POLITECNICO DI TORINO

MASTER DEGREE
in
MATHEMATICAL ENGINEERING

Information Design in Bayesian Routing Games

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Academic year 2022-2023

Abstract

Routing games model the behavior of a large number of strategic users travelling in a transportation network. Each link of the network is endowed with a delay function that describes the travel time as a function of the flow, and users aim at selecting routes with minimum total travel time. An equilibrium of the game is a flow assignment whereby no users have an incentive to deviate from their chosen route. This thesis is focused on Bayesian routing games, in which delays are stochastic functions that depend on a random variable representing the state of the network. In particular, we investigate how a central planner who observes the state of the world should provide information to the users in order to influence their routing behavior, possibly steering it towards a system optimum. This problem is called in the literature Information design.

It is known that revealing the true state of the network, and more in general public signaling policies, are inefficient in minimizing the system cost. Moreover, the signal space can be restricted to route recommendations such that users have no incentive in deviating from the received recommendation, which is known as obedience constraint. For these reasons, this thesis is focused on private signaling, i.e., when users receive different route recommendations at the same time. The fundamental assumption of this Bayesian model is that the prior distribution of the state of the network and the signal policy are common knowledge to the users, who make decisions based on their posterior belief on both the state of the network and the information that other users after receiving the recommendation.

We analyse the properties of the problem, showing that it is convex if the network has two parallel links and the delay functions are affine. This gives us necessary and sufficient condition for optimality. We find sufficient conditions on the support of the unknown parameters of the delay functions and on their moments under which the price of anarchy is minimized. Interestingly, we observe that a large variance of the unknown parameters is beneficial for the system cost under the optimal policy. Moreover, we study the structure of the optimal policy when the price of anarchy is strictly larger than 1, that is when the obedience constraints are not satisfied under the optimal policy that minimizes the objective function without considering the constraints. In particular, we show that it is not possible under the optimal policy that all the users that receive different recommendations are incentivized in deviating at the same time.

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Chapter 1

Introduction

1.1 Motivation

In a lot of fields, it has become important to model the behavior of systems of transporting networks. There are different strategies for persuading strategic users travelling in a network in order to influence them to take the best route for them and for the system. All together, these models are called in the literature Routing games. Consider therefore a network in which each link has a function that describes the travel time or the cost that a user is subject to when goes through that link. These functions are expressed always as functions of the flow on the link. In particular, these functions can depend also on some stochastic variables that represent the state of the world, with specific probability distributions. The problem that studies the behavior of transporting network endowed with these stochastic delay or cost functions is called Bayesian Routing games.

Moreover, consider that on these transportation networks there is a central planner that can be the system itself and it's subject to the total travel time or cost of the network. Its objective is then to minimize its penalty trying to influence users to act in the most profitable way. Suppose that the central planner knows the state of the world and suppose also that users are rational; it is so strictly necessary to convince them of the convenience of a choice. The problem needs therefore to respect some users' obedience constraints. In order to do that, the central planner provides information to persuade users in order to influence their behavoir and it must take into consideration that users will independently evaluate if the received suggestions are optimal for them. The objective is to find an equilibrium that is a flow setting whereby that no users have an incentive to deviate from their chosen route. The main assumption of this Bayesian model is that the prior distribution of the stochastic variables and how the information is made are common knowledge to the users, who make decisions based on their posterior belief on both the state of the world and the information that other users after receiving the information. The mathematical branch that studies this problem is called Information design and it can be seen as a communication signals play (if there is not an informational advantage) or also as a Bayesian persuasion problem (if there is an informational advantage for the system).

Different combinations of the characteristics of the problem of Information design can be made and in this way different settings can be obtained. In some configurations there may be single or multiple receivers, in the same way there may be single or multiple senders (if there is only one it is usually identified with the system). Moreover, multiple receivers can be treated as atomic or non-atomic if they are respectively a discrete and finite set or a continuous and large set. Looking at the environment it can be static or dynamic. Specially, the state of the world is often discrete and more rarely continue. In the end signal from the senders to the users can be private or public. Focusing on the case of a single sender with informational advantage and multiple non-atomic receivers in a static network with stochastic and continuous state of the world, it is well known that public signaling, i.e. giving all the informations about the state of the world is inefficient in optimizing the cost of the system. Moreover, between all the different informations that can be made, one of the most common is giving route recommendations such that users have no incentive in deviating from the received suggestions. In the end, the interesting case is private signal, i.e. when users receive different recommendations at the same time.

Specifically, this work is focused on the study of the particular application of Information design in traffic networks, that is when users of a network have to choose between different routes that can possibly be congested and that can cause various costs to themselves and to the system, but Information design can be applied in a lot of branches and cases. For example, its applications include financial research as price discrimination, routing software problems, medical research and testing.

1.2 Related literature

The problem of information design has been studied in a varied number of literature papers.

One of the first case studied is with a sender with an informational advantage and a single user, the problem in this case takes the name of *Bayesian persuasion* and is analyzed by Kamenica and Gentzkow in 2011 [11]. Recent works correspond instead to information design problems with a sender always with informational advantage and there are many users of the network. References about this case are the works of Bergemann and Morris of 2016 and 2019[3][4].

One importat aspect analysed in Information design is the problem of understand how much information give to users.
Starting with the setting of full information the reference is Aumann (1987) [2] that introduced the concept of *correlated equilibria*: a policy is communicated to fully informed users and it is an equilibrium if users have no incentive to do not follow the recommedations. Correlated equilibria are extended by Bergemann and Morris in 2016 when there is an uncertainty state in the network, with the concept of *Bayes correlated equilibria*, in this case there is a correlation between the signal provided by the sender and the observation of the state.
A first result is in a paper of Das and Kamenica of 2016 [12] that continues Kamenica and Gentzkow [2011] [11] work on Bayesian persuasion by showing that it there are multiple senders and a rich signal space they tends to increment the number of informations revealed. Then an important results is the demonstration that revealing full information is suboptimal and this is done by Acemoglu et al. in 2018 [1], Das and Kamenica in 2017 [8] and by Tavafoghi and Teneketzis always in 2017 [14]. In particular Das and Kamenica [2017] [8] compare with simplified examples with a discrete state of the world the optimal private policy with full information, no information and public information. Instead, Tavafoghi and Teneketzis [2017] [14] goes in the same direction of the work of this thesis but with the assumption of a discrete state of the world. They investigate also a two-step dynamic in which users can learn by their experience.

Another fondamental result that can be found in literature is the *revelation principle*, that states that given an indirect information policy that gives some information to users without directly recommand routes to them, there exists always a direct information policy that induce the same flow and cost on the network and satisfies the constraints. The main reference for this principle is Bergemann and Morris [2016][3], but it is also presented in Zhu and Savla in 2021 [19].

Zhu and Savla [2021] [19] is one the most recent work about Information design. The setting is very similar to the one in this thesis, but the are some differences: the state of the network is discrete, there is a fraction of agents that does not receive information and given the state, the policy is not deterministic and can randomize among different points in the simplex. In this paper it appears the definition of *obedience* to indicate private information policies where users receive recommendations and they do not have incentive in deviating. If there is no obedience, the equilibrium of the game is a *Bayesian Nash equilibrium*. If the equilibrium is obedient, it is a *Bayes correlated equilibrium*.

The situation that is generated when users are not obedient is investigate always by Zhu and Savla [2021] [19], but then also by Wu et al. in 2017 and 2021 [17] [18].

Another interesting paper about Information design is Wu and Amin [2019] [16]. All the results of Kamenica and Gentzkow are formalized and applied in Kamenica and Gentzkow [2019] [10].
Dynamical setting are investigate by Tavafoghi and Teneketzis always in 2019 [15]. A more general approach is studied by Meigs in 2020 [13] where the dynamic information provision in routing games is analysed.

In conclusion, the only papers that present algorithmic approaches to Information design are Dugmhi and Xu [2016] [9] and Zhu and Savla [2021][19].

1.3 Contributions

This thesis investigate Information design in the particular case of a single sender with informational advantage and multiple non-atomic receivers in a static network. One generalization about literature is that the stochastic state of the world is assumed to be a continuous variable. The focus is on private signaling where the informations are directly recommendations about the route to follow with a common knowledge about the prior distribution of the state of the world.

First, the problem is investigated when a generic information policy is implemented in the network and Bayes Wardrop equilbria are analysed, finding a condition for the equilibrium. Therefore, a theorem is deduced from the revelation principle and allows to simplify the problem. Consequently, the optimization problem of Information design is formalized and properties of the problem are studied finding necessary and sufficient condition for optimality.

The problem is deepened in order to find sufficient conditions on the support of the unknown parameters of the delay functions and a necessary and sufficient condition on their moments under which the price of anarchy is minimized. A variation of stochasticity is made in all the coefficients of the delay functions of the network. Moreover, the structure of the optimal policy when the price of anarchy is strictly larger than 1 is investigated, i.e. in the case when the obedience constraints are not satisfied under the optimal policy that minimizes the objective function of the unconstrained optimization problem.

1.4 Organization

First of all, in chapter 2 a background about transportation networks is presented and definitions about flows, cost functions and equilibria are given. In section 2.1 the network is assumed to be deterministic, both in a homogeneous case and in a heterogeneous case, instead in section 2.2 the network is assumed to be stochastic. In each situation definitions of system optimum, Wardrop equilibria and price of anarchy are setted.

Then going through chapter 3, in section 3.1, the problem is at first analysed when there is a general information policy and users optimize their choices only through the prior distribution of the stochastic variables and expected values about the information that other users have received, showing that it becomes a potential game but with a suboptimal equilibrium. Revelation principle is presented and the problem is reduced only to obedient policy. In section 3.2, the problem is formalized when the information policy is chosen as the one that minimizes the system cost and in order to satisfy also users obedience constraints. Properties of the model are presented.

In chapter 4, the problem is therefore investigate with different settings of the network.

- First with a binary network with affine costs with a generic prior in section 4.1, the optimization problem becomes convex and gives necessary and sufficient conditions for optimality. Consequently, the optimum is found with a condition on the support of the stochastic variables in order to have no saturation and with a condition on moments of the random variables in order to respect constraints and gaining the minimum price of anarchy, i.e. 1. An important observation is done noting that a large variance of the stochastic variables is beneficial for the problem under the optimal policy.

- In section 4.2, the problem is studied with the assumption of deterministic linear coefficients of the cost functions, and always with a condition on the support of the stochastic variable in order to have no saturation, it is examined what happens when the price of anarchy is greater that 1. It is shown that is not possible to violate both the constraints, i.e is not possible under the optimal policy that all the users that receive different informations are incentivized in deviating from all of them, consequently the problem is solved when there is one violation finding a new optimum.

- In the end in section 4.3, the problem is analysed with a binary network with affine costs and a uniform prior distribution. In this setting, is it demonstrated that the constraints are never violated and the price of anarchy is always 1.

In conclusion, in chapter 5, future researches are presented involving efforts to generalize the obtained results. Interesting directions could be considering a fraction of population that do not receive informations or going deeper in the analysis of the suboptimal situations of the implementation of the optimal policy without considering the obedience constraints.

Chapter 2

Background on transportation networks and traffic assignment

2.1 Deterministic transportation networks

Homogeneous deterministic transportation networks

Consider a directed multigraphs $\mathcal{G} = (\mathcal{V}, \varepsilon, \theta, \kappa)$ that is defined by a set of nodes $\mathcal{V}$, a set of links ε that connect the nodes and two functions:

- $\theta : \mathcal{V} \to \varepsilon$ that indicates for each link the starting node,

- $\kappa : \mathcal{V} \to \varepsilon$ that indicates for each link the arrival node.

Observation 1. *For ease of notation, we describe the game for directed multigraphs with only one origin node o and one destination node d. However, the arguments can be easily generalized for arbitrary multigraphs.*

In this contest a path is a set of links $p = (e_1, ..., e_n)$ where $\theta(e_1) = o$ and $\kappa(e_n) = d$. The set of all this path is referred as $\Gamma_{(o,d)}$.

Suppose that on each link there is a deterministic delay function $\tau_e : \mathbb{R}_+ \to \mathbb{R}_+$ with the particular assumption that $\tau_e(0) = 0$ and that it is strictly increasing, continuously differentiable and convex. The sum of all the delay functions on the network is called the total travel time of the network or simplier the cost of the network.

Definition 1. *Transportation network is a pair of a directed multigraph* $\mathcal{G} = (\mathcal{V}, \varepsilon, \theta, \kappa)$ *and a family of delay functions* $\tau_e : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \forall e \in \varepsilon$.

On a multigraph a link-node incidence matrix B $\in \mathbb{R}^{\mathcal{V}\times\varepsilon}$, i.e. a matrix whose element ij is equal to 1 if $\theta\,(i) = j$ or it is equal to -1 if $\kappa\,(i) = j$ and it's 0 otherwise, is associated with this graph and it allows to define the flow on the graph. Also it is possible to define on the graph the link-path incidence matrix A $\in [0,1]^{\varepsilon\times\Gamma_{(o,d)}}$, i.e. a matrix whose element ep is equal to 1 if link e is along path p or it is equal to 0 otherwise.

On that graph consider that is defined an exogenous network flow $v \in \mathbb{R}^{\mathcal{V}}$ that gives the unitary mass (without loss of generality) incoming and outgoing from each node. In our contest it is simply the difference between the mass incoming in the origin node and outcoming in the destination node: $\delta^{(0)} - \delta^{(d)}$. A flow f is a vector in $\mathbb{R}_+^{\varepsilon}$ such that it satisfies the flow balance equation $Bf = v$.

Then in the contest of deterministic transportation networks it is possible to define a *flow optimization problem* that is:

$$\min_{f\geq 0, Bf=v} \sum_{e\in\varepsilon} f_e \tau_e\,(f_e) \tag{2.1}$$

whose optimal solution is the **system optimum** (SO) that is the flow f^* that minimize the total travel time for the system given an exogenous network flow.

Consider that on the network there are users that act like a non-atomic population, i.e. their number is large and they are considered as a continous set. Users on the network are strategic, and aim at minizing their travel time without considering the cost for the whole system. This type of behaviour is captured by the notion of Wardrop equilibrium.

A **Wardrop equilibrium** (WE) on a determinisitic transportation network is a flow $f^0 = Az$ where $z \in \mathbb{R}^{\Gamma_{(o,d)}} : z \geq 0, \mathbb{1}'z = v$ such that no one has incentive in deviating from their route. This means that users follow path $p \in \Gamma_{(o,d)}$ if:

$$\sum_{e\in\varepsilon} A_{ep}\tau_e\left(f_e^0\right) \leq \sum_{e\in\varepsilon} A_{eq}\tau_e\left(f_e^0\right) \quad \forall q \in \Gamma_{(o,d)} \tag{2.2}$$

that is: a user chooses a path if the delay on it is lower than the one on every other path.

It can be be demonstrated that this optimization problem is a **potential game**, i.e. it exists a function $\Phi : \mathbb{R}_+^\varepsilon \to \mathbb{R}$ such that for each flows f_i^0 and f_j^0

$$\Rightarrow \sum_{e\in\varepsilon} f_{ie}^0 \tau_e \left(f_{ie}^0\right) - \sum_{e\in\varepsilon} f_{je}^0 \tau_e \left(f_{je}^0\right) = \frac{\partial\Phi(f_i^0)}{\partial f_i^0} - \frac{\partial\Phi(f_j^0)}{\partial f_j^0}.$$

In homogeneous transportation networks the existence of a Wardrop equilibrium is always guaranteed and they are the minimum of the potential function. Moreover, if the delays function are strictly increasing then wardrop equilibrium is also unique.

The inefficiency of Wardrop equilibrium compared to the system optimum can be measured in this deterministic case by the **price of anarchy** (PoA), which is the ratio between the total travel time at the equilibrium and the optimal total travel time, i.e.

$$PoA = \frac{\sum_{e\in\varepsilon} f_e^0 \tau_e \left(f_e^0\right)}{\min_{f\geq 0, Bf=v} \sum_{e\in\varepsilon} f_e \tau_e \left(f_e\right)}. \tag{2.3}$$

[7]

Heterogeneous deterministic transportation networks

Suppose now that on the same graph there are different populations $p \in \mathcal{P}$ of users. The objective is always to minimize the cost of the network for the system and for users. The difference is that each population has its own delay function, always strictly increasing, continuously differentiable and convex, and so it is necessary to define an aggregate network flow on the network, i.e.:

$$f^{agg} = \sum_{p\in\mathcal{P}} f^p$$

Definition 2. *Heterogeneous deterministic transportation network is a triplet of a directed multigraph $\mathcal{G} = (\mathcal{V}, \varepsilon, \theta, \kappa)$ a family of delay functions $\tau_e : \mathbb{R}_+ \to \mathbb{R}_+ \forall e \in \varepsilon$ and a set of populations $\mathcal{P}$ to which users belong.*

In this setting the *system optimum* (SO) is the flow f^* that minimize the total travel time for the system given an exogenous network flow and delay functions that differ for each population:

$$\min_{f\geq 0, Bf=v} \sum_{e\in\varepsilon} \tau_e^p \left(f_e\right) \forall p \in \mathcal{P} \tag{2.4}$$

At the same time the *Wardrop equilibrium* (WE) is the flow f^0 for which users follow path $p \in \Gamma_{(o,d)}$ if:

$$\sum_{e\in\varepsilon} A_{ep} \tau_e^p \left(f_e^0\right) \leq \sum_{e\in\varepsilon} A_{eq} \tau_e^p \left(f_e^0\right) \quad \forall q \in \Gamma_{(o,d)}, \forall p \in \mathcal{P} \tag{2.5}$$

In general the existence and uniqueness of equilibria in heterogeneous network dynamics is not trivial.
In the special case in which every population has same origin and destination and delay functions differ only for a costant, then it can be proved that the game is still potential. If the game is still potential then Wardrop equilibria always existe also if they are not unique in general.
It is then defined that in this situation Wardrop equilibria can be essentially unique that means that can exists different Wardrop equilibra, but they induce on the network the same aggregate network flows. [6]

Several strategies can be used to improve the inefficency of the user equilibrium and reduce the inefficiencies due to the selfish user behaviour. Among these, we mention tolls and information design. In particular, information design is useful when the users have some kind of uncertainty on the network state. In the remainder of this thesis, we shall consider stochastic transportation network games and study information design in this setting.

2.2 Stochastic transportation networks

Consider now that the network depends on the realization of a stochastic continuous variable, that is the state of the world ω, that lives in the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ where Ω is the sample space, $\mathcal{A}$ is the set of events and $\mathbb{P}$ is the probability function.

Definition 3. *Transportation network is a triplet of a directed multigraph* $\mathcal{G} = (\mathcal{V}, \varepsilon, \theta, \kappa)$ *probability space* $(\Omega, \mathcal{A}, \mathbb{P})$ *and a family of delay functions* $\tau_e : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ \forall e \in \varepsilon$.

In this contest different situations may arise.

The first case is if there is **no information** about the state of the world.
The *system optimum* (SO), is the flow f^* that minimize the total travel time for the system given an exogenous network flow with respect to the expected value of the delay functions, i.e.:

$$\min_{f \geq 0, Bf = v} \int_{\Omega} \sum_{e \in \varepsilon} f_e \tau_e (\omega, f_e) \, d\mathbb{P}(\omega) \tag{2.6}$$

At the same time the *Wardrop equilibrium* (WE) is the flow f^0 for which users follow path $p \in \Gamma_{(o,d)}$ if:

$$\int_{\Omega} \sum_{e \in \varepsilon} A_{ep} \tau_e \left(\omega, f_e^0\right) d\mathbb{P}(\omega) \leq \int_{\Omega} \sum_{e \in \varepsilon} A_{eq} \tau_e \left(\omega, f_e^0\right) d\mathbb{P}(\omega) \quad \forall q \in \Gamma_{(o,d)} \tag{2.7}$$

where the optimization occurs on the expected value of the delay functions.

On the contrary, in the second case there can be a **full information** about the state of the world.
Then the *system optimum* (SO) is the flow f^* that minimize the total travel time for the system given an exogenous network flow with complete knowledge about the realization of the stochastic variable ω, i.e.:

$$\min_{f\geq 0, Bf=v} \int_{\Omega} \sum_{e\in\varepsilon} f_e(\omega)\, \tau_e(\omega, f_e(\omega))\, d\mathbb{P}(\omega) \tag{2.8}$$

Instead, the *Wardrop equilibrium* (WE) is the flow f^0 for which users follow path $p \in \Gamma_{(o,d)}$ if:

$$\int_{\Omega} \sum_{e\in\varepsilon} A_{ep}\tau_e\left(\omega, f_e^0(\omega)\right) d\mathbb{P}(\omega) \leq \int_{\Omega} \sum_{e\in\varepsilon} A_{eq}\tau_e\left(\omega, f_e^0(\omega)\right) d\mathbb{P}(\omega) \quad \forall q \in \Gamma_{(o,d)} \tag{2.9}$$

where the optimization occurs for each value of ω.

Consequently, also in this stochastic case it is possible to define the price of anarchy:

- the price of anarchy when users have no knowledge about the state of the world is :

$$PoA_0 = \frac{\int_{\Omega} \sum_{e\in\varepsilon} f_e^0 \tau_e\left(\omega, f_e^0\right)}{\min_{f\geq 0, Bf=v} \int_{\Omega} \sum_{e\in\varepsilon} f_e(\omega)\, \tau_e(\omega, f_e(\omega))\, d\mathbb{P}(\omega)} \tag{2.10}$$

- the price of anarchy when users have full knowledge about the state of the world is :

$$PoA_1 = \frac{\int_{\Omega} \sum_{e\in\varepsilon} f_e^0(\omega)\, \tau_e\left(\omega, f_e^0(\omega)\right)}{\min_{f\geq 0, Bf=v} \int_{\Omega} \sum_{e\in\varepsilon} f_e(\omega)\, \tau_e(\omega, f_e(\omega))\, d\mathbb{P}(\omega)} \tag{2.11}$$

Chapter 3

Information design problem

3.1 Model formulation

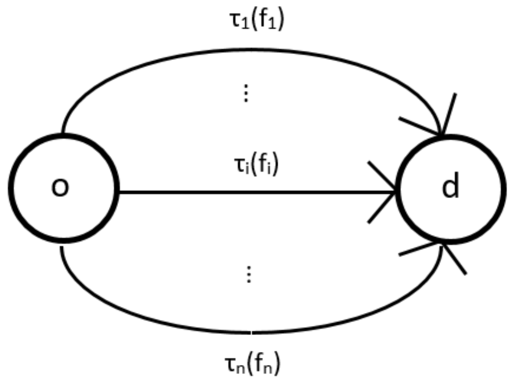

Figure 3.1. Representation of a graph with source and destination nodes with n links and generic delays.

In the general problem of Information design on a stochastic transportation network an omniscient planner, that can be identified as the system, observes the realization of the state of world of the network and sends signals to the users in order to minimize the total travel time of the system. Define the simplex $S_R = \{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}$, as the set of all possible suggestions that the planner can send. We assume that these suggestions coincide with the recommendations of using a certain route $R = i$, where R is the set of possible routes.

Observation 2. *We present results about graphs with only parallel links from the origin node and the destination node, but they can be generalized to generic graphs. With this assumption paths coincide with links.*

Definition 4. ***Information policy*** *is a map* $\pi : \Omega \to S_R$ *that goes from the distribution of the state of the world to the simplex of all possible recommendations.*

Once the signal has been received, users behave like different populations who, on the basis of the indication received, calculate the posterior distribution $d\mathbb{P}_i$ on the random variable ω and on the suggestion $R = i$ using Bayes theorem. Since users are strategic they can deviate from the recommendation. Knowing of the posterior distribution of the random variable and the expected values regarding what the other users have received, users choose their actions in order to reach the Wardrop equilibrium.

From a game theory perspective, both the system and the users search for a combination of actions that is convenient for everyone and that no one has interests to change: this is called reaching the Nash equilibrium.

Let y_{ij} be the fraction of users that after receiving signal of following path i choose to follow path j, such that $y_{ij} \geq 0$ for every i, j and $\sum_j y_{ij} = 1$ for every i. Then, for every i, j such that $\pi_i(\omega) > 0$ for at least a ω and $y_{ij} > 0$, the condition for the equilibrium is:

$$\mathbb{E}\left[\tau_j \mid R = i\right] \leq \mathbb{E}\left[\tau_k \mid R = i\right] \forall k \tag{3.1}$$

that is:

$$\mathbb{E}\left[\tau_j \mid R = i\right] = \frac{\int_\Omega \tau_j \left(\omega, \sum_{r \in R} y_{ri} \pi_r \left(\omega\right)\right) d\mathbb{P}_i \left(\omega\right)}{\int_\Omega \pi_i \left(\omega\right) d\mathbb{P} \left(\omega\right)} \tag{3.2}$$

that contains the posterior distribution calculated with **Bayes theorem**:

$$d\mathbb{P}_i \left(\omega\right) := \frac{\pi_i \left(\omega\right) d\mathbb{P} \left(\omega\right)}{\int_\Omega \pi_i \left(\omega\right) d\mathbb{P} \left(\omega\right)} \tag{3.3}$$

Observe that y depends on the policy π via the posterior $d\mathbb{P}_i(\omega)$. Plugging 3.2 into 3.1, we get the equilibrium condition:

$$\int_\Omega \tau_j \left(\omega, \sum_{r \in R} y_{ri} \pi_r \left(\omega\right)\right) \pi_i \left(\omega\right) d\mathbb{P} \left(\omega\right) \leq \int_\Omega \tau_k \left(\omega, \sum_{r \in R} y_{rk} \pi_r \left(\omega\right)\right) \pi_i \left(\omega\right) d\mathbb{P} \left(\omega\right) \forall i, j, k \tag{3.4}$$

What it turns out is a Wardrop equilibrium in the form of a matrix where y_{ij} is the fraction of users who, after receiving the signal to follow path i, goes on path j. This matrix is therefore stochastic by rows.

Definition 5. *In a stochastic transportation network, given an information policy* π *a* ***Bayes Wardrop equilibrium*** *is a flow* $f^\pi : \Omega \to \mathbb{R}_+^\varepsilon$ *such that it exists a* *stochastic matrix* $y : \Omega \to \mathbb{R}^{R\times R}$ *for which:*

$$f_j(\omega) = \sum_{i\in\Gamma_{(o,d)}} y_{ij}(\pi)\,\pi_i(\omega)\,\forall\omega\in\Omega \tag{3.5}$$

and it holds true 3.4.

If this equilibrium exists and it's unique it is called f_j^π.

The following result gives a characterization of this Bayes Wardrop equilibrium.

Proposition 1. *Given a policy* $\pi(\omega)$*,* $f_j^\pi(\omega)$ *is a Bayes Wardrop equilibrium if and only* $y(\pi)$ *is a solution of the following* ***optimization problem****:*

$$\begin{aligned} &\min_{y:y_{ij}>0} \Phi(y) \\ s.t. \quad &\sum_j y_{ij} = 1 \quad \forall i \end{aligned}$$

where

$$\Phi(y) = \sum_{j\in\Gamma_{(o,d)}} \int_\Omega \int_0^{\sum_{r\in R} y_{rj}\pi_r(\omega)} \tau_j(\omega,s)\,ds d\mathbb{P}(\omega) \tag{3.6}$$

is the potential function.

Proof. Notice that $\Phi(y)$ is convex in y because it is the combination of convex functions $\Rightarrow$ necessary and sufficients conditions for optimality can be found, it follows that y is a minimum of the potential. The partial derivative of this potential function is:

$$\begin{aligned} \frac{\partial\Phi(y)}{\partial y_{ij}} &= \int_\Omega \tau_j\left(\omega, \sum_{r\in R} y_{rj}\pi_r(\omega)\right)\pi_i d\mathbb{P}(\omega) \\ = \int_\Omega \tau_j\left(\omega, \sum_{r\in R} y_{rj}\pi_r(\omega)\right) & d\mathbb{P}_i(\omega) \int_\Omega \pi_i(\omega)\,d\mathbb{P}(\omega) \\ &= \mathbb{E}\left[\tau_j \mid R = i\right] \int_\Omega \pi_i(\omega)\,d\mathbb{P}(\omega) \end{aligned}$$

Then, if $y_{ij} > 0$, $\frac{\partial\Phi(y)}{\partial y_{ik}} - \frac{\partial\Phi(y)}{\partial y_{ij}} \geq 0 \quad \forall k$, which implies $E[\tau_j|R=i] \leq E[\tau_k|R=i]$, which is the definition of Wardrop equilibrium. Since $y_{ij} > 0$, this shows that the solution of the optimization problem is the Wardrop equilibrium $y^*(\pi)$. □

Observation 3. *Note in particular, that it is a weighted potential game, i.e. the derivative of the potential function are the weighted delays of the users.*

Let $y(\pi)$ be the Wardrop equilibrium given a policy π. The cost of a policy π is the expected total travel time at the equilibrium with the flow $f_j = \sum_{i\in\Gamma_{(o,d)}} y_{ij}\pi_i(\omega)$, i.e.

$$C(\pi) = \int_{\Omega} \sum_{j\in\Gamma_{(o,d)}} \tau_j(\omega, f_j(\omega)) f_j(\omega)\, d\mathbb{P}(\omega). \tag{3.7}$$

The system then wants to solve the problem:

$$\pi^* \in \arg\min_{\pi} C(\pi). \tag{3.8}$$

Problem 1. *Given a stochastic transportation network, the strategy of Information design consists in finding the optimal information policy* π^* *such that*

$$\int_{\Omega} \sum_{j\in\Gamma_{(o,d)}} \tau_j\left(\omega, f_j^{\pi^*}(\omega)\right) f_j^{\pi^*}(\omega)\, d\mathbb{P}(\omega) \leq \int_{\Omega} \sum_{j\in\Gamma_{(o,d)}} \tau_j\left(\omega, f_j^{\pi}(\omega)\right) f_j^{\pi}(\omega)\, d\mathbb{P}(\omega)\, \forall\pi \tag{3.9}$$

Theorem 1. *If* π^* *is an optimal solution of problem 1* $\Rightarrow$ *constraints 3.4 are satisfied.*

This theorem is also known as **revelation principle**. [3]
It states that problem 1 can be much simplified because given a policy π, there always exists a policy $\bar{\pi}$ such that $C(\pi) = C(\bar{\pi})$ and $y_{ij}\pi) = \delta_j^{(i)}$. In other words, this means that each user of the network has no interest in deviating from the recommendations it gets. This is also known as obedience constraints.

Observation 4. *The problem of Information design can be restricted only to policy that satisfy the constraints.*

To sum up, narrowing the analysis only to obedient policy, the outline of the process is the following:

- the a priori $d\mathbb{P}$ of ω is known;
- central planner minimizes $\mathbb{E}_{\omega\sim d\mathbb{P}}\tau\left(\omega, f^*\left(d\mathbb{P}_i\right)\right)$;
- central planner chooses a signal π;
- users observe signal π and deduce $d\mathbb{P}_i$;

- users generates a flow $f^0(d\mathbb{P}_i)$ that minimize $\mathbb{E}_{\omega\sim d\mathbb{P}_i}\tau(\omega, f^0)$;
- cost is measured.

Therefore it is possible to write the constrained optimization problem, from a mathematical perspective, with the objective to find the information policy in order to minimize the expected value of the total travel time for the system and in order to respect the obedience constraints for users of the network. The mathematical formulation of the problem of Information design on a network with n parallel links and stochastic delay functions is in the end:

Problem 2.

$$\min_{\pi:\Omega\to S_n} \int_\Omega \left[\sum_{i=1}^{n} \left(\tau_i\left(\omega, \pi_i\left(\omega\right)\right)\right) \pi_i\left(\omega\right)\right] d\mathbb{P}\left(\omega\right)$$

$$\textit{subj. to} \int_\Omega \pi_i\left(\omega\right)\left[\tau_i\left(\omega, \pi_i\left(\omega\right)\right)\right] d\mathbb{P}\left(\omega\right) \leq \int_\Omega \pi_i\left(\omega\right)\left[\tau_j\left(\omega, \pi_j\left(\omega\right)\right)\right] d\mathbb{P}\left(\omega\right) \forall i, j$$

3.2 Model properties

The general problem 2 of Information design on a network with n parallel links and stochastic delay functions has two important properties:

Property 1. *Since delay functions τ are convex then the objective function is convex.*

Property 2. *The optimum of the unconstrained optimization problem can be found for each ω, and so it's indipendent from the probability distribution of the stochastic variable.*

These properties are shown for networks with parallel links but the results can be generalized to arbitrary graphs.

Observation 5. *Constraint functions of the optimization problem are in general not convex , then the resulting optimization problem is neither convex or concave and has also large dimensions. It becomes convex only on a network with two parallel links and affine delay functions.*

Observation 6. *These properties greatly simplify the problem in case the constraints are not violated, because in this case then the optimum can be found by solving a convex optimization problem ω by ω.*

Chapter 4

Information design with two parallel links and affine delay functions

In the next sections we find the optimal solutions of the optimization problem 2 on a network with two parallel links and with affine delay functions in different settings, because in this case the problem becomes convex and, as said in observation 6, optimum can be found ω per ω. First, considering a generic prior distribution of the stochastic variables, then, going through the case with deterministic linear coefficients and in the end, analysing the case with uniform prior disrtibution of the stochastic variables.

4.1 Analysis with generic prior

The optimization problem is reformulated in the case in which there is a network represented by a graph $G = (V, E)$ with only two nodes $V = \{o, d\}$ and only two links $E = \{e_1, e_2\}$ from the origin to the destination. As anticipated, in this specific case delay functions on links are choosen as affine functions of flows on the network and depends on two stochastic random vectors a, b that have a probabily distribution that is generic, also joint, on a space Ω.

$$\tau_1 (a_1, b_1, f_1) = b_1 + a_1 f_1$$
$$\tau_2 (a_2, b_2, f_2) = b_2 + a_2 f_2$$

With this setting the policy function of the signal that the system sends to

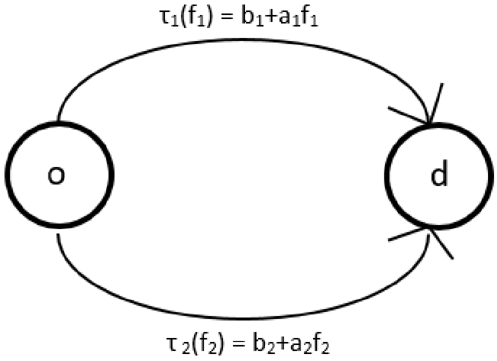

Figure 4.1. Representation of a graph with source and destination nodes with two links and affine delay functions.

users in order to reach an optimality for the network is:

$$\pi_1(a,b) = \mathbb{P}(R = 1 \mid a,b)$$
$$\pi_2(a,b) = \mathbb{P}(R = 2 \mid a,b)$$

and it is such that $\pi_1(a,b) + \pi_2(a,b) = 1$.

Then the optimization problem of information design 2 on a network with 2 links, affine stochastic delay functions and generic prior distribution becomes:

$$\begin{aligned} \min_{\pi_1(a,b)} & \int_\Omega \left[a_2 + b_2 + (-2a_2 + b_1 - b_2)\,\pi_1(a,b) + (a_1 + a_2)\,\pi_1(a,b)^2\right] d\mathbb{P}(a,b) \\ \text{s.t.} & \int_\Omega \left[(-a_2 + b_1 - b_2)\,\pi_1(a,b) + (a_1 + a_2)\,\pi_1(a,b)^2\right] d\mathbb{P}(a,b) \leq 0 \\ & \int_\Omega \left[a_2 + (b_2 - b_1) + (-a_1 - 2a_2 + b_1 - b_2)\,\pi_1(b) + (a_1 + a_2)\,\pi_1(a,b)^2\right] d\mathbb{P}(a,b) \leq 0 \end{aligned} \tag{4.1}$$

Everything is expressed in terms of $\pi_1(a,b)$ because $\pi_1(a,b) + \pi_2(a,b) = 1$, then $\pi_2(a,b) = 1 - \pi_1(a,b)$.

Recall that this problem has an objective function that is convex as discussed in the previous section. Moreover, the two inequality integral constraints that now are also both convex, hence the problem can be defined as a convex optimization problem. This observation gives us the guarantee that the necessary conditions for optimality are also sufficient.

Lemma 2. *The solution of the unconstrained optimzation problem 4.1 is:*

$$\overline{\pi_1}\left(a,b\right) = \left[\frac{2a_2 - b_1 + b_2}{2\left(a_1 + a_2\right)}\right]_0^1 \tag{4.2}$$

$$\overline{\pi_2}\left(a,b\right) = \left[\frac{2a_1 + b_1 - b_2}{2\left(a_1 + a_2\right)}\right]_0^1 \tag{4.3}$$

Proof. Note that notions about the Lagrangian method and duality are needed, it is possible to find details in the appendix A.
The solution of the uncostrained optimization problem can be found easily using the lagrangian function:

$$\mathcal{L}\left(\pi_1\left(a,b\right)\right) = a_2 + b_2 + \left(-2a_2 + b_1 - b_2\right)\pi_1\left(a,b\right) + \left(a_1 + a_2\right)\pi_1^2\left(a,b\right) \tag{4.4}$$

Fixing a particular value of a, b it is it possible to find the optimal value of the lagrangian function deriving it and putting its derivative equal to zero:

$$\frac{\partial\mathcal{L}\left(\pi_1\left(a,b\right)\right)}{\partial\pi_1\left(a,b\right)} = 0 \iff b_1 - b_2 - 2a_2 + 2\left(a_1 + a_2\right)\pi_1\left(a,b\right) = 0$$

This is true if and only if:

$$\overline{\pi_1}\left(a,b\right) = \frac{2a_2 - b_1 + b_2}{2\left(a_1 + a_2\right)}. \tag{4.5}$$

Observe that $\overline{\pi}(a,b)$ is a feasible policy only for a, b such that $0 \leq \overline{\pi}(a,b) \leq 1$. Since $\mathcal{L}\left(\pi_1\left(a,b\right)\right)$ is convex in π, if the stationary point is greater than 1, then $\overline{\pi} = 1$. If instead the stationary point is below 0, then $\overline{\pi} = 0$. □

Note that all depends on the difference between the b variables.

We call $\overline{\pi}\left(a,b\right)$ the solution of the optimization problem defined in Lemma 2, in contrast with $\pi^*\left(a,b\right)$ that is the solution of the constrained optimization problem 4.1. Observe that if $\overline{\pi}\left(a,b\right)$ satisfies the obedience constraints in 4.1, then $\pi^*\left(a,b\right) = \overline{\pi}\left(a,b\right)$.

Moreover, we can define a Price of Anarchy in this setting, which is the ratio between the system cost with optimal policy $\pi^*\left(a,b\right)$ and the optimal system cost that can be achieved if a central planner can dictate the user choices, i.e., the system cost with policy $\overline{\pi}\left(a,b\right)$.

Definition 6.

$$PoA = \frac{\int_\Omega \left[a_2 + b_2 + \left(-2a_2 + b_1 - b_2\right)\overline{\pi_1}\left(a,b\right) + \left(a_1 + a_2\right)\overline{\pi_1}^2\left(a,b\right)\right]d\mathbb{P}\left(a,b\right)}{\int_\Omega \left[a_2 + b_2 + \left(-2a_2 + b_1 - b_2\right)\pi_1^*\left(a,b\right) + \left(a_1 + a_2\right)\pi_1^{*2}\left(a,b\right)\right]d\mathbb{P}\left(a,b\right)}. \tag{4.6}$$

The next result provides sufficient conditions for optimality, i.e., $\pi^*(a,b) = \overline{\pi}(a,b)$ and PoA=1. The idea is to find sufficient conditions under which $\overline{\pi}$ satisfies the obedience constraints. To better formalize the result, let us define the notion of support of a random variable.
Given a random variable X in $\mathbb{R}^n$, its support is the set of values that the random variable can take, i.e.,

$$supp\{X\} = \{x \in \mathbb{R}^n : \mathbb{P}(B(x,\epsilon)) > 0 \forall \epsilon > 0\},$$

where $B(x,\epsilon)$ is the open ball centered in x with radius ϵ.

Theorem 3. *Consider problem 4.1. Assume that:*

$$\begin{cases} \max(supp\{b_1 - b_2\}) \leq 2\min(supp\{a_2\}) \\ \min(supp\{b_1 - b_2\}) \geq -2\max(supp\{a_1\}) \end{cases} \tag{4.7}$$

holds. Then, the optimum is $\pi^*(a,b) = \overline{\pi}(a,b)$ and PoA is equal to 1 if and only if:

$$\begin{cases} \mathbb{E}\left[\frac{2a_2(b_1-b_2)-(b_1-b_2)^2}{(a_1+a_2)}\right] \leq 0 \\ \mathbb{E}\left[\frac{-2a_1(b_1-b_2)-(b_1-b_2)^2}{(a_1+a_2)}\right] \leq 0 \\ \end{cases} \tag{4.8}$$

Proof. The idea of the proof is that under conditions 4.7 and 4.8, $\overline{\pi}$ does not saturate and does not violate the obedience constraints. Plugging $\overline{\pi_1}(a,b)$ into the first constraint of 4.1, it is violated when:

$$\int_\Omega \left[(-a_2 + b_1 - b_2)\left[(2a_2 - (b_1 - b_2))\,\alpha\right] + (a_1 + a_2)\left[(2a_2 - (b_1 - b_2))\,\alpha\right]^2\right] d\mathbb{P}(a,b) > 0$$

which leads to:

$$\mathbb{E}\left[\frac{2a_2(b_1 - b_2) - (b_1 - b_2)^2}{(a_1 + a_2)}\right] > 0 \tag{4.9}$$

Likewise the second constraint of 4.1 is violated when:

$$\int_\Omega a_2 - (b_1 - b_2) + (-a_1 - 2a_2 + b_1 - b_2)\left[(2a_2 - (b_1 - b_2))\,\alpha\right] \\ + (a_1 + a_2)\left[(2a_2 - (b_1 - b_2))\,\alpha\right]^2 d\mathbb{P}(a,b) > 0$$

which leads to:

$$\mathbb{E}\left[\frac{-2a_1(b_1 - b_2) - (b_1 - b_2)^2}{(a_1 + a_2)}\right] > 0 \tag{4.10}$$

□

Observation 7. *Observe that if* $\mathbb{E}\left[b_1 - b_2\right] = 0$ *and* a, b *are independent, then the two constraints are always satisfied and optimality is thus achieved. Instead, this is in general not true if* a, b *are correlated.*

In the next sections, we shall apply Theorem 3 to some special cases. Moreover, we shall consider what happens when optimality is not achieved.

4.2 Analysis with deterministic linear coefficients

The above optimization problem is reformulated in the particular case in which there is a network represented by a graph $G = (V, E)$ with only two nodes $V = \{o, d\}$ and only two links $E = \{e_1, e_2\}$ from the origin to the destination. As anticipated, affine delay functions on links are still choosen as affine functions of flows on the network with the stochastic continuous vector b that has a probabily distribution that is generic, on a space Ω, but with the vector of linear coefficients a instead chosen deterministic and, without loss of generality, also positive:

$$\tau_1\left(b_1, f_1\right) = b_1 + a_1 f_1$$
$$\tau_2\left(b_2, f_2\right) = b_2 + a_2 f_2$$

Then the optimization problem of information design 4.1 on a network with 2 links, affine stochastic delay functions with deterministic linear coefficients and generic prior distribution becomes:

$$\begin{aligned} \min_{\pi_1(b)} & \int_\Omega \left[a_2 + b_2 + \left(-2a_2 + b_1 - b_2\right)\pi_1(b) + \left(a_1 + a_2\right)\pi_1(b)^2\right] d\mathbb{P}(b) \\ \text{s.t.} & \int_\Omega \left[\left(-a_2 + b_1 - b_2\right)\pi_1(b) + \left(a_1 + a_2\right)\pi_1(b)^2\right] d\mathbb{P}(b) \leq 0 \\ & \int_\Omega \left[a_2 + \left(b_2 - b_1\right) + \left(-a_1 - 2a_2 + b_1 - b_2\right)\pi_1(b) + \left(a_1 + a_2\right)\pi_1(b)^2\right] d\mathbb{P}(b) \leq 0 \end{aligned} \tag{4.11}$$

Theorem 4. *Consider problem 4.11. Assume that:*

$$supp\{b_1 - b_2\} \subseteq \left[-2a_1, 2a_2\right] \tag{4.12}$$

holds. Then, the optimum is $\pi_1^*(b) = \overline{\pi_1}(b)$ *and PoA is equal to 1 if and only if:*

$$\begin{cases} \mathbb{E}\left[b_1 - b_2\right] \leq \frac{\mathbb{E}\left[\left(b_1 - b_2\right)^2\right]}{2a_2} \\ \mathbb{E}\left[b_1 - b_2\right] \geq -\frac{\mathbb{E}\left[\left(b_1 - b_2\right)^2\right]}{2a_1} \end{cases} \tag{4.13}$$

Proof. The theorem follows directly from theorem 3, considering that since a is deterministic it can be taken out from the expected value in 4.13. □

Observation 8. *Is is interesting to establish a parallelism with the price of anarchy in the deterministic setting, i.e., if $b_1 - b_2$ can take only 1 value. If $b_1 - b_2 > 0$, then 4.13 and 4.12 state that optimality is achieved if $b_1 - b_2 = 2a_2$. That means $\overline{\pi_1} = 0$, i.e. everyone is on link two. If instead $b_1 - b_2 < 0$, then 4.13 and 4.12 state that optimality is achieved if $b_1 - b_2 = 2a_1$. That means $\overline{\pi_1} = 1$, i.e. everyone is on link one. The conditions on moments is also satisfied if $b_1 - b_2 = 0$. Putting all together, Theorem 4 states that optimality can be reached in the deterministic setting if the two links have equal free-flow delay or if the difference of the two free-flow delays is so large that all users travel on the same link.*
Theorem 4 states that uncertainty is beneficial for the system, i.e. it is possible to achieve optimality, hence PoA=1, even if $\mathbb{E}[b_1 - b_2] \neq 0$. In particular the larger is the variance $\mathbb{E}\left[(b_1 - b_2)^2\right] - \mathbb{E}[b_1 - b_2]^2$, the larger $|\ \mathbb{E}[b_1 - b_2]\ |$ can be.

Theorem 5. *Consider problem 4.11 on a network with two parallel links and stochastic affine delay functions with deterministic linear coefficients. Then:*

1. *if the following condition on moments holds:*

$$-\frac{\mathbb{E}\left[(b_1 - b_2)^2\right]}{2a_1} \leq \mathbb{E}[b_1 - b_2] \leq \frac{\mathbb{E}\left[(b_1 - b_2)^2\right]}{2a_2}$$

   and the condition on the support holds:

$$supp\{b_1 - b_2\} \subseteq [-2a_1, 2a_2]$$

   then, none of the constraints is active, PoA=1 and the solution of the optimal optimization problem is $\pi^*(b) = \overline{\pi}(b)$;

2. *if the following condition on moments holds:*

$$\mathbb{E}[b_1 - b_2] > \frac{\mathbb{E}\left[(b_1 - b_2)^2\right]}{2a_2}$$

   and the condition on the support holds:

$$supp(\omega_1 - \omega_2) \subseteq \left[-2a_1 - a_2 + \sqrt{k}, a_2 + \sqrt{k}\right]$$

where $k = a_2^2 - 2a_2\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]$,
then, first constraint is active and second one is not,

$$PoA = 1 + \frac{1}{2}\frac{\left[a_2 - \sqrt{a_2^2 - 2a_2\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]}\right]^2}{2a_1a_2 + 2(a_1 + a_2)\mathbb{E}[b_2] + 2a_2\mathbb{E}[b_1 - b_2] - \frac{1}{2}\mathbb{E}\left[(b_1 - b_2)^2\right]}$$

and the solution of the optimal optimization problem is

$$\pi_1^*(b) = \frac{\sqrt{a_2^2 - 2a_2\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]} + [a_2 - (b_1 - b_2)]}{2(a_1 + a_2)};$$

3. *if the following condition on moments holds:*

$$\mathbb{E}[b_1 - b_2] < -\frac{\mathbb{E}\left[(b_1 - b_2)^2\right]}{2a_1}$$

and the condition on the support holds:

$$supp(\omega_1 - \omega_2) \subseteq \left[-a_1 - \sqrt{k}, a_1 + 2a_2 - \sqrt{k}\right] \tag{4.14}$$

where $k = a_1^2 + 2a_1\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]$,
then, second constraint is active and first one is not,

$$PoA = 1 + \frac{1}{2}\frac{\left[a_1 - \sqrt{a_1^2 + 2a_1\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]}\right]^2}{2a_1a_2 + 2(a_1 + a_2)\mathbb{E}[b_2] + 2a_2\mathbb{E}[b_1 - b_2] - \frac{1}{2}\mathbb{E}\left[(b_1 - b_2)^2\right]}$$

and the solution of the optimal optimization problem is

$$\pi_1^*(b) = \frac{\sqrt{a_1^2 + 2a_1\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]} + [a_1 + 2a_2 - (b_1 - b_2)]}{2(a_1 + a_2)}.$$

Proof. **Optimal solution**

Case 1 follows directly from theorem 4.

In the second and third case notions about the Lagrangian method and duality are needed again, see A for details.

In case 2, when the first constraint is violated and the second one is satisfied, i.e.

$$\mathbb{E}\left[b_1 - b_2\right] > \frac{\mathbb{E}\left[(b_1 - b_2)^2\right]}{2a_2} \tag{4.15}$$

it is necessary to include this constraint in the lagrangian function as well with a lagragian multiplier λ and then it is necessary to find its optimum again as follows:

$$\begin{aligned}\mathcal{L}\left(\pi_1(b), \lambda\right) = a_2 + b_2 + \left(-2a_2 + b_1 - b_2\right)\pi_1(b) + \left(a_1 + a_2\right)\pi_1(b)^2 \\ +\lambda\left[\left(-a_2 + b_1 - b_2\right)\pi_1(b) + \left(a_1 + a_2\right)\pi_1(b)^2\right]\end{aligned}$$

$$\frac{\partial \mathcal{L}\left(\pi_1(b), \lambda\right)}{\partial \pi_1(b)} = 0 \iff (1+\lambda)\, b_1 - (1+\lambda)\, b_2 - (2+\lambda)\, a_2 + 2\,(1+\lambda)\,(a_1 + a_2)\,\pi_1(b) = 0$$

$$\overline{\pi_1}(b, \lambda) = \left[\frac{(1+\lambda)\, a_2 + a_2 - (1+\lambda)\,(b_1 - b_2)}{2\,(1+\lambda)\,(a_1 + a_2)}\right]_0^1 \tag{4.16}$$

Again to better manipulate this solution the change of variable $\alpha := 2\,(a_1 + a_2)$ is made and the policy therefore becomes:

$$\overline{\pi_1}(b, \lambda) = \left[\alpha\left(\frac{a_2}{(1+\lambda)} + a_2 - (b_1 - b_2)\right)\right]_0^1$$

Let assume that 4.16 does not saturate in $[0,1]^2$.

In order to find the value for the lagrangian multiplier λ it is necessary to solve the dual problem or equivalently solve the first constraint as an equality. This is due to the fact that if the first constraint is violated then this means that it is active and so it is satisfied at equality. It also follows from this that if the first constraint is active then its lagrangian multiplier needs to be strictly different from zero due to the complementary slackness condition. Then the optimal value of the lagrangian multiplier is $\lambda^* > 0$ such that:

$$\int_\Omega \left(-a_2 + b_1 - b_2\right)\overline{\pi_1}(b_1, b_2, \lambda) + \alpha\overline{\pi_1}(b_1, b_2, \lambda)^2 d\mathbb{P}(b) = 0$$

With the policy just found above:

$$\begin{aligned}\int_\Omega \left(-a_2 + (b_1 - b_2)\right)\alpha\left(\frac{a_2}{(1+\lambda)} + a_2 - (b_1 - b_2)\right) + \\ (a_1 + a_2)\left[\alpha\left(\frac{a_2}{(1+\lambda)} + a_2 - (b_1 - b_2)\right)^2\right] d\mathbb{P}(b) = 0\end{aligned}$$

By some computations the result is:

$$(1+\lambda^*)^2 = \frac{a_2^2}{a_2^2 - 2a_2\mathbb{E}\left[b_1-b_2\right] + \mathbb{E}\left[(b_1-b_2)^2\right]}$$

Note that it is possible to demonstrate that λ is well defined because the denominator is greater than $a_2^2 - 2a_2\mathbb{E}\left[b_1-b_2\right] + \mathbb{E}\left[b_1-b_2\right]^2 = \left(a_2^2 - \mathbb{E}\left[b_1-b_2\right]\right)^2$ that is greater than 0. The policy then becomes:

$$\overline{\pi_1}\left(b\right) = \frac{\sqrt{a_2^2 - 2a_2\mathbb{E}\left[b_1-b_2\right] + \mathbb{E}\left[(b_1-b_2)^2\right]} + \left[a_2 - (b_1-b_2)\right]}{2\left(a_1+a_2\right)} \tag{4.17}$$

taking just the positive sqaured root because of the constraint $\lambda^* > 0$. Considering then the boundaries on the policy $\pi\left(b\right) \in [0,1]^2$ it follows that:

$$\overline{\pi_1}\left(b\right) = \left[\frac{\sqrt{a_2^2 - 2a_2\mathbb{E}\left[b_1-b_2\right] + \mathbb{E}\left[(b_1-b_2)^2\right]} + \left[a_2 - (b_1-b_2)\right]}{2\left(a_1+a_2\right)}\right]_0^1$$

$$\overline{\pi_2}\left(b\right) = \left[\frac{\left[2a_1 + a_2 + (b_1-b_2)\right] - \sqrt{a_2^2 - 2a_2\mathbb{E}\left[b_1-b_2\right] + \mathbb{E}\left[(b_1-b_2)^2\right]}}{2\left(a_1+a_2\right)}\right]_0^1$$

The conditions for having no saturation of the policy are:

$$\overline{\pi_1}\left(b\right) \geq 0 \iff \max(supp\{b_1-b_2\}) \leq a_2 + \sqrt{a_2^2 - 2a_2\mathbb{E}\left[b_1-b_2\right] + \mathbb{E}\left[(b_1-b_2)^2\right]}$$

$$\overline{\pi_1}\left(b\right) \leq 1 \iff \min(supp\{b_1-b_2\}) \geq -2a_1 - a_2 + \sqrt{a_2^2 - 2a_2\mathbb{E}\left[b_1-b_2\right] + \mathbb{E}\left[(b_1-b_2)^2\right]}$$

Then this new policy does not saturate exactly in the support:

$$supp(\omega_1 - \omega_2) \subseteq \left[-2a_1 - a_2 + \sqrt{k}, a_2 + \sqrt{k}\right] \tag{4.18}$$

where $k = a_2^2 - 2a_2\mathbb{E}\left[b_1-b_2\right] + \mathbb{E}\left[(b_1-b_2)^2\right]$.

Now it is sure that the first constraint is satisfied by construction. Now we have to make sure that the second constraint is also satisfied. The second constraint is satisfied if:

$$\int_\Omega \left[a_2 + (b_2-b_1) + (-a_1 - 2a_2 + b_1 - b_2)\,\overline{\pi_1}\left(b\right) + (a_1+a_2)\,\overline{\pi_1}\left(b\right)^2\right] d\mathbb{P}\left(b\right) \leq 0$$

with the policy just found including the first constraint:

$$\overline{\pi_1}(b) = \frac{\sqrt{a_2^2 - 2a_2\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]} + [a_2 - (b_1 - b_2)]}{2(a_1 + a_2)}$$

By computation:

$$\mathbb{E}[b_1 - b_2] \geq a_2 - \sqrt{a_2^2 - 2a_2\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]}$$

that is:

$$\mathbb{E}\left[(b_1 - b_2)^2\right] - \mathbb{E}[b_1 - b_2]^2 \geq 0$$

the first member of this inequality is the variance of $b_1 - b_2$ that is always positive independently from the distribution of $b_1 - b_2$ and so the second constraint is still always respected.

Therefore it is possibile to conclude that since $\overline{\pi}(b)$ satisfies both the first and the second constraint of 4.11 and it is a minimum of the Lagrangian function and since the problem is convex, then it is the optimal solution when the condition on the support 4.18 holds.

Going through the case 3, when the second constraint is violated and the first one is satisfied, i.e.

$$\mathbb{E}[b_1 - b_2] < \frac{-\mathbb{E}\left[(b_1 - b_2)^2\right]}{2a_1} \tag{4.19}$$

it is necessary to include that constraint in the lagrangian function as well with a lagragian multiplier λ and then it is necessary to find its optimum again as follows:

$$\mathcal{L}(\pi_1(b), \lambda) = a_2 + b_2 + (-2a_2 + b_1 - b_2)\pi_1(b) + (-a_1 + a_2)\pi_1(b)^2$$
$$+\lambda\left[a_2 + (b_2 - b_1) + (-a_1 - 2a_2 + b_1 - b_2)\pi_1(b) + (a_1 + a_2)\pi_1(b)^2\right]$$

$$\frac{\partial\mathcal{L}(\pi_1(b), \lambda)}{\partial\pi_1(b)} = 0 \iff$$
$$(1+\lambda)b_1 - (1+\lambda)b_2 + \lambda a_1 - 2(1+\lambda)a_2 + 2(1+\lambda)(a_1 + a_2)\pi_1(b) = 0$$

$$\overline{\pi_1}(b, \lambda) = \left[\frac{\lambda a_1 + 2(1+\lambda)a_2 - (1+\lambda)(b_1 - b_2)}{2(1+\lambda)(a_1 + a_2)}\right]_0^1. \tag{4.20}$$

Again to better manipulate this solution the changes of variable $\alpha := 2(a_1 + a_2)$ is made and the policy therefore becomes:

$$\overline{\pi_1}(b, \lambda) = \left[\alpha\left(\frac{\lambda a_1}{(1+\lambda)} + 2a_2 - (b_1 - b_2)\right)\right]_0^1$$

Let assume that 4.20 does not saturate in $[0,1]^2$.

In order to find the value for the lagrangian mutliplier λ it is necessary to solve the dual problem or equivalently solve the second constraint as a equality. This is due to the fact that if the second constraint is violated then this means that it is active and so it is satisfied at equality. It also follows from this that if the second constraint is active then its lagrangian multiplier needs to be strictly different from zero due to the complementary slackness condition. Then the optimal value of the lagrangian multiplier is $\lambda^* > 0$ such that:

$$\int_\Omega a_2 - x + (a_1 - 2a_2 + (b_1 - b_2))\, \overline{\pi_1}(b_1, b_2, \lambda) + (a_1 + a_2)\, \overline{\pi_1}(b_1, b_2, \lambda)^2 d\mathbb{P}(b) = 0$$

With the policy just found above:

$$\int_\Omega a_2 - (b_1 - b_2) + (a_1 - 2a_2 + (b_1 - b_2))\, \alpha\left(\frac{\lambda a_1}{(1+\lambda)} + 2a_2 - b_1, b_2\right) + \\ (a_1 + a_2)\left[(\alpha\left(\frac{\lambda a_1}{(1+\lambda)} + 2a_2 - (b_1 - b_2)\right)\right]^2 d\mathbb{P}(b) = 0$$

By some computations the result is:

$$(1 + \lambda^*)^2 = \frac{a_1^2}{a_1^2 + 2a_1\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]} \tag{4.21}$$

Note that it is possible to demonstrate that λ is well defined because the denominator is greater than $a_2^2 - 2a_2\mathbb{E}[b_1 - b_2] + \mathbb{E}[b_1 - b_2]^2 = (a_2^2 - \mathbb{E}[b_1 - b_2])^2$ that is greater than 0. The policy becomes:

$$\overline{\pi_1}(b) = \frac{\sqrt{a_1^2 + 2a_1\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]} + [a_1 + 2a_2 - (b_1 - b_2)]}{2(a_1 + a_2)} \tag{4.22}$$

taking just the positive sqaured root because of the constraint $\lambda^* > 0$. Considering then the boundaries on the policy $\pi(b, \lambda) \in [0,1]^2$ it follows that:

$$\overline{\pi_1}(b) = \left[\frac{\sqrt{a_1^2 + 2a_1\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]} + [a_1 + 2a_2 - (b_1 - b_2)]}{2(a_1 + a_2)}\right]_0^1$$

$$\overline{\pi_2}(b) = \left[\frac{[a_1 + (b_1 - b_2)] - \sqrt{a_1^2 + 2a_1\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]}}{2(a_1 + a_2)}\right]_0^1$$

The condition for having no saturation of the policy are:

$$\overline{\pi_1}(b) \geq 0 \iff \max(supp\{\omega_1 - \omega_2\}) \leq a_1 + 2a_2 - \sqrt{a_1^2 + 2a_1\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]}$$
$$\overline{\pi_1}(b) \leq 1 \iff \min(supp\{\omega_1 - \omega_2\}) \geq -a_1 - \sqrt{a_1^2 + 2a_1\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]}$$

Then this new policy does not saturate exactly in the support:

$$supp(\omega_1 - \omega_2) \subseteq \left[-a_1 - \sqrt{k}, a_1 + 2a_2 - \sqrt{k}\right] \tag{4.23}$$

where $k = a_1^2 + 2a_1\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]$.

Now it is sure that the second constraint is satisfied by construction. Now we have to make sure that the first constraint is also satisfied. The first constraint is satisfied if:

$$\int_\Omega \left[(-a_2 + b_1 - b_2)\overline{\pi_1}(b) + (a_1 + a_2)\overline{\pi_1}(b)^2\right] d\mathbb{P}(b) \leq 0$$

with the policy just found including the second constraint:

$$\overline{\pi_1}(b) = \frac{\sqrt{a_1^2 + 2a_1\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]} + [a_1 + 2a_2 - (b_1 - b_2)]}{2(a_1 + a_2)}$$

By computation:

$$\mathbb{E}[b_1 - b_2] \leq a_1 + \sqrt{a_1^2 + 2a_1\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]}$$

that is:

$$\mathbb{E}\left[(b_1 - b_2)^2\right] - \mathbb{E}[b_1 - b_2]^2 \geq 0$$

the first member of this inequality is the variance of $b_1 - b_2$ that is always positive independently from the distribution of $b_1 - b_2$ and so the first constraint is still always respected.

Therefore it is possibile to conclude that since $\overline{\pi}(b)$ satisfies both the first and the second constraint of 4.11 and it is a minimum of the Lagrangian function and since the problem is convex, then it is the optimal solution when the condition on the support 4.23 holds.

Value of the price of anarchy

The system cost is the value of the objective function calculated with an optimal value, i.e.:

$$\int_{\Omega} \left[a_2 + b_2 + (-2a_2 + b_1 - b_2)\,\pi_1^*(b) + (a_1 + a_2)\,\pi_1^*(b)^2\right] d\mathbb{P}(b)$$

- using the social optimum, i.e,:

$$\pi_1^*(b) = \frac{2a_2 - (b_1 - b_2)}{2(a_1 + a_2)}$$

  the delay becomes:

$$C_{SO}^* = \alpha\{2a_1a_2 + 2(a_1 + a_2)\,\mathbb{E}[b_2] + 2a_2\mathbb{E}[b_1 - b_2] - \frac{1}{2}\mathbb{E}\left[(b_1 - b_2)^2\right]\}$$

- instead if the first constraint is violated, with the derived policy, i.e.:

$$\overline{\pi_1}(b) = \frac{\sqrt{a_2^2 - 2a_2\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]} + [a_2 - (b_1 - b_2)]}{2(a_1 + a_2)}$$

  the cost becomes:

$$\begin{aligned} C_{UO_1}^* &= \alpha\{2a_1a_2 + 2a_2^2 + 2(a_1 + a_2)\,\mathbb{E}[b_2] \\ &- a_2\sqrt{a_1^2 + 2a_1\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]} + a_2\mathbb{E}[b_1 - b_2]\} \end{aligned}$$

- in the end if the second constraint is violated, with the derived policy, i.e.:

$$\overline{\pi_1}(b) = \frac{-\sqrt{a_1^2 + 2a_1\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]} + [a_1 + 2a_2 - (b_1 - b_2)]}{2(a_1 + a_2)}$$

  the cost becomes:

$$\begin{aligned} C_{UO_2}^* &= \alpha\{2a_1a_2 + 2a_1^2 + 2(a_1 + a_2)\,\mathbb{E}[b_2] \\ &- a_1\sqrt{a_1^2 + 2a_1\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]} + a_2\mathbb{E}[b_1 - b_2]\} \end{aligned}$$

In conclusion, the price of anarchy in case of first or second constraints violation are:

$$PoA_1 = \frac{C_{UO_1}}{C_{SO}}$$

$$= \frac{2a_1a_2 + 2a_2^2 + 2(a_1 + a_2)\,\mathbb{E}[b_2] - a_2\sqrt{a_1^2 + 2a_1\mathbb{E}[b_1 - b_2] + \mathbb{E}\left[(b_1 - b_2)^2\right]} + a_2\mathbb{E}[b_1 - b_2]}{2a_1a_2 + 2(a_1 + a_2)\,\mathbb{E}[b_2] + 2a_2\mathbb{E}[b_1 - b_2] - \frac{1}{2}\mathbb{E}\left[(b_1 - b_2)^2\right]}$$

$$\Rightarrow PoA_1 = 1 + \frac{1}{2}\frac{\left[a_2 - \sqrt{a_2^2 - 2a_2\mathbb{E}\left[b_1 - b_2\right] + \mathbb{E}\left[\left(b_1 - b_2\right)^2\right]}\right]^2}{2a_1a_2 + 2\left(a_1 + a_2\right)\mathbb{E}\left[b_2\right] + 2a_2\mathbb{E}\left[b_1 - b_2\right] - \frac{1}{2}\mathbb{E}\left[\left(b_1 - b_2\right)^2\right]} \tag{4.24}$$

and:

$$PoA_2 = \frac{C_{UO_2}}{C_{SO}}$$

$$= \frac{2a_1a_2 + 2a_1^2 + 2\left(a_1 + a_2\right)\mathbb{E}\left[b_2\right] - a_1\sqrt{a_1^2 + 2a_1\mathbb{E}\left[b_1 - b_2\right] + \mathbb{E}\left[\left(b_1 - b_2\right)^2\right]} + a_2\mathbb{E}\left[b_1 - b_2\right]}{2a_1a_2 + 2\left(a_1 + a_2\right)\mathbb{E}\left[b_2\right] + 2a_2\mathbb{E}\left[b_1 - b_2\right] - \frac{1}{2}\mathbb{E}\left[\left(b_1 - b_2\right)^2\right]}$$

$$\Rightarrow PoA_1 = 1 + \frac{1}{2}\frac{\left[a_1 - \sqrt{a_1^2 + 2a_1\mathbb{E}\left[b_1 - b_2\right] + \mathbb{E}\left[\left(b_1 - b_2\right)^2\right]}\right]^2}{2a_1a_2 + 2\left(a_1 + a_2\right)\mathbb{E}\left[b_2\right] + 2a_2\mathbb{E}\left[b_1 - b_2\right] - \frac{1}{2}\mathbb{E}\left[\left(b_1 - b_2\right)^2\right]} \tag{4.25}$$

□

Observation 9. *In conclusion, observe that in the binary network with affine delay functions and generic prior (with deterministic linear coefficients) it's not possibile to violate both the constraints together if the condition on the support 4.12 holds.*

4.3 Analysis with uniform prior

So far, all our theoretical results rely on an assumption on the support of random vectors a, b that guarantees that under the optimal policy there exists a non-zero flow on both the roads. This assumptions indeed allow to state sufficient and necessary conditions in terms of moments of the random variables that characterize whether the obedience constraints are active or inactive. In the next section we shall consider a case-study that allows explicit computation even relaxing the assumption on the support.

In this section we consider the same setting of Section 4.2, with the additional assumption that the prior of b is uniform in $[0,1]^2$, i.e. we assume

$$dP(b_1, b_2) = \begin{cases} 1 & \text{if } b_1, b_2 \in [0,1]^2 \\ 0 & \text{otherwise.} \end{cases} \tag{4.26}$$

About a, we only assume that those coefficients are non-negative.

The optimization problem of information design 2 on a network with 2 links and affine stochastic delay functions and uniform prior becomes:

$$\begin{aligned} \min_{\pi_1(b)} & \int_{[0,1]^2} \left[a_2 + b_2 + (-2a_2 + b_1 - b_2)\,\pi_1(b) + (a_1 + a_2)\,\pi_1(b)^2\right] db \\ \text{s.t.} & \int_{[0,1]^2} \left[(-a_2 + b_1 - b_2)\,\pi_1(b) + (a_1 + a_2)\,\pi_1(b)^2\right] db \leq 0 \\ & \int_{[0,1]^2} \left[a_2 + (b_2 - b_1) + (-a_1 - 2a_2 + b_1 - b_2)\,\pi_1(b) + (a_1 + a_2)\,\pi_1(b)^2\right] db \leq 0 \end{aligned} \tag{4.27}$$

Theorem 6. *Consider a network with two parallel links and stochastic affine delay functions and uniform prior distribution as in 4.26 with deterministic linear coefficients. Then, PoA=1 for every $a \in \mathbb{R}_+^2$ and*

$$\pi_1^*(b) = \left[\frac{2a_2 - b_1 + b_2}{2(a_1 + a_2)}\right]_0^1. \tag{4.28}$$

Proof. To better manipulate this solution and be able to use it more easily in subsequent calculations, a change of variable is made. Consider the two random variables b_1 and b_2 together and give the name of x to their difference, i.e. $x := b_1 - b_2$. Also consider their sum that is the variable $y := b_1 + b_2$. Moreover call $\alpha := \frac{1}{2(a_1+a_2)}$. This change of variable produces a jacobian that has determinant equal to $\frac{1}{2}$.

Observation 10. *The change of variable means that the integration no longer takes place on the horizontal and vertical axes of the graphic formed by the values of b_1 and b_2 between 0 and 1, but instead integrates on the diagonal lines $x = b_1 - b_2$. If $x > 0$ the integration line is located in the upper triangle of the graphic and grows until it reaches a saturation zone in which $\pi_1(b)$ is zero, i.e. $b_1 >> b_2$ therefore it is not convenient for a user to choose route 1. Vice versa if $x > 0$ the integration line is located in the lower half of the graphic and descends until it reaches a saturation zone in which $\pi_1(b)$ is equal to 1, in fact $b_1 << b_2$ therefore it is always convenient for a user to choose route 1.*

Consider now a general form of the policy, solution of the optimization problem, and apply the change of variable shown above:

$$\overline{\pi_1}(b) = \beta - \alpha(b_1 - b_2) \Rightarrow \overline{\pi_1}(x) = \beta - \alpha x$$

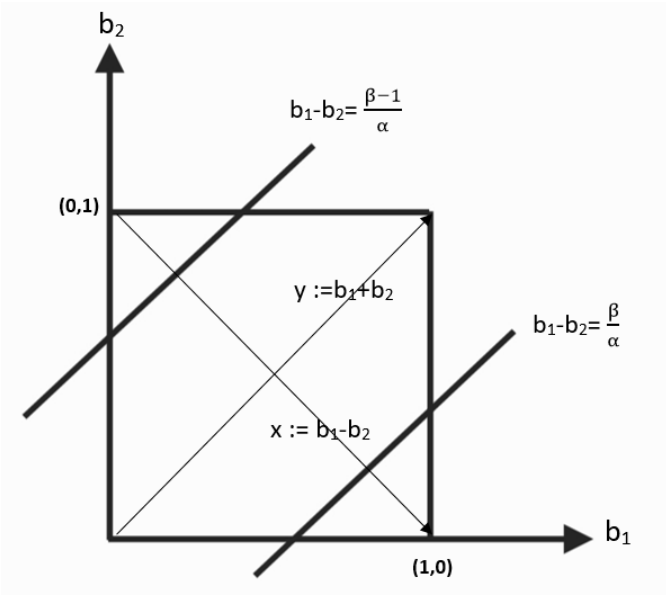

Figure 4.2. Representation of the space of the variables b_1 and b_2.

with the property that $\beta = k\alpha$ that comes from the structure of the policy found before. This general policy saturates between 0 and 1 and so it happens that the optimal solution in b_1 and b_2 is:

$$\overline{\pi_1}\,(b) = \begin{cases} 0 & b_1 - b_2 \geq \frac{\beta}{\alpha} \\ \beta - \alpha\,(b_1 - b_2) & \frac{\beta-1}{\alpha} < b_1 - b_2 < \frac{\beta}{\alpha} \\ 0 & b_1 - b_2 \leq \frac{\beta-1}{\alpha} \end{cases} \tag{4.29}$$

therefore in the new variable x it is:

$$\overline{\pi_1}\,(x) = \begin{cases} 0 & x \geq \frac{\beta}{\alpha} \\ \beta - \alpha x & \frac{\beta-1}{\alpha} < x < \frac{\beta}{\alpha} \\ 0 & x \leq \frac{\beta-1}{\alpha} \end{cases} \tag{4.30}$$

We can split the integration in three areas as shown in figure 4.2 , the first one is the upper triangle T1 where $\overline{\pi_1}$ saturates at 1, the lower triangle T2 where $\overline{\pi_1}$ saturates at 0 and the central area R where it has the explicit form $\overline{\pi_1}\,(x) = \beta - \alpha x$. These three parts are divided by the two straight lines $b_1 - b_2 = \frac{\beta}{\alpha}$ and $b_1 - b_2 = \frac{\beta-1}{\alpha}$. We assume $a_1 \leq a_2$ without loss of generality.

For every function $f(x)$, the integral can be written as:

$$\int_{T_1} f(x)dxdy + \int_R f(x)dxdy + \int_{T_2} f(x)dxdy.$$

It can be shown that the integration on the upper triangle T1 of a function that depends on $b_1 - b_2$ using the change of variable $x := b_1 - b_2$ and $y := b_1 + b_2$ is:

$$\int_{T1} f\left(b_1 - b_2\right) db_1 db_2 = \frac{1}{2}\int_{-1}^{\frac{\beta-1}{\alpha}} \int_{-x}^{x+2} f(x)dydx$$

Also it can be shown that the integration on the upper triangle T2 of a function that depends on $b_1 - b_2$ using the change of variable $x := b_1 - b_2$ and $y := b_1 + b_2$ is:

$$\int_{T2} f\left(b_1 - b_2\right) db_1 db_2 = \frac{1}{2}\int_{\frac{\beta}{\alpha}}^{1} \int_{x}^{-x+2} f(x)dydx$$

Consequently, it can be shown that the integration on the central area R of a function that depends on $b_1 - b_2$ using the change of variable $x := b_1 - b_2$ and $y := b_1 + b_2$ is symmetric and so it is possibile to integrate only on the first half and then multiply everything by two:

$$\int_{R} f\left(b_1 - b_2\right) db_1 db_2 = \int_{0}^{\frac{1-\beta}{\alpha}} \int_{-y}^{+y} f(x)dxdy + \int_{\frac{1-\beta}{\alpha}}^{\frac{\beta}{\alpha}} \int_{\frac{\beta-1}{\alpha}}^{+y} f(x)dxdy + \int_{\frac{\beta}{\alpha}}^{1} \int_{\frac{\beta-1}{\alpha}}^{\frac{\beta}{\alpha}} f(x)dxdy$$

In the end the integral of a function $f(x)$ is:

$$\begin{aligned}\int_{[0,1]^2} f\left(b_1 - b_2\right) db_1 db_2 =& \frac{1}{2}\int_{-1}^{\frac{\beta-1}{\alpha}} \int_{-x}^{x+2} f(x)dydx + \frac{1}{2}\int_{\frac{\beta}{\alpha}}^{1} \int_{x}^{x+2} f(x)dydx + \\ & \int_{0}^{\frac{1-\beta}{\alpha}} \int_{-y}^{+y} f(x)dxdy + \int_{\frac{1-\beta}{\alpha}}^{\frac{\beta}{\alpha}} \int_{\frac{\beta-1}{\alpha}}^{+y} f(x)dxdy + \int_{\frac{\beta}{\alpha}}^{1} \int_{\frac{\beta-1}{\alpha}}^{\frac{\beta}{\alpha}} f(x)dxdy\end{aligned}$$

The first constraint is the integral of the following function:

$$g\left(b_1 - b_2\right) = \left(-a_2 + b_1 - b_2\right) \pi_1\left(b\right) + \left(a_1 + a_2\right) \pi_1\left(b\right)^2$$

and the second constraint is the integral of the following function:

$$h\left(b_1 - b_2\right) = \left(a_2 - b_1 + b_2\right) + \left(-a_1 - 2a_2 + +b_1 - b_2\right) \pi_1\left(b\right) + \left(a_1 + a_2\right) \pi_1\left(b\right)^2$$

considering the generic policy 4.30 found above and its constraint of saturation between 0 and 1, i.e.: Then, the first constraint function $g(x)$, divided in the three areas of the graphic is:

$$\begin{aligned}g_R\left(x\right) &= \left(-a_2 + x\right)\left[\beta - \alpha x\right] + \frac{1}{2\alpha}\left[\beta - \alpha x\right]^2 \\ &= -a_2\beta + \beta^2 2\alpha + a_2\alpha x - \frac{1}{2}\alpha x^2 \\ g_{T1}\left(x\right) &= a_1 + x \\ g_{T2}\left(x\right) &= 0\end{aligned}$$

Right away, the second constraint function $h(x)$, divided in the three areas of the graphic is:

$$\begin{aligned} h_R(x) &= (a_2 - x) + (-a_1 - 2a_2 + x)[\beta - \alpha x] + \frac{1}{2\alpha}[\beta - \alpha x]^2 \\ &= a_2 - (a_1 + 2a_2)\beta + \beta^2 2\alpha + (a_1 + 2a_2)\alpha x - \frac{1}{2}\alpha x^2 \\ h_{T1}(x) &= 0 \\ h_{T2}(x) &= a_2 - x \end{aligned}$$

Three different cases need to be studied:

1. when there's both the upper and the lower saturation, that is when $\frac{\beta-1}{\alpha} > -1$ and $\frac{\beta}{\alpha} < 1$;
2. when there's only the upper saturation, that is when $\frac{\beta-1}{\alpha} > -1$ and $\frac{\beta}{\alpha} \geq 1$;
3. when there's no saturation, that is when $\frac{\beta-1}{\alpha} \leq -1$ and $\frac{\beta}{\alpha} \geq 1$;

In the first case of both upper and lower saturation it turns out that the first constraint is violated when:

$$\begin{aligned} &\frac{1}{2}\int_{-1}^{\frac{\beta-1}{\alpha}}\int_{-x}^{x+2} a_1 + x dy dx + \\ &\int_{0}^{\frac{1-\beta}{\alpha}}\int_{-y}^{+y} -a_2\beta + \frac{\beta^2}{2\alpha} + a_2\alpha x - \frac{1}{2}\alpha x^2 dx dy + \\ &\int_{\frac{1-\beta}{\alpha}}^{\frac{\beta}{\alpha}}\int_{\frac{\beta-1}{\alpha}}^{+y} -a_2\beta + \frac{\beta^2}{2\alpha} + a_2\alpha x - \frac{1}{2}\alpha x^2 dx dy + \\ &\int_{\frac{\beta}{\alpha}}^{1}\int_{\frac{\beta-1}{\alpha}}^{\frac{\beta}{\alpha}} -a_2\beta + \frac{\beta^2}{2\alpha} + a_2\alpha x - \frac{1}{2}\alpha x^2 dx dy + \\ &\frac{1}{2}\int_{\frac{\beta}{\alpha}}^{1}\int_{x}^{-x+2} 0 dy dx > 0, \end{aligned}$$

using the property for which $\beta = k\alpha$ the inequality becomes:

$$\begin{aligned} &\frac{1}{6}(-1 + \alpha + \alpha k)^2[-2 + \alpha(-1 + 2k + 3a_1)] + \\ &\frac{1}{12}(\alpha k - 1)^2\left[\alpha^2 k(5k - 12a_2) + 2\alpha k - 1\right] - \\ &\frac{1}{24}(2\alpha k - 1)\left[8\alpha^3 k^2(k - 2a_2) - 4\alpha^2 k(k - 4a_2) + \alpha(8a_2 - 14k) + 5\right] - \\ &\frac{1}{6}\alpha(k - 1)[3\alpha(k - a_2) - 1] > 0. \end{aligned}$$

With the policy unconstrained 4.28 $k = 2a_2$ and so the condition for having both upper and lower saturations is $a_1 < \frac{1}{2}$ and $a_2 < \frac{1}{2}$. Moreover, using this relationship the first constraint is violated when:

$$2a_1^4 - 2a_2^4 - 4a_1^3 + 2a_2^3 + 4a_1^3 a_2 + 3a_1^2 - 6a_1^2 a_2 - a_1 - a_2 + 3a_1 a_2 > 0. \quad (4.31)$$

It is possible to show that 4.31 is never verified, i.e. the constraint is satisfied as illustrated in figure 4.3.

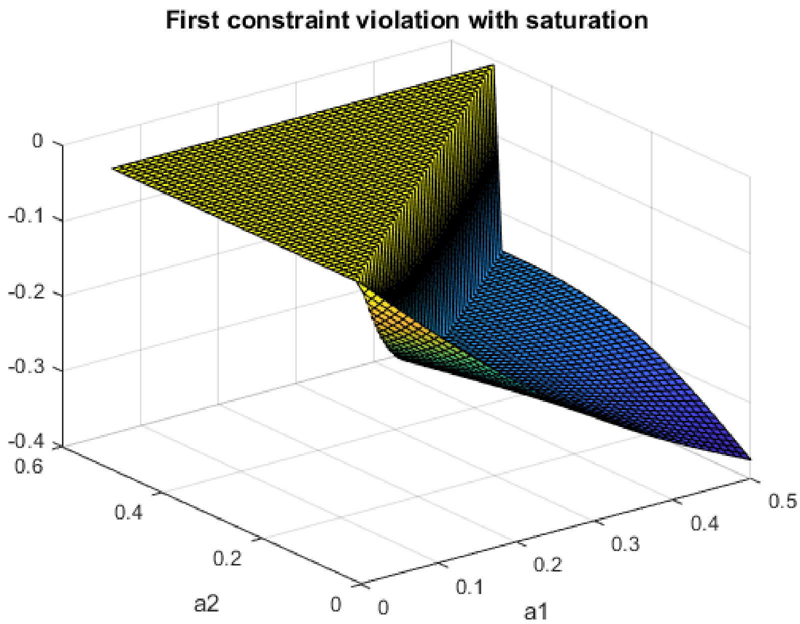

Figure 4.3. Plot of 4.31 in the domain $[0,1]^2$.

Always in the case of both upper and lower saturations, the second constraint is violated when:

$$\frac{1}{2}\int_{-1}^{\frac{\beta-1}{\alpha}}\int_{-x}^{x+2} 0dydx+$$
$$\int_{0}^{\frac{1-\beta}{\alpha}}\int_{-y}^{+y} a_2-(a_1+2a_2)\,\beta+\frac{\beta^2}{2\alpha}+(a_1+2a_2)\,\alpha x-\frac{1}{2}\alpha x^2dxdy+$$
$$\int_{\frac{1-\beta}{\alpha}}^{\frac{\beta}{\alpha}}\int_{\frac{\beta-1}{\alpha}}^{+y} a_2-(a_1+2a_2)\,\beta+\frac{\beta^2}{2\alpha}+(a_1+2a_2)\,\alpha x-\frac{1}{2}\alpha x^2dxdy+$$
$$\int_{\frac{\beta}{\alpha}}^{1}\int_{\frac{\beta-1}{\alpha}}^{\frac{\beta}{\alpha}} a_2-(a_1+2a_2)\,\beta+\frac{\beta^2}{2\alpha}+(a_1+2a_2)\,\alpha x-\frac{1}{2}\alpha x^2dxdy+$$
$$\frac{1}{2}\int_{\frac{\beta}{\alpha}}^{1}\int_{x}^{-x+2} a_2-xdydx>0,$$

using the property for which $\beta=k\alpha$ the inequality becomes:

$$\frac{1}{12}(\alpha k-1)^2\left\{\alpha^2k\left[5k-12\,(a_1+2a_2)\right]+2\alpha\,(k+6a_2)-1\right\}+$$
$$\frac{1}{12}(2\alpha k-1)\left\{\alpha^3\left[-4k^3+4k^2\,(a_1+2a_2)\right]+\alpha^2\left[k^2-4k\,(a_1+2a_2)\right]+\alpha\left[6k-8a_1-4a_2\right]-3\right\}-$$
$$\frac{1}{6}\alpha\,(k-1)\left[3\alpha\,(k+a_1)-2\right]+$$
$$\frac{1}{6}\alpha^3\,(k-1)^2\left[2k-3a_2+1\right]>0.$$

With the unconstrained policy 4.28 $k=2a_2$ and so the condition for having both upper and lower saturations is $a_1<\frac{1}{2}$ and $a_2<\frac{1}{2}$. Moreover, using this relationship the second constraint is violated when:

$$-2a_1^4+2a_2^4+2a_1^3-4a_2^3+4a_1a_2^3+3a_2^2-6a_1a_2^2-a_1-a_2+3a_1a_2>0. \tag{4.32}$$

It is possible to show that 4.32 is never verified, i.e. the constraint is satisfied as illustrated in figure 4.4.

In the second case of only upper saturation it turns out that the first constraint is violated when:

$$\frac{1}{2}\int_{-1}^{\frac{\beta-1}{\alpha}}\int_{-x}^{x+2} a_1+xdydx+$$
$$\int_{0}^{\frac{1-\beta}{\alpha}}\int_{-y}^{+y} -a_2\beta+\frac{\beta^2}{2\alpha}+a_2\alpha x-\frac{1}{2}\alpha x^2dxdy+$$
$$\int_{\frac{1-\beta}{\alpha}}^{1}\int_{\frac{\beta-1}{\alpha}}^{+y} -a_2\beta+\frac{\beta^2}{2\alpha}+a_2\alpha x-\frac{1}{2}\alpha x^2dxdy>0$$

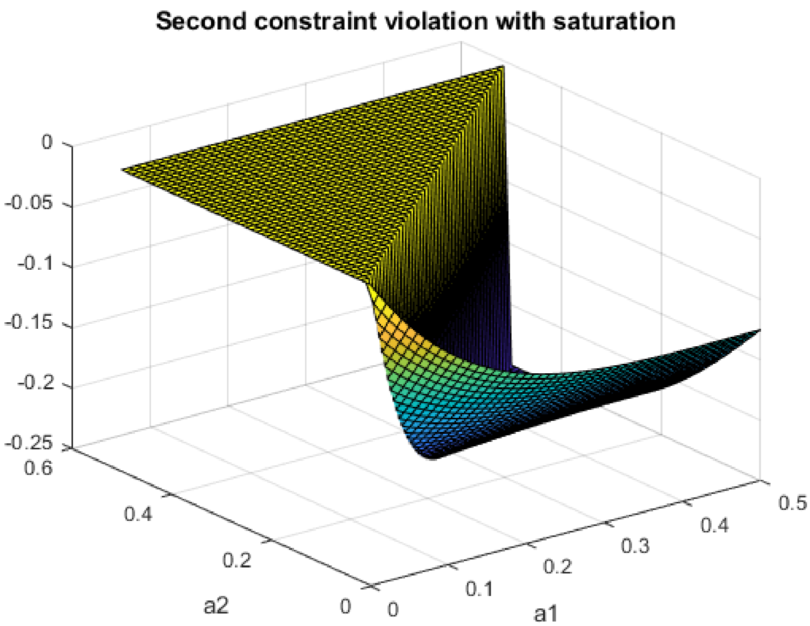

Figure 4.4. Plot of 4.32 in the domain $[0,1]^2$.

usign the property for which $\beta = k\alpha$ the inequality becomes:

$$\frac{1}{6}(-1+\alpha+\alpha k)^2[-2+\alpha(-1+2k+3a_1)]+$$
$$\frac{1}{12}(\alpha k-1)^2\left[\alpha^2 k(5k-12a_2)+2\alpha k-1\right]-$$
$$\frac{1}{24}(\alpha k+\alpha-1)\{\alpha^3\left[13k^3-k^2(28a_2+5)+k(16a_2-1)-4a_2+1\right]+$$
$$\alpha^2\left[-3k^2+k(20a_2-2)-4a_2+1\right]+\alpha[-15k+8a_2+1]+5\}>0$$

Again with the unconstrained policy 4.28 $k = 2a_2$ and so the condition for having only upper saturation is that $a_1 < \frac{1}{2}$ and $a_2 \geq \frac{1}{2}$. Moreover using this relationship the first constraint is violated when:

$$4a_1^4 - 8a_1^3 + 8a_1^3 a_2 + 6a_1^2 - 12a_1^2 a_2 - 2a_1 - a_2 + 6a_1 a_2 - \frac{1}{4} > 0 \quad (4.33)$$

This inequality is solved analitically and it is possibile to prove that it is equivalent to the following relationship:

$$a_2 < \frac{-4a_1^4 + 8a_1^3 - 6a_1^2 + 2a_1 + \frac{1}{4}}{(2a_1 - 1)^3} \quad (4.34)$$

but since $a_1 < \frac{1}{2}$ than the second member of the inequality is always negative and because $a_2 \geq \frac{1}{2}$ then also in this case the first constraint is never violated.

In the second case of only upper saturation it turns out that the second constraint is violated when:

$$\int_0^{\frac{1-\beta}{\alpha}} \int_{-y}^{+y} a_2 - (a_1 + 2a_2)\,\beta + \frac{\beta^2}{2\alpha} + (a_1 + 2a_2)\,\alpha x - \frac{1}{2}\alpha x^2 dxdy +$$
$$\int_{\frac{1-\beta}{\alpha}}^{1} \int_{\frac{\beta-1}{\alpha}}^{+y} a_2 - (a_1 + 2a_2)\,\beta + \frac{\beta^2}{2\alpha} + (a_1 + 2a_2)\,\alpha x - \frac{1}{2}\alpha x^2 dxdy > 0$$

usign the property for which $\beta = k\alpha$ the inequality becomes:

$$\frac{1}{12}(\alpha k - 1)^2 \left\{\alpha^2 k \left[5k - 12\,(a_1 + 2a_2)\right] + 2\alpha\,(k + 6a_2) - 1\right\} -$$
$$\frac{1}{24}(\alpha k + \alpha - 1)\left\{\alpha^3 \left[-3k^3 + k^2(8a_1 + 24a_2 + 17) + k\,(20\,(a_1 + 2a_2) + 5) + 16\,(a_1 + 2a_2) + 1\right] +\right.$$
$$\left.\alpha^2 \left[k^2 + k\,(8a_1 - 100a_2 - 13) - 20a_1 - 52a_2 - 6\right] + \alpha\left[3k - 16a_1 + 156a_2 - 2\right] - 5\right\} > 0$$

Again with the policy unconstrained 4.28 $k = 2a_2$ and so the condition for having only upper saturation is that $a_1 < \frac{1}{2}$ and $a_2 \geq \frac{1}{2}$. Moreover using this relationship the second constraint is violated when:

$$-4a_1^4 + 4a_1^3 - a_1 - \frac{1}{4} > 0 \tag{4.35}$$

This inequality is solved analitically and it possibile to prove that since $a_1 < \frac{1}{2}$ then the inequality is always negative and also in this case the second constraint is never violated.

In case of no saturation it turns out that the first constraint is violated when:

$$\int_0^1 \int_{-y}^{+y} -a_2\beta + \frac{\beta^2}{2\alpha} + a_2\alpha x - \frac{1}{2}\alpha x^2 dxdy > 0$$

usign the property for which $\beta = k\alpha$ the inequality becomes:

$$\left(k^2 - 2a_2 k - \frac{1}{6}\right)\alpha > 0$$

With the policy unconstrained 4.28 $k = 2a_2$ and so the condition for having only upper saturation is that $a_1 \geq \frac{1}{2}$ and $a_2 \geq \frac{1}{2}$. Moreover, using this relationship the first constraint is violated when:

$$\frac{-\frac{1}{6}}{2\,(a_1 + a_2)} > 0 \tag{4.36}$$

This inequality is clearly always negative than also in this case the first constraint is never violated.

In case of no saturation it turns out that the second constraint is violated when:

$$\int_0^1 \int_{-y}^{+y} a_2 - (a_1 + 2a_2)\,\beta + \frac{\beta^2}{2\alpha} + (a_1 + 2a_2)\,\alpha x - \frac{1}{2}\alpha x^2 dxdy > 0$$

usign the property for which $\beta = k\alpha$ the inequality becomes:

$$\left[k^2 - 2\,(a_1 + 2a_2)\,k - \frac{1}{6}\right]\alpha + 2a_2\alpha > 0$$

With the policy unconstrained 4.28 $k = 2a_2$ and so the condition for having only upper saturation is that $a_1 \geq \frac{1}{2}$ and $a_2 \geq \frac{1}{2}$. Moreover, using this relationship the first constraint is violated when:

$$\frac{-\frac{1}{6}}{2\,(a_1 + a_2)} > 0 \tag{4.37}$$

This inequality is clearly always negative than also in this case the second constraint is never violated.

In conclusion in the third case, when there is no saturation the condition of constraints satisfaction is always true and so this is the case of theorem 3, that says that with these assumptions the optimum is the system optimum $\pi^*(b)$ and the price of anarchy is always 1.

□

Observation 11. *To sum up when the model is a network with two parallel links and affine delay functions and prior of the stochastic variable b with uniform distribution then it has been shown that it is possibile to find the unconstrained optimum for each value of b_1 and b_2 and it coincide with the social optimum $\pi^*(b)$. Then it has been proved that even if the policy saturates above and below, or it has only upper saturation or in the end it has no saturations then constraints are never violated. This means that in this particular case users optima always coincide with social optimum and the price of anarchy is always 1.*

Chapter 5

Conclusions

This thesis analyses Information design in Bayesian routing games. The problem investigates how a central planner that knows the stochastic state of the network should provide information to the users in order to influence their routing behavior, possibly leading it towards a system optimum and respecting obedience constraints. The results are focused on private signaling, with a common knowledge on the prior distribution of the state of the network and on the information policy.

We show that the problem is convex if the network is with two parallel links and the delay functions are affine. We find sufficient conditions on the support of the state of the world parameters of the delay functions and necessary and sufficient conditions on their moments under which the price of anarchy is minimized. We observe that a large variance of the state of the world is beneficial for the system cost under the optimal policy. We study the structure of the optimal policy when the price of anarchy is strictly larger than 1, that is when the obedience constraints are not satisfied under the optimal policy that minimizes the objective function without considering the constraints. Moreover, we show that it is not possible under the optimal policy that all the users that receive different recommendations are incentivized in deviating at the same time.

5.1 Future Researches

Future researches related to the work carried out in this thesis are naturally efforts to generalize the results obtained and attempts to expand as much as possible the problems that can be studied and solved.

In particular it would be interesting to go into details of the non convex optimization problem, i.e. the problem in which costraints functions are not convex while instead the objective function can still be convex (since it is convex with the weak hypotesis of convex cost functions). This setting should contain:

- finding the uncostrained optimum solving for each value of the random variables the convex objective function;
- searching for conditions in which constraints are not violated and so for which the price of anarchy is still equal to 1.

An other direction could be studying what happens when there is a fraction of users of the network that do no receive suggestions from the planner. The behavior of this fraction of population would be to observe the paths taken from users that have received suggestions and choose their own actions just following the principle of the Wardrop equilibrium.

Note then that there is the following ranking between these different settings of an Information design problem, from the best to the worst one:

1. flow imposed by the system according to its optimal policy without allowing users to choose;
2. optimal flow both for the system and also for users because it takes into account the constraints;
3. flow proposed by the system according to its optimal policy without considering constraints, but then leaving users who do not obey to look for their own Wardrop optimum.

If the optimum satisfies constraints, then all this situations are equivalent, in general they are not. Connecting with this, an important research shoud be investigating the behavior of users in the third case, when a policy is implemented regardless of the constraints of optimality for users. As already seen, there are two scenarios: if the constraints are respected by this policy then it is already the optimal, if instead the constraints are violated users are looking for Wardrop equilibrium. The goal could be to understand how suboptimal this third situation is with respect to the first and the second one.

Moreover, in order to generalize and validate the obtained results more cases should be analysed:

- with different probability distributions of the prior and with other specific cost functions,
- with different settings of the network or with dynamical environment,

- with multiple senders.

As the problem becomes more general and complex, it may be useful to conduct numerical analysis of the model.

Appendix A

Appendix

A.1 Lagrangian method and duality

[5] The standard form of an optimization problem is:

$$p^* = \min_{x \in \mathcal{R}^n} f(x) \tag{A.1}$$
$$\text{subj. to} \quad g_i(x) \leq 0 \quad \forall i = 1, ..., p \tag{A.2}$$
$$\text{and} \quad h_i(x) = 0 \quad \forall i = 1, ..., q \tag{A.3}$$

where f is the objective function and g and h are respectively the inequality and equality constraints.

Observation 12. *Note that an optimization problem is defined as convex if the objective and the inequality constraints are convex functions and the equality constraints are affine functions. Convex optimization are a small set of optimization problems, but of enormous importance and have the fundamental property for which each local optimum is always also a global optimum for them.*

A ***lagrangian function*** is defined as:

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \sum_{i=1}^{p} \lambda_i g_i(x) + \sum_{i=1}^{q} \mu_i h_i(x) \tag{A.4}$$

where λ and μ are variable called lagrangian mulpliers and are used to add a penalty term proportional to constraints to the objective function.

The objective now is to otpimize the lagrangian function in order to find the optimum of the primal optimization problem. First, it is possible to take the lower extreme of the lagrangian function with respect to variable x and the resulting function is called **lagrange dual function**:

$$g(\lambda, \mu) = \inf_{x \in \mathcal{R}^n} \mathcal{L}(x, \lambda, \mu) \tag{A.5}$$

It follows that $g(\lambda, \mu)$ is always lower or equal than p^*. Second, it is necessary to find the best possible of this lower extreme and this is done calculating the following ***dual problem***:

$$d^* = \max_{\lambda \geq 0, \mu} g(\lambda, \mu) \tag{A.6}$$

Now there are two possible situations:

- it is always true that $d^* \leq p^* \Rightarrow$ weak duality holds;
- instead if the primal problem is convex and other constraints qualification properties subists, then $d^* = p^* \Rightarrow$ strong duality holds.

An example of properties that guarantee strong duality is Slater's condition:

in a convex problem if the first $k \leq p$ function of the inequality constraints are affine and there exists a pont x such that:

$$\begin{aligned} g_i(x) &\leq 0 \quad \forall i = 1:k \\ g_i(x) &< 0 \quad \forall i = k+1:p \\ h_i(x) &= 0 \quad \forall i = 1:q \end{aligned}$$

then the optimum of the primal problem concides with the optimum of the dual problem.

To sum up the **dual property** is:

Lemma 7. *The primal problem can be rewritten as:* $p^* = \min_{x \in \mathcal{R}^n} \max_{\lambda \geq 0, \mu} \mathcal{L}(x, \lambda, \mu)$, *the dual problem can be written as:* $d^* = \max_{\lambda \geq 0, \mu} \min_{x \in \mathcal{R}^n} \mathcal{L}(x, \lambda, \mu)$, *then, if weak duality holds:* $\min_{x \in \mathcal{R}^n} \max_{\lambda \geq 0, \mu} \mathcal{L} \geq \max_{\lambda \geq 0, \mu} \min_{x \in \mathcal{R}^n} \mathcal{L}$ *instead, if strong duality holds:* $\min_{x \in \mathcal{R}^n} \max_{\lambda \geq 0, \mu} \mathcal{L} = \max_{\lambda \geq 0, \mu} \min_{x \in \mathcal{R}^n} \mathcal{L}$.

From this theory follows the ***Karush-Kuhn-Tucker*** conditions that characterize optimality for a generic primal problem:

$$\begin{aligned} \nabla f(x) &= 0 \\ h(x) &= 0 \\ \lambda &\geq 0 \\ \lambda_i f_i(x) &= 0 \quad \forall i = i:m \end{aligned}$$

where the last condition is called complementary slackness condition.
This condition implies that if an inequality constraint is strictly lower than zero this means that the optimum point is in the intern of the boundary and that this constraint is not a problem for the optimization and can be not included in the lagrangian function. To do that then the lagrangian mutliplier asscociated to it it's set uqual to zero. This kind of constraint are called inactive. On the contrary, if the optimum satisfies an inequality constraint on the boundary, i.e. the inequality constraint is solved as an equality and must be included in the lagragian function with a lagrangian mutliplier strinctly bigger than zero. This kind of constraint is called active.

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Acknowledgements

Grazie al Politecnico di Torino per questi inestimabili anni di crescita.

Un grazie sentito al Professor Como e al Professor Cianfanelli per il supporto costante e l'interesse trasmesso.

Alla matematica, per la sua profonda perfezione e bellezza.

Grazie alla mia famiglia per essere stati tutto ciò di cui ho bisogno
e ai miei amici per aver condiviso con me molti dei miei ricordi migliori.

A tutte le persone straordinarie che ho conosciuto negli anni e alle tante preziose conversazioni.

A tutti i momenti, alle occasioni di questi anni, a una ricerca continua e a tante altre soddisfazioni.