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Continuous Opinions and Discrete Actions in social dynamics: analysis of a quantized model

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Summary

Opinion dynamics study the evolution of the ideas of interacting individuals, whether they are people, organizations, or entire communities. Social networks are modeled as graphs, in which every node represents an individual, paired with a real number that models its opinion, and every edge represents an interaction or a social bond. In recent years, interest in the topic has been increasing, and different models have been proposed. The present work focuses on the analysis of a quantized model, which is founded on the premise that people don't have access to the exact opinion of the individuals they are interacting with. Instead, every person can infer an approximation of their interlocutors' opinion based on the behaviour they display.

The model in question is discontinuous in state and piecewise linear. This discontinuity complicates the study, and requires to redefine the concept of solution of a differential equation. To study the model, we focus on Carathéodory solutions. A phenomenon caused by the discontinuity is the presence of extended equilibria ([1]), points that attract some solutions, despite not being proper equilibrium points as we would expect in the continuous case. The first original result presented concerns the basin of attraction of such points. Then, the study focuses on specific graphs, providing two sets of original results. The first set provides analytical expressions of extended equilibria on the line and ring graphs. The second set of results is presented under the assumption of binary action, which means restricting the opinion of the nodes to the interval [0,1], and thus having only two possible behaviours, 0 and 1. In this section, we provide convergence results for the line and ring graphs, an algorithm to find extended equilibria on tree graphs, and show the presence of a Zeno point in the directed cycle graph with 3 nodes.

Acknowledgements

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Contents

Chapter 1

Introduction

When discussing, two people will share their ideas, and influence each other. By learning about each other's point of view, their opinions will grow closer, and they might end up agreeing. If their initial opinions are too different, it might take a lot of time and multiple interactions. Obviously, society requires frequent interactions between many different people, and predicting the development of people's ideas is far from an easy task.

The field that focuses on modeling such social connections is that of opinion dynamics, that has received growing interest in the past decades. Graph theory allows to study networks of interacting individuals, representing every agent as a node, and every interaction (an encounter, discussion, social bond, relationship) as an edge. Every node is associated with an opinion, usually modeled as a real number. The main distinction between different models lies in the equation describing the evolution of these opinions. For instance, time can be modeled as either discrete or continuous, and in the latter case, under which the model studied in this thesis falls, we have a differential equation.

As for most complex problems, models of opinion dynamics have to balance accuracy of the results and complexity of the analysis. The first models proposed (summarized in [9], [10], along with more advanced models), the French-De Groot and the Abelson models, simply have every node move their opinion towards the average of its neighbours', which under very basic assumptions eventually leads to consensus between all the nodes. Although consensus can sometimes be achieved, it is unreasonable to assume that every group of people interacting will eventually agree. More recent works refine these models by integrating realistic assumptions into the update law.

One family of such models is referred to as bounded confidence models (e.g. [3], [4]). Here, the assumption is that, when interacting, people will ignore the other person's opinion if it is too far from their own, as it is considered to be unbelievable or simply wrong. This translates in a graph that varies in time, as any pair of nodes initially connected might stop interacting if their opinions diverge too much.

Another family of models, that contains the model subject of this thesis, goes under the name of continuous opinion and discrete action models (e.g. [1], [5], [6]). The assumption, here, is that people in most cases do not have access to their interlocutor's precise opinion. Instead, they can get an approximate idea based on the behaviour displayed, i.e. the actions taken. Different hypothesis can be added, which lead to different models, for instance [5] assumes discrete time and binary action, i.e. restricts the opinions to the interval [0,1], and adds a weight to model extremists, individuals with opinions close to 0 or 1, who are more unlikely to easily change their mind.

The model studied in this work is referred to as quantized model, in which every node's opinion is defined as a real value $x_i \in \mathbb{R}$, but is seen by every neighbour as rounded to the closest integer. This assumption causes the model to be discontinuous in state, as long the nodes' behaviour is constant the dynamic is linear, but when a node's opinion x_i crosses the middle point between two integers, its quantized value changes, defining a new linear dynamic for its neighbours. The discontinuity requires that we redefine the concept of solution of a differential equation, in this work we focus on Carathéodory solutions, which are the translation of classical solutions to the discontinuous case.

The main phenomenon that we are analyzing is the presence of extended equilibria, points that are attractors, i.e. some solutions asymptotically converge to them, but aren't proper equilibrium points as we would expect in a continuous model, which means they generate non-constant solutions if picked as initial condition. The first part of the study focuses on finding analytical expression for the extended equilibria on particular graphs, the line and ring graph. On the second part, we assume binary action, which simplifies the model while still maintaining many potential applications, as it fits the situations in which there are only two options, like answer yes or no to a question, or do or don't do scenarios. The main results provided are a partial description of the basin of attraction of extended equilibria in the most general case, the analytical expression of extended equilibria for the line and the ring graph, and when binary action is assumed, convergence for the line graph, an algorithm to find extended equilibria on tree graphs, and finally the presence of a Zeno point in a directed cycle graph. The study conducted is mainly analytical, with some support provided by simulations.

The thesis is structured in 4 chapters. Firstly, we go through a brief rundown of the mathematical tools required, concerning graph theory and ordinary differential equations. After that, a quick overview of the models found in the literature takes place. The third chapter is where most of the work done is concentrated, and the results found are shown, proven and explained. Finally, we wrap up the paper with some conclusions, interpretations and open problems.

1.1 Preliminaries

The first section of the work quickly introduces the mathematical elements needed, mainly regarding graph theory and ordinary differential equation.

Definition 1 (Graph)**.** A graph is a pair $G = (V, E)$, where $V = (v_1, v_2, \dots, v_n)$ is the set of nodes, and E is the set of edges, which are ordered pairs of nodes $(i, j) \subset V \times V$, representing the connections between two nodes of the graph.

For every graph, we can define the adjacency matrix, whose generic entry a_{ij} is determined by whether there is an edge between node v_i and node v_j or not.

Definition 2 (Adjacency matrix)**.** Given a graph $G = (V, E)$, its adjacency matrix $A =$

$(a_{ij})_{i,j\in V}$ is defined as

$$a_{ij} = \begin{cases} 1 & if\ (i,j) \in E, \\ 0 & if\ (i,j) \notin E, \end{cases}$$

If all the edges are bidirectional (v_i is connected to $v_j \iff v_j$ is connected to v_i), the graph is called undirected, otherwise it is called directed.

Definition 3 (Undirected graph)**.** Given a graph $G = (V, E)$, G is undirected if and only if its adjacency matrix A is symmetric.

For every node, we can define the set of nodes it is linked to in the following way.

Definition 4 (Neighbourhood)**.** Given a graph $G = (V, E)$, the neighbourhood of a node v_i is defined as

$$N_i = \{v_j | (i,j) \in E\}.$$

The cardinality of N_i, i.e. the number of nodes v_i is connected to, is called out-degree of v_i. Vice versa, the number of nodes connected to v_i is called in-degree.

Definition 5 (Out-degree)**.** Given a graph $G = (V, E)$ with adjacency matrix A, the out-degree of v_i is defined as

$$d_i = \sum_{j=1}^{n} a_{ij}.$$

Definition 6 (In-degree)**.** Given a graph $G = (V, E)$ with adjacency matrix A, the in-degree of v_i is defined as

$$d_i = \sum_{i=1}^{n} a_{ij}.$$

Definition 7 (Degree matrix)**.** Given a graph $G = (V, E)$ with adjacency matrix A, we can define its degree matrix D as

$$D_{ij} = \begin{cases} d_i & if\ i = j, \\ 0 & if\ i \neq j. \end{cases}$$

Definition 8 (Laplacian matrix)**.** Given a graph $G = (V, E)$ with adjacency matrix A, we can define its Laplacian matrix as

$$L = D - A,$$

where D is the degree matrix of the graph. Components-wise, we can write the Laplacian as

$$L_{ij} = \begin{cases} -a_{ij} & if\ i \neq j \\ \sum_{j\neq i} a_{ij} & if\ i = j \end{cases}$$

All the following definitions refers to a given graph $G = (V, E)$ with adjacency matrix A.

Definition 9 (Walk)**.** A walk of length k is a series of nodes $v_1, v_2, \ldots, v_k$ such that (v_i, v_{i+1}) is an edge for all i.

Definition 10 (Cycle)**.** A cycle is a walk such that $v_1 = v_k$, i.e. a walk that returns to the starting point.

Definition 11 (Path)**.** A path is walk such that $v_i \neq v_j$ for $i \neq j$, i.e. a walk with no self-intersections.

Definition 12 (Period of a graph)**.** The graph G is called periodic it has at least one cycle and the length of every cycle can be dived by an integer $p>1$. The maximum such p is called the period of the graph.

Definition 13 (Strongly connected)**.** The graph G is called strongly connected if for all v_i, v_j there exists a walk starting in v_i and ending in v_j.

Definition 14 (Complete graph)**.** G is a complete graph, if for all v_i and v_j, with $i \neq j$, $(v_i, v_j) \in E$.

Complete graphs are usually denominated K_n, where n is the number of nodes.

Definition 15 (Complete bipartite graph)**.** G is a complete bipartite graph, if the nodes can be divided in two subsets $V_1 \subset V$ and $V_2 = V \setminus V_1$, such that (v_i, v_j) is an undirected edge if and only if $v_i \in V_1$ and $v_j \in V_2$.

Complete bipartite graphs are usually denominated K_{n_1,n_2}, where n_i is the number of nodes in V_i.

Definition 16 (Undirected line graph)**.** G is an undirected line graph if its nodes can be numbered in a way such that (v_i, v_{i+1}) is an undirected edge for $i = 1, \ldots, n-1$.

Definition 17 (Directed cycle graph)**.** G is a directed cycle graph if its nodes can be numbered in a way such that (v_i, v_{i+1}) is a directed edge for $i = 1, \ldots, n-1$, and so is (v_n, v_1).

Definition 18 (Ring graph)**.** G is a ring graph if its nodes can be numbered in a way such that (v_i, v_{i+1}) is an undirected edge for $i = 1, \ldots, n-1$, and so is (v_n, v_1).

Definition 19 (Subgraph)**.** $G' = (V', E')$ is a subgraph of G if $V' \subset V$ and $E' \subset E$.

Definition 20 (Tree)**.** G is a tree if it is undirected, connected and has no cycles.

Definition 21 (Spanning tree)**.** A spanning tree of G is a subgraph of G such that $V' = V$ and that is a tree.

1.2 Ordinary Differential Equations

Differential equations are used to describe a huge variety of problems in different fields. For this work, we only need Ordinary Differential Equations, for which we provide some useful definitions and properties in the following pages. Throughout this section, we consider the differential equation

$$\dot{x}(t) = f(x(t)), \tag{1.1}$$

where $t \in I \subset \mathbb{R}$, $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R}^n$.

Definition 22 (Cauchy problem)**.** A Cauchy problem is a pair given by an ordinary differential equation and an initial condition

$$\begin{cases} \dot{x}(t) = f(x(t)), \\ x(t_0) = x_0, \end{cases} \tag{1.2}$$

where $t_0 \in I$.

Definition 23 (Solution)**.** A (classical) solution of the Cauchy problem (1.2) is a function

$$\varphi(t, x_0) : I \times \mathbb{R}^n \to \mathbb{R}^n$$

differentiable in t such that

$$\dot{\varphi}(t, x_0) = f(\varphi(t, x_0)) \quad \forall t \in I,$$

and $\varphi(t_0, x_0) = x_0$.

Definition 24 (Orbit)**.** Given a solution $\varphi(t, x_0)$, its orbit is the set

$$\gamma(x_0) = \{\varphi(t, x_0) | t \in I\}.$$

Definition 25 (Equilibrium point)**.** Given a differential equation, an equilibrium point is a point $x^* \in \mathbb{R}^n$ such that

$$\varphi(t, x^*) = x^* \quad \forall t \in I.$$

We remark that, if f is of class C^1, a point x^* is an equilibrium point if and only if

$$f(x^*) = 0.$$

Definition 26 (Periodic point)**.** x is a periodic point if it is not an equilibrium point and for some $T > 0$, $\varphi(T, x) = x$.

Definition 27 (Cycle)**.** A cycle is a closed curve that is the orbit of a solution starting in a periodic point.

Definition 28 (Lyapunov stability)**.** Given a compact set $M \in \mathbb{R}^n$, M is (Lyapunov) stable if for all ϵ, exists δ such that

$$dist(x, M) \leq \delta \Longrightarrow dist(\varphi(t, x), M) \leq \epsilon \quad \forall t > 0.$$

Remark 1. The previous definition is particularly important when M is made of a single equilibrium point x^*, in which case we say that x^* is stable. Analogously, in the following definitions keep in mind that M will usually be considered to be an equilibrium point.

Given a point $x \in \mathbb{R}^n$ and a set $M \subset \mathbb{R}^n$, we say that

Definition 29. x is attracted by M if

$$\lim_{t \to +\infty} dist(\varphi(t, x), M) = 0.$$

Definition 30 (Basin of attraction)**.**

$$\mathcal{A}(M) = \{x \in \mathbb{R}^n | x \text{ is attracted by } M\}$$

is the basin of attraction of M.

Definition 31 (Attractor)**.** M is an attractor if it has non-null basin of attraction.

Definition 32 (Limit cycle)**.** A limit cycle is a cycle that is an attractor.

Definition 33 (Asymptotic stability)**.** If M is stable and is an attractor, it is said to be locally asymptotically stable. Furthermore, if $\mathcal{A}(M) = \mathbb{R}^n$, it is said to be globally asymptotically stable.

For now we've considered generic differential equation, but in this thesis we will work with linear systems, for which we now give some results.

Definition 34 (Linear system)**.** A linear system is a differential equation of the type

$$\dot{x}(t) = Ax(t) + b,$$

where $x \in \mathbb{R}^n, b \in \mathbb{R}^n$ and A is a $n \times n$ matrix.

Theorem 1. *Given a linear system, the set of equilibria* $X = \{x^* \in \mathbb{R}^n | Ax^* + b = 0\}$ *is a translation of a subspace of* $\mathbb{R}^n$*.*

Theorem 2. *Given a linear system* $\dot{x}(t) = Ax(t) + b$*, and an equilibrium point* x^**, let* $\lambda_1, \dots, \lambda_n$ *be the eigenvalues of the matrix* A*. Then*

- x^* *is asymptotically stable if and only if* $Re(\lambda_i) < 0 \; \forall i$*,*
- x^* *is stable if and only if* $Re(\lambda_i) \leq 0 \; \forall i$*, and the eigenvalues such that* $Re(\lambda_i) = 0$ *are regular, i.e. they have algebraic multiplicity equal to their geometric multiplicity.*

Theorem 3. *In linear systems, if the equilibrium point* x^* *is locally asymptotically stable, then it is also globally asymptotically stable.*

Let us explicitly solve the following differential equation, as it will be significant later in the thesis.

Proposition 1. *Consider the Cauchy problem*

$$\begin{cases} \dot{x}(t) = k - px(t), \\ x(0) = x_0. \end{cases} \tag{1.3}$$

Its solution is

$$x(t) = \left(x_0 - \frac{k}{p}\right)e^{-pt} + \frac{k}{p}. \tag{1.4}$$

Proof. The equation can be solved using the formula for linear differential equations, that is, given the linear equation

$$\dot{x}(t) + a(t)x(t) = b(t),$$

the solution is

$$x(t) = e^{-\int a(t)dt}\Big(\int b(t)e^{\int a(t)dt}dt + c\Big).$$

In the case considered, $a(t) = p, b(t) = k$, so the solution becomes

$$x(t) = e^{-\int pdt}\Big(\int ke^{\int pdt}dt + c\Big) = e^{-pt}\Big(\int ke^{pt}dt\Big) + ce^{-pt} = ke^{-pt}\frac{e^{pt}}{p} + ce^{-pt} =$$

$$= \frac{k}{p} + ce^{-pt}.$$

Applying the initial condition we get

$$x(0) = \frac{k}{p} + c = x_0 \Longrightarrow c = x_0 - \frac{k}{p},$$

so that the solution of the Cauchy problem (1.3) is

$$x(t) = \Big(x_0 - \frac{k}{p}\Big)e^{-pt} + \frac{k}{p}.$$

□

Chapter 2

Opinion dynamics models

2.1 French-DeGroot model

Opinion dynamics models are used to study how information and ideas spread across a network, described by a graph. Every node represents an individual, whose opinion is modeled by a real number that varies in time. Any two individuals interact if there is an edge connecting them. Historically, one of the first opinion dynamics models proposed was the French-DeGroot model, a discrete time model that describes the change of opinion of a node as dependent on the difference with the opinion of its neighbours. In general, the edges of the graph are weighted, to evaluate the strength of the link between two individual (as for instance we might value more the opinion of a close friend, than that of an acquaintance). Given a graph $G = (V, E)$ with n nodes, adjacency matrix A, and weights w such that $\sum_{j=1}^{n} w_{ij} = 1$, for $i = 1 \dots n$, the equations of the model are

$$x(k+1) = Wx(k), \quad k = 0,1 \dots \tag{2.1}$$

or in components

$$x_i(k+1) = \sum_{j=1}^{n} w_{ij} x_j(k), \quad k = 0,1 \dots \tag{2.2}$$

where x is the vector containing the opinion of the n nodes, $k \in \mathbb{N}$ is the time step and $W = D^{-1}A$ is a stochastic matrix, with $D = diag(w)$.

The weight w_{ij} represents the contribution of node j's opinion on the update on i's opinion, while $0 \leq w_{ii} \leq 1$ is i's openness to change its mind: if $w_{ii} = 0$, at every time step node i will update completely relying on its neighbours opinion, disregarding its own, while if $w_{ii} = 1$, the node is *stubborn*, as it will never change value, no matter what the other nodes think. From $\sum_{j=1}^{n} w_{ij} = 1$, we can write $1 - w_{ii} = \sum_{j \neq i} w_{ij}$, which substituted in Equation (2.2) yields

$$x_i(k+1) - x_i(k) = \sum_{j \neq i} w_{ij} [x_j(k) - x_i(k)] \quad \forall i. \tag{2.3}$$

This form of the model that highlights the dependence of the update on the difference of opinions with the neighbours.

This model, under light hypothesis presented in the following theorem, converges to consensus, i.e. a point $x^* = (c, c, \dots, c), c \in \mathbb{R}$ in which all nodes share the same opinion.

Theorem 4. *Given a strongly connected graph G, the model (2.2) converges if and only if G is aperiodic.*

2.2 Abelson model

The linear Abelson model is the translation of the French-DeGroot model in continuous time, indeed starting from Equation (2.3), the left-hand side term represents the increment $\Delta x_i(k)$, that is an approximation of the derivative for the time step that goes to 0. The Abelson model is thus described by the equations

$$\dot{x}_i(t) = \sum_{j \neq i} a_{ij}[x_j(t) - x_i(t)], \tag{2.4}$$

where $a_{ij} \geq 0$ are the entries of the matrix of infinitesimal influence A, which, unlike W, is not necessarily stochastic. The previous equation can be written in matrix form as

$$\dot{x}_i(t) = -L[A]x(t), \tag{2.5}$$

where $L[A]$ is the Laplacian matrix of the graph. The main convergence result of the Abelson model is as follows.

Theorem 5. *The linear Abelson model converges to consensus if and only if G has a directed spanning tree. In that case,*

$$\lim_{t \to \infty} x_i(t) = p_\infty^T x(0),$$

where $p_\infty \in \mathbb{R}^n$ is found by solving $p_\infty^T L[A] = 0$ and $p_\infty^T \mathbb{1}_n = 1$.

2.3 Other models

The main issue with the models above is that the requirements for consensus are extremely loose, and convergence to non-consensus can only happen if the graph is not connected. In reality, though, consensus is not what we expect to see in most cases, thus we look for models that can more accurately describe society. The main phenomenon we would like to observe is the lack of consensus, usually reached in the form of polarization or clusterization of opinions.

Different ideas have been proposed to achieve this goal, for instance the Taylor model [2] adds communication sources, e.g. mass media, that broadcast constant opinions on the nodes, who are influenced by them without being able to influence them back. The presence of multiple sources with different static opinions impairs consensus, as each individual will be influenced in different ways by the external information.

Another approach is used in *bounded confidence* models (e.g. [3], [4]), in which nodes interact only if their opinions don't differ more than a set threshold. The idea behind these models is that when we listen to someone whose opinion is completely different from our own, we might find it unbelievable, consider it wrong and not be affected at all by it. These models can lead to clusterization of opinions, as different sets of nodes can converge to opinions too distant from each other, and thus stop interacting.

The model that will be the subject of this work belong to the family of CODA models, where the acronym stands for "Continuous Opinion, Discrete Action". The idea behind such models is the following assumption: when interacting, nodes know precisely their own opinion, and display a certain discrete behaviour, a sort of approximation of their opinion, that can be seen by their neighbours. This implies that each individual doesn't have access to their neighbours' precise opinion, which impairs communication making consensus harder to achieve.

A typical example helpful to describe this type of models is that of political elections. Each person votes for the candidate whose vision they share the most, so by knowing who someone voted for, it's possible to understand in what range their opinion lays. Although, as every candidate will be voted by a variety of people, all with slightly different ideas, it will be impossible to pinpoint the exact opinion of every voter.

Mathematically, an edge going from v_i to v_j represents v_i observing v_j's behaviour and consequently being influenced by it. The opinion x_i will be adjusted according to a certain h_j, a value somehow close to x_j, but in general not equal, as the information x_j is hidden from v_i's sight.

Different models can be created, with different update laws and different ways of defining the nodes' possible behaviours. For instance, [5] proposes a discrete time model with binary action, that uses the update law

$$p_i(k+1) = p_i(k) + \frac{p_i(k)(1-p_i(k))}{n_i} \sum_{j \in N_i} (q_j(k) - p_i(k)),$$

where $p_i(k), q_i(k) \in [0,1]$ are respectively the opinion and the action of node i at the time step k.

The weight $p_i(k)(1-p_i(k))$ is a way of modeling the fact that people with extreme opinions are usually less inclined to change their mind. In fact, the weight considered goes to 0 as the opinion grows closer to 0 or 1, the edges of the domain, and is maximized when the opinion is in the middle.

Looking at another example, the model proposed in [6] takes a statistical approach, viewing opinion of a node as the probability they think one option is better than the other, and using an application of Bayes theorem to define the update law. The model is then updated ([7]) to introduce the presence of contrarians, individual who tend to oppose the opinion of their neighbours, instead of moving towards them. It is then updated again ([8]) to consider the concept of trust between individual, because as we change our opinion on a certain topic, so does our opinion on the people that influenced us.

Chapter 3

Quantized model

The CODA model analyzed in this work goes under the name of quantized model. The update law resembles the Abelson model, but instead of considering the precise opinion of the neighbours, we round it to the closest integer.

In order to do this, we introduce the quantizer function $q : \mathbb{R} \to \mathbb{Z}$ defined as

$$q(x_j) = \left\lfloor x_j + \frac{1}{2} \right\rfloor, \tag{3.1}$$

i.e. $q(x_j) = h \iff h - \frac{1}{2} \leq x_j < h + \frac{1}{2}$.

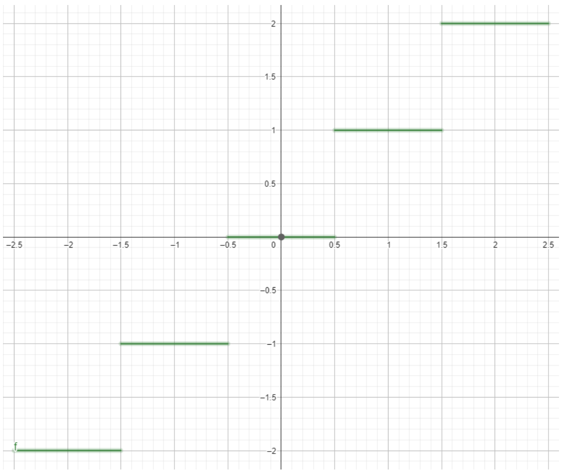

Figure 3.1: The quantizer function $q(x)$.

Notice that, as $x \in \mathbb{R}$, the codomain of $q(x)$ is the entire $\mathbb{Z}$, meaning there are infinite possible behaviours.

With this definition, the equation that describes the model is

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}[q(x_j(t)) - x_i(t)], \quad i = 1, \ldots, n, \tag{3.2}$$

that can be rearranged to be written in matrix form as

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}[q(x_j(t)) - x_i(t)] = \sum_{j=1}^{n} a_{ij}[q(x_j(t)) - x_j(t) + x_j(t) - x_i(t)] =$$

$$= \sum_{j=1}^{n} a_{ij}[q(x_j(t)) - x_j(t)] + \sum_{j=1}^{n} a_{ij}[x_j(t) - x_i(t)] \Longrightarrow$$

$$\dot{\mathbf{x}}(t) = A[q(\mathbf{x}(t)) - \mathbf{x}(t)] - L\mathbf{x}(t),$$

where A is the adjacency matrix and L the Laplacian matrix of the associated graph.

Recalling that $L = D - A$, where D is the degree matrix, we can also write the model in the following form:

$$\dot{\mathbf{x}}(t) = A[q(\mathbf{x}(t))] - D\mathbf{x}(t). \tag{3.3}$$

This simple modification of the Abelson model drastically complicates the analysis, as the system is now discontinuous in state. In fact, the dynamic changes when a node's opinion crosses the middle point between two integer numbers. The dynamic is affine inside hypercubic-shaped domains that we label S_h, properly defined in the following, and changes when a border is crossed.

Definition 35 (Extended equilibrium)**.** Given $h \in \mathbb{Z}^n$, we define $S_h \subset \mathbb{R}^n$ as

$$S_h = \{x \in \mathbb{R}^n | h_i - \frac{1}{2} \leq x_i < h_i + \frac{1}{2}, i = 1, \ldots, n\},$$

i.e. S_h is the set of vectors whose quantized value is equal to h.

Remark 2. We will often need to work with the topological closure of S_h, defined as

$$\overline{S_h} = \{x \in \mathbb{R}^n | h_i - \frac{1}{2} \leq x_i \leq h_i + \frac{1}{2}, i = 1, \ldots, n\}.$$

The discontinuity requires us to generalize the notion of solution.

Definition 36 (Carathéodory solution)**.** Given a differential equation

$$\dot{x}(t) = f(x(t)), \tag{3.4}$$

with initial condition $x(t_0) = x_0$, and $I = [t_0, T) \subset \mathbb{R}$ where $T \leq +\infty$, a Carathéodory solution is a function

$$\varphi(t, x_0) : I \times \mathbb{R}^n \to \mathbb{R}^n$$

such that

$$\dot{\varphi}(t, x_0) = f(\varphi(t, x_0)),$$

for <u>almost all</u> $t \geq t_0$ and $\varphi(t_0, x_0) = x_0$.

Moreover, the solution satisfies

$$\varphi(t, x_0) = x_0 + \int_{t_0}^{t} f(\varphi(\tau, x_0)) d\tau.$$

This type of solution is the natural extension of classical solutions of a differential equation, with the difference of requiring to satisfy the equation only almost everywhere, i.e. everywhere except for a set of points of null measure. When simply referring to a "solution", throughout the thesis, it is to be regarded as a Carathédory solution.

A more generalised type of solutions is given by Krasovskii, taking into account solutions that may slide along the discontinuity.

Definition 37 (Krasovskii solution)**.** Given a differential equation

$$\dot{x}(t) = f(x(t)),$$

with initial condition $x(t_0) = x_0$, and $I = [t_0, T) \subset \mathbb{R}$ where $T \leq +\infty$, a Krasovskii solution is a function

$$\varphi(t, x_0) : I \times \mathbb{R}^n \to \mathbb{R}^n$$

such that

$$\dot{\varphi}(t, x_0) \in \mathcal{K}f(\varphi(t)),$$

where

$$\mathcal{K}f(x) = \bigcap_{\delta>0} \overline{co}\{f(y) \mid y \text{ such that } \|y - x\| < \delta\},$$

where $\overline{co}\{X\}$ is the closure of the set of convex combinations of the elements of X.

The definition of Carathéodory/Krasovskii equilibrium is a natural extension of the classical equilibrium:

Definition 38 (Carathéodory/Krasovskii equilibrium)**.** x^* is a Carathéodory/Krasovskii equilibrium if

$$\varphi(t, x^*) = x^*,$$

for almost all $t \in I$, where $\varphi(t, x^*)$ is a Carathéodory/Krasovskii solution of (3.4).

In [1] was proven that solutions of (3.2) exist, but are not unique. The results on existence of solutions and an example of non-uniqueness are recalled here for convenience.

Theorem 6 (Properties of solutions)**.** *Solutions of* (3.2) *have the following properties.*

1. *For any initial condition, there exist a Carathéodory and a Krasovskii solution of* (3.2)*,*

2. *Any Carathéodory or Krasovskii solution is bounded on its domain,*

3. *Any Carathéodory or Krasovskii solution starting at $t_0 \in \mathbb{R}$ is defined on the set $[t_0, +\infty)$.*

Example 1 (Non-unique Carathéodory solutions)*.* Consider the system

$$\begin{cases} \dot{x}_1 = q(x_2) - x_1, \\ \dot{x}_2 = q(x_1) - x_2, \end{cases} \tag{3.5}$$

with initial conditions $x_1(0) = x_2(0) = \frac{1}{2}$. As $q(\frac{1}{2}) = 1$, we can solve

$$\begin{cases} \dot{x}_1 = 1 - x_1, \\ \dot{x}_2 = 1 - x_2, \end{cases}$$

with the given initial conditions to obtain the solution $x_1(t) = x_2(t) = -\frac{1}{2}e^{-t} + 1$.

However, Carathéodory solutions allow us to choose arbitrary vector fields in single instants. If in $t = 0$, we consider a negative derivative for x_1 and x_2, immediately we enter $S_{(0,0)}$, in which the dynamic is

$$\begin{cases} \dot{x}_1 = -x_1, \\ \dot{x}_2 = -x_2, \end{cases}$$

which yields the solution $x_1(t) = x_2(t) = \frac{1}{2}e^{-t}$, which satisfies (3.5) for all $t \in (0, +\infty)$, that is for almost all $t \in [0, +\infty)$.

This shows that there are two Carathéodory solutions of (3.5) branching from $(x_1, x_2) = (\frac{1}{2}, \frac{1}{2})$.

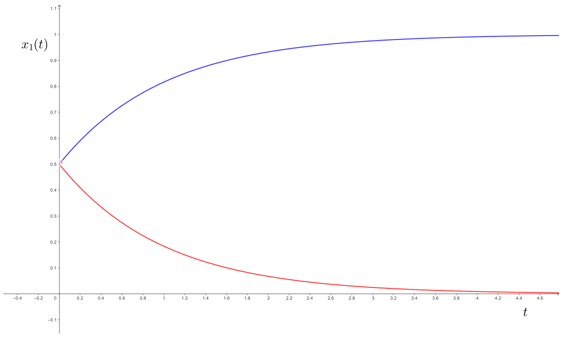

Figure 3.2: The two solutions found in Example 1.

Remark 3. In [1] was found that for the complete and complete bipartite graphs, convergence to consensus is guaranteed for all initial conditions. These graphs have diameter 1 and 2 respectively, and because it is reasonable to assume that a quick flow of information between everyone helps with agreement, a conjecture that could be made is that consensus is always achieved for graphs of diameter ≤ 2. Unfortunately, this conjecture doesn't hold, as shown by the following example, considering a graph of diameter 2.

Consider the graph in Figure (3.3). The dynamic is defined by

$$\begin{cases} \dot{x}_1 = q(x_2) + q(x_5) - 2x_1, \\ \dot{x}_2 = q(x_1) + q(x_3) - 2x_2, \\ \dot{x}_3 = q(x_2) + q(x_4) + q(x_5) - 3x_3, \\ \dot{x}_4 = q(x_3) + q(x_5) - 2x_4, \\ \dot{x}_5 = q(x_1) + q(x_3) + q(x_4) - 3x_5. \end{cases}$$

The point $x^* = (\frac{1}{2}, \frac{1}{2}, \frac{2}{3}, 1, \frac{2}{3})$ is an extended equilibrium on $\overline{S_{(0,0,1,1,1)}}$, as

$$f_{(0,0,1,1,1)}(x^*) = \begin{cases} 0+1-2\cdot\frac{1}{2}=0, \\ 0+1-2\cdot\frac{1}{2}=0, \\ 0+1+1-3\cdot\frac{2}{3}=0, \\ 1+1-2\cdot 1=0, \\ 0+1+1-3\cdot\frac{2}{3}=0. \end{cases}$$

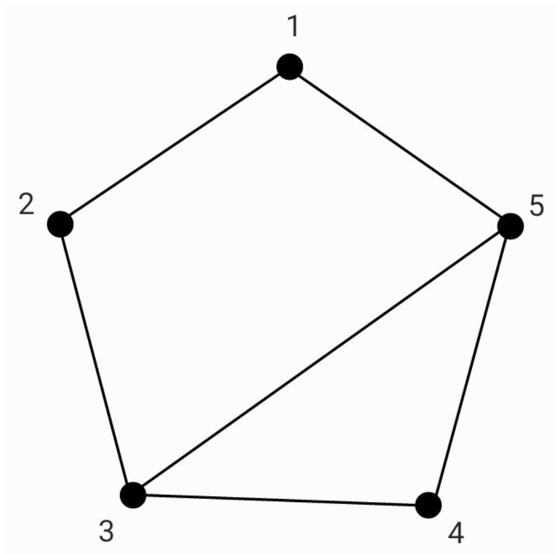

Figure 3.3: The graph of diameter 2 considered in Remark 3.

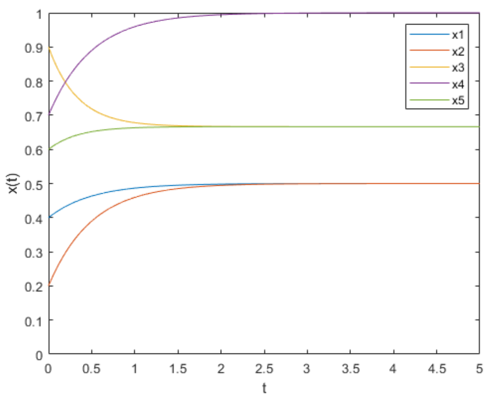

Figure 3.4: Solution converging to the extended equilibrium $x^* = (\frac{1}{2}, \frac{1}{2}, \frac{2}{3}, 1, \frac{2}{3})$ discussed in Remark 3.

3.1 Equilibrium points and extended equilibria

One of the phenomena due to the discontinuity of our model is the existence of what we call *extended equilibria*, points to which solutions can converge to, despite them not being *(Carathéodory) equilibria*. Consider a solution asymptotically converging to a point: as the solution grows closer, its derivative goes to 0; if the dynamic is locally continuous, then the derivative in that point is 0, and we have a regular equilibrium point. On the other hand, if the point is placed exactly on the discontinuity, then the derivative can suddenly jump to a non-null value, and we therefore have a solution asymptotically converging to a non-equilibrium point.

We now rigorously define *extended equilibria* for our model, and provide an example right after.

Definition 39. Let $h \in \mathbb{Z}^n$ and f_h be such that

$$(f_h)_i(x) = \sum_{j \neq i} a_{ij}(h_j - x_i).$$

Then, an extended equilibrium is a point $x^* \in \mathbb{R}^n$ such that there exists $h^* \in \mathbb{Z}^n$ such that $f_{h^*}(x^*) = 0$ and $x^* \in \overline{S_{h^*}}$.

Remark 4. Notice that $f_h(x)$ coincide with $\dot{x}(t)$ on S_h, in fact $x \in S_h \Rightarrow q(x_j) = h_j$. We can write $f_h(x)$ as

$$(f_h)_i(x) = \sum_{j \neq i} a_{ij}(h_j - x_i) = \sum_{j \neq i} a_{ij}h_j - \sum_{j \neq i} a_{ij}x_i = b_i - d_i x_i \Longrightarrow$$

$$f_h(x) = -Dx + b,$$

where b is a vector whose entries b_i are all constant.

Example 2. Consider a line graph with 4 nodes. The dynamic is defined by

$$\begin{cases} \dot{x}_1 = q(x_2) - x_1, \\ \dot{x}_2 = q(x_1) + q(x_3) - 2x_2, \\ \dot{x}_3 = q(x_2) + q(x_4) - 2x_3, \\ \dot{x}_4 = q(x_3) - x_4. \end{cases}$$

The point $x^* = (0, \frac{1}{2}, \frac{1}{2}, 1)$ is an *extended equilibrium*.

In fact, $x^* \in \overline{S_{(0,0,1,1)}}$ and $f_{(0,0,1,1)}(x^*) = 0$, as

$$f_{(0,0,1,1)}(x^*) = \begin{cases} 0 - 0 = 0, \\ 0 + 1 - 2 \cdot \frac{1}{2} = 0, \\ 0 + 1 - 2 \cdot \frac{1}{2} = 0, \\ 1 - 1 = 0. \end{cases}$$

However, x^* is not a *Carathéodory equilibrium*: $\dot{x}_1^* = q(x_2) - x_1 = 1 - 0 \neq 0$.

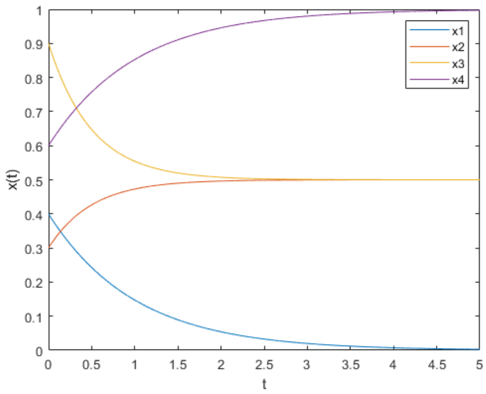

Figure 3.5: Solution converging to the extended equilibrium considered in Example 2.

Remark 5. Extended equilibria are placed on the frontier of different hypercubes, which means that $x^* \in \overline{S_h}$ for multiple values of $h \in \mathbb{Z}^n$, but only one hypercube, the one with $h = h^*$, is such that $f_h(x^*) = 0$. Given x^*, we will refer to S_{h^*} as his corresponding hypercube.
Mind that h^* is not the quantized value of x^*, but for $t \longrightarrow +\infty$ it is the quantized value of the solutions converging to x^*. In fact, x^* is an attractor for all points in S_{h^*}, as we will show in the next section.

3.1.1 Basin of attraction of extended equilibria

The classic concepts of stability don't apply to extended equilibria, as the point itself generates a non constant solution moving away from the equilibrium, meaning that for every small neighbourhood there will be at least one solution making the point unstable.

Despite this, extended equilibria are attractors, and as such we are interested in their basin of attraction. In particular, we want to prove that each extended equilibrium x^* attracts the entire corresponding S_{h^*}.

Proposition 2. *Given an extended equilibrium x^* on a connected graph, consider $h^* \in \mathbb{Z}^n$ for which $f_{h^*}(x^*) = 0$. Then, for all $x_0 \in S_{h^*}$*

$$\varphi(t, x_0) \longrightarrow x^*,$$

for $t \to +\infty$.

Proof. To prove the statement, we need to show that x^* is asymptotically stable for the local dynamic f_{h^*} and that $\varphi(t, x_0)$ cannot leave S_{h^*} for $t > 0$.

Firstly, we prove that every dynamic f_h, for all $h \in \mathbb{Z}^n$, has a single equilibrium, that is globally asymptotically stable. This is easy to see, as from Remark 4, f_h is linear and can be written as

$$f_h(x) = -Dx + b,$$

where the matrix $-D$ is diagonal with negative entries (if the graph is connected), which implies that all the eigenvalues are negative and therefore the only equilibrium is locally asymptotically stable. Since the dynamic is linear, local stability can be extended to global stability, thus the equilibrium is globally asymptotically stable.

Notice that it is not sufficient to extended the local dynamic f_{h^*} to $\mathbb{R}^n$ and prove that x^* is globally asymptotically stable, to conclude that x^* attracts S_{h^*}. Actually, a solution $\varphi_1(t, x_0)$ of $f_{h^*}(x)$ starting from $x_0 \in S_{h^*}$ could in theory not coincide with the solution $\varphi_2(t, x_0)$ of the quantized dynamic. In fact, $\varphi_2(t, x_0)$ could potentially leave S_{h^*}, and consequently stop obeying the dynamic f_{h^*}, drifting away from $\varphi_1(t, x_0)$ (see Figure 3.6), so that $\varphi_1(t, x_0)$ converging to x^* doesn't imply the same for result for $\varphi_2(t, x_0)$.

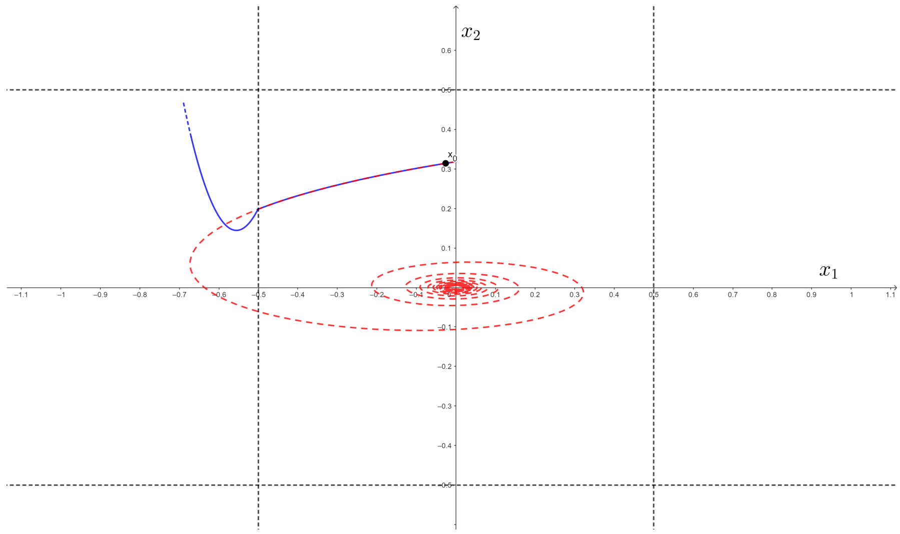

Figure 3.6: Example of a solution that would converge to $x^* = (0,0)$ if the dynamic in $S_{(0,0)}$ were to be extended to $\mathbb{R}^2$, but doesn't in the discontinuous dynamic, as it enters $S_{(-1,0)}$ in which the dynamic is different.

This is true for a generic h, but, as we are about to show, for h^* associated with an extended equilibrium we have that the solution cannot leave S_{h^*}, or in other words

$$x_0 \in S_{h^*} \Longrightarrow \varphi(t, x_0) \in S_{h^*} \ \forall t > 0 \Longrightarrow \varphi(t, x_0) \longrightarrow x^* \text{ for } t \longrightarrow +\infty.$$

Take $x_0 \in S_{h^*}$, we can write the coordinates of the solution starting in x_0 as

$$x_i = h_i^* + \Delta_i,$$

with $\Delta_i \in [-\frac{1}{2}, \frac{1}{2})$.

We can do the same with the equilibrium point,

$$x_i^* = h_i^* + \Delta_i^*,$$

where $\Delta_i^* \in [-\frac{1}{2}, \frac{1}{2}]$.

The dynamic is described by

$$\dot{x}_i = \sum_{j\in N(x_j)} q(x_j) - d_i x_i = \sum_{j\in N(x_j)} h_j^* - d_i(h_i^* + \Delta_i),$$

in which we can substitute the expression $h_i^* = x_i^* - \Delta_i^*$ to obtain

$$\dot{x}_i = \sum_{j\in N(x_j)} h_j^* - d_i(x_i^* - \Delta_i^* + \Delta_i) = (\sum_{j\in N(x_j)} h_j^* - d_i x_i^*) - d_i(-\Delta_i^* + \Delta_i).$$

The first term is the derivative $f_{h^*}(x^*)$ which is null by definition of extended equilibrium. The end result is thus

$$\dot{x}_i = d_i(\Delta_i^* - \Delta_i),$$

where $d_i > 0$ because the graph is connected. This relation shows that x_i asymptotically converges to x_i^*, as x_i moves to the right (has positive derivative) when its placed to the left of x^* ($\Delta_i < \Delta_i^*$), and vice versa x_i moves to the left when placed at the right of the extended equilibrium (see Figure 3.7), with speed that decreases linearly with the distance between the two. This implies that x_i always moves away from the frontier, hence it can never cross it.

Since this holds true for all the coordinates, and all the equations are independent, $x \longrightarrow x^*$ for $t \longrightarrow +\infty$, $\forall x_0 \in S_{h^*}$.

$\square$

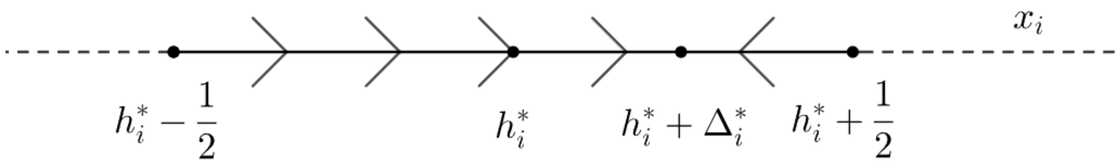

Figure 3.7: For all i, the vector field points toward $x_i^* = h_i^* + \Delta_i^*$.

3.1.2 Line graph

We now move to study particular cases, starting from the line graph. Given n nodes, the line graph is the connected and undirected graph that impairs communication the most, and as such non-consensus is to be expected. The focus of this section is to find the extended equilibria.

The dynamic (3.2) on the line graph becomes, for $i = 2, \dots, n-1$

$$\begin{cases} \dot{x}_1 = q(x_2) - x_1 \\ \dot{x}_i = q(x_{i+1}) + q(x_{i-1}) - 2x_i \\ \dot{x}_n = q(x_{n-1}) - x_n = 0. \end{cases} \tag{3.6}$$

Proposition 3. *Let $k \in \mathbb{Z}^n$ such that $k_1 = k_n = 0, k_j \in \{-1,0,1\}$ for $j = 2, ..., n-1$, and $\sum_{j=2}^{n-1} k_j = 0$.*

Then extended equilibria in a line graph with n nodes can be written in the form

$$x^e = \overline{h} + x^*, \tag{3.7}$$

where $\overline{h} = (h, h, ..., h), h \in \mathbb{Z}$ and x^* is such that

$$x_i^* = \frac{1}{2}k_i + \sum_{j=2}^{i-1}(i-j)k_j.$$

Proof. To find equilibrium points, we set to 0 the derivatives defined in (3.6) and we obtain

$$\begin{cases} q(x_2) - x_1 = 0 \\ q(x_{i+1}) + q(x_{i-1}) - 2x_i = 0 \\ q(x_{n-1}) - x_n = 0. \end{cases}$$

We have that $x_1 = q(x_2)$, which means that x_1 must be an integer.
Let $x_1 = h \in \mathbb{Z}$, we obviously have $q(x_1) = h$, and from the first line of the system we have that $q(x_2) = h$.
Rearranging the term in the second line of the system, we see that

$$x_i = \frac{q(x_{i+1}) + q(x_{i-1})}{2} \quad \text{for} \quad i = 2, \dots, n-1,$$

which means that x_i is half of an integer.
With this information, if we know the quantized value $q(x_i)$, we have 3 possible values of x_i: either $q(x_i) - \frac{1}{2}, q(x_i)$ or $q(x_i) + \frac{1}{2}$ (remember that extended equilibria are on the closure of the hypercubes).
We write this in compact form as $x_i = q(x_i) + \frac{1}{2}k_i$, $k_i \in \{-1,0,1\}$.
We apply this fact to x_2 to obtain $x_2 = h + \frac{1}{2}k_2$.
Let us consider again the second equation of the system, this time centered in x_{i-1}. We have

$$q(x_i) + q(x_{i-2}) - 2x_{i-1} = 0 \Longrightarrow q(x_i) = 2x_{i-1} - q(x_{i-2}),$$

which allows us to find $q(x_i)$, for $i-1 = 2, \dots, n-1 \Rightarrow i = 3, \dots, n$. We can now compute the value of x_3, which can be used to compute x_4 and so on:

$$q(x_3) = 2x_2 - q(x_1) = 2(h + \frac{1}{2}k_2) - h = h + k_2 \Longrightarrow x_3 = h + k_2 + \frac{1}{2}k_3,$$

$$q(x_4) = 2x_3 - q(x_2) = 2(h + k_2 + \frac{1}{2}k_3) - h = h + 2k_2 + k_3 \Longrightarrow x_4 = h + 2k_2 + k_3 + \frac{1}{2}k_4,$$

$$q(x_5) = 2x_4 - q(x_3) = 2(h+2k_2+k_3+\frac{1}{2}k_4) - (h+k_2) = h+3k_2+2k_3+k_4 \Longrightarrow x_5 = h+3k_2+2k_3+k_4+\frac{1}{2}k_5.$$

We now prove by induction that $q(x_i) = h + \sum_{j=2}^{i-1}(i-j)k_j$, and by consequence that $x_i = h + \sum_{j=2}^{i-1}(i-j)k_j + \frac{1}{2}k_i$. From the previous computations, we see that the statement holds true for $i = 3,4,5$.
Moving to the inductive step, we assume that the statement holds for $i-2$ and $i-1$, and we check that it still holds for i:

$$q(x_i) = 2x_{i-1} - q(x_{i-2}) = 2h + 2(\sum_{j=2}^{i-2}(i-1-j)k_j + \frac{1}{2}k_{i-1}) - h - \sum_{j=2}^{i-3}(i-2-j)k_j =$$

$$= h + k_{i-1} + \sum_{j=2}^{i-2} 2(i-1-j)k_j - \sum_{j=2}^{i-3}(i-2-j)k_j =$$

$$= h + k_{i-1} + 2(i-1-i+2)k_{i-2} + \sum_{j=2}^{i-3} 2(i-1-j)k_j - \sum_{j=2}^{i-3}(i-2-j)k_j =$$

$$= h + k_{i-1} + 2k_{i-2} + \sum_{j=2}^{i-3}[2(i-1-j) - (i-2-j)]k_j =$$

$$= h + k_{i-1} + 2k_{i-2} + \sum_{j=2}^{i-3}(i-j)k_j = h + \sum_{j=2}^{i-1}(i-j)k_j.$$

Finally, we consider the third line of the system, in which we substitute the expressions $q(x_{n-1}) = h+\sum_{j=2}^{n-2}(n-1-j)k_j$ and $x_n = h+\sum_{j=2}^{n-1}(n-j)k_j$. Notice that $x_n = q(x_{n-1})$, i.e. x_n is an integer, and consequently $x_n = q(x_n) \Rightarrow k_n = 0$. The condition $q(x_{n-1}) - x_n = 0$ becomes

$$h + \sum_{j=2}^{n-2}(n-1-j)k_j - h - \sum_{j=2}^{n-1}(n-j)k_j = 0 \Longrightarrow$$

$$\sum_{j=2}^{n-2}(n-1-j)k_j - \sum_{j=2}^{n-2}(n-j)k_j - k_{n-1} = 0 \Longrightarrow$$

$$\sum_{j=2}^{n-2}(n-1-j-n+j)k_j - k_{n-1} = 0 \Longrightarrow$$

$$-\sum_{j=2}^{n-2}k_j - k_{n-1} = 0 \Longrightarrow$$

$$\sum_{j=2}^{n-1}k_j = 0.$$

Lastly, both x_1 and x_n must be integer, thus they are equal to their own quantized value, that is $k_1 = k_n = 0$, q.e.d. $\square$

Remark 6. The sum $\sum_{j=2}^{i-1}(i-j)k_j$, present in the expression of the extended equilibria, can equally be written as $\sum_{j=1}^{i}(i-j)k_j$, since

$$\sum_{j=1}^{i}(i-j)k_j = (i-1)k_1 + \sum_{j=2}^{i-1}(i-j)k_j + (i-i)k_i = \sum_{j=2}^{i-1}(i-j)k_j,$$

as $k_1 = 0$. We will often use the second form as it's more intuitive and easier to compute with.

Remark 7. The choice of h is arbitrary, which allows us to assume $h = 0$. This is not restrictive, as every extended equilibrium having $h \neq 0$, can be thought as an extended equilibrium having $h = 0$ in the reference system with origin in $\overline{h} = (h, h, \ldots, h) \in \mathbb{Z}^n$. To simplify the notation, throughout the rest of the thesis we will refer to these points, denoting them and their corresponding hypercube with an asterisk.

Notation 1. $h^* = (h_1^*, h_2^*, \ldots, h_n^*)$ and $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$, where

$$h_i^* = \sum_{j=1}^{i}(i-j)k_j, \tag{3.8}$$

$$x_i^* = \frac{1}{2}k_i + h_i^*. \tag{3.9}$$

Example 3. Let us compute all the extended equilibria (with $h = 0$) on the line graph with 4 nodes defined in Example 2. The only possible values for k are

- $k_a = (0,0,0,0)$,
- $k_b = (0,1,-1,0)$,
- $k_c = (0,-1,1,0)$.

For k_a we simply have $h_a^* = (0,0,0,0), x_a^* = (0,0,0,0)$, as all the term in the sums are null.

For k_b, the corresponding hypercube and extended equilibrium are $h_b^* = (0,0,1,1), x_b^* = (0, \frac{1}{2}, \frac{1}{2}, 1)$. In fact, $h_{b1}^* = 0$ and

$$h_{b2}^* = k_{b1} = 0, \quad h_{b3}^* = 2k_{b1} + k_{b2} = 0 + 1 = 1, \quad h_{b4}^* = 3k_{b1} + 2k_{b2} + k_{b3} = 0 + 2 \cdot 1 - 1 = 1.$$

Then, $x_b^* = h_b^* + \frac{1}{2}k_b = (0, 0 + \frac{1}{2}, 1 - \frac{1}{2}, 1) = (0, \frac{1}{2}, \frac{1}{2}, 1)$.

Analogously, $k_c = -k_b$ leads to $h_c^* = -h_b^* = (0,0,-1,-1)$,and $x_c^* = -x_b^* = (0, -\frac{1}{2}, -\frac{1}{2}, -1)$.

Remark 8. To simplify the computation of h^* in more complex cases, we point out that h_i^* can be obtained from h_{i-1}^* as

$$h_i^* = \sum_{j=1}^{i}(i-j)k_j = \sum_{j=1}^{i-1}(i-j)k_j = \sum_{j=1}^{i-1}(i-j-1)k_j + \sum_{j=1}^{i-1}k_j = h_{i-1}^* + \sum_{j=1}^{i-1}k_j.$$

Example 4. Let us see an example of the computation of one extended equilibrium in a more complicated case, with the help of Remark 8.
Take $n = 13$ and $k = (0,0,1,1,0,-1,-1,0,0,0,1,-1,0)$. We have

$$h_1^* = h_2^* = 0, \quad h_3^* = h_2^* + 0 = 0, \quad h_4^* = h_3^* + \sum_{j=1}^{3} k_j = 0 + 1 = 1,$$

$$h_5^* = h_4^* + \sum_{j=1}^{4} k_j = 1 + 2 = 3, \quad h_6^* = h_5^* + \sum_{j=1}^{5} k_j = 3 + 2 = 5,$$

$$h_7^* = h_6^* + \sum_{j=1}^{6} k_j = 5 + 1 = 6, \quad h_8^* = h_7^* + \sum_{j=1}^{7} k_j = 6 + 0 = 6,$$

$$h_9^* = h_{10}^* = h_{11}^* = 6, \quad as \quad \sum_{j=1}^{m} k_j = 0 \quad \text{for} \quad m = 8,9,10,$$

$$h_{12}^* = h_{11}^* + \sum_{j=1}^{11} k_j = 6 + 1 = 7, \quad h_{13}^* = h_{12}^* + \sum_{j=1}^{12} k_j = 7 + 0 = 7.$$

We have found $h^* = (0,0,0,1,3,5,6,6,6,6,6,7,7)$, the extended equilibrium corresponding to k is therefore

$$x^* = h^* + \frac{1}{2}k = (0,0,\frac{1}{2},\frac{3}{2},3,\frac{9}{2},\frac{11}{2},6,6,6,\frac{13}{2},\frac{13}{2},7).$$

Remark 9. In [1] was found that the furthest extended equilibrium from consensus, i.e. the extended equilibrium with the greatest difference between x_1^* and x_n^*, is such that

$$|x_n^* - x_1^*| = \begin{cases} \frac{(n-2)^2}{4} & \text{if } n \text{ even}, \\ \frac{(n-1)(n-3)}{4} & \text{if } n \text{ odd}. \end{cases}$$

This is achieved for k of the form

1. $k = (0,1,\ldots,1,-1,\ldots,-1,0)$ if n is even,

2. $k = (0,1,\ldots,1,0,-1,\ldots,-1,0)$ if n is odd,

and their opposites. In fact, if $x_1^* = 0$ in the first case we have

$$x_n^* = \sum_{j=2}^{n-1}(n-j)k_j = \sum_{j=2}^{n/2}(n-j) - \sum_{j=n/2+1}^{n-1}(n-j) =$$

$$= (n-2+n-\frac{n}{2})(\frac{n}{4}-\frac{1}{2}) - (n-1-\frac{n}{2}+1)(\frac{n}{4}-\frac{1}{2}) =$$

$$= (n-2+n-\frac{n}{2}-n+1+\frac{n}{2}-1)(n-2)\cdot\frac{1}{4} = \frac{(n-2)(n-2)}{4} = \frac{(n-2)^2}{4},$$

while in the second case

$$x_n^* = \sum_{j=2}^{n-1}(n-j)k_j = \sum_{j=2}^{\frac{n-1}{2}}(n-j) - \sum_{j=\frac{n+3}{2}}^{n-1}(n-j) =$$
$$= (n-2+n-\frac{n-1}{2})(\frac{n-1}{4}-\frac{1}{2}) - (n-\frac{n+3}{2}+1)(\frac{n-1}{4}-\frac{1}{2}) =$$
$$= (n-2+n-\frac{n}{2}+\frac{1}{2}-n+\frac{n}{2}+\frac{3}{2}-1)(n-1-2)\cdot\frac{1}{4} = \frac{(n-1)(n-3)}{4}.$$

3.1.3 Cardinality of the set of extended equilibria

In this subsection we investigate the cardinality of the set of extended equilibria, to see how it scales as n grows larger. The extended equilibria are infinite due to the arbitrary choice of $\overline{h}$ in Equation (3.7), what we are interested in is the cardinality of $E_n^* = \{x^*\}$, with x^* defined in (3.9) for a line graph with n nodes. To this end, as there is a 1 to 1 correspondence between extended equilibria and the k defined in the hypothesis of Proposition 3, it suffices to count in how many different ways k can be written while respecting such hypothesis.

Given a line graph with n nodes, we have that $k_1 = k_n = 0$, therefore the values that can vary are only $m = n-2$. We will find the cardinality by partitioning the set of k in subsets according to the number of cells equal to 0 in k, i.e. X_p^m is the set of k with m cells, of which p cells are null. Let's start by considering m even, and by counting the cases for which $k_i \neq 0$ for all i. Since k cells must sum to 0, k is formed by $\frac{m}{2}$ cells with value 1 and $\frac{m}{2}$ with value -1. The cardinality $\#X_0^m$ of this first subset is then given by a permutation with repetition:

$$\#X_0^m = P_m^{(\frac{m}{2},\frac{m}{2})} = \frac{m!}{\frac{m}{2}!\cdot\frac{m}{2}!}.$$

Consider now the subset of k with 2 cells equal to 0, and consequently $m-2$ cells $\neq 0$. Just like in the previous case, these $m-2$ cells can be ordered in

$$\#X_0^{m-2} = P_{m-2}^{(\frac{m-2}{2},\frac{m-2}{2})} = \frac{(m-2)!}{\frac{m-2}{2}!\cdot\frac{m-2}{2}!}.$$

From there we must add the 2 null cells. From every result of the permutation of the $m-2$ non-null cells, we can add the 2 null cells in $P_m^{(2,m-2)}$ different ways, where in the apex the 2 represents the repetition of the null cells and the $m-2$ the repetition of the non null ones. This mean that

$$\#X_2^m = P_{m-2}^{(\frac{m-2}{2},\frac{m-2}{2})}\cdot P_m^{(2,m-2)}.$$

We can repeat the same process to find k with $p = 4,6,8,\ldots,m$ null cells, to find

$$\#X_p^m = P_{m-p}^{(\frac{m-p}{2},\frac{m-p}{2})}\cdot P_m^{(p,m-p)}.$$

We then have, for m even, that the cardinality we are looking for is the sum of the cardinality of all these subsets, i.e. for $p = 2i$,

$$\begin{aligned}
\#E_n^* = \sum_{i=0}^{\frac{m}{2}} \#X_{2i}^m = \sum_{i=0}^{\frac{m}{2}} P_{m-2i}^{(\frac{m-2i}{2},\frac{m-2i}{2})} \cdot P_m^{(2i,m-2i)} = \\
= \sum_{i=0}^{\frac{m}{2}} \frac{(m-2i)!}{((\frac{m-2i}{2})!)^2} \cdot \frac{m!}{(2i)! \cdot (m-2i)!} = \\
= \sum_{i=0}^{\frac{m}{2}} \frac{m!}{((\frac{m-2i}{2})!)^2 \cdot (2i)!}.
\end{aligned}$$

The case for m odd is similar. Notice that necessarily there is an even number of non null cells, as there must be an equal number of 1s and -1s. Consequently, if m odd there's at least one cell with value 0. Elements in X_1^m can be obtained by adding a null cell to elements in X_0^{m-1}, which can be done in m different ways (notice that $P_m^{(1,m-1)} = m$) per element in X_0^{m-1}, so that

$$\#X_1^m = \#X_0^{m-1} \cdot m = P_{m-1}^{(\frac{m-1}{2},\frac{m-1}{2})} \cdot P_m^{(1,m-1)}.$$

Analogously we can get the cardinality of the sets X_p^m for $p = 3,5,7,\ldots,m$ and by summing them we obtain

$$\begin{aligned}
\#E_n^* = \sum_{i=0}^{\frac{m-1}{2}} \#X_{2i}^{m-1} = \sum_{i=0}^{\frac{m-1}{2}} P_{m-1-2i}^{(\frac{m-1-2i}{2},\frac{m-1-2i}{2})} \cdot P_m^{(2i+1,m-1-2i)} = \\
= \sum_{i=0}^{\frac{m-1}{2}} \frac{(m-1-2i)!}{((\frac{m-1-2i}{2})!)^2} \cdot \frac{m!}{(2i+1)! \cdot (m-1-2i)!} = \\
= \sum_{i=0}^{\frac{m-1}{2}} \frac{m!}{((\frac{m-1-2i}{2})!)^2 \cdot (2i+1)!}.
\end{aligned}$$

The results obtained using these formula are shown in Table 3.1.

Remark 10. Notice that a line graph with 2 or 3 nodes is a complete bipartite graph, for which convergence to consensus is guaranteed. The single extended equilibrium found is the consensus, which is a Carathéodory equilibrium, a particular case of extended equilibrium.

3.1.4 Ring graph

Similar computations can be done for the ring graph, which can be obtained by a line graph by simply adding an edge connecting v_1 and v_n.

Proposition 4. *Let $k \in \mathbb{Z}^n$ such that*

n	m	$\#E_n^*$
2	0	1
3	1	1
4	2	3
5	3	7
6	4	19
7	5	51
8	6	141
9	7	393
10	8	1107

Table 3.1: Computed cardinality of the set of extended equilibria on the line graph with n nodes.

- $k_j \in \{-1,0,1\}$ *for* $j = 1...n$,
- $\sum_{j=1}^{n} k_j = 0$,
- $\sum_{j=1}^{n}(n-j)k_j$ *is a multiple of* n.

Then extended equilibria in a ring graph with n nodes can be written in the form

$$x^e = \overline{h} + x^*,$$

where $\overline{h} = (h, h, ..., h), h \in \mathbb{Z}$ *and* x^* *is such that*

$$x_i^* = \frac{1}{2}k_i + \sum_{j=2}^{i-1}(i-j)k_j.$$

Proof. Extended equilibria must satisfy $q(x_{i-1}) + q(x_{i+1}) - 2x_i = 0$ for $i = 1, \ldots, n$, considering $x_0 := x_n$ and $x_{n+1} := x_1$. This can be rewritten $x_i = \frac{q(x_{i-1})+q(x_{i+1})}{2}$, i.e. x_i is half of an integer, thus we can write it as $x_i = q(x_i) + \frac{1}{2}k_i$. Analogously to the line graph case, by rearranging the terms we get

$$q(x_i) = 2x_{i-1} - q(x_{i-2}) \Longrightarrow$$

$$x_i = \frac{1}{2}k_i + 2x_{i-1} - q(x_{i-2}),$$

meaning that we can find any value x_i if we know the value of the previous two nodes. Let $x_1 = h_1 + \frac{1}{2}k_1, x_n = h_n + \frac{1}{2}k_n$, where $h_i = q(x_i)$. We obtain

$$x_2 = \frac{1}{2}k_2 + 2x_1 - q(x_n) = \frac{1}{2}k_2 + k_1 + 2h_1 - h_n,$$

$$x_3 = \frac{1}{2}k_3 + 2x_2 - q(x_1) = \frac{1}{2}k_3 + k_2 + 2k_1 + 4h_1 - 2h_n - h - 1 = \frac{1}{2}k_3 + k_2 + 2k_1 + 3h_1 - 2h_n,$$

$$x_4 = \frac{1}{2}k_4 + 2x_3 - q(x_2) = \frac{1}{2}k_4 + k_3 + 2k_2 + 4k_1 + 6h_1 - 4h_n - k_1 - 2h_1 + h_n =$$

$$= \frac{1}{2}k_4 + k_3 + 2k_2 + 3k_1 + 4h_1 - 3h_n.$$

By induction we notice that

$$x_i = \frac{1}{2}k_i + \sum_{j=1}^{i}(i-j)k_j + ih_1 - (i-1)h_n.$$

The expression holds for $i = 1,2,3,4$. For the inductive step, let us assume it holds for $i-1$ and $i-2$ and check it holds for i (keep in mind Remark 6 when dealing with the last term of a sum):

$$x_i = \frac{1}{2}k_i + 2x_{i-1} - q(x_{i-2}) =$$

$$\frac{1}{2}k_i + k_{i-1} + 2\sum_{j=1}^{i-1}(i-1-j)k_j + 2(i-1)h_1 - 2(i-2)h_n - \sum_{j=1}^{i-2}(i-2-j)k_j - (i-2)h_1 + (i-3)h_n =$$

$$\frac{1}{2}k_i + k_{i-1} + 2\sum_{j=1}^{i-2}(i-1-j)k_j - \sum_{j=1}^{i-2}(i-2-j)k_j + (2i-2-i+2)h_1 - (2i-4-i+3)h_n =$$

$$\frac{1}{2}k_i + k_{i-1} + \sum_{j=1}^{i-2}(2i-2-2j-i+2+j)k_j + ih_1 - (i-1)h_n =$$

$$\frac{1}{2}k_i + k_{i-1} + \sum_{j=1}^{i-2}(i-j)k_j + ih_1 - (i-1)h_n =$$

$$= \frac{1}{2}k_i + \sum_{j=1}^{i-1}(i-j)k_j + ih_1 - (i-1)h_n =$$

$$= \frac{1}{2}k_i + \sum_{j=1}^{i}(i-j)k_j + ih_1 - (i-1)h_n.$$

Applying the expression to $x_n = \frac{1}{2}k_n + h_n$ yields

$$x_n = \frac{1}{2}k_n + h_n = \frac{1}{2}k_n + \sum_{j=1}^{n}(n-j)k_j + nh_1 - (n-1)h_n \implies$$

$$h_n + (n-1)h_n - nh_1 = \sum_{j=1}^{n}(n-j)k_j \implies$$

$$nh_n - nh_1 = \sum_{j=1}^{n}(n-j)k_j. \tag{3.10}$$

Equation (3.10) can be written as $h_n - h_1 = \frac{1}{n}\sum_{j=1}^{n}(n-j)k_j$, and since h_1 and h_n are integer, necessarily $\sum_{j=1}^{n}(n-j)k_j$ is a multiple of n.

Previously we set $x_1 = \frac{1}{2}k_1 + h_1$, and it must hold $x_1 = \frac{1}{2}k_1 + 2x_n - q(x_{n-1})$, thus

$$x_1 = \frac{1}{2}k_1 + h_1 = \frac{1}{2}k_1 + 2x_n - q(x_{n-1}) \Longrightarrow$$

$$h_1 = k_n + 2\sum_{j=1}^{n}(n-j)k_j + 2nh_1 - 2(n-1)h_n - \sum_{j=1}^{n-1}(n-1-j)k_j - (n-1)h_1 + (n-2)h_n \Longrightarrow$$

$$h_1 = k_n + \sum_{j=1}^{n-1}(2n-2j-n+1+j)k_j + (2n-n+1)h_1 - (2n-2-n+2)h_n \Longrightarrow$$

$$h_1 - (n+1)h_1 + nh_n = k_n + \sum_{j=1}^{n-1}(n-j+1)k_j \Longrightarrow$$

$$nh_n - nh_1 = k_n + \sum_{j=1}^{n-1}(n-j+1)k_j. \tag{3.11}$$

By comparing Equations (3.10) and (3.11), we obtain

$$\sum_{j=1}^{n}(n-j)k_j = k_n + \sum_{j=1}^{n-1}(n-j+1)k_j \Longrightarrow$$

$$k_n + \sum_{j=1}^{n-1}(n-j+1)k_j - \sum_{j=1}^{n-1}(n-j)k_j = 0 \Longrightarrow$$

$$k_n + \sum_{j=1}^{n-1}(n-j+1-n+j)k_j = 0 \Longrightarrow$$

$$k_n + \sum_{j=1}^{n-1}k_j = 0 \Longrightarrow$$

$$\sum_{j=1}^{n}k_j = 0.$$

□

Remark 11. The condition $\sum_{j=1}^{n}(n-j)k_j$ is a multiple of n seems to be a condition of symmetry. Let us assume k to be symmetric and check that $\sum_{j=1}^{n}(n-j)k_j = 0$. If n is even, we have $k_j = k_{n+1-j}$ and

$$\sum_{j=1}^{n}(n-j)k_j = \sum_{j=1}^{n/2}(n-j)k_j + \sum_{j=n/2+1}^{n}(n-j)k_j.$$

Let us write the second sum with the new index $i = n + 1 - j$, considering that by symmetry $k_i = k_j$:

$$\sum_{j=1}^{n/2}(n-j)k_j + \sum_{j=n/2+1}^{n}(n-j)k_j = \sum_{j=1}^{n/2}(n-j)k_j + \sum_{i=1}^{n/2}(i-1)k_i.$$

Having the same index, we can write everything in the same sum,

$$\sum_{j=1}^{n/2}[(n-j)+(j-1)]k_j = \sum_{j=1}^{n/2}(n-1)k_j = (n-1)\sum_{j=1}^{n/2}k_j.$$

Noticing how $\sum_{j=1}^{n/2} k_j = \sum_{j=n/2+1}^{n} k_j = \frac{1}{2}\sum_{j=1}^{n} k_j$, we finally obtain

$$(n-1)\sum_{j=1}^{n/2}k_j = \frac{n-1}{2}\sum_{j=1}^{n}k_j = 0.$$

If n is odd, then k symmetric implies $k_{\frac{n+1}{2}} = 0$. In fact,

$$0 = \sum_{j=1}^{n}k_j = \sum_{j=1}^{\frac{n-1}{2}}k_j + k_{\frac{n+1}{2}} + \sum_{j=\frac{n+3}{2}}^{n}k_j = k_{\frac{n+1}{2}} + 2\sum_{j=1}^{\frac{n-1}{2}}k_j \Longrightarrow$$

$$k_{\frac{n+1}{2}} = -2\sum_{j=1}^{\frac{n-1}{2}}k_j \Longrightarrow k_{\frac{n+1}{2}} = 0$$

since $k_{\frac{n+1}{2}} \in \{-1,0,1\}$ and $2\sum_{j=1}^{\frac{n-1}{2}} k_j$ is an even number. We then have

$$\sum_{j=1}^{n}(n-j)k_j = \sum_{j=1}^{\frac{n-1}{2}}(n-j)k_j + \sum_{j=\frac{n+3}{2}}^{n}(n-j)k_j,$$

which we can treat analogously to the case with n even to reach the same conclusion.

Since we are working on a cycle graph, we can arbitrarily shift all the indexes of the nodes. With the help of an example, we will show that this takes the sum $\sum_{j=1}^{n}(n-j)k_j$ from value 0 to a multiple of n. Let $n = 5$ and $k = (1,-1,0,-1,1)$, the extended equilibrium can be computed to be $x^* = (\frac{1}{2},\frac{1}{2},1,\frac{1}{2},\frac{1}{2})$ with $h^* = (0,1,1,1,0)$. By renaming the nodes through a position shift, we have $k' = (-1,0,-1,1,1)$ associated with $x'^* = (\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $h'^* = (1,1,1,0,0)$. Now $\sum_{j=1}^{n}(n-j)k'_j = -4-2+1 = -5 = -n$. In fact, $h'^*_n - h'^*_1 = \frac{1}{n}\sum_{j=1}^{n}(n-j)k'_j = -\frac{1}{5}\cdot 5 = -1$.

This is true in general, as a shift of k to the left means increasing the coefficient of each k_i for $i = 2,\dots,n$ by 1, and decreasing the coefficient of k_1 by $n-1$ (shown in Table

3.2), which results in an increase/decrease of n (or 0 if $k_1 = 0$). In fact, considering k' as the shifted of k,

$$\sum_{j=1}^{n}(n-j)k'_j = \sum_{j=1}^{n-1}(n-j)k_{j+1} = \sum_{j=2}^{n}(n-j+1)k_j = \sum_{j=2}^{n}(n-j)k_j + \sum_{j=2}^{n}k_j =$$

$$= \sum_{j=1}^{n}(n-j)k_j - (n-1)k_1 - k_1 = \sum_{j=1}^{n}(n-j)k_j - nk_1,$$

meaning that if $\sum_{j=1}^{n}(n-j)k_j$ is a multiple of n, it still will be after an arbitrary amount of shifts.

k_1	k_2	$\dots$	k_{n-1}	k_n
$n-1$	$n-2$	$\dots$	1	0
$k'_1 = k_2$	$k'_2 = k_3$	$\dots$	$k'_{n-1} = k_n$	$k'_n = k_1$
$n-1$	$n-2$	$\dots$	1	0

Table 3.2: Entries of k and relative coefficients before and after the shift.

Example 5. The sum $\sum_{j=1}^{n}(n-j)k_j$ (and consequently the difference between the quantized of two adjacent nodes) cannot be arbitrarily large. Starting from a symmetric k, after every shift the sum will keep growing in modulus as long as $k'_1 \neq 0$ keeps having the same sign. Since there can be at most $\frac{n}{2}$ consecutive 1 in k (if n even, or $\frac{n-1}{2}$ if n odd), $\frac{n}{2}$ marks the higher limit for $|h^*_i - h^*_{i+1}|$. For instance, for $n = 16$, the furthest extended equilibrium from consensus is

- $k = (1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,1,1,1,1)$,
- $h^* = (0,1,3,6,10,13,15,16,16,15,13,10,6,3,1,0)$
- $x^* = (\frac{1}{2},\frac{3}{2},\frac{7}{2},\frac{13}{2},\frac{19}{2},\frac{25}{2},\frac{29}{2},\frac{31}{2},\frac{31}{2},\frac{29}{2},\frac{25}{2},\frac{19}{2},\frac{13}{2},\frac{7}{2},\frac{3}{2},\frac{1}{2})$,

that shifted becomes

- $k = (-1,-1,-1,-1,-1,-1,-1,-1,1,1,1,1,1,1,1,1)$,
- $h^* = (10,13,15,16,16,15,13,10,6,3,1,0,0,1,3,6)$
- $x^* = (\frac{19}{2},\frac{25}{2},\frac{29}{2},\frac{31}{2},\frac{31}{2},\frac{29}{2},\frac{25}{2},\frac{19}{2},\frac{13}{2},\frac{7}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{7}{2},\frac{13}{2})$,

for which in fact $h_1 - h_n = -\frac{1}{n}\sum_{j=1}^{n}(n-j)k_j = -\frac{1}{16}(-15-14-13-12-11-10-9-8+7+6+5+4+3+2+1) = \frac{64}{16} = 4$.

Example 6. **Invisible consensus**: For $n = 8$, $k = (1,-1,-1,1,1,-1,-1,1)$ yields

- $h^* = (0,1,1,0,0,1,1,0)$,
- $x^* = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$

x^* is a non integer consensus extended equilibrium point. It is interesting to notice that, since x^* has a non null basin of attraction, there are solutions for which all the nodes will asymptotically share the same opinion, while having different behaviours. This is an example of an undetectable consensus, as an observer can only see the different actions taken, without knowing the true opinion of each individual.

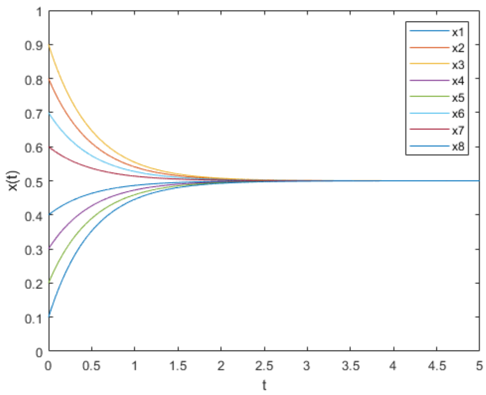

Figure 3.8: Solution with initial conditions in $S_{(0,1,1,0,0,1,1,0)}$, converging to $x^* = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, computed in Example 6.

3.2 Binary actions

To simplify our analysis, we focus on the dynamic in the hypercube $[0,1]^n$, meaning that $q(x_i)$ is either 0 or 1. This simplified version is still of interest, as it models binary behaviours (in general choices between two options, like yes/no or do/don't type of actions). One example could be the decision whether or not to buy a certain product, my opinion will in be in [0,1] while evaluating pros and cons, but my final decision (and what my neighbours will be able to see) will be either 0 or 1 (to buy it or not).
We now show some convergence results for the line graph.

3.2.1 Line graph

Our focus is to find the hypercubes that entirely converge to the consensus $\underline{0} = (0,0,\ldots,0)$, in other words we're looking for a condition on $h \in \mathbb{Z}^n$ in order to have that for all $x_0 \in S_h$, $x(t, x_0) \longrightarrow \overline{h}$ as $t \longrightarrow +\infty$.
Then, by symmetry, the condition for convergence to consensus 1 can also be found.

Proposition 5. *Let $h = \sum_{i=1}^n h_i e_i$ where $h_i \in \{0,1\}$ and e_i is the i-th vector of canonical basis of $\mathbb{R}^n$.*
If for all i :

1. $h_i = 0 \Longrightarrow \frac{1}{d_i} \sum_{j \in N_i} h_j \leq \frac{1}{2}$,

2. $h_i = 1 \Longrightarrow \sum_{j \in N_i} h_j = 0$

then $\forall x_0 \in S_h$, $\lim_{t\to+\infty} x_i(t) \longrightarrow 0 \quad \forall i$.

The two conditions mean respectively that, in the starting hypercube, each node with action 0 has at least one neighbour with action 0, and that every node with action 1 has all neighbours with action 0.

Proof. Firstly, notice how if for instance $q(x_{i-1}) = 0, q(x_i) = 0, q(x_{i+1}) = 1$, then x_i can not leave the interval of quantization 0. In fact, $\dot{x}_i = q(x_{i-1}) + q(x_{i+1}) - 2x_i = 1 - 2x_i$ which asymptotically converges to $\frac{1}{2}$, i.e. x_i keeps getting closer to the frontier, but without ever trespassing in the adjacent interval of quantization. This means that, until the dynamic changes (either x_{i-1} or x_{i+1} change action), we will have $q(x_i) = 0$.

Condition 1 excludes the case in which an x_i such that $q(x_i) = 0$ has all neighbours with action one. The possible cases are therefore as follows:

- x_i is at the edge of the graph (either $i = 1$ or $i = n$, and its only neighbour has action 0,
- x_i is in the middle ($2 \leq i \leq n - 1$) with both neighbours with action 0,
- x_i is in the middle and has one neighbour with action 0 and one with action 1.

In the first two cases, x_i obviously goes to 0, while in the third cases x_i goes to $\frac{1}{2}$ while maintaining $q(x_i) = 0$.
In summary, condition 1 doesn't allow the variables with quantization 0 to swap actions as long as the dynamic doesn't change.

On the other hand, condition 2 grants that every x_i such that $q(x_i) = 1$ will eventually go to 0, as $\dot{x}_i = -d_i x_i$.
Thus, combining the two conditions, we have that we go from the starting hypercube to a hypercube with one cell (or more) swapped from 1 to 0. This new set still satisfies the two conditions, therefore we can recursively apply the same argument until all the cells are equal to 0. Once in $\overline{S_{(0,\dots,0)}}$, we know by Proposition 2 that the solution will converge to $\overline{h} = (0, \dots, 0)$.

□

Remark 12. The hypercubes satisfying the two conditions are such that the second and second to last cell are equal to 0 (otherwise condition 1 would be violated for the node on the edge), and the cells with value 1 are far enough from each other, at least three index numbers away.
For instance (0,0,1,0,0,0,1,0,0,1) satisfies the conditions.

Example 7. Let us consider the binary action dynamic on the line graph with 4 nodes. The starting hypercubes that completely go to consensus 0 are:

- (0,0,0,0),
- (1,0,0,0),
- (0,0,0,1),
- (1,0,0,1).

By symmetry, the hypercubes that completely converge to consensus 1 are

- (1,1,1,1),
- (0,1,1,1),
- (1,1,1,0),
- (0,1,1,0).

There are two extended equilibria, $x_a = (0, \frac{1}{2}, \frac{1}{2}, 1)$, placed on $\overline{S_a} = (0,0,1,1)$, $x_b = (1, \frac{1}{2}, \frac{1}{2}, 0)$, placed on $\overline{S_a} = (1,1,0,0)$. By Proposition 2 $S_{(0,0,1,1)}$ and $S_{(1,1,0,0)}$ completely converge to these points.
The remaining possible sets,

- (0,1,0,0),
- (0,0,1,0),
- (0,1,0,1),
- (1,0,1,0),
- (1,1,0,1),
- (1,0,1,1),

are more difficult to study, In fact, for these cases, different starting condition in the same hypercube will lead to different convergence points. Take for instance $h =(0,1,0,0)$. While x_3 and x_4 cannot leave their interval of quantization, both x_1 and x_2 can. It is then a matter of who reaches the frontier first: if x_1 does, the solution will enter $S_{(1,1,0,0)}$ and will go to the extended equilibrium, while if x_2 is the fastest we enter $S_{(0,0,0,0)}$ and consequently go to the consensus.
In the plane (x_1, x_2), we can compute the analytic expression of the line separating the two areas leading to the two different equilibria. From the dynamic

$$\begin{cases} \dot{x}_1 = 1 - x_1, \\ \dot{x}_2 = -2x_2, \end{cases}$$

with initial conditions $x^0 \in S_h \Rightarrow x_1^0 \in [0, \frac{1}{2}), x_2^0 \in [\frac{1}{2}, 1)$, we can write the solutions

$$\begin{cases} x_1(t) = (x_1^0 - 1)e^{-t} + 1, \\ x_2(t) = x_2^0 e^{-2t}. \end{cases}$$

Let be T_1, T_2 such that

$$x_1(T_1) = \frac{1}{2}, \qquad x_2(T_2) = \frac{1}{2},$$

then the smaller T_i determines which coordinate will cross the frontier first. If $T_1 < T_2$, x_1 switches action first and we enter $h^* = (1,1,0,0)$, on the contrary if $T_2 < T_1$ we end up in $h^* = (0,0,0,0)$. $T_1 = T_2$ marks the separation line we want to find. Computing T_1, T_2, we get

$$(x_1^0 - 1)e^{-T_1} + 1 = \frac{1}{2} \Longrightarrow e^{-T_1} = -\frac{1}{2(x_1^0 - 1)} \Longrightarrow$$

$$-T_1 = \ln\Big(-\frac{1}{2(x_1^0 - 1)}\Big) \Longrightarrow T_1 = \ln(-2(x_1^0 - 1)) \Longrightarrow$$

$$T_1 = \ln(2 - 2x_1^0);$$

$$x_2^0 e^{-2T_2} = \frac{1}{2} \Longrightarrow -2T_2 = \ln\Big(\frac{1}{2x_2^0}\Big) \Longrightarrow T_2 = \frac{1}{2}\ln(2x_2^0) \Longrightarrow$$

$$T_2 = \ln(\sqrt{2x_2^0}).$$

From which $T_2 = T_1$ becomes

$$\ln(\sqrt{2x_2^0}) = \ln(2 - 2x_1^0) \Longrightarrow \sqrt{2x_2^0} = 2 - 2x_1^0 \Longrightarrow$$

$$x_2^0 = \frac{(2 - 2x_1^0)^2}{2}.$$

Finally, we can define

$$\Omega_1 = \{(x_1, x_2) \mid 0 \leq x_1 < \frac{1}{2},\ \frac{1}{2} \leq x_2 \leq 1,\ x_2^0 < \frac{(2 - 2x_1^0)^2}{2}\},$$

$$\Omega_2 = \{(x_1, x_2) \mid 0 \leq x_1 < \frac{1}{2},\ \frac{1}{2} \leq x_2 \leq 1,\ x_2^0 > \frac{(2 - 2x_1^0)^2}{2}\},$$

and conclude that if $(x_1^0, x_2^0) \in \Omega_1$, then the solution will converge to consensus (0,0,0,0), and if $(x_1^0, x_2^0) \in \Omega_2$, then the solution will converge to the extended equilibrium with $h^* = (1,1,0,0)$.

In theory this process can be done for every number of variables that can potentially change actions, although complexity highly increases with the number of switch candidates, as there will be many different hypercubes in which the solution can enter.

Example 8. Consider now the line graph with 7 nodes, and let us pick starting hypercubes that violate one of the two conditions to show that, in these sets, there could be solutions converging to extended equilibria.
Using the interpretation given in Remark 12, the hypercubes violate at least one condition when one of the following occurs:

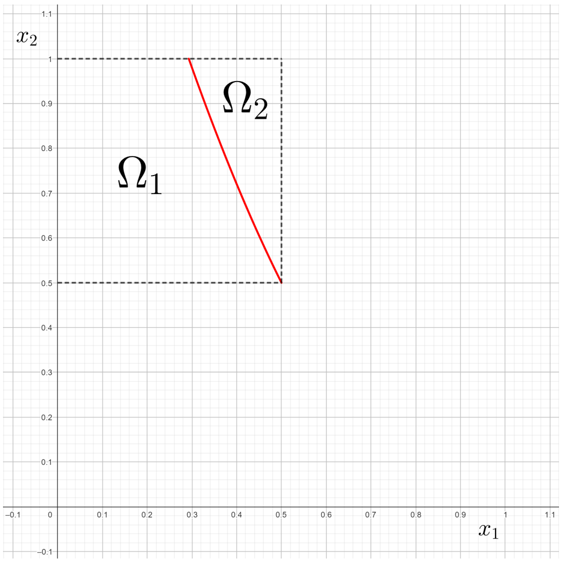

Figure 3.9: The two areas separated by the line computed in Example 7.

1. they have a 1 in the second or second to last cell, e.g. (0,0,0,0,0,1,0),
2. they have two adjacent 1s, e.g. (0,0,1,1,0,0,0),
3. they have two 1s, two index numbers apart, e.g. (0,0,1,0,1,0,0).

In case 1, solutions can either enter hypercube (0,0,0,0,0,0,0) and go to consensus, or enter h^* =(0,0,0,0,0,1,1), which contains the extended equilibrium $x^* = (0,0,0,0,\frac{1}{2},\frac{1}{2},1)$, as $h_i^* = \sum_{j=1}^{i}(i-j)k_j$ for $k = (0,0,0,0,1,-1,0)$.

In case 2 we already find ourselves in a hypercube that is home to an extended equilibrium, that is $x^* = (0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0)$, as $h^* = (0,0,1,1,0,0,0)$ is such that $h_i^* = \sum_{j=1}^{i}(i-j)k_j$ for $k = (0,1,-1,-1,1,0,0)$ (to verify, notice that $\forall i \geq 2, h_i = h_{i-1} + \sum_{j=1}^{i-1} k_j$).

In case 3, x_4 is allowed to change action as its neighbours both have action 1, and if that happens we enter the hypercube (0,0,1,1,1,0,0), that similarly to the previous case respects the condition for the presence of an extended equilibrium, this time for $k = (0,1,-1,0,-1,1,0)$, that generates $x^* = (0,\frac{1}{2},\frac{1}{2},1,\frac{1}{2},\frac{1}{2},0)$.

Proposition 6. *The dynamic on the line graph restricted to the hypercube* $[0,1]^n$ *converges for all initial conditions.*

Proof. Let $h_i = 0$, we've seen that if for instance $h_{i-1} = 0$ and $h_{i+1} = 1$, then h_i can't change value (unless h_{i-1} or h_{i+1} change first) as x_i will asymptotically go to $\frac{1}{2}$. In the case of the dynamic in $[0,1]^n$, if at any time T two adjacent nodes have the same action, then they will stay in that quantization interval $\forall t > T$. In fact, if $h_{i-1} = h_i = 0$, h_i can't reach value 1 unless h_{i-1} does so first, but analogously h_{i-1} can't reach value 1 unless h_i does first.

Let us assume $x_0 \in S_h$, where h is such that $h_{i-1} = h_i$ for some i. Let's show that h_j is definitely constant $\forall j = 1 \ldots, n$. By assumption, and following the previous reasoning, we have that h_{i-1} and h_i are constant, without loss of generality say $h_{i-1} = h_i = 0 \; \forall t \geq 0$. Consider h_{i+1}, if also $h_{i+1} = 0$, then it has an adjacent cell with the same value, which implies h_{i+1} definitely constant. If instead $h_{i+1} = 1$, we have two options: either $h_{i+2} = 0$ for a long enough time for h_{i+1} to reach 0, or $h_{i+2} = 1$ (or $h_{i+2} = 0$ but it switches action before h_{i+1}). In either case, we end up having $h_{i+1} = h_{i+2} \implies h_{i+1}$ and h_{i+2} are definitely constant. Essentially, we have that a string of 2 adjacent nodes with same action leads to a string of $m > 2$ nodes with definitely constant action, which has the last two cells with equal value, allowing to repeat the process until the end of the line. Since x_n always copies x_{n-1}'s action, once h_{n-1} is definitely constant, so will be h_n shortly after. Once we enter the last hypercube, the dynamic becomes linear, and since it's bounded, necessarily it is convergent, concluding the analysis.

If $\nexists i$ such that $h_{i-1} = h_i$, then h is made of alternating 1s and 0s, e.g. $h = (0,1,0,1,0,1,0)$, that is every cell has all neighbours with opposite action, meaning that each coordinate is pulled towards the middle and is allowed to cross the frontier. In most cases, solutions will move from a hypercube to the other through a face, which is represented by a single coordinate switching action (a coordinates reaches $\frac{1}{2}$ before everyone else). If that happens, that coordinate will have the same action of its neighbours, thus we have converge thanks to the previous analysis.

In general, solutions can enter new hypercubes through an edge (more than one coordinate reaches $\frac{1}{2}$ at the same time), which causes the behaviour switch for m coordinates, where $1 < m < n$. This implies that there are at least 2 adjacent cells such that one switches and one does not, causing two adjacent cell to have same action, and therefore leading to the previous case again.

The last possibility is that of all coordinates reaching $\frac{1}{2}$ at the same time, when the solution reaches the vertex of the hypercube. Let's show that there cannot be a switch of all actions. Working with Carathéodory solutions, we can potentially enter any hypercube. When attempting to enter the hypercubes with all coordinates switched, though, we find a vector field that pushes back towards $x = (\frac{1}{2}, \ldots, \frac{1}{2})$, not allowing Carathéodory solutions to enter the hypercube (see Figure 3.10). In fact, let h be such that $h_1 = a \in \{0,1\}$ and

$$h_i = \begin{cases} a, & \text{if } n \text{ odd}, \\ 1 - a, & \text{if } n \text{ even}. \end{cases}$$

By switching all the actions, the hypercube becomes

$$h'_i = 1 - h_i = \begin{cases} 1 - a, & \text{if } n \text{ odd}, \\ a, & \text{if } n \text{ even}. \end{cases}$$

Apparently, a cycle between h and h' could seem possible, but notice that by switching all the action we reverse the vector field. Going from h to h' is only possible through the point $x = (\frac{1}{2}, \ldots, \frac{1}{2})$, but when in x, the solution cannot enter h' as the vector field there opposes it. In summary, not all actions can change at the same time, making a cycle between h and h' impossible.

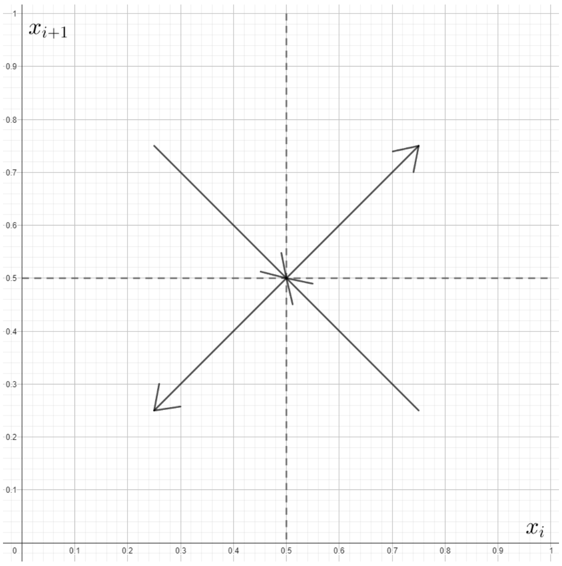

Figure 3.10: Visualisation of the vector fields around $x = (\frac{1}{2}, \dots, \frac{1}{2})$.

□

Remark 13. Proposition (5) and Proposition (6) also hold for the ring graph, that only differ from the line graph for a single edge. Adding the edge connecting v_1 and v_2 means that nodes are indistinguishable, in particular every node has the same degree, which makes the requirement $h_2 = h_{n-1} = 0$ not necessary.

3.2.2 Tree graph

In this section we provide an algorithm to build extended equilibrium on undirected tree graphs, when we consider binary action and opinions in [0,1]. Firstly, let us introduce some useful definitions.

Definition 40 (Level)**.** In an undirected tree graph, one arbitrarily chosen node is defined to be the root. The level of a node is the distance between the root and the node in question.

Definition 41 (Leaf)**.** In a tree graph, the nodes of degree 1 (that are not the root) are called leaves.

To build the extended equilibria, we will work only with the quantized values, i.e. we will build h^* such that there exists an extended equilibrium $x^* \in \overline{S_{h^*}}$. Knowing h^*, it is trivial to find x^*, by simply setting x_i^* equal to the average of its neighbours' quantized values.

To find all the possible h^* let us start by setting $h_1 = 0$ (the cases with $h_1 = 1$ can then be obtained by symmetry), where v_1 is the root. The idea is to show that we can

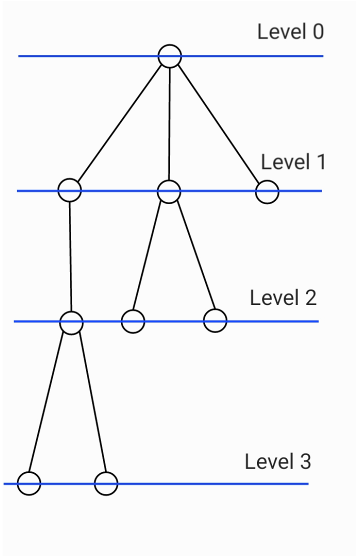

Figure 3.11: Visualisation of the concept of level.

(somewhat arbitrarily) set the values for the level 1 nodes, then for level 2, and so on, without finding ourselves forced to make a choice for h_i that would break the equilibrium of the previously set nodes.

Firstly, consider the set of level 1 nodes, which we label L_1. In order for the root node v_1 to be in equilibrium, we need the average of the quantized values of the v_1's neighbours (i.e. level 1 nodes) to be no greater than $\frac{1}{2}$. To achieve this, we can have at most $\lfloor \frac{d_1}{2} \rfloor$ nodes set to 1, whereas the remaining nodes (up to d_1) will be set to 0. Notice how leaves must necessarily be set to the same quantized value of their only neighbour. In fact, having degree 1, a leaf's dynamic is described by

$$\dot{x}_l = q(x_j) - x_l,$$

from which $\dot{x}_l^* = 0 \Rightarrow x_l^* = q(x_j^*) \Rightarrow h_l^* = h_j^*$.

After setting the level 1 leaves to 0, we need to fill the remaining $d_1 - \#(L_1 \cap \mathcal{L})$, where $\mathcal{L}$ is the set of leaves, thus $\#(L_1 \cap \mathcal{L})$ is the number of level 1 leaves. From a combinatorics point of view, each possible way to assign the remaining values is a result of the extraction of $d_1 - \#(L_1 \cap \mathcal{L})$ elements from a pool containing $\lfloor \frac{d_1}{2} \rfloor$ times the element 1 and $d_1 - \#(L_1 \cap \mathcal{L})$ times the element 0. To write all equilibrium points, we need to write down all possible outcomes.

In general, the i-th step of the algorithm sets the values of level i nodes, for $i = 2, \ldots, l_{max}$, where l_{max} is the depth of the tree. The actions taken are the following:

- for all $v_j \in L_{i-1}$, consider the set $X = N_j \cap L_i$,
- set the leaves in X to the value h_j,
- assuming $h_j = 0$ (if it's not, the results can still be obtained by symmetry), compute all possible outcomes of the extraction of $d_j - 1 - \#(L_i \cap \mathcal{L})$ elements from a pool

containing $d_j - 1 - \#(L_i \cap \mathcal{L})$ times the element 0 and $\lfloor \frac{d_i}{2} \rfloor - h_k$ times the element 1, where h_k is the quantized value of the only $v_k \in N_j \cap L_{i-2}$ (since there can be at most $\lfloor \frac{d_i}{2} \rfloor$ neighbours with value 1, if $h_k = 1$ there is one less available 1 to be set in X).

Notice that, in a tree, a node of level $i \geq 1$ has exactly one neighbour of level $i-1$ and $d_j - 1$ neighbours of level $i+1$. This explains the -1 that appears in the number of values that need to be set through combinatorics, $d_j - 1 - \#(L_i \cap \mathcal{L})$, which can also be seen as $\#(X \setminus \mathcal{L})$. This fact also explains why the algorithm works. When setting the values of the nodes in $N_k \cap L_{i-1}$, the value chosen for h_j doesn't matter (so long as v_j is not a leaf), even if $h_j \neq h_k$ it only takes v_j to have a single other neighbour to ensure its own equilibrium. Since v_j is not a leaf, $d_j \geq 2$ which implies that there are $d_j - 1 \geq 1$ neighbours of v_j in L_i. To summarize, a level i node is set in equilibrium at the i-th step of the algorithm if it is a leaf, and at the $(i+1)$-th step if it isn't. At some point, a level set will only contain leaves, ending the algorithm.

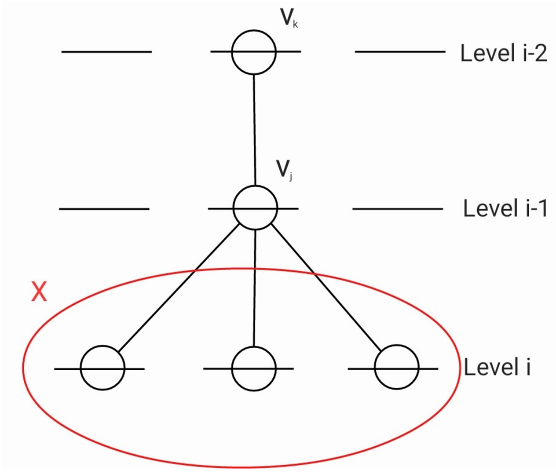

Figure 3.12: The set of nodes of nodes the algorithm works on for all v_j.

Let us apply the algorithm on an example.

Example 9. Consider the tree graph in Figure 3.13. We start by setting $h_1 = 0$. Since v_2 is a leaf, necessarily $h_2 = h_1 = 0$. h_3 can either be set to 0 or 1, as in either case v_1 will have neighbours of average $\leq \frac{1}{2}$. In fact, $d_1 - \#(L_1 \cap \mathcal{L}) = 2 - 1 = 1$ and $\lfloor \frac{d_1}{2} \rfloor = \frac{2}{2} = 1$. Moving on to L_2, we will have $h_4 = h_3$, while for v_5 and v_6 different choices can be made. A total of $\lfloor \frac{d_3}{2} \rfloor - h_1$ of them can be set to 1. If h_3 was set to 0, that becomes $2 - 0 = 2$, i.e. both h_5 and h_6 are allowed to be 1 and we will have four options: $(h_5, h_6) = (0{,}0); (0{,}1); (1{,}0); (1{,}1)$. Otherwise, if $h_3 = 1$, we work by symmetry, pretending $h_3 = 0$ and $h_1 = 1$ from which only one between h_5 and h_6 is allowed to be 1 so that the options are $(h_5, h_6) = (0{,}0); (0{,}1); (1{,}0);$, which, by reverting the symmetry, end up being $(h_5, h_6) = (1{,}1); (1{,}0); (0{,}1)$. The last three nodes are leaves and as such they copy their neighbour's value.

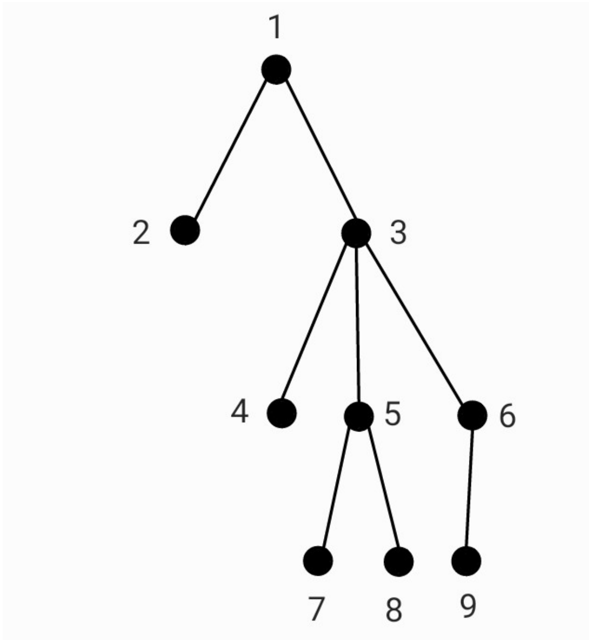

Figure 3.13: The graph considered in Example 9.

To summarise, so far the possible extended equilibria are found when h is one of the ones shown in Table 3.3, where for simplicity the leaves were left out, as they don't add degrees of freedom.

h_1	h_3	h_5	h_6
0	0	0	0
0	0	0	1
0	0	1	0
0	0	1	1
0	1	1	1
0	1	1	0
0	1	0	1

Table 3.3

To obtain all the extended equilibria, by symmetry we list all the options that sprout from the initial choice $h_1 = 1$. In Table 3.4, all the extended equilibria are shown.

Remark 14. A potential way to find extended equilibria on a generic connected graph would be to work on one of the graph's spanning trees, find all the extended equilibria of the spanning tree, then add one by one the missing edges, updating the list of equilibria at each step. Notice that when connecting two nodes that aren't leaves, some extended

h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9
0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	1
0	0	0	0	1	0	1	1	0
0	0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	0	1	1	1	0	1	1	0
0	0	1	1	0	1	0	0	1
1	1	1	1	1	1	1	1	1
1	1	1	1	1	0	1	1	0
1	1	1	1	0	1	0	0	1
1	1	1	1	0	0	0	0	0
1	1	0	0	0	0	0	0	0
1	1	0	0	0	1	0	0	1
1	1	0	0	1	0	1	1	0

Table 3.4: h^* found in Example 9 such that $\overline{S_{h^*}}$ contains extended equilibria.

equilibria can be lost, as the average of the neighbours of the nodes in question can change and leave the interval of equilibrium, $[0,\frac{1}{2}]$ or $[\frac{1}{2},1]$ depending on the quantized value of the nodes. On the other hand, if we add an edge connecting a leaf to another node, the leaf stops having degree 1, and so it will be able to having different values, therefore opening up possibilities for new equilibria.

3.2.3 Directed cycle graph

We now consider a directed cycle graph. The dynamic is

$$\dot{x}_i = q(x_{i+1}) - x_i, \quad \forall i = 1, \dots, n, \tag{3.12}$$

considering $x_{n+1} = x_1$ by definition.

The aim of this section is to investigate the presence of limit cycles for $n = 3$, as in [11] was found a limit cycle for $n = 6$ and we would like to check if they can be found in smaller graphs as well.

For $n = 3$, the dynamic is

$$\begin{cases} \dot{x}_1 = q(x_2) - x_1, \\ \dot{x}_2 = q(x_3) - x_2, \\ \dot{x}_3 = q(x_1) - x_3. \end{cases} \tag{3.13}$$

We start our analysis from the initial conditions $x^0 = (\frac{1}{2}+\delta, \frac{1}{2}-\epsilon, \frac{1}{2})$ with $\delta, \epsilon > 0$. This choice is due to the fact that we must not be in a consensus cube, which would imply the convergence to the consensus point. In this case we chose our initial cube to be $S_{(1,0,1)}$,

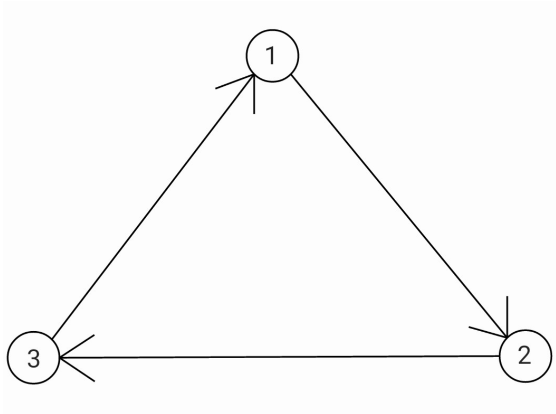

Figure 3.14: Directed cycle graph with $n = 3$.

as despite the arbitrary choice of the value of $q(x_3^0)$, x_3 is pointing towards x_1 which has action 1, so $x_3(t) > \frac{1}{2}, q(x_3(t)) = 1$ for $t \in (0, T_1)$. However the choice doesn't matter, as any non consensus cube can be obtained from (1,0,1) by simple changes of variables (rotation of indexes and/or symmetry with respect to $\frac{1}{2}$).

Moreover, one of the variables is picked to start with value $\frac{1}{2}$. The dynamic inside any hypercube is linear, not allowing cycles inside a single cube, which means that any cycle at some point will necessarily cross the border between different cubes, i.e. $x_i(T) = \frac{1}{2}$ for some i and some T. We choose $T_0 = 0$, and define $T_1, T_2, \dots$ the times at which the other infinite frontier crossings of the hypothetical cycle will happen.

We start by solving Equation (3.12), using Equation (1.4), in the starting cube, i.e. we consider $q(x_i)$ constant for all i. We have

$$\dot{x}_i = q(x_{i+1}) - x_i \Longrightarrow x_i = [x_i^0 - q(x_{i+1})]e^{-t} + q(x_{i+1}),$$

which applied to (3.13) with $x^0 = (\frac{1}{2} + \delta, \frac{1}{2} - \epsilon, \frac{1}{2})$ yields

$$\begin{cases} x_1(t) = (\frac{1}{2} + \delta)e^{-t}, \\ x_2(t) = (-\frac{1}{2} - \epsilon)e^{-t} + 1, \\ x_3(t) = -\frac{1}{2}e^{-t} + 1. \end{cases}$$

Now we compute T_1, the time at which the solution crosses into a new cube. Either x_1 or x_2 can reach value $\frac{1}{2}$ and therefore switch actions. Notice that if x_2 does so first, we enter $S_{(1,1,1)}$, which can never be left. Since we are interested in the case of infinite switches, we consider the case in which x_1 crosses first, entering $S_{(0,0,1)}$, that happens under the condition $\delta < \epsilon$. T_1 is found as follows:

$$x_1(T_1) = \left(\frac{1}{2} + \delta\right)e^{-T_1} = \frac{1}{2} \Longrightarrow e^{-T_1} = \frac{1}{2(\frac{1}{2} + \delta)} \Longrightarrow$$

$$-T_1 = \ln\left(\frac{1}{1+2\delta}\right) \Longrightarrow T_1 = \ln(1+2\delta).$$

Computing $x_2(T_1)$ and $x_3(T_1)$ we get

$$x_2(T_1) = \Big(-\frac{1}{2}-\epsilon\Big)e^{-T_1}+1 = \Big(-\frac{1}{2}-\epsilon\Big)e^{ln(\frac{1}{1+2\delta})}+1 =$$

$$= \frac{-\frac{1}{2}-\epsilon}{1+2\delta}+\frac{1}{2}+\frac{1}{2} = \frac{1}{2}+\frac{-1-2\epsilon+1+2\delta}{2(1+2\delta)} = \frac{1}{2}-\frac{\epsilon-\delta}{1+2\delta}$$

and

$$x_3(T_1) = -\frac{1}{2}e^{-T_1}+1 = -\frac{1}{2}e^{\ln(\frac{1}{1+2\delta})}+1 =$$

$$= -\frac{1}{2(1+2\delta)}+\frac{1}{2}+\frac{1}{2} = \frac{1}{2}+\frac{1+2\delta-1}{2(1+2\delta)} = \frac{1}{2}+\frac{\delta}{1+2\delta}.$$

The solution at time T_1 is then

$$\begin{cases} x_1(T_1) = \frac{1}{2}, \\ x_2(T_1) = \frac{1}{2}-\frac{\epsilon-\delta}{1+2\delta}, \\ x_3(T_1) = \frac{1}{2}+\frac{\delta}{1+2\delta}, \end{cases}$$

which has a similar structure to the initial condition: one variable has value $\frac{1}{2}$ and the other two have disagreeing actions (remember that $\delta < \epsilon$). To make it clear, we apply the change of variables

$$\begin{cases} x_1'(t) = 1-x_2(t), \\ x_2'(t) = 1-x_3(t), \\ x_3'(t) = 1-x_1(t), \end{cases}$$

which, dropping the apostrophes, results in

$$\begin{cases} x_1(T_1) = \frac{1}{2}+\frac{\epsilon-\delta}{1+2\delta} = \frac{1}{2}+\delta', \\ x_2(T_1) = \frac{1}{2}-\frac{\delta}{1+2\delta} = \frac{1}{2}-\epsilon', \\ x_3(T_1) = \frac{1}{2}, \end{cases}$$

with $\delta' = \frac{\epsilon-\delta}{1+2\delta} > 0$ and $\epsilon' = \frac{\delta}{1+2\delta} > 0$.

We are in the same configuration as where we started from, we can reiterate the reasoning to obtain the same results, this time in function of δ', ϵ': x_1 crosses first if $\delta' < \epsilon'$, which happens at time $T_2 = T_1 + \ln(1+2\delta')$, yielding

$$\begin{cases} x_1(T_2) = \frac{1}{2}+\delta'', \\ x_2(T_2) = \frac{1}{2}-\epsilon'', \\ x_3(T_2) = \frac{1}{2}, \end{cases}$$

with $\delta'' = \frac{\epsilon'-\delta'}{1+2\delta'}$ and $\epsilon'' = \frac{\delta'}{1+2\delta'}$. This can be iterated i times to find $\delta^{(i)}, \epsilon^{(i)}$ as a function of $\delta^{(i-1)}, \epsilon^{(i-1)}$ (and in cascade as a function of δ, ϵ) with the law

$$\delta^{(i)} = \frac{\epsilon^{(i-1)} - \delta^{(i-1)}}{1 + 2\delta^{(i-1)}}, \epsilon^{(i)} = \frac{\delta^{(i-1)}}{1 + 2\delta^{(i-1)}}. \tag{3.14}$$

To go to step $i + 1$, it must hold

$$\delta^{(i)} < \epsilon^{(i)}, \tag{3.15}$$

otherwise the solution will enter the consensus cube.

Our goal is to investigate the behaviour as the number of possible switches goes to infinity. Looking for an expression of $\delta^{(i)}, \epsilon^{(i)}$, we start by computing δ'', ϵ'':

$$\delta'' = \frac{\epsilon' - \delta'}{1 + 2\delta'} = \frac{\frac{\delta}{1+2\delta} - \frac{\epsilon-\delta}{1+2\delta}}{1 + 2(\frac{\epsilon-\delta}{1+2\delta})} = \frac{\frac{2\delta-\epsilon}{1+2\delta}}{\frac{1+2\delta+2\epsilon-2\delta}{1+2\delta}} = \frac{2\delta - \epsilon}{1 + 2\epsilon},$$

$$\epsilon'' = \frac{\delta'}{1 + 2\delta'} = \frac{\frac{\epsilon-\delta}{1+2\delta}}{1 + 2(\frac{\epsilon-\delta}{1+2\delta})} = \frac{\frac{\epsilon-\delta}{1+2\delta}}{\frac{1+2\delta+2\epsilon-2\delta}{1+2\delta}} = \frac{\epsilon - \delta}{1 + 2\epsilon}.$$

One way to make computation easier is to write $\delta^{(i)} = \frac{N_\delta^{(i)}}{D^{(i)}}, \epsilon^{(i)} = \frac{N_\epsilon^{(i)}}{D^{(i)}}$ (notice that the denominator is always the same for δ and ϵ), so that we can write

$$\delta^{(i)} = \frac{\epsilon^{(i-1)} - \delta^{(i-1)}}{1 + 2\delta^{(i-1)}} = \frac{\frac{N_\epsilon^{(i-1)}}{D^{(i-1)}} - \frac{N_\delta^{(i-1)}}{D^{(i-1)}}}{1 + 2\frac{N_\delta^{(i-1)}}{D^{(i-1)}}} = \frac{\frac{N_\epsilon^{(i-1)} - N_\delta^{(i-1)}}{D^{(i-1)}}}{\frac{D^{(i-1)} + 2N_\delta^{(i-1)}}{D^{(i-1)}}} = \frac{N_\epsilon^{(i-1)} - N_\delta^{(i-1)}}{D^{(i-1)} + 2N_\delta^{(i-1)}} = \frac{N_\delta^{(i)}}{D^{(i)}},$$

$$\epsilon^{(i)} = \frac{\delta^{(i-1)}}{1 + 2\delta^{(i-1)}} = \frac{\frac{N_\delta^{(i-1)}}{D^{(i-1)}}}{1 + 2\frac{N_\delta^{(i-1)}}{D^{(i-1)}}} = \frac{\frac{N_\delta^{(i-1)}}{D^{(i-1)}}}{\frac{D^{(i-1)} + 2N_\delta^{(i-1)}}{D^{(i-1)}}} = \frac{N_\delta^{(i-1)}}{D^{(i-1)} + 2N_\delta^{(i-1)}} = \frac{N_\epsilon^{(i)}}{D^{(i)}}.$$

The update laws are therefore

$$N_\epsilon^{(i)} = N_\delta^{(i-1)}, \tag{3.16}$$

$$N_\delta^{(i)} = -N_\delta^{(i-1)} + N_\epsilon^{(i-1)} = -N_\delta^{(i-1)} + N_\delta^{(i-2)}, \tag{3.17}$$

$$D^{(i)} = D^{(i-1)} + 2N_\delta^{(i-1)}, \tag{3.18}$$

which can be used to compute the values shown in Table 3.5. Notice also that the condition $\delta^{(i)} < \epsilon^{(i)}$, necessary to avoid the convergence to consensus for one more step, can be written as $N_\delta^{(i)} < N_\epsilon^{(i)} = N_\delta^{(i-1)} \Rightarrow N_\delta^{(i)} - N_\delta^{(i-1)} < 0$, as the denominator is always positive.

We notice the appearance of Fibonacci coefficients, by induction we can write

step	$N_\delta^{(i)}$	$N_\epsilon^{(i)}$	$D^{(i)}$	condition for next step
$i = 0$	δ	ϵ	1	$\delta < \epsilon$
$i = 1$	$-\delta + \epsilon$	δ	$1 + 2\delta$	$2\delta > \epsilon$
$i = 2$	$2\delta - \epsilon$	$-\delta + \epsilon$	$1 + 2\epsilon$	$3\delta < 2\epsilon$
$i = 3$	$-3\delta + 2\epsilon$	$2\delta - \epsilon$	$1 + 4\delta$	$5\delta > 3\epsilon$
$i = 4$	$5\delta - 3\epsilon$	$-3\delta + 2\epsilon$	$1 - 2\delta + 4\epsilon$	$8\delta < 5\epsilon$
$i = 5$	$-8\delta + 5\epsilon$	$5\delta - 3\epsilon$	$1 + 8\delta - 2\epsilon$	$13\delta > 8\epsilon$
...	...	...	...	...

Table 3.5

$$N_\delta^{(i)} = F_{-i-1}\delta + F_{-i}\epsilon, \tag{3.19}$$

where F_i is the i-th term of the Fibonacci sequence, considering that the Fibonacci sequence, defined by $F_0 = 0, F_1 = 1$, $F_i = F_{i-1} + F_{i-2}$, can also be extended towards $-\infty$, by writing $F_{i-2} = -F_{i-1} + F_i$. This yields the sequence

$$\ldots 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13 \ldots$$

In fact, Expression (3.19) holds for the $i \geq 0$ results shown in Table 3.5, and

$$N_\delta^{(i)} = -N_\delta^{(i-1)} + N_\delta^{(i-2)} = -F_{-i}\delta - F_{-i+1}\epsilon + F_{-i+1}\delta + F_{-i+2}\epsilon =$$

$$= (-F_{-i} + F_{-i+1})\delta + (-F_{-i+1} + F_{-i+2})\epsilon =$$

$$= F_{-i-1}\delta + F_{-i}\epsilon.$$

Condition (3.15) can then be written, through a similar computation (since $N_\delta^{(i)} - N_\delta^{(i-1)} = -N_\delta^{(i+1)}$), as

$$N_\delta^{(i)} - N_\delta^{(i-1)} < 0 \implies -N_\delta^{(i+1)} = -F_{-i-2}\delta - F_{-i-1}\epsilon < 0 \implies -F_{-i-2}\delta < F_{-i-1}\epsilon.$$

Notice how $F_{-i} = -F_i$ for i even and $F_{-i} = F_i$ for i odd. Thanks to this fact we can write

$$\implies \begin{cases} F_{i+2}\delta < F_{i+1}\epsilon & \text{if } i \text{ even,} \\ -F_{i+2}\delta < -F_{i+1}\epsilon & \text{if } i \text{ odd.} \end{cases}$$

$$\implies \begin{cases} \frac{F_{i+2}}{F_{i+1}}\delta < \epsilon & \text{if } i \text{ even,} \\ \frac{F_{i+2}}{F_{i+1}}\delta > \epsilon & \text{if } i \text{ odd.} \end{cases}$$

Taking i even and combining conditions $\delta^{(i)} < \epsilon^{(i)}$ and $\delta^{(i+1)} < \epsilon^{(i+1)}$ we obtain

$$\frac{F_{i+2}}{F_{i+1}}\delta < \epsilon < \frac{F_{i+3}}{F_{i+2}}\delta.$$

The sequence $\frac{F_{i+2}}{F_{i+1}}$ converges to $\varphi = \frac{1+\sqrt{5}}{2}$ for $i \to +\infty$ (from below for i even and from above for i odd), thus we can conclude that the system goes through infinite switches if and only if $\epsilon = \varphi\delta$. Let us show that $\delta^{(i)}, \epsilon^{(i)} \to 0$ for $i \to +\infty$. Assume $\delta^{(i)} \to \overline{\delta}, \epsilon^{(i)} \to \overline{\epsilon}$. It must hold

$$\begin{cases} \overline{\delta} = \frac{\overline{\epsilon}-\overline{\delta}}{1+2\overline{\delta}} \\ \overline{\epsilon} = \frac{\overline{\delta}}{1+2\overline{\delta}} \end{cases} \implies \overline{\delta} = \frac{\frac{\overline{\delta}}{1+2\overline{\delta}} - \overline{\delta}}{1+2\overline{\delta}} \implies \overline{\delta} = \frac{\overline{\delta} - \overline{\delta} - 2\overline{\delta}^2}{(1+2\overline{\delta})^2} \implies$$

$$\overline{\delta}(1+2\overline{\delta})^2 + 2\overline{\delta}^2 = 0 \implies \overline{\delta}(1+4\overline{\delta}+4\overline{\delta}^2+2\overline{\delta}) = 0 \implies \overline{\delta} = 0$$

as the second factor is always strictly positive. Also $\overline{\epsilon} = 0$ by substitution.

Knowing that the solution asymptotically grows closer to $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, the last step necessary to complete the analysis is to understand if that happens for a finite or infinite time. Taking $T_i = \ln(1+2\delta^{(i)})$ and $\epsilon = \varphi\delta$, let us compute the time required to switch infinite times, i.e.

$$\sum_{i=1}^{\infty} T_i = \sum_{i=1}^{\infty} \ln(1+2\delta^{(i)}) \approx 2\sum_{i=1}^{\infty} \delta^{(i)} = 2\sum_{i=1}^{\infty} \frac{N_\delta^{(i)}}{D^{(i)}} < 2\sum_{i=1}^{\infty} N_\delta^{(i)}.$$

Expression (3.19), when $\epsilon = \varphi\delta$, becomes

$$N_\delta^{(i)} = F_{-i-1}\delta + F_{-i}\epsilon = F_{-i-1}\delta + F_{-i}\varphi\delta = \delta(F_{-i}\varphi + F_{-i-1}).$$

A property of φ states that $\varphi^n = F_n\varphi + F_{n-1}$, which can be used in the previous expression to obtain

$$N_\delta^{(i)} = \delta\varphi^{-i},$$

from which

$$\sum_{i=1}^{\infty} T_i < 2\sum_{i=1}^{\infty} N_\delta^{(i)} = 2\delta\sum_{i=1}^{\infty} \varphi^{-i} < +\infty.$$

What we've found is that $x^* = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is a Zeno point, i.e. a point for which $x(t) \to x^*$ for $t \to T^* < +\infty$, meaning that the solution converges in finite time, while the dynamic goes through infinite switches. If needed, the solution can be extended by defining it in $[T^*, +\infty)$ as the solution having $x(T^*) = x^*$ as initial condition. In Figures 3.15 and 3.16 simulation results are shown, that will be discusses in a dedicated section later.

Remark 15. The previous proof implies the convergence for all initial conditions for the directed cycle with 3 nodes. Indeed, we've seen that the only initial conditions that don't cause the solution to end up in a consensus cube after a finite number of switches converge to the Zeno point. The proof only took in consideration solutions crossing from a cube to the other through a face (i.e. only one variable equal to $\frac{1}{2}$ at the time). This is due to the fact that if a solution crosses through an edge or a vertex, it necessarily enters a

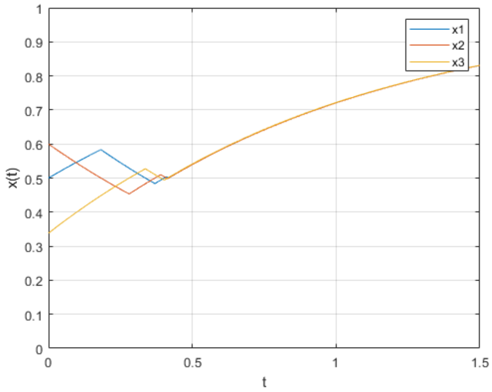

Figure 3.15: Plot of $x_1(t), x_2(t), x_3(t)$ on the directed cycle, when the Zeno point arises.

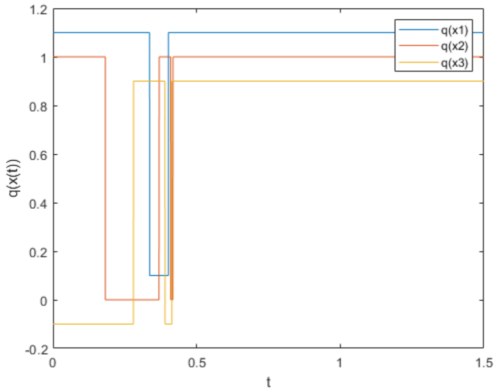

Figure 3.16: Plot of the quantized values (translated up and down to avoid overlap and grant better clarity). The vertical lines represent the switch of a quantized value, and consequently of the dynamic. Many switches appear in the correspondence of the Zeno point.

consensus cube right after, completing the convergence proof. In fact, in the first case the crossing point will be of the type $x_0 = (\frac{1}{2}, \frac{1}{2}, x)$, meaning that x_2 will move towards x, instantly copying their action. Consequently, also x_1, moving towards x_2, will copy x's action, as such we reach a consensus cube. Similarly, in the last case $x_0 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, once we arbitrarily decide the action of one of the nodes, the other two will instantly copy it, ending the proof.

Remark 16. As mentioned earlier, in [11] a cycle was found on the directed cycle graph with $n = 6$, starting from the initial conditions in $\overline{S_{(1,0,0,0,1,1)}}$

$$x_0 = (\frac{1}{2} + \delta, \frac{1}{2}, \frac{1}{2} - \beta, \frac{1}{2} - \delta, \frac{1}{2}, \frac{1}{2} + \beta),$$

where $\delta = \frac{1}{4}(\sqrt{5} - 1), \beta = \frac{1}{4}(3 - \sqrt{5})$.

We can rearrange the terms to show the dependence on φ. In fact,

$$\delta = \frac{1}{4}(\sqrt{5}-1) = -\frac{1-\sqrt{5}}{4} = -\frac{\psi}{2} = \frac{\varphi}{2} - \frac{1}{2},$$

$$\beta = \frac{1}{4}(3-\sqrt{5}) = \frac{1}{2} - \delta = 1 - \varphi,$$

which allows us to write

$$x_0 = (\frac{\varphi}{2}, \frac{1}{2}, \frac{\varphi}{2} - \frac{1}{2}, 1 - \frac{\varphi}{2}, \frac{1}{2}, \frac{3}{2} - \frac{\varphi}{2}).$$

We highlight the distance from the quantized value each node is looking at, i.e. the distance from $h = (0,0,0,1,1,1)$, and then factor out $\frac{1}{2}$ to show the presence of powers of φ.

$$\begin{aligned} x_0 &= (\frac{\varphi}{2}, \frac{1}{2}, \frac{\varphi}{2} - \frac{1}{2}, 1 - \frac{\varphi}{2}, \frac{1}{2}, \frac{3}{2} - \frac{\varphi}{2}) = \\ &= h + (\frac{\varphi}{2}, \frac{1}{2}, \frac{\varphi}{2} - \frac{1}{2}, -\frac{\varphi}{2}, -\frac{1}{2}, \frac{1}{2} - \frac{\varphi}{2}) = \\ &= h + \frac{1}{2}(\varphi, 1, \varphi - 1, -\varphi, -1, 1 - \varphi) = \\ &= h + \frac{1}{2}(\varphi^1, \varphi^0, \varphi^{-1}, -\varphi^1, -\varphi^0, -\varphi^{-1}). \end{aligned}$$

It is interesting to see how exponential solutions strictly depend on the golden ratio. Furthermore, simulations seem show that the cycle is attractive for initial conditions that keep that symmetry, i.e.

$$x_0' = h + \frac{1}{2}(a, b, c, -a, -b, -c),$$

even when dropping the restriction of binary actions. Some examples are shown in Figure 3.19.

To investigate the cause let us consider the cycle, and focus on x_2, x_3 and x_4, the variables quantized to 0, as by symmetry everything will also work for the remaining variables.

What happens in the cycle is that at a certain time T there is a switch in the values of the variables, i.e. $x_i(T) = x_{i+1}(0)$. In particular, for the considered variables, we have

$$x_2(T) = x_3(0), \quad x_3(T) = x_4(0), \quad x_4(T) = x_5(0) = \frac{1}{2} = x_2(0).$$

To have this happen in general we would need 3 points a, b, c, such that

$$x_2(T) = ae^{-T} = b, \quad x_3(T) = be^{-T} = c, \quad x_4(T) = (c - 2a)e^{-T} + 2a = a.$$

Since we want the cycle to revolve around $\frac{1}{2}$, it must be $a = \frac{1}{2}$, we can then compute

$$\frac{1}{2}e^{-T} = b \Longrightarrow -T = \ln(2b),$$

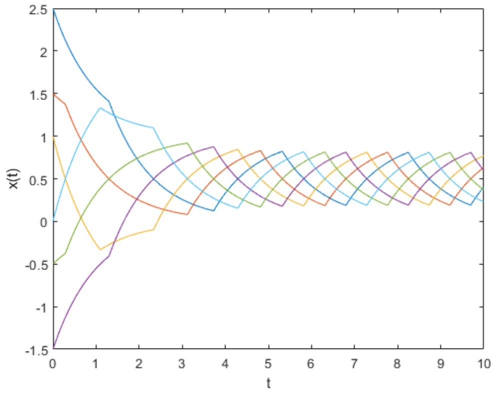

Figure 3.17: $x_0' = h + \frac{1}{2}(5{,}3{,}2, -5, -3, -2)$.

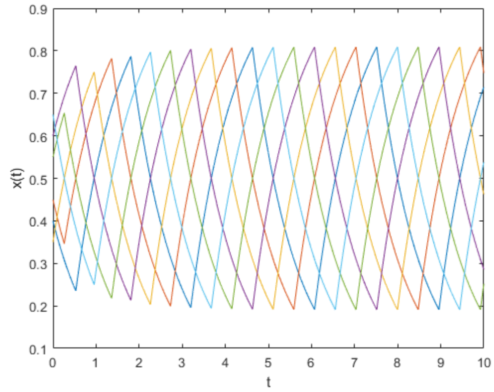

Figure 3.18: $x_0' = h + \frac{1}{2}(0.8{,}0.9{,}0.7, -0.8, -0.9, -0.7)$.

Figure 3.19: Some examples of solutions converging to the limit cycle.

$$be^{-T} = c \Longrightarrow c = 2b^2,$$

$$(c-1)e^{-T} = -\frac{1}{2} \Longrightarrow (2b^2 - 1) \cdot 2b + \frac{1}{2} = 0 \Longrightarrow 4b^3 - 2b + \frac{1}{2} = 0.$$

One solution of this equation is negative (see the plot in Figure 3.20) and thus unacceptable, while the other two are $b = \frac{1}{2}$, (trivial solution, all three points coincide so they "swap" places at time $T = 0$), and $b = \frac{\varphi^{-1}}{2}$, which yields the cycle. We can check this fact by substitution:

$$4\left(\frac{1}{2}\right)^3 - 2 \cdot \frac{1}{2} + \frac{1}{2} = 4 \cdot \frac{1}{8} - 1 + \frac{1}{2} = \frac{1}{2} - 1 + \frac{1}{2} = 0,$$

$$4\left(\frac{\varphi^{-1}}{2}\right)^3 - 2\cdot\frac{\varphi^{-1}}{2} + \frac{1}{2} = \frac{1}{2}\varphi^{-3} - \varphi^{-1} + \frac{1}{2} =$$
$$= \frac{1}{2}(2\varphi - 3) - (\varphi - 1) + \frac{1}{2} = \varphi - \frac{3}{2} - \varphi + 1 + \frac{1}{2} = 0.$$

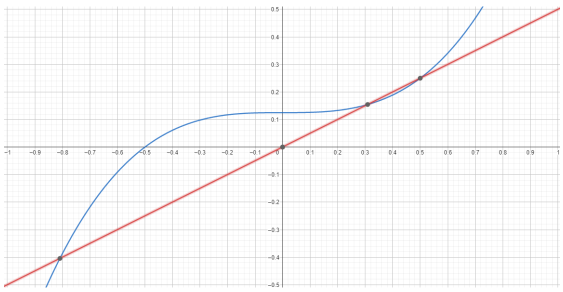

Figure 3.20: Solution of the equation $x^3 + \frac{1}{8} = \frac{1}{2}x$.

The importance of φ can also be seen through the following construction. Take the function $x(t) = e^t$, and the subset of the domain $\mathcal{A} \subset \mathbb{R}, \mathcal{A} = \{j \ln\varphi, j \in \mathbb{Z}\}$. We have $cod\mathcal{A} = \{\varphi^j, j \in \mathbb{Z}\}$, as
$$x(j\ln\varphi) = e^{j\ln\varphi} = e^{\ln\varphi^j} = \varphi^j.$$

What we see is that by taking a linear sequence of times a_i, we obtain a sequence of positions $x(a_i)$ for which $x(a_i) = x(a_{i-1}) + x(a_{i-2})$.

This is due to the property of φ

$$\varphi^n = \varphi^{n-1} + \varphi^{n-2},$$

that can be proven using the relation $\varphi^n = F_n\varphi + F_{n-1}$, as

$$\varphi^{n-1} + \varphi^{n-2} = F_{n-1}\varphi + F_{n-2} + F_{n-2}\varphi + F_{n-3} =$$
$$= (F_{n-1} + F_{n-2})\varphi + F_{n-2} + F_{n-3} = F_n\varphi + F_{n-1} = \varphi^n.$$

This implies that, if we consider the family of solutions of the Cauchy Problems

$$\begin{cases} \dot{x}_j(t) = x_j(t), \\ x_j(0) = \varphi^j, \end{cases} \tag{3.20}$$

with $j \in \mathbb{Z}$, we have $x_j(T) = x_{j+1}(0)$ for $T = \ln\varphi$, i.e. every solution takes the place of the following one. Notice that the solution of the Cauchy Problem for a generic j is $x_j(t) = \varphi^j e^t = e^{t+j\ln\varphi}$, so that studying the solutions is equivalent to studying the evolution of the sequence of the elements of $\mathcal{A}$, that all change in time according to the law $x(t) = e^t$.

Our aim was to find initial conditions from which one exponential solution would cover as much space as the other two combined in the same time. The same time requirement explains the linear sequence of the elements of $\mathcal{A}$, while φ appears to satisfy the other condition. In fact, we want

$$x(a_i) = x(a_{i-1}) + x(a_{i-2}),$$

but it also holds that

$$x(a_i) = x(a_{i-1}) + \int_{a_{i-1}}^{a_i} x(t)dt.$$

Consequently, imposing this condition means requiring $x(a_{i-2}) = \int_{a_{i-1}}^{a_i} x(t)dt$, which from a generic succession $b_i = i \ln b$, yields

$$x(b_{i-2}) = \int_{b_{i-1}}^{b_i} e^t dt \iff e^{b_{i-2}} = e^{b_i} - e^{b_{i-1}} \iff b^i - b^{i-1} = b^{i-2} \iff b^2 - b - 1 = 0$$
$$\implies b = \varphi$$

since $b > 0$.

In brief, φ seems to appear due to the presence in the cases considered of three variables, one of which must be the sum of the other two.

3.3 Comments on simulations and numerical issues

In the previous sections, some simulation results were shown. Simulating discontinuous equations is a delicate matter, however the numerical approach to discontinuous equations is not in the scope of this thesis. The simulations were run mostly in order to provide graphs to help show or explain the analytical results found, and in that matter the simulations have always been coherent.

The simulations in question were run on the software Matlab, using the Implicit Euler method. The time interval considered was $t \in [0,10]$ which allowed to show all the phenomena of interest, while the time step of the numeric method was set to be $dt = 10^{-4}$.

Regarding the method used, the update law used by Implicit Euler to solve the ordinary differential equation $\dot{x}(t) = f(x(t))$ is

$$x(k+1) = x(k) + dt \cdot f(x(k+1)),$$

which applied to Equation (3.2) becomes

$$x_i(k+1) = x_i(k) + dt \cdot \sum_{j=1}^{n} a_{ij}[q(x_j(k+1)) - x_i(k+1)],$$

where k is the index of the current time step.

Notice how this would require the use of $x_j(k+1)$, that is yet to be computed for $j > i$. For this reason, the method was coded to use the updated value $x_i(k+1)$ for x_i and the non-updated value $x_j(k)$ for x_j, resulting in the update law

$$x_i(k+1) = x_i(k) + dt \cdot \sum_{j=1}^{n} a_{ij}[q(x_j(k)) - x_i(k+1)].$$

The graphs in Figure 3.15 show the behaviour of the solution closing in on the Zeno point. However, due to the approximating nature of numerical methods, it is obviously impossible to see all the infinite switches of the quantized values, as they happen at time intervals of length decreasing to 0, which quickly become shorter than the time step chosen. This makes spotting the presence of the Zeno point impossible from simulations alone, so that we require an analytical study to prove its existence. However, its presence can become more evident by running the simulation with smaller time steps, as more switches become visible, as shown in Figure 3.23, where $q(x_1)$ and $q(x_3)$ can be seen to go through an additional switch when using the refined time step $dt = 10^{-7}$. The switches appear as vertical lines, as the variables stay in the quantized interval 0 for an extremely short time.

Another issue that can be caused by numerical approximation can arise when simulating the previously mentioned cycle, as that requires a perfect switch of the the initial conditions, thus small errors can add up in the long run, driving the cycle to break. This problem can be avoided by choosing an adequately small time step for the interval considered. In Figure 3.26 can be seen the comparison of the results when using two different value for the time step dt, 10^{-3} and 10^{-4}, when focusing on the interval $t \in [0{,}10]$.

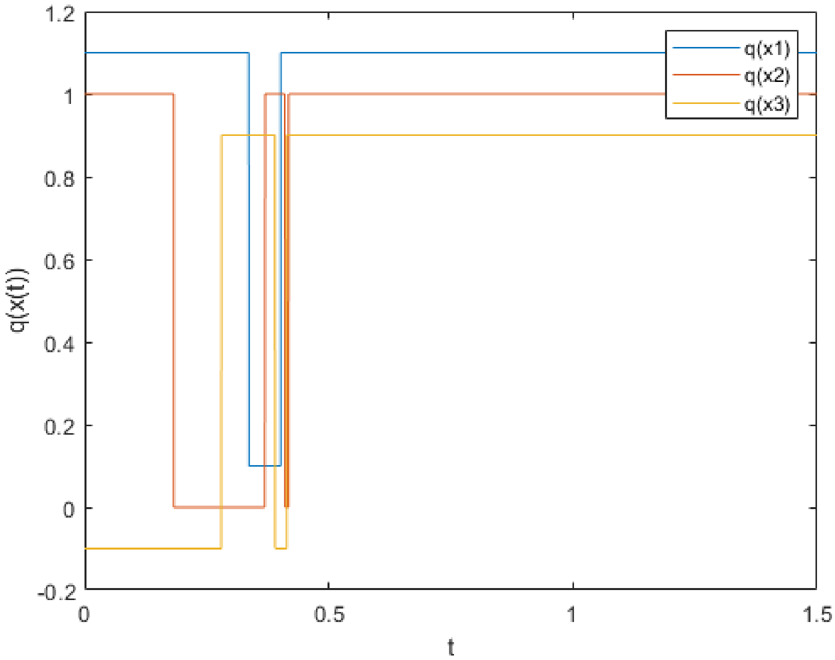

Figure 3.21: $dt = 10^{-4}$.

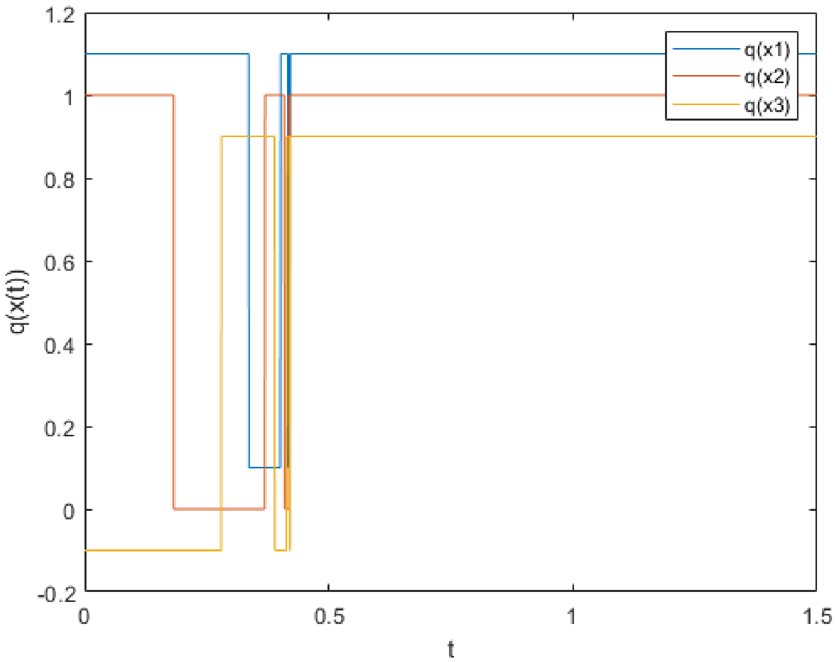

Figure 3.22: $dt = 10^{-7}$.

Figure 3.23: Quantized values of the variables around the Zeno point, for different time step sizes.

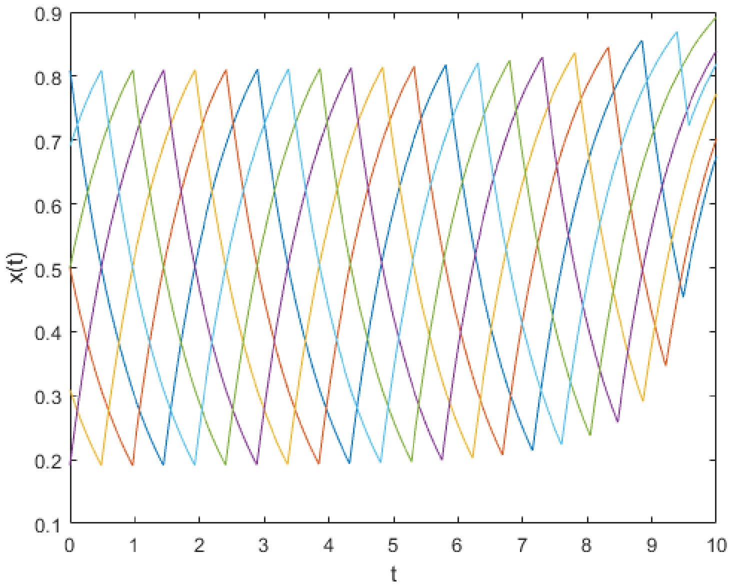

Figure 3.24: $dt = 10^{-3}$.

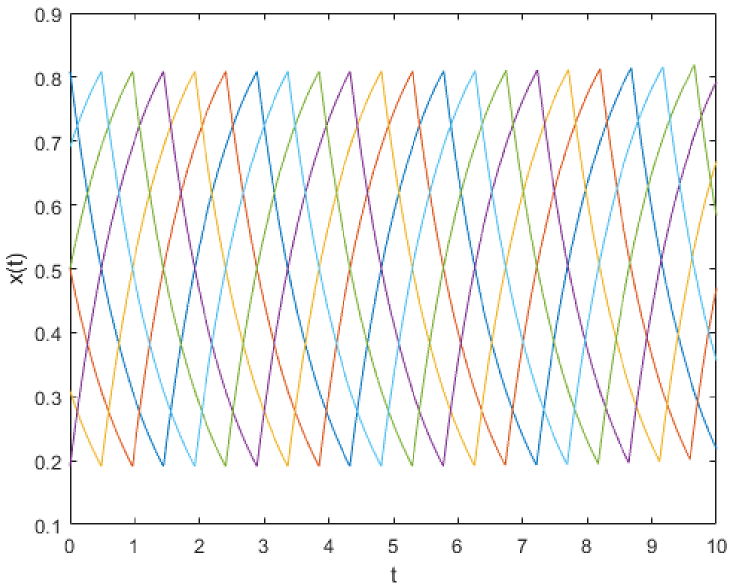

Figure 3.25: $dt = 10^{-4}$.

Figure 3.26: Comparison of the effect of numerical errors on the cycle for different time step sizes.

Chapter 4

Conclusion

In this thesis we have provided a variety of results for Carathéodory solutions of the model (3.2). Firstly, Proposition 2 states that every extended equilibrium, in the general case, attracts the entire hypercube it is defined on. Secondly, we expanded on the work on extended equilibria done in [1], by providing analytical expressions for extended equilibria on the line and ring graphs, as well as formula for the cardinality of the set of extended equilibria on the line graph.

After introducing the binary action hypothesis, we provided a proof of convergence for all initial conditions on the line graph, that trivially translates to the ring graph as well from Remark 13. Furthermore, we described an algorithm to find extended equilibria on tree graphs. Finally, we recalled from [11] the existence of a cyclic solution on a directed cycle graph with $n = 6$ nodes, and we tried to obtain a similar result for $n = 3$, which lead to the discovery of a Zeno point instead.

We now conclude the thesis by discussing some possible interpretations of the model in a social setting, and later listing the main problems still unanswered.

4.1 Interpretation

The model proposed shows a variety of phenomena. We've seen how disagreement is the norm, how clusters of adjacent nodes with the same action are the expected outcome, even how in some cases opinions keep changing without ever settling down.

Some other interesting considerations can be made. Consider, as an example, the extended equilibrium $x^* = (0, \frac{1}{2}, \frac{1}{2}, 1) \in \overline{S_{(0,0,1,1)}}$ on the line graph with 4 nodes. This example shows that a node's behaviour doesn't depend only on its opinion, but also on its past history. In fact, x_2 and x_3 share the same opinion, but behave differently. An interpretation is that these individuals influence each other to the point where they agree, but they both continue doing what they were doing before out of force of habit. This effect is further amplified in Example 6, in which all nodes agree but show different behaviours.

Another aspect worth highlighting is the influence of the degree of the nodes. We know that consensus is always achieved in the complete and complete bipartite graphs, both of which model networks with high level of connection. Indeed, in these cases the average degree of the nodes scales linearly with the number of nodes n, while on the line graph the maximum degree is 2 no matter the size of the graph. It is realistic to think this fact impairs the flow of information, supported by the wide variety of non-consensus attractive points the line graph presents.

The degree of the nodes is not the only factor, though, as the structure of the graph also matters. For instance, consider the following example.

Example 10. Take a complete graph K'_{50} at consensus 0, a complete graph K''_{50} at consensus 20, then create an undirected edge going from a node of K'_{50} to a node K''_{50}, in a one to one correspondence. The resulting graph is already in his final hypercube, as every node of K'_{50} converges towards the average of its neighbours' actions which is equal to $\frac{20}{50} < \frac{1}{2}$, hence it doesn't change action (the same can be said for K''_{50}).

We have created a graph that is in equilibrium with 2 sets of nodes that have drastically different opinions, and this difference can be arbitrarily increased when increasing the number n of nodes in the graph. Notice that each node has a considerable degree $d_i = \frac{n}{2}$, which is the same of the case of the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$, which instead always converges to consensus. This suggests that when the degree is high, non-consensus can still arise due to the lack of communication between different communities. This is realistic as modern day society is characterized by very high degrees of interaction, and yet disagreement is still the normality.

4.2 Open problems

The present work focused mainly on the study of convergence of the solutions, particularly on finding extended equilibria, as convergence to these points appear to be the most frequent outcome. The results presented were obtained on specific graphs, thus the main open problem is to find a generalisation to any graph, that is finding analytical expression for extended equilibria on a general graph, and determining conditions under which convergence is achieved. Notice that we have examples of non convergence of the solutions (limit cycle in the directed cycle graph with 6 nodes), but most cases seem to converge. It could be interesting to find large classes of graphs for which convergence is guaranteed, good candidates might for instance be undirected graphs and acyclic graphs.

It could also be interesting to investigate other graphs that might manifest cyclic behaviours, namely the directed cycle graph with a number of nodes different from 3 and 6. Finding examples on other kind of graphs would be useful to try and understand what condition allows cyclic solutions to exist.

Another problem is the translation of the results found for binary action to the general case $q(x_i) \in \mathbb{Z}$. The restriction to binary action drastically simplifies computations, but could be not necessary in many cases, likely for example for the convergence on the line discussed in section 3.2.1.

Finally, the topic of numerical simulations could be studied, to understand which of the multiple solutions is computed and in general to formally justify the reliability of the numerical results.

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