POLITECNICO DI TORINO

Master's degree
in Mathematical Engineering

Master's Degree Thesis

The spread of fake news and the interplay with personal competence: Boltzmann-type kinetic models and Monte Carlo simulations

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A.Y. 2021-2022

A nonna Diana

Abstract

Nowadays, the Internet is becoming the primary source of information, and the use of social networks makes it possible to come every day in contact with a great deal of news, much of which turns out to be fake. In recent years, for this reason, more and more fake news has been studied, trying to understand how to recognize them, how they propagate and how they affect people, especially their knowledge.
In the first part of this thesis, it is analyzed through kinetic theory how individual's competence varies in the interaction with a fake news, characterized by a degree of falsehood. This analysis shows that, over long periods of time, there is a polarization in agents' knowledge toward the highest degree of competence (those who are perfectly able to discern between true and false news) and toward the lowest degree of competence (those who are unable in any way to recognize the reliability of a news). This analytically obtained behavior is also verified numerically through Monte Carlo simulations.
The second part is devoted to the study of the popularity of fake news on a social network. At first, we make the assumption that, who is not sufficiently competent, will share the news. From the study of the evolution of the popularity conditional on a given degree of falsity of the news, the results show that, a totally false news, tends in a long time to completely lose popularity, since no one will share it, while, if the news is true, the average popularity conditional on that degree of falsity will not cancel out. In the end, we propose 3 alternative models for the popularity of the fake news.

Ringraziamenti

Risulta difficile per me realizzarlo ma questo lavoro corona la fine di un grande percorso, che ho vissuto a pieno in questi anni e che mai scorderò. Dietro a questo traguardo c'è il lavoro di tante persone che hanno fatto moltissimo per me: insegnandomi, coccolandomi, facendomi ridere, tranquilizzandomi o essendo presenti nei momenti più bui.

Un sentito ringraziamento va al *Prof. Andrea Tosin*, il quale mi ha seguita passo passo nella realizzazione di questo eleborato, ascoltando le mie idee e cercando di svilupparle insieme. Il suo modo di fare, sempre disponibile e pronto a trasmettere conoscenza, senza mai far sentire inferiore chi ha davanti, mi ha stimolata molto nella realizzazione di questo lavoro. Mi ritengo molto fortunata ad aver potuto lavorare con lui.

Un grazie enorme va alla *mia famiglia*, la quale mi ha fatto il regalo più grande, permettendomi di studiare e di intraprendere questo percorso sia formativo che personale, comprendendo i tempi di cui avevo bisogno e dandomi un supporto quotidiano.
Ringrazio con tutto il cuore *mia madre*, la quale mi ha spronata sempre ad essere una combattente e a lottare per quello a cui tengo. Quando mi facevo prendere dallo sconforto, mi guardava chiedendosi, e chiedendomi, come mai io non riuscissi ad entrare nell'ottica che avrei potuto fare tutto. Mi ha spinta a stringere i denti, affrontando l'ansia che mi faceva sentire incapace di poter raggiungere questo grande traguardo.
Ringrazio moltissimo *mio padre*, il quale, nonostante la distanza, è sempre stato presente con il cuore e non solo. Ha fatto in modo di non farmi mancare mai nulla, così che io mi godessi al massimo questo bellissimo percorso. Forse è una delle persone che ha creduto più in me e sono sicura di averlo reso orgoglioso.
Un grazie va anche a *mia sorella Monica*, la quale per me è stata d'esempio per capire che con il duro lavoro si può arrivare dovunque si voglia, anche da soli.
Ultimo membro della famiglia, ma non per importanza, che vorrei ringraziare è *mio fratello Francesco*. Lui mi ha insegnato ad andare "dove mi esplode il cuore" senza curarmi troppo di quello che gli altri vorrebbero per me, ma focalizzandomi sempre su quello che io vorrei per me stessa. Mi ha insegnato a coltivare ogni giorno la curiosità e a non smettere mai di imparare dagli altri e da tutto quello che ho intorno.
Sono molto grata anche alla mia *famiglia calabrese* (che ormai tanto calabrese non è più visto che siamo tutti sparsi per il mondo, ma il sangue quello è e quello rimarrà), zii e cugini, a cui sono molto legata; per loro sono "la scienziata" della famiglia e mi hanno fatto capire veramente quanto "la distanza separa i corpi ma non i cuori".

Ringrazio *Carlo*, il quale in questi anni mi ha vissuta a 360 gradi ed è sempre stato in grado di farmi spuntare un sorriso. Tutte le esperienze vissute con lui, dai viaggi allo studio, mi hanno arricchita tantissimo. É sempre stato al mio fianco anche quando l'agitazione aveva il sopravvento e, seppur non capisse il motivo per il quale mi disperassi tanto per cose facilmente risolvibili, passava ore a tranquillizzarmi. Mi ha aiutata sempre e il suo "lo spacchiamo" prima di ogni esame non mi ha fatto sentire mai sola. Questo traguardo e anche un po' suo.

Ringrazio i miei compagni, che mi hanno vissuta nei momenti felici, in cui hanno riso e scherzato con me, e nei momenti tristi, in cui sono sempre stati pronti a dire una parola di conforto e a farmi sentire compresa. Passare questi anni con voi è stato tanto bello che ogni volta che passo davanti al Politencnico sento già nostalgia delle risate a lezione e delle pause caffè a parlare di qualsiasi cosa.
Ringrazio Giulia e Scola, con i quali ho condiviso tutti i giorni negli ultimi mesi. Hanno alleggerito quelle giornate eterne davanti al computer, con tanti pettegolezzi (in sardo si dice "ciaramiddare" per chi non lo sapesse), tante battute e tanti dolci. Mi hanno ascoltata ore e ore, sono stati pazienti quando li assillavo per sapere se secondo loro la tesi andasse bene e mi hanno dato la forza fino ad oggi.
Ringrazio *Lisa T. e Francesco*, con i quali sono cresciuta. Non so come sia possibile che in così poco tempo (o forse a me sembra poco) siamo passati dalle nuotate al mare con i braccioli ad oggi, in cui siamo adulti e l'argomento di conversazione più gettonato è il lavoro. Mi rende molto contenta sapere che, nonostante le persone cambino crescendo, esistono persone come voi che rimangono.
Ringrazio *Carolina, Lisa S., Katia e Nicola*, i quali in questi ultimi anni non sono potuti essere presenti fisicamente come prima ma, nonostante le strade diverse intraprese, hanno fatto in modo che io li sentissi sempre vicini con il cuore, e che mi arrivasse comunque il loro supporto e affetto.

Un ultimo grazie va ai luoghi che mi hanno accolta in questi anni. Torino per me è stata la città perfetta, mi ha fatto sentire a casa dal primo giorno. Non so dove mi porterà la vita, ma so che per sempre un pezzo di cuore rimarrà qua.
Infine, è impossibile per me non ringraziare la *Comala*. Ho passato più ore qui che a casa mia. Dopo la pandemia, mi ha dato la possibilità di riprendere una vita sociale e ha alleggerito tutti i momenti di studio, con tante risate e qualche spritz.

Amici, questa volta posso dire: *"Martina batte l'ansia 1 a 0"* .
Se stai leggendo questi ringraziamenti, probabilmente è arrivata l'ora di brindare e festeggiare.
Alla mia, alla nostra.

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Intoduction

More and more, in recent years, the use of the Internet and social networks has become extremely widespread and with it the number of fake news. Social networks, and, in general, the Internet are within the reach of all and we make extensive use of them, also as a primary source of information. Identifying if a news is false is not an easy task and for this reason, nowadays, it has become necessary to study the effect of fake news on individuals' knowledge and how they tend to become popular. In particular, this need has become even more urgent following the arrival of the Covid-19 pandemic, which, due to the huge number of fake news circulating online (mainly in the phase when there was little official data), caused a slowdown of the vaccination campaign all over the world.

In the first part of this work, we use kinetic theory to study, through the Boltzmann-type equation, the effect of the interaction between the individual and fake news on the ability of the agents to recognize the reliability of the news and, therefore, to distinguish the real news from the false ones. By this approach, we have studied the evolution of the average of the agents' competence and the cases in which it remains constant over time. Afterward, in the regime of quasi-invariant interactions, the stationary distributions, to which competence tends for a long time, were found. The result obtained shows that, with and without stochastic fluctuation, the dynamics tends toward a disappearance of intermediate level of knowledge and a consequent polarization towards extreme values of competence: the maximum, which represents individuals who can discern between false and true news, regardless of the degree of falsity, and the minimum, which characterizes the individuals who are not competent and who are not able at all to recognize fake news. This theoretical result was also verified through a numerical simulation by the Monte Carlo method implemented in Matlab.

In the second part, we analyze, with the theory of multi-agent systems, the spreading of the popularity of fake news on a social network and how it changes according to the degree of falsity of the news. At the begging of this part, in the chosen model, it was made the simplistic assumption that the agents, which are not competent enough to recognize false news, share the news on the social network. For this model, we focus on studying the trend of average popularity over time, and then on the behavior of the dynamics of popularity conditioned to the degree of falsity of the news, to be able to compare results for different levels of falsity. To do that, we have derived a Boltzmann-type equation specific for the conditional popularity distribution at a determined level of falsehood of the news.

Furthermore, through the deduced equation, it was possible to analyze the variation over time of the average news popularity with a fixed level of falsehood and, under the limit of quasi-invariant interactions, derive the Fokker-Planck equation to find the stationary distribution.
In conclusion, three alternative models for the study of popularity have been proposed. In all these models, the hypothesis on the sharing of the news is relaxed, allowing the individuals with a degree of competence close to the degree of falsity to share the news anyway. Hence, even if the agent is more competent and is able to recognize the news as false, he can still share it as long as the difference between the competence of the individual and the degree of the falsity of the news is sufficiently small. The definition of sufficiently small changes according to the model chosen.

Part I

Competence's changes due to the interaction with the fake news

Chapter 1

The epidemiological approach in the study of fake news

In many scientific articles, in which the topic of fake news is discussed, an epidemiological approach is used. As an example, Daley and Kendall in [2] proposed a SIR-type model to treat fake news as an infectious disease. The model considers agents labeled as ignorants, spreaders and stiflers, which constitute the susceptible, infected and recovered, respectively, of the famous epidemiological model. Later, other epidemiological models such as SIS and SIRS were used for the study of fake news. More advanced models have also been proposed, such as SEIZ, for studying the evolution of susceptible, exposed, infected, and skeptical groups over time.

1.1 SEIR model with individual competence

In this chapter, we will analyze the SEIR model proposed by J.Franceschi and L.Pareschi in [4] , in which they considered a structure of the population based on the concept of competence, which represents the agent's ability to evaluate the news with which he or she came in contact.
In the model, the population is divided into:

- Susceptible agents (S), those who are unaware of the news;
- Exposed agents (E), those who have come into contact with the news and will move into an active class;
- Infectious agents (I), those who spread the false news;
- Removed agents (R), those who are aware of the news and decide not to share the news.

The dynamics of the SEIR model can be described as follows: a susceptible comes into contact with the news through an infectious agent. After contact, the agent evaluates the information, becoming an exposed. Finally, he can decide between: not sharing the news, removing himself from the news propagation process and changing his label to removed agents, or sharing it becoming part of the infectious class (see Figure 1.1).

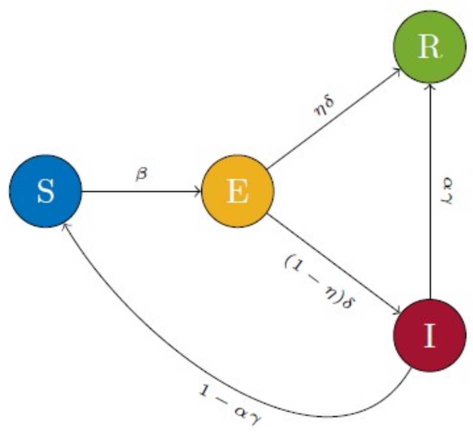

Figure 1.1. SEIR model.

We define $f_S(x,t)$, $f_E(x,t)$, $f_I(x,t)$, $f_R(x,t)$, respectively the distribution of competence at time $t > 0$ of susceptible, exposed, infected, and removed. Assuming no one leaves the social network, we have:

$$\int_X (f_S(x,t) + f_E(x,t) + f_I(x,t) + f_R(x,t))dx = 1.$$

From the definition of densities it is possible to derive the fraction of individuals belonging to each class at time t:

$$S(t) = \int_X f_S(x,t)dx, \qquad E(t) = \int_X f_E(x,t)dx,$$
$$I(t) = \int_X f_I(x,t)dx, \qquad R(t) = \int_X f_R(x,t)dx.$$

Obviously, it will be worth: $S + E + I + R = 1$.
Assuming that this dynamics is independent of knowledge, the system can be expressed with the system of ODEs:

$$\begin{cases} \frac{dS}{dt} &= -\beta SI + (1-\alpha)\gamma I \\ \frac{dE}{dt} &= \beta SI - \delta E \\ \frac{dI}{dt} &= (1-\eta)\delta E - \gamma I \\ \frac{dR}{dt} &= \eta\delta E + \alpha\gamma I. \end{cases} \tag{1.1}$$

Let us look at each ODE in more detail.
The first tells us that an agent stops being susceptible if it comes into contact with an

infected person, the rate at which this happens is β. It also tells us that an infected can become susceptible: this occurs if the agent loses interest in the news and forgets its content. The γ denotes the rate at which spreaders lose interest in sharing the news, while α is related to the probability that the individual will remember the fake news.
The change in the exposed class can be related to an increase during the interaction between susceptible and infected, in which the susceptible automatically becomes exposed, or a decrease related to whether the exposed agent makes the decision to share or not the news, and the rate at which a person makes the decision is δ.
The number of infected agents may decrease if the spreaders lose interest in sharing the news or forget it, while they may increase if an exposed makes the decision to share, which occurs with the rate $(1-\eta)\delta$, where $1-\eta$ indicates the portion of agents that become infected.
Finally, the removals can only increase. The factors that make possible an increase in that class are either that an exposed person decides not to share the news (rate $\eta\delta$) or if an infected person forgets and loses interest in the news (rate $\alpha\gamma$).

We will now focus on how, in [4], the SEIR model was combined with the dynamics of competence. Let's denote the degree of competence that the agent acquires or loses in each interaction with the background with $z \in \mathbb{R}_+$ and its probability distribution with $C(z)$.
If a susceptible agent has competence level x and interacts with an agent with competence x^*, their knowledge after the interaction will be given by:

$$\begin{cases} x' &= (1-\lambda_S(x))x + \lambda_{CJ}(x)x_* + \lambda_{BS}(x)z + k_{SJ}x \\ x'_* &= (1-\lambda_J(x_*))x_* + \lambda_{CS}(x_*)x + \lambda_{BJ}(x_*)z + \tilde{k}_{SJ}x_*, \end{cases} \tag{1.2}$$

for $J \in \{S, E, I, R\}$. In (1.2) it appears:

- $\lambda_S(\cdot)$: the competence lost thanks to forgetfulness;
- $\lambda_{BS}(\cdot)$: the competence earned by the background;
- λ_{CJ}: the knowledge gained in the interaction with an agent of class J;
- k_{EJ} and $\tilde{k}_{EJ}$: the independent stochastic fluctuations with zero mean and variance $\sigma(t)$.

Similarly to (1.2), we can define the interactions for the other classes.
For exposed agents the post-interaction competence will be:

$$\begin{cases} x' &= (1-\lambda_E(x))x + \lambda_{CJ}(x)x_* + \lambda_{BE}(x)z + k_{EJ}x \\ x'_* &= (1-\lambda_J(x_*))x_* + \lambda_{CE}(x_*)x + \lambda_{BJ}(x_*)z + \tilde{k}_{EJ}x_*. \end{cases} \tag{1.3}$$

For the infectious class:

$$\begin{cases} x' &= (1-\lambda_I(x))x + \lambda_{CJ}(x)x_* + \lambda_{BI}(x)z + k_{IJ}x \\ x'_* &= (1-\lambda_J(x_*))x_* + \lambda_{CI}(x_*)x + \lambda_{BJ}(x_*)z + \tilde{k}_{IJ}x_*. \end{cases} \tag{1.4}$$

For the removed agents:

$$\begin{cases} x' &= (1-\lambda_R(x))x + \lambda_{CJ}(x)x_* + \lambda_{BR}(x)z + k_{RJ}x \\ x'_* &= (1-\lambda_J(x_*))x_* + \lambda_{CR}(x_*)x + \lambda_{BJ}(x_*)z + \tilde{k}_{RJ}x_*. \end{cases} \tag{1.5}$$

The Boltzmann-type theory was used to combine the effect of the dynamics of competence with the dynamics of the epidemiological model by introducing the interaction operator $Q_{HJ}(\cdot,\cdot)$.
Let $\phi(x)$ be an observable of competence. We can analyze the effect of $Q_{HJ}(f_H, f_J)(x,t)$ on $\phi(x)$. If x' follows the interaction rule (1.2), i.e., the agent is in the class of susceptible:

$$\int_{\mathbb{R}_+} Q_{SJ}(f_S, f_J)\phi(x)dx = \langle \int_{\mathbb{R}_+^2} f_S(x,t)f_J(x_*,t)(\phi(x') - \phi(x))dx_*dx \rangle. \tag{1.6}$$

Otherwise, if the agent is part of the class of exposed, x' is defined as in (1.3), and we have:

$$\int_{\mathbb{R}_+} Q_{EJ}(f_E, f_J)\phi(x)dx = \langle \int_{\mathbb{R}_+^2} f_E(x,t)f_J(x_*,t)(\phi(x') - \phi(x))dx_*dx \rangle. \tag{1.7}$$

For infectious agents (x' defined as (1.4)):

$$\int_{\mathbb{R}_+} Q_{IJ}(f_I, f_J)\phi(x)dx = \langle \int_{\mathbb{R}_+^2} f_I(x,t)f_J(x_*,t)(\phi(x') - \phi(x))dx_*dx \rangle. \tag{1.8}$$

Lastly, for the agent in the removed class, x' follows (1.5) and the Boltzmann-type equation is:

$$\int_{\mathbb{R}_+} Q_{RJ}(f_R, f_J)\phi(x)dx = \langle \int_{\mathbb{R}_+^2} f_R(x,t)f_J(x_*,t)(\phi(x') - \phi(x))dx_*dx \rangle. \tag{1.9}$$

Choosing $\phi(\cdot) = 1$ we have that, for all of the last four equations, the term on the right-hand side will become zero. Hence, the number of agents in each class will remain the same.
Let's define the function:

$$K(x,t) = \int_{\mathbb{R}_+} \beta(x, x_*)f_I(x_*,t)dx_*, \tag{1.10}$$

where the contact rate is expressed as a function of both x and x_*, because individuals interact mostly with agents with a similar level of knowledge. Let's use (1.10) to rewrite (1.1) as:

$$
\begin{cases}
\frac{\partial f_S(x,t)}{\partial t} & = -K(x,t)f_S(x,t) + (1-\alpha(x))\gamma(x)f_I(x,t) + \sum_{J\in\{S,E,I,R\}} Q_{SJ}(f_S, f_J)(x,t) \\
\frac{\partial f_E(x,t)}{\partial t} & = K(x,t)f_S(x,t) - \delta(x)f_E(x,t) + \sum_{J\in\{S,E,I,R\}} Q_{EJ}(f_E, f_J)(x,t) \\
\frac{\partial f_I(x,t)}{\partial t} & = \delta(x)(1-\eta(x))f_E(x,t) - \gamma(x)f_I(x,t) + \sum_{J\in\{S,E,I,R\}} Q_{IJ}(f_I, f_J)(x,t) \\
\frac{\partial f_R(x,t)}{\partial t} & = \delta(x)\eta(x)f_E(x,t) + \alpha(x)\gamma(x)f_I(x,t) + \sum_{J\in\{S,E,I,R\}} Q_{RJ}(f_R, f_J)(x,t).
\end{cases}
\tag{1.11}
$$

It is reasonable to express $\delta(x)$ as a function of the competence since it represents the rate at which the agent decides to spread or not the news, and, of course, a more competent agent spends a lot of time analyzing the reliability of the news. Also η, which marks the decision of individuals to share the news, depends on x. Instead, γ and α do not have a strict dependence on x.

1.1.1 The quasi-invariant regime and the Fokker-Planck equation

Let's study the behavior for $t \to \infty$ using the quasi-invariant limit. To have small changes during the interaction it is necessary to scale, through a coefficient small and positive ϵ, the following quantities:

$$\lambda_{BJ} \to \epsilon\lambda_{BJ}, \quad \lambda_{CJ} \to \epsilon\lambda_{CJ}, \quad \lambda_J \to \epsilon\lambda_J, \quad \sigma \to \epsilon\sigma,$$

with $J \in \{S, E, I, R\}$, and also:

$$\beta(x, x_*) \to \epsilon\beta(x, x_*), \quad \delta(x) \to \epsilon\delta(x), \quad \gamma(x) \to \epsilon\gamma(x), \quad \eta(x) \to \epsilon\eta(x).$$

To analyze the large-time behavior it is important to scale the time, hence, we define $\tau = \epsilon t$.

Since ϵ has to be chosen small enough, it is possible to rewrite $\phi(x')$ with the Taylor expansion centered in x:

$$\phi(x') = \phi(x) + (x'-x)\phi'(x) + \frac{(x'-x)^2}{2}\phi''(x) + \mathcal{O}(\epsilon^2).$$

According to the definition in (1.2), the Taylor expansion for susceptible agents is:

$$\phi(x') = \phi(x) + (-\lambda_S(x)x + \lambda_{CJ}(x)x_* + \lambda_{BS}(x)z)\phi'(x) + \frac{\sigma}{2}x^2\phi''(x) + \mathcal{O}(\epsilon^2),$$

where we exploit the fact that the mean of the stochastic fluctuation is zero and the variance is σ.
Similarly, we can evaluate the Taylor expansion for the other classes. For exposed we get:

$$\phi(x') = \phi(x) + (-\lambda_E(x)x + \lambda_{CJ}(x)x_* + \lambda_{BE}(x)z)\phi'(x) + \frac{\sigma}{2}x^2\phi''(x) + \mathcal{O}(\epsilon^2).$$

For the class of infectious:

$$\phi(x') = \phi(x) + (-\lambda_I(x)x + \lambda_{CJ}(x)x_* + \lambda_{BI}(x)z)\phi'(x) + \frac{\sigma}{2}x^2\phi''(x) + \mathcal{O}(\epsilon^2),$$

and finally for removed agents, we have:

$$\phi(x') = \phi(x) + (-\lambda_R(x)x + \lambda_{CJ}(x)x_* + \lambda_{BR}(x)z)\phi'(x) + \frac{\sigma}{2}x^2\phi''(x) + \mathcal{O}(\epsilon^2).$$

Replacing these results in (1.6) - (1.9), we obtain:

$$\begin{aligned}
\frac{1}{\epsilon}\int_{\mathbb{R}_+} Q^{\epsilon}_{SJ}(f_S, f_J)\phi(x)dx &\approx \int_{\mathbb{R}_+}\left[-\phi(x)'(\lambda_S xJ - \lambda_{CJ}m_J - \lambda_{BS}m_B J) + \frac{\sigma}{2}\phi(x)''x^2 J\right] f_S(x,t)\,dx\\
\frac{1}{\epsilon}\int_{\mathbb{R}_+} Q^{\epsilon}_{EJ}(f_E, f_J)\phi(x)dx &\approx \int_{\mathbb{R}_+}\left[-\phi(x)'(\lambda_E xJ - \lambda_{CJ}m_J - \lambda_{BE}m_B J) + \frac{\sigma}{2}\phi(x)''x^2 J\right] f_E(x,t)\,dx\\
\frac{1}{\epsilon}\int_{\mathbb{R}_+} Q^{\epsilon}_{IJ}(f_I, f_J)\phi(x)dx &\approx \int_{\mathbb{R}_+}\left[-\phi(x)'(\lambda_I xJ - \lambda_{CJ}m_J - \lambda_{BI}m_B J) + \frac{\sigma}{2}\phi(x)''x^2 J\right] f_I(x,t)\,dx\\
\frac{1}{\epsilon}\int_{\mathbb{R}_+} Q^{\epsilon}_{RJ}(f_R, f_J)\phi(x)dx &\approx \int_{\mathbb{R}_+}\left[-\phi(x)'(\lambda_R xJ - \lambda_{CJ}m_J - \lambda_{BR}m_B J) + \frac{\sigma}{2}\phi(x)''x^2 J\right] f_R(x,t)\,dx,
\end{aligned} \tag{1.12}$$

where $Q^{\epsilon}_{HJ}(\cdot,\cdot)$ represent the scaled version of $Q_{HJ}(\cdot,\cdot)$.
In (1.12), we used the following definitions for a generic class $J \in \{S, E, I, R\}$:

$$J = \int_{\mathbb{R}_+} f_J(x,t)dx, \qquad m_J = \int_{\mathbb{R}_+} x f_J(x,t)dx.$$

We also denoted the mean of the competence gained or lost in the interaction with the background with m_B, i.e.,

$$m_B = \int_{\mathbb{R}_+} zC(z)dz.$$

Taking advantage of the results obtained in (1.6)-(1.8) and pass to the quasi-invariant regime (imposing $\epsilon \to 0$), we can rewrite (1.11) to obtain the Fokker-Planck system. Proceeding as said and integrating by part the right-hand side of (1.12), we reach:

$$
\begin{aligned}
\frac{\partial f_S(x,t)}{\partial t} =& - K(x,t)f_S(x,t) + (1-\alpha(x))\gamma(x)f_I(x,t) + \frac{\partial}{\partial x}[(x\lambda_S - \bar{m}(t) - \lambda_{BS}m_B)f_S(x,t)] + \\
&+ \frac{\sigma}{2}\frac{\partial^2}{\partial x^2}(x^2 f_S(x,t)) \\
\frac{\partial f_E(x,t)}{\partial t} =& K(x,t)f_S(x,t) - \delta(x)f_E(x,t) + \frac{\partial}{\partial x}[(x\lambda_E - \bar{m}(t) - \lambda_{BE}m_B)f_E(x,t)] + \\
&+ \frac{\sigma}{2}\frac{\partial^2}{\partial x^2}(x^2 f_E(x,t)) \\
\frac{\partial f_I(x,t)}{\partial t} =& \delta(x)(1-\eta(x))f_E(x,t) - \gamma(x)f_I(x,t) + \frac{\partial}{\partial x}[(x\lambda_I - \bar{m}(t) - \lambda_{BI}m_B)f_I(x,t)] + \\
&+ \frac{\sigma}{2}\frac{\partial^2}{\partial x^2}(x^2 f_I(x,t)) \\
\frac{\partial f_R(x,t)}{\partial t} =& \delta(x)\eta(x)f_E(x,t) + \alpha(x)\gamma(x)f_I(x,t) + \frac{\partial}{\partial x}[(x\lambda_R - \bar{m}(t) - \lambda_{BR}m_B)f_R(x,t)] + \\
&+ \frac{\sigma}{2}\frac{\partial^2}{\partial x^2}(x^2 f_R(x,t)),
\end{aligned}
$$

in which $\bar{m}(t)$ is defined as follows:

$$
\bar{m}(t) = \lambda_{CS}m_S(t) + \lambda_{CE}m_E(t) + \lambda_{CI}m_I(t) + \lambda_{CR}m_R(t).
$$

Chapter 2

Definition of the model

In the following chapters, we proposed an alternative approach to the epidemiological one, to studying fake news and the interplay with competence. The model that we propose allows a more aggregate modeling approach, omitting a subdivision of the population in classes. Furthermore, unlike the model introduced in Chapter 1, we consider for fake news a distribution based on the level of the falsehood permitting a more accurate description of the news, that does not just classify news as true or false.
The approach used in this work is based on Statistical Mechanics. This subject allows studying multi-agent systems without focusing on the interactions of each individual particle, which would require the study of an extremely large number of coupled differential equations. To do this, it resorts to a single instrument: the probability density, creating a clear parallel between physical and mathematical concepts.
In 1880 Ludwing Boltzmann began to apply this theory to gases, to then generalize this method to different systems, giving rise to Statistical Mechanics. This approach permits a good analysis of the dynamics with a macroscopic description of it and has the huge advantage of having a very wide field of application: from social systems (opinion dynamics, voter model, etc.) to the vehicular traffic or crowd dynamics. It is important to notice that, to give an aggregate statistical description of our system, whatever it is, it is necessary to resort to the hypothesis of indistinguishable agents.

First of all, we define the *interaction rule* for our system and then we deduce the *Boltzmann-type equation* and use it to study our dynamics.

2.1 Interaction rule

Let X and Y be random variables that respectively represent the individual's knowledge (viz. the ability to recognize the reliability of news) and the degree of falsity of the news. The post-interaction competence is a random variable, whose realization is given by the following expression:

$$x^* = x + \lambda(x, y) + D(x)\eta, \tag{2.1}$$

where λ is a function of x and y defined in the following way:

$$\begin{cases} c_2(1-x) & \text{if} \quad x \geq 1-y \\ -c_1 x & \text{if} \quad x < 1-y. \end{cases} \tag{2.2}$$

The choice of this interaction rule comes out from the fact that, if an individual has a knowledge greater than or equal to the degree of truthfulness of the news ($x \geq 1-y$), then the agent recognizes the reliability of the news. Hence, his knowledge will tend to increase, since he will not only neglect its content but will have a superior ability to recognize other fake news with content similar to the one he came into contact with. The increase is considered proportional to the lack of knowledge needed to achieve the maximum level of competence, that is $1-x$, and the constant of proportionality will be denoted by c_2.
Conversely, if the individual does not have sufficient knowledge to recognize the accuracy of the fake news, which means that his knowledge is less than the degree of truthfulness of the news ($x < 1-y$), the competence will decrease, since the person will acquire information that does not represent the truth. The decrease is proportional to the degree of knowledge x and the constant of proportionality is indicated with c_1. Both the constants c_1 and c_2 assume values in the interval $[0.1]$.
The *stochastic fluctuation*, denoted by η, by definition, has $\mathbb{E}[\eta] = 0$ and $\mathrm{Var}[\eta] = \sigma^2 > 0$, since, if the variance was zero, the fluctuation would be an identically zero constant.
The function $D(x)$ represents the diffusion coefficient and adjusts the intensity of stochastic fluctuation. Obviously, it must be chosen properly, to have a post-interaction knowledge that falls within the interval $[0.1]$.

2.2 Derivation of the Boltzmann-type equation

To derive the Boltzmann equation we start from the discrete case. We take into account a time interval $\Delta t > 0$ and we consider a pair of one agent and one fake news. In the time interval Δt, the couple may or may not interact. If it interacts, then the knowledge evolves according to (2.1), otherwise, it remains unchanged, maintaining the same value of competence as time t. To model this, we consider a random variable $\Theta \sim Bernoulli(\frac{\Delta t}{\tau})$ which, based on the returned value, tells us whether, in the interval Δt, the interaction between agent and news occurs ($\Theta = 1$) or not ($\Theta = 0$). The interaction probability is proportional to the duration of the time interval, and the proportionality constant is the interaction frequency $\frac{1}{\tau}$. Hence, $\tau > 0$ represents the characteristic time between two interactions. Since $\frac{\Delta t}{\tau}$ is a probability, it has to be less than or equal to 1, from which follows the constraint $\Delta t \leq \tau$. This condition does not turn out to be particularly stringent since, to be possible to go from the discrete to the continuous case, it is necessary to choose a small Δt. Consequently, we have that the knowledge at time $t + \Delta t$ will be:

$$\begin{aligned} x_{t+\Delta t} &= (1-\Theta)x_t + \Theta x_t^* \\ &= (1-\Theta)x_t + \Theta(x_t + \lambda(x,y) + D\sigma Y), \end{aligned} \tag{2.3}$$

where the definition of x_t^* follows from (2.1) and Y represents the "normalized" fluctuation with zero mean and variance equal to 1.

Let us now consider an observable of the system (viz., any quantity that can be calculated as a function of the microscopic state of the agent) and apply it to (2.3). Then, we analyze the average changes in the observable:

$$\langle \varphi(x_{t+\Delta t}) \rangle = \langle \varphi((1-\Theta)x_t + \Theta x_t^*) \rangle.$$

Since the Bernoulli is a discrete probability distribution that takes only two values, we can easily calculate the mean as follows:

$$\begin{aligned} \langle \varphi(x_{t+\Delta t}) \rangle &= \langle \varphi(x_t^*) \rangle \mathbb{P}(\Theta = 1) + \langle \varphi(x_t) \rangle \mathbb{P}(\Theta = 0) \\ &= \langle \varphi(x_t^*) \rangle \frac{\Delta t}{\tau} + \langle \varphi(x_t) \rangle \left(1 - \frac{\Delta t}{\tau}\right). \end{aligned}$$

From which, bringing $\varphi(x_t)$ to the other side of equal, it follows:

$$\frac{\langle \varphi(x_{t+\Delta t}) \rangle - \langle \varphi(x_t) \rangle}{\Delta t} = \frac{1}{\tau}[\langle \varphi(x_t^*) \rangle - \langle \varphi(x_t) \rangle].$$

We pass to continuous time by making the limit for $\Delta t \to 0$ and, consequently, the incremental ratio will tend to the derivative in time of the mean of $\varphi(x_t)$:

$$\frac{d}{dt}\langle \varphi(x_t) \rangle = \frac{1}{\tau}[\langle \varphi(x_t^*) \rangle - \langle \varphi(x_t) \rangle],$$

and therefore, by the arbitrariness of φ, we obtain the Boltzmann equation in weak form:

$$\frac{d}{dt}\int_0^1 f(x,t)\varphi(x)dx = \frac{1}{\tau}\langle \int_{[0,1]^2} (\varphi(x^*) - \varphi(x)) f(x,t) g(y) dx dy \rangle \qquad \forall \varphi : [0,1] \to [0,1]. \tag{2.4}$$

The meaning of the Boltzmann equation is: the variation over time of the mean of the observable φ coincides with the mean of the variation of φ in a generic binary interaction.

2.3 Study of the first moment

We define $f(x,t)$, the density of the agents characterized by a degree of knowledge $x \in [0,1]$ at time $t \geq 0$, and $g(y)$, the probability distribution of the degree of falsity of the news $y \in [0,1]$. We introduce $\varphi(x)$, any observable quantity of the competence.
The law of evolution of $f(x,t)$ follows the *Boltzmann-type equation in weak form* in (2.4). Choosing $\tau = 1$, the following expression comes out:

$$\frac{d}{dt}\int_0^1 f(x,t)\varphi(x)dx = \langle \int_{[0,1]^2} (\varphi(x^*) - \varphi(x)) f(x,t) g(y) dx dy \rangle, \tag{2.5}$$

where x^* obeys to the interaction rules (2.1).
Since $f(x,t)$ is a probability density, it is known that $\int_0^1 f(x,t)dx = 1$. An evidence of this can be easily obtained by choosing $\varphi(x) = 1$ in (2.5):

$$\frac{d}{dt}\int_0^1 f(x,t)dx = \int_{[0,1]^2}(1-1)f(x,t)g(y)dxdy = 0,$$

with which it is proved that $\int_0^1 f(x,t)dx$ must be a constant.
In this section, we aim to study the first moment and how it varies over time. This is possible choosing $\varphi(x) = x$ and replacing it in (2.5):

$$\frac{d}{dt}\int_0^1 xf(x,t)dx = \langle\int_{[0,1]^2}(x^* - x)f(x,t)g(y)dxdy\rangle.$$

From equation (2.1) we find:

$$\langle x^* - x\rangle = \langle x + \lambda(x,y) + D(x)\eta - x\rangle = \lambda(x,y),$$

in which the second equality derives from $\langle\eta\rangle = 0$.
Going back to the calculation of the variation in time of the first moment, we have:

$$\frac{d}{dt}\int_0^1 xf(x,t)dx = \int_0^1\int_{1-x}^1 c_2(1-x)g(y)f(x,t)dydx - \int_0^1\int_0^{1-x} c_1 xg(y)f(x,t)dydx.$$

Let's define $G_Y(\cdot)$ as the cumulative distribution function of the random variable Y and $M_X(t)$ the mean of the degree of competence X. Noting, as we did for $f(x,t)$, that $\int_0^1 g(y)dy = 1$, we obtain:

$$\begin{aligned}\frac{dM_X(t)}{dt} &= \int_0^1 c_2(1-x)[1-G_Y(1-x)]f(x,t)dx - \int_0^1 c_1 xG_Y(1-x)f(x,t)dx\\ &= c_2 - c_2\int_0^1 G_Y(1-x)f(x,t)dx - c_2M_X(t) + c_2\int_0^1 xG_Y(1-x)f(x,t)dx+\\ &- c_1\int_0^1 xG_Y(1-x)f(x,t)dx,\end{aligned}$$

where the definition of $M_X(t)$ and the fact that $f(x,t)$ is a probability density have been exploited.
For simplicity, we assume that Y has a uniform distribution over the interval [0,1]. As a consequence, we have $G_Y(x) = x$ and the computation is reduced to:

$$\begin{aligned}\frac{dM_X(t)}{dt} &= c_2 - c_2\int_0^1(1-x)f(x,t)dx - c_2M_X(t) + c_2\int_0^1 x(1-x)f(x,t)dx+\\ &- c_1\int_0^1 x(1-x)f(x,t)dx.\end{aligned}$$

By developing the calculations and simplifying, we obtain:

$$\frac{dM_X(t)}{dt} = (c_2 - c_1)M_X(t) + (c_1 - c_2)M_X^{(2)},$$

in which $M_X^{(2)}$ denotes the second moment, i.e the energy of the system.
Clearly the evolution of M_X depends on $M_X^{(2)}$, hence, to determine the evolution of a

given moment, it will be necessary to always use the one of higher order, not allowing to determine an evolution in closed form for the moments of X. To overcome this problem, the interaction rule (2.1) can be further simplified by setting $c_1 = c_2$, obtaining the following equation for the evolution of the first moment:

$$\frac{dM_X(t)}{dt} = 0,$$

which shows that $M_X(t)$ will remain constant and equal to $M_X(0)$ for all time t.

2.4 Remark: why is forgetfulness omitted in our case?

Many models that deal with the evolution of knowledge based on the interaction with news or any information, take into account forgetfulness, i.e., the natural process according to which an individual tends in a long time to forget what he has come into contact with. However, assuming in our case the existence of this process leads to a not very interesting result, as we will show in this section.

Given the existence of a forgetting process, the interaction rule becomes:

$$x^* = (1 - \lambda_{dim}(x))x + \lambda(x, y) + D(x)\eta. \tag{2.6}$$

We can evaluate $\langle x^* - x\rangle$ under (2.6):

$$\langle x^* - x\rangle = \langle(1 - \lambda_{dim}(x))x + \lambda(x, y) + D(x)\eta - x\rangle = -\lambda_{dim}(x)x + \lambda(x, y).$$

Substituting this result in the Boltzmann equation in the weak form (2.5) and rearranging the terms, we get:

$$\begin{aligned}\frac{d}{dt}\int_0^1 xf(x,t)dx = -\int_0^1 x\lambda_{dim}(x)f(x,t)dx\int_0^1 g(y)dy + \int_0^1\int_{1-x}^1 c_2(1-x)g(y)f(x,t)dydx+\\ -\int_0^1\int_0^{1-x} c_1 xg(y)f(x,t)dydx.\end{aligned}$$

Following the steps done in 2.3, we define $G_Y(\cdot)$ the cumulative distribution function of the random variable Y and, then, we assume that the distribution of Y is uniform over the interval [0,1]:

$$\begin{aligned}\frac{dM_X(t)}{dt} &= -\int_0^1 x\lambda_{dim}(x)f(x,t)dx + \int_0^1 c_2(1-x)[1-G_Y(1-x)]f(x,t)dx+\\ &- \int_0^1 c_1 x G_Y(1-x)f(x,t)dx\\ &= -\int_0^1 x\lambda_{dim}(x)f(x,t) + c_2 - c_2\int_0^1 G_Y(1-x)f(x,t)dx - c_2 M_X(t)+\\ &+ c_2\int_0^1 xG_Y(1-x)f(x,t)dx - c_1\int_0^1 xG_Y(1-x)f(x,t)dx\\ &= -\int_0^1 x\lambda_{dim}(x)f(x,t) + c_2 - c_2\int_0^1 (1-x)f(x,t)dx - c_2 M_X(t)+\\ &+ c_2\int_0^1 x(1-x)f(x,t)dx - c_1\int_0^1 x(1-x)f(x,t)dx.\end{aligned}$$

Performing the calculations we have:

$$\frac{dM_X(t)}{dt} = -\int_0^1 x\lambda_{dim}(x)f(x,t)dx + (c_2-c_1)M_X(t) + (c_1-c_2)M_X^{(2)}.$$

We need again to request $c_1 = c_2$ to get a closed-form equation for the first moment, from which it follows:

$$\frac{dM_X(t)}{dt} = -\int_0^1 x\lambda_{dim}(x)f(x,t).$$

Under the further assumption that λ_{dim} is a constant in the range between 0 and 1, we get:

$$\frac{dM_X(t)}{dt} = -\lambda_{dim}M_X(t).$$

Therefore we reach the following equation in closed form for the mean of the knowledge:

$$M_X(t) = M_X(0)e^{-\lambda_{dim}t}.$$

This result basically tells us that on average, due to forgetfulness, agents tend to completely lose their competence (because $M_X \to 0$ for $t \to +\infty$). More generally, the whole population completely loses its competence, since, if the average of a random variable X, which is limited between 0 and 1, tends to 0, we have that the random variable X itself will tend to become deterministically null. To prove this, it is useful to notice that, since x is in the interval between 0 and 1, then $x^2 \leq x$:

$$0 \leq \mathbb{E}[X^2] = \int_0^1 x^2 f(x,t)dx \leq \int_0^1 xf(x,t)dx = M_X(t).$$

This tells us that, if $M_X \to 0$, then $\mathbb{E}[X^2]$ is also forced to go to zero.
To conclude it is necessary to show that the variance also tends to zero, since an identically zero random variable must necessarily have both mean and variance equal to zero:

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \longrightarrow 0.$$

Chapter 3

Model without stochastic fluctuation

3.1 Study of energy without stochastic fluctuation and forgetfulness

Suppose now to omit the forgetting process. If we assume $c_1 = c_2$, as derived in 2.3, the mean remains constant and equal to its initial value, viz. $M_X(t) = M_X(0)$.
In this chapter, we neglect the stochastic fluctuation by setting $D(x) = 0$. Under these hypotheses, we are going to study the variation of the energy of the system, i.e., of the second moment, by setting $\varphi(x) = x^2$ in (2.5):

$$\frac{d}{dt}\int_0^1 x^2 f(x,t)dx = \langle \int_{[0,1]^2} [(x^*)^2 - x^2] f(x,t) g(y) dx dy \rangle. \tag{3.1}$$

Therefore, we define the *post-interaction competence* x^* of the simplified dynamics:

$$x^* = x + \lambda(x,y), \tag{3.2}$$

with $\lambda(x,y)$ defined as in (2.2).

We first verify that, under the interaction rule (3.2), we obtain a value of x^* in the range of interest. To have x^* greater than or equal to 0, it is sufficient to verify that this is true for the smallest value of $\lambda(x,y)$, since, if it holds for this value, it will be verified also for larger values. Hence, by replacing $\lambda(x,y)$ with the value $-c_1 x$, we find the following condition that makes $x^* \geq 0$:

$$x - c_1 x \geq 0 \iff (1-c_1)x \geq 0 \iff c_1 \leq 1,$$

given that x is, by definition, greater than 0.
To ensure that $x^* \leq 1$, it is enough to verify the condition with the largest value of $\lambda(x,y)$. By replacing $\lambda(x,y)$ with the value $c_2(1-x)$, we find the following constraint that makes $x^* \leq 1$:

$$x + c_2(1-x) \leq 1 \iff (1-x)(c_2-1) \leq 0 \iff c_2 \leq 1,$$

considering that x, by definition, is less then 1.
Both conditions are always satisfied since c_1 and c_2 stay in the interval [0,1], because they represent, in a certain way, the percentage of knowledge that is respectively lost and gained in the interaction. We can therefore proceed with the study of the second moment.

From (3.2) we obtain:

$$(x^*)^2 - x^2 = 2\lambda(x,y)x + \lambda^2(x,y) = \begin{cases} 2c_2x(1-x) + c_2^2(1-x)^2 & \text{if} \quad x \geq 1-y \\ -2c_1x^2 + c_1^2x^2 & \text{if} \quad x < 1-y. \end{cases}$$

Replacing these values in (3.1) we note that, since no stochastic fluctuation term appears, it is possible to omit the brackets $\langle\cdot\rangle$:

$$\begin{aligned} \frac{dM^{(2)}(t)}{dt} &= \int_0^1 \int_{1-x}^1 2c_2x(1-x) + c_2^2(1-x)^2 g(y)f(x,t)dydx + \\ &+ \int_0^1 \int_0^{1-x} (-2c_1x^2 + c_1^2x^2)g(y)f(x,t)dydx. \end{aligned}$$

As we did in the previous chapters, we define G_Y the cumulative distribution function of Y, obtaining:

$$\begin{aligned} \frac{dM^{(2)}(t)}{dt} &= \int_0^1 [(c_2^2 - 2c_2)x^2 + 2x(c_2 - c_2^2) + c_2^2][1 - G_Y(1-x)]f(x,t)dx + \\ &+ \int_0^1 (-2c_1x^2 + c_1^2x^2)G_Y(1-x)f(x,t)dx. \end{aligned}$$

Let's impose $c_1 = c_2$ and again exploit the property of probability density.

$$\begin{aligned} \frac{dM^{(2)}(t)}{dt} &= \int_0^1 [(c_2^2 - 2c_2)x^2 + 2x(c_2 - c_2^2) + c_2^2]f(x,t)dx - \int_0^1 [(c_2^2 - 2c_2)x^2 + \\ &+ 2x(c_2 - c_2^2) + c_2^2]G_Y(1-x)f(x,t)dx \\ &+ \int_0^1 (-2c_2x^2 + c_2^2x^2)G_Y(1-x)f(x,t)dx. \end{aligned}$$

Simplifying the terms and assuming that the distribution, which governs the degree of falsity of the news, is a uniform distribution over the interval [0,1]:

$$\begin{aligned} &= \int_0^1 [(c_2^2 - 2c_2)x^2 + 2x(c_2 - c_2^2) + c_2^2 - 2x(c_2 - c_2^2) - c_2^2 + 2x^2(c_2 - c_2^2) + c_2^2x)]f(x,t)dx \\ &= -c_2^2 \int_0^1 x^2 f(x,t)dx + c_2^2 \int_0^1 xf(x,t)dx = c_2^2[M_X - M_X^{(2)}]. \end{aligned} \tag{3.3}$$

Based on what we achieved in 2.3, if $c_1 = c_2$ and $\lambda_{dim} = 0$ then $M_X(t) = M_X(0)\ \forall t$. This allows us to derive the following formula for the second moment, imposing an initial condition $M_X^{(2)}(0)$:

$$M_X^{(2)} = M_X(0) + (M_X^{(2)}(0) - M_X(0))e^{-c_2^2 t}.$$

We note that, for $t \to \infty$, we have that the second term will go to zero and $M_X^{(2)}$ will tend to $M_X(0)$. This result tells us that, unlike in many other models where stochastic fluctuation is not considered, the variance does not tend to vanish.

3.2 Study of the asymptotic behavior through the Fokker-Planck equation

To study the asymptotic behavior (in time) of the solutions of the Boltzmann equation, we use an asymptotic procedure which is known by the name of *limit of quasi-invariant interactions* and which allows us to pass from the Boltzmann equation to the *Fokker-Planck equation*, a kinetic equation simpler to handle. If the interactions produce small changes, the equilibria of the Fokker-Planck equation coincide with those of the Boltzmann equation. Then, we can focus on the study of the simpler equation, remembering however that, if we are not under the hypothesis of the quasi-invariant interactions, usually the two equations behave differently.
As we mentioned, the limit of quasi-invariant interactions is based on the fact that, near the equilibrium, the interactions make small contributions to the variation of the microscopic state of the particles. To pass to the quasi-invariant regime, we need to lead both the deterministic and the stochastic part (which in this case is null) of order ϵ, scaling $\lambda(x, y)$ as follows:

$$\lambda(x, y) \to \epsilon\lambda(x, y),$$

where $\epsilon > 0$ is sufficiently small.
To study the trend near equilibrium, it is necessary to consider a longer time scale by setting $\tau = \epsilon t$. Considering these rescalings, (2.5) becomes:

$$\frac{d}{d\tau}\int_0^1 \varphi(x)f(x,\tau)dx = \frac{1}{\epsilon}\int_{[0,1]^2} \langle\varphi(x') - \varphi(x)\rangle f(x,\tau)g(y)dxdy, \tag{3.4}$$

in which, x' indicates the scaled post-interaction knowledge, defined in the following way:

$$x' = x + \epsilon\lambda(x, y).$$

Let's introduce $\varphi \in C^3([0,1])$ and $\tilde{x} \in (min\{x, x'\}, max\{x, x'\})$ to be able to write the Taylor expansion up to order 3 (with Peano remainder) of $\varphi(x')$ centered in x:

$$\begin{aligned}\langle\varphi(x') - \varphi(x)\rangle &= \langle\varphi(x) + \varphi'(x)\epsilon\lambda(x,y) + \frac{1}{2}\varphi''(x)\epsilon^2\lambda^2(x,y) + \frac{1}{6}\varphi'''(\tilde{x})\epsilon^3\lambda^3(x,y) - \varphi(x)\rangle \\ &= \varphi'(x)\epsilon\lambda(x,y) + \frac{1}{2}\varphi''(x)\epsilon^2\lambda^2(x,y) + \frac{1}{6}\varphi'''(\tilde{x})\epsilon^3\lambda^3(x,y).\end{aligned}$$

Replacing this result in (3.4), we obtain:

$$\frac{d}{d\tau}\int_0^1 \varphi(x)f(x,\tau)dx = \int_{[0,1]^2}\bigg[\varphi'(x)\lambda(x,y)+\frac{1}{2}\varphi''(x)\epsilon\lambda^2(x,y)+ \\ +\frac{1}{6}\varphi'''(\tilde{x})\epsilon^2\lambda^3(x,y)\bigg]f(x,\tau)g(y)dxdy.$$

Passing to the limit of quasiinvariant interactions, $\epsilon \to 0^+$:

$$\frac{d}{d\tau}\int_0^1 \varphi(x)f(x,\tau)dx = \int_0^1 \varphi'(x)\bigg(\int_0^1 \lambda(x,y)g(y)dy\bigg)f(x,\tau)dx.$$

If we further assume that the test function has compact support, i.e. $\varphi \in C_C^3([0,1])$, integrating by parts the term on the right of the equal, we obtain:

$$\frac{d}{d\tau}\int_0^1 \varphi(x)f(x,\tau)dx = -\int_0^1 \varphi(x)\partial_x\bigg[\bigg(\int_0^1 \lambda(x,y)g(y)dy\bigg)f(x,\tau)\bigg]dx.$$

Due to the arbitrariness of φ, we get the following strong form:

$$\partial_\tau f = -\partial_x\bigg[\bigg(\int_0^1 \lambda(x,y)g(y)dy\bigg)f\bigg], \tag{3.5}$$

then, we proceed with the computation of the integral in (3.5):

$$\begin{aligned}\int_0^1 \lambda(x,y)g(y)dy &= \int_{1-x}^1 c_2(1-x)g(y)dy - \int_0^{1-x} c_1 x g(y)dy \\ &= c_2(1-x)[1-G_Y(1-x)] - c_1 x G_Y(1-x).\end{aligned}$$

If we consider Y uniformly distributed in the interval [0,1]:

$$\int_0^1 \lambda(x,y)g(y)dy = c_2(1-x)x - c_1x(1-x) = (c_2-c_1)x(1-x).$$

Hence (3.5) becomes:

$$\partial_\tau f = -\partial_x[(c_2-c_1)x(1-x)f].$$

It is immediate to see that, if $c_1 = c_2$, we have $\partial_\tau f = 0$, therefore the density of knowledge remains unchanged over time. This is because the fraction of knowledge gained coincides with the one that is lost.
The case $c_1 \neq c_2$ is more interesting. The stationary distribution, which we denote by $f^\infty(x)$, is the limit for $\tau \to \infty$ of $f(x,\tau)$ and must not depend on time. To obtain $f^\infty(x)$, it is sufficient to derive the equilibria by setting $\partial_\tau f = 0$:

$$(c_2-c_1)x(1-x)f^\infty = 0.$$

Under the hypothesis of $c_2 - c_1 \neq 0$, it is possible to simplify the term containing the coefficients, obtaining:

$$x(1-x)f^{\infty} = 0. \tag{3.6}$$

Surely, the product is null if $f^{\infty} = 0$, but this is not acceptable as the integral on [0,1] is different from 1. The acceptable solutions are given by a linear combination of the Dirac delta centered in 0, $\delta(x)$, and the Dirac delta centered at 1, $\delta(x-1)$.
The Dirac delta centered in x_0 is defined as follows:

$$\int_{-\infty}^{+\infty} \delta(x-x_0)\phi(x) = \phi(x_0).$$

Therefore, we derive the following expression for the stationary distribution:

$$f^{\infty}(x) = a\delta(x) + b\delta(x-1). \tag{3.7}$$

In fact, if $x = 0$ or $x = 1$, the equation (3.6) will be verified since $x(1-x)$ will assume value 0, while, for all other values, both $\delta(x)$ and $\delta(x-1)$, for the definition of the delta, will vanish again, returning (3.6) equal to zero.
This result tells us that, the model without diffusion, tends to extreme levels of knowledge for a long time, returning a fraction of individuals with maximum knowledge and the remaining fraction of individuals with zero knowledge. As a consequence, for long t, there will be no agents with an opinion in the range of (0,1).
Since $f^{\infty}(x)$ is a probability density, we need to impose that its integral, over the definition interval of x, must be equal to 1:

$$\int_0^1 f^{\infty}(x)dx = a + b = 1.$$

In other words, the density, that we are looking for, must be a *convex combination* of the Dirac delta centered in 0 and the Dirac delta centered in 1, and therefore, if we indicate the coefficient of the first delta with a, as a consequence, b must be equal to $1-a$.

From the theory, we know that, since this is the solution of the Fokker-Planck equation, it will coincide with the solution of the Boltzmann equation in the quasi- invariant regime. However, we can try to see if (3.7) represents an exact stationary distribution for the Boltzmann equation (3.4) for each pair of coefficients c_1 and c_2. Substituting (3.7) in (2.5) we have:

$$\frac{d}{dt}\int_0^1 f^{\infty}(x)\varphi(x)dx = \langle \int_{[0,1]^2} (\varphi(x^*) - \varphi(x))(a\delta(x) + b\delta(x-1))g(y)dxdy\rangle. \tag{3.8}$$

Exploiting the definition of x^* in (3.2):

$$\frac{d}{dt}\int_0^1 f^\infty(x)\varphi(x)dx = \int_{[0,1]^2}[\varphi(x+\lambda(x,y))-\varphi(x)][a\delta(x)+b\delta(x-1)]g(y)dxdy \tag{3.9}$$

$$= \int_0^1\int_{1-x}^1[\varphi(x+c_2(1-x))-\varphi(x)][a\delta(x)+b\delta(x-1)]g(y)dydx+$$

$$+\int_0^1\int_0^{1-x}[\varphi(x-c_1x)-\varphi(x)][a\delta(x)+b\delta(x-1)]g(y)dydx$$

$$=\int_0^1\{[\varphi(x+c_2(1-x))-\varphi(x)][1-G_Y(1-x)]\}[a\delta(x)+b\delta(x-1)]dx+$$

$$+\int_0^1\{[\varphi(x-c_1x)-\varphi(x)]G_Y(1-x)\}[a\delta(x)+b\delta(x-1)]dx$$

$$=a[\varphi(c_2)-\varphi(0)][1-G_Y(1)]+b[\varphi(1)-\varphi(1)][1-G_Y(0)]+$$

$$+a[\varphi(0)-\varphi(0)]G_Y(1)+b[\varphi(1-c_1)-\varphi(1)]G_Y(0)=0,$$

since the cumulative distribution function of a random variable with support [0,1] in 0 is null, and in 1 is equal to 1.
We have proved that we are in the special case in which f^∞ is an exact stationary distribution for the Boltzmann equation for any choice of c_1 and c_2.
It is important to highlight that this result does not tell us that (3.7) constitutes the only solution of (2.5), but that it is one of them.
In the case $c_1 = c_2$ we can also impose the conservation of the first moment, consistent with what we obtained in Section 2.3, to derive the exact value of a. Hence:

$$M_X(0)=\int_0^1 xf^\infty(x)dx = a\int_0^1 x\delta(x)\,dx + (1-a)\int_0^1 x\delta(x-1)dx = 1-a. \tag{3.10}$$

Going to replace (3.10) in (3.7), we have our stationary distribution for competence in absence of stochastic fluctuation and under the assumption that c_1 and c_2 are equal:

$$f^\infty(x) = (1-M_X(0))\delta(x)+M_X(0)\delta(x-1). \tag{3.11}$$

Thus, we have a *polarization* towards the two extreme levels of competence, the maximum one and the null one, in which the fraction of individuals with maximum knowledge is given by the value of $M_X(0)$, which is reasonable since individuals with a null competence will have no influence in the calculation of the average. We note that f^∞ does not depend on c_1 and c_2 in the form, so the two constants will only affect the time needed to

reach the stationary distribution.
The solution (3.11) shows that, if the initial population has the maximum possible knowledge, then they will not lose competence, conversely, if initially all agents are totally ignorant, then the level of knowledge will remain null. As obtained in 3.1, under the assumption that $c_1 = c_2$, the energy is also equal to $M_X(0)$ for $t \to \infty$, therefore we can check that this is verified by our f^∞:

$$M_X^{(2)} = \int_0^1 x^2 f^\infty(x)dx = (1 - M_X(0)) \int_0^1 x^2 \delta(x)\, dx + M_X(0) \int_0^1 x^2 \delta(x-1)dx = M_X(0).$$

Chapter 4

Model with stochastic fluctuation

4.1 Study of the dynamics with stochastic fluctuation

In chapter 3, a simplified model was carried out, which does not take into account the stochastic fluctuation. In this chapter, we consider the stochastic fluctuation by choosing $D(x) = \min\{x, 1-x\}$ as the non-zero diffusion coefficient.
First of all, it is necessary to verify that the choice of the above diffusion coefficient allows having $x^* \in [0,1]$.
Therefore, we define the post-interaction knowledge x^* as in (2.1), and we verify under what conditions on η the competence x^* assumes values greater than 0 and less than 1.
To have x^* greater than or equal to zero, it is sufficient to verify that this holds for the smallest values assumed by $\lambda(x, y)$, that is $-c_1 x$, since, if for this value the inequality is satisfied, then, necessarily, it will be satisfied for bigger values of $\lambda(x, y)$. Thus we get:

$$x - c_1 x + D(x)\eta \geq 0 \iff D(x)\eta \geq (c_1 - 1)x. \tag{4.1}$$

Conversely, to get $x^* \leq 1$ we can impose the condition for the largest values assumed by $\lambda(x, y)$, that is $c_2(1-x)$:

$$x + c_2(1-x) + D(x)\eta \leq 1 \iff D(x)\eta \leq (1-x)(1-c_2). \tag{4.2}$$

If we combine (4.1) and (4.2), we obtain:

$$(c_1 - 1)x \leq D(x)\eta \leq (1-x)(1-c_2), \tag{4.3}$$

from which we conclude, replacing the two possible values of $D(x)$, that we have acceptable values for x^* under the following condition for the random variable:

$$c_1 - 1 \leq \eta \leq 1 - c_2. \tag{4.4}$$

Assuming that (4.4) holds, we can proceed with the study of our dynamics.

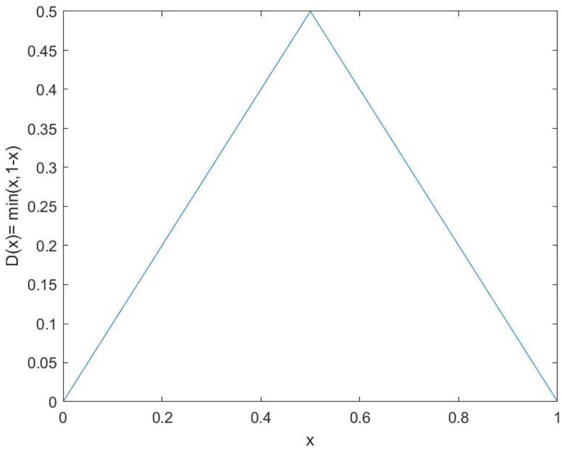

Figure 4.1. Diffusion coefficient: $D(x) = \min\{x,1-x\}$.

From (2.1) we reach:

$$\begin{aligned}\langle (x^*)^2 - x^2 \rangle &= \lambda^2(x,y) + D^2(x)\langle \eta^2 \rangle + 2x\lambda(x,y) + D(x)\langle \eta \rangle x + 2\lambda(x,y)D(x)\langle \eta \rangle \\ &= \lambda^2(x,y) + D^2(x)\langle \eta^2 \rangle + 2x\lambda(x,y), \end{aligned} \tag{4.5}$$

in which the last passage derives from the fact that, by definition of stochastic fluctuation, $\langle \eta \rangle = 0$ and, as a consequence, $\langle \eta^2 \rangle$ coincides with $\text{Var}(\eta) = \sigma^2$, which we impose different from zero, to not fall into the deterministically null case.
Replacing (4.5) in (3.1) we achieve:

$$\frac{dE(t)}{dt} = \int_{[0,1]^2} [\lambda^2(x,y) + 2x\lambda(x,y) + D^2(x)\sigma^2] g(y) f(x,t) dy dx,$$

in which, to distinguish from the previous case, we use $E(t)$ to indicate the energy.
We immediately notice that, the only difference with respect to the case without stochastic fluctuation, lies in the term in which the diffusion coefficient appears, consequently, for the linearity of the integral, we can focus only on that:

$$\int_{[0,1]^2} D^2(x)\sigma^2 g(y) f(x,y) dy dx.$$

Since the diffusion coefficient does not depend on y, we can apply the property of the integral of the probability distributions, reducing our two-dimensional integral to a mono-dimensional one.
A further observation concerns the definition of $D(x)$, that we can write as:

$$D(x) = \min\{x,1-x\} = \begin{cases} x & \text{if} \quad 0 \le x < \frac{1}{2} \\ 1-x & \text{if} \quad \frac{1}{2} \le x \le 1. \end{cases}$$

Therefore, we can proceed with the calculation of the integral:

$$\sigma^2\int_0^1 D^2(x)f(x,y)dx = \sigma^2\Big[\int_0^{\frac{1}{2}} x^2 f(x,t)dx + \int_{\frac{1}{2}}^1 f(x,t)dx - 2\int_{\frac{1}{2}}^1 xf(x,t)dx + \int_{\frac{1}{2}}^1 x^2 f(x,t)dx\Big].$$

Since we have not fixed a particular probability distribution for the random variable X, we are not able to explicitly evaluate the integrals $\int_{\frac{1}{2}}^1 xf(x,t)dx$ and $\int_{\frac{1}{2}}^1 x^2 f(x,t)dx$. The only thing we can do is looking for an estimate.
To find an estimate from below, we recall the fact that x cannot be greater than 1 and that the integral of the density on a subset of [0,1] will necessarily be less than or equal to 1, obtaining the following lower bound:

$$\begin{aligned}\sigma^2\int_0^1 D^2(x)f(x,y)dx =& \sigma^2\Big[E(t) + \int_{\frac{1}{2}}^1 f(x,t)dx - 2\int_{\frac{1}{2}}^1 xf(x,t)dx\Big] \geq \sigma^2\Big[E(t) - \int_{\frac{1}{2}}^1 f(x,t)dx\Big] \geq \\ &\geq \sigma^2(E(t)-1).\end{aligned} \tag{4.6}$$

Let us now focus on the upper bound. Since x varies between 0 and 1, then $x \geq x^2$ and also $\int_{\frac{1}{2}}^1 f(x,t)dx \leq \int_0^1 f(x,t)dx = 1$, from which it follows:

$$\begin{aligned}\sigma^2\int_0^1 D^2(x)f(x,y)dx \leq& \sigma^2\Big[\int_0^{\frac{1}{2}} x^2 f(x,t)dx + \int_{\frac{1}{2}}^1 x^2 f(x,t)dx - 2\int_{\frac{1}{2}}^1 x^2 f(x,t)dx + 1\Big] = \\ &= \sigma^2\Big[E(t) - 2\int_{\frac{1}{2}}^1 x^2 f(x,t)dx + 1\Big] \leq \sigma^2[E(t)+1].\end{aligned} \tag{4.7}$$

Recalling now that in Chapter 3 we found:

$$\frac{dM^{(2)}(t)}{dt} = c_2^2[M_X(0) - M_X^{(2)}(t)]. \tag{4.8}$$

Combining (4.8) with the estimates in (4.6) and (4.7), we have the following bounds for the variation of the energy over time:

$$E(t)(\sigma^2 - c_2^2) + c_2^2 M_X(0) - \sigma^2 \leq \frac{dE(t)}{dt} \leq E(t)(\sigma^2 - c_2^2) + c_2^2 M_X(0) + \sigma^2.$$

It is easy to see that, if $\sigma^2 > c_2^2$, then necessarily $E(t)$ will explode. Under the opposite hypothesis $\sigma^2 \leq c_2^2$, we know that the energy will be bounded from below and from above. Let's focus first on the lower bound by solving the following differential equation:

$$\dot{E} = E(\sigma^2 - c_2^2) + c_2^2 M_X(0) - \sigma^2,$$

from which we obtain:

$$E(t) = \frac{c_2^2 M_X(0) - \sigma^2}{c_2^2 - \sigma^2} + \left(E(0) - \frac{c_2^2 M_X(0) - \sigma^2}{c_2^2 - \sigma^2}\right)e^{(\sigma^2 - c_2^2)t}. \tag{4.9}$$

Let's move on to the analysis of the upper bound:

$$\dot{E} = E(\sigma^2 - c_2^2) + c_2^2 M_X(0) + \sigma^2,$$

from which we get:

$$E(t) = \frac{c_2^2 M_X(0) + \sigma^2}{c_2^2 - \sigma^2} + \left(E(0) - \frac{c_2^2 M_X(0) + \sigma^2}{c_2^2 - \sigma^2}\right) e^{(\sigma^2 - c_2^2)t}. \tag{4.10}$$

If we combine the results obtained in (4.9) and (4.10) and make the limit for $t \to \infty$ we get that $e^{(\sigma^2 - c_2^2)t} \to 0$, as we are under the hypothesis $\sigma^2 - c_2^2 \leq 0$ so, as a consequence:

$$\frac{c_2^2 M_X(0) - \sigma^2}{c_2^2 - \sigma^2} \leq E^\infty \leq \frac{c_2^2 M_X(0) + \sigma^2}{c_2^2 - \sigma^2}.$$

If we consider $\sigma^2 = 0$, i.e. the case in which the stochastic fluctuation is zero, we get $M_X(0) \leq E^\infty \leq M_X(0)$ and therefore $E^\infty = M_X(0)$, which is consistent with the result obtained in Chapter 3.

4.2 Study of the asymptotic behavior through the Fokker Planck equation

To derive the behavior of the solution of the Boltzmann-type equation for long times, and therefore close to equilibrium, we pass to the *quasi-invariant regime*, in which the interactions produce small changes in the state of the particles. In order to have little changes, we must perform the following scaling: make the deterministic part of the interaction rule and the variance of the stochastic fluctuation of order ϵ (small and positive) and change the time to have a larger scale. It must be notice that, to make the variance of the fluctuation of order ϵ, we need to multiply the stochastic fluctuation by $\sqrt{\epsilon}$.
Then, we consider the scaled post-interaction knowledge defined below:

$$x' = x + \epsilon\lambda(x, y) + \sqrt{\epsilon} D(x) Z\sigma,$$

where Z represents the "standardized" stochastic fluctuation, such that $\langle Z \rangle = 0$ and $\langle Z^2 \rangle = \langle Z \rangle^2 = 1$. As a consequence, we can write η as σZ.
Now let's scale the time by setting $\tau = \epsilon t$. With the aforementioned scaling, the Boltzmann equation (in weak form) becomes as in (3.4).

Let's introduce $\varphi \in C^3([0,1])$ and analyze the Taylor expansion of $\varphi(x')$ centered in x:

$$\begin{aligned}
\langle \varphi(x') - \varphi(x) \rangle &= \langle \varphi(x) + \varphi'(x)[\epsilon\lambda(x,y) + \sqrt{\epsilon}\sigma D(x) Z] + \frac{1}{2}\varphi''(x)[\epsilon\lambda(x,y) + \sqrt{\epsilon}\sigma D(x) Z]^2 + \\
&+ \frac{1}{6}\varphi'''(\tilde{x})[\epsilon\lambda(x,y) + \sqrt{\epsilon}\sigma D(x) Z]^3 - \varphi(x) \rangle \\
&= \varphi'(x)\epsilon\lambda(x,y) + \frac{1}{2}\varphi''(x)\epsilon^2\lambda^2(x,y) + \frac{1}{2}\varphi''(x)\epsilon\sigma^2 D^2(x)\langle Z^2\rangle + \\
&+ \varphi''(x)\epsilon^{\frac{3}{2}}\sigma\lambda(x,y)D(x)\langle Z\rangle + \frac{1}{6}\varphi'''(\tilde{x})[\epsilon^3\lambda^3(x,y) + 3\epsilon^2\lambda(x,y)\sigma^2 D^2(x) + \\
&+ 3\epsilon^{\frac{5}{2}}\lambda^2(x,y)\sigma D(x)\langle Z\rangle + \epsilon^{\frac{3}{2}}\sigma^3 D^3(x)\langle Z^3\rangle] \\
&= \varphi'(x)\epsilon\lambda(x,y) + \frac{1}{2}\varphi''(x)\epsilon^2\lambda^2(x,y) + \frac{1}{2}\varphi''(x)\epsilon\sigma^2 D^2(x) + \\
&+ \frac{1}{6}\varphi'''(\tilde{x})[\epsilon^3\lambda^3(x,y) + 3\epsilon^2\lambda(x,y)\sigma^2 D^2(x) + \epsilon^{\frac{3}{2}}\sigma^3 D^3(x)\langle Z^3\rangle],
\end{aligned}$$

replacing the result in (3.4), we have:

$$\begin{aligned}
\frac{d}{d\tau}\int_0^1 \varphi(x) f(x,t)dx = \int_0^1\int_0^1 [\varphi'(x)\lambda(x,y) + \frac{1}{2}\varphi''(x)\epsilon\lambda^2(x,y) + \frac{1}{2}\varphi''(x)\sigma^2 D^2(x) + \\
+ \frac{1}{6}\varphi'''(\tilde{x})[\epsilon^2\lambda^3(x,y) + 3\epsilon\lambda(x,y)\sigma^2 D^2(x) + \sqrt{\epsilon}\sigma^3 D^3(x)\langle Z^3\rangle] f(x,\tau) g(y) dx dy.
\end{aligned}$$

Let us assume that η has finite third moment, i.e., $|\langle \eta^3\rangle| < +\infty$ and, consequently, also $|\langle Y^3\rangle| < +\infty$. By passing to the limit $\epsilon \to 0^+$ (limit of quasi-invariant interactions), we achieve the global behavior of the Boltzmann equation for small changes:

$$\frac{d}{d\tau}\int_0^1 \varphi(x) f(x,\tau)dx = \int_0^1\int_0^1 [\varphi'(x)\lambda(x,y) + \frac{\sigma^2}{2}\varphi''(x)D^2(x)] f(x,\tau) g(y) dx dy,$$

exploiting the fact that $\lambda(x,y)$ and $D(x)$ are bounded in the range [0,1] to conclude that all other terms to the right of the equal tend to zero. Rearranging the terms we get:

$$\frac{d}{d\tau}\int_0^1 \varphi(x) f(x,\tau)dx = \int_0^1 \varphi'(x)\left(\int_0^1 \lambda(x,y)g(y)dy\right) f(x,\tau)dx + \frac{\sigma^2}{2}\int_0^1 \varphi''(x)D^2(x)f(x,\tau)dx.$$

If we assume that φ has also a compact support in [0,1], i.e., $\varphi \in C^3_C([0,1])$, by integrating by parts we obtain:

$$= -\int_0^1 \varphi(x)\partial_X\left[\left(\int_0^1 \lambda(x,y)g(y)dy\right) f\right] + \frac{\sigma^2}{2}\int_0^1 \varphi(x)\partial_X^2[D^2(x)f].$$

Due to the arbitrariness of $\varphi(x)$, we can pass to the strong form:

$$\partial_\tau f = -\partial_X\left[\left(\int_0^1 \lambda(x,y)g(y)dy\right) f\right] + \frac{\sigma^2}{2}\partial_X^2[D^2(x)f]. \tag{4.11}$$

The equation (4.11) is called the *Fokker-Planck equation*. From it we can explicitly derive the stationary profile of the asymptotic distribution of competence $f^\infty(x)$, which is obtained by making τ tends to ∞. The distribution $f^\infty(x)$ must be an equilibrium configuration, hence it has to solve (4.11) with $\partial_\tau f = 0$.
Then, by exploiting the results obtained in Section 3.2, we achieve the following ODE in the independent variable x:

$$\partial_X \Big[-(c_2 - c_1)x(1-x)f^\infty + \frac{\sigma^2}{2}\partial_X(D^2(x)f^\infty)\Big] = 0.$$

The term inside the square brackets represents the *flow of the Fokker-Planck equation*. Since at equilibrium the flow is not only constant, but in particular it is null, we solve:

$$-(c_2 - c_1)x(1-x)f^\infty + \frac{\sigma^2}{2}\partial_X(D^2(x)f^\infty) = 0. \tag{4.12}$$

If $c_1 = c_2$, the expression becomes:

$$\partial_X(D^2(x)f^\infty) = 0,$$

which is verified if f^∞ vanishes at the points where $D^2(x) \neq 0$. Since $D^2(x)$ is null only in $x = 0$ and $x = 1$, we can define f^∞ as a linear combination of the Dirac delta centered in 0 and the Dirac delta centered in 1, and, because also in this case the mean remains constant, we can proceed in a similar way to Section 3.2, obtaining the stationary distribution in (3.11).

If we focus on the case $c_1 \neq c_2$, the fact that $D(x)$ is piecewise-defined can create complications.
Let's try to choose a different diffusion coefficient:

$$D(x) = \sqrt{x(1-x)}, \tag{4.13}$$

showed in Figure 4.2.

We note that this diffusion coefficient does not satisfy the condition (4.3) to have x^* in the domain of interest. In fact, if we replace (4.13) in (4.3), we reach:

$$\frac{(c_1-1)x}{\sqrt{x(1-x)}} \leq \eta \leq \frac{(1-x)(1-c_2)}{\sqrt{x(1-x)}},$$

in which, if $x \to 1$ then the upper bound tends to 0, thus, η should be defined only for negative or null values, but this is not possible because η has zero mean and positive variance. Therefore, there is no way to satisfy the constraint. Consequently, it is not possible to study the Boltzmann equation with a diffusion coefficient of this type. However, as shown in [10], we can study the Fokker-Planck equation thanks to the fact that it exists a family of functions that at the limit returns the Fokker-Planck equation for $D(x) = \sqrt{x(1-x)}$.

Replacing the new diffusion coefficient in (4.12), we obtain:

$$-(c_2 - c_1)x(1-x)f^\infty + \frac{\sigma^2}{2}\partial_X(x(1-x)f^\infty) = 0.$$

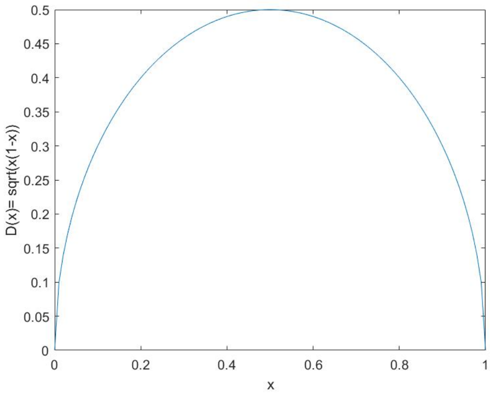

Figure 4.2. Diffusion coefficient: $D(x) = \sqrt{x(1-x)}$.

If we define $g(x) := x(1-x)f^{\infty}$, the previous equation becomes:

$$-(c_2 - c_1)g(x) + \frac{\sigma^2}{2}\partial_X g(x) = 0.$$

Rearranging the terms, we get:

$$D[ln(g(x)] = \frac{\partial_X g(x)}{g(x)} = (c_2 - c_1)\frac{2}{\sigma^2}.$$

By integrating on both sides:

$$ln(g(x)) = \frac{2}{\sigma^2}(c_2 - c_1)x.$$

Applying the exponential function, we achieve the following expression for $g(x)$:

$$g(x) = Ke^{\frac{2}{\sigma^2}(c_2-c_1)x}, \tag{4.14}$$

in which K indicates the normalization constant. From (4.14) we find out the following definition of f^{∞}:

$$f^{\infty} = \frac{g(x)}{x(1-x)} = \frac{Ke^{\frac{2}{\sigma^2}(c_2-c_1)x}}{x(1-x)}.$$

We note that f^{∞} is a solution of the Fokker-Planck equation, but it cannot be a probability density, since the integral on the interval [0,1] is different from 1.
However, we can observe that a generic diffusion coefficient $D(x)$ must necessarily assume a null value for $x = 0$ and $x = 1$, otherwise, the fluctuation returns values for the post-interaction competence x^* that are not acceptable. This observation makes it clear that, if we consider $f^{\infty} = a\delta(x) + b\delta(x-1)$ and going to replace it in (4.12), we get:

$$-(c_2 - c_1)x(1-x)[a\delta(x) + (1-a)\delta(x-1)] + \frac{\sigma^2}{2}\partial_X[D^2(x)(a\delta(x) + (1-a)\delta(x-1))] = 0,$$

which is necessarily null, since for $x = 0$ or $x = 1$ both the term $(c_2 - c_1)x(1-x)$ and $D^2(x)$ become zero, while, for intermediate values, the Dirac deltas will assume a null value. So, even in the presence of diffusion, for any choice of an admissible diffusion coefficient, $f^\infty = a\delta(x) + b\delta(x-1)$ is a solution of the Fokker-Planck equation.
Let us check, as we did for the model without fluctuation, if this probability distribution constitutes, in general, an asymptotic solution of the Boltzmann-type equation. Substituting (2.1) into (2.5), we obtain:

$$\frac{d}{dt}\int_0^1 f^\infty(x)\varphi(x)dx = \langle \int_{[0,1]^2} (\varphi(x + \lambda(x,y) + D(x)\eta) - \varphi(x))f^\infty(x)g(y)dxdy\rangle. \quad (4.15)$$

Choosing f^∞ equal to (3.7):

$$\begin{aligned}
=&\langle \int_0^1 \int_{1-x}^1 [\varphi(x + c_2(1-x) + D(x)\eta) - \varphi(x)][a\delta(x) + b\delta(x-1)]g(y)dydx\rangle +\\
&+ \langle \int_0^1 \int_0^{1-x} [\varphi(x - c_1 x + D(x)\eta) - \varphi(x)][a\delta(x) + b\delta(x-1)]g(y)dydx\rangle\\
=&\langle \int_0^1 \{[\varphi(x + c_2(1-x) + D(x)\eta) - \varphi(x)][1 - G_Y(1-x)]\}[a\delta(x) + b\delta(x-1)]dx\rangle +\\
+&\langle \int_0^1 \{[\varphi(x - c_1 x + D(x)\eta) - \varphi(x)]G_Y(1-x)\}[a\delta(x) + b\delta(x-1)]dx\rangle.
\end{aligned}$$

Since $D(0) = D(1) = 0$, we get the same calculations performed for the case without stochastic fluctuation in (3.9) and, consequently, we achieve that, even in the presence of stochastic fluctuation, the definition of the stationary distribution as a convex combination of Dirac deltas concentrated in 0 and 1 is a solution of the Boltzmann equation, for any choice of the diffusion coefficient. Since $\langle\eta\rangle = 0$, the trend of the mean will be identical to the case without diffusion and, therefore, we can reason as in part I and observe that, if $c_1 = c_2$, the mean will be conserved and we can explicitly derive the value of parameters a and b:

$$f^\infty(x) = (1 - M_X(0))\delta(x) + M_X(0)\delta(x-1).$$

In the case $c_1 \neq c_2$, the mean depends on the second moment and we have no way to easily derive a condition for the coefficients.

It might be surprising that the model with diffusion returns a solution with a polarization toward the extreme values, since, often, the results obtained with the inclusion of a fluctuation are less concentrated than in the case without a fluctuation. This is not the case, because the Fokker-Planck diffusion tends to push toward the most extreme values of the defining interval, but, since our model admits a solution for the fluctuation-free case that is already concentrated on 0 and 1, the solution of the model with fluctuation will also tend to stay on those extreme values.

Chapter 5

Numerical illustration with Monte Carlo

In general, the weak form of the differential equations is not used directly to find the solution, anyway it is possible to pass to the strong form and then use a quadrature formula to discretize the integral. However, this approach is usually not used, because the accuracy is not good enough compared to the computational cost involved in solving multidimensional integrals using quadrature formulas, particularly in high dimensions.
To solve the Boltzmann equation numerically, a widely used method is to simulate the stochastic particle system with the *Monte Carlo method.* This method has the characteristic of being very compact and non-intrusive, i.e., the way in which it is implemented does not depend on how the interaction rules that characterize the system are made. Hence, in the case in which someone wants to choose a different interaction rule, it is not necessary to change the algorithm from the beginning.

5.1 Algorithm of Monte Carlo simulation

1. Fix N, the total number of particles to be simulated. Since we want to provide a good reconstruction of the statistical distribution we need many samples. Usually, a good order of magnitude is 10^6. However, in our model the interactions are not between agents but between agent and fake news, hence it is enough to consider 10^4 agents which interact with an equal number of fake news, represented with their level of falsity.

2. Generate an initial sample of N competences, X, and a sample of N degrees of falsehood for news, F. Generating a random sample is itself a complex problem, but there are predefined routines in Matlab that generate a vector of values distributed according to the best-known probability distributions. The values $x_1^0, \dots, x_N^0$ must be sampled from the probability distribution $f_0(x)$ that has to be assigned as the initial condition. In many cases, when the cumulative distribution function of the chosen distribution is known and not particularly difficult to invert, Matlab uses

a method called *Inverse Transform Method*, that consists of sampling from a uniform distribution on the interval [0,1] through a pseudo-random number generation method (e.g., linear congruential generators LCGs), and then obtain the desired sample by applying the inverse of the cumulative distribution function to the uniform sample. In fact, it can be easily shown, by exploiting the monotonicity of the cumulative distribution function and the properties of the uniform distribution, that the sample X thus obtained follows the desired distribution:

$$\mathbb{P}\{X \leq x\} = \mathbb{P}\{F^{-1}(U) \leq x\} = \mathbb{P}\{U \leq F(x)\} = F(x),$$

in which, the fact that U is uniformly distributed was exploited.
It is interesting to note that, for sampling from a normal distribution, since it has a cumulative distribution function that is not particularly easy to invert, Matlab uses ad hoc methods (e.g., the Polar approach and Box-Muller approach).

3. We set $\Delta t > 0$ sufficiently small, so that we have a good approximation of (2.5).

4. We start the iterations for $n = 0, \ldots, N$, where N denotes the number of steps in the time interval analyzed:

   - we randomly permute the elements of the vector X at each iteration to make the sample of agents and the sample of fake news statistically independent, otherwise, the agent will always continue to interact with the same news and the knowledge of the particle will depend only on the value of that news;
   - for each pair, we sample $\Theta \sim Bernoulli(\frac{\Delta t}{\tau})$, obtained from the routine for the Binomial distribution, to see if the pair interacts;
   - we sample the normalized stochastic fluctuation Y from the distribution we prefer. We note that the choice of the stochastic fluctuation $\eta = \sigma Y$ can affect the solution of the Boltzmann equation, but in the quasi-invariant interactions regime the specific law of η is not important, because the stochastic fluctuation fits into the Fokker-Planck equation only through the variance σ^2. Consequently, by choosing a distribution with zero mean and variance σ^2 and under the assumption of the quasi-invariant limit of iterations, the solution will not change;
   - we apply the interaction rule:

     $$x_i^{n+1} = (1-\Theta)x_i^n + \Theta[x_i^n + \lambda(x_i^n, y) + D\sigma Y];$$

   - we create the new vector x^{n+1};
   - we approximate $f(x, t^{n+1})$ (where $t^n = n\Delta t$) with an histogram constructed from the values contained in x^{n+1}.

The computational time depends on two factors: the particle number, which, however, needs to be large to obtain a statistically significant result, and the number of interactions, which depends on Δt. Also, for efficiency reasons, it is preferable to use a vector form instead of implementing particle by particle.

Below we report the code to simulate the quasi-invariant regime for the model with diffusion coefficient equal to $D(x) = \sqrt{x(1-x)}$ and with $N = 10^4$ agents interacting with an identical number of news.
Let us first see the case in which the sample of the degrees of falsity and competences are distributed uniformly over the interval [0,1].

```
N=1e4;
eps=1e-2; % small to be in the quasi-invariant regime
dt=eps;
Tfin=1e3; %final time
nmax=floor(Tfin/dt);
sigma=sqrt(eps);
tau=eps;

X=rand(N,1);
%X= betarnd(2,5,N,1); %mean less than 0.5
F=rand(N,1);
u=mean(X); % if c1=c2 the mean is constant in time

hbar=waitbar(0,'','Name','Iterazioni');
for n=1:nmax
    waitbar(n/nmax,hbar,sprintf('$n$ = %d / %d',n,nmax));
    X=X(randperm(N)); %random permutation of X
    Theta=binornd(1,dt/tau,N,1); % a value for each couple
    Y=sqrt(12)*(-0.5 + rand(N,1)); %mean 0 and variance 1
    eta=sigma*Y;
    c1=0.4;
    c2=0.4;
    diff= X.*(1-X);
    D=sqrt(diff).*(diff>=0);
    lambda = c2.*(1-X).*heaviside(X-1+F)- c1.*X.*heaviside(1-F-X);
    Xnew=(1-Theta).*X+Theta.*(X+eps*lambda+D.*eta);
    X=Xnew;
end
close(hbar)

figure(1)
hold on
set(0,'DefaultTextInterpreter','latex')
set(0,'DefaultAxesFontSize',18)
set(0,'DefaultLineLineWidth',1.2);
set(gca,'TickLabelInterpreter','latex')
histogram(X,'Normalization','probability','LineStyle','none','FaceColor
    ','#9ECB73');
y1=yline(u,'-.b','y = M(0)','LineWidth',2);
yl.LabelHorizontalAlignment = 'center';
xlabel('$x$')
ylabel('$f^\infty(x)$')
```

Note that $dt = \tau$ has been chosen, so that the parameter of Θ is 1, which is equivalent to saying that each chosen pair interacts. This makes it easier to reach the steady state and thus it reduces the computational time.

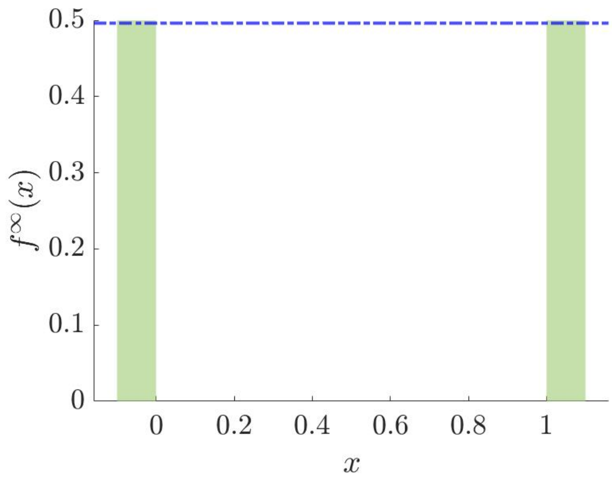

Figure 5.1. Solution of the Boltzmann-type equation in the quasi-invariant regime with stochastic fluctuation and initial competence uniformly distributed, in the case $c_1 = c_2$.

If we set $c_1 = c_2$, the theoretical result suggests that the distribution f^∞ is defined as follows:

$$f^\infty(x) = (1 - M_X(0))\delta(x) + M_X(0)\delta(x-1).$$

Since a uniform distribution over the interval [0,1] was taken as the initial distribution of competence, the histogram shows two peaks of height 0.5 in $x = 0$ and $x = 1$, as shown in Figure 5.1. This outcome is consistent with the analytical approach because we choose an initial distribution with $M_X(0) = 0.5$.
Next, to analyze a more interesting case, we simulated the trend by changing the initial distribution of X.

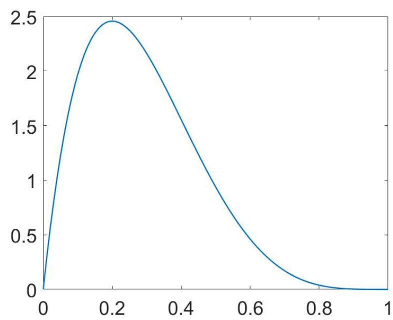

Figure 5.2. Probability density function of B(2,5).

In particular, the probability distribution Beta with parameters $\alpha = 2$ and $\beta = 5$ was chosen, whose density is reported in Figure 5.2. As can be seen, it has a mean of less than 0.5 and cancels out at the extreme values 0 and 1. Leaving the other parameters unchanged and replacing line 10 with line 9 in the script, the result obtained is the histogram shown in 5.4. The figure 5.3 shows the evolution of $f(x,t)$ over time. The transition from the continuous distribution to the discrete one, centered in $x = 0$ and $x = 1$, occurs in the following way: the population, initially distributed as a B(2,5) (see picture on the left in Figure (5.3)), tends to move toward the lower level of knowledge, and this happens because the mean is closer to 0 and, therefore, most of the population has a low competence (see center picture in 5.3), finally, the agents with an intermediate value of knowledge tend to move, more and more, toward the extreme values (see pictures on the right in 5.3), until the levels of competence different from the maximum and the minimum ones tend to vanish and the distribution assumes the asymptotic trend shown in 5.4.

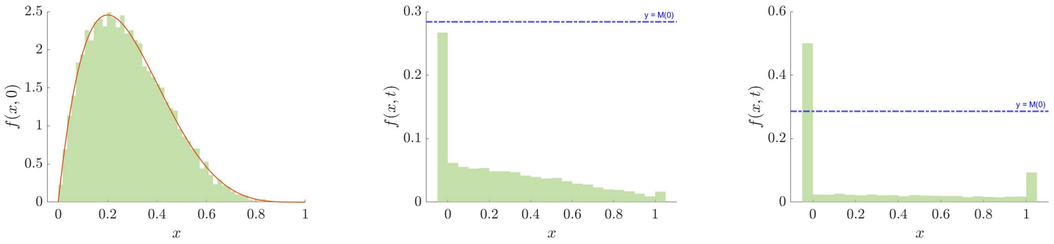

Figure 5.3. Evolution of $f(x,t)$ over time.

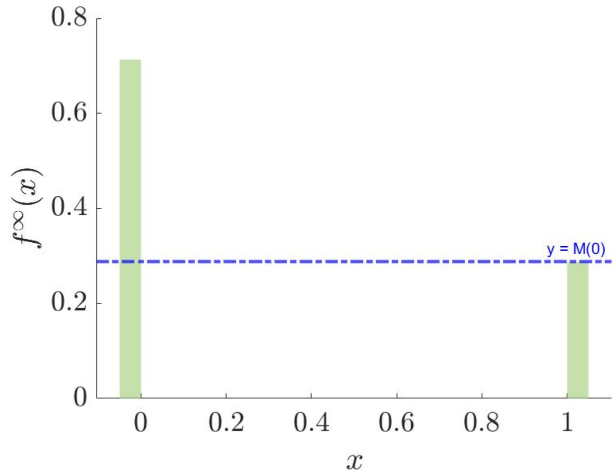

Figure 5.4. Solution of the Boltzmann-type equation in the quasi-invariant regime for the model with stochastic fluctuation, with competence initially distributed as a B(2,5) and $c_1 = c_2$.

The horizontal dashed line indicates the value of $M_X(0)$ and we see that, even with this choice of initial distribution, the results are consistent with those found analytically, in fact the fraction of agents with maximum competence coincides with $M_X(0)$.
It is also important to notice that both the histograms 5.1 and 5.4 are defined out of the range (0,1) and the only values, within the definition interval of the competence that are part of the histogram, are 0 and 1. This makes the numerical approximation very precise, because it perfectly shows that the solution is defined only in the two extreme values.

As we saw in (3.9) the convex combination of $\delta(x)$ and $\delta(x-1)$ is also an asymptotic solution of the Boltzmann equation. To check this result with the Monte Carlo simulation, we need only to choose $\epsilon = 1$ in the script, avoiding any scaling.
We simulate the asymptotic solution of (2.5) under the assumption of competence uniformly distributed in two different cases: $c_1 = c_2$ and $c_1 \neq c_2$.

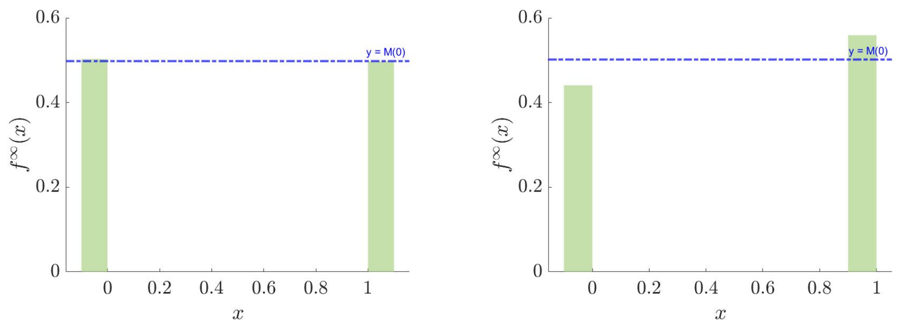

Figure 5.5. Solution of the Boltzmann-type equation for the model with stochastic fluctuation. On the left we have the solution for $c_1 = c_2$, and on the right for $c_1 \neq c_2$.

As we can see in Figure 5.5, the picture on the left shows a performance equal to Figure 5.1, in which the value assumed for $x = 1$ is on the line of $M_X(0)$. Instead, in the case $c_1 \neq c_2$ (in particular we choose $c_1 = 0.4$ and $c_2 = 0.6$), as we saw in the theory, the value of b, in

$$f^\infty = a\delta(x) + b\delta(x-1),$$

is not equal to $M_X(0)$ since the mean is not conserved.

Choosing $D = 0$, it is possible to check the solution of the Boltzmann-type equation for the model without stochastic fluctuation. Also in this case, we consider the initial competence uniformly distributed. As we can see in Figure 5.6, the solution for $c_1 = c_2$ is the same as the picture on the left of Figure 5.5, consistent with the solution found in (3.7). Instead, if $c_1 \neq c_2$, the values of a and b differ from the case with a non-zero diffusion coefficient.

Both the results obtained in Figure 5.5 and Figure 5.6 suggest to us an important conclusion that could not be derived from the theory: the convex combination of $\delta(x)$ and $\delta(x-1)$ is the *unique* solution of the Boltzmann-type equation (2.5).

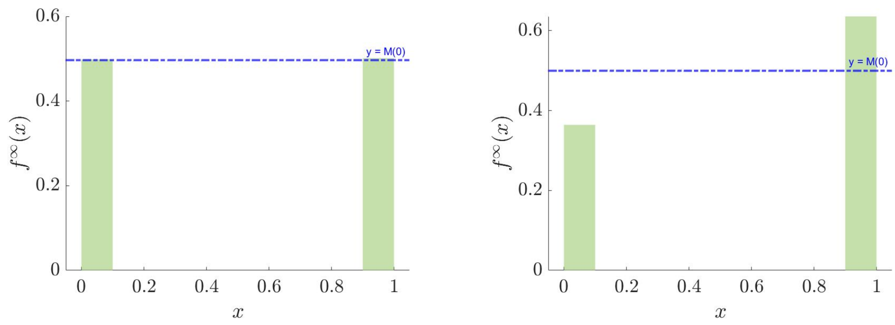

Figure 5.6. Solution of the Boltzmann-type equation for the model without stochastic fluctuation. On the left we have the solution for $c_1 = c_2$, and on the right for $c_1 \neq c_2$.

Now let's analyze the result related to the solution of the Boltzmann-type equation for the model without stochastic fluctuation under the quasi-invariant regime. The results obtained in 3.2 show a different asymptotic behavior based on the value of the two constants c_1 and c_2. In the case $c_1 \neq c_2$, the analysis shows that the solution is given by the convex combination of the delta exactly as in the cases mentioned before. From theory, however, it was not possible to find the coefficients since the mean, for these values of the constants, is not conserved. Also this time, the simulations help us to find a behavior that with the analytical approach was not evident: as it is shown in Figure 5.7, in this case, there are not two picks, instead the dynamics tend to a unique value. In particular, we notice that if $c_1 > c_2$ (graphic on the right of the Figure 5.7), i.e the proportion of competence loss for the incapacity to recognize the reliability of the news is greater than the proportion of competence gained, all the population tends to a null value of competence in the long-run. Otherwise, if $c_1 < c_2$ (graphic on the left of the Figure 5.7), everybody will reach the maximum level of competence. Thanks to these outcomes, we know that in the quasi-invariant interactions regime the mean M_X, which could not be determined analytically, takes value 1 if $c_1 < c_2$, and value 0 if $c_1 > c_2$. Moreover, this is consistent with the asymptotic theory of the quasi-invariant regime developed in Section 3.2, because the two peaks that the theory generally predicts in f^{∞} are reduced to only one if the asymptotic mean is known to be equal to 0 or 1.
We note that there is a difference between the outcome obtained by simulating the solution of the Boltzmann-type equation not in the quasi-invariant regime and with $c_1 \neq c_2$ (picture on the right of Figure 5.6) and the pictures in Figure 5.7. When $c_1 \neq c_2$ and without stochastic fluctuation, only in the quasi-invariant regime there is a concentration of the population in one of the two extreme knowledge classes: this perfectly emphasizes the fact that the quasi-invariant regime is not, in general, representative of any regime treatable by the general Boltzmann equation.

Instead, in the case $c_1 = c_2$ and $D(x) = 0$, in Section 3.2 we obtained that the distribution remains constant over time. We can choose as the initial distribution for the competence the Beta distribution with parameters $\alpha = 2$ and $\beta = 5$ (the same holds for the uniform distribution), and check if the competence continues to have the same trend.

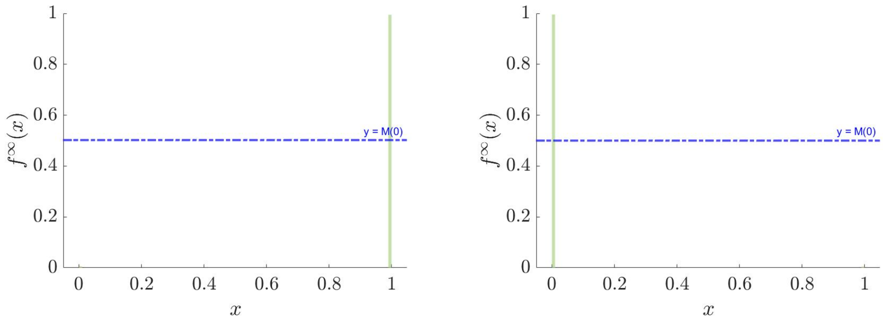

Figure 5.7. Solution of the Boltzmann-type equation in the quasi-invariant regime for the model without stochastic fluctuation. On the left we have the solution for $c_1 < c_2$, otherwise on the right $c_1 > c_2$.

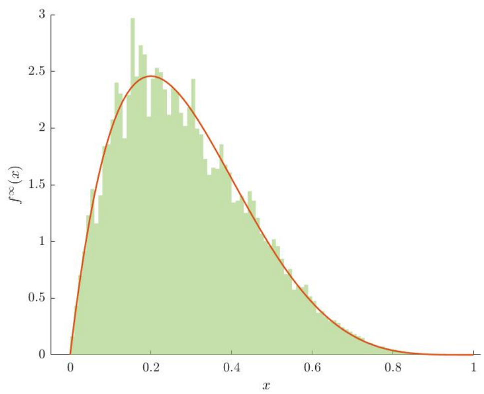

Figure 5.8. Solution of the Boltzmann-type equation in the quasi- invariant regime for the model without stochastic fluctuation with initial distribution B(2,5) and $c_1 = c_2$.

To simulate this result, it is necessary to choose $D = 0$ in the Matlab code and, in order to have a better approximation of the result, we choose a greater number of agents (10^5) and a smaller ϵ (10^{-4}). Furthermore, to reduce the number of iterations, we choose a smaller final time (10^2). In addition, to have a comparison with the density of the initial distribution, we wrote the following lines to the final part of the code:

```
z=0:0.01:1;
b=betapdf(z,2,5);
plot(z,b)
```

As can be seen in figure 5.8, the histogram shows a trend similar to that of the initially assigned distribution. Obviously, the shown graph represents a discrete approximation and, therefore, it is less smooth and cannot be able to perfectly fit the continuous distribution.

Part II

Kinetic description of the fake news popularity dynamics

Chapter 6

Dynamics of the popularity of a product on a social network

In this part, we will analyze the approach used in [9] to study the spread of a product's popularity on a social network. The term "product" means any information that can reach the users of the social network (such as news, videos, advertisements, etc.). Starting from this, we will modify the interaction creating a model that is more suitable for studying the spread of fake news.
The Boltzmann approach is very powerful because it can be also used to study all the types of dynamics on a network. In this chapter, we will explore, through the Boltzmann-type equation, the dynamics of the popularity of any product that can reach the member of a social network.
Let $v \in \mathbb{R}_+$ be a variable that represents the popularity of a product. The evolution of the popularity of this product depends on the interactions with agents. Every agent has a microscopic state described by the opinion-degree of connection pair (w, c). In [9] the following interaction rule was proposed:

$$v' = (1 - \mu)v + P(w, c) + D_{pop}(v)\xi. \tag{6.1}$$

Let's analyze better the terms that appear in this equation:

- $\mu \in (0,1)$ is a rate that expresses the natural decline in the popularity of a product when it is not reposted;
- $P : [-1,1] \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a function that expresses the increase in popularity when the product is reposted;
- ξ is a random variable with zero mean and finite variance $\hat{\sigma}^2 > 0$, which models a stochastic fluctuation in popularity;
- $D_{pop}(v) \geq 0$ is the diffusion coefficient, that has the role of regulating the strength of the fluctuation, which depends on the popularity itself.

A possible choice of the function $P(w,c)$ can be:

$$P(w,c) = \nu c\chi(|w-\hat{w}| \in [0,\Delta]), \tag{6.2}$$

where $\chi(|w-\hat{w}| \in [0,\Delta])$ indicates the characteristic function on the interval $[0,\Delta]$, i.e.,:

$$\chi(|w-\hat{w}| \in [0,\Delta]) = \begin{cases} 1 & \text{if} \quad |w-\hat{w}| \in [0,\Delta] \\ 0 & \text{otherwise.} \end{cases}$$

The $\hat{w} \in [-1{,}1]$ denotes the target opinion, i.e., the opinion to which the product is mostly direct. From (6.2) we have that the increase in popularity of a product depends on whether an individual belonging to the network decides to share the product. This happens if his opinion is close enough to the threshold of the target opinion. If this occurs, then the increase in popularity is proportional to the degree of connection of the individual, viz. to the number of followers reached when the product is reposted.
At the beginning of the chapter, we specified that the product popularity variable must belong to $\mathbb{R}_+$:, for this reason, we know that v', defined as in (6.1), must satisfy the condition $v' \geq 0$. Since $P(w,c) \geq 0$ by definition, it is enough to require that $D_{pop}(v)\xi \geq (\mu-1)v$, which is satisfied if the following inequalities hold: $\xi \geq \mu - 1$ and $D_{pop}(v) \leq v$. From here we deduce that the stochastic fluctuation ξ can assume negative values, while from the other equation combined with the request that $D_{pop}(v) \geq 0$ for the definition of the diffusion coefficient, we need $D_{pop}(0) = 0$.. To choose D_{pop} in such a way that the spread gets bigger and bigger as popularity grows, one possible choice would be $D_{pop} = v$. Other choices could be:

- $D_{pop}(v) = \min\{v, V_0\}$, if we want to express a saturation of the diffusion coefficient for large values of v ($v > V_0$);
- $D_{pop}(v) = \min\{v, \frac{V_0^2}{v}\}$ that, unlike the previous case, only expresses a decline in popularity for products that exceed a popularity threshold equal to V_0.

6.1 Dynamics of the popularity of fake news

In the case of fake news, however, not all the agents who have an opinion close to the target one will share the fake news. To overcome this, we can hypothesize, for simplicity, that, if the individual has insufficient knowledge to recognize the fake news, he will share the news, bringing to the news a growth in its popularity proportional to the agent's degree of connection. Conversely, if the agent can recognize the reliability of the news, he will not share the news. Hence, the interaction rule is:

$$v' = (1-\mu)v + P(x,y,c) + D_{pop}(v)\xi. \tag{6.3}$$

In this case, P could be defined, as follows:

$$P(x,y,c) = \nu c\chi(x < 1-y), \tag{6.4}$$

The probability densities that we need for the model are:

- $h = h(t, v, y)$: the fraction of products with popularity in the range $[v, v + dv]$ at time t and having degree of falsity $[y, y + dy]$;
- $p(t, x, c)$: the density of agents with a microscopic state described by the pair knowledge-degree of connection (x, c).

In this model, we will have two dynamics that evolve over time: one relating to the dynamics of knowledge on the social network and one relating to the dynamics of the popularity of the product. The weak form of the Boltzmann equation for product popularity is given by the equation:

$$\frac{d}{dt}\int_{\mathbb{R}_+}\int_0^1 \varphi(v,y)h(t,v,y)\,dy\,dv = \tag{6.5}$$
$$\left\langle \int_{\mathbb{R}_+}\int_0^1\int_0^1\int_{\mathbb{R}_+}[\varphi(v',y)-\varphi(v,y)]h(t,v,y)p(t,x,c)\,dv\,dx\,dy\,dc\right\rangle.$$

In the previous equation, $\langle\cdot\rangle$ denotes the mean respect to the random variable ξ and $\varphi : \mathbb{R}_+ \to \mathbb{R}$ indicates the test function, which represents any observable amount of the popularity v and the degree of falsity y.
Since in the equation appears also $p(t, x, c)$, it would be necessary to solve the following coupled system:

$$\begin{cases} \frac{d}{dt}\int_{\mathbb{R}_+}\varphi(x,c)p(t,x,c)dx &= \int_0^1\int_{\mathbb{R}_+}\langle\varphi(x',c)-\varphi(x,c)\rangle p(t,x,c)\,dc\,dx \\ \frac{d}{dt}\int_{\mathbb{R}_+}\varphi(v,y)h(t,v,y)dv &= \int_{\mathbb{R}_+}\int_0^1\int_0^1\int_{\mathbb{R}_+}\langle\varphi(v',y)-\varphi(v,y)\rangle h(t,v,y)p(t,x,c)\,dv\,dx\,dy\,dc, \end{cases}$$

solving the first equation, we should obtain $p(t, x, c)$ which we can then replace in the second equation to find $h(t, v, y)$. Doing all this, however, could be complicated, therefore, we can introduce the hypothesis that the dynamics of knowledge is much faster than that of the popularity of news. Under this assumption, it is possible to rewrite (6.5) replacing $p(t, x, c)$ with the asymptotic distribution $p^\infty(x, c)$, which can be easily obtained through the Fokker-Planck equation.

Also in this case, it is interesting to analyze the distribution of some significant statistical moments, in particular, we will focus on the average of the popularity.

Substituting $\varphi(v, y) = v$ in (6.5) and maintaining the hypothesis that the opinion dynamics is faster than the popularity spread process, we have:

$$\frac{dM_V}{dt} = \int_{\mathbb{R}_+}\int_0^1\int_0^1\int_{\mathbb{R}_+}\langle v'-v\rangle h(t,v,y)p^\infty(x,c)\,dv\,dx\,dy\,dc. \tag{6.6}$$

Similarly to the previous chapter $\langle v'-v\rangle = -\mu v + P(x,c,y) + D_{pop}(v)\langle\xi\rangle = -\mu v + P(x,c,y)$, since for definition $\langle\xi\rangle = 0$, hence (6.6) can be rewritten as follows:

$$\frac{dM_V}{dt} = \int_{\mathbb{R}_+}\int_0^1\int_0^1\int_{\mathbb{R}_+}(-\mu v + P(x,c,y))h(t,v,y)p^\infty(x,c)\,dv\,dx\,dy\,dc.$$

Chapter 7

A Boltzmann-like equation for the distribution of popularity conditional on the degree of falsity

Up to now, we have considered the joint distribution of the popularity and the degree of falsity of the news, but, it is more interesting to study the popularity conditional on the degree of untruth, to be able to see, once a certain degree of falsity is fixed, what is the distribution of conditional popularity. To do this, however, it is necessary to find a Boltzmann-type equation for the conditional distribution, that we denote by $h_y(t, v)$. As it is well known, the joint distribution of v and c can be rewritten as the product of the distribution of v conditional on y by the marginal of y, viz. $h(t, v, y) = h_y(t, v)g(y)$. In our model, we assume that the probability density of the connection $g(y)$ does not depend on time. Hence the equation (6.5) becomes:

$$\frac{d}{dt}\int_{\mathbb{R}_+}\int_0^1 \varphi(v, y)h_y(t, v)g(y)\,dy\,dv = \int_{\mathbb{R}_+}\int_0^1\int_0^1\int_{\mathbb{R}_+}\langle\varphi(v', y) - \varphi(v, y)\rangle h_y(t, v)g(y)$$
$$p(t, x, c)\,dv\,dx\,dy\,dc.$$

Under the further hypothesis that $\varphi(v, y)$ can be factored into the product of two functions: one dependent only on v, $\phi(v)$, and the other dependent only on y, $\psi(y)$:

$$\frac{d}{dt}\int_0^1 \psi(y)\left(\int_{\mathbb{R}_+}\phi(v)h_y(t, v)\,dv\right)g(y)\,dy = \Big\langle\int_{\mathbb{R}_+}\int_0^1\int_0^1\int_{\mathbb{R}_+}\psi(y)[\phi(v') - \phi(v)]h_y(t, v)g(y)$$
$$p(t, x, c)\,dv\,dx\,dy\,dc\Big\rangle.$$

We also assume, as done above for the joint distribution, that the dynamics relating to the knowledge of the individual is faster than the dynamics of popularity, thus we replace

$p(t,x,c)$ with its asymptotic distribution which, for consistency of notation, we denote by $p^{\infty}(x,c)$:

$$\frac{d}{dt}\int_0^1 \psi(y)\bigg(\int_{\mathbb{R}_+}\phi(v)h_y(t,v)\,dv\bigg)g(y)\,dy = \tag{7.1}$$
$$\bigg\langle \int_0^1 \psi(y)\bigg[\int_0^1\int_{\mathbb{R}_+}\bigg(\int_{\mathbb{R}_+}[\phi(v')-\phi(v)]h_y(t,v)\,dv\bigg)p^{\infty}(x,c)\,dc\,dx\bigg]g(y)\,dy\bigg\rangle.$$

By defining the following functions:

$$A(t,y) = \int_{\mathbb{R}_+}\phi(v)\partial_t h_y(t,v)\,dv,$$

$$B(t,y) = \bigg\langle \int_0^1\int_{\mathbb{R}_+}\bigg(\int_{\mathbb{R}_+}[\phi(v')-\phi(v)]h_y(t,v)\,dv\bigg)p^{\infty}(x,c)\,dc\,dx\bigg\rangle,$$

equation (7.1) turn to:

$$\int_0^1 \psi(y)A(t,y)g(y)dy = \int_0^1 \psi(y)B(t,y)g(y)dy.$$

For the arbitrariness of ψ, then the equality is valid if and only if:

$$A(t,y) = B(t,y).$$

By imposing this equality and taking out the derivative in time from the integral in $A(t,y)$, we obtain a Boltzmann-like equation for the conditional distribution $h_y(t,v)$, which is a function of y:

$$\frac{d}{dt}\int_{\mathbb{R}_+}\phi(v)h_y(t,v)\,dv = \bigg\langle \int_0^1\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}[\phi(v')-\phi(v)]h_y(t,v)p^{\infty}(x,c)\,dv\,dc\,dx\bigg\rangle. \tag{7.2}$$

This allows us to study the mean of the popularity dynamics and its asymptotic distribution conditional on the value of the degree of falsity y.

7.1 Average distribution of popularity conditional on the degree of falsity of the fake news

To calculate the derivative in time of the average of the distribution of popularity conditional on the degree of falsehood, we consider $\phi(v) = v$ in (7.2):

$$\frac{dM_{V|Y}}{dt} = \frac{d}{dt}\int_{\mathbb{R}_+} v h_y(t,v)\,dv = \int_0^1\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\langle v'-v\rangle h_y(t,v)p^{\infty}(x,c)\,dv\,dc\,dx.$$

As we saw in Section 6.1, $\langle v' - v \rangle = -\mu v + P(x, c, y) + D_{pop}(v)\langle \xi \rangle = -\mu v + P(x, c, y)$, in which $P(x, c, y)$ is defined as in (6.4), therefore we get:

$$\begin{aligned} \frac{dM_{V|Y}}{dt} &= \int_0^1 \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} (-\mu v + P(x, y, c)) h_y(t, v) p^\infty(x, c)\, dv\, dc\, dx \\ &= -\mu M_{V|Y} + \nu \int_0^1 \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} c\chi(x < 1 - y) h_y(t, v) p^\infty(x, c)\, dc\, dv\, dx. \end{aligned} \tag{7.3}$$

Let us now focus on calculating the integral. Since $h_y(t, v)$ is a probability density, then it must be worth $\int_0^1 h_y(t, v) = 1$. Since no term in the integral depends on v, we have:

$$\begin{aligned} \int_0^1 \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} c\chi(x < 1 - y) h_y(t, v) p^\infty(x, c)\, dc\, dv\, dx &= \int_0^1 \int_{\mathbb{R}_+} c\chi(x < 1 - y) p^\infty(x, c)\, dc\, dx \\ &= \int_0^{1-y} \int_{\mathbb{R}_+} c p^\infty(x, c)\, dc\, dx. \end{aligned} \tag{7.4}$$

From which we obtain, replacing in (7.3):

$$\frac{dM_{V|Y}}{dt} = -\mu M_{V|Y} + \nu \int_{\mathbb{R}_+} \int_0^{1-y} c p^\infty(x, c)\, dx\, dc. \tag{7.5}$$

We note that there is an upper bound for (7.4):

$$\begin{aligned} \int_{\mathbb{R}_+} \int_0^{1-y} c p^\infty(x, c)\, dx\, dc \le \int_{\mathbb{R}_+} \int_0^1 c p^\infty(x, c)\, dx\, dc = \int_{\mathbb{R}_+} c \left(\int_0^1 p^\infty(x, c) dx \right) dc = \\ = \int_{\mathbb{R}_+} c \tilde{g}^\infty(c) dc = M_C, \end{aligned} \tag{7.6}$$

where $\tilde{g}^\infty(c)$ is the asymptotic probability distribution of c and M_C indicates the mean of the distribution of the network connection. Hence, in general, we have:

$$\frac{dM_{V|Y}}{dt} \le -\mu M_{V|Y}(y) + \nu M_C.$$

We note that, if we analyze the news with a null degree of falsity and therefore totally true ($y = 0$), the inequality in (7.6) will be valid with the equal and therefore the conditional mean will take on the following value:

$$\frac{dM_{V|y=0}}{dt} = -\mu M_{V|y=0} + \nu M_C,$$

from which it follows, since M_C is constant over time:

$$M_{V|y=0} = \frac{\nu M_C}{\mu} + \left(M_{V|y=0}(0) - \frac{\nu M_C}{\mu} \right) e^{-\mu t}, \tag{7.7}$$

where $M_{V|y=0}(0)$ denotes the average popularity of totally true news at the initial time. It is easy to see that, for $t \to \infty$, we have $M_{V|y=0} \to \frac{\nu M_C}{\mu}$.

While, if $y = 1$, which means for news with a maximum degree of falsehood, that everyone will recognize as false and no one will repost, (7.4) becomes null because the integration extremes will coincide and consequently we will simply have:

$$\frac{dM_{V|y=1}}{dt} = -\mu M_{V|y=1},$$

which will return the following result:

$$M_{V|y=1} = M_{V|y=1}(0)e^{-\mu t}, \tag{7.8}$$

so, for $t \to \infty$, the average popularity of news that are totally fake will go to zero due to the process of popularity loss related to non-reposting.
We note that the estimation in (7.6) causes a loss of information about the competence x. To avoid this issue, we assume that the distribution of competence and connectivity are independent. Under this hypothesis, it is possible to factorize the joint distribution of x and c as follows:

$$p^\infty(x,c) = f^\infty(x)\tilde{g}^\infty(c) = (a\delta(x) + b\delta(x-1))\tilde{g}^\infty(c), \tag{7.9}$$

in which we assume that $\tilde{g}^\infty$ is given, and we exploit the result obtained in the part I to assign the distribution $f^\infty(x)$. Consequently, equation (7.4), under the independence assumption, becomes:

$$\begin{aligned} \int_0^{1-y}\int_{\mathbb{R}_+} cp^\infty(x,c) &= \int_{\mathbb{R}_+} c\tilde{g}^\infty(c)dc \int_0^{1-y}[a\delta(x) + (1-a)\delta(x-1)]dx = \\ &= M_C \int_0^{1-y}[a\delta(x) + (1-a)\delta(x-1)]dx. \end{aligned} \tag{7.10}$$

The solution of the integral changes according to the level of the degree of falsity y:

- if $y = 0$, i.e., the news is totally true, we have the integral of f^∞ over the entire defining interval of the random variable. Thus (7.10) assumes value M_C, which leads to the solution obtained in (7.7);
- if $0 < y \le 1$ then $0 \le 1-y < 1$, we will have that the Dirac delta concentrated in 0 will fall within the defining range of the integral, while the Dirac delta concentrated in 1 will remain outside, so the result of (7.10) is aM_C. Consequently, the conditional mean, in this case, will be given by:

  $$M_{V|y} = \frac{\nu a M_C}{\mu} + \left(M_{V|y}(0) - \frac{\nu a M_C}{\mu}\right)e^{-\mu t}. \tag{7.11}$$

  Recalling that, in the case where the constants c_1 and c_2 coincide, the mean is conserved over time and, then, $a = 1 - M_X(0)$, where $M_X(0)$ denotes the initial mean of competence, (7.11) becomes:

$$M_{V|y} = \frac{\nu(1 - M_X(0))M_C}{\mu} + \left(M_{V|y}(0) - \frac{\nu(1 - M_X(0))M_C}{\mu} \right) e^{-\mu t}. \tag{7.12}$$

We note that, from (7.12), taking the limit for $t \to \infty$, we get $M_{V|y} \to \frac{\nu(1-M_X(0))M_C}{\mu}$, since $M_X(0)$ must take value in [0,1], we can fall into the following cases:

1. if $M_X(0) = 0$, i.e., at the beginning all agents have zero knowledge, then we have the same behavior as when $y = 0$ in (7.7), viz. the behavior is the same as when all news are totally true and therefore everyone reposts except those with maximum knowledge ($x < 1$);
2. if $M_X(0) = 1$, i.e., at the beginning all agents have maximal competence, then we fall back into the same behavior as when $y = 1$ in (7.8), so no one reposts and, as a consequence, the news loses popularity;
3. If $0 < M_X(0) < 1$, then, for $t \to \infty$, we will reach an average in the interval $\left(0, \frac{\nu M_C}{\mu}\right)$.

So, regardless of the average degree of competence, a news that is not true ($y \neq 0$) will have an asymptotic mean smaller or equal then the one of the news with $y = 0$.

Chapter 8

Study of the asymptotic behavior

Let's analyze the asymptotic behavior of the density function of popularity conditional on news falsity by exploiting, as we have already seen in 3.2, the limit of the quasi-invariant interactions. To proceed with this analysis, it is important to move to a dilated time scale and to multiply the various terms by a very small factor ϵ ($0 < \epsilon \ll 1$). Then, we define $\mu = \mu_0\epsilon$, $\nu = \nu_0\epsilon$ with ν_0, $\mu_0 > 0$ and $P(x,y,c) = \epsilon P_0(x,y,c)$ with $P_0(x,y,c) = \nu_0 c\chi(x < 1-y)$.
Finally we define the distribution on the dilated time scale $\tilde{h}_y(\tau,v) = h_y(\tau/\epsilon,v)$ which, for (7.2), must satisfy:

$$\frac{d}{d\tau}\int_{\mathbb{R}_+}\phi(v)\tilde{h}_y(\tau,v)\,dv = \frac{1}{\epsilon}\left\langle \int_0^1\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}[\phi(v')-\phi(v)]\tilde{h}_y(\tau,v)p^\infty(x,c)\,dv\,dc\,dx\right\rangle. \quad (8.1)$$

For simplicity, we analyze the case without stochastic fluctuation and thus $D(v_{pop}) = 0$. We consider $\phi \in C^3_C(0{,}1)$ and proceed to make the Taylor series expansion centered in v, with Peano remainder, of the term $\phi(v') - \phi(v)$:

$$\phi(v') - \phi(v) = \phi'(v)(v'-v) + \frac{1}{2}\phi''(v)(v'-v)^2 + \frac{1}{6}\phi'''(\bar{v})(v'-v)^3,$$

with $\bar{v} \in (min\{v,v'\}, max\{v,v'\})$.
Since we are neglecting the stochastic fluctuation, there will be no random terms and, therefore, we can omit the brackets $\langle\cdot\rangle$ obtaining:

$$\begin{aligned} v' - v &= -\mu v + P(x,y,c), \\ (v'-v)^2 &= \mu^2 v^2 + P^2(x,y,c) - 2\mu v P(x,y,c). \end{aligned}$$

Replacing these results in (8.1) and exploiting the definitions given at the beginning of the chapter, we get:

$$\begin{aligned}\frac{d}{d\tau}\int_{\mathbb{R}_+}\phi(v)\tilde{h}_y(\tau,v)dv &= \int_0^1\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\phi'(v)(-\mu_0 v+P_0)\tilde{h}_y(\tau,v)p^\infty(x,c)\,dv\,dc\,dx+\\&+\frac{\epsilon}{2}\int_0^1\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\phi''(v)(\mu_0^2v^2+P_0^2-2\mu_0 vP_0)\tilde{h}_y(\tau,v)p^\infty(x,c)\,dv\,dc\,dx\\&+\hat{R}(\phi),\end{aligned}$$

in which:

$$\hat{R}(\phi):=\frac{1}{6}\int_0^1\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\phi'''(\bar{v})[-\mu v+P(x,y,c)]^3\tilde{h}_y(\tau,v)p^\infty(x,c)\,dv\,dc\,dx.$$

If we consider the limit of quasi-invariant interactions, $\epsilon\to 0^+$, we get:

$$\frac{d}{d\tau}\int_{\mathbb{R}_+}\phi(v)\tilde{h}_y(\tau,v)dv=\int_0^1\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\phi'(v)[-\mu_0 v+P_0(x,y,c)]\tilde{h}_y(\tau,v)p^\infty(x,c)\,dv\,dc\,dx.$$

Integrating by parts and taking advantage of the fact that, being ϕ with compact support, it will be zero at the extremes of the interval [0,1], we get:

$$\begin{aligned}\frac{d}{d\tau}\int_{\mathbb{R}_+}\phi(v)\tilde{h}_y(\tau,v)dv &= -\int_0^1\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\phi(v)\partial_v[(-\mu_0 v+P_0(x,y,c))\tilde{h}_y(\tau,v)]p^\infty(x,c)\,dv\,dc\,dx\\&=\int_{\mathbb{R}_+}\phi(v)\partial_v\left[\left(\mu_0 v-\int_0^1\int_{\mathbb{R}_+}P_0(x,y,c)dc\,dx\right)\tilde{h}_y(\tau,v)\right]p^\infty(x,c)\,dv.\end{aligned}$$

Introducing the following definition:

$$\mathcal{P}_0:=\int_0^1\int_{\mathbb{R}_+}P_0(x,y,c)p^\infty(x,c)dcdx,$$

and, by exploiting the arbitrariness of ϕ, we deduce the following Fokker-Planck equation:

$$\partial_\tau\tilde{h}_y=\partial_v[(\mu_0 v-\mathcal{P}_0)\tilde{h}_y]. \tag{8.2}$$

We derive the stationary distribution by imposing $\partial_\tau\tilde{h}_y=0$. Furthermore, since at steady state the flux of any conservation law is necessarily zero, we obtain:

$$(\mu_0 v-\mathcal{P}_0)\tilde{h}_y^\infty=0.$$

This result suggest us that, if $v\neq\frac{\mathcal{P}_0}{\mu_0}$, $\tilde{h}_y^\infty$ must be zero. We again assume that knowledge and degree of connection are independent, so that we can rewrite $p^\infty(x,c)$ as in (7.9). With calculations similar to those performed in (7.10), we have:

$$\mathcal{P}_0=\nu_0 M_C\int_0^{1-y}[a\delta(x)+(1-a)\delta(x-1)]\,dx.$$

Exactly as observed for (7.10), if $y = 0$ then, since $\int_0^1 f^\infty(x)\,dx = 1$, we will have that $\mathcal{P}_0 = \nu_0 M_C$. Whereas, if $y \in (0,1]$, the delta centered in 1 will not fall in the defining interval of the integral, therefore $\mathcal{P}_0 = \nu_0 a M_C$. If we divide these values by μ_0, the ϵ is simplified and we get, in all three cases, the asymptotic values of the mean conditional on the degree of falsity. Then the asymptotic distribution, in the case without fluctuation, coincides with a Dirac delta centered on the value of the conditional mean for $t \to \infty$. Hence, formally, we have, if $y = 0$:

$$\tilde{h}^\infty_{y=0} = \delta\Big(x - \frac{\nu M_C}{\mu}\Big), \tag{8.3}$$

otherwise:

$$\tilde{h}^\infty_{y\neq 0} = \delta\Big(x - \frac{\nu a M_C}{\mu}\Big), \tag{8.4}$$

where $a = 1 - M_X(0)$ if $c_1 = c_2$.

8.1 Numerical simulation

In this section, following the steps mentioned in Chapter 5, we provide a numerical simulation for the dynamics of the news popularity, in which we avoid the stochastic fluctuation.

This is the code used to simulate the dynamics:

```
%FIRST PART
N_x=1e4;
eps1=1;
dt1=eps1;
Tfin=1e2; %final time
nmax1=floor(Tfin/dt1);
sigma1=sqrt(eps1);
tau1=eps1;

X=rand(N_x,1);
F=rand(N_x,1);
for n=1:nmax1
    X=X(randperm(N_x)); %random permutation of X
    Theta1=binornd(1,dt1/tau1,N_x,1); %a value for each couple
    %all pairs interact
    Y1=sqrt(12)*(-0.5 + rand(N_x,1)); %mean=0, variance=1
    eta=sigma1*Y1;
    c1=0.4;
    c2=0.4;
    diff= X.*(1-X);
    D1=sqrt(diff).*(diff>=0);
    lambda = c2.*(1-X).*heaviside(X-1+F)- c1.*X.*heaviside(1-F-X);
    Xnew=(1-Theta1).*X+Theta1.*(X+eps1*lambda+D1.*eta);
    X=Xnew;
end
```

```
%SECOND PART

N_v=N_x;
eps2=1e-2; %small for the quasi-invariant regime
dt2=eps2; %delta t = tau
nmax2=floor(Tfin/dt2);
D2=0;
tau2=eps2;

X = X.*heaviside(X-1e-20);
C=rand(N_x,1);
V=rand(N_v,1);

mu_0 = 0.2;
nu_0 = 0.3;
q = (nu_0*(1-mean(X))*mean(C))/mu_0; % if y different from 0
q_0 = (nu_0*mean(C))/mu_0; %if y=0

sigma2 = 0.5;
hbar=waitbar(0,'','Name','Iterazione 2');
for n=1:nmax2
    waitbar(n/nmax2,hbar,sprintf('$n$ = %d / %d',n,nmax2));
    V=V(randperm(N_v)); %random permutation of V
    Theta2=binornd(1,dt2/tau2,N_v,1); %all the couple interact
    Y2=sqrt(12)*(-0.5 + rand(N_v,1)); %mean=0 e variance=1
    eta2=sigma2*Y2;
    f=0; %fixed degree of falsity
    P_0 = nu_0.*C.*heaviside(1-f-X+1e-10);
    Vnew=(1-Theta2).*V+Theta2.*(V - eps2*mu_0*V+eps2*P_0+D2.*eta2);
    V=Vnew;
end
close(hbar)

figure(1)
hold on
set(0,'DefaultTextInterpreter','latex')
set(0,'DefaultAxesFontSize',18)
set(0,'DefaultLineLineWidth',1.2);
set(gca,'TickLabelInterpreter','latex')
histogram(V,'Normalization','probability','LineStyle','none','FaceColor','#9ECB73');
y1=xline(q_0,'-.b',' y = M^\infty_{y=0}(v)','LineWidth',2);
y2=xline(q,'-.r',' y = M^\infty_{y \neq 0}(v)','LineWidth',2);
xlabel('$v$')
ylabel('$h^\infty_Y (v)$')
```

The first part of the code aims to obtain a sample that is distributed as (3.11). To create this we used the same code wrote in Chapter 5. In particular, we implemented the solution of the Boltzmann-type equation for the model with stochastic diffusion and $c_1 = c_2$. We choose an initial distribution for the competence uniformly distributed, in order to have an half of the population with $x = 0$ and the other half with $x = 1$. Looking at the sample X, we saw that for agents with zero knowledge, the values of x returned were very small (order 10^{-22}) and they did not coincide with zero. To overcome this problem we added a tolerance threshold that would replace very small values with zero.

In the second part we simulate the dynamics of the popularity, through the following

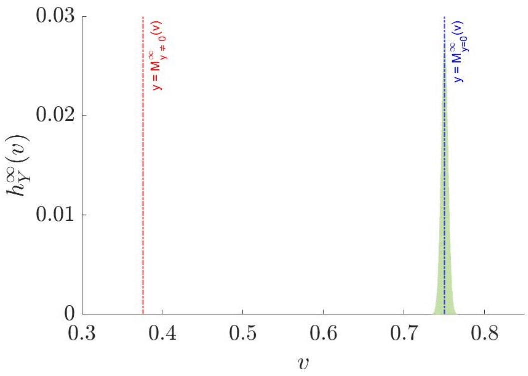

Figure 8.1. Asymptotic behavior of the popularity of the fake news conditional to the minimum degree of falsity.

rule for the popularity of the agent i in the time interval $n + 1$:

$$v_i^{n+1} = (1 - \Theta)v_i^n + \Theta[(1 - \mu)v_i^n + P(x_i^n, y, c)], \tag{8.5}$$

and we plotted the histogram that approximates the asymptotic behavior of the distribution of the popularity conditional to the value of the degree of falsity of the news. According with what we found in Chapter 8, the dynamics tends to a delta centered in the value of the mean for $t \to \infty$.

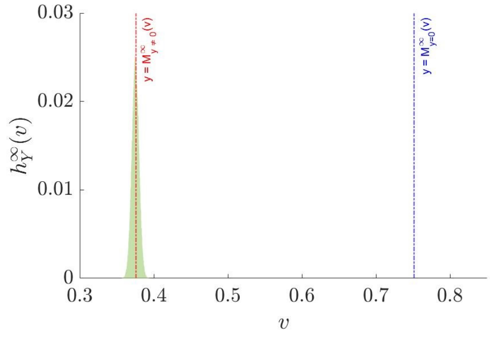

Figure 8.2. Asymptotic behavior of the popularity of the fake news conditional to a degree of falsity different from zero.

In 8.1 and 8.2 there are respectively the behaviors when the degree of falsity is 0 and when it is 1. The red and the blue dashed lines represent respectively the limit of the mean

of the popularity conditional on $y \neq 0$, that is, since we choose $c_1 = c_2$, $\frac{\nu(1-M_X(0))M_C}{\mu}$, and $y = 0$, which is $\frac{\nu M_C}{\mu}$.
It is evident that the simulations are consistent with the theory. In fact, both the results show, in a clear way, that the asymptotic distribution is defined on the value of the mean. Moreover, comparing the two histograms it is obvious that the popularity of the totally true news is greater than the one of the news that are completely false. If we fix a degree of falsity different from zero, we obtain histogram that is equivalent to 8.2. This is due to the fact that, if $y \neq \{0,1\}$, the average popularity conditional on the degree of falsehood is given by (7.11), exactly as $y = 1$.

Chapter 9

Alternative models for the dynamics of popularity

The model considered in the previous section requires that anyone with sufficient competence to recognize fake news will not share it again. This assumption turns out to be very stringent, and in order to obtain a model more similar to reality, we can relax this constraint by analyzing alternative models. As an example, we can consider a model in which, when the difference between the competence and the level of falsity is very small, the agent decides to share the news, even if the individual turns out to be competent enough to recognize the falsity of the news.
To study this model, we need to change the definition of $P(x, y, c)$, given in (6.4), replacing it with:

$$P_1(x, y, c) = \nu c \chi(y - d < x < y + d), \tag{9.1}$$

with $d \leq y$ e $d \leq 1 - y$. We proceed to calculate the change over time of the mean of the distribution of popularity conditional on the degree of falsity, under this first alternative model, substituting in (7.3) the new function (9.1):

$$\begin{aligned} \frac{dM_{V|Y}}{dt} &= \int_0^1 \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} [-\mu v + P_1(x, y, c)] h_y(t, v) p^\infty(x, c) \, dv \, dc \, dx \\ &= -\mu M_{V|Y} + \nu \int_0^1 \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} c \chi(y - d < x < y + d) h_y(t, v) p^\infty(x, c) \, dc \, dv \, dx. \end{aligned} \tag{9.2}$$

Let's focus on the calculation of the integral:

$$\begin{aligned} &\int_0^1 \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} c \chi(y - d < x < y + d) h_y(t, v) p^\infty(x, c) \, dc \, dv \, dx = \\ &\qquad = \int_0^1 \int_{\mathbb{R}_+} c \chi(y - d < x < y + d) p^\infty(x, c) \, dc \, dx \\ &\qquad = \int_{y-d}^{y+d} \int_{\mathbb{R}_+} c p^\infty(x, c) \, dc \, dx. \end{aligned}$$

If we again assume independence between knowledge and connection, recalling the definition of f^∞ found in (3.7), we get:

$$\int_{y-d}^{y+d} \int_{\mathbb{R}_+} cp^\infty(x,c)\, dc\, dx = \int_{\mathbb{R}_+} c\tilde{g}^\infty(c)dc \int_{y-d}^{y+d} [a\delta(x) + (1-a)\delta(x-1)]dx. \tag{9.3}$$

If the interval $[y-d, y+d]$ contains zero and not one, then $y = d$ and $d < \frac{1}{2}$, the integral will take value aM_C, and so we will have:

$$M_{V|y} = \frac{\nu a M_C}{\mu} + \left(M_{V|y}(0) - \frac{\nu a M_C}{\mu}\right)e^{-\mu t}. \tag{9.4}$$

If we are also in the case where c_1 and c_2 coincide, then we have $a = 1 - M_X(0)$ and thus it results in (7.12).
On the other hand, if it contains the one but not the zero, then $d = 1 - y$ and $d < \frac{1}{2}$, the integral will return $(1-a)M_C$ and so we have:

$$M_{V|y} = \frac{\nu(1-a)M_C}{\mu} + \left(M_{V|y}(0) - \frac{\nu(1-a)M_C}{\mu}\right)e^{-\mu t}, \tag{9.5}$$

with $1 - a = M_X(0)$ if $c_1 = c_2$.
We notice that the result in (9.5) does not appear in the initial model.
Obviously, it makes no sense to choose a d that covers the entire range of definition of the competence, since it would mean that, whatever the degree of falsity of the news, the agent will share the news.

Another alternative model comes from choosing a popularity increase function done in the following way:

$$P_2(x,y,c) = \nu c\chi(x < 1 - y + \gamma) \qquad \forall\gamma \quad \text{s.t.} \quad \gamma \geq 0. \tag{9.6}$$

The choice of this function models the case in which an agent might repost fake news even if his competence is slightly higher. Hence, even if an agent recognizes the falsity of the news, he still decides to repost it because it is close to his level of competence. Again, we can analyze the average and see if the obtained behavior is similar to the initially chosen model.
Under the assumption that the new function expressing the increase in popularity is (9.6), the change over time of the average popularity conditional on the degree of falsity will be:

$$\begin{aligned} \frac{dM_{V|Y}}{dt} &= \int_0^1 \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} (-\mu v + P_2(x,y,c))h_y(t,v)p^\infty(x,c)\, dv\, dc\, dx \\ &= -\mu M_{V|Y} + \nu \int_0^1 \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} c\chi(x < 1 - y + \gamma)h_y(t,v)p^\infty(x,c)\, dv\, dc\, dx. \end{aligned} \tag{9.7}$$

Let's evaluate the integral:

$$
\begin{aligned}
&\int_0^1 \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} c\chi(x < 1 - y + \gamma) h_y(t,v) p^\infty(x,c)\, dv\, dc\, dx = \\
&\qquad = \int_0^1 \int_{\mathbb{R}_+} c\chi(x < 1 - y + \gamma) p^\infty(x,c)\, dc\, dx \\
&\qquad = \int_0^{1-y+\gamma} \int_{\mathbb{R}_+} c p^\infty(x,c)\, dc\, dx.
\end{aligned}
$$

Under the assumption of independence, we get:

$$
\int_0^{1-y+\gamma} \int_{\mathbb{R}_+} c p^\infty(x,c)\, dc\, dx = \int_{\mathbb{R}_+} c\tilde{g}^\infty(c) dc \int_0^{1-y+\gamma} [a\delta(x) + (1-a)\delta(x-1)] dx. \tag{9.8}
$$

If $\gamma < y$ then the Dirac delta centered in 1 does not fall within the definition interval of the integral, therefore, the integral (9.8) returns as a solution aM_C, hence we obtain the expression for the mean defined in (7.11). Otherwise, for $y \leq \gamma$, the integral will be defined over the whole [0,1], and thus the conditional mean will be defined as in the case of the original model with $y = 0$ (see (7.7)). This result suggests that, under this model, the news that are not excessively false (whose threshold is given by γ) will behave as the totally true one in the first model.

A third alternative model comes out by combining two models. It is reasonable to assume that the more competent an agent is, the less likely he or she will be to share the news. Consequently, it is possible to define the γ in the previous model not as a constant equal for all, but as a function of competence x. This function must take smaller and smaller values as x increases. One idea might be to define $\gamma(x) = r(1-x)$, with $r > 0$. We note that, with the choice of this function, we have that, if $x = 1$ we fall back into the starting model studied at the beginning of part II, while if $x = 0$, we relax the constraint by falling back into the last model discussed. For intermediate values, we get that, as $\gamma(x)$ decreases, it becomes increasingly difficult for an agent with competence above the degree of falsity of the news to repost the news.
In this case, the repost-related popularity growth function becomes:

$$
P_3(x,y,c) = \nu c\chi(x < 1 - y + r(1-x)). \tag{9.9}
$$

The change over time of the mean conditional on the falsity of the news under this model is:

$$
\begin{aligned}
\frac{dM_{V|Y}}{dt} &= \int_0^1 \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} (-\mu v + P_3(x,y,c)) h_y(t,v) p^\infty(x,c)\, dv\, dc\, dx \qquad (9.10)\\
&= -\mu M_{V|Y} + \nu \int_0^1 \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} c\chi(x < 1 - y + r(1-x)) h_y(t,v) p^\infty(x,c)\, dc\, dv\, dx,
\end{aligned}
$$

in which:

$$\int_0^1 \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} c\chi(x < 1 - y + r(1-x)) h_y(t,v) p^\infty(x,c)\, dc\, dv\, dx =$$

$$= \int_0^1 \int_{\mathbb{R}_+} c\chi(x < 1 - y + r(1-x)) p^\infty(x,c)\, dc\, dx$$

$$= \int_0^{\frac{1-y+r}{1+r}} \int_{\mathbb{R}_+} cp^\infty(x,c)\, dc\, dx.$$

If we again assume independence:

$$\int_0^{\frac{1-y+r}{1+r}} \int_{\mathbb{R}_+} cp^\infty(x,c)\, dc\, dx = \int_{\mathbb{R}_+} c\tilde{g}^\infty(c) dc \int_0^{\frac{1-y+r}{1+r}} [a\delta(x) + (1-a)\delta(x-1)] dx.$$

We note that, for each choice of $y \in [0,1]$, we have $\frac{1-y+r}{1+r} \leq 1$. If $y = 0$, we get the mean (7.7). On the other hand, if $y > 0$, the upper extreme of the definition interval of the integral will be less than 1 and thus the mean will be given by (7.11). Both results are consistent with the original model.

Conclusion

In this work, two models were proposed concerning two distinct aspects of fake news.
The first model deals with the study of the changes in personal competence due to the interactions with the news, which is characterized by a certain degree of falsehood. This model provides an aggregate view of the population, without distinguishing individuals into groups as is often done in the epidemiological approach.
The results obtained theoretically, with kinetic theory, were then checked by a numerical approach and both the methods gave evidence of a polarization toward extreme degrees of competence, nullifying over long timescales the intermediate values of knowledge.
In the second model, another crucial aspect of news was studied: the spread of their popularity on a social network. In the starting model, it was assumed that anyone who is not competent enough to understand the reliability of the news, would share it.
Specifically, we focused on studying the trend of popularity conditional on a fixed value of falsehood, to give a comparison based on different levels of falsity. This approach showed that, regardless of the initial average degree of popularity, a totally true news will have greater or equal popularity than news with non-zero degrees of falsity. Furthermore, we analyzed the behavior for t that tends to infinity and, unlike the initial model that asymptotically showed a pull toward two different values, this one returned that the dynamics, in the long run and avoiding the presence of a stochastic fluctuation, will tend toward a single value: the asymptotic conditional mean.

These models provide a starting point for future developments: for example, in the study of the evolution of knowledge, the model could be improved by adding a term related to the variation in competence due to the interaction with the surrounding environment (see [4] and [7]). In our model we worked, for simplicity, with the degree of falsehood uniformly distributed, however, this assumption is overly simplistic since it is reasonable to think that the number of news with a small level of falsity is higher than the ones that are completely untrue. Thus, another possible way to enrich the model is to consider a less trivial distribution for the degree of falsity.
Regarding the popularity model, it is realistic to think that social network connections change over time, hence, further studies in which the assumption of a constant degree of connection is avoided can be carried out. This can be done using an approach similar to the one used in the entire thesis, writing a Boltzmann-type equation for $g(c,t)$ and then deriving, through the Fokker-Planck equation, the asymptotic distribution $g^{\infty}(c)$, to be substituted into the popularity model proposed in Section 6.1.

Furthermore, we studied the asymptotic behavior for the popularity model omitting the stochastic fluctuation. It could be interesting to analyze how the model changes in presence of a stochastic fluctuation, exactly as we did for the model of the competence in Section 4.2.

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