, as shown by the first considerations on phenotypic switching [8]. Indeed, the problem settings result to be the same, but with a different meaning, as shown by the similar "Pareto fronts" in Figure 1.

Subsequently, with the tools of **KKT maximization** [15] we analysed the analytical behaviour of the extinction probability with respect to the wealth and with the same tools we introduced the cases for a Kelly's non-diagonal model with a game-theoretic approach. The main results of the last section, indeed, concern **the possible reduction of the game to an essential part of it** [2], which keeps only the useful information about the bookmaker odds and the environment fluctuations.

The reduction ends up to be possible and important to apply the game-theoretic approach to games which are non fully-mixing, extending the possibility to treat optimization problems in which the gambler is allowed to play not necessarily fully-mixing strategies. We showed how the reduction to a fully-mixing game, indeed, is possible under certain conditions. This would mean that the gambler (or the population in a phenotypic context) could wait, without investing, for the first few runs as shown in [6], using inference to study the behaviour of the environment [1], to discover that a certain horse (environmental condition) i will never win ($p_i = 0$). For a non-diagonal case then, the optimization problem could be solved removing the trivial and useless information about horse i directly from the matrix of the game, that is specifying for the stochasticity of the environment.

We suspect there could be a way to reduce the problem even to a 2x3 matrix when an horse for strategy p has a zero probability of winning while the gambler strategy doesn't have any, thus reducing the problem to a fully-mixing rectangular one. In any case, our results and considerations allow the game-theoretic fully-mixing approach to find the same correct solutions as Kelly's maximization problem, meaning it could be used in new and different frameworks, to reduce game-theoretic or optimization problems, from biology to chemistry or finance [8], [16], [6].

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A Appendix

A.1 On the multiplicative Kelly's process

Here we show the detailed calculation for the average $log - capital$ $\langle W \rangle$ in the case $N \gg 1$. Introducing Kelly's process one has:

$$\mathrm{C}_N = o_{x_N} b_{x_N} C_{N-1} = \prod_{n=1}^{N} o_{x_n} b_{x_n} C_0$$

$$\Longleftrightarrow$$

$$\log C_N = \log \left(\prod_{n=1}^{N} o_{x_n} b_{x_n} C_0 \right) = \sum_{n=1}^{N} \log \left(o_{x_n} b_{x_n} \right) + \log C_0$$

$$\Longleftrightarrow$$

$$\langle W \rangle \doteq \lim_{N\to\infty} \frac{1}{N} \log C_N = \lim_{N\to\infty} \left(\frac{1}{N} \sum_{n=1}^{N} \log \left(o_{x_n} b_{x_n} \right) + \cancelto{0}{\frac{1}{N} \log C_0} \right)$$

$$\xrightarrow{CLT} \sum_x p_x \log \left(o_x b_x \right)$$

□

A.2 Survival probability calculation

In this subsection we show the detailed calculation of $S(t)$, once the constrained $P(x,t)$ has been obtained through image method:

$$S(t) = \int_B^{+\infty} P(x,t)\; dx =$$

$$= \frac{1}{\sqrt{2\pi\sigma_W^2 t}} \left[\int_B^{+\infty} e^{-\frac{(x-\langle W \rangle t - x_0)^2}{2\sigma_W^2 t}} - e^{-\frac{2\langle W \rangle (x_0 - B)}{\sigma_W^2}} \int_B^{+\infty} e^{-\frac{(x-\langle W \rangle t - 2B + x_0)^2}{2\sigma_W^2 t}} \right] =$$

$$= \frac{1}{2} \left[\mathrm{erf}\left(+\infty\right) - \mathrm{erf}\left(\frac{B - x_0 - \mu t}{\sqrt{2\sigma_W^2 t}} \right) \right] - \frac{1}{2} e^{-\frac{2\langle W \rangle (x_0 - B)}{\sigma_W^2}} \left[\mathrm{erf}\left(+\infty\right) - \mathrm{erf}\left(\frac{-B + x_0 - \mu t}{\sqrt{2\sigma_W^2 t}} \right) \right] =$$

$$= \frac{1}{2} \left[1 - \mathrm{erf}\left(\frac{B - x_0 - \mu t}{\sqrt{2\sigma_W^2 t}} \right) \right] - \frac{1}{2} e^{-\frac{2\langle W \rangle (x_0 - B)}{\sigma_W^2}} \left[1 + \mathrm{erf}\left(\frac{B - x_0 + \mu t}{\sqrt{2\sigma_W^2 t}} \right) \right] =$$

$$= \frac{1}{2} \mathrm{erfc}\left(\frac{B - x_0 - \mu t}{\sqrt{2\sigma_W^2 t}} \right) - \frac{1}{2} e^{-\frac{2\langle W \rangle (x_0 - B)}{\sigma_W^2}} \mathrm{erfc}\left(\frac{x_0 - B - \mu t}{\sqrt{2\sigma_W^2 t}} \right)$$

□

A.3 Proof of $\mathbf{E}\left(\left(\frac{b_x}{r_x}\right)^{-\lambda}\right) \leq 1 \implies \mathbf{P}(W_{min} < B) < \beta$

Remembering λ being defined as $\lambda = \frac{\log \beta}{\log B} \in (+\infty, 0]$ as $\beta \in (0,1)$ and following the proof by [6], one begins introducing the Lemma 1:

Lemma 1. *Consider an I.I.D. sequence $X_1, X_2, ..., X_n$ from a probability measure $\mathbf{p}$ such that its random walk is $S_n = X_1 + X_2 + ... + X_n$, τ is its stopping time and its cumulative generating function*

$$\Psi(\lambda) = \log \mathbf{E}\left(e^{-\lambda X}\right) = \log \int e^{-\lambda X} \; d\mathbf{p}(X)$$

Then, it's true that:

$$\mathbf{E}\left[e^{-\lambda S_\tau - \tau \Psi} | \tau < \infty\right] \boldsymbol{P}(\tau < \infty) \leq 1$$

The proof of this can be found in [6].

Defining the stopping time τ as

$$\tau = \inf\{t \geq 1 | w_t < B\}$$

Note that $\tau < \infty \iff W_{min} < B$. From Lemma 1 one gets

$$1 \geq \mathbf{E}\left[e^{-\lambda \log w_\tau - \tau \log \mathbf{E}\left(\left(\frac{b_x}{r_x}\right)^{-\lambda}\right)} | \tau < \infty\right] \mathbf{P}\left(W_{min} < B\right)$$

Since $-\tau \log \mathbf{E}\left(\left(\frac{b_x}{r_x}\right)^{-\lambda}\right) \geq 0$ when $\tau < \infty$, one has

$$1 \geq \mathbf{E}\left[e^{-\lambda \log w_\tau | \tau < \infty}\right] \mathbf{P}\left(W_{min} < B\right)$$

But since $w_\tau < B$ when $\tau < \infty$ one gets

$$1 < e^{-\lambda \log B} \mathbf{P}\left(W_{min} < B\right)$$

Ending up with

$$\mathbf{P}\left(W_{min} < B\right) < B^\lambda \doteq \beta$$

□

A.4 Convexity of the diagonal optimization problem

In this section of the appendix, we want to show that the maximization problem in its diagonal case can be analysed through the study of just its first derivative. Hence, we want to show the problem, without proper constraints, to be convex down i.e. concave everywhere. To do this, we remind the function we need to maximize with respect to the vector $\mathbf{b}$:

$$\langle W \rangle = \sum_x p_x \log\left(\frac{b_x}{r_x}\right)$$

with derivatives

$$\frac{\partial \langle W \rangle}{\partial b_x} = \frac{p_x}{b_x}$$

$$\frac{\partial^2 \langle W \rangle}{\partial b_x^2} = -\frac{p_x}{b_x^2} \qquad \frac{\partial^2 \langle W \rangle}{\partial b_x \partial b_y} = 0$$

Thus, the Hessian will be

$$\begin{bmatrix} -\frac{p_1}{b_1^2} & 0 & \cdots & 0 \\ 0 & -\frac{p_2}{b_2^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\frac{p_2}{b_2^2} \end{bmatrix}$$

This matrix is diagonal and its eigenvalues are all evidently non-positive for $b_x \in (0, 1]$. This means that the matrix is negative semi-definite, and the function is concave for all $b_x \in (0, 1]$ for $x = 1, ..., n$. □

A.5 Convexity of the non-diagonal optimization problem

As just done in section A.4, we need to show the non-diagonal average growth rate to be concave everywhere. The function to maximize would be:

$$\langle W \rangle = \sum_i p_i \log \left(\sum_j o_{ij} b_j \right)$$

with derivatives

$$\frac{\partial \langle W \rangle}{\partial b_x} = \sum_i p_i \frac{o_{ix}}{\sum_j o_{ij} b_j}$$

$$\frac{\partial^2 \langle W \rangle}{\partial b_x^2} = -\sum_i p_i \left(\frac{o_{ix}}{\sum_j o_{ij} b_j} \right)^2 \qquad \frac{\partial^2 \langle W \rangle}{\partial b_x \partial b_y} = -\sum_i p_i \frac{o_{ix} o_{iy}}{\left(\sum_j o_{ij} b_j \right)^2}$$

Hence, the Hessian will be:

$$\begin{bmatrix} -\sum_i p_i \left(\frac{o_{i1}}{\sum_j o_{ij} b_j} \right)^2 & -\sum_i p_i \frac{o_{i1} o_{i2}}{\left(\sum_j o_{ij} b_j \right)^2} & \cdots & -\sum_i p_i \frac{o_{i1} o_{in}}{\left(\sum_j o_{ij} b_j \right)^2} \\ -\sum_i p_i \frac{o_{i2} o_{i1}}{\left(\sum_j o_{ij} b_j \right)^2} & -\sum_i p_i \left(\frac{o_{i2}}{\sum_j o_{ij} b_j} \right)^2 & \cdots & -\sum_i p_i \frac{o_{2x} o_{in}}{\left(\sum_j o_{ij} b_j \right)^2} \\ \vdots & \vdots & \ddots & \vdots \\ -\sum_i p_i \frac{o_{in} o_{i1}}{\left(\sum_j o_{ij} b_j \right)^2} & -\sum_i p_i \frac{o_{in} o_{i2}}{\left(\sum_j o_{ij} b_j \right)^2} & \cdots & -\sum_i p_i \left(\frac{o_{in}}{\sum_j o_{ij} b_j} \right)^2 \end{bmatrix}$$

This matrix has all negative elements and it's symmetric. However, this is not enough to conclude that all its eigenvalues are non-positive, i.e. we have no sufficient information to conclude that this Hessian is negative semi-definite and, furthermore, the analytical treatment to study its eigenvalues wouldn't be trivial at all. We then proceed in a different way, following the proof in [14].
Proving the convexity of $\langle W \rangle = \sum_i p_i \log \left((\mathbf{O}\mathbf{b})_i \right) = \sum_i p_i \log \left(\sum_j o_{ij} b_j \right)$ means showing that for all $\mathbf{b_1}$ and $\mathbf{b_2}$ and for all $t \in [0, 1]$

$$\langle W \left(t\mathbf{b_1} + (1-t)\mathbf{b_2} \right) \rangle \geq t \langle W \left(\mathbf{b_1} \right) \rangle + (1-t) \langle W \left(\mathbf{b_2} \right) \rangle$$

Knowing the logarithm to be concave for any value of its variable, one has:

$$\log\left(t\left(\mathbf{Ob_1}\right)_i+(1-t)\left(\left(\mathbf{Ob_2}\right)_i\right)\geq t\log\left(\left(\mathbf{Ob_1}\right)_i\right)+(1-t)\log\left(\left(\mathbf{Ob_2}\right)_i\right)$$

Since p_i's are probabilities, i.e. positive quantities, we can multiply each inequality i by the respective p_i, not modifying the concavity, and then sum all the factors in order to obtain the required result: our function is indeed concave and we just need to study its first derivative to find the maximum point $b*$ we need.

□