POLITECNICO DI TORINO

Master's Degree in Mathematical Engineering

Master's Degree Thesis

Geometric variational problems in the setting of sets of finite perimeter

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Summary

The theory of sets of finite perimeter provides, in the broader framework of Geometric Measure Theory, a particularly wellsuited framework for studying the existence, regularity, and structure of singularities of minimizers in those geometric variational problems in which surface area is minimized under a volume constraint. To this end, the class of sets of finite perimeter satisfy these requirements:

1. a class $\mathcal{F}$ of sets $\mathbb{E} \subset \mathbb{R}^n$ endowed with a topology with good compactness properties so that sets with smooth boundaries belong to this class and are dense;

2. a notion of *perimeter* $P(E)$ for every $E \in \mathcal{F}$ so that $E \mapsto P(E)$ is lower-semicontinuous on $\mathcal{F}$, and P extends the usual notion of perimeter $\mathcal{H}^{n-1}$ of the boundary ∂E; more precisely we require that $P(E) = \mathcal{H}^{n-1}(\partial E)$ for every set E with smooth boundary and for every $E \in \mathcal{F}$ there exists a sequence $E_h \to E$ with smooth boundaries and satisfying $\mathcal{H}^{n-1}(\partial E_h) \to P(E)$.

The methods and ideas introduced are applied to study the classical Plateau problem, which consists in finding surfaces with minimal area and prescribed boundary, and variational problems concerning confined liquid drops, briefly denoted as *capillarity* problems. The equilibrium shape of the liquid drop is given by the non-trivial interaction between the surface tension, which depends on the perimeter of the free surface of the drop inside the container, the contact surface between the drop and the container and the potential energy acting on the drop, for instance gravity.
A typical problem is the following: *find the domain $D \subset \mathbb{R}^3$ which minimizes*

$$\text{Area}(\partial D) + \underbrace{\int_D f(x)dx}_{\text{additional integral term}} + \underbrace{\text{additional constraint}}_{\text{e.g. Volume}(D)\text{ is prescribed}}.$$

Thus, as usually done in the calculus of variations, the semicontinuity and compactness method is used for proving the existence of minimizers. Geometric properties of the minimizers are deduced by performing first variations of minimizers; for instance, *Young's law* comes out naturally as a stationarity condition of the minimizers in the capillarity problems.

Acknowledgements

Scriveva Isaac Newton in una lettera:

"If I have seen further it is by standing on the shoulders of Giants."

Per me i giganti sono tutte le persone che mi hanno insegnato qualcosa nella vita. La mia famiglia mi ha insegnato i valori che mi guidano. I miei professori mi hanno insegnato e trasmesso il loro sapere. I miei amici mi hanno insegnato a sorridere insieme nel bene e nel male. Infine, la mia ragazza mi sta insegnando cosa è l'amore, quello che alla fine carica di significato tutte le cose. A loro va il ringraziamento.

"Mathematics, rightly viewed, possesses not only truth,
but supreme beauty—a beauty cold and austere, like that of sculpture."
Bertrand Russell

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Chapter 1

Radon measures

In this first chapter we introduce the basic results of measure theory about Radon measures. The important results we are going to use later are Riesz's representation theorem and the notion of weak-star convergence of Radon measures. The first permits to work on perimeters of sets as (vector-valued) Radon measures; the second, together with the compactness and the lower semicontinuity, permits to exploit the Direct method in variational problems involving functionals of measures.

1.1 Outer measures, Radon measures

Denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all subsets of $\mathbb{R}^n$. An **outer measure** on $\mathbb{R}^n$ is a set function on $\mathbb{R}^n$ with values in $[0,\infty]$, $\mu : \mathcal{P}(\mathbb{R}^n) \to [0,\infty]$, $\mu(\emptyset) = 0$ and with

$$E \subset \bigcup_{h\in\mathbb{N}} E_h \implies \mu(E) \le \sum_{h\in\mathbb{N}} \mu(E_h).$$

The last property, called **σ-subadditivity**, implies the **monotonicity** of μ,

$$E \subset F \implies \mu(E) \le \mu(F).$$

Example 1.1. The **Lebesgue measure** of a set $E \subset \mathbb{R}^n$ is defined as

$$\mathcal{L}^n(E) = |E| = \inf_{\mathcal{F}} \sum_{Q\in\mathcal{F}} r(Q)^n \tag{1.1}$$

where $\mathcal{F}$ is a countable covering of E by cubes with sides parallel to the coordinate axes, and $r(Q)$ denotes the side length of Q (the cubes Q are not assumed to be open, nor closed).

By Carathéodory's theorem, if μ is an outer measure on $\mathbb{R}^n$ then it becomes a

measure on the family of sets $\mathcal{M}(\mu)$ consisting of those sets $E \subset \mathbb{R}^n$ sucht that

$$\mu(F) = \mu(E \cap F) + \mu(F \backslash E), \quad \forall F \subset R^n,$$

namely μ is σ-additive on the σ-algebra $\mathcal{M}(\mu)$. The elements of $\mathcal{M}(\mu)$ are called **μ-measureable sets**.
An outer measure is called a **Borel measure** if $\mathcal{B}(\mathbb{R}^n) \subset \mathcal{M}(\mu)$, where with $\mathcal{B}(\mathbb{R}^n)$ we denote the family of Borel sets of $\mathbb{R}^n$, i.e. the smallest σ-algebra generated by the open sets of $\mathbb{R}^n$. The following theorem provides an useful characterization of Borel measures on $\mathbb{R}^n$.

Theorem 1.1. (Carathéodory's criterion) *If μ is an outer measure on $\mathbb{R}^n$, then μ is a Borel measure on $\mathbb{R}^n$ if and only if*

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$$

for every $E_1, E_2 \subset \mathbb{R}^n$ such that $\operatorname{dist}(E_1, E_2) > 0$.

Finally, we say that a Borel measure μ is **regular** if for every $F \subset \mathbb{R}^n$ there exists a Borel set E such that

$$F \subset E, \quad \mu(F) = \mu(E).$$

An outer measure μ on $\mathbb{R}^n$ is **locally finite** if $\mu(K) < \infty$ for every compact set $K \subset \mathbb{R}^n$.

Definition 1.1. An outer measure μ is a **Radon measure** on $\mathbb{R}^n$ if it is locally finite and Borel regular. By Borel regularity, for a Radon measure it also holds that

$$\mu(E) = \inf\{\mu(A) : E \subset A, A \textit{ open}\} \tag{1.2}$$
$$= \sup\{\mu(K) : K \subset E, K \textit{ compact}\}, \tag{1.3}$$

for every Borel set $E \subset \mathbb{R}^n$.

Thus, by Borel regularity, a Radon measure μ is characterized on $\mathcal{M}(\mu)$ by its behaviour on compact (or open) sets.
One way to obtain a Radon measure from a Borel regular measure μ on $\mathbb{R}^n$ is to restrict μ to a set $E \in \mathcal{M}(\mu)$ such that $\mu \llcorner E$ is locally finite. Thus the restriction $\mu \llcorner E$ is a Radon measure on $\mathbb{R}^n$, where $\mu \llcorner E$ is defined as

$$\mu \llcorner E(F) = \mu(E \cap F), \quad F \subset \mathbb{R}^n.$$

Proposition 1.1. *If $\{E_t\}_{t\in I}$ is a disjoint family of Borel sets in $\mathbb{R}^n$, indexed over some set I, and μ is a Radon measure on $\mathbb{R}^n$, then $\mu(E_t) > 0$ for at most countably many $t \in I$.*

Proof. If $I_k = \{t \in I : \mu(E_t \cap B_k) > k^{-1}\}$, then $\{t \in I : \mu(E_t) > 0\} = \bigcup_{k\in\mathbb{N}} I_k$. But I_k is finite, with $\#(I_k) \le k\mu(B_k)$: indeed, if $J \subset I_k$ is finite, then

$$\mu(B_k) \ge \mu\left(\bigcup_{t\in I} \mathbb{E}_t \cap B_k\right) \ge \mu\left(\bigcup_{t\in J} E_t \cap B_k\right) = \sum_{t\in J} \mu(E_t \cap B_k) \ge \frac{\#(J)}{k}.$$

□

1.2 Riesz's representation theorem for Radon measures

If μ is a Radon measure on $\mathbb{R}^n$, then the linear functional $L : C_c^0(\mathbb{R}^n) \to \mathbb{R}$,

$$\langle L, \varphi \rangle = \int_{\mathbb{R}^n} \varphi \, d\mu, \quad \varphi \in C_c^0(\mathbb{R}^n),$$

is positive ($\varphi \geq 0$ implies $\langle L, \varphi \rangle \geq 0$) or, equivalently, monotone ($\varphi_1 \leq \varphi_2$ implies $\langle L, \varphi_1 \rangle \leq \langle L, \varphi_2 \rangle$). As a consequence, L is continuous with respect to the following notion of convergence on $C_c^0(\mathbb{R}^n)$: $\varphi_h \to \varphi$ in $C_c^0(\mathbb{R}^n)$ if $\varphi_h \to \varphi$ uniformly on $\mathbb{R}^n$ and, for a compact set $K \subset \mathbb{R}^n$,

$$\operatorname{spt}(\varphi) \cup \bigcup_{h \in \mathbb{N}} \operatorname{spt}(\varphi_h) \subset K.$$

Indeed, $\varphi_h \to \varphi$ in $C_c^0(\mathbb{R}^n)$ implies $\langle L, \varphi_h \rangle \to \langle L, \varphi \rangle$, as we have

$$\sup \Big\{ \langle L, \varphi \rangle : \varphi \in C_c^0(\mathbb{R}^n), |\varphi| \leq M, \operatorname{spt}(\varphi) \subset K \Big\} \leq M\mu(K) < \infty,$$

for every compact set $K \subset \mathbb{R}^n$ and $M > 0$. In other words, once it is fixed a compact set K, the linear functional L is bounded, hence continuous on $C_c^0(\mathbb{R}^n)$. Therefore if L is integration with respect to a Radon measure μ on $\mathbb{R}^n$, then L is a linear bounded functional on $C_c^0(\mathbb{R}^n)$, with the additional property of being monotone.

Using this point of view, we want to introduce the important notion of vector-valued Radon measures. Indeed, we can consider a linear functional $L : C_c^0(\mathbb{R}^n; \mathbb{R}^m) \to \mathbb{R}$, which by linearity is continuous with respect to the convergence in $C_c^0(\mathbb{R}^n; \mathbb{R}^m)$ if and only if it is **bounded**, in the sense that, for every compact $K \subset \mathbb{R}^n$,

$$\sup \Big\{ \langle L, \varphi \rangle : \varphi \in C_c^0(\mathbb{R}^n; \mathbb{R}^m), |\varphi| \leq 1, \operatorname{spt}(\varphi) \subset K \Big\} < \infty. \tag{1.4}$$

We can construct one simple example.

Example 1.2. If μ is a Radon measure on $\mathbb{R}^n$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n, \mu; \mathbb{R}^m)$, we can define a bounded linear functional $f\mu : C_c^0(\mathbb{R}^n; \mathbb{R}^m) \to \mathbb{R}$ setting

$$\langle f\mu, \varphi \rangle = \int_{\mathbb{R}^n} (\varphi \cdot f) \, d\mu, \quad \varphi \in C_c^0(\mathbb{R}^n; \mathbb{R}^m).$$

Riesz's theorem ensures that conversely every bounded linear functional on $C_c^0(\mathbb{R}^n; \mathbb{R}^m)$ can be represented as a product $f\mu$. In particular, the Radon measure μ can be characterized in terms of L as follows. Define the **total variation** $|L|$ of a linear functional L on $C_c^0(\mathbb{R}^n;\mathbb{R}^m)$ as the set function $|L| : \mathcal{P}(\mathbb{R}^n) \to [0,\infty]$ such that, for every $A \subset \mathbb{R}^n$ open,

$$|L|(A) = \sup\left\{\langle L, \varphi\rangle : \varphi \in C_c^0(A;\mathbb{R}^m), |\varphi| \le 1\right\} \tag{1.5}$$

and, for $E \subset \mathbb{R}^n$ arbitrary,

$$|L|(E) = \inf\{|L|(A) : E \subset A, A \text{ is open}\} \tag{1.6}$$

Theorem 1.2. (Riesz's theorem) *If $L : C_c^0(\mathbb{R}^n;\mathbb{R}^m) \to \mathbb{R}$ is a bounded linear functional, then its total variation $|L|$ is a Radon measure on $\mathbb{R}^n$ and there exists a $|L|$-measurable function $g : \mathbb{R}^n \to \mathbb{R}^m$ with $|g| = 1$ $|L|$-a.e. on $\mathbb{R}^n$ and*

$$\langle L, \varphi\rangle = \int_{\mathbb{R}^n} (\varphi \cdot g)\, d|L|, \qquad \forall \varphi \in C_c^0(\mathbb{R}^n;\mathbb{R}^m), \tag{1.7}$$

that is, $L = g|L|$. Moreover, for every open set $A \subset \mathbb{R}^n$,

$$|L|(A) = \sup\left\{\int_{\mathbb{R}^n} (\varphi \cdot g)\, d|L| : \varphi \in C_c^0(A;\mathbb{R}^m), |\varphi| \le 1\right\}. \tag{1.8}$$

Remark 1.1. When $L = f\mu$ as in Example 1.2, then the total variation $|f\mu|$ and the vector field g in the statement of Riesz's theorem satisfy

$$|f\mu| = |f|\mu, \quad g = \frac{f}{|f|} \ \ |f|\ \mu\text{-a.e. on } \mathbb{R}^n.$$

Remark 1.2. (Bounded linear functionals and vector-valued set functions) Let $\mathcal{B}_b(\mathbb{R}^n)$ denote the family of bounded Borel sets of $\mathbb{R}^n$, and $\mathcal{B}(E)$ the family of Borel sets contained in $E \subset \mathbb{R}^n$. If L is a bounded linear functional on $C_c^0(\mathbb{R}^n;\mathbb{R}^m)$, then L induces a $\mathbb{R}^m$- valued set function $\nu : \mathcal{B}_b(\mathbb{R}^n) \to \mathbb{R}^m$,

$$\nu(E) = \int_E g d|L|, \quad E \in \mathcal{B}_b(\mathbb{R}^n), \tag{1.9}$$

that enjoys the σ-additivity property

$$\nu\left(\bigcup_{h\in\mathbb{N}} E_h\right) = \sum_{h\in\mathbb{N}} \nu(E_h)$$

on every disjoint sequence $\{E_h\}_{h\in\mathbb{N}} \subset \mathcal{B}(K)$, for some compact set $K \subset \mathbb{R}^n$. Thus, *bounded linear functionals on $C_c^0(\mathbb{R}^n;\mathbb{R}^m)$ naturally induce $\mathbb{R}^m$-valued set functions on $\mathbb{R}^m$ that are σ-additive on bounded Borel sets.*
Taking into account this, we define $\mathbb{R}^m$-**valued Radon measures on** $\mathbb{R}^n$ as the

bounded linear functionals on $C_c^0(\mathbb{R}^n;\mathbb{R}^m)$. Consequently, denoting μ as an arbitrary vector-valued Radon measure (instead of L), we set

$$\langle \mu, \varphi \rangle = \int_{\mathbb{R}^n} \varphi \cdot d\mu \tag{1.10}$$

to denote the value of μ at $\varphi \in C_c^0(\mathbb{R}^n;\mathbb{R}^m)$.

We now proceed to prove Riesz's theorem. We first need a few lemmas.

Lemma 1.1. *If L is a bounded linear functional on $C_c^0(\mathbb{R}^n;\mathbb{R}^m)$, then its total variation $|L|$ is a Radon measure on $\mathbb{R}^n$.*

Proof. Step one: We prove that $|L|$ is an outer measure. Let us first show that

$$|L|(A) \le \sum_{h\in\mathbb{N}} |L|(A_h), \tag{1.11}$$

for $A = \cup_{h\in\mathbb{N}} A_h$, A_h open. Indeed, let $\varphi \in C_c^0(A;\mathbb{R}^m)$ with $|\varphi| \le 1$. Since $\mathrm{spt}(\varphi) \subset A$ is compact, there exists $N \in \mathbb{N}$ such that $\mathrm{spt}(\varphi) \subset \cup_{h=1}^N A_h$. We consider the corresponding partition of unity, that is

$$\varphi_h \in C_c^0(A_h), \quad 0 \le \varphi_h \le 1, \quad \sum_{h=1}^N \varphi_h = 1 \text{ on } \mathrm{spt}(\varphi).$$

Since $\varphi = \sum_{h=1}^N \varphi\varphi_h$ and $\varphi\varphi_h \in C_c^0(A;\mathbb{R}^m)$ with $|\varphi\varphi_h| \le 1$, we have (by definition 1.5)

$$\langle L, \varphi \rangle = \sum_{h=1}^N \langle L, \varphi\varphi_h \rangle \le \sum_{h=1}^N |L|(A_h) \le \sum_{h\in\mathbb{N}} |L|(A_h),$$

and taking the supremum over all admissible φ on the left hand side we find (1.11). We now consider $E \subset \cup_{h\in\mathbb{N}} E_h$, and prove that

$$|L|(E) \le \sum_{h\in\mathbb{N}} |L|(E_h).$$

Given $\varepsilon > 0$ and $h \in \mathbb{N}$, by definition of $|L|$ we find A_h open with $E_h \subset A_h$ and $|L|(A_h) \le |L|(E_h) + \varepsilon/2^h$. Hence, by 1.11

$$|L|(E) \le |L|\left(\bigcup_{h\in\mathbb{N}} A_h\right) \le \sum_{h\in\mathbb{N}} |L|(A_h) \le \sum_{h\in\mathbb{N}} |L|(E_h) + \varepsilon.$$

Step two: By Theorem 1.1, $|L|$ is a Borel measure if $\mathrm{dist}(E_1, E_2) > 0$ implies

$$|L|(E_1 \cup E_2) \ge |L|(E_1) + |L|(E_2).$$

When E_1, E_2 are open, it follows from the definition of $|L|$. In the general case, since $0 < \mathrm{dist}(E_1, E_2) = \mathrm{dist}(\overline{E_1}, \overline{E_2})$, there exist open sets A_1, A_2 such that $\overline{E_j} \subset A_j$ and $\mathrm{dist}(A_1, A_2) > 0$. If A is open and $E_1 \cup E_2 \subset A$, then $\mathrm{dist}(A_1 \cap A, A_2 \cap A) > 0$ and $E_j \subset A_j \cap A$, so that the inequality above on open sets implies

$$|L|(A) \geq |L|\left((A_1 \cap A) \cup (A_2 \cap A)\right) \geq |L|(A_1 \cap A) + |L|(A_2 \cap A) \geq |L|(E_1) + |L|(E_2).$$

As A is arbitrary, taking the infimum the result follows for generic E. Hence $|L|$ is a Borel measure, locally finite thanks to

$$\sup\Big\{\langle L, \varphi\rangle : \varphi \in C^0_c(\mathbb{R}^n; \mathbb{R}^m), |\varphi| \leq 1, \mathrm{spt}(\varphi) \subset K\Big\} < \infty.$$

Finally, $|L|$ is a Borel regular measure (thus a Radon measure), since, if $E \subset \mathbb{R}^n$, $|L|(E) < \infty$ and $\{A_h\}_{h\in\mathbb{N}}$ are open sets with $E \subset A_h$ and $|L|(A_h) \to |L|(E)$, then $F = \cap_{h\in\mathbb{N}} A_h$ is a Borel set with $E \subset F$ and $|L|(E) = |L|(F)$. □

By the elementary Riesz's representation theorem on Hilbert spaces, if μ is a Radon measure on $\mathbb{R}^n$, and $L : L^2(\mathbb{R}^n, \mu) \to \mathbb{R}$ is a linear functional with

$$\sup\Big\{\langle L, u\rangle : u \in L^2(\mathbb{R}^n, \mu), ||u||_{L^2(\mathbb{R}^n,\mu)} = 1\Big\} = C < \infty$$

then there exists $v \in L^2(\mathbb{R}^n, \mu)$ such that $||v||_{L^2(\mathbb{R}^n,\mu)} = C$ and

$$\langle L, u\rangle = \int_{\mathbb{R}^n} uv \; d\mu, \qquad \forall u \in L^2(\mathbb{R}^n, \mu).$$

Bounded linear functionals on $L^1(\mathbb{R}^n, \mu)$ are then addressed as follows.

Lemma 1.2. (Riesz's representation theorem in L^1) *If μ is a Radon measure on $\mathbb{R}^n$ and $L : L^1(\mathbb{R}^n, \mu) \to \mathbb{R}$ is a linear functional such that*

$$\sup\Big\{\langle L, u\rangle : u \in L^1(\mathbb{R}^n, \mu), ||u||_{L^1(\mathbb{R}^n,\mu)} = 1\Big\} = C < \infty, \tag{1.12}$$

then there exists a function $v \in L^\infty(\mathbb{R}^n, \mu)$ with $||v||_{L^\infty(\mathbb{R}^n,\mu)} = C$ and

$$\langle L, u\rangle = \int_{\mathbb{R}^n} uv \; d\mu, \qquad \forall u \in L^1(\mathbb{R}^n, \mu). \tag{1.13}$$

Proof. Setting $E_h = B_{h+1} \backslash \overline{B_h}$, $h \in \mathbb{N}$, let $\{t_h\}_{h\in\mathbb{N}} \subset (0, \infty)$ be such that $w = \sum_{h\in\mathbb{N}} t_h 1_{E_h} \in L^2(\mathbb{R}^n, \mu)$. The linear functional $L_0 : L^2(\mathbb{R}^n, \mu) \to \mathbb{R}$ defined as

$$\langle L_0, u\rangle = \langle L, wu\rangle, \qquad u \in L^2(\mathbb{R}^n, \mu),$$

is continuous on $L^2(\mathbb{R}^n, \mu)$, since wu is in $L^1(\mathbb{R}^n, \mu)$ by Cauchy-Schwarz,

$$|\langle L_0, u\rangle| = |\langle L, wu\rangle| \leq C||wu||_{L^1(\mathbb{R}^n,\mu)} \leq C||w||_{L^2(\mathbb{R}^n,\mu)}||u||_{L^2(\mathbb{R}^n,\mu)},$$

with norm bounded by $C||w||_{L^2(\mathbb{R}^n,\mu)}$. By Riesz's representation theorem on $L^2(\mathbb{R}^n,\mu)$, there exists $z \in L^2(\mathbb{R}^n,\mu)$ such that

$$\langle L, wu\rangle = \int_{\mathbb{R}^n} uz \, d\mu, \qquad \forall u \in L^2(\mathbb{R}^n,\mu). \tag{1.14}$$

Since $w > 0$ on $\mathbb{R}^n$. the μ-measurable function $v = z/w$ has the required properties. Indeed, as w is uniformly positive on compact sets, if $u \in C^0_c(\mathbb{R}^n)$, then $u/w \in L^2(\mathbb{R}^n,\mu)$. By (1.14) we thus find

$$\langle L, u\rangle = \int_{\mathbb{R}^n} uv \, d\mu, \qquad \forall u \in C^0_c(\mathbb{R}^n). \tag{1.15}$$

To show that $v \in L^\infty(\mathbb{R}^n,\mu)$ with $||v||_{L^\infty(\mathbb{R}^n,\mu)} \le C$, assume on the contrary that

$$\mu\left(\{x \in \mathbb{R}^n : |v(x)| > C\}\right) > 0,$$

so that $|v| > C$ on a Borel set F with $0 < \mu(F) < \infty$. Testing (1.14) with

$$u_0 = 1_{F\cap\{v>C\}} - 1_{F\cap\{v<-C\}} \in L^2(\mathbb{R}^n,\mu),$$

we would then find the following contradiction

$$C\int_F w < \int_F |z| = \int_F u_0 z = L(wu_0) \le C\int_{\mathbb{R}^n} w|u_0| = C\int_F w.$$

Since $v \in L^\infty(\mathbb{R}^n,\mu)$, (1.15) defines a continuous functional on $L^1(\mathbb{R}^n,\mu)$. Since L is continuous on $L^1(\mathbb{R}^n,\mu)$, by density of $C^0_c(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n,\mu)$, we deduce (1.13) from (1.15). Finally, $||v||_{L^\infty(\mathbb{R}^n,\mu)} < C$ would contradict (1.15) and (1.12). □

Now we can finally prove Riesz's theorem.

Proof. (of Riesz's theorem 1.2) By Lemma 1.1, $|L|$ is a Radon measure on $\mathbb{R}^n$. Let us now define a functional $M : C^0_c(\mathbb{R}^n;[0,\infty)) \to [0,\infty)$ as

$$\langle M, \varphi\rangle = \sup\left\{\langle L, \psi\rangle : \psi \in C^0_c(\mathbb{R}^n;\mathbb{R}^m), |\psi| \le \varphi\right\}, \quad \varphi \in C^0_c(\mathbb{R}^n;[0,\infty)).$$

In step one we show that M is additive, positively homogeneous of degree one, and monotone on $C^0_c(\mathbb{R}^n;[0,\infty))$. In step two, we show the inequality

$$\langle M, \varphi\rangle \le \int_{R^n} \varphi d|L|, \qquad \forall \varphi \in C^0_c(\mathbb{R}^n;[0,\infty)). \tag{1.16}$$

Finally, in step three, we combine (1.16) with Riesz's representation theorem in $L^1(\mathbb{R}^n, |L|)$ in order to conclude the proof.
Step one: We show that, whenever $\varphi_1, \varphi_2 \in C^0_c(\mathbb{R}^n; [0, \infty))$ and $c \geq 0$, we have

$$\begin{aligned} \langle M, \varphi_1 + \varphi_2 \rangle &= \langle M, \varphi_1 \rangle + \langle M, \varphi_2 \rangle, \\ \langle M, c\varphi_1 \rangle &= c\langle M, \varphi_1 \rangle, \\ \langle M, \varphi_1 \rangle &\leq \langle M, \varphi_2 \rangle, \qquad \text{if } \varphi_1 \leq \varphi_2. \end{aligned}$$

The second and the third are easily proved, as well as the inequality $\geq$ in the first (just noting that for any $|\psi_1| \leq \varphi_1$ and $|\psi_2| \leq \varphi_2$, then $|\psi_1 + \psi_2| \leq \varphi_1 + \varphi_2$)). Now let $\psi \in C^0_c(\mathbb{R}^n; \mathbb{R}^m)$ be such that $|\psi| \leq \varphi_1 + \varphi_2$, and set

$$\psi_h = \frac{\varphi_h}{\varphi_1 + \varphi_2}\psi \qquad \text{on } \{\varphi_1 + \varphi_2 > 0\}, \qquad \psi_h = 0 \text{ elsewhere,}$$

for $h = 1{,}2$. Since $\psi_h \in C^0_c(\mathbb{R}^n; \mathbb{R}^m)$ with $|\psi_h| \leq \varphi_h$ and $\psi = \psi_1 + \psi_2$,

$$\langle L, \psi \rangle = \langle L, \psi_1 \rangle + \langle L, \psi_2 \rangle \leq \langle M, \varphi_1 \rangle + \langle M, \varphi_2 \rangle,$$

and complete the proof by arbitrariness of ψ.
Step two: Given $\varphi \in C^0_c(\mathbb{R}^n; [0, \infty))$ and $\varepsilon > 0$, let $\{t_h\}_{h=0}^N \subset \mathbb{R}$ such that

$$t_0 < 0 < t_1 < \ldots < t_{N-1} < \sup_{\mathbb{R}^n} \varphi < t_N, \qquad t_{h+1} - t_h \leq \varepsilon$$

and consider the partition $\{E_h\}_{h=1}^N$ of $\mathrm{spt}(\varphi)$ by disjoint Borel sets, defined as

$$E_h = \{x \in \mathrm{spt}(\varphi) : t_{h-1} < \varphi(x) \leq t_h\}, \qquad 1 \leq h \leq N.$$

Since $|L|$ is a Radon measure, there exist open sets A_h with $E_h \subset A_h$ and

$$|L|(A_h) \leq |L|(E_h) + \frac{\varepsilon}{N}, \qquad 1 \leq h \leq N.$$

If necessary replacing A_h with the open set $\{x \in A_h : \varphi(x) < t_h + \epsilon\}$ (intersect A_h with the set $\{\varphi < t_h + \varepsilon\}$, which is open by continuity of φ), we can also assume

$$\varphi < t_h + \varepsilon \quad \text{on } A_h.$$

Finally, let $\{\xi_h\}_{h=1}^N$ be a partition of unity subordinated to the open covering $\{A_h\}_{h=1}^N$ of the compact set $\mathrm{spt}(\varphi)$, namely $\xi_h \in C^0_c(A_h), 0 \leq \xi_h \leq 1$, and $\sum_{h=1}^N \xi_h = 1$ on spt (φ). Since $\varphi = \sum_{h=1}^N \xi_h \varphi$, by step one and $\varphi < t_h + \varepsilon$ on A_h, we find that

$$\langle M, \varphi \rangle = \sum_{h=1}^N \langle M, \xi_h \varphi \rangle \leq \sum_{h=1}^N (t_h + \varepsilon) \langle M, \xi_h \rangle.$$

If $\psi \in C_c^0(\mathbb{R}^n;\mathbb{R}^m)$ and $|\psi| \le \xi_h$, then $\mathrm{spt}(\psi) \subset A_h$ and $|\psi| \le 1$. Hence $\langle M, \xi_h \rangle \le |L|(A_h)$ and, by the inequality on $|L|(A_h)$ of above, we find that

$$\begin{aligned}
\langle M, \varphi \rangle &\le \sum_{h=1}^{N} (t_h + \varepsilon) \left(|L|(E_h) + \frac{\varepsilon}{N} \right) \\
\text{(by } t_h \le t_{h-1} + \varepsilon\text{)} \qquad &\le \sum_{h=1}^{N} (t_{h-1} + 2\varepsilon) \left(|L|(E_h) + \frac{\varepsilon}{N} \right) \\
\text{(by } t_{h-1} \le \varphi \text{ on } E_h\text{)} \qquad &\le \int_{\mathbb{R}^n} \varphi d|L| + t_N \varepsilon + 2\varepsilon |L|(\mathrm{spt}(\varphi)) + 2\varepsilon^2 \\
\text{(by } t_N \le \sup_{\mathbb{R}^n} \varphi + \varepsilon\text{)} \qquad &\le \int_{\mathbb{R}^n} \varphi d|L| + \varepsilon \left(\sup_{\mathbb{R}^n} \varphi + \varepsilon + 2|L|(\mathrm{spt}(\varphi)) + 2\varepsilon \right).
\end{aligned}$$

Let $\varepsilon \to 0^+$ to prove (1.16).

Step three: Given $e \in S^{m-1}$, we define $L_e : C_c^0(\mathbb{R}^n) \to \mathbb{R}$ by

$$\langle L_e, \varphi \rangle = \langle L, \varphi e \rangle, \qquad \varphi \in C_c^0(\mathbb{R}^n).$$

By (1.16), we find that, for every $\varphi \in C_c^0(\mathbb{R}^n)$ (using $|\varphi e| \le |\varphi|$),

$$\langle L_e, \varphi \rangle \le \sup \left\{ \langle L, \psi \rangle : \psi \in C_c^0(\mathbb{R}^n;\mathbb{R}^m), |\psi| \le |\varphi| \right\} = \langle M, |\varphi| \rangle \le \int_{\mathbb{R}^n} |\varphi| d|L|.$$

By density of $C_c^0(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n, |L|)$ we can extend L_e as a linear functional on $L^1(\mathbb{R}^n, |L|)$ such that $|\langle L_e, u \rangle| \le \int_{\mathbb{R}^n} |u| d|L|$. Thus, by Lemma 1.2, there exists $g_e \in L^\infty(\mathbb{R}^n, |L|)$ such that

$$\langle L, ue \rangle = \int_{\mathbb{R}^n} u g_e d|L|, \qquad \forall u \in L^1(\mathbb{R}^n, |L|).$$

If we set $g : \mathbb{R}^n \to \mathbb{R}^m$, $g^{(i)} = g_{e_i}$, then g is bounded and $|L|$-measurable, with

$$\langle L, \varphi \rangle = \sum_{h=1}^{m} \langle L_{e_i}, \varphi \cdot e_i \rangle = \sum_{h=1}^{m} \int_{\mathbb{R}^n} (\varphi \cdot e_i)\, g^{(i)} d|L| = \int_{\mathbb{R}^n} (\varphi \cdot g)\, d|L|,$$

for every $\varphi \in C_c^0(\mathbb{R}^n;\mathbb{R}^m)$. Moreover, $|g(x)| = 1$ for $|L|$-a.e. $x \in \mathbb{R}^n$. Indeed,

$$|L|(A) = \sup \left\{ \int_{\mathbb{R}^n} (\varphi \cdot g)\, d|L| : \varphi \in C_c^0(A;\mathbb{R}^m), |\varphi| \le 1 \right\} \tag{1.17}$$

for every open set $A \subset \mathbb{R}^n$. By (1.17), $|L|(A) \le \int_A |g| d|L|$ for every bounded open set A. Hence, $|g| > 0$ $|L|$-a.e. on $\mathbb{R}^n$ and $1_{\{|g|>0\}}\,(g/|g|) \in L^1(A, |L|;\mathbb{R}^m)$. By density there exists $\{\varphi_h\}_{h \in N} \subset C_c^0(A;\mathbb{R}^m)$ such that $|\varphi_h| \le 1$ and $\varphi_h \to 1_{\{|g|>0\}}\,(g/|g|)$ in $L^1(A, |L|;\mathbb{R}^m)$. Thus, $\varphi_h \cdot g \to |g|$ in $L^1(A, |L|)$, and

$$|L|(A) \ge \int_{\mathbb{R}^n} (\varphi_h \cdot g)\, d|L| \to \int_A |g| d|L| \ge |L|(A)$$

on every open set $A \subset \mathbb{R}^n$. Hence, $|g(x)| = 1$ for $|L|$-a.e. $x \in \mathbb{R}^n$. □

1.3 Weak-star convergence and compactness

We established the correspondence between Radon measures on $\mathbb{R}^n$ with values in $\mathbb{R}^m$ and continuous linear functionals on $C^0_c(\mathbb{R}^n;\mathbb{R}^m)$. Thus we can endowe the space of Radon measures with the usual weak-star convergence of functionals (as elements of a dual space).

Definition 1.2. Let $\{\mu_h\}_{h\in\mathbb{N}}$ and μ be Radon measures on $\mathbb{R}^n$ with values in $\mathbb{R}^m$. We say that μ_h **weak-star converges** to μ, $\mu_h \overset{*}{\rightharpoonup} \mu$, if

$$\int_{\mathbb{R}^n} \varphi \cdot d\mu = \lim_{h\to\infty} \int_{\mathbb{R}^n} \varphi \cdot d\mu_h, \qquad \forall \varphi \in C^0_c(\mathbb{R}^n;\mathbb{R}^m).$$

Example 1.3. (Concentration of mass) The "n-dimensional" measure $\mu_h = h^n \mathcal{L}^n \llcorner (0,h^{-1})^n$ weak-star converges to the "zero-dimensional" measure $\mu = \delta_0$,

$$\int_{\mathbb{R}^n} \varphi \, d\mu_h = h^n \int_{(0,1/h)^n} \varphi(x)dx \to \varphi(0) = \int_{\mathbb{R}^n} \varphi \, d\mu, \qquad \forall \varphi \in C^0_c(\mathbb{R}^n).$$

Example 1.4. (Spreading of mass) An increasingly diffused lower dimensional distribution of mass may weak-star converge to a "higher-dimensional" measure. If we set $\mu_h = \sum_{h=1}^k h^{-1}\delta_{k/h}$, then $\mu_h \overset{*}{\rightharpoonup} \mathcal{L}^1 \llcorner (0,1)$, as

$$\int_{\mathbb{R}} \varphi \, d\mu_h = \sum_{h=1}^{k} \frac{\varphi(k/h)}{h} \to \int_{(0,1)} \varphi(x)dx, \qquad \forall \varphi \in C^0_c(\mathbb{R}).$$

Example 1.5. (Tangent space to a smooth curve) A fundamental idea in Geometric Measure Theory is formulating the existence of tangent spaces in terms of weak-star convergence of Radon measures. Let Γ be a smooth curve in $\mathbb{R}^n$, that is $\Gamma = \gamma((a,b))$ for $\gamma : (a,b) \to \mathbb{R}^n$ smooth and injective. Given $t_0 \in (a,b)$, the tangent space to Γ at $x_0 = \gamma(t_0)$ is the line $\pi = \{s\gamma'(t_0) : s \in \mathbb{R}\}$. Consider now Γ as a Radon measure, looking at $\mu = \mathcal{H}^1 \llcorner \Gamma$, and define the **blow-ups** $\mu_{x_0,r}$ of μ at x_0, setting

$$\mu_{x_0,r} = \frac{1}{r} \left(\Phi_{x_0,r}\right)_{\#} \left(\mathcal{H}^1 \llcorner \Gamma\right) = \mathcal{H}^1 \llcorner \left(\frac{\Gamma - x_0}{r}\right),$$

where $\Phi_{x_0,r}(y) = (y - x_0)/r, y \in \mathbb{R}^n$. The fact that π is the tangent space to Γ at x_0 implies that $\mu_{x_0,r} \overset{*}{\rightharpoonup} \mathcal{H}^1 \llcorner \pi$ as $r \to 0^+$. Indeed, if $\varphi \in C^0_c(\mathbb{R}^n)$, then by definition of push-forward of measures we find that

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi \, d\mu_{x_0,r} &= \frac{1}{r} \int_{\Gamma} \varphi \left(\frac{y - x_0}{r}\right) d\mathcal{H}^1(y) = \frac{1}{r} \int_a^b \varphi \left(\frac{\gamma(t) - \gamma(t_0)}{r}\right) |\gamma'(t)| \, dt \\
&= \frac{1}{r} \int_{-(t_0-a)/r}^{(b-t_0)/r} \varphi \left(\frac{\gamma(t_0 + rs) - \gamma(t_0)}{r}\right) |\gamma'(t_0 + rs)| \, ds \\
&\to \int_{\mathbb{R}} \varphi \left(s\gamma'(t_0)\right) |\gamma'(t_0)| ds = \int_{\pi} \varphi \, d\mathcal{H}^1, \quad \text{as } r \to 0^+.
\end{aligned}$$

Proposition 1.2. *If $\{\mu_h\}_{h\in\mathbb{N}}$ and μ are Radon measures on $\mathbb{R}^n$, then the following three statements are equivalent.*

(i) $\mu_h \overset{*}{\rightharpoonup} \mu$.

(ii) If K is compact and A is open, then

$$\mu(K) \geq \limsup_{h\to\infty} \mu_h(K)$$
$$\mu(A) \leq \liminf_{h\to\infty} \mu_h(A).$$

(iii) If E is a bounded Borel set with $\mu(\partial E) = 0$, then

$$\mu(E) = \lim_{h\to\infty} \mu_h(E).$$

Moreover, if $\mu_h \overset{*}{\rightharpoonup} \mu$, then for every $x \in \operatorname{spt}(\mu)$, there exists $\{x_h\}_{h\in\mathbb{N}} \subset \mathbb{R}^n$ with $\lim_{h\to\infty} x_h = x$ and $x_h \in \operatorname{spt}(\mu_h)$, $\forall h \in \mathbb{N}$.

One important feature of weak-star convergence is the **weak-star lower semicontinuity of the total variation** of a vector-valued Radon measure. We recall that

$$|\mu|(A) = \sup\left\{\int_{\mathbb{R}^n} \varphi \cdot d\mu : \varphi \in C_c^0(A;\mathbb{R}^m), |\varphi| \leq 1\right\}.$$

Proposition 1.3. *If μ_h and μ are vector-valued Radon measures with $\mu_h \overset{*}{\rightharpoonup} \mu$, then for every open set $A \subset \mathbb{R}^n$ we have*

$$|\mu|(A) \leq \liminf_{h\to\infty} |\mu_h|(A). \tag{1.18}$$

Proof. Given $\varphi \in C_c^0(A;\mathbb{R}^m)$ with $|\varphi| \leq 1$, by $\mu_h \overset{*}{\rightharpoonup} \mu$ and thanks to the definition of $|\mu|(A)$,

$$\int_{\mathbb{R}^n} \varphi \cdot d\mu = \lim_{h\to\infty} \int_{\mathbb{R}^n} \varphi \cdot d\mu_h \leq \liminf_{h\to\infty} |\mu_h|(A).$$

By the arbitrariness of φ, taking the supremum on the left hand side, we find (1.18).

□

An important advantage of weak-star convergence is that compactness is had relatively easy.

Theorem 1.3. (**Compactness criterion for Radon measures**) *If $\{\mu_h\}_{h\in\mathbb{N}}$ is a sequence of Radon measures on $\mathbb{R}^n$ such that, for every compact set K in $\mathbb{R}^n$,*

$$\sup_{h\in\mathbb{N}} \mu_h(K) < \infty$$

then there exists a Radon measure μ on $\mathbb{R}^n$ and a sequence $h(k) \to \infty$ as $k \to \infty$ such that $\mu_{h(k)} \overset{*}{\rightharpoonup} \mu$.

Finally, we can regularize a Radon measure with a regularization kernel in the same way as for functions. Recall that given $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a regularization kernel ρ_ε, we define the ε-**regularization** of u as

$$u_\varepsilon(x) = (u \star \rho_\varepsilon)(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)u(y)dy, \quad x \in \mathbb{R}^n.$$

If $u \in C^0_c(\mathbb{R}^n)$, then $u_\varepsilon \to u$ in $C^0_c(\mathbb{R}^n)$.
If now μ is a $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^n$, then we define the *functions* $(u \star \rho_\varepsilon) : \mathbb{R}^n \to \mathbb{R}^m$ as

$$(\mu \star \rho_\varepsilon)(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)d\mu(y), \quad x \in \mathbb{R}^n. \tag{1.19}$$

This function is in $C^\infty(\mathbb{R}^n;\mathbb{R}^m)$ for every $\varepsilon > 0$ (the gradient is applied to ρ_ε).
The ε-**regularization** μ_ε of μ is the $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^n$,

$$\langle \mu_\varepsilon, \varphi \rangle = \int_{\mathbb{R}^n} \varphi(x) \cdot (\mu \star \rho_\varepsilon)(x)dx, \qquad \varphi \in C^0_c(\mathbb{R}^n;\mathbb{R}^m).$$

Equivalently, for every bounded Borel set $E \subset \mathbb{R}^n$, we set

$$\mu_\varepsilon(E) = \int_E (\mu \star \rho_\varepsilon)(x)dx.$$

Theorem 1.4. *If μ is a $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^n$, then, as $\varepsilon \to 0^+$,*

$$\mu_\varepsilon \overset{*}{\rightharpoonup} \mu, \quad |\mu_\varepsilon| \overset{*}{\rightharpoonup} |\mu|.$$

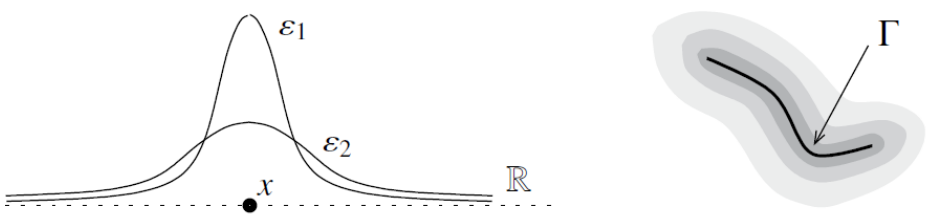

Figure 1.1: On the left, the functions $\mu \star \rho_\varepsilon$ relative to $\varepsilon_1 < \varepsilon_2$ for the measure $\mu = \delta_x$. On the right, a level set representation of $\mu \star \rho_\varepsilon$ for $\mu = \mathcal{H}^1 \llcorner \Gamma$. Thus the regularization μ_ε can be "higher-dimensional" with respect to the measure μ [1].

Chapter 2

Hausdorff measures

This chapter introduces the notion of Hausdorff measure and the related area formula for Lipschitz functions. The Hausdorff measures provide an important source of examples of Radon measures and formalize the idea of "lower dimensional" measure in $\mathbb{R}^n$, for example the surface measure in $\mathbb{R}^3$.
Moreover, in Chapter 3 we are going to exploit rewrite the classical Gauss-Green theorem in a more general framework, thanks to the $(n-1)$-Hausdorff measure in $\mathbb{R}^n$, which formalizes the idea of boundary of (regular) sets.

2.1 Hausdorff measure

Definition 2.1. Given $n, k \in \mathbb{N}, \delta > 0$, the **$k$-dimensinal Hausdorff measure of step** δ of a set $E \subset \mathbb{R}^n$ is defined as

$$\mathcal{H}^k_\delta(E) = \inf_{\mathcal{F}} \sum_{F \in \mathcal{F}} \omega_k \left(\frac{\operatorname{diam}(F)}{2} \right)^k ,$$

where $\mathcal{F}$ is a countable covering of E by sets $F \subset \mathbb{R}^n$ such that $\operatorname{diam}(F) < \delta$ and ω_k is the Lebesgue measure of the unit ball in $\mathbb{R}^k$; see Figure 2.1.

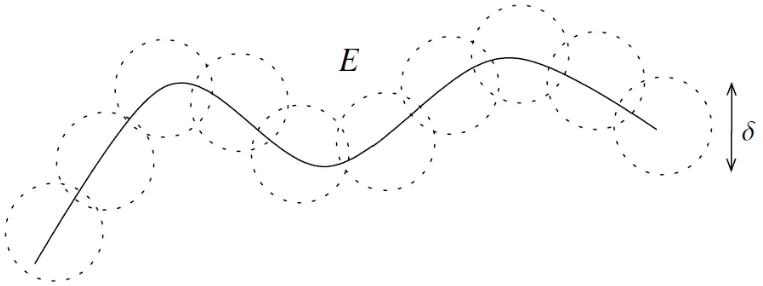

Figure 2.1: When computing $\mathcal{H}^k_\delta(E)$ one sums up, corresponding to each element F of a covering $\mathcal{F}$ of E, the k-dimensional measure of a k-dimensional ball of diameter $\mathrm{diam}(F)$ [1].

The ***k*-dimensional Hausdorff measure** of $E \subset \mathbb{R}^n$ is then

$$\mathcal{H}^k(E) = \sup_{\delta \in (0,\infty]} \mathcal{H}^k_\delta(E) = \lim_{\delta \to 0^+} \mathcal{H}^k_\delta(E). \tag{2.1}$$

Given a family of sets A_i and a covering $\cup_{j\in\mathbb{N}} C^i_j$ with $\mathrm{diam}(C^i_j) < \delta$ of each set A_i, the union of these coverings $\cup_{j,i\in\mathbb{N}} C^i_j$ covers the union $\cup_{i\in\mathbb{N}} A_i$. Thus

$$\mathcal{H}^k_\delta\left(\bigcup_{i\in\mathbb{N}} A_i\right) \leq \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \omega_k \left(\frac{\mathrm{diam}(C^i_j)}{2}\right)^k$$

By taking infima on the right hand side, for each $\delta \in (0,\infty]$, $\mathcal{H}^k_\delta$ is an outer measure. As an immediate consequence of taking the supremum, $\mathcal{H}^k$ is an outer measure too:

$$\mathcal{H}^k_\delta\left(\bigcup_{i\in\mathbb{N}} A_i\right) \leq \sum_{i\in\mathbb{N}} \mathcal{H}^k_\delta(A_i) \leq \sum_{i\in\mathbb{N}} \mathcal{H}^k(A_i),$$

then let $\delta \to 0$.
We can introduce a measure-theoretic notion of dimension. Given $E \subset \mathbb{R}^n$ we define the **Hausdorff dimension** of E as

$$\dim(E) = \inf\left\{k \in [0,\infty) : \mathcal{H}^k(E) = 0\right\}$$

Its use as a notion of dimension is justified by the following statements that we state without proof [1].

(i) If $E \subset \mathbb{R}^n$, then $\dim(E) \in [0,n]$. Moreover $\mathcal{H}^s(E) = \infty$ for every $s < \dim(E)$ and $\mathcal{H}^s(E) \in (0,\infty)$ implies $s = \dim(E)$.

(ii) $\mathcal{H}^0$ is the counting measure.

(iii) If E is a curve, then $\mathcal{H}^1(E)$ coincides with the classical length of E.

(iv) If $1 \leq k \leq n-1$, $k \in \mathbb{N}$, and E is a k-dimensional C^1-surface, then $\mathcal{H}^k(E)$ coincides with the classical k-dimensional area of E.

(v) If $E \subset \mathbb{R}^n$, then $\mathcal{H}^n(E) = \mathcal{L}^n(E)$.

(vi) If $s > n$, then $\mathcal{H}^s = 0$.

(vii) If A is an open set in $\mathbb{R}^n$, then $\dim(A) = n$.

Proposition 2.1. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function, then*

$$\mathcal{H}^s(f(E)) \leq \mathrm{Lip}(f)^s \mathcal{H}^s(E), \tag{2.2}$$

for every $s \in [0, \infty)$ and $E \subset \mathbb{R}^n$. In particular $\dim(f(E)) \leq \dim(E)$.

Remark 2.1. By Proposition 2.1 we find that Hausdorff measures are decreased under projection over an affine subspace of $\mathbb{R}^n$. Indeed, if H is an affine subspace of R^n and $f : \mathbb{R}^n \to \mathbb{R}^n$ is the projection of $\mathbb{R}^n$ over H, then $\mathrm{Lip}(f) = 1$.

2.2 Area formula

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be an injective Lipschitz function, where $1 \leq n \leq m$. The **Jacobian** of f is the bounded Borel function $Jf : \mathbb{R}^n \to [0, \infty]$,

$$\begin{cases} \sqrt{\det(\nabla f(x)^* \nabla f(x))}, & \text{if } f \text{ is differentiable at } x \\ +\infty & \text{if } f \text{ is not differentiable at } x. \end{cases}$$

Thus, by Rademacher's theorem, the set of points $x \in \mathbb{R}^n$ such that $Jf < \infty$ (i.e. where f is differentiable) has full Lebesgue measure on $\mathbb{R}^n$.

Theorem 2.1. (Area formula for injective maps) *If $f : \mathbb{R}^n \to \mathbb{R}^m$ $(1 \leq n \leq m)$ is an injective Lipschitz function and $E \subset \mathbb{R}^n$ is Lebesgue measurable, then*

$$\mathcal{H}^n(f(E)) = \int_E Jf(x)dx, \tag{2.3}$$

and $\mathcal{H}^n \llcorner f(\mathbb{R}^n)$ is a Radon measure on $\mathbb{R}^n$.

Remark 2.2. By Proposition 2.1, $f(E)$ is (at most) n-dimensional in $\mathbb{R}^m$.

Remark 2.3. We can also improve the area formula integrating Borel functions over the image of the Lipschitz map f (for example when we want to integrate over a surface).
If $g : \mathbb{R}^m \to [-\infty, \infty]$ is a Borel measurable function on $\mathbb{R}^m$ and either $g \geq 0$ or $g \in L^1(\mathbb{R}^m, \mathcal{H}^n \llcorner f(\mathbb{R}^n))$, then $g \circ f$ is Borel measurable on $\mathbb{R}^n$ and

$$\int_{f(\mathbb{R}^n)} g \, d\mathcal{H}^n = \int_{\mathbb{R}^n} g\,(f(x))\, Jf(x) dx. \tag{2.4}$$

Indeed, if $g \geq 0$, then $g = \sum_{h\in\mathbb{N}} c_h \, 1_{F_h}$, where $c_h \geq 0$ and $F_h \in \mathcal{B}(\mathbb{R}^m)$, $h \in N$. Setting $E_h = f^{-1}(F_h)$, then $g \circ f = \sum_{h \in N} c_h \, 1_{E_h}$, and by (2.3),

$$\int_{\mathbb{R}^m} g \, d\mathcal{H}^n = \sum_{h\in\mathbb{N}} c_h \mathcal{H}^n(F_h) = \sum_{h\in\mathbb{N}} c_h \int_{E_h} Jf = \int_{\mathbb{R}^n} (g \circ f) Jf.$$

If $g \in L^1(\mathbb{R}^m, \mathcal{H}^n \llcorner f(\mathbb{R}^n))$, then it suffices to notice that $g = g^+ - g^-$.

The proof of the area formula can be first done for linear functions, namely for $T \in \mathbb{R}^m \otimes \mathbb{R}^n$, for which it takes the form

$$\mathcal{H}^n(T(E)) = JT\,|E|, \qquad E \subset \mathbb{R}^n.$$

Then, one proves that Lipschitz immersions can be "linearized" in the sense that there exists a partition of $\mathbb{R}^n$ into Borel sets $\{F_h\}_{h\in\mathbb{N}}$ where f is arbitrarily close to a linear function T_h, for each $h \in \mathbb{N}$, on the set $\{Jf > 0\}$. The singular set $\{Jf = 0\}$ is also mapped by f into an $\mathcal{H}^n$-negligible set (this is a necessary condition for (2.3) to hold). This idea is due to Federer [2], [1]. We do not present here the details being mostly technicalities.

Theorem 2.2. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ $(1 \leq n \leq m)$ is a Lipschitz function, then*

$$\mathcal{H}^n(f(E)) = 0,$$

where $E = \{x \in \mathbb{R}^n : Jf(x) = 0\}$.

We also state the area formula for Lipschitz function which are not injective, taking into account the *multiplicities*.

Theorem 2.3. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ $(1 \leq n \leq m)$ is a Lipschitz function, and E is a Lebesgue measurable set of $\mathbb{R}^n$, then the* multiplicity function $M : \mathbb{R}^m \to \mathbb{N} \cup \{+\infty\}, M(y) = \mathcal{H}^0\,(E \cap \{f = y\})$ *of f over E is $\mathcal{H}^n$-measurable on $\mathbb{R}^m$, and*

$$\int_{\mathbb{R}^m} \mathcal{H}^0\,(E \cap \{f = y\})\, d\mathcal{H}^n(y) = \int_E Jf(x) dx. \tag{2.5}$$

2.2.1 Area of a graph of codimension one

Given $u : \mathbb{R}^{n-1} \to \mathbb{R}$ and $G \subset \mathbb{R}^{n-1}$, we define the **graph of** u **over** G as (see the **Notation** at the beginning of Chapter 3)

$$\Gamma(u; G) = \{x \in \mathbb{R}^n : \mathbf{q}x = u(\mathbf{p}x), \mathbf{p}x \in G\},$$

and set for brevity $\Gamma(u) = \Gamma(u; \mathbb{R}^{n-1})$. As a simple consequence of the area formula we find the following theorem which we are going to use in the rest.

Theorem 2.4. (Area of graph of codimension one) *If* $u : \mathbb{R}^{n-1} \to \mathbb{R}$ *is a Lipschitz function, then for every Lebesgue measurable set* G *in* $\mathbb{R}^{n-1}$,

$$\mathcal{H}^{n-1}\Big(\Gamma(u; G)\Big) = \int_G \sqrt{1 + |\nabla' u(z)|^2}\, dz. \tag{2.6}$$

In fact, $\mathcal{H}^{n-1} \llcorner \Gamma(u)$ *is a Radon measure on* $\mathbb{R}^n$*, and for every* $\varphi \in C^0_c(\mathbb{R}^n)$*,*

$$\int_{\Gamma(u)} \varphi \, d\mathcal{H}^{n-1} = \int_{\mathbb{R}^{n-1}} \varphi(z, u(z))\sqrt{1 + |\nabla' u(z)|^2}\, dz. \tag{2.7}$$

Proof. If $v \neq 0$, then $v = |v| w_1$, $|w_1| = 1$, and introducing an orthonormal basis $\{w_i\}_{i=1}^n$ of $\mathbb{R}^n$, we find $\mathrm{Id} + v \otimes v = (1 + |v|^2) w_1 \otimes w_1 + \sum_{i=2}^n w_i \otimes w_i$; thus

$$\det(\mathrm{Id} + v \otimes v) = 1 + |v|^2, \qquad \forall v \in \mathbb{R}^n.$$

Now let $f : \mathbb{R}^{n-1} \to \mathbb{R}^n$ be the injective Lipschitz function defined as

$$f(z) = (z, u(z)), \qquad z \in \mathbb{R}^{n-1}.$$

Since $\Gamma(u; G) = f(G)$ for every $G \subset \mathbb{R}^n$, $\mathcal{H}^{n-1} \llcorner \Gamma(u)$ is a Radon measure by Theorem 2.1. We now show that $Jf = \sqrt{1 + |\nabla' u|^2}$ on $\mathbb{R}^{n-1}$, so that (2.6) and (2.7) will follow from (2.3) and (2.4) respectively. To this end, we compute

$$\nabla f = \sum_{i=1}^{n-1} (e_i + (\partial_i u) e_n) \otimes e_i,$$

and recall that $(a \otimes b)(c \otimes d) = (b \cdot c)(a \otimes d)$, for $a, b, c, d \in \mathbb{R}^n$, to find

$$\begin{aligned}(\nabla f)^*(\nabla f) &= \left(\sum_{i=1}^{n-1} e_i \otimes (e_i + (\partial_i u) e_n)\right)\left(\sum_{j=1}^{n-1} (e_j + (\partial_j u) e_n) \otimes e_j\right) \\ &= \sum_{i,j=1}^{n-1} (\delta_{i,j} + (\partial_i u)(\partial_j u))\, e_i \otimes e_j = \mathrm{Id} + (\nabla' u) \otimes (\nabla' u).\end{aligned}$$

By $\det(\mathrm{Id} + v \otimes v) = 1 + |v|^2$, we conclude that $Jf = \sqrt{1 + |\nabla' u|^2}$. □

Chapter 3

Sets of finite perimeter

The classical Gauss-Green theorem on open sets with C^1-boundary plays a fundamental role in the theory of sets of finite perimeter. The starting point of this theory is indeed a generalization of the Gauss-Green theorem based on the notion of vector-valued Radon measures. The key observation behind the definition of a set of finite perimeter is that the boundary the boundary of a set is related to the *distributional derivative* of its characteristic function. In this chapter we use the notions introduced in chapters 1 and 2 to perform this generalization.

Notation: Given n and $1 \leq k \leq n-1$, we denote by $\mathbf{p} : \mathbb{R}^n \to \mathbb{R}^k \times \{0\} = \mathbb{R}^k$ and $\mathbf{q} : \mathbb{R}^n \to \{0\} \times \mathbb{R}^{n-k} = \mathbb{R}^{n-k}$ the horizontal and vertical projections, so that $x = (\mathbf{p}x, \mathbf{q}x) \in \mathbb{R}^n$. We then introduce the cylinder of center $x \in \mathbb{R}^n$ and radius $r > 0$,

$$\mathbf{C}(x,r) = \{y \in \mathbb{R}^n : |\mathbf{p}(y-x)| < r, \, |\mathbf{q}(y-x)| < r\},$$

and the k-dimensional ball of center $z \in \mathbb{R}^k$ and radius $r > 0$,

$$\mathbf{D}(x,r) = \left\{w \in \mathbb{R}^k : |z - w| < R\right\}.$$

When $k = n-1$, we set $\mathbf{p}x = x'$ and $\mathbf{q}x = x_n$, so that $x = (x', x_n)$. We also set $\nabla' = (\partial_1, ..., \partial_{n-1})$.

3.1 Gauss-Green theorem on smooth sets

Let E be an open set in $\mathbb{R}^n$ and let $k \in \mathbb{N} \cup \{\infty\}, k \geq 1$. We say that E has C^k-**boundary** (or **smooth boundary** if $k = \infty$) if for every $x \in \partial E$ there exist $r > 0$ and $\psi \in C^k(B(x,r))$ with $\nabla\psi(y) \neq 0$ for every $y \in B(x,r)$ and

$$B(x,r) \cap E = \{y \in B(x,r) : \psi(y) < 0\} \tag{3.1}$$

$$B(x,r) \cap \partial E = \{y \in B(x,r) : \psi(y) = 0\} \tag{3.2}$$

The **outer unit normal** ν_E to E is then defined *locally* as

$$\nu_E(y) = \frac{\nabla\psi(y)}{|\nabla\psi(y)|}, \qquad \forall y \in B(x,r) \cap \partial E.$$

This definition is independent of the choice of ψ and r, therefore ν_E can be considered as a vector field on the whole ∂E, with $\nu_E \in C^{k-1}(\partial E; S^{n-1})$.

Remark 3.1. If E is an open set with C^1-boundary, then $\mathcal{H}^{n-1} \llcorner \partial E$ is a Radon measure on $\mathbb{R}^n$. Indeed, by the implicit function theorem, if $x \in \partial E$ and $r > 0$ is the same as in (3.1) and (3.2), then there exist $s > 0$ and a function $u \in C^1(\mathbf{D}(\mathbf{p}x, s))$ such that $\mathbf{C}(x,s) \subset B(x,r)$ and, up to a rotation,

$$\mathbf{C}(x,s) \cap E = \{y \in \mathbf{C}(x,s) : \mathbf{q}y > u(\mathbf{p}y)\},$$
$$\mathbf{C}(x,s) \cap \partial E = \{y \in \mathbf{C}(x,s) : \mathbf{q}y = u(\mathbf{p}y)\}.$$

Hence, $\mathcal{H}^{n-1} \llcorner (\mathbf{C}(x,s) \cap \partial E) = \mathcal{H}^{n-1} \llcorner \Gamma(u; \mathbf{D}(\mathbf{p}x, s))$, where we denote $\Gamma(u, G) = \{x \in \mathbb{R}^n : \mathbf{q}x = u(\mathbf{p}x), \mathbf{p}x \in G\}$, for every $G \subset \mathbb{R}^{n-1}$. Since we can define the injective Lipschitz immersion $f : \mathbb{R}^{n-1} \to \mathbb{R}$ given by

$$f(z) = (z, u(z)),$$

it holds that $\Gamma(u; G) = f(G)$, for every $G \subset \mathbb{R}^{n-1}$. Thus, for $G = \mathbf{D}(\mathbf{p}x, s)$, by Theorem 2.1, $\mathcal{H}^{n-1} \llcorner \Gamma(u, \mathbf{D}(\mathbf{p}x, s))$ is a Radon measure. By a partition of unity, it is easily seen that $\mathcal{H}^{n-1} \llcorner \partial E$ is a Radon measure on $\mathbb{R}^n$. Let us also notice that, having expressed $\mathbf{C}(x,s) \cap E$ as the epigraph of u over $\mathbf{D}(\mathbf{p}x, s)$, by the chain rule we infer the following formula for the outer unit normal ν_E of E:

$$\nu_E(y) = \frac{(\nabla' u(\mathbf{p}y), -1)}{\sqrt{1 + |\nabla' u(\mathbf{p}y)|^2}}, \qquad \forall y \in \mathbf{C}(x,s) \cap \partial E.$$

Theorem 3.1. *If E is an open set with C^1-boundary, then for every $\varphi \in C_c^1(\mathbb{R}^n)$,*

$$\int_E \nabla\varphi(x) dx = \int_{\partial E} \varphi\, \nu_E \, d\mathcal{H}^{n-1}. \tag{3.3}$$

Equivalently, the divergence theorem holds true:

$$\int_E \operatorname{div} T(x) dx = \int_{\partial E} T \cdot \nu_E \, d\mathcal{H}^{n-1}, \qquad \forall\, T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n). \tag{3.4}$$

A useful first generalization of Gauss-Green theorem is given on sets $E \subset \mathbb{R}^n$ with **almost C^1-boundary**, namely there exists a closed set $M_0 \subset \partial E$ with

$$\mathcal{H}^{n-1}(M_0) = 0,$$

and for every $x \in \partial E \backslash M_0 = M$, (3.1) and (3.2) hold. We call M the **regular part** of ∂E. The outer unit normal to E is defined as a continuous vector field $\nu_E \in C^0(M; S^{n-1})$, through the same local representations.

Theorem 3.2. *If E is an open set in $\mathbb{R}^n$ with almost C^1-boundary, and M is the regular part of ∂E, then for every $\varphi \in C^1_c(\mathbb{R}^n)$*

$$\int_E \nabla\varphi = \int_M \varphi\, \nu_E \, d\mathcal{H}^{n-1}.$$

3.2 Sets of finite perimeter

Let E be a Lebesgue measurable set in $\mathbb{R}^n$. We say that E is a **set of locally finite perimeter** in $\mathbb{R}^n$ if for every compact set $K \subset \mathbb{R}^n$ we have

$$\sup\left\{\int_E \operatorname{div} T(x)dx : T \in C^1_c(\mathbb{R}^n;\mathbb{R}^m),\ \operatorname{spt}(T) \subset K,\ \sup_{\mathbb{R}^n}|T| \leq 1\right\} < \infty \qquad (3.5)$$

If this quantity is bounded independently of K, then we say that E is a **set of finite perimeter**.

Proposition 3.1. *If E is a Lebesgue measurable set in $\mathbb{R}^n$, then E is a set of locally finite perimeter if and only if there exists a $\mathbb{R}^n$-valued Radon measure μ_E on $\mathbb{R}^n$ such that*

$$\int_E \operatorname{div} T = \int_{\mathbb{R}^n} T \cdot d\mu_E, \qquad \forall\, T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n). \qquad (3.6)$$

Moreover, E is a set of finite perimeter if and only if $|\mu_E|(\mathbb{R}^n) < \infty$.

Remark 3.2. The formula (3.6) is equivalent to

$$\int_E \nabla\varphi = \int_{\mathbb{R}^n} \varphi\, d\mu_E, \qquad \forall\, \varphi \in C^1_c(\mathbb{R}^n). \qquad (3.7)$$

by using (3.6) n times, with $T_i = (0, .., 0, \underbrace{\varphi}_{\text{i-th}}, 0, .., 0)$, for $i = 1, .., n$.

We call μ_E the **Gauss-Green measure** of E, and define the **relative perimeter** of E in $F \subset \mathbb{R}^n$, and the **perimeter** of E, as

$$P(E; F) = |\mu_E|(F), \quad P(E) = |\mu_E|(\mathbb{R}^n).$$

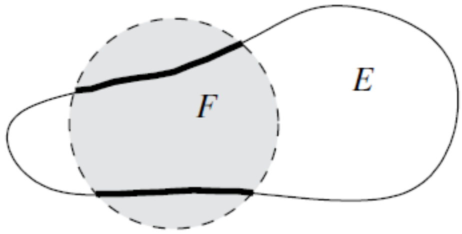

Figure 3.1: The perimeter $P(E; F)$ of E relative to F is the $(n-1)$-dimensional measure of the intersection of the (reduced) boundary of E with F [1].

We notice that the definition of set of locally finite perimeter is equivalent to saying that the distributional gradient $D1_E$ of $1_E \in L^1_{\text{loc}}(\mathbb{R}^n)$ can be represented as the integration with respect to the $\mathbb{R}^n$-valued Radon measure $-\mu_E$. Therefore we can speak of *distributional perimeter* of a set E.

Proof. (*of Proposition 3.1*) Let E be a set of locally finite perimeter in $\mathbb{R}^n$, and consider the linear functional $L : C^1_c(\mathbb{R}^n; \mathbb{R}^n) \to \mathbb{R}$ defined by

$$\langle L, T \rangle = \int_E \operatorname{div} T(x) dx.$$

For every compact set $K \subset \mathbb{R}^n$, by (3.5) there exists $C(K) \in \mathbb{R}$ such that $\langle L, T \rangle \leq C(K) \sup_{\mathbb{R}^n} |T|$ whenever T is supported inside K. Hence, L can be extended by density to a continuous linear functional on $C^0_c(\mathbb{R}^n; \mathbb{R}^m)$, and the existence of μ_E follows by Riesz's theorem (Theorem 1.2). If E is a set of finite perimeter, then for example by taking increasing closed balls we find that $|\mu_E|(\mathbb{R}^n) < \infty$.
Conversely, if $K \subset \mathbb{R}^n$ is compact, $T \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$ with $|T| \leq 1$ on $\mathbb{R}^n$ with $\operatorname{spt}(T) \subset K$, then by (3.6) we have $\int_E \operatorname{div} T(x) dx \leq |\mu_E|(K) < \infty$, so that by taking the supremum over T, E is a set of locally finite perimeter. □

Example 3.1. By the Gauss-Green theorem, if $E \subset \mathbb{R}^n$ is an open (not necessarily bounded) set with C^1 boundary, then

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial E$$

namely $\nu_E \mathcal{H}^{n-1} \llcorner \partial E$ is a $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^n$ such that (3.6) and (3.7) hold true, and E is a set of locally finite perimeter with Gauss-Green measure $\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial E$. In the language of Riesz's Theorem 1.2, $g = \nu_E$ and $|\mu_E| = \mathcal{H}^{n-1} \llcorner \partial E$. Indeed for $|\mu_E|$-almost every point, $|g| = |\nu_E| = 1$. Also we have that

$$P(E) = \mathcal{H}^{n-1}(\partial E), \quad P(E; F) = \mathcal{H}^{n-1}(F \cap \partial E),$$

for every $F \subset \mathbb{R}^n$.

Remark 3.3. If E is a set of (locally) finite perimeter in $\mathbb{R}^n$ and $|E \triangle F| = 0$, then F is a set of (locally) finite perimeter and $\mu_E = \mu_F$; the converse is also true. This is because in the definition (3.6) the test function T does not detect modifications in the domain of integration by a negligible (in the Lebesgue measure sense) set, so $\int_E \operatorname{div} T = \int_F \operatorname{div} T$. In particular, the perimeter $P(E)$ of E is invariant by modifications of E on and/or by a set of measure zero, although these modifications may wildly affect the size of its topological boundary (for example, removing lines here and there in a 2-dimensional set, see Figure 3.2). Moreover, every set of Lebesgue measure zero is of finite perimeter and has perimeter zero.

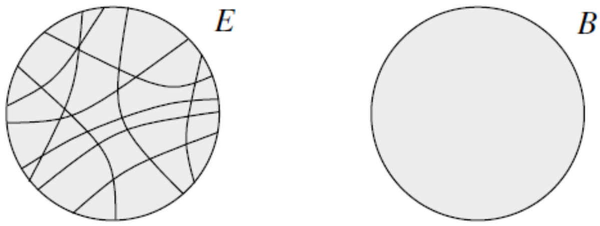

Figure 3.2: The set $E \subset \mathbb{R}^2$ is equivalent to the unit disk B. They both have distributional perimeter 2π, although $\mathcal{H}^1(\partial E)$ is much greater than 2π [1].

We provide other useful examples, to show compliance with the geometric intuition.

Example 3.2. If E is an open set with almost C^1 boundary in $\mathbb{R}^n$, and if M is the regular part of ∂E, then, by Theorem 3.2, E is a set of locally finite perimeter, with $\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner M$ and, for every $F \subset \mathbb{R}^n$,

$$P(E; F) = \mathcal{H}^{n-1}(F \cap M) = \mathcal{H}^{n-1}(F \cap \partial E).$$

Example 3.3. (Scaling and translation) If $\lambda > 0, x \in \mathbb{R}^n$ and E is a set of finite perimeter in $\mathbb{R}^n$ then $x + \lambda E$ is a set of finite perimeter with

$$P(x + \lambda E) = \lambda^{n-1} P(E).$$

This comes from the simple change of variables $y = x + \lambda z, z \in E$, in the integral $\int_{x+\lambda E} \nabla \varphi(y)\, dy$, which brings the factor λ^{n-1}.

Example 3.4. (Complement) If E is a set of locally finite perimeter, then $\mathbb{R}^n \setminus E$ is a set of locally finite perimeter with

$$\mu_{\mathbb{R}^n \setminus E} = -\mu_E, \quad P(E) = P(\mathbb{R}^n \setminus E). \tag{3.8}$$

Indeed, we know from the fundamental theorem of calculus that for $\varphi \in C_c^1(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \nabla\varphi = 0.$$

Writing then $\mathbb{R}^n = E \cup (\mathbb{R}^n \setminus E)$, we have the chain of equalities

$$\int_{\mathbb{R}^n} \varphi \, d\mu_{\mathbb{R}^n \setminus E} = \int_{\mathbb{R}^n \setminus E} \nabla\varphi = -\int_E \nabla\varphi = -\int_{\mathbb{R}^n} \varphi \, d\mu_E, \qquad \forall\, \varphi \in C_c^1(\mathbb{R}^n).$$

Of course the value of the sup in the definition of $P(E)$ is not affected by the change of sign, by linearity (instead of φ, take $-\varphi$), and so we have $P(E) = P(\mathbb{R}^n \setminus E)$.

3.2.1 Lower semicontinuity of perimeter

By (1.5) and Proposition 3.1, if A is an open set in $\mathbb{R}^n$ and E is a set of locally finite perimeter in $\mathbb{R}^n$, then

$$P(E;A) = \sup\left\{\int_E \operatorname{div} T(x)dx : T \in C_c^1(A;\mathbb{R}^n),\ \sup_{\mathbb{R}^n}|T| \le 1\right\}, \tag{3.9}$$

(note that this quantity can be infinite, if A is not bounded. It is always finite if E is of finite perimeter). By density we may also take $T \in C_c^\infty(A;\mathbb{R}^n)$.
Given Lebesgue measurable sets $\{E_h\}_{h\in\mathbb{N}}$ and $E \subset \mathbb{R}^n$, we say that E_h **locally converges** to E, and write $E_h \overset{\text{loc}}{\to} E$, if

$$\lim_{h\to\infty} \Big|K \cap (E \triangle E_h)\Big| = 0, \qquad \forall\, K \subset \mathbb{R}^n \text{ compact},$$

or, in other words, when the L^1_{loc} convergence of the indicator functions hold:

$$\lim_{h\to\infty} \int_K |1_{E_h} - 1_E| = 0, \qquad \forall\, K \subset \mathbb{R}^n \text{ compact}.$$

We say simply that E_h **converges** to E, and write $E_h \to E$, if

$$\lim_{h\to\infty} |E \triangle E_h| = 0.$$

Proposition 3.2. (Lower semicontinuity of perimeter) *If $\{E_h\}_{h\in\mathbb{N}}$ is a sequence of sets of locally finite perimeter in $\mathbb{R}^n$, with*

$$E_h \overset{\text{loc}}{\to} E, \qquad \limsup_{h\to\infty} P(E_h;K) < \infty, \tag{3.10}$$

for every compact set K in $\mathbb{R}^n$, then E is of locally finite perimeter in $\mathbb{R}^n$, $\mu_{E_h} \overset{*}{\rightharpoonup} \mu_E$ and, for every open set $A \subset \mathbb{R}^n$, we have

$$P(E;A) \le \liminf_{h\to\infty} P(E_h;A). \tag{3.11}$$

Proof. If A is open, $T \in C_c^1(A;\mathbb{R}^n)$ and $|T| \le 1$ on $\mathbb{R}^n$, then by (3.9)

$$\int_E \operatorname{div} T(x)dx = \lim_{h\to\infty}\int_{E_h} \operatorname{div} T(x)dx \le \liminf_{h\to\infty} P(E_h;A).$$

By (3.9), (3.10), and by applying the formula above with A bounded (such that $\overline{A}$ is compact), we see that E is of locally finite perimeter in $\mathbb{R}^n$, and that (3.11) holds true (even if A is unbounded). By (3.7) and since $E_h \overset{\text{loc}}{\to} E$, we have that

$$\lim_{h\to\infty}\int_{\mathbb{R}^n} \varphi\, d\mu_{E_h} = \lim_{h\to\infty}\int_{E_h} \nabla\varphi = \int_E \nabla\varphi = \int_{\mathbb{R}^n} \varphi\, d\mu_E, \qquad \forall\, \varphi \in C_c^1(\mathbb{R}^n).$$

Now we use the density of $C_c^1(\mathbb{R}^n)$ into $C_c^0(\mathbb{R}^n)$ and the uniform boundedness (3.10): fix $\varphi \in C_c^0(\mathbb{R}^n)$ and take a sequence $\{\varphi_j\}_{j\in\mathbb{N}} \subset C_c^1(\mathbb{R}^n), \varphi_j \to \varphi$ in $C_c^0(\mathbb{R}^n)$, so that $\operatorname{spt}(\varphi_j) \subset K$, $\operatorname{spt}(\varphi) \subset K$ for a fixed compact set K. Then for any $h \in \mathbb{N}$ we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} \varphi\, d(\mu_{E_h} - \mu_E)\right| \\ &\quad \le \left|\int_{\mathbb{R}^n} (\varphi - \varphi_j) d\mu_{E_h}\right| + \left|\int_{\mathbb{R}^n} \varphi_j\, d(\mu_{E_h} - \mu_E)\right| + \left|\int_{\mathbb{R}^n} (\varphi_j - \varphi) d\mu_E\right|.\end{aligned}$$

The first term is arbitrarily small because $P(E_h;K)$ is bounded uniformly in h:

$$\left|\int_{\mathbb{R}^n} (\varphi - \varphi_j) d\mu_{E_h}\right| \le \sup_{\mathbb{R}^n} |\varphi - \varphi_j|\, |\mu_{E_h}|(K) = \sup_{\mathbb{R}^n} |\varphi - \varphi_j|\, P(E_h;K)$$

and let $j \to \infty$. The second term goes to 0 uniformly in j because φ_j is in $C_c^1(\mathbb{R}^n)$ and we let $h \to \infty$ as above. Finally, the third term goes to 0 because E is a set of locally finite perimeter:

$$\left|\int_{\mathbb{R}^n} (\varphi_j - \varphi) d\mu_E\right| \le \sup_{\mathbb{R}^n} |\varphi - \varphi_j|\, |\mu_E|(K) = \sup_{\mathbb{R}^n} |\varphi - \varphi_j|\, P(E;K)$$

and let $j \to \infty$. □

Example 3.5. If E is a Lebesgue measurable set in $\mathbb{R}^n$, $\{u_h\}_{h\in\mathbb{N}} \subset C_c^1(\mathbb{R}^n)$, $u_h \to 1_E$ in $L^1_{\text{loc}}(\mathbb{R}^n)$, and, for every compact set K in $\mathbb{R}^n$,

$$\limsup_{h\to\infty}\int_K |\nabla u_h| < \infty$$

then E is of locally finite perimeter, with

$$P(E;A) \le \liminf_{h\to\infty}\int_A |\nabla u_h|, \qquad \text{for every } A \subset \mathbb{R}^n \text{ open.} \tag{3.12}$$

Proof. Fix a compact set K. For every $T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ with $\mathrm{spt}(T) \subset K$, $|T| \le 1$ we have

$$\int_E \operatorname{div} T = \lim_{h\to\infty} \int_{\mathbb{R}^n} u_h \operatorname{div} T = -\lim_{h\to\infty} \int_{\mathbb{R}^n} T \cdot \nabla u_h \le \limsup_{h\to\infty} \int_K |\nabla u_h| < \infty$$

by arbitrariness of K and T, E is of locally finite perimeter. Apply the same calculations of above with $T \in C^1_c(A;\mathbb{R}^n)$ and lim inf to obtain (3.12). □

3.2.2 Topological boundary and Gauss-Green measure

As seen in Remark 3.3, the topological boundary may differ between two sets with the same Gauss-Green perimeter measure. We want here to give the precise relation between the support of the perimeter measure, $\mathrm{spt}(\mu_E)$, and the topological boundary ∂E. Recall that the support of a measure is defined as

$$\mathbb{R}^n \setminus \mathrm{spt}(\mu) = \{x \in \mathbb{R}^n : \mu(B(x,r)) = 0 \text{ for some } r > 0\}.$$

or equivalently, $\mathrm{spt}(\mu) = \{x \in \mathbb{R}^n : \mu(B(x,r)) > 0 \text{ for all } r > 0\}$.

Proposition 3.3. *If E is a set of locally finite perimeter in $\mathbb{R}^n$, then*

$$\mathrm{spt}(\mu_E) = \{x \in \mathbb{R}^n : 0 < |E \cap B(x,r)| < \omega_n r^n \ \ \forall r > 0\} \subset \partial E \tag{3.13}$$

Proof. If $x \in \mathbb{R}^n$ is such that $|E \cap B(x,r)| = 0$ for some $r > 0$, then

$$0 = \int_E \nabla\varphi = \int_{\mathbb{R}^n} \varphi\, d\mu_E, \qquad \forall\, \varphi \in C^1_c(B(x,r)).$$

Thus, by taking the sup over these φ, $|\mu_E|(B(x,r)) = 0$ and $x \in \mathrm{spt}(\mu_E)$ by definition of support. Similarly, if $x \in \mathbb{R}^n$ and $|E \cap B(x,r)| = |B(x,r)|$, for some $r > 0$, then $x \notin \mathrm{spt}(\mu_E)$, since by the fundamental theorem of calculus

$$0 = \int_{B(x,r)} \nabla\varphi = \int_E \nabla\varphi = \int_{\mathbb{R}^n} \varphi\, d\mu_E, \qquad \forall\, \varphi \in C^1_c(B(x,r)).$$

Finally, if $x \notin \mathrm{spt}(\mu_E)$, then $|\mu_E|(B(x,r)) = 0$ for some $r > 0$, and

$$0 = \int_{\mathbb{R}^n} \varphi\, d\mu_E = \int_E \nabla\varphi = \int_{\mathbb{R}^n} 1_E \nabla\varphi, \qquad \forall\, \varphi \in C^\infty_c(B(x,r)).$$

By the fundamental lemma of calculus of variations, there exists $c \in \mathbb{R}$ such that $1_E = c$ a.e. on $B(x,r)$. Necessarily, $c \in \{0,1\}$ and, correspondingly, $|E \cap B(x,r)| \in \{0, \omega_n r^n\}$ and this proves (3.13). □

Corollary 3.1. *If E is a set of locally finite perimeter in $\mathbb{R}^n$, then there exists a Borel set F such that*

$$|E \triangle F| = 0, \quad \mathrm{spt}(\mu_F) = \partial F.$$

3.2.3 Regularization

Sets of finite perimeter are of course Lebesgue measurable sets, so that the characteristic function $1_E \in L^1_{\text{loc}}(\mathbb{R}^n)$. Thus we can consider the regularization of the characteristic functions with regularizing kernels.
Consider the ε-regularization $(1_E \star \rho_\varepsilon)$ of 1_E,

$$(1_E \star \rho_\varepsilon)(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x-y) 1_E(y) dy = \int_{E \cap B(x,r)} \rho_\varepsilon(x-y) dy, \quad x \in \mathbb{R}^n.$$

Clearly, we have $0 \leq (1_E \star \rho_\varepsilon) \leq 1$, and, moreover,

$$(1_E \star \rho_\varepsilon)(x) = \begin{cases} 1, & \text{if } |B(x,\varepsilon) \setminus E| = 0, \\ 0, & \text{if } |B(x,\varepsilon) \cap E| = 0, \end{cases}$$

see Figure 3.3. If E is an open set with smooth boundary, then we expect $\nabla(1_E \star \rho_\varepsilon)$ to satisfy

$$(1_E \star \rho_\varepsilon)(x) \approx -\varepsilon^{-1} \nu_E(\text{projection of } x \text{ on } \partial E), \quad \text{if } \operatorname{dist}(x, \partial E) < \varepsilon,$$

and $\nabla(1_E \star \rho_\varepsilon)(x) = 0$ if $\operatorname{dist}(x, \partial E) > \varepsilon$. Hence, as $\varepsilon \to 0$, it should hold that

$$\int_{\mathbb{R}^n} |\nabla(1_E \star \rho_\varepsilon)(x)| dx \approx \frac{|\{y \in \mathbb{R}^n : \operatorname{dist}(y, \partial E) < \varepsilon\}|}{\varepsilon} \approx \frac{\varepsilon \mathcal{H}^{n-1}(\partial E)}{\varepsilon} = P(E).$$

In the next Proposition this formula is proved to be true if E is a set of locally finite perimeter, whether $P(E)$ is finite or not.

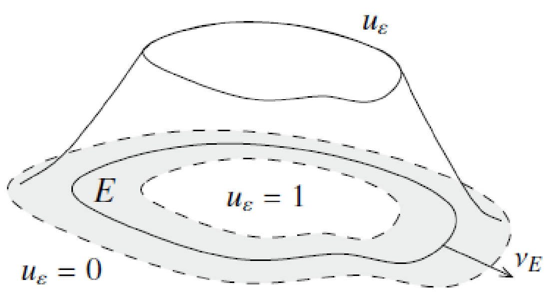

Figure 3.3: The ε-regularization of the characteristic function of an open set with smooth boundary.The ε-neighborhood of ∂E is painted in gray and corresponds to the set of those x such that $0 < u_\varepsilon(x) < 1$. Correspondingly $\nabla u_\varepsilon(x)$ is approximately $-(1/\varepsilon)\nu_E$ evaluated at the projection of x over ∂E [1].

Proposition 3.4. *If E is a set of locally finite perimeter in $\mathbb{R}^n$, then*

$$(\mu_E)_\varepsilon = -\nabla(1_E \star \rho_\varepsilon)\mathcal{L}^n, \qquad \forall \varepsilon > 0, \tag{3.14}$$

$$-\nabla(1_E \star \rho_\varepsilon)\mathcal{L}^n \stackrel{*}{\rightharpoonup} \mu_E, \qquad |\nabla(1_E \star \rho_\varepsilon)|\mathcal{L}^n \stackrel{*}{\rightharpoonup} |\mu_E|, \tag{3.15}$$

as $\varepsilon \to 0^+$. If, conversely, E is a Lebesgue measurable set in $\mathbb{R}^n$ such that

$$\limsup_{\varepsilon \to 0^+} \int_K |\nabla(1_E \star \rho_\varepsilon)(x)| dx < \infty, \tag{3.16}$$

for every compact set $K \subset \mathbb{R}^n$, then E is of locally finite perimeter.

Proof. By (3.7) and the definition of ε-regularization of μ_E as in (1.19) we have that, for every $x \in \mathbb{R}^n$,

$$(\mu_E \star \rho_\varepsilon)(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x-y) d\mu_E(y) = -\int_E \nabla \rho_\varepsilon(x-y) dy = -\nabla(1_E \star \rho_\varepsilon)(x).$$

Since $(\mu_E)_\varepsilon = (\mu_E \star \rho_\varepsilon)$, (3.14) follows. By Theorem 1.4 applied to μ_E together with (3.14), we directly deduce (3.15).
Conversely, if E is a Lebesgue measurable set in $\mathbb{R}^n$ such that (3.16) holds true, then by the compactness Theorem 1.3, applied to $|(\mu_E)_\varepsilon|$ indicized by $\varepsilon > 0$, there exist a $\mathbb{R}^n$-valued Radon measure μ on $\mathbb{R}^n$ and a sequence $\varepsilon_h \to 0^+$, such that $-\nabla(1_E \star \rho_{\varepsilon_h})\mathcal{L}^n \stackrel{*}{\rightharpoonup} \mu$. In particular, if $\varphi \in C_c^1(\mathbb{R}^n)$, then we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi \, d\mu &= -\lim_{h\to\infty} \int_{\mathbb{R}^n} \varphi(x) \nabla(1_E \star \rho_{\varepsilon_h})(x) dx \\
&= -\lim_{h\to\infty} \int_{\mathbb{R}^n} \varphi(x) \int_{\mathbb{R}^n} 1_E(y) \nabla \rho_{\varepsilon_h}(x-y) dy \, dx \\
&= -\lim_{h\to\infty} \int_E \int_{\mathbb{R}^n} \varphi(x) \nabla \rho_{\varepsilon_h}(x-y) dx \, dy \\
&= \lim_{h\to\infty} \int_E \int_{\mathbb{R}^n} \nabla\varphi(x) \rho_{\varepsilon_h}(x-y) dx \, dy = \lim_{h\to\infty} \int_E (\nabla\varphi)_{\varepsilon_h}(y) dy = \int_E \nabla\varphi.
\end{aligned}$$

This proves that E is of locally finite perimeter in $\mathbb{R}^n$, since $\mu_E = \mu$. □

Remark 3.4. From $|(\mu_E)_\varepsilon| = |\nabla(1_E \star \rho_\varepsilon)|\mathcal{L}^n \stackrel{*}{\rightharpoonup} |\mu_E|$, we have that

$$|(\mu_E)_\varepsilon|(\mathbb{R}^n) \to |\mu_E|(\mathbb{R}^n), \qquad \text{as } \varepsilon \to 0^+.$$

which is

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n} |\nabla(1_E \star \rho_\varepsilon)|(x) dx = P(E). \tag{3.17}$$

As an application of (3.4), we obtain the useful result concerning unions and intersections of sets of finite perimeter.

Lemma 3.1. *If E and F are sets of (locally) finite perimeter in $\mathbb{R}^n$, then $E \cup F$ and $E \cap F$ are sets of (locally) finite perimeter in $\mathbb{R}^n$, and for $A \subset \mathbb{R}^n$ open,*

$$P(E \cup F; A) + P(E \cap F; A) \leq P(E; A) + P(F; A), \tag{3.18}$$

which is

$$|\mu_{E\cup F}|(A) + |\mu_{E\cap F}|(A) \leq |\mu_E|(A) + |\mu_F|(A).$$

Proof. We use the regularization. If $u_\varepsilon = 1_E \star \rho_\varepsilon$, $v_\varepsilon = 1_F \star \rho_\varepsilon$, then $0 \leq u_\varepsilon, v_\varepsilon \leq 1$, $u_\varepsilon v_\varepsilon \to 1_{E\cap F}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and $w_\varepsilon = u_\varepsilon + v_\varepsilon - u_\varepsilon v_\varepsilon \to 1_{E\cup F}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover,

$$\int_A |\nabla(u_\varepsilon v_\varepsilon)| \leq \int_A v_\varepsilon |\nabla u_\varepsilon| + u_\varepsilon |\nabla v_\varepsilon|,$$
$$\int_A |\nabla w_\varepsilon| \leq \int_A (1 - v_\varepsilon)|\nabla u_\varepsilon| + (1 - u_\varepsilon)|\nabla v_\varepsilon|,$$

whenever A is an open bounded set in $\mathbb{R}^n$. Adding up the two inequalities,

$$\int_A |\nabla(u_\varepsilon v_\varepsilon)| + \int_A |\nabla w_\varepsilon| \leq \int_A |\nabla u_\varepsilon| + |\nabla v_\varepsilon|,$$

where the upper limit as $\varepsilon \to 0^+$ of the right-hand side is bounded above by $P(E; \overline{A}) + P(F; \overline{A}) < \infty$. By Example 3.5, $E \cap F$ and $E \cup F$ are of locally finite perimeter in $\mathbb{R}^n$, with

$$P(E \cup F; A) + P(E \cap F; A) \leq P(E; \overline{A}) + P(F; \overline{A}), \tag{3.19}$$

for every bounded open set A. Now let A be any open set in $\mathbb{R}^n$, set $A_k = \{x \in \mathbb{R}^n : A \cap B_k : \mathrm{dist}(x, \partial A) < k^{-1}\}, k \in \mathbb{N}$, and apply (3.19) to each A_k, to find

$$P(E \cup F; A_k) + P(E \cap F; A_k) \leq P(E; A) + P(F; A)$$

Letting $k \to \infty$, the left hand side converges to $P(E \cup F; A) + P(E \cap F; A)$. □

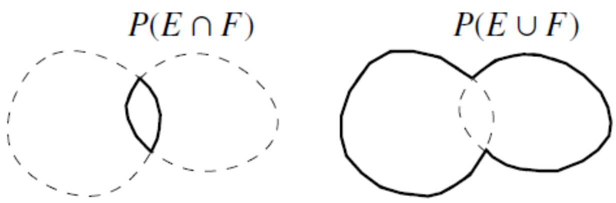

Figure 3.4: If the boundaries of E and F intersect on a set of null $(n-1)$-dimensional measure, then inequality (3.18) is an equality [1].

3.2.4 Compactness from perimeter bounds

Theorem 3.3. *If $R > 0$ and $\{E_h\}_{h\in\mathbb{N}}$ are sets of finite perimeter in $\mathbb{R}^n$, with*

$$\sup_{h\in\mathbb{N}} P(E_h) < \infty \tag{3.20}$$

$$E_h \subset B_R, \qquad \forall\, h \in \mathbb{N}, \tag{3.21}$$

then there exist E of finite perimeter in $\mathbb{R}^n$ and a subsequence $E_{h_k}, k \in \mathbb{N}$ such that

$$E_{h_k} \to E, \qquad \mu_{E_{h_k}} \overset{*}{\rightharpoonup} \mu_E, \qquad E \subset B_R.$$

Proof. The proof is done by approximation, so we will use Proposition 3.4.
Step one: We show that if $Q(x,r) = x + (0,r)^n$ and $u \in C^1(\mathbb{R}^n)$, then

$$\int_{Q(x,r)} \left|u - (u)_{Q(x,r)}\right| \le \sqrt{n} r \int_{Q(x,r)} |\nabla u|, \tag{3.22}$$

where $(u)_{Q(x,r)} = r^{-n} \int_{Q(x,r)} u$. By a change of variables and up to adding a constant to u, we reduce to considering the case $Q(x,r) = (0{,}1)^n = Q$ and $(u)_Q = 0$ (zero mean). Finally, since $\sum_{i=1}^n |x_i| \le \sqrt{n}\sqrt{\sum_{i=1}^n x_i^2}$, it suffices to show

$$\int_Q |u| \le \sum_{i=1}^n \int_Q |\partial_i u|.$$

We prove it by induction. In the case $n = 1$, by the mean value theorem for integrals there exists $x_0 \in Q$ such that $u(x_0) = (u)_Q = 0$, so that $|u(x)| = |u(x) - u(x_0)| \le \int_Q |u'|$ for every $x \in Q$ and we have the claim by integrating in Q (which has measure 1).
Let now $n \ge 2$, set $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$, and define $v(x_1) = \int_{(0,1)^{n-1}} u(x_1, x')dx'$. Since $\int_{(0,1)} v = \int_Q u = 0$, and $v'(x_1) = \int_{(0,1)^{n-1}} \partial_1 u(x_1, x')dx'$, arguing by induction (the case $n-1$ on $u(x_1, x')$ and $n = 1$ on $v(x_1)$) we find

$$\begin{aligned}
\int_Q |u| &= \int_{(0,1)} dx_1 \int_{(0,1)^{n-1}} |u(x)|dx' \\
&\le \int_{(0,1)} dx_1 \int_{(0,1)^{n-1}} |u(x_1,x') - v(x_1)|dx' + \int_{(0,1)} |v(x_1)|dx_1 \\
&\le \int_{(0,1)} dx_1 \sum_{i=2}^n \int_{(0,1)^{n-1}} |\partial_i u|dx' + \int_{(0,1)} |v'(x_1)|dx_1 \\
&\le \sum_{i=2}^n \int_Q |\partial_i u| + \int_{(0,1)} \int_{(0,1)^{n-1}} |\partial_1 u(x_1,x')|dx'dx_1 = \sum_{i=1}^n \int_Q |\partial_i u|.
\end{aligned}$$

Step two: If E is a set of finite perimeter in $\mathbb{R}^n$ with $|E| < \infty$, then for every $r > 0$ there exists a finite union T of disjoint cubes of side length r with

$$|E \triangle T| \le \sqrt{n} r P(E);$$

see Figure 3.5.

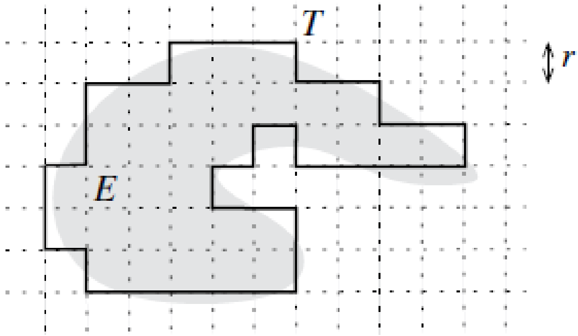

Figure 3.5: We obtain a set T from a partition of $\mathbb{R}^n$ into cubes of side length r as the union of those cubes Q such that $|E \cap Q| \ge |Q|/2$. This set is finite because $|E| < \infty$ [1].

Indeed, let $\{Q_h\}_{h\in\mathbb{N}}$ be a disjoint family of open cubes of side length r such that $\cup_{h\in\mathbb{N}} \overline{Q_h} = \mathbb{R}^n$. If $\varepsilon > 0$ and $u = (1_E \star \rho_\varepsilon)$ (here we use the approximation of 1_E to have a C^1 function as in step one), then by step one

$$\int_{\mathbb{R}^n} |\nabla u| = \sum_{h\in\mathbb{N}} \int_{Q_h} |\nabla u| \ge \frac{1}{\sqrt{n} r} \int_{Q_h} \Big|u - (u)_{Q_h}\Big|.$$

Letting $\varepsilon \to 0$, recalling that by the regularization $|P(E)| = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} |\nabla(1_E \star \rho_\varepsilon)|$, we find that

$$\begin{aligned}
\sqrt{n} r P(E) &\ge \sum_{h\in\mathbb{N}} \int_{Q_h} \Big|1_E - (1_E)_{Q_h}\Big| = \sum_{h\in\mathbb{N}} \int_{Q_h} \left|1_E - \frac{|Q_h \cap E|}{r^n}\right| \\
&= \sum_{h\in\mathbb{N}} |E \cap Q_h| \left|1 - \frac{|Q_h \cap E|}{r^n}\right| + |Q_h \setminus E| \left|\frac{|Q_h \cap E|}{r^n}\right| \\
&= 2 \sum_{h\in\mathbb{N}} \frac{|E \cap Q_h|\, |Q_h \setminus E|}{r^n}.
\end{aligned}$$

Since $|E| < \infty$, $|Q_h \cap E| \ge r^n/2$ for at most finitely many cubes Q_h. Up to a permutation we can assume that these cubes are exactly the first N elements of the sequence $\{Q_h\}_{h\in\mathbb{N}}$, that is we may assume that

$$|Q_h \cap E| \ge \frac{r^n}{2}, \text{ if } 1 \le h \le N, \qquad |Q_h \setminus E| \ge \frac{r^n}{2}, \text{ if } h \ge N+1.$$

As a consequence, if we let $T = \cup_{h=1}^{N} Q_h$, then we find as required

$$\sqrt{n} r P(E) \geq \sum_{h=1}^{N} |Q_h \setminus E| + \sum_{h=N+1}^{\infty} |Q_h \cap E| = |T \setminus E| + |E \setminus T| = |T \triangle E|.$$

Step three: The set $X = \{E \in \mathcal{M}(\mathcal{L}^n) : |E| < \infty\}$ is a complete metric space endowed with the distance $d(E, F) = |E \triangle F| = ||1_E - 1_F||_{L^1(\mathbb{R}^n)}$. We now claim that each set $Y_{R,p} \subset X$ defined as

$$Y_{R,p} = \{E \in \mathcal{M}(\mathcal{L}^n) : E \subset B_R, P(E) \leq p\}, \quad R, p \in (0, \infty)$$

is d-compact. By lower semicontinuity of perimeter (Proposition 3.2), $Y_{R,p}$ is closed (recall also that, by assumption, $\sup_{h \in \mathbb{N}} P(E_h) < \infty$). Thus, to ensure compactness, we are left to prove that $Y_{R,p}$ is totally bounded: for every $\delta > 0$, there exist $M \in \mathbb{N}$ and a finite family $\{T_j\}_{j=1}^{M} \subset X$ such that $Y_{R,p}$ is covered by the finite M balls of radius δ, namely it holds that

$$\forall\, E \in Y_{R,p},\ \exists\, T_j \text{ for some } j = 1, ..., M, \text{ such that } d(E, T_h) \leq \delta.$$

To prove this we are going to use the fact that all the sets E_h are uniformly contained in B_R. Indeed, for fixed $\delta > 0$, let $r > 0$ be such that $\sqrt{n}\, r\, p \leq \delta$, and let $\{Q_h\}_{h \in \mathbb{N}}$ be the family of cubes associated with the side length r as in step two. Only a finite number of cubes Q_h intersect B_R, and we denote this finite family $\{S_h\}_{h=1}^{N}$. Thus, taking all the possible finite unions of cubes from $\{S_h\}_{h=1}^{N}$ gives a finite family too, and we denote such a family as $\{T_j\}_{j=1}^{M}$.
By step two, for every $E \in Y_{R,p}$ there exists T_j (we know that such T of step two associated to E is one of the T_j because $E \subset B_R$ and T_j are constructed from B_R) such that

$$|E \triangle T_h| \leq \sqrt{n}\, r\, p \leq \delta,$$

as required.
Step four: By assumption (3.20) and (3.21), $\{E_h\}_{h \in \mathbb{N}} \subset Y_{R,p}$, for some $R, p > 0$. By step three, there exists $E \subset B_R$ and a subsequence such that $E_{h_k} \to E$, in the $L^1(\mathbb{R}^n)$ topology. Finally, using Proposition 3.2, E is a set of finite perimeter in $\mathbb{R}^n$ and $\mu_{E_h} \overset{*}{\rightharpoonup} \mu_E$. □

Remark 3.5. The assumption $E_h \subset B_R$ for all $h \in \mathbb{N}$ is necessary, because we cannot conclude the compactness of a sequence of sets from the perimeter bound (3.20) only. An easy counterexample is a sequence of unit balls with centers $\{x_h\}_{h \in \mathbb{N}}$ such that $|x_h| \to \infty$. This sequence $E_h = B(x_h, 1)$ satisfied $P(E_h) = n\omega_n$ for every $h \in \mathbb{N}$, however for every Lebesgue measurable set E we have

$$|E \triangle E_h| = |E| + \omega_n$$

for h large enough (because x_h goes to infinity), thus $\{E_h\}_{h\in\mathbb{N}}$ can not admit any converging subsequence.
However, the sequence locally converges to the empty set because

$$\lim_{h\to\infty}\left|K\cap(\emptyset\triangle E_h)\right| = \lim_{h\to\infty}\int_K |1_{E_h}| = 0 \qquad \forall\, K\subset\mathbb{R}^n \text{ compact},$$

so that compactness with respect to the local convergence still holds. Consequently, it is often useful to consider sequences of sets that are only of locally finite perimeter (Theorem 3.3 is for sets of finite perimeter), which are expected to converge at most locally. The following Corollary is particularly useful.

Corollary 3.2. *If $\{E_h\}_{h\in\mathbb{N}}$ are sets of locally finite perimeter in $\mathbb{R}^n$ with*

$$\sup_{h\in\mathbb{N}} P(E_h;B_R) < \infty, \qquad \forall\, R>0, \tag{3.23}$$

then there exist E of locally finite perimeter and a subsequence $\{E_{h_k}\}_{k\in\mathbb{N}}$ such that

$$E_{h_k} \overset{\text{loc}}{\to} E, \qquad \mu_{E_{h_k}} \overset{*}{\rightharpoonup} \mu_E.$$

Proof. Step one: If E is of locally finite perimeter and $R>0$, then

$$P(E\cap B_R) \le P(E;B_R) + P(B_R). \tag{3.24}$$

Indeed, given $R'<R$, let $v_\varepsilon\in C_c^\infty(B_{R'})$ be such that $0\le v_\varepsilon\le 1$, $v_\varepsilon\to 1_{B_{R'}}$ in $L^1(\mathbb{R}^n)$, and $\int_{\mathbb{R}^n}|\nabla v_\varepsilon| \to P(B_{R'})$ as $\varepsilon\to 0^+$, and let $u_\varepsilon = 1_E\star\rho_\varepsilon$. First we note that

$$u_\varepsilon v_\varepsilon \to 1_{E\cap B_{R'}}$$

in $L^1_{\text{loc}}(\mathbb{R}^n)$. Therefore by Example 3.5 it holds that

$$P(E\cap B_{R'}) \le \liminf_{\varepsilon\to 0^+}\int_{\mathbb{R}^n}|\nabla(u_\varepsilon v_\varepsilon)|.$$

Now using that $u_\varepsilon\le 1$ and Example 3.5 again to u_ε and v_ε we find that

$$\begin{aligned}
P(E\cap B_{R'}) &\le \liminf_{\varepsilon\to 0^+}\int_{\mathbb{R}^n}|\nabla(u_\varepsilon v_\varepsilon)| \le \limsup_{\varepsilon\to 0^+}\int_{\mathbb{R}^n}u_\varepsilon|\nabla(v_\varepsilon)| + \int_{\mathbb{R}^n}v_\varepsilon|\nabla(u_\varepsilon)|\\
&\le \limsup_{\varepsilon\to 0^+}\int_{\mathbb{R}^n}|\nabla(v_\varepsilon)| + \int_{\mathbb{R}^n}v_\varepsilon|\nabla(u_\varepsilon)|\\
&\le P(B_{R'}) + \lim_{\varepsilon\to 0^+}\int_{B_{R'}}|\nabla(u_\varepsilon)| \le P(B_{R'}) + P(E;\overline{B_{R'}})\\
&\le P(B_R) + P(E;B_R).
\end{aligned}$$

Since $E \cap B_{R'} \overset{\text{loc}}{\to} E \cap B_R$ as $R' \to R$, by Proposition 3.2 we find (3.24).
Step two: By (3.23) and (3.24), and given $j \in \mathbb{N}$, we may apply the compactness Theorem 3.3 to $\{E_h \cap B_j\}_{h\in\mathbb{N}}$. By a standard diagonal argument, we find a common subsequence $\{E_{h_k}\}_{k\in\mathbb{N}}$ such that, for each $j \in \mathbb{N}$, there exists a set of finite perimeter F_j and $E_{h_k} \cap B_j \to F_j$ as $k \to \infty$. By the fact that $E_h \cap B_j \subset E_h \cap B_{j+1}$, up to null sets $F_j \subset F_{j+1}$, so that E_{h_k} locally converges to $E = \bigcup_{j\in\mathbb{N}} F_j$. By Proposition (3.2), E is a set of locally finite perimeter and $\mu_{E_{h_k}} \overset{*}{\rightharpoonup} \mu_E$. □

Chapter 4

Existence of minimizers in geometric variational problems

Finite perimeter sets are particularly suited for solving problems related to area (i.e. perimeter) minimization which are slightly different from Plateau's problem. A typical problem is: *find the domain* $D \subset \mathbb{R}^3$ *which minimizes*

$$\text{Area}(\partial D) + \underbrace{\int_D f(x)dx}_{\text{additional integral term}} + \underbrace{\text{additional constraint}}_{\text{e.g. Volume}(D)\text{ is prescribed}}.$$

Some geometric variational problems of this type are Plateau-type problems, relative isoperimetric problems, prescribed mean curvature problems and capillarity, i.e. equilibrium of liquid drops confined in a given container. We want to solve this problem by the usual "compactness and semicontinuity" approach, i.e. the direct method. To this end, we would like to use:

1. a class $\mathcal{F}$ of sets $\mathbb{E} \subset \mathbb{R}^n$ endowed with a topology with good compactness properties so that sets with smooth boundaries belong to this class and are dense;

2. a notion of *perimeter* $P(E)$ for every $E \in \mathcal{F}$ so that $E \mapsto P(E)$ is lower-semicontinuous on $\mathcal{F}$, and P extends the usual notion of perimeter $\mathcal{H}^{n-1}$ of the boundary ∂E; more precisely we require that $P(E) = \mathcal{H}^{n-1}(\partial E)$ for every set E with smooth boundary and for every $E \in \mathcal{F}$ there exists a sequence $E_h \to E$ with smooth boundaries and satisfying $\mathcal{H}^{n-1}(\partial E_h) \to P(E)$.

We know from Chapter 3 that sets of finite perimeter exactly satisfy these requirements. We recall that given a $E \subset \mathbb{R}^n$ measurable set,

$$\text{Perimeter of } E = P(E) = ||D1_E|| = |\mu_E|(\mathbb{R}^n) = \sup_{|T|\leq 1} \int_{\mathbb{R}^n} T \cdot d\mu_E = \sup_{|T|\leq 1} \int_E \operatorname{div} T,$$

$$d(E, E') = |E \triangle E'| = ||1_E - 1_{E'}||_{L^1(\mathbb{R}^n)},$$

($D1_E$ denotes the distributional derivative of 1_E).

4.1 Plateau-type problem: finding surfaces with minimal area and prescribed boundary

The classical Plateau problem consists in minimizing the area among surfaces passing through a given curve. Generalized formulations of this problem are more properly conceived, for example, in the setting of currents and varifolds; however a simple formulation is possible in the framework of sets of finite perimeter.

Given a set $\Omega \subset \mathbb{R}^3$ which is open, bounded and convex, let Γ be a curve on $\partial\Omega$ and Σ_0 a subset of $\partial\Omega$ such that its relative boundary to $\partial\Omega$ agrees with Γ.

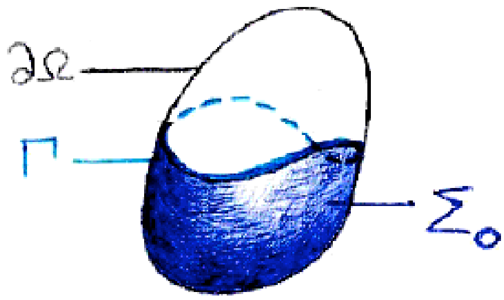

Figure 4.1: The prescribed curve Γ is given as the relative boundary to $\partial\Omega$ of a set $\Sigma_0 \subset \partial\Omega$ [3].

We now construct a smooth, bounded, open set E_0 in $\mathbb{R}^3 \setminus \Omega$ such that $\partial E_0 \cap \partial\Omega = \Sigma_0$. The set E_0 is surrounding Ω covering the set Σ_0, see Figure 4.2 (in section).

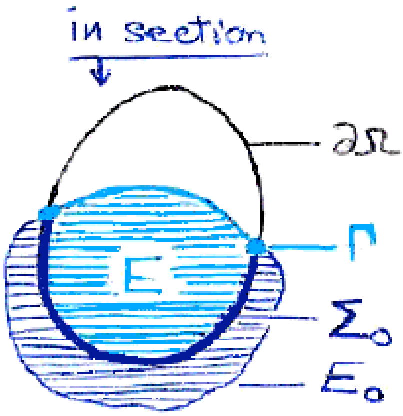

Figure 4.2: The open set E_0 is such that $\partial E_0 \cap \partial\Omega = \Sigma_0$. We minimize $P(E)$ among all sets E with finite perimeter in $\mathbb{R}^3$ such that $E \setminus \Omega = E_0$ [3].

Then we minimize $P(E)$ among all sets E with finite perimeter in $\mathbb{R}^3$ such that $E \setminus \Omega = E_0$, which is roughly speaking imposing Σ_0 as the "boundary condition" for the admissible sets E, which turns out to impose Γ as the contour of the part of ∂E which is inside Ω (see Figure 4.2).
The constraint $E \setminus \Omega = E_0$ is equivalently written as

$$|(E \setminus \Omega)\triangle E_0| = \mathcal{L}^3\left((E \setminus \Omega)\triangle E_0\right) = 0$$

and this constraint is closed in the $L^1(\mathbb{R}^3)$ convergence. In fact, we consider

$$\gamma(\Omega, E_0) = \inf\left\{P(E) : E \subset \mathbb{R}^3,\ E \setminus \Omega = E_0\right\}.$$

Existence of minimizers is then addressed as follows.

Proposition 4.1. (Existence of minimizers for the Plateau-type problem).
Let $\Omega \subset \mathbb{R}^3$ be a bounded, open set and let E_0 be a set of finite perimeter in $\mathbb{R}^n$. Then there exists a set of finite perimeter E such that $E \setminus \Omega = E_0$ and $P(E) \leq P(F)$ for every F such that $F \setminus \Omega = E_0$. In particular, E is a minimizer in the variational problem

$$\gamma(\Omega, E_0) = \inf\left\{P(E) : E \subset \mathbb{R}^3,\ E \setminus \Omega = E_0\right\}. \tag{4.1}$$

Moreover, if Ω is convex, then $S = \partial E \cap \overline{\Omega}$ minimizes the area among all surfaces with boundary Γ.

Proof. Since E_0 itself is admissible in (4.1), we have $\gamma = \gamma(\Omega, E_0) < \infty$. Let us now consider a minimizing sequence $\{E_h\}_{h\in\mathbb{N}}$ in (4.1),

$$E_h \setminus \Omega = E_0, \qquad P(E_h) \leq P(E_0), \qquad \lim_{h\to\infty} P(E_h) = \gamma.$$

Since $\overline{\Omega} \cup E_0$ is bounded and $E_h \subset \overline{\Omega} \cup E_0$, then by Theorem 3.3 (using that $P(E_h) \leq P(E_0) < \infty$) there exists a set of finite perimeter E such that, up to extracting a subsequence, we have $E_h \to E$, namely $1_{E_h} \to 1_E$ in $L^1(\mathbb{R}^3)$. In particular $E \setminus \Omega = E_0$ (E is admissible), because the constraint is closed in the $L^1(\mathbb{R}^3)$ convergence. Indeed,

$$|(E \setminus \Omega) \triangle E_0| = \int_{\mathbb{R}^n} |1_{E \setminus \Omega} - 1_{E_0}| \leq \int_{\mathbb{R}^n} |1_{E \setminus \Omega} - 1_{E_h \setminus \Omega}| + \int_{\mathbb{R}^n} |1_{E_h \setminus \Omega} - 1_{E_0}|,$$

the first term is arbitrarily small because $E_h \to E$ and thus $E_h \setminus \Omega = E_h \cap \Omega^c \to E \cap \Omega^c$, whereas the second is equal to 0 being E_h an admissible set. Moreover, by Proposition 3.2,

$$\gamma \leq P(E) \leq \liminf_{h \to \infty} P(E_h) = \gamma,$$

so E is a minimizer in (4.1).
To prove that $S = \partial E \cap \overline{\Omega}$ minimizes the area among all surfaces with boundary Γ, it can be first proved by regularity theory [3] that ∂E is smooth inside Ω and Lipschitz on $\partial\Omega$, so that S is an admissible (smooth) surface with boundary Γ. Another competitor S' (with less area) must be contained in the convex set Ω, otherwise projecting S' on Ω would reduce the area without modifying the boundary. Secondly, the surface $S' \cup (\partial E_0 \setminus \overline{\Omega})$ turns out to be a compact Lipschitz surface without boundary in $\mathbb{R}^3$, and therefore it is oriented and bounds a set E' with finite perimeter such that $E' \setminus \Omega = E_0$. This set E' would satisfy $P(E') < P(E)$ (because S' has less area than S), thus contradicting the minimality of E in the original problem (4.1). □

Remark 4.1. This approach imposes strong constraints on the geometry of the bounding curve Γ. The point is that finite perimeter sets are not really suited for Plateau's classical problem. This same approach can be extended to higher dimensions to obtain minimal hypersurfaces with prescribed boundary. However regularity can not be expected if $d \geq 8$, as in Bernstein's problem [4].

Remark 4.2. One might wonder why we did not follow a simpler way, namely taking the set E which minimizes the perimeter among all sets E contained in $\overline{\Omega}$ such that

$$\partial^* E \cap \partial\Omega = \Sigma_0.$$

(With $\partial^* E$ we denote the reduced boundary). The reason is that the constraint $\partial^* E \cap \partial\Omega = \Sigma_0$, or better $\mathcal{H}^2\left((\partial^* E \cap \partial\Omega) \triangle \Sigma_0\right) = 0$ is not closed in the $L^1(\mathbb{R}^3)$ convergence. To see this, consider the following example in Figure 4.3.

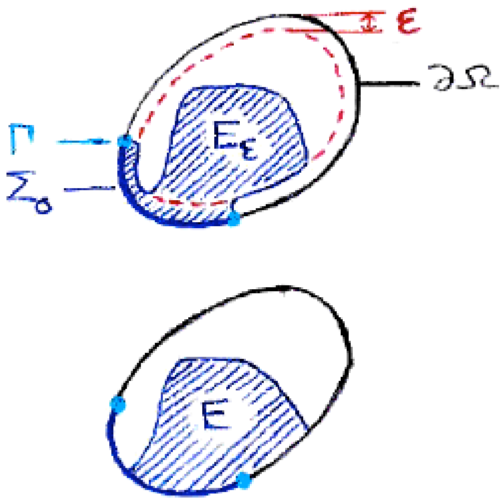

Figure 4.3: The sets E_ε are such that $\partial^* E_\varepsilon \cap \partial\Omega = \Sigma_0$, but for the limit E as $\varepsilon \to 0$, $\partial^* E \cap \partial\Omega \neq \Sigma_0$, so the condition is not closed [3].

This can be reduced to the fact that the trace operator on the space $BV(\mathbb{R}^3)$ (which contains sets of finite perimeter, i.e. the characteristic function is a function in $BV(\mathbb{R}^3)$) is well-defined but, unlike what happens with Sobolev spaces, is only continuous with the norm topology, not the dual topology, i.e. it is not weak-$*$ continuous [3], [5].

Remark 4.3. The lack of closure of the constraint $\partial^* E \cap \partial\Omega = \Sigma_0$, or its measure theoretic version $\mathcal{H}^2\left((\partial^* E \cap \partial\Omega)\triangle\Sigma_0\right) = 0$ means in particular that if we take a minimizing sequence E_h, there is no way to ensure that the limit E still satisfies the constraint, that is the surface $\partial E \cap \partial\Omega$ has boundary equal to Γ. This is not only a technical problem but corresponds to a real phenomenon. Indeed let us consider Γ as the boundary of two coaxial discs of radius 1, and Ω bounded by two parallel planes (which contain the two discs respectively) at distance h. The union of the two (open) discs is Σ_0, see Figure 4.4 on the left. It can be proved that for h sufficiently small the catenoid is the absolute minimizer of the area among all the surfaces with boundary Γ; when h is sufficiently large, the two discs are already the minimizer, namely $S = \Sigma_0$.
Let us consider the case with h large. If we minimize $P(E)$ among all sets $E \subset \overline{\Omega}$ such that

$$\partial^* E \cap \partial\Omega = \Sigma_0,$$

then every minimizing sequence $\{E_n\}_{n\in\mathbb{N}}$ (see Figure 4.4 on the right) satisfies

$$E_n \xrightarrow{L^1} \emptyset = E, \quad P(E) = 0,$$

so there exists no minimizer. Instead if we proceed in the same way as before with the surrounding set E_0, the minimizer is exactly the set E_0 (namely, there is no "extra" part inside Ω) and the corresponding minimal surface is indeed $\partial E_0 \cap \partial\Omega = \Sigma_0$.

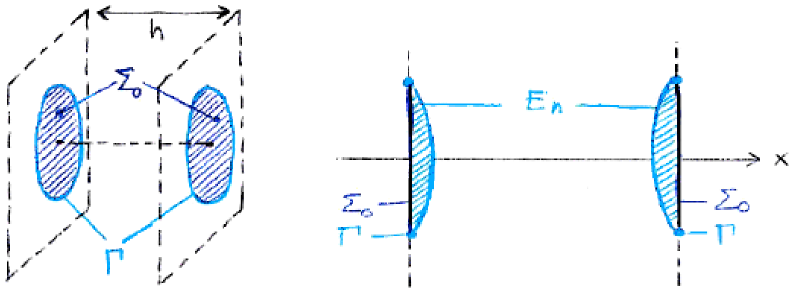

Figure 4.4: Two coaxial and parallel rings at distance h (on the left) and a section of the minimizing sequence $\{E_n\}_{n\in\mathbb{N}}$ (on the right) [3], [4].

4.2 Relative isoperimetric problem and potential energy

Given an open set $A \subset \mathbb{R}^n$, the **relative isoperimetric problem in** A consists in the volume-constrained minimization of the relative perimeter in A, namely

$$\alpha(A, m) = \inf\{P(E; A) : E \subset A, |E| = m\}, \tag{4.2}$$

where $m \in (0, |A|)$ (we allow $|A| = \infty$); see Figure 4.5.

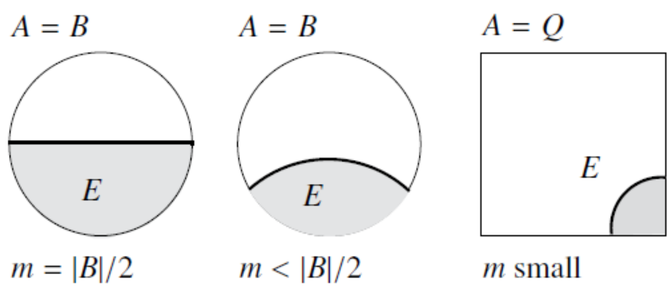

Figure 4.5: Some relative isoperimetric problems in the plane. We cannot expect uniqueness of minimizers [1].

A minimizer E in (4.2), normalized to have $\mathrm{spt}(\mu_E) = \partial E$ according to Corollary 3.1, is called **relative isoperimetric set in** A. The case $A = \mathbb{R}^n$ corresponds to the Euclidean isoperimetric problem. Apart from their geometric interest, relative isoperimetric problems are also strictly related to the study of equilibrium shapes of a liquid confined in a given container.
When A is bounded and has finite perimeter, the existence of minimizers is proved by the Direct Method.

Proposition 4.2. (Existence of relative isoperimetric sets). *If A is an open bounded set of finite perimeter and $m \in (0, |A|]$, then there exists a set of finite perimeter $E \subset A$ such that $P(E; A) = \alpha(A; m)$ and $|E| = m$. In particular, E is a minimizer in the variational problem*

$$\alpha(A, m) = \inf \{P(E; A) : E \subset A, |E| = m\}.$$

Proof. Let $E_t = A \cap \{x : x_1 < t\}$ $(t \in \mathbb{R})$. By a continuity argument, there exists $t \in \mathbb{R}$ such that $|E_t| = m$. By Lemma 3.1 and the definition of $E_t \subset A$ we have that

$$P(E_t; A) \leq P(A) + P(\{x : x_1 < t\}; A) < \infty$$

(since the half space $\{x : x_1 < t\}$ is of locally finite perimeter and A is bounded), and therefore $\alpha = \alpha(A; m) < \infty$. Now let $\{E_h\}_{h \in \mathbb{N}}$ be a minimizing sequence in (4.2) that is

$$E_h \subset A, \quad |E_h| = m, \quad \lim_{h \to \infty} P(E_h; A) = \alpha.$$

We now notice that

$$P(E_h) = P(E_h \subset A) \leq P(E_h; A) + P(A), \tag{4.3}$$

(in the case A is a ball, this was proved in (3.24); in the general case it follows from set operations on Gauss-Green measures using Federer's theorem, see Theorem 16.3 in [1]). By (4.3), we deduce that $\sup_{h \in \mathbb{N}} P(E_h) < \infty$. Since A is bounded, by Theorem 3.3 there exists a set of finite perimeter $E \subset \mathbb{R}^n$ such that, up to extracting a subsequence, $E_h \to E$. In particular $E \subset A$ (by the Theorem) and $|E| = \lim_{h \to \infty} |E_h| = m$, so that by Proposition 3.2,

$$\alpha \leq P(E; A) \leq \liminf_{h \to \infty} P(E_h; A) = \alpha.$$

□

Problems involving potential energies: Interesting variational problems arise from the interaction between perimeter and potential energy terms. Given a Lebesgue measurable function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, we define the **potential energy** of E associated with g on the Lebesgue measurable set E simply as

$$\mathcal{G}(E) = \int_E g(x)dx.$$

The minimization of the potential energy alone is easy to understand. When the function g is in $L^1(\mathbb{R}^n)$ the functional is naturally continuous (strongly) with respect to the convergence $E_h \to E$, for example by dominated convergence applied to $g\,1_{E_h}$; by Fatou's lemma we just need $g^- \in L^1(\mathbb{R}^n)$ to have lower semicontinuity. The volume constraint is obviously closed, since we work with convergence of Lebesgue integrals.
One example is the action of gravity on subsets of $\mathbb{R}^n$ lying above the horizontal plane $\{x_3 = 0\}$, and we set

$$g(x) = \begin{cases} x_3, & \text{if } x_3 > 0 \\ \infty, & \text{if } x_3 < 0. \end{cases}$$

A problem of geometric nature is the **prescribed mean curvature problem** associated with a Lebesgue measurable function $g : \mathbb{R}^n \to \mathbb{R}$ and an open set $A \subset \mathbb{R}^n$,

$$\inf \{P(E) + \mathcal{G}(E) : E \subset A\}. \tag{4.4}$$

The terminology used here arises from the fact that, if $g \in C^0(A)$, E is a minimizer in (4.4), and $A \cap \partial E$ is a C^2-hypersurface, then the mean curvature H_E of E is equal to $-g$ in A; see Chapter 5. If g is positive then the problem is trivial and the solutions is the empty set, because otherwise we could choose the set $E = \{g > 0\}$ and obtain a positive value of the functional. If, however, g takes negative values, then the problem will possess, in general, non trivial minimizers. If $g \in L^1(A)$ and A is bounded, then the existence of minimizers is easily obtained by the Direct Method. One has only to take the following proposition into account.

Proposition 4.3. *If $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a Lebesgue measurable function with $g^- \in L^1(F)$ for a Lebesgue measurable set $F \subset \mathbb{R}^n$ (possibly $F = \mathbb{R}^n$), and $E_h \to E$, then*

$$\int_{E\cap F} g(x)dx \le \liminf_{h\to\infty} \int_{E_h\cap F} g(x)dx.$$

Proof. By Fatou's lemma and since $g^- \in L^1(F)$ we easily find that

$$\int_{E\cap F} g^+(x)dx \le \liminf_{h\to\infty} \int_{E_h\cap F} g^+(x)dx,$$

$$\int_{E\cap F} g^-(x)dx = \lim_{h\to\infty} \int_{E_h\cap F} g^-(x)dx.$$

We conclude by $g = g^+ - g^-$. □

We can improve last Proposition with just the local convergence $E_h \overset{\text{loc}}{\to} E$ if g is non-negative.

Proposition 4.4. *If $g : \mathbb{R}^n \to [0, \infty]$ is measurable and $E_h \overset{\text{loc}}{\to} E$, then*

$$\mathcal{G}(E) \leq \liminf_{h\to\infty} \mathcal{G}(E_h).$$

Proof. For every compact set $K \subset \mathbb{R}^n$, by last Proposition applied to $E_h \cap K \to E \cap K$ we have

$$\int_{K\cap E} g(x)dx \leq \liminf_{h\to\infty} \int_{K\cap E_h} g(x)dx \leq \liminf_{h\to\infty} \int_{E_h} g(x)dx$$

where in the last inequality we used the positivity of g. Using now $K_j = B_j, j \in \mathbb{N}$ by the monotone convergence theorem we get

$$\mathcal{G}(E) = \lim_{j\to\infty} \int_{B_j\cap E} g(x)dx \leq \liminf_{h\to\infty} \mathcal{G}(E_h).$$

□

Finally we can prove the existence of a minimizer for the following problems.

Proposition 4.5. *If A is a bounded, open set of finite perimeter, $m \in (0, |A|)$, $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a Lebesgue measurable function with $g \in L^1(A)$, then the following variational problem*

$$\inf \{P(E; A) + \mathcal{G}(E) : E \subset A, |E| = m\} \tag{4.5}$$

admits a minimizer.

Proof. The proof is similar to the one of Proposition 4.2. Indeed, let $\{E_h\}_{h\in\mathbb{N}}$ be a minimizing sequence in (4.5) that is

$$E_h \subset A, \quad |E_h| = m, \quad \lim_{h\to\infty} P(E_h; A) + \mathcal{G}(E_h) = \alpha,$$

($\alpha < \infty$ as proved in Proposition 4.2, now also using $g \in L^1(A)$ and $E_t \subset A$). For each $h \in \mathbb{N}$, since $g \in L^1(A)$ and $E_h \subset A$, we have that

$$\int_{E_h} g(x)dx \geq -\int_{E_h} |g(x)|dx \geq -\int_A |g(x)|dx = -c$$

with $c < +\infty$, and therefore we can bound

$$P(E_h; A) - c \leq P(E_h; A) + \mathcal{G}(E_h)$$

$$\lim_{h\to\infty} P(E_h; A) \le \lim_{h\to\infty} P(E_h; A) + \mathcal{G}(E_h) + c = \alpha + c.$$

From this we deduce $\sup_{h\in\mathbb{N}} P(E_h) < \infty$ as in Proposition 4.2. Since A is bounded, by Theorem 3.3 there exists a set of finite perimeter $E \subset \mathbb{R}^n$ such that, up to extracting a subsequence, $E_h \to E$. In particular $E \subset A$ (by the Theorem) and $|E| = \lim_{h\to\infty} |E_h| = m$, so that by Proposition 3.2 and Proposition 4.3,

$$\alpha \le P(E; A) + \mathcal{G}(E) \le \liminf_{h\to\infty} P(E_h; A) + \mathcal{G}(E_h) = \alpha$$

(where we used the superadditivity of $\liminf$). This proves that E is a minimizer of the variational problem (4.5). $\square$

Chapter 5

Capillarity

An interesting problem for which sets of finite perimeter are a good framework is the *capillarity*, which consists in studying the equilibrium shapes of a *drop* of liquid confined in a given container. Mathematically speaking, the drop E will be a set of finite perimeter and the container A an open set with sufficiently smooth boundary, with $E \subset A$. In Sections 5.1 and 5.2 we give the necessary analytical tools for taking into account the intrinsic geometric nature of the problem, most notably the Gauss-Green theorem on surfaces, where the mean curvature is involved. In Section 5.3 first variations are computed in order to be able to find stationarity conditions for perimeter minimizers. In Section 5.4 the problem is finally studied through the minimization of the *Gauss free energy functional*.

5.1 Tangential differentiability and the area formula on surfaces

Let M be a k-dimensional C^1-surface in $\mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is **tangentially differentiable with respect to** M **at** x if there exists a linear function $\nabla^M f(x) \in \mathbb{R}^m \otimes T_x M$ (the vector space of linear maps from $T_x M$ to $\mathbb{R}^m$) such that, uniformly on $\{v \in T_x M : |v| = 1\}$,

$$\lim_{h \to 0} \frac{f(x + tv) - f(x)}{h} = \nabla^M f(x)v. \tag{5.1}$$

In other words, the restriction of f to $x + T_x M$ is differentiable at x. The **tangential Jacobian** of f with respect to M at x is then defined by

$$J^M f(x) = \sqrt{\det(\nabla^M f(x)^* \nabla^M f(x))}.$$

Remark 5.1. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ may fail to be differentiable at every $x \in M$ while being tangentially differentiable with respect to M at every $x \in M$. For example, let $M=\{x_n=0\}\subset\mathbb{R}^n$, $\varphi\in C^1_c(\mathbb{R}^{n-1};\mathbb{R}^m)$, and set $f(x)=\varphi(x')+|x_n|e$, for $e\in\mathbb{R}^m$ and $x=(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}=\mathbb{R}^n$. In this case

$$\nabla^M f(x)=\sum_{i=1}^{n-1}\partial_i\varphi(x')\otimes e_i,\qquad J^M f(x)=J\varphi(x'),\qquad \forall x\in M.$$

Remark 5.2. If $f\in C^1(\mathbb{R}^n;\mathbb{R}^m)$, M is a k-dimensional C^1-surface in $\mathbb{R}^n$, and $x\in M$, then f is tangentially differentiable at x and $\nabla^M f(x)$ is the restriction of $\nabla f(x)$ at T_xM: thus, if $\{\tau_h\}_{h=1}^k$ is an orthonormal basis of T_xM and $\{\nu_h\}_{h=1}^{n-k}$ is an orthonormal basis of $(T_xM)^\perp$, then

$$\nabla^M f(x)=\sum_{h=1}^k(\nabla f(x)\tau_h)\otimes\tau_h=\nabla f(x)-\sum_{h=1}^{n-k}(\nabla f(x)\nu_h)\otimes\nu_h.$$

Theorem 5.1. (Area formula on surfaces) *If $M\subset\mathbb{R}^n$ is a k-dimensional C^1-surface and $f\in C^1(\mathbb{R}^n;\mathbb{R}^m)$ $(m\ge k)$ is injective, then*

$$\mathcal{H}^k(f(M))=\int_M J^M f\,d\mathcal{H}^k.$$

Proof. Step one: If V is a k-dimensional subspace of $\mathbb{R}^n$, $T_1\in\mathbb{R}^n\otimes\mathbb{R}^k$ is such that $T_1(\mathbb{R}^k)=V$, and $T_2\in\mathbb{R}^m\otimes V$ (so that $T_2T_1\in\mathbb{R}^m\otimes\mathbb{R}^k$), then

$$J(T_2T_1)=JT_2\,JT_1. \tag{5.2}$$

This can be proved using the polar decompositions $T_1=P_1S_1$ and $T_2=P_2S_2$, where $P_1\in\mathbf{O}(k,n),S_1\in\mathbf{Sym}(k),P_2\in\mathbf{O}(V,m)$, and $S_2\in\mathbf{Sym}(V)$. Then by computing $(T_2T_1)^*T_2T_1=S_1US_1$, with $U=P_1^*S_2^2P_1$, and using the spectral theorem for $S_2^2=\sum_{h=1}^k\mu_hv_h\otimes v_h$, with $\mu_h\ge 0$ and $\{v_h\}_{h=1}^k$ orthonormal basis of V, one proves the formula.

Step two: Since M is a k-dimensional C^1-surface, there exist $A_h\subset\mathbb{R}^k$ open and $g_h\in C^1(\mathbb{R}^k;\mathbb{R}^n)$ injective such that $M=\bigcup_{h\in\mathbb{N}}g_h(A_h)$. Since $T_xM=T_x(g_h(A_h))$ when $x\in M\cap g_h(A_h)$, we can directly assume $M=g(A)$ for $A\subset\mathbb{R}^k$ open and $g\in C^1(\mathbb{R}^k;\mathbb{R}^n)$ injective. Applying the area formula (2.3) to $f\circ g\in C^1(\mathbb{R}^k;\mathbb{R}^m)$,

$$\mathcal{H}^k(f(M))=\int_A J(f\circ g)(z)dz.$$

If $z\in A$, then $\nabla g(z)(\mathbb{R}^k)=T_{g(z)}M$, and, in particular,

$$\nabla(f\circ g)(z)=\nabla f(g(z))\nabla g(z)=\nabla^M f(g(z))\nabla g(z).$$

By step one, $J(f \circ g) = ((J^M f) \circ g)Jg$ on A. Hence, again by the area formula (2.3),

$$\mathcal{H}^k(f(M)) = \int_A \Big((J^M f) \circ g\Big)\, Jg = \int_{g(A)} J^M f \, d\mathcal{H}^k = \int_M J^M f \, d\mathcal{H}^k.$$

□

We now improve the same formula on rectifiable sets, so we are going to use the definitions and the results from Appendix A.
Let M be a locally $\mathcal{H}^k$-rectifiable set in $\mathbb{R}^n$ and let $x \in M$ be such that the approximate tangent space T_xM exists. As in the C^1-case, we say that $f : \mathbb{R}^n \to \mathbb{R}^m$ is **tangentially differentiable with respect to** M **at** x if the restriction of f to $x + T_xM$ is differentiable at x.

Lemma 5.1. *If M is a locally $\mathcal{H}^k$-rectifiable set, and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map, then $\nabla^M f(x)$ exists at $\mathcal{H}^k$-a.e. $x \in M$.*

Proof. By Theorem A.2 we can directly assume that $M = g(E)$ is a regular Lipschitz image. If $M = g(E)$ is a k-dimensional regular Lipschitz image in $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function, and $f \circ g$ is differentiable at $z \in E$, then f is tangentially differentiable with respect to M at $x = g(z)$, with

$$\nabla^M f(x) = \nabla(f \circ g)(z) \nabla g(z)^{-1}, \qquad \text{on } T_xM = dg_z(\mathbb{R}^k). \tag{5.3}$$

Here we have denoted by $\nabla g(z)^{-1}$ the inverse of $\nabla g(z)$ seen as an isomorphism between $\mathbb{R}^k$ and $T_xM = dg_z(\mathbb{R}^k)$.
By Lemma A.1, M admits the approximate tangent space $T_xM = dg_z(\mathbb{R}^k)$ at $x = g(z)$. If $w \in \mathbb{R}^k$ and $v = \nabla g(z)w$, then

$$\begin{aligned} \lim_{t \to 0^+} \frac{f(x+tv) - f(x)}{t} &= \lim_{t \to 0^+} \frac{f(g(z) + t\nabla g(z)w) - f(g(z))}{t} \\ &= \lim_{t \to 0^+} \frac{f(g(z+tw)) - f(g(z))}{t}, \end{aligned}$$

since f is a Lipschitz function and since $|g(z+tw) - g(z) - t\nabla g(z)w| = o(t)$. Since $f \circ g$ is differentiable at z by Rademacher's theorem A.1, we thus find that f admits directional derivatives at x along directions $v \in T_xM$, with

$$\frac{\partial f}{\partial v}(x) = \nabla(f \circ g)(z)w, \qquad w = \nabla g(z)^{-1}. \tag{5.4}$$

Since $\nabla g(z)$ is a linear isomorphism between $\mathbb{R}^k$ and T_xM we find that

$$v \in T_xM \mapsto \frac{\partial f}{\partial v}(x)$$

is a linear map (for each $v \in T_xM$, there is a unique $w \in \mathbb{R}^k$). Since f is a Lipschitz function it follows that

$$\lim_{t\to 0^+} \frac{f(x+tv)-f(x)}{t} = \frac{\partial f}{\partial v}(x), \qquad \text{uniformly on } v \in T_xM, |v| = 1,$$

which, by (5.4), proves (5.3), so we have the thesis of the Lemma. □

Using Lemma 5.1, one can prove the following theorem. By Theorem A.2 it is enough to consider $M = g(E)$ a regular Lipschitz image and use Lemma A.1 to deduce the existence of $T_xM = \nabla g(z)(\mathbb{R}^k)$. Then one proceeds in the same way as Theorem 5.1.

Theorem 5.2. *If M is a locally $\mathcal{H}^k$-rectifiable set and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map with $1 \le k \le m$, then*

$$\int_{\mathbb{R}^m} \mathcal{H}^0 \left(M \cap \{f = y\}\right) d\mathcal{H}^k(y) = \int_M J^M f \, d\mathcal{H}^k. \tag{5.5}$$

A useful corollary we are going to use is the following.

Corollary 5.1. *If $S \subset \mathbb{R}^{n-1}$ is a $\mathcal{H}^{n-2}$-rectifiable set in $\mathbb{R}^{n-1}$, $u : \mathbb{R}^{n-1} \to \mathbb{R}$ a Lipschitz function, $\Gamma = \{(z, u(z)) \in \mathbb{R}^n : z \in S\}$, $g : \mathbb{R}^n \to [-\infty, \infty]$ is a Borel function, and either $g \ge 0$ or $g \in L^1(\mathbb{R}^n, \mathcal{H}^{n-2} \llcorner \Gamma)$, then*

$$\int_\Gamma g \, d\mathcal{H}^{n-2} = \int_S \overline{g} \sqrt{1 + |\nabla^S u|^2} \, d\mathcal{H}^{n-2}, \tag{5.6}$$

where we have set $\overline{g}(z) = g(z, u(z)), z \in \mathbb{R}^{n-1}$.

Proof. Consider the Lipschitz function $f : \mathbb{R}^{n-1} \to \mathbb{R}^n$ defined by $f(z) = (z, u(z))$, $z \in \mathbb{R}^{n-1}$, so that $\Gamma = f(S)$ is trivially a locally $\mathcal{H}^{n-2}$-rectifiable set in $\mathbb{R}^n$. By Theorem 5.2, we only have to prove that

$$J^S f = \sqrt{1 + |\nabla^S u|^2}, \qquad \mathcal{H}^{n-2}\text{-a.e. on } S.$$

Indeed, since $\nabla f = \mathrm{Id}_{\mathbb{R}^{n-1}} + e_n \otimes \nabla' u$ and $\nabla^S u = \nabla' u - (\nabla' u \cdot \nu_S)\nu_S$, where $\nu_S(z) \in (T_zS)^\perp$ for $\mathcal{H}^{n-2}$-a.e. $z \in S$, we have

$$\begin{aligned} \nabla^S f &= \nabla f - (\nabla f \nu_S) \otimes \nu_S \\ &= \mathrm{Id}_{\mathbb{R}^{n-1}} + e_n \otimes \nabla' u - \nu_S \otimes \nu_S - (\nabla' u \cdot \nu_S) e_n \otimes \nu_S \\ &= \mathrm{Id}_{\nu_S^\perp} + e_n \otimes \nabla^S u, \end{aligned}$$

so that $(\nabla^S f)^*(\nabla^S f) = \mathrm{Id}_{\nu_S^\perp} + (\nabla^S u) \otimes (\nabla^S u)$. Using $\det(\mathrm{Id} + v \otimes v) = 1 + |v|^2$, we conclude. □

5.2 Gauss-Green theorem on surfaces

We finally introduce the natural extension of the Gauss-Green theorem to hypersurfaces in $\mathbb{R}^n$. The resulting formula will prove useful in understanding the geometric meaning of the first variation formula for perimeter in Section 5.3, and will play a crucial role in establishing an important necessary "boundary condition for minimality", known as Young's law, in Section 5.4.
If $M \subset \mathbb{R}^n$ is a k-dimensional C^1-surface and $T \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$ we shall say that T is **tangential** to M if $T(x) \in T_x M$ for every $x \in M$, and that T is **normal to** M if, instead, $T(x) \in (T_x M)^\perp$ for every $x \in M$.

Theorem 5.3. (Gauss-Green theorem on surfaces) *If $M \subset \mathbb{R}^n$ is a C^2-hypersurface with boundary Γ, then there exists a normal vector field $\mathbf{H}_M \in C^0(M; \mathbb{R}^n)$ to M and a normal vector field $\nu_\Gamma^M \in C^1(\Gamma; S^{n-1})$ to Γ such that*

$$\int_M \nabla^M \varphi \, d\mathcal{H}^{n-1} = \int_M \varphi \mathbf{H}_M \, d\mathcal{H}^{n-1} + \int_\Gamma \varphi \nu_\Gamma^M d\mathcal{H}^{n-2}, \tag{5.7}$$

for every $\varphi \in C^1_c(\mathbb{R}^n)$. Moreover, if $T \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$ is normal to M, then

$$T \cdot \nu_\Gamma^M = 0 \quad \text{on } \Gamma. \tag{5.8}$$

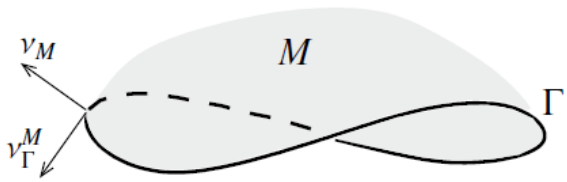

Figure 5.1: The normal to the boundary Γ of M [1].

Before proving Theorem 5.3, we make some remarks.

Remark 5.3. We say that M is a C^h-hypersurface with boundary Γ if M is a C^h-hypersurface, Γ is the relative boundary of M, and, for every $x \in \Gamma$, there exist $r > 0$, an open set $E \subset \mathbb{R}^{n-1}$ with C^h-boundary, and a function $u \in C^h(\mathbb{R}^{n-1})$ such that, up to rotation and with Notation of Chapter 3,

$$\mathbf{C}(x, r) \cap M = \{(z, u(z)) : z \in \mathbf{D}(\mathbf{p}x, r) \cap E\},$$
$$\mathbf{C}(x, r) \cap \Gamma = \{(z, u(z)) : z \in \mathbf{D}(\mathbf{p}x, r) \cap \partial E\}.$$

As it turns out from the implicit function theorem, Γ is an $(n-2)$-dimensional C^h-surface (with empty relative boundary in $\mathbb{R}^n$). We also notice that at every relative interior point of M, that is, at every $x \in M$, the first condition above holds true with $E = \mathbb{R}^{n-1}$.

Remark 5.4. (Mean curvature vector) The vector field $\mathbf{H}_M$ is called the **mean curvature vector** to M. The definition of the **scalar mean curvature** $H_M : M \to \mathbb{R}$ of M depends on the mean curvature vector and the explicit choice of a unit normal vector field $\nu_M : M \to S^{n-1}$ to M through the formula

$$\mathbf{H}_M = H_M \nu_M.$$

Remark 5.5. By condition (5.8), ν_Γ^M is tangential to M, that is

$$\nu_\Gamma^M \cdot \nu_M = 0 \quad \text{on } \Gamma,$$

see Figure 5.1.

Remark 5.6. (Divergence theorem on surfaces) Given a vector field $T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$, we define the **tangential divergence** of T on M by the formula

$$\operatorname{div}^M T = \operatorname{div} T - (\nabla T \nu_M) \cdot \nu_M = \operatorname{trace}(\nabla^M T), \tag{5.9}$$

where $\nu_M : M \to S^{n-1}$ is any unit normal vector field to M. Discontinously switching ν_M to $-\nu_M$ on part of M leaves $\operatorname{div}^M T$ unchanged, therefore it is always $\operatorname{div}^M T \in C^0(M)$ even if M is not orientable. The Gauss-Green formula on surfaces (5.7) is equivalently formulated in a "divergence-type formula" as follows: for every $T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$

$$\int_M \operatorname{div}^M T \, d\mathcal{H}^{n-1} = \int_M T \cdot \mathbf{H}_M \, d\mathcal{H}^{n-1} + \int_\Gamma (T \cdot \nu_\Gamma^M) d\mathcal{H}^{n-2}. \tag{5.10}$$

Proof. (of Theorem 5.3) By using partitions of unity, and up to a rigid motions and homotheties, it suffices to prove (5.7) for $\varphi \in C_c^1(\mathbf{C})$, assuming that

$$\mathbf{C} \cap M = \{(z, u(z)) : z \in \mathbf{D} \cap E\},$$
$$\mathbf{C} \cap \Gamma = \{(z, u(z)) : z \in \mathbf{D} \cap \partial E\},$$

where $u \in C^2(\mathbb{R}^{n-1})$ and E is an open set with C^2-boundary in $\mathbb{R}^{n-1}$ (possibly, $E = \mathbb{R}^{n-1}$). An orientation of the C^2-surface $\mathbf{C} \cap M$ is then given by the vector field $\nu_M \in C^1(\mathbf{C} \cap M; S^{n-1})$, defined locally as

$$\overline{\nu_M} = \frac{(-\nabla' u, 1)}{\sqrt{1 + |\nabla' u|^2}}, \qquad \text{on } \mathbf{D} \cap E, \tag{5.11}$$

where, if $g : \mathbb{R}^n \to \mathbb{R}$, then we set $\overline{g} : \mathbb{R}^{n-1} \to \mathbb{R}$ as $\overline{g}(z) = g(z, u(z))$ $(z \in \mathbb{R}^{n-1})$, namely the restriction of g over the graph of u. Since $\mathbf{H}_M = H_M \nu_M$, we define

$\mathbf{H}_M \in C^0(\mathbf{C}\cap M;\mathbb{R}^n)$ and $H_M \in C^0(\mathbf{C}\cap M)$ locally taking into account (5.11) and thus by setting

$$\overline{H_M} = -\mathrm{div}'\left(\frac{\nabla' u}{\sqrt{1+|\nabla' u|^2}}\right) \qquad \text{on } \mathbf{D}\cap E. \tag{5.12}$$

We now notice that $\overline{\varphi} \in C_c^1(\mathbf{D})$, with $\overline{\nabla\varphi} = (\overline{\nabla'\varphi}, \overline{\partial_n\varphi})$ and with

$$\overline{\nabla\varphi\cdot\nu_M} = \frac{-\overline{\nabla'\varphi}\cdot\nabla' u + \overline{\partial_n\varphi}}{\sqrt{1+|\nabla' u|^2}} \qquad \text{on } \mathbf{D}\cap E.$$

Since $\nabla^M\varphi = \nabla\varphi - (\nabla\varphi\cdot\nu_M)\nu_M$, by using Theorem 2.4 for each component (horizontals e_k, for $k=1,\dots,n-1$ and vertical e_n) and by using the expressions of $\overline{\nu_M}$ and $\overline{\nabla\varphi\cdot\nu_M}$, we find

$$e_n\cdot\int_M \nabla^M\varphi\, d\mathcal{H}^{n-1} = \int_{\mathbf{D}\cap E}\left(\overline{\partial_n\varphi} + \frac{(\overline{\nabla'\varphi}\cdot\nabla' u - \overline{\partial_n\varphi})}{1+|\nabla' u|^2}\right)\sqrt{1+|\nabla' u|^2}, \tag{5.13}$$

$$e_k\cdot\int_M \nabla^M\varphi\, d\mathcal{H}^{n-1} = \int_{\mathbf{D}\cap E}\left(\overline{\partial_k\varphi} - \frac{(\overline{\nabla'\varphi}\cdot\nabla' u - \overline{\partial_n\varphi})}{1+|\nabla' u|^2}\partial_k u\right)\sqrt{1+|\nabla' u|^2}. \tag{5.14}$$

Vertical component: Concerning (5.13), $\nabla'\overline{\varphi} = \nabla'\varphi(z,u(z)) = \overline{\nabla'\varphi} + \overline{\partial_n\varphi}\nabla' u$ gives

$$\left(\overline{\partial_n\varphi} + \frac{(\overline{\nabla'\varphi}\cdot\nabla' u - \overline{\partial_n\varphi})}{1+|\nabla' u|^2}\right)\sqrt{1+|\nabla' u|^2} = \frac{\nabla' u}{\sqrt{1+|\nabla' u|^2}}\cdot\nabla'\overline{\varphi},$$

and thus, by the classical divergence theorem and since $\overline{\varphi}=0$ on $\partial\mathbf{D}$,

$$\begin{aligned} e_n\cdot\int_M &\nabla^M\varphi\, d\mathcal{H}^{n-1} \\ &= -\int_{\mathbf{D}\cap E}\overline{\varphi}\,\mathrm{div}'\left(\frac{\nabla' u}{\sqrt{1+|\nabla' u|^2}}\right) + \int_{\mathbf{D}\cap\partial E}\overline{\varphi}\frac{\nabla' u\cdot\nu_E}{\sqrt{1+|\nabla' u|^2}}d\mathcal{H}^{n-2} \\ &= e_n\cdot\int_M \varphi\mathbf{H}_M\, d\mathcal{H}^{n-1} + e_n\cdot\int_\Gamma \varphi\nu_\Gamma^M d\mathcal{H}^{n-2}, \end{aligned}$$

provided we define $e_n\cdot\overline{\nu_\Gamma^M}$ on $\mathbf{C}\cap\Gamma$ by the formula

$$e_n\cdot\overline{\nu_\Gamma^M} = \frac{\nabla' u\cdot\nu_E}{\sqrt{1+|\nabla' u|^2}\sqrt{1+|\nabla^S u|^2}} \tag{5.15}$$

where $S = \mathbf{D}\cap\partial E$ and Corollary 5.1 has been taken into account for the last equality (the integral over Γ).

Horizontal components: Again by Theorem 2.4, the formula (5.12) and the divergence theorem,

$$
\begin{aligned}
e_k \cdot \int_M \varphi \, \mathbf{H}_M \, d\mathcal{H}^{n-1} &= \int_{\mathbf{D}\cap E} \overline{\varphi} \, \partial_k u \, \mathrm{div}' \left(\frac{\nabla' u}{\sqrt{1+|\nabla' u|^2}} \right) \\
&= -\int_{\mathbf{D}\cap E} \frac{\nabla' u}{\sqrt{1+|\nabla' u|^2}} \cdot \nabla'(\overline{\varphi} \, \partial_k u) + \int_{\mathbf{D}\cap\partial E} \overline{\varphi} \, \partial_k u \frac{\nabla' u \cdot \nu_E}{\sqrt{1+|\nabla' u|^2}} \\
&= -\int_{\mathbf{D}\cap E} \overline{\varphi} \, \partial_k \left(\sqrt{1+|\nabla' u|^2} \right) + \partial_k u \frac{\overline{\nabla' \varphi} \cdot \nabla' u}{\sqrt{1+|\nabla' u|^2}} + \partial_k u \, \overline{\partial_n \varphi} \frac{|\nabla' u|^2}{\sqrt{1+|\nabla' u|^2}} \\
&\quad + \int_{\mathbf{D}\cap\partial E} \overline{\varphi} \, \partial_k u \frac{\nabla' u \cdot \nu_E}{\sqrt{1+|\nabla' u|^2}}
\end{aligned}
$$

From (5.14) and by the Gauss-Green theorem we thus conclude that

$$
\begin{aligned}
& e_k \cdot \int_M (\nabla^M \varphi - \varphi \, \mathbf{H}_M) \, d\mathcal{H}^{n-1} \\
&= \int_{\mathbf{D}\cap E} (\overline{\partial_k \varphi} + \overline{\partial_n \varphi} \, \partial_k u) \sqrt{1+|\nabla' u|^2} + \overline{\varphi} \, \partial_k \left(\sqrt{1+|\nabla' u|^2} \right) \\
&\quad - \int_{\mathbf{D}\cap\partial E} \overline{\varphi} \, \partial_k u \frac{\nabla' u \cdot \nu_E}{\sqrt{1+|\nabla' u|^2}} \\
&= \int_{\mathbf{D}\cap E} \partial_k \left(\overline{\varphi} \sqrt{1+|\nabla' u|^2} \right) - \int_{\mathbf{D}\cap\partial E} \overline{\varphi} \, \partial_k u \frac{\nabla' u \cdot \nu_E}{\sqrt{1+|\nabla' u|^2}} \\
&= \int_{\mathbf{D}\cap\partial E} \overline{\varphi} \left(\sqrt{1+|\nabla' u|^2} e_k - \frac{\partial_k u \nabla' u}{\sqrt{1+|\nabla' u|^2}} \right) \cdot \nu_E \, d\mathcal{H}^{n-2} = e_k \cdot \int_\Gamma \varphi \, \nu_\Gamma^M \, d\mathcal{H}^{n-2},
\end{aligned}
$$

provided we defined $e_k \cdot \nu_\Gamma^M$ on $\mathbf{C} \cap \Gamma$ and for $k = 1, ..., n-1$ as

$$
e_k \cdot \overline{\nu_\Gamma^M} = \left(\sqrt{1+|\nabla' u|^2} e_k - \frac{\partial_k u \nabla' u}{\sqrt{1+|\nabla' u|^2}} \right) \cdot \frac{\nu_E}{\sqrt{1+|\nabla^S u|^2}}, \tag{5.16}
$$

and once again we used Corollary 5.1 in the last equality.
Geometric properties of ν_Γ^M: by a direct computation, using the expressions (5.15) and (5.16) together with the formula for $\overline{\nu_M}$, it is easily checked that ν_Γ^M is a unit vector and it is orthogonal to ν_M and normal to Γ. □

5.3 First variation of perimeter

In order to derive valuable information about perimeter minimizers, we want to construct a curve of competitors "passing through" the candidate minimizer E, as usually done in the Calculus of Variations to derive the Euler-Lagrange equations (first order necessary minimality conditions).

We recall that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a **diffeomorphism of** $\mathbb{R}^n$ if f is smooth, bijective and has a smooth inverse $g = f^{-1}$. If E is an open set with C^1-boundary, then $f(E)$ is still an open set with C^1-boundary. In the notation of Section 3.1, from $f(\{\psi = 0\}) = \{\psi \circ g = 0\}$ and $\nabla(\psi \circ g) = (\nabla g)^*[(\nabla \psi) \circ g]$, we find

$$\nu_{f(E)}(y) = \frac{\nabla g(y)^* \nu_E(g(y))}{|\nabla g(y)^* \nu_E(g(y))|}, \qquad \forall y \in \partial f(E) = f(\partial E).$$

Similar conclusions hold for sets of locally finite perimeter. From now on we are going to use the results from Appendix B, i.e. the structure theorems to deal with (diffeomorphic images of) sets of finite perimeter.

Proposition 5.1. (Diffeomorphic images of sets of finite perimeter) *If E is a set of locally finite perimeter in $\mathbb{R}^n$ and f is a diffeomorphism of $\mathbb{R}^n$ with $g = f^{-1}$, then $f(E)$ is a set of locally finite perimeter in $\mathbb{R}^n$ with*

$$\mathcal{H}^{n-1}\Big(f(\partial^* E) \triangle \partial^* f(E)\Big) = 0, \tag{5.17}$$

$$\int_{\partial^* f(E)} \varphi \, \nu_{f(E)} d\mathcal{H}^{n-1} = \int_{\partial^* E} (\varphi \circ f) Jf (\nabla g \circ f)^* \, \nu_E \, d\mathcal{H}^{n-1}, \tag{5.18}$$

for every $\varphi \in C^0_c(\mathbb{R}^n)$, which may take also the form

$$\mu_{f(E)} = f_{\#} \left(Jf (\nabla g \circ f)^* \, \mu_E \right).$$

In particular, for every Borel set $F \subset \mathbb{R}^n$,

$$\mathcal{H}^{n-1}\Big(F \cap \partial^* f(E)\Big) = \int_{g(F) \cap \partial^* E} Jf \Big| (\nabla g \circ f)^* \nu_E \Big| \, d\mathcal{H}^{n-1}. \tag{5.19}$$

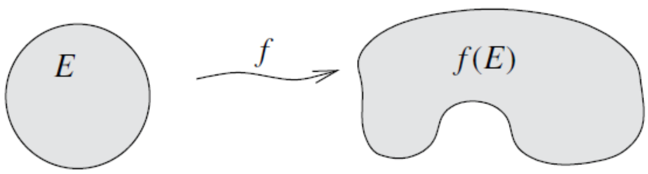

Figure 5.2: Deforming a set of finite perimeter E by a diffeomorphism f. By (5.17) the image of the reduced boundary of E is $\mathcal{H}^{n-1}$-equivalent to the reduced boundary of $f(E)$ [1].

Proof. We first remark that if $u_h \to u$ in $L^1_{\rm loc}(\mathbb{R}^n)$, then $u_h \circ g \to u \circ g$ in $L^1_{\rm loc}(\mathbb{R}^n)$. Indeed, if K is compact, then $g(K)$ is compact, and thus the area formula (writing $K = f(g(K))$) implies

$$\int_K |u_h \circ g - u \circ g| = \int_{g(K)} |u_h - u| Jf \le \mathrm{Lip}\,(f; g(K))^n \int_{g(K)} |u_h - u|.$$

We used the fact that for a Lipschitz function f with Lipschitz constant L, then it holds that $Jf \le L^n$ (being polynomial in the entries of the gradient matrix).
Now define a tensor field $Gf \in C^\infty(\mathbb{R}^n; \mathbb{R}^n \otimes \mathbb{R}^n)$ as

$$Gf = Jf(\nabla g \circ f)^*, \qquad g = f^{-1}.$$

If $u \in C^1(\mathbb{R}^n)$, $T \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$, $v = u \circ g$, and $S = T \circ g$, then by the area formula (2.4)

$$\begin{aligned}\int_{\mathbb{R}^n} v \,\mathrm{div}\, S &= -\int_{\mathbb{R}^n} S \cdot \nabla v = -\int_{\mathbb{R}^n} S \cdot \Big((\nabla g)^* (\nabla u \circ g) \Big) \\ &= -\int_{\mathbb{R}^n} T(x) \cdot \Big(Gf(x) \nabla u(x) \Big) dx.\end{aligned}$$

If $u = (1_E) \star \rho_\varepsilon$, then, by our initial remark, $v = u \circ g \to 1_{f(E)}$ in $L^1_{\rm loc}(\mathbb{R}^n)$ as $\varepsilon \to 0^+$. Since by the regularization Proposition 3.4 we have $-\nabla u \overset{*}{\rightharpoonup} \mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E$, by the identity above we find

$$\int_{f(E)} \mathrm{div}\, S = \int_{\partial^* E} T(x) \cdot \Big(Gf(x) \nu_E(x) \Big) d\mathcal{H}^{n-1}(x)$$

so that, by (3.9), $f(E)$ is of locally finite perimeter with

$$\int_{\partial^* f(E)} S \cdot \nu_{f(E)} d\mathcal{H}^{n-1} = \int_{\partial^* E} T(x) \cdot \Big(Gf(x) \nu_E(x) \Big) d\mathcal{H}^{n-1}(x).$$

Take now $S = (\varphi e) \star \rho_\varepsilon$ in the formula above for $\varphi \in C^0_c(\mathbb{R}^n)$ and $e \in S^{n-1}$; letting $\varepsilon \to 0^+$,

$$e \cdot \int_{\partial^* f(E)} \varphi \, \nu_{f(E)} d\mathcal{H}^{n-1} = e \cdot \int_{\partial^* E} (\varphi \circ f) Gf \, \nu_E \, d\mathcal{H}^{n-1}, \qquad \forall e \in S^{n-1},$$

which is (5.18). Now by (5.18) and an approximation argument,

$$\int_{F \cap \partial^* f(E)} \nu_{f(E)} \, d\mathcal{H}^{n-1} = \int_{g(F) \cap \partial^* E} Gf \, \nu_E \, d\mathcal{H}^{n-1},$$

for every Borel set $F \subset \mathbb{R}^n$. Taking total variations in the formula above, we prove (5.19). We now prove that $f(E^{(1)}) = f(E)^{(1)}$ and $f(E^{(0)}) = f(E)^{(0)}$, so as to deduce $\partial^e f(E) = f(\partial^e E)$, and thus, by Federer's theorem B.3, the equivalence (5.17). As $(\mathbb{R}^n \setminus E)^{(0)} = E^{(1)}$ and f is a bijection, it suffices to show $f(E^{(0)}) = f(E)^{(0)}$. If we set

$$L_x = \mathrm{Lip}(g; B(f(x),1)), \qquad M_x = \mathrm{Lip}(f; B(x, L_x)), \qquad x \in \mathbb{R}^n,$$

then by a direct computation $g(B(f(x), r)) \subset B(x, L_x r)$ for every $r < 1$. Thus, by the area formula,

$$|f(E) \cap B(f(x), r)| = \int_{E \cap g(B(f(x),r))} Jf \leq M_x^n |E \cap B(x, L_x r)|,$$

for every $x \in \mathbb{R}^n$, $r < 1$. From the last inequality we deduce that $x \in E^{(0)}$ implies $f(x) \in f(E)^{(0)}$, so that $f(E^{(0)}) \subset f(E)^{(0)}$. Reversing the roles of f and g and repeating the same calculations, we get the equality. □

The following lemma provides the first order Taylor's expansion of the determinant close to the identity.

Lemma 5.2. *If* $Z \in \mathbb{R}^n \otimes \mathbb{R}^n$, $\mathrm{Id} = \mathrm{Id}_{\mathbb{R}^n}$, *then*

$$(\mathrm{Id} + tZ)^{-1} = \mathrm{Id} - tZ + \mathrm{O}(t^2),$$

$$\det(\mathrm{Id} + tZ) = 1 + t\,\mathrm{trace}(Z) + \mathrm{O}(t^2).$$

We now construct the variations.
A **one parameter family of diffeomorphisms of** $\mathbb{R}^n$ is a smooth function

$$(x,t) \in \mathbb{R}^n \times (-\varepsilon, \varepsilon) \mapsto f(t,x) = f_t(x) \in \mathbb{R}^n, \qquad \varepsilon > 0,$$

such that, for each fixed $|t| < \varepsilon$, $f_t : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism of $\mathbb{R}^n$. Given an open set $A \subset \mathbb{R}^n$, we say that $\{f_t\}_{|t|<\varepsilon}$ is a **local variation in** A if it defines a one-parameter family of diffeomorphisms such that

$$f_0(x) = x, \qquad \forall x \in \mathbb{R}^n,$$
$$\{x \in \mathbb{R}^n : f_t(x) \neq x\} \subset\subset A, \qquad \forall |t| < \varepsilon.$$

From the last properties it is easily seen that, if $\{f_t\}_{|t|<\varepsilon}$ is a local variation in A, then

$$f_t(E) \triangle E \subset\subset A, \forall\, E \subset \mathbb{R}^n,$$

and the following Taylor's expansion holds uniformly in $\mathbb{R}^n$ (since the set $\{f_t \neq \mathrm{Id}\}$ is compact),

$$f_t(x) = x + t\,T(x) + \mathrm{O}(t^2), \qquad \nabla f_t(x) = \mathrm{Id} + t\nabla T(x) + \mathrm{O}(t^2),$$

where $T \in C_c^\infty(A;\mathbb{R}^n)$ is the **initial velocity of** $\{f_t\}_{|t|<\varepsilon}$,

$$T(x) = \frac{\partial f}{\partial t}(0,x), \qquad x \in \mathbb{R}^n.$$

Conversely, starting from an initial velocity $T \in C_c^\infty(A;\mathbb{R}^n)$, from standard ODE theory $f(t,x)$ is given as the solution of the following Cauchy problem (parametrized with respect to the initial condition $x \in \mathbb{R}^n$)

$$\frac{\partial}{\partial t} f(t,x) = T(f(t,x)), \qquad x \in \mathbb{R}^n,$$
$$f(0,x) = x, \qquad x \in \mathbb{R}^n,$$

for small values of t. The solution $\{f_t\}_{|t|<\varepsilon}$ is a **local variation associated with** T.
We now compute the **first variation of perimeter** (relative to the open set A) with respect to the local variation $\{f_t\}_{|t|<\varepsilon}$ in A, that is, we aim to compute

$$\left.\frac{d}{dt}\right|_{t=0} P(f_t(E);A), \qquad \text{for } T \in C_c^\infty(A;\mathbb{R}^n) \text{ given.}$$

Theorem 5.4. (First variation of perimeter) *If A is an open set in $\mathbb{R}^n$, E is a set of locally finite perimeter in $\mathbb{R}^n$, and $\{f_t\}_{|t|<\varepsilon}$ is a local variation in A, then*

$$P(f_t(E);A) = P(E;A) + t\int_{\partial^* E} \operatorname{div}_E T \, d\mathcal{H}^{n-1} + \mathrm{O}(t^2), \tag{5.20}$$

where T is the initial velocity of $\{f_t\}_{|t|<\varepsilon}$ and $\operatorname{div}_E T : \partial^* E \to \mathbb{R}$, as done in Remark 5.6,

$$\operatorname{div}_E T(x) = \operatorname{div} T(x) - \nu_E(x) \cdot \nabla T(x) \nu_E(x), \qquad x \in \partial^* E, \tag{5.21}$$

*is a Borel function called the **boundary divergence of** T **on** E.*

Remark 5.7. (Mean curvature vector and perimeter) The results from Section 5.2 provide an important geometric insight into the first variation formula of perimeter (5.20). Indeed, if E is an open set with C^2-boundary, applying Theorem 5.3 to $M = \partial E$,

$$\int_{\partial E} \operatorname{div}^{\partial E} T \, d\mathcal{H}^{n-1} = \int_{\partial E} T \cdot \mathbf{H}_{\partial E} \, d\mathcal{H}^{n-1}, \qquad \forall T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n), \tag{5.22}$$

where, recalling Remark 5.6, $\operatorname{div}^{\partial E}$ denotes the tangential divergence of T with respect to $M = \partial E$, and where $\mathbf{H}_{\partial E}$ is the mean curvature vector to ∂E, so $\mathbf{H}_{\partial E} = H_{\partial E} \nu_E$. Of course it holds that $\operatorname{div}_E T = \operatorname{div}^{\partial E} T$ on ∂E, and $\mathbf{H}_E = \mathbf{H}_{\partial E}$. With these conventions, the first variation of perimeter on open sets with C^2-boundary takes the form

$$\left.\frac{d}{dt}\right|_{t=0} P(f_t(E);A) = \int_{\partial E} (T \cdot \nu_E) H_E \, d\mathcal{H}^{n-1}. \tag{5.23}$$

Remark 5.8. If E is of locally finite perimeter, then the distributional mean curvature vector of E in A open is the functional $\mathbf{H}_E : C_c^\infty(A;\mathbb{R}^n) \to \mathbb{R}$ defined by the formula

$$\langle \mathbf{H}_E, T\rangle = \int_{\partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1}, \qquad \forall\, T \in C_c^\infty(A;\mathbb{R}^n).$$

We say that E has (locally summable) distributional (scalar) mean curvature in A, if there exists $H \in L^1_{\mathrm{loc}}(A \cap \partial^* E; \mathcal{H}^{n-1})$ such that

$$\int_{\partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} = \int_{\partial^* E} (T \cdot \nu_E)\, H \, d\mathcal{H}^{n-1}, \qquad \forall T \in C_c^\infty(A;\mathbb{R}^n). \tag{5.24}$$

Proof. (of Theorem 5.4) By Proposition 5.1, in particular by (5.19),

$$P(f_t(E);A) = \int_{A\cap\partial^* E} Jf_t \Big|(\nabla g_t \circ f_t)^* \, \nu_E\Big| \, d\mathcal{H}^{n-1}, \qquad g_t = (f_t)^{-1},$$

so that $P(f_t(E);A)$ is a smooth function of t in a neighbourhood of $t=0$. Since $\nabla f_t = \mathrm{Id} + t\nabla T + \mathrm{O}(t^2)$, by Lemma 5.2 we have

$$\begin{aligned} \nabla g_t \circ f_t = (\nabla f_t)^{-1} &= \mathrm{Id} - t\nabla T + \mathrm{O}(t^2) \\ Jf_t &= 1 + t\,\mathrm{div}\, T + \mathrm{O}(t^2), \end{aligned}$$

uniformly on $\mathbb{R}^n$ as $t \to 0$. In particular,

$$\begin{aligned} |(\nabla g_t \circ f_t)^* \, \nu_E|^2 = |\nu_E - t(\nabla T)^*\nu_E|^2 + \mathrm{O}(t^2) &= 1 - 2t\,\nu_E \cdot \Big((\nabla T)^*\nu_E\Big) + \mathrm{O}(t^2) \\ &= 1 - 2t\,\nu_E \cdot (\nabla T \nu_E) + \mathrm{O}(t^2), \end{aligned}$$

and thus we conclude, as required, that

$$Jf_t \Big|(\nabla g_t \circ f_t)^* \, \nu_E\Big| = 1 + t\Big(\mathrm{div}\, T - \nu_E \cdot (\nabla T \nu_E)\Big) + \mathrm{O}(t^2),$$

noting that $\mathrm{div}\, T - \nu_E \cdot (\nabla T\nu_E) = \mathrm{div}_E T$ on $\partial^* E$. □

We now compute the first variation of the potential energy $\mathcal{G}(E) = \int_E g(x)dx$.

Proposition 5.2. (First variation of potential energy) *If E is a set of locally finite perimeter in $\mathbb{R}^n$, $|E| < \infty$, $g \in C^0(\mathbb{R}^n)$, A is open, and $\{f_t\}_{|t|<\varepsilon}$ is a local variation in A with initial velocity T, then*

$$\int_{f_t(E)} g = \int_E g + t \int_{\partial^* E} g(T\cdot \nu_E) d\mathcal{H}^{n-1} + \mathrm{o}(t).$$

In fact, in the case of Lebesgue measure $g = \mathrm{Id}$, we find that

$$|f_t(E)| = |E| + t \int_{\partial^* E} (T \cdot \nu_E) d\mathcal{H}^{n-1} + \mathrm{O}(t^2). \tag{5.25}$$

Proof. If $g \in C^1(\mathbb{R}^n)$, then by the area formula we have $\mathcal{G}(f_t(E)) = \int_E (g \circ f_t)\, J f_t$. Thus by Lemma 5.2,

$$\begin{aligned}
\mathcal{G}(f_t(E)) - \mathcal{G}(E) &= \int_E g\left(x + t\,T(x) + \mathrm{O}(t^2)\right) J f_t(x) - g(x)\,dx \\
&= \int_E (g + t\nabla g \cdot T)(1 + t\,\mathrm{div}\,T) - g + \mathrm{O}(t^2) \\
&= t\int_E \mathrm{div}(gT) + \mathrm{O}(t^2) = t\int_{\partial^* E} g(T\cdot \nu_E) d\mathcal{H}^{n-1} + \mathrm{O}(t^2).
\end{aligned}$$

Now let $g \in C^0(\mathbb{R}^n)$, $\tilde{g} \in C^1(\mathbb{R}^n)$, and $\sigma = \sup_{\mathbb{R}^n} |g - \tilde{g}| > 0$. Then, setting $\mathcal{G}(E) = \int_E g$,

$$\left| \frac{\mathcal{G}(f_t(E)) - \mathcal{G}(E)}{t} - \frac{1}{t}\left(\int_{f_t(E)} \tilde{g} - \int_E \tilde{g} \right) \right| \leq \frac{|f_t(E) \triangle E|}{|t|}\sigma.$$

By Lemma 5.3 below, there exist positive constants C and $\varepsilon_0 < \varepsilon$ such that $|f_t(E)\triangle E| \leq C|t|$ whenever $|t| < \varepsilon_0$. Thus for $|t| < \varepsilon_0$ it holds that

$$\left| \frac{\mathcal{G}(f_t(E)) - \mathcal{G}(E)}{t} - \frac{1}{t}\left(\int_{f_t(E)} \tilde{g} - \int_E \tilde{g} \right) \right| \leq C\,\sigma, \tag{5.26}$$

so that the relative error between the two ratios is at most $C\,\sigma$. Since $\tilde{g} \in C^1(\mathbb{R}^n)$, the Taylor expansion of above holds and we find

$$\lim_{t\to 0^+} \frac{1}{t}\left(\int_{f_t(E)} \tilde{g} - \int_E \tilde{g} \right) = \int_{\partial^* E} \tilde{g}(T \cdot \nu_E) d\mathcal{H}^{n-1}.$$

Then, by (5.26),

$$\begin{aligned}
\int_{\partial^* E} \tilde{g}(T\cdot\nu_E) d\mathcal{H}^{n-1} - C\,\sigma &\leq \liminf_{t\to 0^+} \frac{\mathcal{G}(f_t(E)) - \mathcal{G}(E)}{t} \\
&\leq \limsup_{t\to 0^+} \frac{\mathcal{G}(f_t(E)) - \mathcal{G}(E)}{t} \leq \int_{\partial^* E} \tilde{g}(T\cdot\nu_E) d\mathcal{H}^{n-1} + C\,\sigma
\end{aligned}$$

As $\sigma \to 0^+$, since $\mathrm{spt}(T)$ is compact we have by dominated convergence that $\int_{\partial^* E} \tilde{g}(T\cdot \nu_E) d\mathcal{H}^{n-1} \to \int_{\partial^* E} g(T\cdot\nu_E) d\mathcal{H}^{n-1}$, and from this we conclude. □

Lemma 5.3. *If E is a set of locally finite perimeter in $\mathbb{R}^n$, A is open, and $\{f_t\}_{|t|<\varepsilon}$ is a local variation in A, then there exist positive constants C and $\varepsilon_0 < \varepsilon$ such that, if K is a compact set with $\{x \neq f_t(x)\} \subset K \subset A$, then*

$$|f_t(E)\triangle E| \leq C|t| P(E; K). \tag{5.27}$$

Proof. Set $g_t = f_t^{-1}$, and let $S \in C_c^\infty(A;\mathbb{R}^n)$ denote the initial velocity of the local variation $\{g_t\}_{|t|<\varepsilon}$. If $\Phi_{s,t}(x) = s\,x + (1-s)g_t(x)$ for $x \in \mathbb{R}^n$, $s \in (0,1)$, $|t| < \varepsilon$, then positive constants C and $\varepsilon_0 < \varepsilon$ exist such that

$$J\Phi_{s,t}(x) \geq \frac{1}{2}, \qquad |x - g_t(x)| \leq C|t|, \qquad \forall x \in \mathbb{R}^n,\ |t| < \varepsilon_0,$$

and $\{\Phi_{s,t}\}_{|t|<\varepsilon_0}$ is a local variation in A. The first bound comes from $\nabla\Phi_{s,t} = s\,\mathrm{Id} + (1-s)\,(\mathrm{Id} + t\nabla S + \mathrm{O}(t^2)) = \mathrm{Id} + t(1-s)\nabla S + \mathrm{O}(t^2)$, and by Lemma 5.2, $J\Phi_{s,t} = 1 + t(1-s)\mathrm{div}\, S$, so that by compact support of $\mathrm{div}\, S$, $s \in (0,1)$, for t small enough it holds $t(1-s)\mathrm{div}\, S \geq -1/2$. The second bound holds because $\partial_t(g_t) = S \circ g_t$ and S is bounded. By the fundamental theorem of calculus we have

$$\begin{aligned} u(g_t(x)) &= u(x) + \int_0^1 \frac{\partial}{\partial s}[u(sx + (1-s)g_t(x))]ds \\ &= \int_0^1 \nabla u\,(\Phi_{s,t}(x)) \cdot (x - g_t(x))ds, \end{aligned}$$

and thus by Fubini's theorem and the area formula we find

$$\begin{aligned} \int_{\mathbb{R}^n} |u(x) - u(g_t(x))|dx &\leq C|t| \int_K dx \int_0^1 |\nabla u(\Phi_{s,t}(x))|ds \\ &= C|t| \int_0^1 ds \int_K \frac{|\nabla u(y)|}{J\Phi_{s,t}(\Phi_{s,t}^{-1}(y))} dy \leq 2C|t| \int_K |\nabla u|. \end{aligned}$$

We now set $u = u_\varepsilon \star \rho_\varepsilon$ and let $\varepsilon \to 0^+$ to deduce (5.27) by dominated convergence, Proposition 3.4 and Proposition 1.2 *(ii)*. □

5.4 Equilibrium shapes of liquids, Young's law

In this section we study the equilibrium shapes of a liquid confined in a given container. The study of this problem provides a significant and instructive application of the various methods and ideas developed so far.
The problem is studied through the introduction of a free energy functional. Precisely, if a liquid occupies a region E inside a given container A (mathematically, E will be a set of finite perimeter and A an open set with sufficiently smooth boundary), then its free energy is given by

$$\sigma_{LF}P(E;A) - \sigma_{LS}P(E;\partial A) + \int_E g(x)dx;$$

see Figure 5.3. Here, $\sigma_{LF} > 0$ denotes the surface tension at the *liquid free surface*, interface between the liquid and the other medium (be it another liquid or gas) filling A. The term $-\sigma_{LS}P(E;\partial A)$ is called the **wetting energy** and it is responsible for the interaction of the liquid with the wall of the container, the *liquid-solid* interface. Finally, the third term denotes a potential energy acting on the liquid, which is typically assumed to be the gravitational energy $g(x) = g\rho\, x_n$, where g is the acceleration of gravity and ρ the (constant) density of the incompressible liquid. The free surface is denoted as Σ^f, the contact surface as Σ^c.

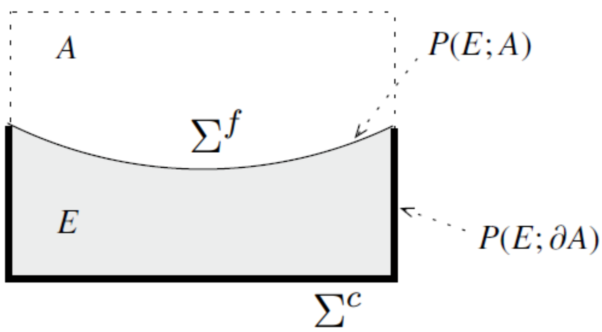

Figure 5.3: The equilibrium shape of a liquid inside a container A [1]. The free surface of the liquid is denoted as Σ^f, the contact surface between liquid and solid container as Σ^c.

For reasons to be soon clarified, the coefficients σ_{LF} and σ_{LS} are assumed to satisfy the "wetting conditions", which is

$$|\sigma_{LS}| \leq \sigma_{LF}$$

or also denoting $\beta = -\sigma_{LS}/\sigma_{LF}$, $|\beta| \leq 1$. The coefficient β is called the **relative adhesion coefficient**. With this notation and normalizing $\sigma_{LF} = 1$ for simiplicity, we study the free energy functional

$$P(E;A) - \beta P(E;\partial A) + \int_E g(x)dx.$$

with the condition $|\beta| \le 1$. The free energy functional is usually minimized under a prescribed volume constraint $|E| = m$.

5.4.1 Lower semicontinuity and existence of minimizers

Given $\beta \in [-1,1]$, an open set $A \subset \mathbb{R}^n$ and a set of finite perimeter $E \subset A$, we shall set

$$\mathcal{F}_\beta(E;A) = P(E;A) - \beta P(E;\partial A) \tag{5.28}$$

for the total surface energy term, and denote by

$$\mathcal{G}(E) = \int_E g(x)dx \tag{5.29}$$

the potential energy associated with a given Borel function $g : \mathbb{R}^n \to \mathbb{R}$. The free energy functional we are going to study is

$$\mathcal{F}_\beta(E;A) + \mathcal{G}(E).$$

We start by discussing the lower semicontinuity of the total surface energy $\mathcal{F}_\beta$ with respect to the L^1-convergence. A necessary condition for the lower semicontinuity is $|\beta| \le 1$; see Remark 5.9. For $\beta \in [-1,0]$, the semicontinuity holds quite trivially, whereas for $\beta \in (0,1]$ we use an additional assumption on A, see Proposition 5.4.

Proposition 5.3. *If $\beta \in [-1,0]$, A is an open set of finite perimeter in $\mathbb{R}^n$, $\{E_h\}_{h\in\mathbb{N}}$ and E are sets of finite perimeter contained in A, and $E_h \to E$, then*

$$\mathcal{F}_\beta(E;A) \le \liminf_{h\to\infty} \mathcal{F}_\beta(E_h;A). \tag{5.30}$$

Proof. By Proposition B.2, for every $E \subset A$ we have

$$P(E) = P(E;A) + P(E;\partial A), \qquad P(E;\partial A) \le P(A), \tag{5.31}$$

$$P(E) \le \mathcal{F}_\beta(E;A) + (1 + |\beta|)P(A). \tag{5.32}$$

Without loss of generality let us assume the right-hand side of (5.30) to be finite. By (5.32) applied to the (minimizing) sequence $\{E_h\}_{h\in\mathbb{N}}$, $E_h \subset A$, we thus find that $\sup_{h\in\mathbb{N}} P(E_h)$ is finite. For this reason, by the compactness Theorem 3.3, the convergence of the E_h to E implies that $\mu_{E_h} \overset{*}{\rightharpoonup} \mu_E$ and, by Proposition 3.2,

$$\liminf_{h\to\infty} P(E_h) \ge P(E), \qquad \liminf_{h\to\infty} P(E_h;A) \ge P(E;A). \tag{5.33}$$

We are thus left to exploit the identity

$$\begin{aligned}\mathcal{F}_\beta(E;A) &= (1+\beta)P(E;A) - \beta\Big(P(E;A) + P(E;\partial A)\Big)\\ &= (1+\beta)P(E;A) - \beta P(E)\end{aligned}$$

and the non-negativity of $1+\beta$ and $-\beta$ to deduce (5.30) from (5.33). □

Remark 5.9. (Loss of semicontinuity when $|\beta| > 1$) If $|\beta| > 1$, then the lower semicontinuity inequality (5.30) may fail. Indeed, let us set $A = \Omega$ the half plane. If $\beta < -1$, which is $\sigma_{LS} > \sigma_{LF}$, take the following sequence E_ε in Figure 5.4,

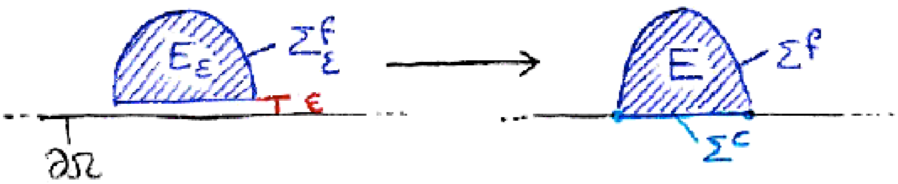

Figure 5.4: When $\beta < -1$, the drop never touches the wall of the container, because it is always more convenient (energy-wise) to insert a thin layer of air to detach the drop from the container, because $\sigma_{LS} > \sigma_{LF}$ [3].

We have, for every $\varepsilon > 0$, $P(E_\varepsilon;\Omega) = \mathcal{H}^{n-1}(\Sigma^f_\varepsilon)$ and $P(E_\varepsilon;\partial\Omega) = 0$, and since $E_\varepsilon \to E$ as $\varepsilon \to 0^+$, the lower semicontinuity fails because

$$\begin{aligned}\mathcal{F}_\beta(E_\varepsilon;\Omega) = \mathcal{H}^{n-1}(\Sigma^f_\varepsilon) \to \mathcal{H}^{n-1}(\Sigma^f) + \mathcal{H}^{n-1}(\Sigma^c) &= P(E;\Omega) + P(E;\partial\Omega)\\ &< P(E;\Omega) - \beta P(E;\partial\Omega)\\ &= \mathcal{F}_\beta(E;\Omega).\end{aligned}$$

Viceversa, if $\beta > 1$, that is $-\sigma_{LS} > \sigma_{LF}$, let us take the following sequence E_ε in Figure 5.5,

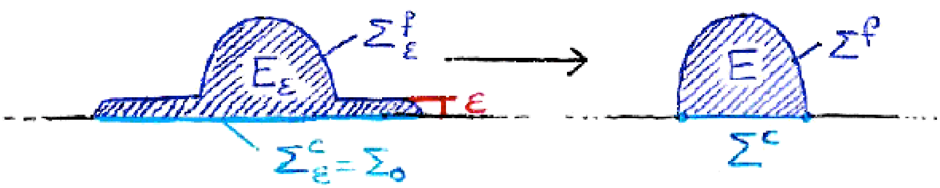

Figure 5.5: When $\beta > 1$, it is always convenient to cover the wall of the container with a thin film of liquid, and separate it from the air [3].

In this case, since $\mathcal{H}^{n-1}(\Sigma^f_\varepsilon) = \mathcal{H}^{n-1}(\Sigma^f) + \mathcal{H}^{n-1}(\Sigma_0 \setminus \Sigma^c) + \mathrm{o}(1)$ as $\varepsilon \to 0^+$, we have that

$$\begin{aligned}\mathcal{F}_\beta(E_\varepsilon;\Omega) = \mathcal{H}^{n-1}(\Sigma^f_\varepsilon) - \beta\mathcal{H}^{n-1}(\Sigma^c_\varepsilon) &\to \mathcal{H}^{n-1}(\Sigma^f) + \mathcal{H}^{n-1}(\Sigma_0 \setminus \Sigma^c) - \beta\mathcal{H}^{n-1}(\Sigma_0)\\ &< \mathcal{H}^{n-1}(\Sigma^f) + \beta\mathcal{H}^{n-1}(\Sigma_0 \setminus \Sigma^c) - \beta\mathcal{H}^{n-1}(\Sigma_0)\\ &= \mathcal{H}^{n-1}(\Sigma^f) - \beta\mathcal{H}^{n-1}(\Sigma^c)\\ &= P(E;\Omega) - \beta P(E;\partial\Omega) = \mathcal{F}_\beta(E;\Omega).\end{aligned}$$

Proposition 5.4. *Let A be a bounded open set with finite perimeter with the property that, for sufficiently small $\delta > 0$, a compactly supported Lipschitz vector field $T_\delta : \mathbb{R}^n \to \mathbb{R}^n$ exists such that $|T_\delta| \le 1$ on $\mathbb{R}^n$ and*

$$T_\delta \cdot \nu_A = 1 \ \text{ on } \partial A, \qquad T_\delta = 0 \ \text{ on } A \setminus \overline{A_\delta}, \tag{5.34}$$

where $A_\delta = \{x \in A : \operatorname{dist}(x,\partial A) < \delta\}$ (if A has C^2-boundary, T_δ may be constructed starting from the gradient of the signed distance function from A). Then the same conclusion of Proposition 5.3 holds true with $\beta \in (0,1]$ too.

Proof. If F is a set of finite perimeter contained in A, then by the divergence theorem (applied first to $T_{\delta,\varepsilon} = T_\delta \star \rho_\varepsilon$ and then, by approximation, as $\varepsilon \to 0^+$ the limit holds uniformly) and Proposition B.2,

$$\int_F \operatorname{div} T_\delta = \int_{A\cap\partial^* F} T_\delta \cdot \nu_F \, d\mathcal{H}^{n-1} + \int_{\partial A\cap\partial^* F} T_\delta \cdot \nu_A \, d\mathcal{H}^{n-1}.$$

Exploiting (5.34), by first putting $\int_{A\cap\partial^* F} T_\delta \cdot \nu_F \, d\mathcal{H}^{n-1}$ on the left-hand side, we thus find

$$P(F;\partial A) \le P(F;A_\delta) + C(\delta)|F|, \qquad F \subset A, \tag{5.35}$$

(where $C(\delta) = \sup_{\mathbb{R}^n} |\nabla T_\delta| \to \infty$ as $\delta \to 0^+$). If now $\{E_h\}_{h\in\mathbb{N}}$ are sets of finite perimeter contained in A and $E_h \to E$, then, as in the proof of Proposition 5.3, we find $\mu_{E_h} \overset{*}{\rightharpoonup} \mu_E$. Applying (5.35) to $F = E_h \triangle E$, while taking also into account Proposition B.1 and Proposition B.2, we find

$$\begin{aligned}|P(E_h;\partial A) - P(E;\partial A)| &\le \mathcal{H}^{n-1}\big(\partial A \cap (\partial^* E_h \triangle \partial^* E)\big) = P(E_h \triangle E;\partial A)\\ &\le P(E_h \triangle E; A_\delta) + C(\delta)|E_h \triangle E|\\ &\le P(E_h;A_\delta) + P(E;A_\delta) + C(\delta)|E_h \triangle E|.\end{aligned}$$

In particular, if $0 \le \beta \le 1$,

$$\begin{aligned}\mathcal{F}_\beta(E_h;A) - \mathcal{F}_\beta(E;A) &\ge P(E_h;A) - P(E;A) - |P(E_h;\partial A) - P(E;\partial A)|\\ &\ge P(E_h;A \setminus \overline{A_\delta}) - P(E;A) - P(E;A_\delta) - C(\delta)|E_h \triangle E|.\end{aligned}$$

By Proposition 3.2, since $A \setminus \overline{A_\delta}$ is open, letting $h \to \infty$, we find

$$\liminf_{h\to\infty} \mathcal{F}_\beta(E_h; A) \geq \mathcal{F}_\beta(E; A) + P(E; A \setminus \overline{A_\delta}) - P(E; A) - P(E; A_\delta),$$

where the right-hand side converges to $\mathcal{F}_\beta(E; A)$ as $\delta \to 0^+$. □

Finally we can state the main theorem of existence of minimizers in bounded containers.

Theorem 5.5. (Existence of minimizers in bounded containers)
If $|\beta| \leq 1$, $g \in L^1_{\text{loc}}(\mathbb{R}^n)$, A is an open bounded set of finite perimeter in $\mathbb{R}^n$ (satisfying (5.34) *in the case $\beta > 0$), and $m \in (0, |A|)$, then there exists a minimizer in*

$$\gamma = \inf \left\{ \mathcal{F}_\beta(E; A) + \mathcal{G}(E) : E \subset A,\, |E| = m \right\}. \tag{5.36}$$

Proof. As shown in Proposition 4.2, the competition class in non-empty, so that $\gamma < \infty$. In fact, we have $\gamma \in \mathbb{R}$, since by $P(E; A) \geq 0$, $P(E; \partial A) \leq P(A)$, and $g \in L^1_{\text{loc}}(\mathbb{R}^n)$,

$$\mathcal{F}_\beta(E; A) + \mathcal{G}(E) \geq -|\beta| P(A) - \int_A |g(x)| dx.$$

Let now $\{E_h\}_{h\in\mathbb{N}}$ be a minimizing sequence in (5.36). As done in Proposition 5.3, by (5.32) we deduce that $\sup_{h\in\mathbb{N}} P(E_h)$ is finite. Since A is bounded, by the compactness Theorem 3.3, there exists a set $E \subset A$ such that, up to subsequences, $E_h \to E$, so that, evidently, $|E| = m$. We now combine Proposition 4.3, Proposition 5.3 and Proposition 5.4 to conclude by lower semicontinuity that E is a minimizer. □

5.4.2 Stationarity conditions

Adopting the methods from Section 5.3, we are now going to prove that the mean curvature of the interior interface of an equilibrium configuration equals the potential energy plus a constant additive factor. The wetting energy plays no role in this result, and indeed no restriction on β is required.

We first apply first variation arguments (from Section 5.3) to study minimizers in *relative isoperimetric problems* defined in (4.2). Given A open and E of finite perimeter in $\mathbb{R}^n$, we say that E is a **volume-constrained perimeter minimizer in** A, if $\mathrm{spt}(\mu_E) = \partial E$ and

$$P(E; A) \leq P(F; A), \tag{5.37}$$

whenever $|E \cap A| = |F \cap A|$ and $E \triangle F \subset\subset A$. Let us now show that minimizers in the relative isoperimetric problem (4.2) are also volume-constrained perimeter minimizers. Indeed, let E be a minimizer in the relative isoperimetric problem (4.2) and F be such that $|E \cap A| = |F \cap A|$ and $E \triangle F \subset\subset A$. By $|E \cap A| = |F \cap A|$ we deduce that $m = |E| = |E \cap A| = |F \cap A|$; by $E \triangle F \subset\subset A$ and $E \subset A$ we deduce that also $F \subset A$, so $|F| = m$ and therefore F is a competitor in the (4.2). Thus it holds that

$$P(E; A) \leq P(F; A)$$

and E is a volume-constrained perimeter minimizer. We now prove that volume-constrained perimeter minimizers have *constant distributional mean curvature* (as defined in Remark 5.8).

Theorem 5.6. (Constant mean curvature) *If E is a volume-constrained minimizer in the open set A, then there exists $\lambda \in \mathbb{R}$ such that*

$$\int_{\partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} = \lambda \int_{\partial^* E} (T \cdot \nu_E) d\mathcal{H}^{n-1}, \qquad \forall T \in C_c^\infty(A; \mathbb{R}^n). \tag{5.38}$$

In particular, by $\mathbf{H}_E = H_E \nu_E$, E has scalar distributional mean curvature H_E in A constantly equal to λ.

The following lemma is the key technical tool to obtain the result of above. Given a set of finite perimeter E and an open set A with $A \cap \partial^* E \neq \emptyset$, we change the volume of E by a prescribed (suitably small) amount, at the cost of a proportional perimeter variation; see Figure 5.6.

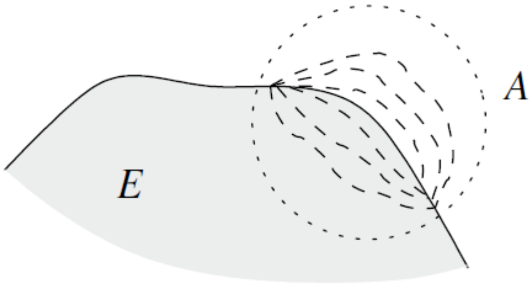

Figure 5.6: The situation in Lemma 5.4. Since $P(E;A) \neq 0$, there exists a vector field $T \in C_c^\infty(A;\mathbb{R}^n)$ that we can use to "move" $A \cap \partial^* E$. The local variations $f_t(x) = x + tT(x)$ associated with T allow us to increase or decrease volume by a certain maximal amount σ_0 which depends on E and A through T. The corresponding perimeter variations are proportional to the volume variations, through a constant C that, again, depends on E and A through T [1].

Lemma 5.4. (Volume-fixing variations) *If E is a set of finite perimeter and A is an open set such that $\mathcal{H}^{n-1}(A \cap \partial^* E) > 0$, then there exist $\sigma_0 = \sigma_0(E, A) > 0$ and $C = C(E, A) < \infty$ such that for every $\sigma \in (-\sigma_0, \sigma_0)$ we can find a set of finite perimeter F with $F \triangle E \subset\subset A$ and*

$$|F| = |E| + \sigma, \quad |P(F;A) - P(E;A)| \leq C|\sigma|.$$

Proof. Since $\mathcal{H}^{n-1}(A \cap \partial^* E) > 0$ there exists $T \in C_c^\infty(A;\mathbb{R}^n)$ such that

$$\gamma = \int_{\partial^* E} (T \cdot \nu_E) d\mathcal{H}^{n-1} > 0.$$

Let $\{f_t\}_{|t|<\varepsilon}$ be a local variation associated with T. Recalling (5.20) and (5.25), namely

$$P(f_t(E);A) = P(E;A) + t \int_{\partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} + \mathrm{O}(t^2),$$
$$|f_t(E)| = |E| + t \int_{\partial^* E} (T \cdot \nu_E) d\mathcal{H}^{n-1} + \mathrm{O}(t^2),$$

since $\gamma > 0$ and T has compact support, we may find $\varepsilon_0 > 0$ such that $f_t(E) \triangle E \subset\subset A$ and $|f_t(E)| = |E| + t\gamma + \mathrm{O}(t^2)$ is increasing on $t \in (-\varepsilon_0, \varepsilon_0)$, with

$$\Big| |f_t(E)| - |E| \Big| \geq \frac{\gamma}{2}|t|, \qquad \forall |t| < \varepsilon_0,$$
$$\Big| P(f_t(E);A) - P(E;A) \Big| \leq 2 \left| \int_{\partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} \right| |t|, \qquad \forall |t| < \varepsilon_0.$$

If $\sigma_0 > 0$ is such that $(|E| - \sigma_0, |E| + \sigma_0) \subset \{|f_t(E)| : t \in (-\varepsilon_0, \varepsilon_0)\}$, and

$$C = \frac{4}{\gamma}\left|\int_{\partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1}\right|,$$

then for every $|\sigma| < \sigma_0$, since $|f_t(E)|$ is increasing on $t \in (-\varepsilon_0, \varepsilon_0)$ there exists $|t| < t_0$ such that $F = f_t(E)$ has all the required properties. □

Proof. (of Theorem 5.6)
Step one: We prove that there exists $r_0 > 0$ such that if $T \in C_c^\infty(A; \mathbb{R}^n)$ with $\mathrm{spt}(T) \subset\subset B(x, r_0)$ for some $x \in A$, and

$$\int_{\partial^* E} (T \cdot \nu_E) d\mathcal{H}^{n-1} = 0, \tag{5.39}$$

then

$$\int_{\partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} = 0. \tag{5.40}$$

In other words: if T produces a zero first order volume variation of E, then, by volume-constrained minimality, T produces a zero first order perimeter variation of E. Indeed, let us consider $r_0 > 0$ to be such that

$$(A \cap \partial^* E) \setminus B(z, r_0) \neq \emptyset, \qquad \forall z \in A. \tag{5.41}$$

Given $T \in C_c^1(A; \mathbb{R}^n)$ with $\mathrm{spt}(T) \subset B(x, r_0)$ for some $x \in A$ and with (5.39) in force, by Proposition 1.1, applied to the disjoint family of Borel sets $\partial^* E \cap \partial B(x, t)$ indexed over $t < r_0$, we find $r < r_0$ such that

$$\mathrm{spt}(T) \subset\subset B(x, r), \qquad \mathcal{H}^{n-1}\Big(\partial^* E \cap \partial B(x, r)\Big) = 0. \tag{5.42}$$

By (5.41) with $z = x$ and, again by Proposition 1.1, there exists $y \in A \cap \partial^* E$ and $s > 0$ with $B(y, s) \cap B(x, r) = \emptyset$ and

$$\mathcal{H}^{n-1}\Big(\partial^* E \cap \partial B(y, s)\Big) = 0. \tag{5.43}$$

Now let σ_0 and C denote the constants associated by Lemma 5.4 with E in the open set $B(y, s)$, and let $\{f_t\}_{|t|<\varepsilon}$ be a local variation in $B(x, r)$ associated with T. By (5.20), (5.25) and (5.39), we find that

$$|f_t(E)| = |E| + \mathrm{O}(t^2), \tag{5.44}$$

$$P(f_t(E); A) = P(E; A) + t \int_{\partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} + \mathrm{O}(t^2). \tag{5.45}$$

If we set $\sigma(t) = |E| - |f_t(E)|$, then, up to decreasing ε, by (5.44) we find $|\sigma(t)| < \sigma_0$ for every $|t| < \varepsilon$. Hence, by Lemma 5.4, for every volume fraction $\sigma(t)$ corresponding to $|t| < \varepsilon$ we can construct F_t with $E \triangle F_t \subset\subset B(y,s)$ and

$$|F_t(E)| - |E| = \sigma(t) = |E| - |f_t(E)|, \tag{5.46}$$
$$|P(F_t; B(y,s)) - P(E; B(y,s))| \leq C|\sigma(t)| = \mathrm{O}(t^2). \tag{5.47}$$

We note that $f_t(E)$ gives a volume deficit, whereas F_t increases the volume compensating this deficit. We finally test the volume-constrained minimality of E against the competitors

$$E_t = \Big(f_t(E) \cap B(x,r)\Big) \cup \Big(F_t \cap B(y,s)\Big) \cup \Big(E \setminus \Big(B(x,r) \cup B(y,s)\Big)\Big),$$

defined for $|t| < \varepsilon$; see Figure 5.7.

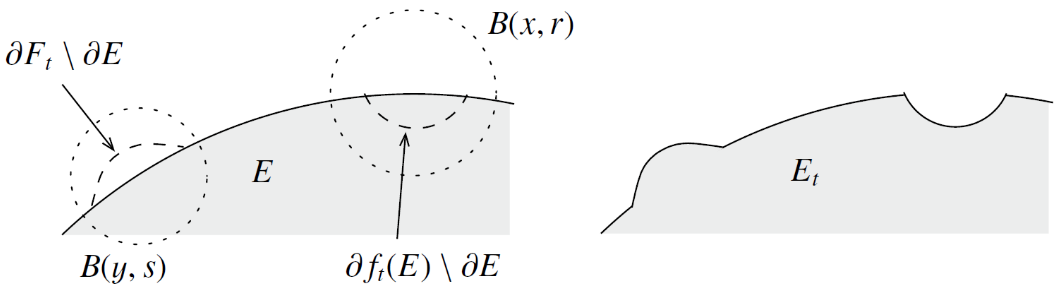

Figure 5.7: A comparison set E_t used in the proof of Theorem 5.6. The set F_t is a variation of E supported in $B(y,s)$ which compensates the volume deficit between $f_t(E)$ and E [1].

These sets are indeed competitors for E, as $|E| = |E_t|$ by (5.46):

$$\begin{aligned} |E_t| - |E| &= |E_t \cap B(x,r)| - |E \cap B(x,r)| + |E_t \cap B(y,s)| - |E \cap B(y,s)| \\ &= |f_t(E) \cap B(x,r)| - |E \cap B(x,r)| + |F_t \cap B(y,s)| - |E \cap B(y,s)| \\ &= |f_t(E)| - |E| + |F_t| - |E| = 0. \end{aligned}$$

By the volume-constrained minimality of E, the definition of E_t and the formulas (5.45), (5.47), we find

$$\begin{aligned} 0 &\leq P(E_t; A) - P(E; A) \\ &\leq P(f_t(E); B(x,r)) - P(E; B(x,r)) + P(F_t; B(y,s)) - P(E; B(y,s)) \\ &= t \int_{\partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} + \mathrm{O}(t^2) \end{aligned}$$

which trivially gives (5.40).
Step two: Up to further decreasing the value of r_0, we may assume that

$$\Big(A\cap\partial^*E\Big)\setminus\Big(B(x,r_0)\cup B(y,r_0)\Big)\neq\emptyset,\qquad \forall x,y\in A. \tag{5.48}$$

Now let $T_1,T_2\in C_c^\infty(A;\mathbb{R}^n)$ such that, for $h=1,2$,

$$\mathrm{spt}(T_h)\subset\subset B(x_h,r_0),\qquad \int_{\partial^*E}(T_h\cdot\nu_E)d\mathcal{H}^{n-1}\neq 0.$$

By Proposition 1.1, we may find $r<r_0$ such that, for $h=1,2$,

$$\mathrm{spt}(T_h)\subset\subset B(x_h,r),\qquad \mathcal{H}^{n-1}\Big(\partial^*E\cap\Big(\partial B(x_1,r)\cup\partial B(x_2,r)\Big)\Big)=0. \tag{5.49}$$

Finally, define $T\in C_c^\infty(A;\mathbb{R}^n)$ by setting

$$T=T_1-\frac{\int_{\partial^*E}(T_1\cdot\nu_E)d\mathcal{H}^{n-1}}{\int_{\partial^*E}(T_2\cdot\nu_E)d\mathcal{H}^{n-1}}T_2.$$

By definition of T we have $\int_{\partial^*E}(T\cdot\nu_E)d\mathcal{H}^{n-1}=0$. Hence, exploiting (5.48) and (5.49), and up to replacing $B(x,r)$ with $B(x_1,r)\cup B(x_2,r)$ everywhere, we may repeat the exact same argument of step one to prove that $\int_{\partial^*E}\mathrm{div}_E T\,d\mathcal{H}^{n-1}$, that is

$$\frac{\int_{\partial^*E}\mathrm{div}_E T_1\,d\mathcal{H}^{n-1}}{\int_{\partial^*E}(T_1\cdot\nu_E)d\mathcal{H}^{n-1}}=\frac{\int_{\partial^*E}\mathrm{div}_E T_2\,d\mathcal{H}^{n-1}}{\int_{\partial^*E}(T_2\cdot\nu_E)d\mathcal{H}^{n-1}}.$$

Therefore, by arbitrariness of T_1,T_2, there exists $\lambda\in\mathbb{R}$ such that (5.38) holds true for every $T\in C_c^\infty(A;\mathbb{R}^n)$ such that $\mathrm{spt}(T)\subset\subset B(x,r_0)$ for some $x\in A$. Now let T be a generic vector field in $C_c^\infty(A;\mathbb{R}^n)$, and let $\{B(z_k,r_0)\}_{k=1}^N$ be a finite cover of $\mathrm{spt}(T)$ by open balls centered in A. Using a partition of unity $\{\zeta_k\}_{k=1}^N$, with $\zeta_k\in C_c^\infty(B(z_k,r_0))$ and $\sum_{k=1}^N\zeta_k=1$ on an open neighbourhood of $\mathrm{spt}(T)$, and exploiting the lineary of the boundary divergence operator, we thus find

$$\begin{aligned}\int_{\partial^*E}\mathrm{div}_E T\,d\mathcal{H}^{n-1}&=\sum_{k=1}^N\int_{\partial^*E}\mathrm{div}_E(\zeta_k\,T)d\mathcal{H}^{n-1}\\&=\lambda\sum_{k=1}^N\int_{\partial^*E}\zeta_k(T\cdot\nu_E)d\mathcal{H}^{n-1}=\lambda\int_{\partial^*E}(T\cdot\nu_E)d\mathcal{H}^{n-1}.\end{aligned}$$

□

As already mentioned, using Theorem 5.6 we are going to prove that the mean curvature of the interior interface of an equilibrium configuration equals the potential energy plus a constant additive factor.

Theorem 5.7. (Interior stationarity condition) *If $\beta \in \mathbb{R}$, A is open, $g \in C^0(A)$, $E \subset A$ has finite perimeter and measure, and*

$$\mathcal{F}_\beta(E;A) + \mathcal{G}(E) \leq \mathcal{F}_\beta(F;A) + \mathcal{G}(F)$$

for every $F \subset A$ with $|E| = |F|$, then there exists a constant $\lambda \in \mathbb{R}$ such that

$$\int_{A \cap \partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} = \int_{A \cap \partial^* E} (-g + \lambda)(T \cdot \nu_E) d\mathcal{H}^{n-1}, \tag{5.50}$$

for every $T \in C^1_c(A;\mathbb{R}^n)$. In particular, there exists $\lambda \in \mathbb{R}$ such that E has distributional mean curvature equal to $-g + \lambda$ in A.

Proof. If $T \in C^\infty_c(A;\mathbb{R}^n)$ and $\{f_t\}_{|t|<\varepsilon}$ is a local variation associated with T, then $\{x \in \mathbb{R}^n : x \neq f_t(x)\} \subset\subset A$ gives, for ε small enough, $f_t(E) \subset A$ for every $|t| < \varepsilon$. By Proposition 5.1, $\partial^*(f_t(E))$ is $\mathcal{H}^{n-1}$-equivalent to $f_t(\partial^* E)$, while $f_t(\partial^* E) \cap \partial A = \partial^* E \cap \partial A$, since we have $\{x \in \mathbb{R}^n : x \neq f_t(x)\} \subset\subset A$. We thus find $P(f_t(E);\partial A) = P(E;\partial A)$ for every $|t| < \varepsilon$, that is, the wetting energy (i.e. the term $-\beta P(E;\partial A)$) is constant along $\{f_t(E)\}_{|t|<\varepsilon}$.
Now, taking into account Proposition 5.2, in particular the formula

$$\mathcal{G}(f_t(E)) = \mathcal{G}(E) + t \int_{\partial^* E} g(T \cdot \nu_E) \, d\mathcal{H}^{n-1} + \mathrm{o}(t),$$

it is sufficient to argue as in the proof of Theorem 5.6. The difference is that the minimality condition to exploit has now the potential energy term, namely

$$\mathcal{F}_\beta(E;A) + \mathcal{G}(E) \leq \mathcal{F}_\beta(f_t(E);A) + \mathcal{G}(f_t(E)), \qquad \forall |t| < \varepsilon,$$

considering also that the wetting energy is constant for $|t| < \varepsilon$, so it holds that $-\beta P(E;\partial A) = -\beta P(f_t(E);\partial A)$. With the same assumptions and calculations of step one in the proof of Theorem 5.6, one finds that

$$\begin{aligned} 0 &\leq \mathcal{F}_\beta(f_t(E);A) + \mathcal{G}(f_t(E)) - \mathcal{F}_\beta(E;A) - \mathcal{G}(E) \\ &= P(f_t(E);A) - P(E;A) + \mathcal{G}(f_t(E)) - \mathcal{G}(E) \\ &\leq t \int_{\partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} + t \int_{\partial^* E} g(T \cdot \nu_E) \, d\mathcal{H}^{n-1} + \mathrm{o}(t), \end{aligned}$$

thus if T produces a zero first order volume variation of E, namely if

$$\int_{\partial^* E} (T \cdot \nu_E) d\mathcal{H}^{n-1} = 0,$$

then

$$\int_{\partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} = -\int_{\partial^* E} g(T \cdot \nu_E) d\mathcal{H}^{n-1}. \tag{5.51}$$

Now, let $T_1, T_2 \in C_c^\infty(A;\mathbb{R}^n)$ be as in step two in the proof of Theorem 5.6, and define $T \in C_c^\infty(A;\mathbb{R}^n)$ by setting

$$T = T_1 - \frac{\int_{\partial^* E}(T_1 \cdot \nu_E)d\mathcal{H}^{n-1}}{\int_{\partial^* E}(T_2 \cdot \nu_E)d\mathcal{H}^{n-1}} T_2.$$

Since we trivially have $\int_{\partial^* E}(T \cdot \nu_E)d\mathcal{H}^{n-1} = 0$, by (5.51), the definition of T and the linearity of the tangential divergence we find that

$$\begin{aligned}
&\int_{\partial^* E} \mathrm{div}_E T_1 \, d\mathcal{H}^{n-1} - \frac{\int_{\partial^* E}(T_1 \cdot \nu_E)d\mathcal{H}^{n-1}}{\int_{\partial^* E}(T_2 \cdot \nu_E)d\mathcal{H}^{n-1}} \int_{\partial^* E} \mathrm{div}_E T_2 \, d\mathcal{H}^{n-1} \\
&= -\int_{\partial^* E} g(T \cdot \nu_E)d\mathcal{H}^{n-1} \\
&= -\int_{\partial^* E} g(T_1 \cdot \nu_E)d\mathcal{H}^{n-1} + \frac{\int_{\partial^* E}(T_1 \cdot \nu_E)d\mathcal{H}^{n-1}}{\int_{\partial^* E}(T_2 \cdot \nu_E)d\mathcal{H}^{n-1}} \int_{\partial^* E} g(T_2 \cdot \nu_E)d\mathcal{H}^{n-1},
\end{aligned}$$

which can be written as

$$\frac{\int_{\partial^* E} (\mathrm{div}_E T_1 + g(T_1 \cdot \nu_E)) \, d\mathcal{H}^{n-1}}{\int_{\partial^* E}(T_1 \cdot \nu_E)d\mathcal{H}^{n-1}} = \frac{\int_{\partial^* E} (\mathrm{div}_E T_2 + g(T_2 \cdot \nu_E)) \, d\mathcal{H}^{n-1}}{\int_{\partial^* E}(T_2 \cdot \nu_E)d\mathcal{H}^{n-1}}.$$

Therefore there exists $\lambda \in \mathbb{R}$ such that

$$\int_{\partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} = \int_{\partial^* E} (-g + \lambda)(T \cdot \nu_E)d\mathcal{H}^{n-1}$$

holds true for every $T \in C_c^\infty(A;\mathbb{R}^n)$ with $\mathrm{spt}(T) \subset\subset B(x, r_0)$ for some $x \in A$ and $r_0 > 0$. Finally, using a standard partition of unity exactly as in the last part of the proof of Theorem 5.6, we conclude that there exists a constant $\lambda \in \mathbb{R}$ such that

$$\int_{A \cap \partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} = \int_{A \cap \partial^* E} (-g + \lambda)(T \cdot \nu_E)d\mathcal{H}^{n-1},$$

for every $T \in C_c^1(A;\mathbb{R}^n)$. □

Remark 5.10. (Laplace's law) By (5.24), the equation (5.50) can be rewritten as

$$\int_{A \cap \partial^* E} (-H - g + \lambda)(T \cdot \nu_E)d\mathcal{H}^{n-1} = 0,$$

for every $T \in C_c^1(A;\mathbb{R}^n)$. Since $A \cap \partial^* E = \Sigma^f$, i.e. the free surface of E inside A, by arbitrariness of T we have

$$H + g = \lambda = const \quad \text{on } \Sigma^f.$$

The last equation is known as *Laplace's law*: it can be interpreted in terms of forces as

$$\underbrace{\text{surface tension}}_{\approx H} + \underbrace{\text{volume forces}}_{\approx g} = \underbrace{\text{difference of pressure at the two sides of } \Sigma^f}_{\approx \lambda}$$

Remark 5.11. (Lagrange multiplier) The equation (5.50) could also be found by the method of Lagrange multipliers. The constraint is naturally the volume constraint, which is fixed. By constrained minimality it holds that

$$\frac{d}{dt}\Big(\mathcal{F}_\beta(f_t(E);A)+\mathcal{G}(f_t(E))\Big)\Big|_{t=0} - \underbrace{\lambda\frac{d}{dt}(|f_t(E)|)\Big|_{t=0}}_{\substack{\text{Lagrange multiplier}\\ \text{due to volume constraint}}} = 0.$$

Taking into account the first variation of perimeter (5.20), Proposition 5.2 and recalling that the wetting energy is constant along $\{f_t(E)\}_{|t|<\varepsilon}$, the equation of above gives

$$\int_{\partial^* E}\mathrm{div}_E T\, d\mathcal{H}^{n-1} + \int_{\partial^* E} g(T\cdot\nu_E)d\mathcal{H}^{n-1} - \lambda\int_{\partial^* E}(T\cdot\nu_E)d\mathcal{H}^{n-1} = 0$$

which is the same of Theorem 5.7. Note that we used vector fields T compactly supported in A, therefore they do not produce any variation of the contact surface $\partial^* E\cap\partial A = \Sigma^c$.

Having found the *interior* stationarity condition, we now proceed to discuss the behaviour at *boundary points*, in order to derive a stationarity condition known as **Young's law**. For its derivation we will need to assume the regularity of the interior interface up to the boundary. The proof will use local variations associated with vector fields which act *tangentially* on ∂A (so not anymore compactly supported in A), as well as Theorem 5.3.

Theorem 5.8. (Young's law) *If $\beta\in\mathbb{R}$, $g\in L^1(\mathbb{R}^n)$, A is an open set with C^1-boundary in $\mathbb{R}^n$, $E\subset A$ is an open set with finite perimeter and measure, $A\cap\partial E$ is a C^2-hypersurface with boundary, and*

$$\mathcal{F}_\beta(E;A)+\mathcal{G}(E)\le\mathcal{F}_\beta(F;A)+\mathcal{G}(F), \tag{5.52}$$

for every $F\subset A$ with $|F|=|E|$, then

$$\nu_E\cdot\nu_A = -\beta, \qquad \textit{on } \mathrm{bdry}(A\cap\partial E). \tag{5.53}$$

In particular, necessarily $|\beta|\le 1$.

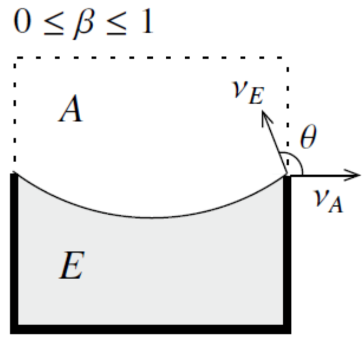

Figure 5.8: As a consequence of the constant mean curvature condition and Young's law, the free surface of E meets ∂A at a fixed angle θ such that $\cos\theta = -\beta$ [1].

Proof. Step one: We show that if $T \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ is tangent to ∂A and preserves volume at first order, that is,

$$T(x)\cdot\nu_A(x) = 0, \qquad \forall x \in \partial A, \tag{5.54}$$

$$\int_{\partial^* E} (T\cdot\nu_E)\, d\mathcal{H}^{n-1} = 0, \tag{5.55}$$

(recall Proposition 5.2), then there exists a one-parameter family of diffeomorphisms $h : (-\varepsilon,\varepsilon)\times\mathbb{R}^n \to \mathbb{R}^n$ having T as initial velocity, such that

$$h_t(E) \subset A, \qquad |h_t(E)| = |E|, \qquad \forall |t| < \varepsilon.$$

To show this we start by constructing the usual local variation $\{f_t\}_{|t|<\varepsilon}$ having T as its initial velocity, by solving the Cauchy's problems,

$$\begin{aligned} \frac{\partial}{\partial t} f(t,x) &= T(f(t,x)), \qquad && x \in \mathbb{R}^n, \\ f(0,x) &= x, && x \in \mathbb{R}^n. \end{aligned}$$

Since, locally, ∂A is the level set of a scalar function, we deduce from (5.54) that $f_t(\partial A) \subset \partial A$ for every $|t| < \varepsilon$ (otherwise, it would not be true anymore that $T\cdot\nu_A = 0$). Exploiting the uniqueness in the Cauchy problem, we see that $f_t(A) \subset A$, and, in particular, $f_t(E) \subset A$, for every $|t| < \varepsilon$.
We now want to modify $\{f_t\}_{|t|<\varepsilon}$ into a volume-preserving local variation, without losing the confinement property in A. We first consider a vector field which, at first order, increases the measure of E, specifically, we consider $S \in C_c^\infty(A;\mathbb{R}^n)$ such that

$$\int_{A\cap\partial E} (S\cdot\nu_E)\, d\mathcal{H}^{n-1} > 0. \tag{5.56}$$

The existence of S follows by the assumption that $A\cap\partial E\neq\emptyset$ is a C^2-hypersurface, which implies $A\cap\partial E\neq\emptyset$ and $\mathcal{H}^{n-1}(A\cap\partial E)>0$. Up to decreasing ε, we may define a two-parameter family of diffeomorphisms $g:(-\varepsilon,\varepsilon)^2\times\mathbb{R}^n\to\mathbb{R}^n$, setting

$$g(t,s,x)=f(t,x)+s\,S(x)=g_{t,s}(x),\qquad |t|,|s|<\varepsilon,\ x\in\mathbb{R}^n.$$

Notice that, since $\mathrm{spt}(S)\subset\subset A$, for ε small enough we may assume that

$$g_{t,s}(E)\subset A,\qquad \forall|t|,|s|<\varepsilon. \tag{5.57}$$

Correspondingly, let us consider the map $\psi:(-\varepsilon,\varepsilon)^2\to(0,\infty)$,

$$\psi(t,s)=|g_{t,s}(E)|=\int_E Jg_{t,s}(x)dx,\qquad |t|,|s|<\varepsilon.$$

Clearly $\psi(0{,}0)=|E|$. Since we have

$$\nabla g_{t,s}(x)=\mathrm{Id}+t\,T(x)+s\,S(x)+\mathrm{o}\left(\sqrt{t^2+s^2}\right), \tag{5.58}$$

uniformly on $\mathbb{R}^n$, by Lemma 5.2, (5.55) and (5.56) we find

$$\frac{\partial\psi}{\partial t}(0{,}0)=\int_E \mathrm{div}\,T=\int_{\partial^*E}T\cdot\nu_E\,d\mathcal{H}^{n-1}=0, \tag{5.59}$$

$$\frac{\partial\psi}{\partial s}(0{,}0)=\int_E \mathrm{div}\,S=\int_{\partial^*E}S\cdot\nu_E\,d\mathcal{H}^{n-1}>0. \tag{5.60}$$

By the implicit function theorem, up to further decreasing the value of ε, there exists a smooth function $\gamma:(-\varepsilon,\varepsilon)\to\mathbb{R}$ such that, for every $|t|<\varepsilon$,

$$\psi(t,\gamma(t))=|E|,\qquad \gamma(0)=0. \tag{5.61}$$

In particular, differentiating in t

$$0=\frac{\partial\psi}{\partial t}(0{,}0)+\frac{\partial\psi}{\partial s}(0{,}0)\gamma'(0),$$

which, by (5.59) and (5.60), gives $\gamma'(0)=0$. If we set $h(t,x)=g(t,\gamma(t),x)$, then by (5.61) we have $|h_t(E)|=|E|$, and by (5.57) we have $h_t(E)\subset A$, both for every $|t|<\varepsilon$. Moreover, by $g(t,s,x)=f(t,x)+s\,S(x)$ and $\gamma'(0)=0$, we find

$$\frac{\partial h}{\partial t}(0,x)=\frac{\partial f}{\partial t}(0,x)+\gamma'(0)\,S(x)=T(x),$$

so $\{h_t\}_{|t|<\varepsilon}$ has initial velocity T.

Step two: Given T and h_t as in step one, we now apply the minimality inequality (5.52) to deduce that

$$\frac{d}{dt}\left(\mathcal{F}_\beta(h_t(E);A)+\int_{h_t(E)} g\right)\Bigg|_{t=0}=0. \tag{5.62}$$

As usual by Theorem 5.4 and Proposition 5.2 we find

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}\int_{h_t(E)} g&=\int_{\partial^* E} g(T\cdot\nu_E)d\mathcal{H}^{n-1},\\ \frac{d}{dt}\bigg|_{t=0} P(h_t(E);A)&=\int_{A\cap\partial^* E}\mathrm{div}_E T\,d\mathcal{H}^{n-1}.\end{aligned}$$

Let us now consider the C^2-hypersurface (with boundary) $M=A\cap\partial E$, and the C^1-hypersurface (with boundary) $N=\partial E\cap\partial A$. Since $\mathrm{bdry}\,M=\mathrm{bdry}\,N$, we set $\Gamma=\mathrm{bdry}\,M=\mathrm{bdry}\,N$ for the common boundary of M and N. If we use ν_E and ν_A to define the orientation of M and N respectively, and denote by ν_Γ^M and ν_Γ^N the induced orientations on Γ, then by Theorem 5.3 (recall that $\mathrm{div}_E T=\mathrm{div}^M T$ on ∂E, see Remark 5.7),

$$\begin{aligned}\int_{A\cap\partial E}\mathrm{div}_E T\,d\mathcal{H}^{n-1}&=\int_M \mathrm{div}^M T\,d\mathcal{H}^{n-1}\\ &=\int_M H_M(T\cdot\nu_E)d\mathcal{H}^{n-1}+\int_\Gamma T\cdot\nu_\Gamma^M\,d\mathcal{H}^{n-2}\\ &=\int_{A\cap\partial E} H_E(T\cdot\nu_E)d\mathcal{H}^{n-1}+\int_\Gamma T\cdot\nu_\Gamma^M\,d\mathcal{H}^{n-2}.\end{aligned}$$

At the same time, taking into account that T is tangential to ∂A by (5.54), we deduce that $\int_N H_N(T\cdot\nu_E)d\mathcal{H}^{n-1}=0$ ($\nu_E=\nu_A$ on N) and thus applying Theorem 5.3 to N we find

$$\int_{\partial A\cap\partial E}\mathrm{div}_E T\,d\mathcal{H}^{n-1}=\int_N \mathrm{div}^N T\,d\mathcal{H}^{n-1}=\int_\Gamma T\cdot\nu_\Gamma^N\,d\mathcal{H}^{n-2},$$

which is interesting to us because

$$\frac{d}{dt}\bigg|_{t=0} P(h_t(E);\partial A)=\int_{\partial A\cap\partial E}\mathrm{div}_E T\,d\mathcal{H}^{n-1}.$$

Therefore, from (5.62), we deduce that

$$0=\int_{A\cap\partial E}(H_E+g)(T\cdot\nu_E)d\mathcal{H}^{n-1}+\int_\Gamma T\cdot(\nu_\Gamma^M-\beta\,\nu_\Gamma^N)d\mathcal{H}^{n-2}.$$

By Theorem 5.7, and in particular by Remark 5.10, $H_E + g$ is constant on $A \cap \partial E$. Thus by (5.55) we find

$$\int_\Gamma T \cdot (\nu_\Gamma^M - \beta\, \nu_\Gamma^N) d\mathcal{H}^{n-2} = 0, \tag{5.63}$$

whenever $T \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ satisfies (5.54) and (5.55). We now remark that for every $T_0 \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ satisfying $T_0 \cdot \nu_A = 0$ on ∂A, there exists $s > 0$ and $S_0 \in C_c^\infty(A; \mathbb{R}^n)$ such that $T = S_0 + s\, T_0 \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ satisfies (5.54) and (5.55). This is true because $\mathcal{H}^{n-1}(A \cap \partial E) > 0$ gives the existence of $S_0 \in C_c^\infty(A; \mathbb{R}^n)$ such that $\int_{A \cap \partial E} S_0 \cdot \nu_E \, d\mathcal{H}^{n-1} \neq 0$; then, since $\int_{\partial^* E} T_0 \cdot \nu_E \, d\mathcal{H}^{n-1} \neq 0$, it is sufficient, according to the relative signs, to take S_0 or $-S_0$ and adapt $s > 0$ accordingly.
By (5.63) we thus conclude that

$$\int_\Gamma T_0 \cdot (\nu_\Gamma^M - \beta\, \nu_\Gamma^N) d\mathcal{H}^{n-2} = 0, \tag{5.64}$$

whenever $T_0 \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ and $T_0 \cdot \nu_A$ on $\partial A = 0$. In particular, for every such vector field, we have $T_0 \cdot \nu_\Gamma^M = T_0 \cdot ((\nu_\Gamma^M \cdot \nu_\Gamma^N)\nu_\Gamma^N)$, so that (5.64) combined with the fundamental lemma of the calculus of variations implies that

$$\beta = \nu_\Gamma^M \cdot \nu_\Gamma^N = -\nu_E \cdot \nu_A \qquad \text{on } \Gamma.$$

□

Remark 5.12. Young's law is insensitive to the presence of the potential energy. The contact angle of a liquid drop at equilibrium on a horizontal plane is determined by the capillarity effects only, which turn out to be much stronger than gravity effects. Gravity, of course, influences the equilibrium shape away from the contact plane, for example, by flattening the liquid drop at its top.

Remark 5.13. (Regularity and Young's law) In Theorem 5.8 we assumed $A \cap \partial E$ to be a C^2-hypersurface. Using finite perimeter sets one obtains easy existence result for minimizers of the capillary energy with prescribed volume. Then one should prove that these minimizers are smooth enough in order to obtain that they actually satisfy the equilibrium conditions we derived in the smooth setting; see Theorem 6.3.

Chapter 6

Further developments

Here we provide some additional results with no proof. In particular we give an interesting characterization in the case when A is a half-space and provide the important theorem of regularity of minimizers, which, among other things, justifies the assumption of Theorem 5.8 and Remark 5.13.

6.1 Characterization of liquid drops on half spaces

We first characterize equilibrium shapes of liquid drops confined in a half-space in the absence of gravity. We thus consider the variational problems

$$\psi(\beta) = \inf \{\mathcal{F}_\beta(E; H) : E \subset H, P(E) < \infty, |E| = 1\}, \tag{6.1}$$

where $H = \{x_n > 0\}$ and $\mathcal{F}_\beta$ was defined in (5.28).

Theorem 6.1. (Liquid drops in the absence of gravity) *For every $\beta \in (-1,1)$, there exists a unique $\sigma(\beta) > 0$ with the following property: a set of finite perimeter $E \subset H$ with $|E| = 1$ is a minimizer in the variational problem (6.1) if and only if, up to horizontal translation, E is equivalent to the set*

$$G_\beta = B(s\, e_n, r) \cap H,$$

where $s \in \mathbb{R}$ and $r > 0$ are uniquely determined by the constraints

$$|G_\beta| = 1, \qquad P(G_\beta, \partial H) = \sigma(\beta).$$

Moreover,

$$\nu_{G_\beta} \cdot e_n = \beta, \qquad \text{on } \operatorname{bdry}(H \cap \partial G_\beta).$$

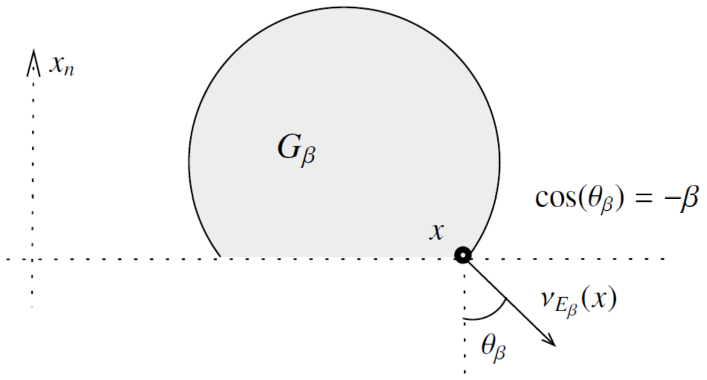

Figure 6.1: Given $\beta \in (-1,1)$, a minimizer (unique up to horizontal translations) G_β is obtained by suitably intersecting a ball with center on the e_n-axis with the half-space H [1].

Without the action of gravity, the equilibrium shape is a ball intersected with the half-space. The mean curvature in this case is constant, as stated in Theorem 5.6, and equal to the mean curvature of the ball. The contact surface depends on β: for instance, when $\beta \to (-1)^+$, minimizers converge to balls contained in H and tangent to ∂H.
Now we finally add the action of gravity, therefore we state the equilibrium problem for a liquid drop sitting on a horizontal (hyper)plane under the action of gravity.

Theorem 6.2. (Sessile liquid drops) *If $\beta \in (-1,1)$, $g > 0$ and $m > 0$, them there exists a minimizer in the variational problem*

$$\inf\left\{\mathcal{F}_\beta(E; H) + g\int_E x_n\,dx : E \subset H, P(E) < \infty, |E| = m\right\}.$$

Every such minimizer is equivalent to a boundet set, which, up to translation, it is equivalent to its Schwartz symmetrization [1], [5].

The proof of this theorem relies on the properties of Schwartz symmetrization of a set. Given a Lebesgue measurable set $E \subset \mathbb{R}^n$, with $|E| < \infty$, we denote by

$$E_t = \{z \in \mathbb{R}^{n-1} : (z,t) \in E\}, \qquad t \in \mathbb{R}$$

the horizontal slices of E, and consider the function $v_E \in L^1(\mathbb{R})$ defined for $t \in \mathbb{R}$ as $v_E(t) = \mathcal{H}^{n-1}(E_t)$. We define the Lebesgue measurable set

$$E^* = \left\{x \in \mathbb{R}^n : |\mathbf{p}x| < \left(\frac{v_E(\mathbf{q}x)}{\omega_{n-1}}\right)^{1/(n-1)}\right\},$$

known as the Schwartz symmetrization E^* of E; see [1] and [5] for details.

6.2 Regularity theory and analysis of singularities

In [1], Part III is discussed the regularity of the boundaries those sets of finite perimeter which arise as minimizers in some of the variational problems which we considered. The deep theorem is the following.

Theorem 6.3. *If $n \geq 2$, A is an open set in $\mathbb{R}^n$, and E is a local perimeter minimizer in A, then $A \cap \partial^* E$ is an analytic hypersurface with vanishing mean curvature which is relatively open in $A \cap \partial E$, while the* ***singular set*** *of E in A,*

$$\Sigma(E; A) = A \cap (\partial E \setminus \partial^* E),$$

satisfies the following properties:

(i) if $2 \leq n \leq 7$, then $\Sigma(E; A)$ is empty;

(ii) if $n = 8$, then $\Sigma(E; A)$ has no accumulation points in A;

(iii) if $n \geq 9$, then $\mathcal{H}^s(\Sigma(E; A)) = 0$ for every $s > n - 8$.

These assertions are sharp: there exists a perimeter minimizer E in $\mathbb{R}^8$ such that $\mathcal{H}^0(\Sigma(E; \mathbb{R}^8)) = 1$; moreover, if $n \geq 9$, then there exists a perimeter minimizer E in $\mathbb{R}^n$ such that $\mathcal{H}^{n-8}(\Sigma(E; \mathbb{R}^n)) = \infty$.

The proof of this theorem is essentially divided into two parts. The first one concerns the regularity of the reduced boundary in A and, precisely, it consists of proving that the locally $\mathcal{H}^{n-1}$-rectifiable set $A \cap \partial^* E$ is, in fact, a $C^{1,\gamma}$-hypersurface for every $\gamma \in (0,1)$ (its analiticity follows from standard elliptic regularity theory.) The second part of the argument is devoted to the analysis of the structure of the singular set $\Sigma(E; A)$. Roughly speaking, the blow-ups $E_{x,r}$ of E at points $x \in \Sigma(E; A)$ will have to converge to *cones* which are local perimeter minimizers in $\mathbb{R}^n$, and which have their vertex at a singular point. Starting from this result, and discussing the possible existence of such singular minimizing cones, we shall prove the claimed estimates.

Appendix A

Rectifiability

We first recall that given a function $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, its **distributional gradient** Du is defined as the linear functional $Du : C^\infty_c(\mathbb{R}^n) \to \mathbb{R}^n$,

$$\langle Du, \varphi \rangle = -\int_{\mathbb{R}^n} u \nabla \varphi, \qquad \varphi \in C^\infty_c(\mathbb{R}^n).$$

Whenever Du is representable as integration of the test function φ against a L^1_{loc} vector field, that is, if there exists a vector field $T \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} u \nabla \varphi = -\int_{\mathbb{R}^n} \varphi T, \qquad \forall \varphi \in C^\infty_c(\mathbb{R}^n),$$

we say that u has a **weak gradient** on $\mathbb{R}^n$.
Lipschitz functions play a special role in Geometric Measure Theory, because they are "measure-theoretically C^1". In particular they admit bounded weak gradients and they are a.e. classically differentiable (Rademacher's theorem).

Proposition A.1. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function, then $f \in L^\infty_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ and f admits a weak gradient $\nabla f \in L^\infty_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m \otimes \mathbb{R}^n)$.*

Theorem A.1. (Rademacher's theorem). *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function and x is a Lebesgue point of the weak gradient ∇f, then f is differentiable in x (in particular, f is differentiable a.e. on $\mathbb{R}^n$), with*

$$df_x[\tau] = \nabla f(x)[\tau], \qquad \forall \tau \in \mathbb{R}^n$$

where $df_x \in \mathbb{R}^m \otimes \mathbb{R}^n$ is the differential of f at x.

We now introduce the notion of rectifiable set, which provides a generalization of the notion of surface of primary importance in the study of geometric variational problems. In the following we fix $k \in \mathbb{N}$, with $1 \le k \le n-1$.

Definition A.1. Given a $\mathcal{H}^k$-measurable set $M \subset \mathbb{R}^n$, we say that M is **countably $\mathcal{H}^k$-rectifiable** if there exists countably many Lipschitz maps $f_h : \mathbb{R}^k \to \mathbb{R}^n$ such that

$$\mathcal{H}^k\left(M \setminus \bigcup_{h\in\mathbb{N}} f_h(\mathbb{R}^k)\right) = 0;$$

we say that M is **locally $\mathcal{H}^k$-rectifiable** provided $\mathcal{H}^k(K \cap M) < \infty$ for every compact set $K \subset \mathbb{R}^n$; finally, if $\mathcal{H}^k(M) < \infty$, then M is simply called **$\mathcal{H}^k$-rectifiable**. Moreover, $\mathcal{H}^k \llcorner M$ is a Radon measure if and only if M is locally $\mathcal{H}^k$-rectifiable.

Rectifiable sets are decomposable in the following way. Given a Lipschitz function $f : \mathbb{R}^k \to \mathbb{R}^n$, and a bounded Borel set $E \subset \mathbb{R}^k$, we say that the pair (f, E) defined a **regular Lipschitz image** $f(E)$ in $\mathbb{R}^n$ if

(i) f is injective and differentiable on E, with $Jf(x) > 0$ for every $x \in E$;

(ii) every $x \in E$ is a point of density 1 for E, namely

$$\lim_{r\to 0^+} \frac{|E \cap B(x,r)|}{\omega_k r^k} = 1;$$

(iii) every $x \in E$ is a Lebesgue point of ∇f. Since $\nabla f \in L^\infty_{\text{loc}}(\mathbb{R}^k; \mathbb{R}^n \otimes \mathbb{R}^k)$ and $T \in \mathbb{R}^n \otimes \mathbb{R}^k \mapsto JT$ is continuous, then x is also a Lebesgue point of Jf, which is

$$\lim_{r\to 0^+} \int_{B(x,r)} |Jf(z) - Jf(x)| dz = 0.$$

It can be shown then that we can always decompose a countably $\mathcal{H}^k$-rectifiable set by means of regular Lipschitz images.

Theorem A.2. (Decomposition of rectifiable sets) *If M is countably $\mathcal{H}^k$-rectifiable in $\mathbb{R}^n$, then there exists a Borel set $M_0 \subset \mathbb{R}^n$, countably many Lipschitz maps $f_h : \mathbb{R}^k \to \mathbb{R}^n$ and bounded Borel sets $E_h \subset \mathbb{R}^k$ such that*

$$M = M_0 \cup \bigcup_{h\in\mathbb{N}} f_h(E_h), \quad \mathcal{H}^k(M_0) = 0.$$

Each pair (f_h, E_h) defines a regular Lipschitz image.

Theorem A.2 allows us to prove the existence (in a measure-theoretic sense) of tangent spaces to rectifiable sets, thus making rectifiable sets a measure-theoretic generalization of smooth surfaces. Define $\Phi_{x,r} : \mathbb{R}^n \to \mathbb{R}^n$ as $\Phi_{x,r}(y) = (y - x)/r$, $y \in \mathbb{R}^n$, so that, if μ is a Radon measure on $\mathbb{R}^n$ and E is a Borel set, then

$$\frac{(\Phi_{x,r})_{\#}\mu(E)}{r^k} = \frac{\mu(x + rE)}{r^k}$$

Theorem A.3. (Existence of approximate tangent spaces) *If $M \subset \mathbb{R}^n$ is a locally $\mathcal{H}^k$-rectifiable set, then for $\mathcal{H}^k$-a.e. $x \in M$ there exists a unique k-dimensional plane π_x such that, as $r \to 0^+$,*

$$\frac{(\Phi_{x,r})_{\#}(\mathcal{H}^k \llcorner M)}{r^k} = \mathcal{H}^k \llcorner \left(\frac{M - x}{r}\right) \stackrel{*}{\rightharpoonup} \mathcal{H}^k \llcorner \pi_x, \tag{A.1}$$

that is

$$\lim_{r \to 0^+} \frac{1}{r^k} \int_M \varphi\left(\frac{y - x}{r}\right) d\mathcal{H}^k(y) = \int_{\pi_x} \varphi d\mathcal{H}^k, \qquad \forall \varphi \in C_c^0(\mathbb{R}^n).$$

In particular it holds that

$$\lim_{r \to 0^+} \frac{\mathcal{H}^k(M \cap B(x,r))}{\omega_k r^k} = 1, \qquad \mathcal{H}^k\text{-a.e. } x \in M. \tag{A.2}$$

Remark A.1. If a k-dimensional plane π_x satisfies (A.1), then we set $\pi_x = T_x M$ and name it the **approximate tangent space to M at x**. The set of points $x \in M$ such that (A.1) holds true depends only on the Radon measure $\mu = \mathcal{H}^k \llcorner M$. It is a locally $\mathcal{H}^k$-rectifiable set in $\mathbb{R}^n$, which is left unchanged if we modify M on and by $\mathcal{H}^k$-null sets.

The proof of Theorem A.3, relies on measure theory results (differentiation of Radon measures), Theorem A.2 and the following Lemma.

Lemma A.1. *If $M = f(E)$ is a k-dimensional regular Lipschitz image in $\mathbb{R}^n$ and $z \in E$, then*

$$T_x M = \nabla f(z)(\mathbb{R}^k), \quad x = f(z).$$

Example A.1. (Tangent space to a graph) If $u : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function, and we define $f : \mathbb{R}^{n-1} \to \mathbb{R}^n$ as $f(z) = (z, u(z)), z \in \mathbb{R}^{n-1}$, then $\Gamma = f(\mathbb{R}^{n-1})$ is locally $\mathcal{H}^{n-1}$-rectifiable and, for a.e. $z \in \mathbb{R}^{n-1}$,

$$T_{f(z)}\Gamma = \nu(z)^{\perp}, \qquad \nu(z) = (-\nabla' u(z),1).$$

This is easily inferred by Lemma A.1 noting that $\nabla f(z)(\tau) = (\tau, \tau \cdot \nabla' u) \in \nu(z)^{\perp}$, for $\tau \in \mathbb{R}^{n-1}$.

Appendix B

Structure theorem for sets of finite perimeter

Structure theorems address to what extent a (locally) finite perimeter set resembles a regular one, i.e. in some suitable sense it possesses a $(n-1)$-dimensional boundary and an outer unit normal. The objective is to extend the classical Gauss-Green theorem (with the $\mathcal{H}^{n-1}$ measure) to sets of (locally) finite perimeter.
The key notion to consider in order to understand the geometric structure of sets of finite perimeter is that of *reduced boundary*, which may be explained as follows. If E is an open set with C^1-boundary, then the continuity of the outer unit normal ν_E allows us to characterize $\nu_E(x)$ in terms of the Gauss-Green measure $\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial E$ as

$$\nu_E(x) = \lim_{r\to 0^+} \fint_{B(x,r)\cap \partial E} \nu_E \, d\mathcal{H}^{n-1} = \lim_{r\to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))}, \quad \forall x \in \partial E.$$

If now E is a generic set of locally finite perimeter, then $|\mu_E(B(x,r))| > 0$ for every $x \in \operatorname{spt}(\mu_E)$ and $r > 0$, we can make the following definition.

Definition B.1. The **reduced boundary** $\partial^* E$ of a set of locally finite perimeter E in $\mathbb{R}^n$ is the set of those $x \in \operatorname{spt}(\mu_E)$ such that the limit

$$\lim_{r\to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))} \qquad \text{exists and belongs to } S^{n-1}. \tag{B.1}$$

We define the Borel function $\nu_E : \partial^* E \to S^{n-1}$ by setting

$$\nu_E(x) = \lim_{r\to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))}, \qquad x \in \partial^* E.$$

We call ν_E the **measure-theoretic outer unit normal to** E. By the Lebesgue-Besicovitch differentiation theorem [6], we have

$$\mu_E = \nu_E \, |\mu_E| \llcorner \partial^* E, \tag{B.2}$$

so that the distributional Gauss-Green theorem (3.7) takes the form

$$\int_E \nabla\varphi = \int_{\partial^* E} \varphi \, \nu_E \, d|\mu_E|, \qquad \forall \varphi \in C^1_c(\mathbb{R}^n).$$

Remark B.1. By example 3.1, if E is an open set with C^1-boudnary, then $\partial^* E = \partial E$ and the measure-theoretic outer unit normal coincides with the classical notion of outer unit normal.

Example B.1. If $E \subset \mathbb{R}^2$ is a square with sides parallel to the coordinate axes, then the limit $\nu_(x)$ exists for every $x \in \partial E$. However $|\nu(x)| = 1$ if and only if x is not a vertex of E: indeed, if x is a vertex, then $|\nu(x)| = |(e_1 + e_2)/2| < 1$ and thus $\partial^* E$ is equal to ∂E minus the four vertexes of E.

The fundamental results about reduced boundaries describe their local tangential properties (Theorem B.1) and their structure of generalized hypersurface (Theorem B.2). Local properties are studied by looking at the **blow-ups** $E_{x,r}$ of E:

$$E_{x,r} = \frac{E - x}{r} = \Phi_{x,r}(E), \qquad x \in \mathbb{R}^n, r > 0,$$

where, as usual, $\Phi_{x,r} = \frac{(y-x)}{r}$, $y \in \mathbb{R}^n$. By Lebesgue's points theorem [5],

$$\begin{aligned} x \in E^{(1)} \qquad &\text{if and only if} \qquad E_{x,r} \overset{\text{loc}}{\to} \mathbb{R}^n \qquad \text{as } r \to 0^+, \\ x \in E^{(0)} \qquad &\text{if and only if} \qquad E_{x,r} \overset{\text{loc}}{\to} \emptyset \qquad \text{as } r \to 0^+, \end{aligned}$$

where we set $E^{(t)}$, the **set of points of density** t **of** E, as

$$E^{(t)} = \left\{ x \in \mathbb{R}^n : \lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{\omega_n r^n} = t \right\}.$$

Theorem B.1. (Tangential properties of the reduced boundary) *If E is a set of locally finite perimeter in $\mathbb{R}^n$, and $x \in \partial^* E$, then*

$$E_{x,r} \overset{loc}{\to} H_x = \{ y \in \mathbb{R}^n : y \cdot \nu_E(x) \le 0 \} \qquad \text{as } r \to 0^+. \tag{B.3}$$

Similarly, if $\pi_x = \partial H_x = \nu_E(x)^\perp$, then, as $r \to 0^+$,

$$\mu_{E_{x,r}} \overset{*}{\rightharpoonup} \nu_E(x) \mathcal{H}^{n-1} \llcorner \pi_x, \qquad |\mu_{E_{x,r}}| \overset{*}{\rightharpoonup} \mathcal{H}^{n-1} \llcorner \pi_x. \tag{B.4}$$

Theorem B.2. (De Giorgi's structure theorem) *If E is a set of locally finite perimeter in $\mathbb{R}^n$, then the Gauss-Green measure μ_E of E satisfies*

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E, \qquad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E, \tag{B.5}$$

and the generalized Gauss-Green formula holds true:

$$\int_E \nabla \varphi = \int_{\partial^* E} \varphi \, \nu_E \, d\mathcal{H}^{n-1}, \qquad \forall \varphi \in C^1_c(\mathbb{R}^n). \tag{B.6}$$

Moreover, there exist countably many C^1-hypersurfaces M_h in $\mathbb{R}^n$, compact sets $K_h \subset M_h$, and a Borel set F with $\mathcal{H}^{n-1}(F) = 0$, such that

$$\partial^* E = F \cup \bigcup_{h \in \mathbb{N}} K_h,$$

and, for every $x \in K_h$, $\nu_E(x)^\perp = T_x M_h$, the tangent space to M_h at x.

Theorem B.2 is fundamental because it asserts that reduced boundaries have the structure of generalized hypersurfaces (up to a $\mathcal{H}^{n-1}$-null set F), thus leading to a geometrically expressive reformulation of the distributional Gauss-Green theorem.

Corollary B.1. (of Theorem B.1) *If E is a set of locally finite perimeter in $\mathbb{R}^n$ and $x \in \partial^* E$, then*

$$\lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{\omega_n r^n} = \frac{1}{2}, \tag{B.7}$$

$$\lim_{r \to 0^+} \frac{P(E; B(x,r))}{\omega_{n-1} r^{n-1}} = 1. \tag{B.8}$$

In particular, $\partial^* E \subset E^{(1/2)}$, the set of points of density one-half of E

Let us now introduce the **essential boundary** $\partial^e E$ of a Lebesgue measurable set $E \subset \mathbb{R}^n$,

$$\partial^e E = \mathbb{R}^n \setminus \Big(E^{(0)} \cup E^{(1)} \Big).$$

We obviously have $E^{(1/2)} \subset \partial^e E$. The content of the next Theorem, due to Federer [2], consists in the $\mathcal{H}^{n-1}$-equivalence of the reduced boundary, the set of points of density one-half and the essential boundary.

Theorem B.3. (Federer's theorem) *If E is a set of locally finite perimeter in $\mathbb{R}^n$, then $\partial^* E \subset E^{(1/2)} \subset \partial^e E$, with*

$$\mathcal{H}^{n-1}\Big(\partial^e E \setminus \partial^* E \Big) = 0. \tag{B.9}$$

Combining Federer's theorem with Theorem B.1, one can characterize the Gauss-Green measures of $E \cap F$, $E \setminus F$ and $E \cup F$ (that, by Lemma 3.1 are sets of locally finite perimeter provided E and F are) in terms of μ_E and μ_F. For simplicity we write $M_1 \approx M_2$ if M_1 and M_2 are Borel sets such that $\mathcal{H}^{n-1}(M_1 \triangle M_2) = 0$. By (B.9), $\partial^* E \approx E^{(1/2)} \approx \partial^e E$ and

$$M \approx \Big(M \cap E^{(1)}\Big) \cup \Big(M \cap E^{(0)}\Big) \cup \Big(M \cap E^{(1/2)}\Big),$$

for every Borel set M and for every set of locally finite perimeter E.
In particular, we are going to use the characterization of the Gauss-Green measures of the symmetric difference $E \triangle F$ and in the case when $E \subset F$.

Proposition B.1. (Gauss-Green measure of the symmetric difference) *If E and F are sets of locally finite perimeter in $\mathbb{R}^n$, then $E \triangle F$ is of locally finite perimeter and*

$$\mu_{E \triangle F} = \mu_E \llcorner F^{(0)} + \mu_F \llcorner E^{(0)} - \mu_E \llcorner F^{(1)} - \mu_F \llcorner E^{(1)}. \tag{B.10}$$

In particular, for every Borel set $G \subset \mathbb{R}^n$,

$$P(E \triangle F; G) = P(E; G \setminus \partial^* F) + P(F; G \setminus \partial^* E) \leq P(E; G) + P(F; G).$$

Proposition B.2. *If E and F are sets of locally finite perimeter with $E \subset F$, then $\mu_E = \mu_F$ on $\partial^* E \cap \partial^* F$ and*

$$\mu_E = \mu_E \llcorner F^{(1)} + \nu_F \llcorner \Big(\partial^* F \cap \partial^* E\Big).$$

In particular, $P(E) = P(E; F^{(1)}) + P(E; F^{(1/2)})$. In the case $F = A$ is an open set, then $P(E) = P(E; A) + P(E; \partial A)$.

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