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Statistical Mechanics of the Tzitzéica-Bullough-Dodd Integrable Field Theory

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1 Introduction

The theory of statistical mechanics is born at the end of the 19th century to describe the equilibrium behaviour of macroscopic systems. Thanks to the work of giants like Boltzmann, Maxwell and Gibbs nowadays there is a well-defined mathematical and physical framework within which one can predict not only the thermodynamical averages of observables but also their associated fluctuations, typical of a probabilistic description of nature. In many situations these deviations can be considered rather boring but they are actually responsible for the existence of phase transitions and all the interesting physics related to them.

Originally the goal of what is now referred to as statistical physics, was to explain the thermodynamics of macroscopic systems starting from a more fundamental microscopic point of view. Given the enormous number of degrees of freedom the only feasible, and interesting, description is a statistical one in which the perfect determinism of Hamiltonian mechanics is lost. With just a few assumptions is then possible to derive the central results of statistical mechanics.

The interesting quantities to measure in a system subjected to thermal fluctuations are the time average of physical observables. The Classical Dynamical Average (CDA) for an observable $F[\{q_i\}, \{p_i\}]$ is defined as

$$\langle F[\{q_i\}, \{p_i\}]\rangle_{CDA} \equiv \lim_{t\to\infty} \frac{1}{t} \int_0^t F[\{q_i(t)\}, \{p_i(t)\}]\, dt \tag{1}$$

The ergodic hypothesis states that the CDA of an observable can be computed equivalently as an ensemble average i.e. an average over the system's phase space weighted by a certain probability distribution $\rho(\{q_i\}, \{p_i\})$. ρ depends on a few parameters determined by the initial condition at $t = 0$ and by the coupling of the system under consideration with its environment. Two celebrated ensembles are the microcanonical and the canonical ensemble. With microcanonical ensemble one refers to a system that is closed and totally isolated from its environment, in this case the probability density is democratically spread on all the states compatible with the initial energy of the system. This ensemble is fundamental since all the others can be derived from this one, dividing the system in two subsystems and considering one as the environment of the other. The canonical ensembles describes a system that can exchange energy with the environment and is in equilibrium with it. The probability distribution is given by the Gibbs measure $\rho(\{q_i\}, \{p_i\}) \propto e^{-\beta\mathcal{H}(\{q_i\},\{p_i\})}$, where β stands for the inverse temperature and $\mathcal{H}$ is the Hamitonian of the system.

It is possible to show that the probability densities associated to each ensemble can be viewed as the result of Maximum Entropy Inference. In this picture the coupling with the environment is encoded in constraints on the average of those physical quantities that are shared between the system and its surrounding. The associated Lagrange multipliers represent well known thermodynamics quantities like temperature, pressure and chemical potential.

Despite the great success of statistical physics there are issues that are not completely solved yet. For example it is not completely clear how a reversible microscopic dynamics can give rise to a macroscopic behaviour that is strongly irreversible. This problem has been addressed by Boltzmann himself who derived the famous H-theorem to give a microscopic foundation to the second law of thermodynamics.

The framework of statistical physics is very general and it can be adapted to systems that have continuous degrees of freedom (field theories) and to quantum mechanical systems. Here I will be particularly interested in the equilibration of isolated extended systems with a continuous number of degrees of freedom. In this case one can expect the thermalization of local quantities for which the rest of the system acts as a thermal bath.

There exist a special class of field theories that is important in mathematics, classical and quantum physics: the integrable field theories. Integrability is a property of certain dynamical systems that possess a "high enough" number of conserved quantities, such that its dynamics is strongly constrained with respect to the whole phase space. If the number of conserved charges is high enough it is possible to find the action-angle coordinates for the system. In this coordinates the dynamic is trivial, since the action variables are constant and the angles vary linearly with time. This kind of systems is opposed to the chaotic dynamics that are usually encountered in physics. The simplest example of an integrable hamiltonian is the one of a collection of uncoupled harmonic oscillators. In this system in addition to the total energy also the energy of each one of the oscillators is conserved making its dynamics particularly simple and solvable. Even though this example seems quite trivial one may note that a free field theory can be regarded exactly as a collection of infinitely many harmonic oscillators.

The importance of this class of theories stems from the fact they are often solvable, at least at a formal level. Furthermore the exact methods that have been developed to solve them are very interesting themselves, since they combine areas of mathematics as far from each other as the theory of non-linear differential equations and algebraic topology. One of this method that will appear again in the following is the Inverse Scattering Method.

Integrable models are relevant also for statistical physics since they break the assumption of ergodicity. As already mentioned, in fact, the dynamics have to meet the constraints imposed by the conserved quantities and so it cannot involve all the states with a certain energy. By taking in account all the constraints one can assume that the expected equilibrium probability distribution for an integrable theory has to contain one generalized inverse temperature for each charge Q_i as a mark of its conservation in the whole isolated system. The Gibbs measure is then replaced with the Generalized Gibbs Ensemble (GGE).

$$\rho_{GGE} \propto e^{\sum_i \beta_i Q_i} \tag{2}$$

The aim of this work is to adapt the method developed in [1] to predict the equilibrium properties of the Sinh-Gordon model to another important integrable theory, the one named after Tzitzéica-Bullough-Dodd (TBD). This technique can be used to predict the classical dynamical averages of a classical integrable field theory's observables in the thermodynamic limit. The thermodynamic limit is obtain by taking a field defined on a 1-dimensional ring of length L and taking the limit $L \to \infty$. The whole procedure relies on the fact that for a quantum integrable theory one can compute the equilibrium average of observables by means of the Leclair-Mussardo formula [2] . From there one can implement the semiclassical limit $\hbar \to 0$ and get the interesting quantites for the classical case. This method is itself far from trivial and theoretically very interesting since it brings together semiclassical approximations, form factors computation, the thermodynamic Bethe Ansatz and the inverse scattering method. The complexity of the goal here stems from the fact that the implementation of the inverse scattering technique for the Sinh-Gordon cannot be blindly applied to the TBD model since, as it will be clear later, the latter is linked to $3x3$ matrices while the former relies on an auxiliary system that is made of $2x2$ matrices. For this reason here I will concentrate on adapting some parts of the procedure while the complete tailoring of the method to the TBD theory is left as a future objective.

2 Tzitzéica-Bullough-Dodd model

The Tzitzéica-Bullough-Dodd model is a classical integrable model that can be written in Lagrangian/Hamiltonian formalism as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{m^2}{2g^2}(2e^{g\Phi} + e^{-2g\Phi}) \tag{3}$$

$$\mathcal{H} = \frac{\pi^2}{2} + \frac{\Phi_x^2}{2} + \frac{m^2}{2g^2}(2e^{g\Phi} + e^{-2g\Phi}) \tag{4}$$

Performing the following change of variables

$$\begin{aligned} \phi &= g\Phi \\ \tilde{x}^\mu &= mx^\mu \end{aligned} \tag{5}$$

One gets

$$\mathcal{L} = \frac{m^2}{g^2}\hat{\mathcal{L}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}(2e^\phi + e^{-2\phi}) \tag{6}$$

The corresponding Euler-Lagrange equation of motion is

$$\partial_t^2\phi = \partial_x^2\phi + e^{-2\phi} - e^\phi \tag{7}$$

The TBD equation appears in many fields of mathematics and physics. In differential geometry it is used to study surfaces of constant affine curvature. In field theory it represents an important example of integrable model and in contrast to many other theories, such as the Sinh-Gordon and the Sin-Gordon, the interaction term is not even. In this area the TBD model belongs to the class of affine Toda field theories. It is worth mentioning also applications to gas dynamics and soliton theory.
The energy-momentun tensor related to the TBD Lagrangian is

$$T^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi - \eta^{\mu\nu}\mathcal{L} \tag{8}$$

$$T = \begin{pmatrix} -\mathcal{H} & -\partial_t\varphi\partial_x\varphi \\ -\partial_t\varphi\partial_x\varphi & \mathcal{H} - \left(2e^\varphi + e^{-2\varphi}\right) \end{pmatrix} \tag{9}$$

2.1 Some stationary solutions

It can be interesting to derive some analytic solutions of (7), if nothing else, to check the correctness of its numerical integration. Here I will focus on a particular family of stationary solutions, namely the functions $\varphi(x)$ that satisfy

$$
\begin{aligned}
\varphi_{xx} &= -\frac{dV}{d\varphi} \\
V(\varphi) &= -\left(e^{\varphi} + \frac{e^{-2\varphi}}{2}\right)
\end{aligned} \tag{10}
$$

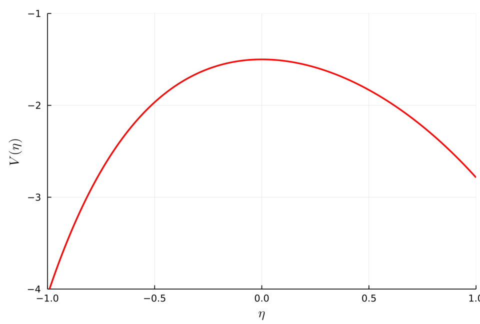

Figure 1: Energy potential of the 1D dynamical system

This can be seen as a 1-dimensional dynamical system for a point particle of unit mass where x represents the time, φ the space coordinate and $V(\varphi)$ the energy potential. Equation (10) can be then rewritten as

$$
\varphi_x = \pm\sqrt{2E + 2e^{\varphi} + e^{-2\varphi}} \tag{11}
$$

The parameter E represents the energy of the fictitious dynamical system. Equation (11) is likely to have no explicit solution for a generic value of E but here I specialize to $E = -\frac{3}{2}$. This is a special value since it corresponds to the maximum of the concave potential.

Integrating (11) one gets another parameter, let's call it C, that simply represents a translation of the function $\varphi(x)$ along the x-axis. Of course every initial value problem associated to (10) can be mapped to a certain value of the couple of parameters (E, C). The sign appearing in (11) represents the time-reversibility typical of the Hamiltonian dynamical systems.

Given the special value chosen for the energy the solutions of the above equation can be divided in two according to the sign of $\varphi(x)$. Indeed the trajectories cannot cross the point $\varphi(x) = 0$ and will conserve the sign for any x.

After a tedius calculation the explicit expressions for these solutions are

$$
\varphi_+(x) = ln\left[1 + \frac{3}{8}\left(tanh\left(\frac{\sqrt{3}}{4}x\right) - tanh\left(\frac{\sqrt{3}}{4}x\right)^{-1}\right)^2\right] \tag{12}
$$

$$
\varphi_-(x) = ln\left[1 - 6\frac{e^{-\sqrt{3}x}}{\left(1 + e^{-\sqrt{3}x}\right)^2}\right] \tag{13}
$$

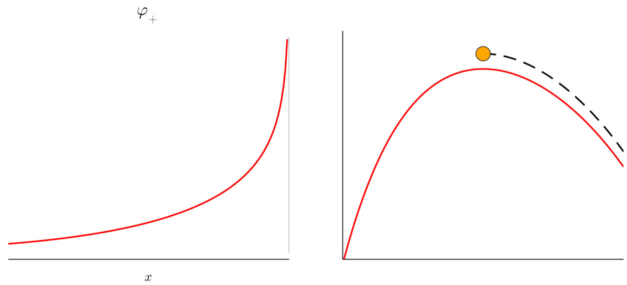

Figure 2: On the left the function $\varphi_+(x)$, on the right a pictorial representation of the associated trajectory

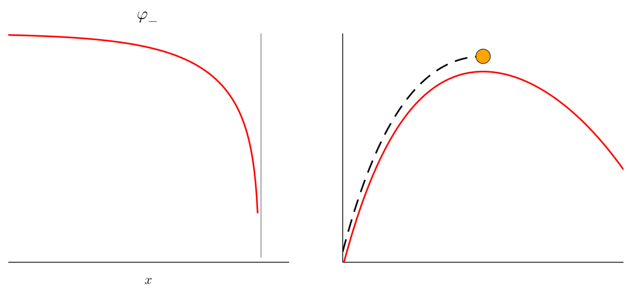

Figure 3: On the left the function $\varphi_-(x)$, on the right a pictorial representation of the associated trajectory

As already mentioned these 2 solutions can be mapped into 2 families of solutions by applying parity transformations and space-translations. $\varphi_\pm(x)$ represent somehow two special solutions of the dynamical system. In fact, using x as time and φ as space coordinate, one can note that in both solutions there is a part where a finite interval of space is covered in an infinite time (the approach to the maximum of the potential) and a part where it takes a finite time to cover an infinite amount of space (divergence).

3 Dynamics and thermalization of classical field theories

3.1 Time evolution of the field

The thermalization dynamics of a classical field theory can be observed by integrating numerically its corresponding equation of motion. One possible approach is to discretize and limit both the space and the time of the theory. In order to do so one needs to introduce 4 parameters:

a_x	space discretization
a_t	time discretization
N_x	total number of space-steps
N_t	total number of time-steps

The discretized version of (7) is

$$\phi(t+a_t,x) = \left(\frac{a_t}{a_x}\right)^2 (\phi(t,x+a_x)+\phi(t,x-a_x)) - \phi(t-a_t,x) + 2\left(1-\left(\frac{a_t}{a_x}\right)^2\right)\phi(t,x) + a_t^2\left(e^{-2\phi(t,x)} - e^{\phi(t,x)}\right) \quad (14)$$

One needs as initial conditions the field configuration and its time derivative at a certain time t_0, from these the time evolution is carried out by applying iteratively (14). An important numerical check for the integration of this hamiltonian systems is to compute the energy as a function of the time-step. Since this is a conserved quantity for the exact equation of motion but not for the discretized version one can use the energy fluctuations as a measure of the quality of the results. It is therefore useful to introduce the discretized version of the Hamiltonian.

$$H = \frac{a_x}{2}\sum_{i=0}^{N_x-1}\left[\left(\frac{\phi(t,ia_x)-\phi(t-a_t,ia_x)}{a_t}\right)^2 + \left(\frac{\phi(t,ia_x)-\phi(t,(i-1)a_x)}{a_x}\right)^2 + e^{\phi(t,ia_x)} + e^{-2\phi(t,ia_x)}\right] \quad (15)$$

In the case of integrable field theories there are many other conserved quantities other than the energy. Checking some of these could be a good idea too.

To perform the time evolution of the field one has to choose the boundary conditions. In this case we are interested in periodic boundary conditions so $\phi(t,x+L) \equiv \phi(t,(x)$ where $L = N_x * a_x$. Since in this work we are interested in the thermalization of the fied theory, there are no special initial conditions and one could generate them randomly. Here we follow the protocol put forward in [1] which consists in choosing $\phi(t_0,x)$ and $\phi_t(t_0,x)$ respectively as:

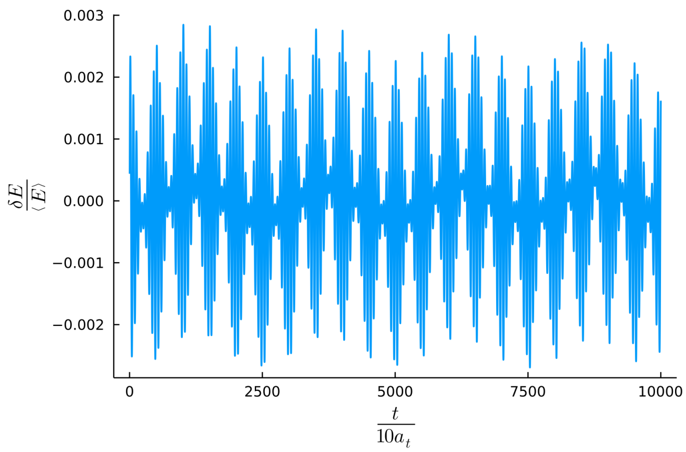

Figure 4: Example of normalized energy fluctuations of a numerical run with parameters: $N_t = 10^5$, $N_x = 10^3$, $a_t = 10^{-4}$, $a_x = 10^{-3}$;

$$\phi(t_0, x) = A \sum_{n=1}^{n_{max}} c_n cos\left(\frac{2\pi}{L}x + 2\pi\gamma_n\right)$$
$$\phi_t(t_0, x) = B \sum_{n=1}^{n_{max}} d_n cos\left(\frac{2\pi}{L}x + 2\pi\delta_n\right) \quad (16)$$

Where c_n, d_n, γ_n and δ_n are parameters drawn from some probability distribution and A, B and n_{max} can be chosen to set the energy, the highest occupied mode and possibly the momentum of the field configuration.

The choice of the parameters is important both for the purpose of investigating the equilibrium and out-of-equilibrium properties of a macroscopic system (thermodynamic limit) and for the stability and precision of the algorithm implemented. From a mathematical point of view one would need $N_x, N_t \to \infty$ and $a_x, a_t \to 0$. The first constraint is more physically rephrased as N_x and N_t "sufficiently" big. L(N_x) has to be large enough so that the assumption of statistical mechanics become realistic and the equilibrium properties of the system (at least for local observables) are given by the, possibly Generalized, Gibbs Ensamble. T(N_t) should be long enough so that the system has reached equilibrium. For the stability of the algorithm instead one should require that $\frac{a_t}{a_x} < 1$ and that they are small enough to limit the energy fluctuations associated with the time evolution.

3.2 Virial theorem

The Virial thereom can give exact expression for the CDA of local physical quantities at equilibrium. This results are very general since they depend only on the fact that the time average of a bounded function's total derivative vanishes in the limit $t \to \infty$.

Take an observable $I[\varphi(x,t)] = \frac{1}{L}\int_0^L f(\varphi(x,t))\,dx$ of a classical field theory with periodic boundary conditions. Assume for now that it is possible to prove that $\partial_t f[\varphi(x,t)]$ is a bounded function for $x \in [0,L]$, $t \in [0,+\infty)$. It is then possible to say that

$$\left\langle \partial_t^2 I[\varphi(x,t)] \right\rangle_{CDA} = 0 \tag{17}$$

Developing explicitly the time derivative

$$\partial_t^2 I[\varphi(x,t)] = \left[\frac{d^2 f}{d\varphi^2}\left(\partial_t\varphi\right)^2\right] + \left[\frac{df}{d\varphi}\partial_t^2\varphi\right] \tag{18}$$

Where the square brackets are a shorthand for the space integration procedure. Exploiting the TBD equation of motion and performing an integration by parts one arrives at the final expression

$$\left\langle \left[\frac{d^2 f}{d\varphi^2}\left(\partial_t\varphi\right)^2\right]\right\rangle_{CDA} = \left\langle \left[\frac{d^2 f}{d\varphi^2}\left(\partial_x\varphi\right)^2\right]\right\rangle_{CDA} + \left\langle \left[\frac{df}{d\varphi}\left(e^{\varphi} - e^{-2\varphi}\right)\right]\right\rangle_{CDA} \tag{19}$$

Taking as an example $f[\varphi(x,t)] = cosh(\varphi(x,t))$ the above expression takes the form

$$\left\langle \left[cosh(\varphi)\left(\partial_t\varphi\right)^2\right]\right\rangle_{CDA} = \left\langle \left[cosh(\varphi)\left(\partial_x\varphi\right)^2\right]\right\rangle_{CDA} + \frac{1}{2}\left\langle \left[e^{2\varphi} - 1 - e^{-\varphi} + e^{-3\varphi}\right]\right\rangle_{CDA} \tag{20}$$

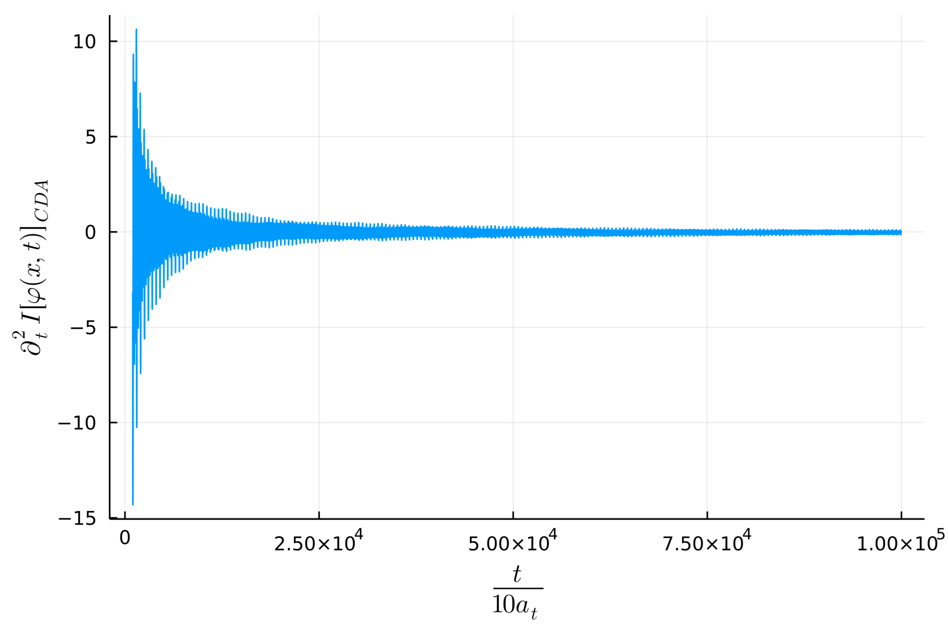

Figure 5: Example of a typical numerical run that shows that $\left\langle \partial_t^2 I[\varphi(x,t)] \right\rangle_{CDA}$ thermalizes towards 0

In Figure 5 it is possible to see the thermalization of the quantity defined above. Even if the fluctuations seem very small it is not possible to say that the system is equilibrated, since it is known that integrable system exhibit extremely long thermalization scales. It is worth stressing that the validity of the Virial theorem is not limited to equilibrium but it holds whenever the time t that enters the CDA is "long enough", like in the stationary quasi-thermal state of the above example.

4 Inverse scattering method

The inverse scattering method is a powerful technique to solve, at least formally, non-linear PDEs related to integrable theories. The method was devised in the late 60s to solve the dynamics of the KdV equation and since then it has been generalized to solve many other integrable models. An introduction to this technique and some examples can be found in [3]. The example of the rapidly decaying solutions of the KdV is the most pedagogical since one can make a nice physical analogy to understand the working principle of the inverse scattering method.

The starting point consist in writing a 1-dimensional Schrödinger equation in which the field $\varphi(x, t_0)$ figures as the potential, this is called auxiliary equation. Here $\varphi(x, t)$ is the solution of the KdV equation. Since the field is rapidly decreasing one can divide the eigenstates of the Hamiltonian in bound and scattering states. The subsequent part is about the inverse system, here one wants to prove that the potential can be reconstructed from the knowledge of the so-called scattering data. With the help of complex analysis is possible to write down a system of integral equations that depends only on the above-mentioned scattering data and whose solution can lead to the original potential. Physically this is equivalent to the many imaging techniques employed to analyse a target by making particles or radiation collide with it. For instance the shape of the atomic nucleus has been investigated in this manner.

The other fundamental part relies on the use of a particular relation between the Schrödinger operator and its time derivative. This is called Lax representation and allows to get the time dependence of the scattering coefficients.

With these two ingredients one can make up the inverse scatteing method:

- Take the initial configuration $\varphi(x, t_0)$ and find the scattering data of the associated auxiliary equation;

- Consider the dynamics of the scattering data which is often trivial and compute them at time t;

- Exploit the knowledge of the scattering coefficients at time t to compute $\varphi(x, t)$ using the inverse method;

In the case of the TBD model the method is much more elaborate because the auxiliary system is a 3x3 matrix equation.

In the rest of this section the inverse scattering method for rapidly decreasing fields obeying the TBD equation will be presented, large part of this treatment has been taken from [4] and adapted to laboratory-frame coordinates.

4.1 Lax representation

A very general form for the Lax representation is given by a matrix equation of the type

$$\partial_\tau U - \partial_\sigma V + [U, V] = 0 \tag{21}$$

This representation is usually simpler to find considering light-cone coordinates that are defined as

$$\sigma = \tfrac{x+t}{2} \qquad \tau = \tfrac{x-t}{2} \tag{22}$$

In this coordinates system the equation of motion of the TBD model takes the following form

$$\partial_\sigma \partial_\tau \phi = e^{\phi} - e^{-2\phi} \tag{23}$$

A possible choice for $U(\sigma, \tau, \lambda)$ and $V(\sigma, \tau, \lambda)$ compatible with (23) is

$$U = \begin{pmatrix} -\phi_\sigma & 0 & \lambda \\ \lambda & \phi_\sigma & 0 \\ 0 & \lambda & 0 \end{pmatrix} \tag{24}$$

$$V = \begin{pmatrix} 0 & \frac{e^{-2\phi}}{\lambda} & 0 \\ 0 & 0 & \frac{e^{\phi}}{\lambda} \\ \frac{e^{\phi}}{\lambda} & 0 & 0 \end{pmatrix} \tag{25}$$

The Lax representation (21) can be regarded as the compatibility condition for the following system of equations

$$\begin{aligned} \partial_\sigma \varphi &= U\varphi \\ \partial_\tau \varphi &= V\varphi \end{aligned} \tag{26}$$

As already mentioned here I want to work in laboratory coordinates. The corresponding matrices are found by performing a simple change of variable.

$$\begin{aligned} \partial_x \varphi &= \tilde{U}\varphi \\ \partial_t \varphi &= \tilde{V}\varphi \end{aligned} \tag{27}$$

$$\begin{aligned} \tilde{U} &= \tfrac{U+V}{2} \\ \tilde{V} &= \tfrac{U-V}{2} \end{aligned} \tag{28}$$

$$\begin{aligned} \tilde{U} &= \frac{1}{2} \begin{pmatrix} -(\phi_t + \phi_x) & \frac{e^{-2\phi}}{\lambda} & \lambda \\ \lambda & (\phi_t + \phi_x) & \frac{e^{\phi}}{\lambda} \\ \frac{e^{\phi}}{\lambda} & \lambda & 0 \end{pmatrix} \\ \tilde{V} &= \frac{1}{2} \begin{pmatrix} -(\phi_t + \phi_x) & -\frac{e^{-2\phi}}{\lambda} & \lambda \\ \lambda & (\phi_t + \phi_x) & -\frac{e^{\phi}}{\lambda} \\ -\frac{e^{\phi}}{\lambda} & \lambda & 0 \end{pmatrix} \end{aligned} \tag{29}$$

Henceforth I will concentrate on the derivations necessary to build the inverse system at a fixed time t. For this reason all the time depenendencies will be dropped and then reintroduced at the end of the section where I am going to discuss the time evolution.

Let's define now some quantities that will be useful in the following:

$$\tilde{U}_\infty = \lim_{x \to \pm\infty} \tilde{U}(x) \quad \tilde{V}_\infty = \lim_{x \to \pm\infty} \tilde{V}(x) \tag{30}$$

$$S = \begin{pmatrix} 1 & -\frac{1}{2}\left(1+i\sqrt{3}\right) & -\frac{1}{2}\left(1-i\sqrt{3}\right) \\ 1 & -\frac{1}{2}\left(1-i\sqrt{3}\right) & -\frac{1}{2}\left(1+i\sqrt{3}\right) \\ 1 & 1 & 1 \end{pmatrix} \tag{31}$$

$$K(x,\lambda) = S^{-1}\tilde{U}(x,\lambda)S$$

$$\begin{aligned} K_\infty(\lambda) &= S^{-1}\tilde{U}_\infty(\lambda)S \\ (K_\infty)_{ij}(\lambda) &= k_i(\lambda)\delta_{ij} \end{aligned} \tag{32}$$

$$k_1(\lambda) = \frac{\lambda+\lambda^{-1}}{2}, \quad k_2(\lambda) = \frac{\lambda e^{\frac{i2\pi}{3}}+\lambda^{-1}e^{-\frac{i2\pi}{3}}}{2}, \quad k_3(\lambda) = \frac{\lambda e^{-\frac{i2\pi}{3}}+\lambda^{-1}e^{\frac{i2\pi}{3}}}{2},$$

Where S and $K_\infty(\lambda)$ represent respectively the matrix of the eigenvectors and of the eigenvalues of $\tilde{U}_\infty(\lambda)$. Solving the equation

$$\partial_x\psi = \tilde{U}_\infty\psi \tag{33}$$

One gets 3 independent solutions

$$S^{-1}\psi_i(x,\lambda) = e^{k_i(\lambda)x}\hat{u}_i, \qquad i = 1,2,3 \tag{34}$$

Or equivalently in the form of the fundamental matrix

$$\psi(x,\lambda) = Se^{K_\infty(\lambda)x} \tag{35}$$

4.2 Symmetries

Given P and $\tilde{P}$ two permutations matrices

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \qquad \tilde{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \tag{36}$$

From the previous expressions for the k_i one can derive the following relations

$$\begin{aligned} K_\infty(\lambda) &= PK_\infty(\lambda e^{i\frac{2}{3}\pi})P^{-1} \\ S^{-1}\psi(x,\lambda) &= PS^{-1}\psi(x,\lambda e^{i\frac{2}{3}\pi})P^{-1} \end{aligned} \tag{37}$$

$$\begin{aligned} K_\infty(\lambda) &= \tilde{P}\overline{K_\infty(\overline{\lambda})}\tilde{P}^{-1} \\ S^{-1}\psi(x,\lambda) &= \tilde{P}\overline{S^{-1}\psi(x,\overline{\lambda})}\tilde{P}^{-1} \end{aligned} \tag{38}$$

Another important relation can be identified by exploiting the fact that $\tilde{U}_\infty$ is traceless

$$\epsilon_{ijk}(S^{-1}\psi_i(x,\lambda)) = (S^{-1}\psi_j(x,-\lambda)) \times (S^{-1}\psi_k(x,-\lambda)) \tag{39}$$

where $\times$ stands for the usual vector product.

4.3 Transfer Matrix

The system of equation (27) admits solutions that exhibit an asymptotic behaviour at $x \to \pm\infty$ equivalent to (34) since $\phi(x,t)$ is rapidly decreasing. It is then possible to define the Jost functions $f_i^{(-)}(x,\lambda)$, $f_i^{(+)}(x,\lambda)$ as the solutions of the first equation of (27) that are asymptotically equivalent at $\pm\infty$ to $\psi_i(x)$. $f^{(\pm)}(x,\lambda)$ are the matrices whose columns correspond to the Jost functions just defined. One can now define the transfer matrix T(λ) as:

$$f^{(-)}(x,\lambda) = f^{(+)}(x,\lambda)T(\lambda) \tag{40}$$

$$T(\lambda) = \begin{pmatrix} a_{11}(\lambda) & a_{12}(\lambda) & a_{13}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) & a_{23}(\lambda) \\ a_{31}(\lambda) & a_{32}(\lambda) & a_{33}(\lambda) \end{pmatrix}, \quad R(\lambda) = T(\lambda)^{-1} = \begin{pmatrix} b_{11}(\lambda) & b_{12}(\lambda) & b_{13}(\lambda) \\ b_{21}(\lambda) & b_{22}(\lambda) & b_{23}(\lambda) \\ b_{31}(\lambda) & b_{32}(\lambda) & b_{33}(\lambda) \end{pmatrix}, \tag{41}$$

The transfer matrix is x-independent since both $f^(\pm)(x,\lambda)$ are fundamental matrices of the same differential equation.

One can exploit the previously-stated symmetries to reduce the number of independent coefficients of $T(\lambda)$, if the transformations act on the Jost functions as they act on their associated ψ_i. This is indeed the case for (37) and (38) since the following relations hold

$$K(\lambda) = PK(\lambda e^{i\frac{2}{3}\pi})P^{-1} \tag{42}$$

$$K(\lambda) = \tilde{P}\overline{K(\overline{\lambda})}\tilde{P}^{-1} \tag{43}$$

As far as (39) is concerned one should check that given $\varphi_1(x,\lambda)$ and $\varphi_2(x,\lambda)$ two generic solutions of (27) then

$$\begin{aligned} h(x,\lambda) &= \frac{3\Omega}{det(S)}\left(\varphi_1(x,-\lambda) \times \varphi_2(x,-\lambda)\right) \\ \Omega &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{44}$$

it is still a solution of (27). The matrix Ω and the numerical coefficient come from the change of basis performed by S. As mentioned above this symmetries can be used to deduce some interesting equations for the coefficients of $T(\lambda)$

$$\begin{gathered} T(\lambda) = PT(\lambda e^{i\frac{2}{3}\pi})P^{-1} \\ a_{12}(\lambda) = a_{31}(\lambda e^{-i\frac{2}{3}\pi}), \quad a_{22}(\lambda) = a_{11}(\lambda e^{-i\frac{2}{3}\pi}), \quad a_{32}(\lambda) = a_{21}(\lambda e^{-i\frac{2}{3}\pi}), \\ a_{13}(\lambda) = a_{21}(\lambda e^{i\frac{2}{3}\pi}), \quad a_{23}(\lambda) = a_{31}(\lambda e^{i\frac{2}{3}\pi}), \quad a_{33}(\lambda) = a_{11}(\lambda e^{i\frac{2}{3}\pi}), \end{gathered} \tag{45}$$

$$\begin{gathered} T(\lambda) = \tilde{P}\overline{T(\overline{\lambda})}\tilde{P}^{-1} \\ a_{11}(\lambda) = \overline{a_{11}(\overline{\lambda})} \quad a_{31}(\lambda) = \overline{a_{21}(\overline{\lambda})} \end{gathered} \tag{46}$$

One can now rewrite the matrix $T(\lambda)$ as a function of $a_{11}(\lambda)$ and $a_{21}(\lambda)$ only

$$T(\lambda) = \begin{pmatrix} a_{11}(\lambda) & \overline{a_{21}(\overline{\lambda}e^{i\frac{2}{3}\pi})} & a_{21}(\lambda e^{i\frac{2}{3}\pi}) \\ a_{21}(\lambda) & a_{11}(\lambda e^{-i\frac{2}{3}\pi}) & \overline{a_{21}(\overline{\lambda}e^{-i\frac{2}{3}\pi})} \\ \overline{a_{21}(\overline{\lambda})} & a_{21}(\lambda e^{-i\frac{2}{3}\pi}) & a_{11}(\lambda e^{i\frac{2}{3}\pi}) \end{pmatrix} \tag{47}$$

Some other relations can be proved using the Wronskian. The Wronskian of three functions $(f/g/h) : \mathbb{R} \to \mathbb{R}^3$ is defined as

$$W(f,g,h) = \begin{vmatrix} f^{(1)} & f^{(2)} & f^{(3)} \\ g^{(1)} & g^{(2)} & g^{(3)} \\ h^{(1)} & h^{(2)} & h^{(3)} \end{vmatrix} \tag{48}$$

Taken f,g and h solutions of the first equation of (27)

$$\partial_x W(f,g,h) = Tr(\tilde{U})W(f,g,h) = 0 \tag{49}$$

As far as the Wronskian of the three Jost functions is concerned

$$W(f_1^{(-)}, f_2^{(-)}, f_3^{(-)}) = W(f_1^{(+)}, f_2^{(+)}, f_3^{(+)}) = det(S) = 3\sqrt{3}i \tag{50}$$

It is possible to write the scattering coefficients as a function of the Jost functions

$$a_{ij}(\lambda) = \frac{1}{2det(S)}\epsilon_{jmn}W(f_i, g_m, g_n) \tag{51}$$

Moreover from (40) one gets that $\det T(\lambda) = 1$.

Another identity can be derived using transformation (44)

$$\begin{aligned} f_1^{(+/-)}(x,\lambda) &= \frac{3\Omega}{det(S)}\left(f_2^{(+/-)}(x,-\lambda) \times f_3^{(+/-)}(x,-\lambda)\right) \\ f_2^{(+/-)}(x,\lambda) &= \frac{3\Omega}{det(S)}\left(f_3^{(+/-)}(x,-\lambda) \times f_1^{(+/-)}(x,-\lambda)\right) \\ f_3^{(+/-)}(x,\lambda) &= \frac{3\Omega}{det(S)}\left(f_1^{(+/-)}(x,-\lambda) \times f_2^{(+/-)}(x,-\lambda)\right) \end{aligned} \tag{52}$$

Or exploiting the matrix formalism

$$f^{(+/-)}(x,\lambda) = 6\Omega\left(f^{(+/-)}(x,-\lambda)^{\top}\right)^{-1} \tag{53}$$

Given that $\tilde{U}(x,\lambda)$ is real one can say that $f^{(+/-)}(x,-\lambda) = \overline{f^{(+/-)}(x,-\overline{\lambda})}\tilde{P}$. And therefore

$$f^{(+/-)}(x,\lambda) = 6\Omega\left(f^{(+/-)}(x,-\overline{\lambda})^{\dagger}\right)^{-1}\tilde{P} \tag{54}$$

From which one can derive a relation between the transfer matrix and its inverse

$$R(\lambda) = T(-\overline{\lambda})^{\dagger}\tilde{P} \tag{55}$$

4.4 Time evolution of scattering coefficients

It is possible to prove that starting at time t_0 with a rapidly decreasing field configuration $\phi(x, t_0)$ and evolving it with (7), $\phi(x,t)$ remains infinitesimal at $\pm\infty$ at any instant of time. As a consequence all the inverse scattering method developed here is valid for any t and therefore it makes sense to define a time dependent transfer matrix $T(\lambda, t)$. To get the time evolution of the scattering coefficients it is useful to look at the time evolution of the Jost functions in the case $\phi(x,t) = 0$. Equation (27) becomes

$$\begin{aligned} \partial_x \varphi &= \tilde{U}_\infty \varphi \\ \partial_t \varphi &= \tilde{V}_\infty \varphi \end{aligned} \tag{56}$$

$$\tilde{V}_\infty(\lambda) = S H_\infty(\lambda) S^{-1} \tag{57}$$

$$H_\infty(\lambda) = \frac{1}{2} \begin{pmatrix} \lambda - \lambda^{-1} & 0 & 0 \\ 0 & \lambda e^{i\frac{2}{3}\pi} - \lambda^{-1} e^{-i\frac{2}{3}\pi} & 0 \\ 0 & 0 & \lambda e^{-i\frac{2}{3}\pi} - \lambda^{-1} e^{i\frac{2}{3}\pi} \end{pmatrix} \tag{58}$$

The solutions of (56) are

$$\psi(x,t,\lambda) = S e^{K_\infty(\lambda)x + H_\infty(\lambda)t} \tag{59}$$

One can extend (40) to be valid at any time instant as

$$f^{(-)}(x,t,\lambda) = f^{(+)}(x,t,\lambda) T(\lambda) \tag{60}$$

Here $T(\lambda)$ doesn't depend on time because the time dependence is entirely absorbed by the two groups of Jost functions that represent two fundamental matrices of (27). Furthermore one can observe that

$$f^{(+/-)}(x,t,\lambda) = f^{(+/-)}(x,\lambda) e^{H_\infty(\lambda)t} \tag{61}$$

By comparing the last two equations one gets that

$$T(\lambda, t) = e^{H_\infty(\lambda)t} T(\lambda) e^{-H_\infty(\lambda)t} \tag{62}$$

4.5 Analitic properties

From now on every quantity will be expressed in the basis of the eigenvectors S.

For the purpose of showing the analytic properties of the Jost functions it's useful to divide the complex plane in six domains

$$Z_j = \left\{ \lambda \in \mathbb{C} | arg(\lambda) \in \left((j-1)\frac{\pi}{3}, j\frac{\pi}{3} \right) \right\} \quad j = 1, 2, .., 6 \tag{63}$$

Each domain possess a well specified hierarchy between the real parts of the k_i's

$$
\begin{aligned}
&Z_1 = \{\lambda \in \mathbb{C} | Re(k_2(\lambda)) < Re(k_3(\lambda)) < Re(k_1(\lambda))\} \quad && Z_4 = \{\lambda \in \mathbb{C} | Re(k_1(\lambda)) < Re(k_3(\lambda)) < Re(k_2(\lambda))\} \\
&Z_2 = \{\lambda \in \mathbb{C} | Re(k_2(\lambda)) < Re(k_1(\lambda)) < Re(k_3(\lambda))\} \quad && Z_5 = \{\lambda \in \mathbb{C} | Re(k_3(\lambda)) < Re(k_1(\lambda)) < Re(k_2(\lambda))\} \\
&Z_3 = \{\lambda \in \mathbb{C} | Re(k_1(\lambda)) < Re(k_2(\lambda)) < Re(k_3(\lambda))\} \quad && Z_6 = \{\lambda \in \mathbb{C} | Re(k_3(\lambda)) < Re(k_2(\lambda)) < Re(k_1(\lambda))\}
\end{aligned}
\tag{64}
$$

Now rewrite the first equation of (27) as

$$
\begin{aligned}
&J(x,\lambda)\varphi(x,\lambda) = X(x,\lambda)\varphi \\
&J(x,\lambda) := \mathbb{1}\partial_x - K_\infty(\lambda) \\
&X(x,\lambda) := K(x,\lambda) - K_\infty(\lambda)
\end{aligned}
\tag{65}
$$

In particular one needs a Green function for the operator J

$$
J(x,\lambda)_{ij} G(x-y,\lambda)_{jk} = \delta_{ik}\delta(x-y) \tag{66}
$$

$$
J(x,\lambda) = \begin{pmatrix} \partial_x - k_1(\lambda) & 0 & 0 \\ 0 & \partial_x - k_2(\lambda) & 0 \\ 0 & 0 & \partial_x - k_3(\lambda) \end{pmatrix} \tag{67}
$$

Starting the discussion with the "minus" Jost functions, the $f^{(-)}$s, I choose the appropriate Green function to be

$$
G(x-y,\lambda) = \Theta(x-y)e^{K_\infty(x-y)} \tag{68}
$$

Where Θ is the step function. One can now write formally this class of Jost functions as

$$
f_i^{(-)}(x,\lambda) = e^{k_i(\lambda)x}\hat{u}_i + \int_{-\infty}^{x} G(x-y,\lambda)X(y,\lambda)f_i(y,\lambda)\,dy \tag{69}
$$

$$
\chi_i^{(-)}(x,\lambda) = f_i^{(-)}(x,\lambda) \cdot e^{-k_i(\lambda)x} \tag{70}
$$

$$
\chi_i^{(-)}(x,\lambda) = \hat{u}_i + \int_{-\infty}^{x} e^{-k_i(\lambda)x} G(x-y,\lambda)X(y,\lambda)\chi_i^{(-)}(y,\lambda)\,dy \tag{71}
$$

Where $\chi_i^{(-)}(x,\lambda)$ are called modified Jost functions. One can know prove what is the domain of analyticity of $f_i^{(-)}(x,\lambda)$.

The X matrix (A.1) is an analytic function for any λ different from zero and all its x-dependency is contained in $\phi(x,t)$ that shows a rapidly decreasing behaviour at $\pm\infty$. The term that one should worry about is $e^{-k_i(\lambda)x}G(x-y,\lambda)$. In the case $i=1$

$$
e^{-k_1(\lambda)x}G(x-y,\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{(k_2(\lambda)-k_1(\lambda))(x-y)} & 0 \\ 0 & 0 & e^{(k_3(\lambda)-k_1(\lambda))(x-y)} \end{pmatrix} \tag{72}
$$

One should ensure that the exponentials present in (72) make the integral converge in order to prove the analyticity of $\chi_1^{(-)}(x,\lambda)$ (see [4] for a more rigorous proof). Since $(x-y)$ is positive and diverging in the domain of integration one should impose that $Re(k_i(\lambda) - k_1(\lambda)) < 0$, for $i = 2, 3$. This condition is verified in Z_1 and Z_6 as it is clear from their definition. Similarly one can obtain the analyticity domain for $f_2^{(-)}(x, \lambda)$ and $f_3^{(-)}(x, \lambda)$.

$f_1^{(-)}(x, \lambda)$	$Z_1 \cup Z_6$
$f_2^{(-)}(x, \lambda)$	$Z_4 \cup Z_5$
$f_3^{(-)}(x, \lambda)$	$Z_2 \cup Z_3$

(73)

This result can also be derived by knowing the domains of analyticity for one of the three Jost functions and then use (37) and (38) to get the other two. In a similar fashion one can derive the regions where the "plus" Jost functions, $f_i^{(+)}$ are analytic. Choosing the appropriate Green function and defining $\chi_i^{(+)}(x, \lambda)$

$$G(x - y, \lambda) = -\Theta(y - x)e^{K_\infty(\lambda)(x-y)} \tag{74}$$

$$f_i^{(+)}(x, \lambda) = e^{k_i(\lambda)x}\hat{u}_i + \int_x^{+\infty} G(x - y, \lambda)X(y, \lambda)g_i(y, \lambda)\, dy \tag{75}$$

$$\chi_i^{(+)}(x, \lambda) = f_i^{(+)}(x, \lambda) \cdot e^{-k_i(\lambda)x} \tag{76}$$

$$\chi_i^{(+)}(x, \lambda) = \hat{u}_i + \int_x^{+\infty} e^{-k_i(\lambda)x}G(x - y, \lambda)X(y, \lambda)\chi_i^{(+)}(y, \lambda)\, dy \tag{77}$$

$f_1^{(+)}(x, \lambda)$	$Z_3 \cup Z_4$
$f_2^{(+)}(x, \lambda)$	$Z_1 \cup Z_2$
$f_3^{(+)}(x, \lambda)$	$Z_5 \cup Z_6$

(78)

Again this domains could have been derived knowing the analytic domains for the $f_i^{(-)}$s and using (44).

It is also important to discuss the analytic properties of the scattering coefficient. One can write $a_{ii}(\lambda)$ as $\lim_{x\to+\infty}(\chi_i^{(-)}(x, \lambda))_i$, therefore

$$a_{ii}(\lambda) = 1 + \int_{-\infty}^{+\infty} X(y, \lambda)_{ij}(\chi_i^{(-)}(y, \lambda))_j\, dy \tag{79}$$

From this expression one can use an argument similar to the one used previously to prove that the coefficient $a_{ii}(\lambda)$ is analytic in the same domain in which $\chi_i^{(-)}(x, \lambda)$ is analytic.
Similarly one can note that $b_{ii}(\lambda) = \lim_{x\to-\infty}(\chi_i^{(+)}(x, \lambda))_i$ therefore, as before, the scattering coefficients are analytic in the domain where the associated modified Jost function is analytic.
As discussed in [5] the inverse problem linked with the linear system (27) can be written as a Riemann-Hilbert problem. For this purpose it is necessary to define six different matrices $M_n(x, \lambda), n = 1, 2..., 6$, each of which is well defined and analytic in the associated Z_i.

$$(M_n(x, \lambda))_{ij} := e^{k_i(\lambda)x}\delta_{ij} + \int_{s_{ij}^n * \infty}^{x} \left(e^{K_\infty(\lambda)(x-y)}X(y, \lambda)M_n(y, \lambda)\right)_{ij}\, dy \tag{80}$$

Where s^n selects the right interval to ensure the analyticity. In particular

$$s_{ij}^n = \begin{cases} +1 \text{ if } Re(k_i(\lambda) - k_j(\lambda)) > 0 \\ -1 \text{ if } Re(k_i(\lambda) - k_j(\lambda)) < 0 \end{cases} \tag{81}$$

The M_ns are solutions of (27)'s first equation by construction and hence they can be written as a linear function of the fundamental matrices $f^{(-)}(x, \lambda)$ and $f^{(+)}(x, \lambda)$

$$\begin{aligned} M_n(x, \lambda) &= f^{(+)}(x, \lambda) S_n(\lambda) \\ M_n(x, \lambda) &= f^{(-)}(x, \lambda) T_n(\lambda) \end{aligned} \tag{82}$$

A relation between the matrices $S_n(\lambda)$ and $T_n(\lambda)$ can be derived using (40)

$$S_n(\lambda) = T(\lambda) T_n(\lambda) \tag{83}$$

In the following I will briefly discuss why the M_ns are well defined and how to compute the elements of the other two matrices introduced above. Consider the case $n = 1$

$$s^1 = \begin{pmatrix} -1 & +1 & +1 \\ -1 & -1 & -1 \\ -1 & +1 & -1 \end{pmatrix} \tag{84}$$

In equation (80) it is clear that the columns of M_n are independent one from each other. Starting from the first column of M_1 one can see that the equation is identical to the one for $f_1^{(-)}(x, \lambda)$ and therefore $(M_1)_{j1}$ is well defined. It is then straightforward to say that

$$\begin{aligned} (S_1)_{j1}(\lambda) &= T_{j1}(\lambda) \\ (T_1)_{j1}(\lambda) &= \delta_{j1} \end{aligned} \tag{85}$$

The second and third columns require a bit more effort. As far as the second one is concerned one can check that the associated equations are compatible with the following asymptotic behaviours (the αs are numerical prefactors)

$$(M_1)_{j2}(x, \lambda) \to \begin{pmatrix} 0 \\ \alpha_1 e^{k_2(\lambda)x} \\ 0 \end{pmatrix} x \to +\infty, \quad (M_1)_{j2}(x, \lambda) \to e^{k_2(\lambda)x} \begin{pmatrix} \alpha_2 \\ 1 \\ \alpha_3 \end{pmatrix} x \to -\infty, \tag{86}$$

Which means that

$$(S_1)_{12}(\lambda) = (S_1)_{32}(\lambda) = 0, \quad (T_1)_{22}(\lambda) = 1, \tag{87}$$

And using (83)

$$(S_1)_{22}(\lambda) = b_{22}(\lambda)^{-1} \tag{88}$$

similarly for the third column

$$(M_1)_{j3}(x,\lambda) \to e^{k_3(\lambda)x} \begin{pmatrix} 0 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} x \to +\infty, \quad (M_1)_{j3}(x,\lambda) \to e^{k_3(\lambda)x} \begin{pmatrix} \alpha_6 \\ 0 \\ 1 \end{pmatrix} x \to -\infty, \tag{89}$$

$$(S_1)_{13}(\lambda) = 0 \quad (S_1)_{23}(\lambda) = -\frac{b_{23}(\lambda}{a_{11}(\lambda)} \quad (S_1)_{33}(\lambda) = \frac{b_{22}(\lambda}{a_{11}(\lambda)} \tag{90}$$

One can apply similar arguments also for $n = 2, ..., 6$. All the matrices $S_n(\lambda)$ and $T_n(\lambda)$ can be found in (A.2). Through relations (82) one can also write the M_ns as a function of the "plus" and "minus" Jost functions ((A.3)).

4.6 Asymptotic behaviour

Consider now $\chi_1^{(+)}(x,\lambda)$ in the limit $|\lambda| \to \infty$, $\lambda \in Z_3 \cup Z_4$

$$\chi_i^{(+)} = \hat{u}_i - \int_x^{+\infty} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-(k_2-k_1)(y-x)} & 0 \\ 0 & 0 & e^{-(k_3-k_1)(y-x)} \end{pmatrix} X(y,\lambda)\chi_i^{(+)}(y,\lambda)\,dy \tag{91}$$

$$k_2(\lambda) - k_1(\lambda) = \frac{\lambda}{2}\left(e^{i\frac{2}{3}\pi} - 1\right) + \mathcal{O}\left(\lambda^{-1}\right) = -\frac{\sqrt{3}\lambda}{2}e^{-i\frac{\pi}{6}} + \mathcal{O}\left(\lambda^{-1}\right) \tag{92}$$

$$k_3(\lambda) - k_1(\lambda) = \frac{\lambda}{2}\left(e^{-i\frac{2}{3}\pi} - 1\right) + \mathcal{O}\left(\lambda^{-1}\right) = -\frac{\sqrt{3}\lambda}{2}e^{i\frac{\pi}{6}} + \mathcal{O}\left(\lambda^{-1}\right) \tag{93}$$

It is clear from the discussion about analytic properties that the real parts of (92) and (93) are positive. For the purpose of computing the asymptotic expansion I introduce new coefficients α and β such that

$$X(x,\lambda)_{ij} = \frac{\alpha_{ij}(x)}{\lambda} + \beta_{ij}(x) \tag{94}$$

Now replace $\chi_1^{(+)}$ in (91) with the following expansion in $\frac{1}{\lambda}$

$$(\chi_1^{(+)}(x,\lambda))_j = \delta_{1j}\gamma_0 + \sum_{n=1}^{\infty}\gamma_{jn}(x)\lambda^{-n} \tag{95}$$

It is possible to verify that $\gamma_0 = 1$, from which follows that if $|\lambda| \to \infty$, $\lambda \in Z_3 \cup Z_4$ then $a(\lambda) \to 1$, i.e. there is no scattering between the Jost functions with different index.

$$\chi_i^{(+)} - \hat{u}_i = -\int_x^{+\infty} \begin{pmatrix} \frac{\alpha_{11}}{\lambda} & \frac{\alpha_{12}}{\lambda} + \beta_{12} & \frac{\alpha_{13}}{\lambda} + \beta_{13} \\ e_2(\frac{\alpha_{21}}{\lambda} + \beta_{21}) & e_2(\frac{\alpha_{22}}{\lambda}) & e_2(\frac{\alpha_{23}}{\lambda} + \beta_{23}) \\ e_3(\frac{\alpha_{31}}{\lambda} + \beta_{31}) & e_3(\frac{\alpha_{32}}{\lambda} + \beta_{32}) & e_3(\frac{\alpha_{33}\lambda}{)} \end{pmatrix} \chi_i^{(+)}\,dy \tag{96}$$

Where for brevity I have replaced $e^{-(k_2-k_1)(y-x)}$ with e_2 and $e^{-(k_3-k_1)(y-x)}$ with e_3

$$\gamma_{21}(x) = -\lim_{\lambda\to\infty}\lambda\int_x^{+\infty}\beta_{21}(y)e^{\frac{\sqrt{3}\lambda}{2}e^{-i\frac{\pi}{6}}(y-x)}\,dy \tag{97}$$

$$\gamma_{21}(x) = -\lim_{\lambda\to\infty}\int_0^{+\infty}\beta_{21}\left(x+\frac{y}{\lambda}\right)e^{\frac{\sqrt{3}}{2}e^{-i\frac{\pi}{6}}y}\,dy \tag{98}$$

$$\gamma_{21}(x) = -\beta_{21}(x)\frac{2e^{i\frac{\pi}{6}}}{\sqrt{3}} \tag{99}$$

Similarly

$$\gamma_{31}(x) = -\beta_{31}(x)\frac{2e^{-i\frac{\pi}{6}}}{\sqrt{3}} \tag{100}$$

$$\beta_{21}(x) = \overline{\beta_{31}(x)} \Rightarrow \gamma_{21}(x) = \overline{\gamma_{31}(x)} \tag{101}$$

In order to extract the field $\phi(x,t)$ we need to compute also the term $\gamma_{11}(x)$

$$\gamma_{11}(x) = -\int_x^{+\infty}\alpha_{11}(x) + \beta_{12}(x)\gamma_{21}(x) + \beta_{13}(x)\gamma_{31}(x)\,dy \tag{102}$$

$$\frac{d\gamma_{11}}{dx} = \alpha_{11}(x) + \beta_{12}(x)\gamma_{21}(x) + \beta_{13}(x)\gamma_{31}(x) \tag{103}$$

Noting that $\beta_{31}(x) = \overline{\beta_{13}(x)}$, $\beta_{12}(x) = \overline{\beta_{21}(x)}$ and writing explicitely $\alpha_{11}(x)$

$$\frac{d\gamma_{11}}{dx} = \frac{1}{6}e^{-2\phi} + \frac{1}{3}e^{\phi} - \frac{1}{2} + \left(\frac{\sqrt{3}e^{-i\frac{\pi}{6}}}{2}|\gamma_{21}(x)|^2 + c.c.\right) \tag{104}$$

$$\begin{aligned} &h^3(x) + 3h(x)\left(-2\frac{d\gamma_{11}}{dx} + 3|\gamma_{21}(x)|^2 - 1\right) + 2 = 0 \\ &h(x) := e^{-\phi(x)} \\ &\phi(x,t) = -ln(h_*(x,t)) \end{aligned} \tag{105}$$

Assuming that (105) has a unique solution, one can reconstruct the field $\phi(x,t)$ knowing the asymptotic behaviour of the modified Jost functions. In general (105) has more than one solution. One could overcome this problem by checking which solution $h_*(x)$ agrees with the initial conditions and then for continuity the right solution to (7) will be the time evolution of that zero.

One can derive an asymptotic expansion also for the fundamental matrix of (27). The equation for the fundamental matrix in the S basis is

$$\partial_x\varphi(x,\lambda) = (K_\infty(\lambda) + X(x,\lambda))\varphi(x,\lambda) \tag{106}$$

Now introduce the following functional form for the asymptotic expansion of $\varphi(x,\lambda)$

$$\begin{aligned} \varphi(x,\lambda) &= \chi(x,\lambda)e^{K_\infty(\lambda)x} \\ \chi(x,\lambda) &= \sum_{n=0}^{+\infty}\chi^{(n)}(x)\lambda^{-n} \end{aligned} \tag{107}$$

Substitute now this in the above equation to get

$$\partial_x \chi(x,\lambda) = [K_\infty(\lambda), \chi(x,\lambda)] + X(x,\lambda)\chi(x,\lambda) \tag{108}$$

To write a recursive expression for the $\chi^{(n)}(x)$ is useful to introduce

$$X(x,\lambda) = \frac{A(x)}{\lambda} + B(x) \tag{109}$$

Where $B(x)$ is off-diagonal. Expanding the asymptotic series

$$\begin{aligned}\sum_{n=0}^{+\infty} \partial_x \chi(x)_{ij}^{(n)} \lambda^{-n} = \frac{1-\delta_{ij}}{2} \left(\sum_{n=0}^{+\infty} \Delta_{ij} \chi_{ij}^{(n)}(x)\lambda^{-n+1} + \sum_{n=0}^{+\infty} 3\Delta_{ij}^{-1} \chi_{ij}^{(n)}(x)\lambda^{-n-1} \right) \\ + \sum_{n=0}^{+\infty} B(x)_{io}\chi_{oj}^{(n)}(x)\lambda^{-n} + \sum_{n=0}^{+\infty} A(x)_{io}\chi_{oj}^{(n)}(x)\lambda^{-n-1}\end{aligned} \tag{110}$$

Where Δ is a matrix of coefficient both x and λ independent that can be easily computed. Comparing the sums term-by-term one gets

$$\begin{aligned}&\chi_{ij}^{(0)}(x) = \delta_{ij} c_i \\ &\chi_{ij}^{(1)}(x) = -\frac{2B_{ij}(x)\chi_{jj}^{(0)}}{\Delta_{ij}},\ i \neq j\end{aligned} \tag{111}$$

And for $n \geq 1$

$$\begin{aligned}&\partial_x \chi_{ii}^{(n)}(x) = B_{io}(x)\chi_{oi}^{(n)}(x) + A_{io}(x)\chi_{oi}^{(n-1)}(x) \\ &\frac{\Delta_{ij}}{2}\chi_{ij}^{(n+1)}(x) = \partial_x \chi_{ij}^{(n)}(x) - \frac{3}{2\Delta_{ij}}\chi_{ij}^{(n-1)}(x) - B_{io}(x)\chi_{oj}^{(n)}(x) - A_{io}(x)\chi_{oj}^{(n-1)}(x),\ i \neq j\end{aligned} \tag{112}$$

Chosen the three constants $c_i, i = 1, 2, 3$, all the other coefficients of the expansion can be computed solving the above equations. If one takes $\chi^{(0)}(x) = \mathbb{1}$ then

$$\lim_{|\lambda| \to \infty} \varphi(x,\lambda) = \psi(x,\lambda) \tag{113}$$

And

$$\lim_{|\lambda| \to \infty} T(\lambda) = \mathbb{1} \tag{114}$$

It is interesting to note that the series of equation (107) is likely to have a radius of convergence equal to 0. It is easy to check, in fact, that the behaviour of the expansion at $x \to \pm\infty$ is always proportional to $e^{K_\infty(\lambda)x}$. This implies that the transfer matrix is diagonal and lambda-independent in the series' convergence domain, behaviour that I don't expect to be true in any domain.

From (A.3), exploiting the above asymptotic expansion, it is easy to get the large-λ-behaviour of the M_ns

$$\lim_{|\lambda|\to\infty} M_i(x,\lambda)e^{-K_\infty(\lambda)x} = \mathbb{1},\ i = 1,2,3,4,5,6 \tag{115}$$

To write the inverse problem is necessary to see what happens also when $\lambda \to 0$. This is much easier exploiting light-cone coordinates. All the derivations of this section can be adapted straightforwardly to this different choice of coordinates. One can compare (60) for the two different systems of coordinates.

$$\begin{aligned} f^{(-)}(x,t,\lambda) &= f^{(+)}(x,t,\lambda)T(\lambda) \\ g^{(-)}(\sigma,\tau,\lambda) &= g^{(+)}(\sigma,\tau,\lambda)T^{(LC)}(\lambda) \end{aligned} \tag{116}$$

Where $g^{(\pm)}(\sigma,\tau,\lambda)$ are solutions of the system (26) and exhibit asymptotic behaviour for $\sigma \to \pm\infty$ equivalent to $e^{(K_\infty(\lambda)+H_\infty(\lambda))\sigma+(K_\infty(\lambda)-H_\infty(\lambda))\tau}$. Given that system (26) and system (27) are equivalent both the gs are are also solutions of the equation in laboratory coordinates. To get the relation between the fs and the gs let analyse the asymptotic behaviour of the latter for fixed t and $x \to \pm\infty$.

$$\begin{aligned} &x \to \pm\infty, t = t_0 \Rightarrow \sigma \to \pm\infty \\ &g^{(\pm)}(\sigma(x,t),\tau(x,t),\lambda) \approx e^{(K_\infty(\lambda)+H_\infty(\lambda))\sigma+(K_\infty(\lambda)-H_\infty(\lambda))\tau} = e^{K_\infty(\lambda)x+H_\infty(\lambda)t} \end{aligned} \tag{117}$$

That means that $g^{(\pm)}(\sigma,\tau,\lambda) = f^{(\pm)}(x,t,\lambda)$. From (116) it follows that $T^{(LC)}(\lambda) = T(\lambda)$. Let's now concentrate on

$$f^{(\pm)}(\sigma,\tau,\lambda)e^{-(K_\infty(\lambda)+H_\infty(\lambda))\sigma-(K_\infty(\lambda)-H_\infty(\lambda))\tau} = \chi^{(\pm)}(\sigma,\tau,\lambda)e^{-(K_\infty(\lambda)-H_\infty(\lambda))\tau} = \chi^{(\pm)}(\sigma,\lambda) \tag{118}$$

This quantity is τ independent since the τ evolution of the Jost functions is obtained by multiplying on the right for the $e^{(K_\infty(\lambda)-H_\infty(\lambda))\tau}$. $\chi(\sigma,\lambda)$ is then a fundamental matrix of the first of equations (26) with asymptotic behaviour equivalent to $\mathbb{1}$ for $x \to \pm\infty$. Differently from $\tilde{U}$, The matrix $U(\sigma,\lambda)$ is a analytic function of λ for $\lambda \to 0$. Therefore it is fairly easy to prove that its solutions are well behaved for values of λ in the neighbourhood of 0. This is then true also for $\chi^{(\pm)}(x,\lambda)$ defined as

$$\chi^{(\pm)}(x,\lambda) = f^{(\pm)}(x,t,\lambda)e^{-K_\infty(\lambda)x-H_\infty(\lambda)t} \tag{119}$$

4.7 Discrete spectrum

For the development of the inverse scattering method is fundamental to characterize the so-called discrete spectrum of (27). In the present case with discrete spectrum I mean the solutions of (27) that become infinitesimal at $\pm\infty$ (bound states). It's useful to define the following domains in the complex plane

$$\tilde{Z}_i = \left\{\lambda \in \mathbb{C} | arg(\lambda) \in \left(\frac{\pi}{6} + (i-1)\frac{\pi}{3}, \frac{\pi}{6} + i\frac{\pi}{3}\right)\right\} \quad i = 1,2,..,6 \tag{120}$$

One can discuss the bound states in one of this domains, say $\tilde{Z}_1$, and all the other will follow similarly.
In the region Z_1 one has that $Re(k_1(\lambda)) < 0$, $Re(k_2(\lambda)) > 0$ and $Re(k_3(\lambda)) < 0$. If $\varkappa(x,\lambda_b)$ is a bound state of (27) it can be written both as a linear combination of the $f^{(+)}$s and as a linear combination of the $f^{(-)}$s. $\varkappa(x,\lambda_b)$ must be proportional to $f_2^{(-)}(x,\lambda_b)$ since it is the only Jost functions f that is bounded at $-\infty$. Analogously the bound state has to be a linear combination of $f_1^{(+)}(x,\lambda_b)$ and $f_3^{(-)}(x,\lambda_b)$. From (40) one can deduce that $a_{22}(\lambda_b) = 0$.
In this manner bound states in each domain Z_i are in one to one correspondence with the zeros of the diagonal elements of either $T(\lambda)$ or $R(\lambda)$

$\tilde{Z}_1$	$b_{22}(\lambda)$		$\tilde{Z}_4$	$a_{22}(\lambda)$
$\tilde{Z}_2$	$a_{33}(\lambda)$		$\tilde{Z}_5$	$b_{33}(\lambda)$
$\tilde{Z}_3$	$b_{11}(\lambda)$		$\tilde{Z}_6$	$a_{11}(\lambda)$

(121)

One can tell much more about the discrete spectrum using the M_ns defined in a previous section. For instance using (82) one can say that

$$(M_1)_{j2}(x,\lambda) = (M_2)_{j2}(x,\lambda) = \frac{f_2^{(+)}(x,\lambda)}{b_{22}(\lambda)} \tag{122}$$

Since both the numerator and the denominator of the above expression are analytic functions in the associated complex domain, it means that $b_{22}(\lambda)$ cannot have any zero for $\lambda \in Z_1 \cup Z_2$. The same reasonment can be applied to $b_{11}(\lambda)$ and $b_{33}(\lambda)$ in their relative analytic domains. Exploiting (55) one can prove the same for the three diagonal coefficients of the transfer matrix $a_{11}(\lambda), a_{22}(\lambda)$ and $a_{33}(\lambda)$. As a result these six coefficients can possibly have zeros only on the boundaries of the associated analytic domains.

$$\Sigma_i = \left\{\lambda \in \mathbb{C} | arg(\lambda) = (i-1)\frac{\pi}{3}\right\} i = 1, 2, ..., 6 \tag{123}$$

$a_{11}(\lambda)$	$\Sigma_6 \cup \Sigma_1 \cup \Sigma_2$		$b_{11}(\lambda)$	$\Sigma_3 \cup \Sigma_4 \cup \Sigma_5$
$b_{22}(\lambda)$	$\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$		$a_{22}(\lambda)$	$\Sigma_4 \cup \Sigma_5 \cup \Sigma_6$
$a_{33}(\lambda)$	$\Sigma_2 \cup \Sigma_3 \cup \Sigma_4$		$b_{33}(\lambda)$	$\Sigma_5 \cup \Sigma_6 \cup \Sigma_1$

(124)

One can make use of the relations previously derived between the scattering coefficients to link their zeros. In fact it is sufficient to know all the zeros of one coefficient, say $a_{11}(\lambda)$, to derive the whole discrete spectrum. Henceforth I will assume that all the zeros of the scattering coefficients are simple.

4.8 Riemann-Hilbert problem

The Riemann-Hilbert problem is one of the main boundary value problems in analytic function theory. It owes its name to the fact that it was studied for the first time by Riemann during his PhD and it was generalized at a later time by Hilbert. In its interesting formulation for integrable systems its statement is:
Given $\Gamma \subset \mathbb{C}$ a closed curve that divides the complex plane $\mathbb{C} \setminus \Gamma$ in two domains $D_{\pm}$, find an analytic function $\Theta(x,\lambda)$ such that

$$\Theta(x,\lambda) = \Theta_{\pm}(x,\lambda), \quad \lambda \in D_{\pm} \tag{125}$$

It extends smoothly to Γ and

$$\begin{aligned} &\Theta_{+}(x,\lambda) = \Theta_{-}(x,\lambda)\Lambda(x,\lambda), \quad \lambda \in \Gamma \\ &\Theta(x,\lambda) \to \mathbb{1} \quad as\ \lambda \to \infty \end{aligned} \tag{126}$$

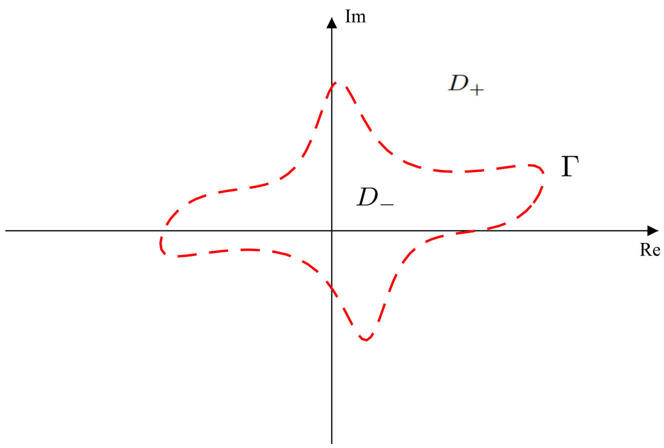

Figure 6: Graphical example of the setting of a Riemann-Hilbert problem

Surprisingly the R-H problem has proved to be the natural formulation for many inverse problems associated to integrable theories [5]. This problem can be usually turned into a system of integrable equations by means of the Cauchy projector. The formulation of the R-H problem for the TBD equation and the effort to write it as a system of coupled integral equations are left as a future task.

5 Transfer matrix calculation

One of the key steps to implement the procedure set up in [1] is the computation of the transfer matrix. This is the case because all the information about the field configuration (and hence about the conserved charges) is contained in the scattering data that are functions of the transfer matrix elements.

5.1 Analytic equations

There are two different paths that one can take to compute the transfer matrix:

1. Define $\tilde{T}(x,\lambda)$ by means of the equation below

$$f^{(-)}(x,\lambda) = \psi(x,\lambda)\tilde{T}(x,\lambda) \tag{127}$$

   From the asymptotic limit of $f^{(-)}(x,\lambda)$ one gets that

$$\lim_{x\to-\infty} \tilde{T}(x,\lambda) = \mathbb{1} \tag{128}$$

   Knowing the differential equations obeyed by $f^{(-)}(x,\lambda)$ one gets a differential equation for $\tilde{T}(x,\lambda)$

$$\begin{aligned} &\partial_x\tilde{T}(x,\lambda) = M(x,\lambda)\tilde{T}(x,\lambda) \\ &M(x,\lambda) = \psi^{-1}(x,\lambda)\left(K(x,\lambda) - K_\infty(\lambda)\right)\psi(x,\lambda) = e^{-K_\infty(\lambda)x}X(x,\lambda)e^{K_\infty(\lambda)x} \end{aligned} \tag{129}$$

   This procedure allows to compute the transfer matrix since $\lim_{x\to\infty}\tilde{T}(x,\lambda) = T(\lambda)$. This follows from the definition of $\tilde{T}(x,\lambda)$ and the fact that $\psi(x,\lambda) \approx f^{(+)}(x,\lambda), x\to\infty$.

2. The second method relies on the first equation of (27). From the definition of the transfer matrix (40) one can infer that

$$T(\lambda) = (f^{(+)}(x,\lambda)^{-1}f^{(-)}(x,\lambda) \underset{x\to+\infty}{\approx} \psi^{-1}(x,\lambda)f^{(-)}(x,\lambda) \tag{130}$$

   Now define $W(x,y,\lambda)$ as the space propagator associated to (27)

$$\varphi(x,\lambda) = W(x,y,\lambda)\varphi(y,\lambda) \tag{131}$$

   W is the solution of the following differential equation

$$\begin{aligned} &\partial_x W(x,y,\lambda) = K(x,\lambda)W(x,y,\lambda) \\ &W(y,y,\lambda) = \mathbb{1} \end{aligned} \tag{132}$$

   The equation for the transfer matrix then reads

$$T(\lambda) = \lim_{x\to+\infty}\lim_{y\to-\infty}\psi^{-1}(x,\lambda)W(x,y,\lambda)\psi(y,\lambda) \tag{133}$$

As shown in [1], In the thermodynamic limit the interesting quantities to compute are the coefficients of the transfer matrix relative to the truncated field i.e. the piecewise-defined function that is equal to the field $\phi(t,x)$ in the interval $[0,L]$ and zero outside.

In this setting $\tilde{T}(x,\lambda)$ is the identity matrix for any $x<0$ and is equal to T for any $x>L$. In the same spirit $f^{(\pm)}(x,\lambda)$ is equal to $\psi(x,\lambda)$ for $x \gtrless 0$ and so (133) becomes

$$T(\lambda) = \psi^{-1}(L,\lambda)W(L,0,\lambda)\psi(0,\lambda) \tag{134}$$

To find the transfer matrix is therefore sufficient to solve (129) (method 1) or (132) (method 2) for $x \in [0,L]$.

5.2 Numerical algorithms

Here I will devise the algorithms specifically for the first method nonetheless it is straightforward to adapt them to the second one since both rely on a differential equation of the same king. A simple numerical algorithm to solve a matrix first order linear differential equation like (129) consists in propagating in space $\tilde{T}(\lambda)$ in steps of a_x approximating the propagator with $e^{a_x M(x,\lambda)}$. Explicitly

$$T(x,\lambda) = \prod_{i=0}^{N_x-1} e^{a_x M(ia_x,\lambda)} \tag{135}$$

Where the product is space-ordered, that means that it has to be carried out starting on the right in increasing order of i. There are some interesting numerical checks that are worth mentioning. First of all it is possible to prove that $M(x,\lambda)$ is invariant under a transformation analogous to (42)

$$M(x,\lambda) = PM(x,\lambda e^{i\frac{2}{3}\pi})P^{-1} \tag{136}$$

The same has to be valid for the transfer matrix computed with the above algorithm and so one can exploit this to check the error introduced by the numerical calculations.
From (127) one can prove that the determinant of $\tilde{T}$ should be equal to 1 for every x because $f(x,\lambda)$ and $\psi(x,\lambda)$ have the same determinant. This can be useful to evaluate qualitatively the error introduced by the chosen approximation for the propagator.

A slightly more sophisticated algorithm to solve the type of differential equation I am interested here is based on cubic splines. This method consists in writing the unknown matrix $\tilde{T}(x,\lambda)$ as a third degree polynomial in x with matrix coefficients.

$$T(x,\lambda) = \sum_{n=0}^{3} T_j^{(n)}(\lambda)\frac{(x-ja_x)^n}{n!}, \quad x \in [ja_x,(j+1)a_x] \tag{137}$$

This expansion, and therefore its coefficients, are specific for each space interval of length a_x between 0 and L. The strategy here is to choose $T_j^{(n)}(\lambda), n=0,1,2,$ to be equal to the corresponding coefficients of the Taylor expansion of $\tilde{T}(x,\lambda)$ centered in $x=ja_x$, while $T_j^{(3)}$ will be chosen imposing that the polynomial is a solution of the exact differential equation in $x=(j+1)a_x$. The solution procedure start from the left and propagates the solution in space. For $x \in [0,a_x]$ one can easily find the first three coefficients $\tilde{T}(0,\lambda), \partial_x\tilde{T}(0,\lambda)$ and $\partial_x^2\tilde{T}(0,\lambda)$. These can be computed using the initial condition $\tilde{T}(0,\lambda)=\mathbb{1}$ and the differential equation.

$$
\begin{aligned}
&\partial_x \tilde{T}(0,\lambda) = M(0,\lambda)\tilde{T}(0,\lambda) \\
&\partial_x^2 \tilde{T}(0,\lambda) = \partial_x M(0,\lambda)\tilde{T}(0,\lambda) + M(0,\lambda)\partial_x \tilde{T}(0,\lambda)
\end{aligned}
\tag{138}
$$

The equation for the remaining coefficient can be derived enforcing the constraint mentioned before. As far as the others space intervals are concerned the first three coefficients can be computed knowing the function in the previous space intervals and the remaining one again applying the same procedure. The final solution will be a $C^2([0,L])$ piecewise-defined function that satisfies the exact differential equation in the set of discrete point $\{ja_x, j = 1, .., N_x\}$. An estimation of the error and more details about this technique can be found in [6].

Both these algorithms are actually quite sensitive to the choice of the parameters, expecially to the value of λ.

6 Conclusions

This work should be seen as the beginning of the attemp to adapt the whole procedure developed in [1] to the TBD model. Here I introduced the model, some numerical techniques to compute its time evolution and its transfer matrix and laid the foundations for the inverse scattering method procedure. There is still much left to do: overcome the various problem shown by the numerical algorithms, finish the part relative to the inverse scattering method and then, starting from the quantum version of the TBD, compute the form factors and their semiclassical limit.

Appendix A Matrices

$$
\begin{aligned}
X_{j1} &= \frac{1}{6}\begin{pmatrix} -\frac{3-e^{-2\phi}-2e^{\phi}}{\lambda} \\ \frac{e^{-i\frac{\pi}{3}}\left(e^{\phi}-e^{-2\phi}\right)}{\lambda} - i\sqrt{3}\left(\phi_t+\phi_x\right) \\ \frac{e^{i\frac{\pi}{3}}\left(e^{\phi}-e^{-2\phi}\right)}{\lambda} + i\sqrt{3}\left(\phi_t+\phi_x\right) \end{pmatrix} \\
X_{j2} &= \frac{1}{6}\begin{pmatrix} \frac{e^{-i\frac{\pi}{3}}\left(e^{\phi}-e^{-2\phi}\right)}{\lambda} + i\sqrt{3}\left(\phi_t+\phi_x\right) \\ \frac{e^{i\frac{\pi}{3}}\left(3-2e^{\phi}-e^{-2\phi}\right)}{\lambda} \\ \frac{e^{-2\phi}-e^{\phi}}{\lambda} - i\sqrt{3}\left(\phi_t+\phi_x\right) \end{pmatrix} \\
X_{j3} &= \frac{1}{6}\begin{pmatrix} \frac{e^{i\frac{\pi}{3}}\left(e^{\phi}-e^{-2\phi}\right)}{\lambda} - i\sqrt{3}\left(\phi_t+\phi_x\right) \\ \frac{e^{-2\phi}-e^{\phi}}{\lambda} + i\sqrt{3}\left(\phi_t+\phi_x\right) \\ \frac{e^{-i\frac{\pi}{3}}\left(3-2e^{\phi}-e^{-2\phi}\right)}{\lambda} \end{pmatrix}
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
S_1(\lambda) &= \begin{pmatrix} a_{11}(\lambda) & 0 & 0 \\ a_{21}(\lambda) & \frac{1}{b_{22}(\lambda)} & -\frac{b_{23}(\lambda)}{a_{11}(\lambda)} \\ a_{31}(\lambda) & 0 & \frac{b_{22}(\lambda)}{a_{11}(\lambda)} \end{pmatrix}, \quad T_1(\lambda) = \begin{pmatrix} 1 & \frac{b_{12}(\lambda)}{b_{22}(\lambda)} & -\frac{a_{13}(\lambda)}{a_{11}(\lambda)} \\ 0 & 1 & 0 \\ 0 & \frac{b_{32}(\lambda)}{b_{22}(\lambda)} & 1 \end{pmatrix}, \\
S_2(\lambda) &= \begin{pmatrix} \frac{b_{22}(\lambda)}{a_{33}(\lambda} & 0 & a_{13}(\lambda) \\ -\frac{b_{21}(\lambda)}{a_{33}(\lambda} & \frac{1}{b_{22}(\lambda)} & a_{23}(\lambda) \\ 0 & 0 & a_{33}(\lambda) \end{pmatrix}, \quad T_2(\lambda) = \begin{pmatrix} 1 & \frac{b_{12}(\lambda)}{b_{22}(\lambda)} & 0 \\ 0 & 1 & 0 \\ -\frac{a_{31}(\lambda)}{a_{33}(\lambda)} & \frac{b_{32}(\lambda)}{b_{22}(\lambda)} & 1 \end{pmatrix}, \\
S_3(\lambda) &= \begin{pmatrix} \frac{1}{b_{11}(\lambda)} & -\frac{b_{12}(\lambda)}{a_{33}(\lambda)} & a_{13}(\lambda) \\ 0 & \frac{b_{11}(\lambda)}{a_{33}(\lambda)} & a_{23}(\lambda) \\ 0 & 0 & a_{33}(\lambda) \end{pmatrix}, \quad T_3(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{b_{21}(\lambda)}{b_{11}(\lambda)} & 1 & 0 \\ \frac{b_{31}(\lambda)}{b_{11}(\lambda)} & -\frac{a_{32}(\lambda)}{a_{33}(\lambda)} & 1 \end{pmatrix}, \\
S_4(\lambda) &= \begin{pmatrix} \frac{1}{b_{11}(\lambda)} & a_{12}(\lambda) & -\frac{b_{13}(\lambda)}{a_{22}(\lambda)} \\ 0 & a_{22}(\lambda) & 0 \\ 0 & a_{32}(\lambda) & \frac{b_{11}(\lambda)}{a_{22}(\lambda)} \end{pmatrix}, \quad T_4(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{b_{21}(\lambda)}{b_{11}(\lambda)} & 1 & -\frac{a_{23}(\lambda)}{a_{22}(\lambda)} \\ \frac{b_{31}(\lambda)}{b_{11}(\lambda)} & 0 & 1 \end{pmatrix}, \\
S_5(\lambda) &= \begin{pmatrix} \frac{b_{33}(\lambda)}{a_{22}(\lambda)} & a_{12}(\lambda) & 0 \\ 0 & a_{22}(\lambda) & 0 \\ -\frac{b_{31}(\lambda)}{a_{22}(\lambda)} & a_{32}(\lambda) & \frac{1}{b_{33}(\lambda} \end{pmatrix}, \quad T_5(\lambda) = \begin{pmatrix} 1 & 0 & \frac{b_{13}(\lambda)}{b_{33}(\lambda)} \\ -\frac{a_{21}(\lambda)}{a_{22}(\lambda)} & 1 & \frac{b_{23}(\lambda)}{b_{33}(\lambda)} \\ 0 & 0 & 1 \end{pmatrix}, \\
S_6(\lambda) &= \begin{pmatrix} a_{11}(\lambda) & 0 & 0 \\ a_{21}(\lambda) & \frac{b_{33}(\lambda)}{a_{11}(\lambda)} & 0 \\ a_{31}(\lambda) & -\frac{b_{32}(\lambda)}{a_{11}(\lambda)} & \frac{1}{b_{33}(\lambda)} \end{pmatrix}, \quad T_6(\lambda) = \begin{pmatrix} 1 & -\frac{a_{12}(\lambda)}{a_{11}(\lambda)} & \frac{b_{13}(\lambda)}{b_{33}(\lambda)} \\ 0 & 1 & \frac{b_{23}(\lambda)}{b_{33}(\lambda)} \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
M_1(x,\lambda) &= \left(f_1^{(-)}(x,\lambda), \frac{f_2^{(+)}(x,\lambda)}{b_{22}(\lambda)}, f_3^{(-)}(x,\lambda) - \frac{a_{13}(\lambda)}{a_{11}(\lambda)} f_1^{(-)}(x,\lambda) \right) \\
M_2(x,\lambda) &= \left(f_1^{(-)}(x,\lambda) - \frac{a_{31}(\lambda)}{a_{33}(\lambda)} f_3^{(-)}(x,\lambda), \frac{f_2^{(+)}(x,\lambda)}{b_{22}(\lambda)}, f_3^{(-)}(x,\lambda) \right) \\
M_3(x,\lambda) &= \left(\frac{f_1^{(+)}(x,\lambda)}{b_{11}(\lambda)}, f_2^{(-)}(x,\lambda) - \frac{a_{32}(\lambda)}{a_{33}(\lambda)} f_3^{(-)}(x,\lambda), f_3^{(-)}(x,\lambda) \right) \\
M_4(x,\lambda) &= \left(\frac{f_1^{(+)}(x,\lambda)}{b_{11}(\lambda)}, f_2^{(-)}(x,\lambda), f_3^{(-)}(x,\lambda) - \frac{a_{23}(\lambda)}{a_{22}(\lambda)} f_2^{(-)}(x,\lambda) \right) \\
M_5(x,\lambda) &= \left(f_1^{(-)}(x,\lambda) - \frac{a_{21}(\lambda)}{a_{22}(\lambda)} f_2^{(-)}(x,\lambda), f_2^{(-)}(x,\lambda), \frac{f_3^{(+)}(x,\lambda)}{b_{33}(\lambda)} \right) \\
M_6(x,\lambda) &= \left(f_1^{(-)}(x,\lambda), f_2^{(-)}(x,\lambda) - \frac{a_{12}(\lambda)}{a_{11}(\lambda)} f_1^{(-)}(x,\lambda), \frac{f_3^{(+)}(x,\lambda)}{b_{33}(\lambda)} \right)
\end{aligned}
\tag{A.3}
$$

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