POLITECNICO DI TORINO

Corso di Laurea Magistrale
in Ingegneria Matematica

Tesi di Laurea Magistrale

Numerical investigation on the eigenvalue problem of the quantum Rabi model

Relatori

prof. Francesco Antonino Raffa

prof. Gianni Coppa

Candidato

Alessandro Fissore

Anno Accademico 2021-2022

Summary

Historically, light was considered as an electromagnetic wave and wave propagation was embodied into the Maxwell equations. This classical approach cannot explain all optical phenomena: indeed, some of them requires the use of the quantum theory and the description of light as a stream of photons. These phenomena are studied by quantum optics.

The first approach to the study of the light-matter interaction was classical, where the atom was considered as a dipole and the light as an electromagnetic wave. Successively, a semi-classical approach was introduced, where semi-classical means that the behaviour of the atom is described using the quantum theory, whereas the light is represented as a wave. In the second half of the twentieth century a quantum approach was applied also to the light introducing the concept of photon, so that quantum optics was born, as it is known today.

Quantum optics finds applications in different fields, for example: laser, cavity QED (Quantum Electrodynamics), quantum computing through photons, etc. In this thesis the focus is on the quantum optics application to the matter-radiation interaction.

In the world of quantum optics, the semi-classical model firstly proposed by Rabi (1937) has great relevance. It was successively modified resorting to the quantum theory. In this thesis the fully quantum version of the model is investigated. The quantum Rabi model describes the simplest interaction between matter and light, as it considers one mode of the quantum electromagnetic field and matter is described in the form of a two level atom.

Obviously, diagonalizing the Hamiltonian of the Rabi model is of great importance, since knowing its spectrum means knowing the behaviour of the system. However, at

present, analytical expressions for its eigenvalues and eigenvectors are not known. Therefore, one introduces simplified models whose closed form solutions are known. The models presented in this thesis are the Jaynes-Cummings and anti Jaynes-Cummings models.

In the following, the Hamiltonians of both models are derived from the Rabi Hamiltonian. For these models explicit expressions of both eigenvalues and eigenvectors are available. They are thoroughly discussed in the present work. The role of the rotating-wave approximation in the definition of both Jaynes-Cummings and anti Jaynes-Cummings models is carefully analyzed.

For the full Rabi Hamiltonian, approximated eigenvalues are obtained. The eigenvalues are numerically computed using two different methods. Both methods are derived in the framework of the thesis.

Emphasis is given to the comparison between the Rabi eigenvalues and those of the Jaynes-Cummings and anti Jaynes-Cummings models. The comparison of the eigenvalues gives the validity conditions for the two simplified models.

Acknowledgements

Vorrei ringraziare la mia famiglia, per avermi supportato sempre in questi anni e per avermi dato la possibilità di crescere in un ambiente meraviglioso. Ringrazio mia mamma, perché è stata la prima a credere in me e a spronarmi nei momenti di difficoltà, e mio papà, per avermi mostrato l'importanza dell'impegno e della costanza che mi hanno permesso di raggiungere questo traguardo.

Ringrazio moltissimo i miei fratelli, Alberto e Samuele. Alberto, perché ha saputo essere un esempio da seguire e una persona con cui confidarsi, e Samuele, perché ha saputo portare gioia e spensieratezza nelle mie giornate, come solo i bambini sanno fare.

Voglio ringraziare i miei amici, per esserci sempre stati. Voglio ringraziarli per i loro consigli e per il tempo che mi hanno dedicato, per i momenti di gioia e di felicità che ho condiviso con loro. E' con loro che riesco sempre a ritrovare il sorriso e a dimenticare le preoccupazioni.

Vorrei ringraziare i professori Franceso Antonino Raffa e Gianni Coppa, relatori di questa tesi, per i loro preziosi consigli e per avermi guidato attentamente nella stesura di questo lavoro.

Contents

Chapter 1

Definition of the Rabi model

1.1 Introduction

The Rabi model gives a fundamental theoretical description of light-matter interaction, Specifically, the Rabi Hamiltonian describes the interaction of a two-level atom with a single mode of the electromagnetic radiation via a dipole interaction.
This first chapter describes the steps leading to the various terms of the Rabi Hamiltonian. Since Lie algebras play a crucial role in the derivation of the Hamiltonian this chapter opens with a summary of the Lie algebras. In the second section the free Hamiltonian is presented. The free Hamiltonian describes the non-interacting system and it is the first part of the Rabi Hamiltonian. In the third section the interaction Hamiltonian is described. Interaction Hamiltonian is the second part of the Rabi Hamiltonian and describes the interaction between the matter, in the form of a two-level atom, and a single mode of the electromagnetic field. Since, by its very definition, the interaction Hamiltonian lives in a tensor product space, the main properties of this kind of space and its elements are outlined in section three.

1.2 Mathematical treatment of Lie algebras

In this section Lie algebras are presented. For convenience, other mathematical structures, namely, sets, groups, vector spaces are introduced with reference to the exposition reported in [1].

Definition 1.2.1 (Set)**.** A set is a collection of objects that do not necessarily have any additional structure or properties.

Definition 1.2.2 (Group)**.** A group G is a set $g_1, g_2, \ldots, g_n \in \mathsf{G}$ equipped with an operation, called group multiplication ($\circ$) such that

1. $g_i \in \mathsf{G}, g_j \in \mathsf{G} \implies g_i \circ g_j \in \mathsf{G}$ closure
2. $g_i \circ (g_j \circ g_k) = (g_i \circ g_j) \circ g_k$ associativity
3. $g_1 \circ g_i = g_i = g_i \circ g_1 \ \forall g_i$ existence of identity
4. $g_k \circ g_l = g_l \circ g_k = g_1$ unique inverse $g_l = g_k^{-1}$

A group is called abelian or commutative if it obeys also the following postulate

5. $g_i \circ g_j = g_j \circ g_i \ \forall g_i, g_j$ commutativity

In an abelian group the group multiplication operator is denoted as $(+)$ instead of $(\circ)$. An important example of group is given by the set of the real $n \times n$ non-singular matrices under matrix multiplication. This group is called general linear group $\mathsf{GL}(n, \mathbb{R})$. It satisfies the conditions 1 to 4 for a group but it does not satisfy the condition 5, therefor it is not an abelian group.

Definition 1.2.3 (Field)**.** A field F is a set of elements $f_0, f_1, \ldots,$ together with two operations

i. $+$ called addition

ii. $\circ$ called scalar multiplication

such that

i. F is an abelian group under $+$, with f_0 the identity.

ii.
1. $f_i \circ f_j \in F$ — closure
2. $f_i \circ (f_j \circ f_k) = (f_i \circ f_j) \circ f_k$ — associativity
3. $f_i \circ 1 = 1 \circ f_i = f_i$ — identity
4. $f_i \circ f_i^{-1} = 1 = f_i^{-1} \circ f_i,\ f_i \neq f_0$ — inverse, except for f_0
5. $f_i \circ (f_j \circ f_k) = f_i \circ f_j + f_i \circ f_k$
   $(f_i \circ f_j) \circ f_k = f_i \circ f_k + f_j \circ f_k$ — distributive law

A field is said to be commutative if it satisfies the further condition

6. $f_i \circ f_j = f_j \circ f_i$ — commutativity

The most used fields are real and complex numbers and the quaternions. A complex number can be represented in the form

$$c = a1 + ib$$

where the units 1 and $i = \sqrt{-1}$ obey

$$1 \cdot 1 = 1; \qquad i \cdot 1 = 1 \cdot i = i; \qquad i \cdot i = -1$$

and $a,\ b$ are real numbers. Each quaternion can be represented in the form

$$q = q_0\lambda_0 + q_1\lambda_1 + q_2\lambda_2 + q_3\lambda_3$$

where q_i for $i = 0,1,2,3$ are real numbers and λ_i obey

$$\lambda_0\lambda_i = \lambda_i\lambda_0 = \lambda_i, \qquad \lambda_i\lambda_i = -\lambda_0, \qquad i = 0,1,2,3$$
$$\lambda_1\lambda_2 = -\lambda_2\lambda_1 = \lambda_3, \qquad \lambda_2\lambda_3 = -\lambda_3\lambda_2 = \lambda_1, \qquad \lambda_3\lambda_1 = -\lambda_1\lambda_3 = \lambda_2,$$

where in particular λ_0 is the unit, $\lambda_0 = 1$. For quaternions and complex numbers complex conjugation can be defined

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3)^\dagger = (\lambda_0, -\lambda_1, -\lambda_2, -\lambda_3)$$
$$(1, i)^\dagger = (1, -i).$$

Before introducing the Lie algebras the definition of vector space must be given.

Definition 1.2.4 (Linear vector space)**.** A linear vector space V is

a. a collection $\mathbf{v}_0, \mathbf{v}_1, \ldots, \in V$ called vectors

b. a collection $f_0, f_1, f_2, \ldots, \in F$, a field

together with two operations

i. vector addition $+$

ii. scalar multiplication $\circ$

such that

i. $(V, +)$ is an abelian group, V satisfies all the five propositions of Definition 1.2.2.

ii.

1.	$f_i \in \mathrm{F},\ \mathbf{v}_j \in V \implies f_i \mathbf{v}_j \in V$	closure
2.	$f_i \circ (f_j \circ \mathbf{v}_k) = (f_i \circ f_j) \circ \mathbf{v}_k$	associativity
3.	$1 \circ \mathbf{v}_i = \mathbf{v}_i = \mathbf{v}_i \circ 1$	identity
4.	$f_i \circ (\mathbf{v}_k + \mathbf{v}_l) = f_i \circ \mathbf{v}_k + f_i \circ \mathbf{v}_l$ $(f_i + f_j) \circ \mathbf{v}_k = f_i \circ \mathbf{v}_k + f_j \circ \mathbf{v}_k$	bilinearity

One supplements the definition of vector space with the the following features: The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N$ are linearly independent if

$$\sum_{i=1}^{N} \alpha_i \mathbf{v}_i = 0 \implies \alpha_i = 0 \qquad i = 1,2,\ldots,N.$$

For an N-dimensional vector space it is always possible to find a set of N nonzero linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N$ such that every set of $N+1$ nonzero vectors are linearly dependent. Any such maximal set of vectors is called a basis for the vector space and any vector $\mathbf{v}$ in this space can be expanded in terms of a basis

$$\mathbf{v} = \sum_{i=1}^{N} \alpha_i \mathbf{v}_i.$$

For an N-dimensional vector space a canonical representation exists. The canonical representation is indicated with V_N and its basis vectors are given by the $N \times 1$ vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \ \mathbf{e}_N = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

The simplest example is V_3, the canonical representation in the $\mathbb{R}^3$ space

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ldots, \ \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Finally it is feasible to introduce the definition of algebras.

Definition 1.2.5 (Linear algebra)**.** A linear algebra A consists of

a. a collection $\mathbf{v}_1, \mathbf{v}_2, \ldots, \in V$, called vectors

b. a collection $f_0, f_1, f_2, \ldots, \in F$, a field.

together with three kind of operations

i. vector addition, $+$

ii. scalar multiplication, $\circ$

iii. vector multiplication, $\square$

such that

i. $(V, +)$ is an abelian group

ii. 1. $f_i \in \mathrm{F}, \mathbf{v}_j \in V \implies f_i \mathbf{v}_j \in V$ closure

2. $f_i \circ (f_j \circ \mathbf{v}_k) = (f_i \circ f_j) \circ \mathbf{v}_k$ associativity

3. $1 \circ \mathbf{v}_i = \mathbf{v}_i = \mathbf{v}_i \circ 1$ identity

4. $f_i \circ (\mathbf{v}_k + \mathbf{v}_l) = f_i \circ \mathbf{v}_k + f_i \circ \mathbf{v}_l$
   $(f_i + f_j) \circ \mathbf{v}_k = f_i \circ \mathbf{v}_k + f_j \circ \mathbf{v}_k$ bilinearity

iii. 1. $\mathbf{v}_1, \mathbf{v}_2 \in V \implies \mathbf{v}_1 \square \mathbf{v}_2 \in V$ closure

2. $(\mathbf{v}_1 + \mathbf{v}_2)\square\mathbf{v}_3 = \mathbf{v}_1\square\mathbf{v}_3 + \mathbf{v}_2\square\mathbf{v}_3$
   $\mathbf{v}_1\square(\mathbf{v}_2 + \mathbf{v}_3) = \mathbf{v}_1\square\mathbf{v}_2 + \mathbf{v}_1\square\mathbf{v}_3$ bilinearity

Different varieties of algebras may be obtained, depending on which additional postulate are also satisfied. Two properties are particularly important

3. $\mathbf{v}_1\square\mathbf{v}_2 = -\mathbf{v}_2\square\mathbf{v}_1$ antisymmetry
4. $\mathbf{v}_1\square(\mathbf{v}_2\square\mathbf{v}_3) = (\mathbf{v}_1\square\mathbf{v}_2)\square\mathbf{v}_3 + \mathbf{v}_2\square(\mathbf{v}_1\square\mathbf{v}_3)$ derivative property

Example 1.2.1. An important example of a linear algebra is given by the set of $n \times n$ real antisymmetric matrices $A^T = -A$, in component $A_{ij} = -A_{ji}$. The vector multiplication is defined as

$$A\square B = [A, B] = AB - BA \tag{1.1}$$

$$[A, \beta B + \gamma C] = \beta[A, B] + \gamma[A, C]. \tag{1.2}$$

All the conditions given in Definition 1.2.5 are satisfied by this set of matrices under this operation, this system is a linear algebra and the operation is called commutator.

Definition 1.2.6 (Lie algebra)**.** A linear algebra with commutation relation that satisfies the antisymmetric condition is called Lie algebra. The derivative property is satisfied by a Lie algebra and it is rewritten as

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

called Jacobi's identity.

Linear algebras are defined as vector spaces, therefore it is convenient to introduce basis vectors. As an example, a basis for the general Lie algebra presented in the example above is derived.

Example 1.2.2. Let M_{ij} be the $n \times n$ matrix with 1 in the ith row and jth column and zeroes elsewhere,

$$M_{ij} = \begin{bmatrix} & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \end{bmatrix}.$$

There are n^2 matrices of this form. Since this Lie algebra is the set of antisymmetric matrices, a basis formed by antisymmetric matrices must be found.

$$A_{ij} = M_{ij} - M_{ji} = -A_{ji}.$$

These matrices A_{ij} are the required basis. The subscripts ij in this case do not indicate an element of the matrix but they represent a well defined matrix. As example, an element of the basis in the 3×3 space is

$$A_{12} = M_{12} - M_{21} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} = -A_{21}$$

To describe an algebra, finding its generators it is not enough, commutation between them must be considered because it describes the structure of the algebra. In this example

$$[A_{ij}, A_{kl}] = A_{ij}A_{kl} - A_{kl}A_{ij}.$$

When an N-dimensional vector space is studied it is useful to map it into the canonical vector space V_N and different maps exist:

- The operation of mapping one algebraic structure into a similar one is called a homomorphism if it preserves the operations associated with that structure;
- If the mapping operation is one to one, or faithful, so that an inverse is well defined and exists one has an isomorphism;
- When the mapping is into an algebraic structure that can be written down concretely it is called realization and if it is into a set of matrices it is called representation.

An example of map is the exponential map. It is defined as

$$\exp(\mathbf{X}) = \sum_{n=0}^{\infty} \frac{\mathbf{X}^n}{n!}$$

with $\mathbf{X}$ a matrix.
A general property holds for this map. The exponential of an element in a vector space on which a Lie algebra structure is defined is an element of some group. Therefore, it is possible to study the Lie algebras in analogy with groups using the exponential map. Some examples of groups are presented and their algebras are found using this map.

Example 1.2.3 (Special linear group)**.** The first group is the special liner group indicated with $\mathsf{SL}(n, \mathbb{R})$ if it is defined on the field of real numbers or $\mathsf{SL}(n, \mathbb{C})$ if it is defined on the field of complex numbers. All the matrices $\mathbf{A}$ in this group satisfy the condition

$$\det(\mathbf{A}) = 1$$

The special linear group is a subgroup of the general linear group $\mathsf{GL}(n)$. The importance of this group is linked to the change of basis. A matrix $\mathbf{A}$ in $\mathsf{SL}(n)$ and a set of basis vectors of the $\mathbb{R}^n$ vector space $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ are given. The product $e_1 \times e_2 \times \cdots \times e_n$ is called volume element associated with the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. If $\mathbf{A}$ is an $n \times n$ invertible matrix, a new set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ can be found with the relation

$$\mathbf{e}'_j = \sum_{i=1}^{n} A_{ji} \mathbf{e}_i.$$

The new set of vectors is a basis for $\mathbb{R}^3$ and the following relationship holds

$$e'_1 \times e'_2 \times \cdots \times e'_n = \big(\det(\mathbf{A})\big)\ e_1 \times e_2 \times \cdots \times e_n$$

where the determinant of $\mathbf{A}$ is called the determinant of the transformation.
If the matrix $\mathbf{A}$ is in the special linear group this change of basis preserves the volume, in particular

$$e'_1 \times e'_2 \times \cdots \times e'_n = e_1 \times e_2 \times \cdots \times e_n$$

.

To continue the classification of the classical groups it is necessary to introduce the metric.

Definition 1.2.7 (Metric)**.** A metric function on a vector space V is a mapping of a pair of vectors into a number in the field F associated with the vector space

$$(\mathbf{v}_1, \mathbf{v}_2) = f \qquad \mathbf{v}_1, \mathbf{v}_2 \in V,\ f \in F.$$

This mapping obeys

$$(\mathbf{v}_1, \alpha\mathbf{v}_2 + \beta\mathbf{v}_3) = \alpha(\mathbf{v}_1, \mathbf{v}_2) + \beta(\mathbf{v}_1, \mathbf{v}_3) \tag{1.3}$$

and

$$(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2, \mathbf{v}_3) = (\mathbf{v}_1, \mathbf{v}_3)\alpha + (\mathbf{v}_2, \mathbf{v}_3)\beta \tag{1.4}$$

or

$$(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2, \mathbf{v}_3) = (\mathbf{v}_1, \mathbf{v}_3)\alpha^\dagger + (\mathbf{v}_2, \mathbf{v}_3)\beta^\dagger. \tag{1.5}$$

Metrics obeying condition (1.3) and (1.4) are called bilinear metrics, those obeying (1.3) and (1.5) are called sesquilinear. The metric is completely specified by its action on each pair of basis vectors.

$$(\mathbf{e}_i, \mathbf{e}_j) = g_{ij}.$$

The transformation of the metric function g_{ij} under a change of basis $\mathbf{e}'_j = \sum_r A_{jr}\mathbf{e}_r$ is given by

$$g'_{ij} = (\mathbf{e}'_i, \mathbf{e}'_j) = (\sum_r A_{ir}\mathbf{e}_r, \sum_s A_{js}\mathbf{e}_s) = \sum_r \sum_s A_{js} g_{rs} A^\dagger_{ir}.$$

Groups preserving bilinear symmetric metrics are called orthogonal. They are denoted $\mathsf{O}(n, \mathbb{R})$ or $\mathsf{O}(n, \mathbb{C})$ respectively if the matrices are defined on the real numbers field or complex numbers field. The orthogonal group is a subgroup of the general linear group. A matrix $\mathbf{Q}$ is in $\mathsf{O}(n, F)$ if it is orthogonal, which means that it is invertible and satisfies

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbb{I}$$

where $\mathbf{Q}^T$ is the transpose of the matrix $\mathbf{Q}$. A subgroup of $\mathsf{O}(n, F)$ is given by the orthogonal matrices with determinant one. This group is called the special orthogonal

group $\mathsf{SO}(n, F)$. Since the condition $\det(\mathbf{A}) = 1$ defines the special linear group, the special orthogonal group can be seen as the intersection of the orthogonal group and the special linear group

$$\mathsf{SO}(n, F) = \mathsf{O}(n, F) \cap \mathsf{SL}(n, F).$$

Groups preserving sesquilinear symmetric metrics are called unitary. They are denoted by $\mathsf{U}(n, \mathbb{R})$ or $\mathsf{U}(n, \mathbb{C})$ respectively if the matrices are defined on $\mathbb{R}$ or $\mathbb{C}$. Unitary group is a subgroup of $\mathsf{GL}(n, F)$. If a matrix $\mathbf{U}$ satisfies

$$\mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U} = \mathbb{I},$$

it is called unitary and it belongs to the unitary group. Also in this case the group of unitary matrices which satisfy $\det(\mathbf{U}) = 1$ forms a subgroup of $\mathsf{U}(n, F)$ called special unitary group $\mathsf{SU}(n, F)$. The special unitary group is the result of the following intersection

$$\mathsf{SU}(n, F) = \mathsf{U}(n, F) \cap \mathsf{SL}(n, F).$$

One notes that $\mathsf{U}(n, \mathbb{R})$ and $\mathsf{O}(n, \mathbb{R})$ are the same group because in the real numbers field the conjugate transpose operation † reduces to the transpose operation.
Using the exponential map it is possible to find a Lie algebra for each group. As an example, an unitary matrix $\mathbf{U}$ is given, so that $\mathbf{U}^\dagger = \mathbf{U}^{-1}$. Using the exponential map $\mathbf{U}$ can be written as $\mathbf{U} = e^{\alpha\mathbf{X}}$ with α a real number. Thus, $e^{\alpha\mathbf{X}}$ is unitary if and only if

$$e^{\alpha\mathbf{X}^\dagger} = e^{-\alpha\mathbf{X}}$$

and this equivalence holds for all real α if $\mathbf{X}^\dagger = -\mathbf{X}$. Then the Lie algebra of $\mathsf{U}(n)$ is the vector space of matrices $\mathbf{X}$ such that $\mathbf{X}^\dagger = -\mathbf{X}$ and it is denoted as $\mathfrak{u}(n)$.
If $\mathbf{U}$ is in the special unitary group it also satisfies the condition $\det(\mathbf{U}) = 1$, so that using the exponential map

$$\det e^{\alpha\mathbf{X}} = e^{\alpha \operatorname{tr}\mathbf{X}} = 1 \implies \operatorname{tr}(\mathbf{X}) = 0$$

Then the Lie algebra $\mathfrak{su}(n)$ of $\mathsf{SU}(n)$ is the vector space of matrices that satisfy both $\mathbf{X}^\dagger = -\mathbf{X}$ and $\operatorname{tr}(\mathbf{X}) = 0$.

1.3 Free Hamiltonian

The free Hamiltonian describes the non interacting components of the system, H_A and H_F. H_A is the Hamiltonian that describes the atom in its approximation as a two-level system. H_F is the Hamiltonian of the electromagnetic field. Specifically H_F is only an approximation of the actual electromagnetic Hamiltonian, since it contains only one mode of the field. H_F and H_A are introduced following the study reported in [2]

To study the free Hamiltonian two vector spaces $\mathcal{E}_A$ and $\mathcal{E}_F$ are introduced. $\mathcal{E}_A$ is a two-dimensional vector space and it is used to describe the behaviour of the two-level atom, whereas $\mathcal{E}_F$ is an infinite-dimensional vector space and it is used to describe the electromagnetic field.

At this point the tensor product space must be introduced because the physical system consists of two different components that live in two different spaces, it is defined as $\mathcal{E} := \mathcal{E}_A \otimes \mathcal{E}_F$ and each term of the Hamiltonian is written as an element of $\mathcal{E}$.

A mathematical description of the tensor product space is developed in section 1.4.1.

1.3.1 Two-level atom approximation

The interaction between light and atoms is usually developed in terms of the two-level atom approximation. Bohr postulated that a quantum light of angular frequency ω is emitted or absorbed when an atom jumps between two quantized energy level

$$E_2 - E_1 = \hbar\omega, \tag{1.6}$$

where E_1 and E_2 are the energies of the two quantized levels. The atom has several quantum levels and there will be many optic transitions between them. In the two-level atom approximation only the transition that satisfies (1.6) is considered and all the other levels are ignored. The validity of this approximation is based on the resonance phenomenon. In the classical picture, light beam induces dipole oscillation in the atom. If the light frequency and the natural frequency of the atom are similar, the interaction between them will be strong and the magnitude of the dipole will be large. When the two frequencies are far away the system is off-resonance and both the oscillation magnitude

of the dipole and the interaction will be small. It follows that it is a good approximation to ignore the interactions between light and the atom in off-resonance conditions.

In this approximation the atom can be found in two states denoted by $|g\rangle$ and $|e\rangle$, the ground and excited state, respectively. In $\mathcal{E}_A$ the states can be represented as

$$|g\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, |e\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

They form an ON basis in $\mathcal{E}_A$

$$\langle g|e\rangle = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 = \langle e|g\rangle ;$$

$$\langle g|g\rangle = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 = \langle e|e\rangle .$$

To derive the Hamiltonian of a two-level system it is necessary to introduce the Pauli matrices σ_1, σ_2 and σ_3;

$$\sigma_1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_2 \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_3 \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.7}$$

From Pauli matrices the spin operators can be derived. Since natural units $\hbar = 1$ are used throughout, the spin operators are defined as

$$S_k = \frac{1}{2}\sigma_k \qquad k = 1,2,3.$$

The relevance of spin operators in this analysis is due to the definition of the operators S_+ and S_-:

$$S_+ = S_1 + iS_2, \qquad S_- = S_+^\dagger.$$

Indeed, in a group-theoretic approach the matrices $\{S_+, S_-, S_3\}$ are the generators of $\mathfrak{su}(2)$ Lie algebra with relationship:

$$[S_+, S_-] = 2S_3 \qquad [S_3, S_\pm] = \pm S_\pm.$$

It is readily verified that the operators S_+ and S_- can be written also in terms of atom eigenstates

$$S_+ = |e\rangle\langle g| \qquad S_- = |g\rangle\langle e| . \tag{1.8}$$

Physically, S_+ describes the transition from the ground state $|g\rangle$ to the excited state $|e\rangle$ whereas S_- describes the opposite transition from the state $|e\rangle$ to the state $|g\rangle$. These two operators are called ladder operators because they describe the transition between discrete states. To derive the Hamiltonian it is necessary to observe that the zero energy level for the atom is set halfway between the two states, therefore H_A satisfies

$$H_A |g\rangle = -\frac{1}{2}\omega_0 |g\rangle \qquad H_A |e\rangle = \frac{1}{2}\omega_0 |e\rangle . \tag{1.9}$$

The action of S_3 on the eigenstates is

$$S_3 |g\rangle = -\frac{1}{2} |g\rangle \qquad S_3 |e\rangle = \frac{1}{2} |e\rangle . \tag{1.10}$$

It follows that in $\mathcal{E}_A$ the Hamiltonian of the two-level atom is written as

$$H_A = \omega_0 S_3,$$

while in the product space $\mathcal{E} = \mathcal{E}_A \otimes \mathcal{E}_F$ it takes the form

$$H_A = \omega_0 S_3 \otimes \mathbb{I}$$

where $\mathbb{I}$ is the infinite-dimensional identity in $\mathcal{E}_F$.

1.3.2 Electromagnetic field

The quantum theory of light is based on the quantum harmonic oscillator which is studied in analogy with the classical harmonic oscillator. The Hamiltonian in the classical case is well known,

$$H = \frac{p}{2m} + \frac{1}{2}m\omega^2 x^2,$$

where p is momentum of the harmonic oscillator, m is its mass and ω is its frequency.
To study the quantum harmonic oscillator x and p are replaced by two operators $\hat{x}$ and $\hat{p}$ defined as

$$\hat{x} = x \qquad \hat{p}_x = -i\frac{\partial}{\partial x}$$

and the Hamiltonian can be rewritten as

$$\hat{H} = \frac{\hat{p}_x}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

Since only the Hamiltonian of the quantum harmonic oscillator is considered here, the hat symbol ($\hat{\cdot}$) will be omitted for the sake of simplicity. In this study only one mode of the electromagnetic field is considered, though the formulation can be generalized to several modes.

For each mode two operators are defined. In units $\hbar = 1$ they are

$$a = \frac{1}{(2m\omega)^{\frac{1}{2}}}(m\omega x + ip_x), \qquad a^\dagger = \frac{1}{(2m\omega)^{\frac{1}{2}}}(m\omega x - ip_x). \tag{1.11}$$

They are called annihilation or lowering operator and creation or rising operator, respectively. Using these new operators the Hamiltonian of the single mode-field can be written as

$$H_F = \omega\left(\hat{n} + \frac{1}{2}\right) \tag{1.12}$$

where $\hat{n} = a^\dagger a$. The commutation relations between H_F and the operators a, $a^\dagger$ are

$$\left[H_F, a^\dagger\right] = \omega a^\dagger, \qquad [H_F, a] = -\omega a.$$

From the eigenvalue equation for H_F in the Fock space $\mathfrak{F} \equiv \{|n\rangle \text{ s.t. } \hat{n}\,|n\rangle = n\,|n\rangle\, \forall n \geq 0\}$

$$H_F\,|n\rangle = E_n\,|n\rangle\,, \tag{1.13}$$

where $|n\rangle$ is an eigenvector of H_F corresponding to the nth quantized energy level E_n and n is the number of excited quanta on this level.

If the operator $H_F a^\dagger$ is applied to $|n\rangle$ one has

$$H_F a^\dagger\,|n\rangle = (\omega + E_n)a^\dagger\,|n\rangle\,,$$

then also $a^\dagger\,|n\rangle$ is an eigenvector of H_F with energy eigenvalue $(E_n + \omega)$. Similarly,

$$H_F a\,|n\rangle = (-\hbar\omega + E_n)a\,|n\rangle\,,$$

so that also $a\,|n\rangle$ is an eigenvector with energy eigenvalue $(E_n - \omega)$.

These two equations prove that the energy spectrum of the harmonic oscillator is discrete and the energy levels are equally spaced. The operators a and $a^\dagger$ work on a set of discrete states then they are called ladder operators. The difference between these two operators

and the operators S_+ and S_- is that a and $a^\dagger$ work on an unbounded set of states while S_+ and S_- work on a finite set composed of only two states. The energy of the quantum harmonic oscillator must be always positive, zero is a lower bound for the energy states. If $|0\rangle$ is the vacuum state in Fock space, then one postulates

$$a\,|0\rangle = 0. \tag{1.14}$$

Using Eq. (1.14) the energy of the ground state can be easily found

$$H_F\,|0\rangle = \omega(\hat{n} + \frac{1}{2})\,|0\rangle = \frac{1}{2}\omega\,|0\rangle = E_0\,|0\rangle\,,$$

where

$$E_0 = \frac{1}{2}\omega.$$

This implies that the energy E_n of the nth level is

$$E_n = n\omega + E_0 = \omega(n + \frac{1}{2}). \tag{1.15}$$

The action of the operators a and $a^\dagger$ is described by their action on $|n\rangle$

$$a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,. \tag{1.16}$$

The set $\{\mathbb{I}, a, a^\dagger, \hat{n}\}$ constitutes a linear algebra and its structure is defined by the commutation relation

$$[\mathbb{I}, \bullet] = 0, \qquad \left[a, a^\dagger\right] = \mathbb{I}, \qquad [\hat{n}, a] = -a, \qquad \left[\hat{n}, a^\dagger\right] = a^\dagger.$$

In conclusion the Hamiltonian of the field H_F expressed in the product space $\mathcal{E}$ is

$$H_F = \mathbb{I}_2 \otimes \left(\hat{n} + \frac{1}{2}\right)\omega$$

where $\mathbb{I}_2$ is the two-dimensional identity in $\mathcal{E}_A$.

1.4 Interaction Hamiltonian

In this section the interaction Hamiltonian H_{int} is presented. It describes the interaction between the atom and the electromagnetic field in their relative approximations. This

Hamiltonian lives in the tensor product space $\mathcal{E} = \mathcal{E}_A \otimes \mathcal{E}_F$, therefore a description of the tensor product space and its properties is firstly given. Successively, the interaction Hamiltonian is derived and the full Rabi Hamiltonian is presented.

1.4.1 Tensor product space

Given two vector spaces V_N^1 and V_M^2, where N and M denote the dimensions of the spaces $N = \dim V_N^1$ and $M = \dim V_M^2$, it is desirable to construct a new vector space as the direct product of these two vectors spaces. This new vector space is called direct product vector space and it is denoted $V_N^1 \otimes V_M^2$.
Let $\{\mathbf{e}_1, \mathbf{e}_2 \ldots, \mathbf{e}_N\}$ be basis vectors for V_N^1 and $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_M\}$ be basis vectors for V_M^2. The $N \cdot M$ basis vectors for the direct product space $V_N^1 \otimes V_M^2$ are $\{\mathbf{e}_i \otimes \mathbf{f}_j\}$ for $i = 1,2,\ldots,N$ $j = 1,2,\ldots,M$. The tensor product space has the following property

$$\dim(V_N^1 \otimes V_M^2) = \dim V_N^1 \cdot \dim V_M^2.$$

Given two vectors $\mathbf{v}^1 \in V_N^1$ and $\mathbf{v}^2 \in V_M^2$, their product $\mathbf{v} = \mathbf{v}^1 \otimes \mathbf{v}^2$ belongs to the vector product space. The opposite statement is not true in general, a vector $\mathbf{w} \in V_N^1 \otimes V_M^2$ cannot be constructed as the direct product of a vector $\mathbf{w}^1 \in V_N^1$ and a vector $\mathbf{w}^2 \in V_M^2$.

An arbitrary vector $\mathbf{v}$ in the direct product vector space $V_N^1 \otimes V_M^2$ has component v_{ij} with respect to the basis vectors $\{\mathbf{e}_i \otimes \mathbf{f}_j\}$

$$\mathbf{v} = \sum_{i=1}^{N} \sum_{j=1}^{M} v_{ij} \mathbf{e}_i \otimes \mathbf{f}_j$$

Changes of basis $\mathbf{e}' = A\mathbf{e}$, $\mathbf{f}' = B\mathbf{f}$ in V_N^1 and V_M^2 respectively, induce a change of basis in the direct product space. In components

$$\mathbf{e}'_i \otimes \mathbf{f}'_j = (\sum_{r=1}^{N} A_{ir} \mathbf{e}_r) \otimes (\sum_{s=1}^{M} B_{js} \mathbf{f}_s) = \sum_{r=1}^{N} \sum_{s=1}^{M} A_{ir} B_{js} (\mathbf{e}_r \otimes \mathbf{f}_s).$$

1.4.2 Interaction Hamiltonian and Rabi Hamiltonian

As it was done for the Hamiltonian of the electromagnetic field also for the interaction Hamiltonian an analogy with the classical case proves useful. In the classical case the

interaction between an electron and the electromagnetic field, studied in its dipole approximation, is described by

$$H_{int} = -\mathbf{d} \cdot \mathbf{E}(\mathbf{r}, t)$$

where $\mathbf{E}(\mathbf{r}, t)$ is the electric field and $\mathbf{d} = e\mathbf{r}$ is the electric dipole. Replacing the electric field and the dipole with their respective operators the Hamiltonian operator is obtained

$$\hat{H}_{int} = -\hat{\mathbf{d}} \cdot \hat{\mathbf{E}}. \tag{1.17}$$

In this section the hat symbol ($\hat{\cdot}$) above the operators is manteined for clarity, because a few comparisons are presented between the operators and their classic counterparts.
Since only one mode of the electromagnetic field is considered it is seen as confined in a cavity and a simplified scheme of which is shown in figure 1.1. The scheme for the electromagnetic field is reported in figure 1.2. The field is considered as a plane wave that propagates along z-axis, the electric field is polarized along x-axis and the magnetic field along the y-axis. Only the equation for the electric field is taken into account,

$$\mathbf{E}(\mathbf{r}, t) = E_x(z, t)\mathbf{e}_x.$$

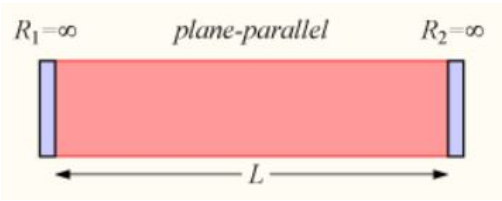

Figure 1.1. Scheme of a a planar cavity of length L within two parallel mirrors of reflectivity $R1$ and $R2$, respectively. The cavity acts as an interferometer when light of wavelength λ is introduced through one of the end mirror. Light confined in the cavity reflects multiple times, producing standing waves for certain resonance frequencies.

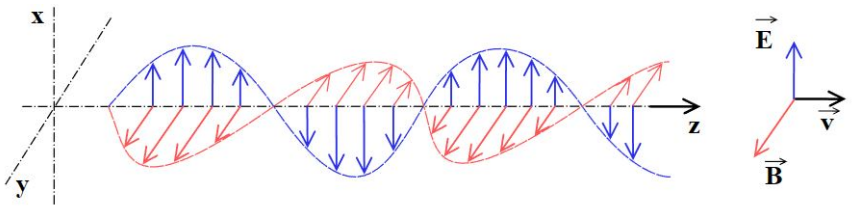

Figure 1.2. Scheme of electric and magnetic field, $E_x(z, t)$ and $B_y(z, t)$, considered as plane waves which propagates along z-axis.

The electric field is confined along z between two levels at distance L. On these levels the electric field must be zero for all t, namely the field satisfies the boundary conditions

$$E_x(z=0,t) = E_x(z=L,t) = 0.$$

These conditions are satisfied by a plane wave of the form

$$E_x(z,t) = Q(t)\sin(kz) \tag{1.18}$$

and the parameter $k \neq 0$ verifies

$$\sin(kL) = 0 \implies kL = m\pi \text{ for } m = 1,2,3\ldots$$

Replacing the equation (1.18) in the Ampère-Maxwell and Faraday equations in SI units in absence of sources

$$\nabla \times \mathbf{B} = \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

one gets

$$\frac{\partial B_y}{\partial z} = -\frac{1}{c^2}\frac{dQ}{dt}(t)\sin(kz), \tag{1.19}$$
$$\frac{\partial B_y}{\partial t} = -kQ(t)\cos(kz). \tag{1.20}$$

From (1.19)

$$B_y(z,t) = \frac{1}{kc^2}\frac{dQ}{dt}(t)\cos(kz)$$

and replacing B_y in the equation (1.20) it follows that $Q(t)$ satisfies the equation

$$\frac{d^2Q}{dt^2}(t) + c^2k^2Q(t) = 0.$$

This is the equation of an harmonic oscillator with unit mass and $\omega = ck$.
Using the equation for the energy of the electromagnetic field in SI units, a normalization for $Q(t)$ is given

$$\begin{aligned} U_{field} &= \frac{\epsilon_0}{2}\int_V (\mathbf{E}^2 + c^2\mathbf{B}^2)dV \\ &= \frac{1}{V}\int_V \left[\omega^2x^2(t)\sin^2(kz) + p^2(t)\cos^2(kz)\right]dxdydz \\ &= \frac{1}{2}(p^2 + \omega^2x^2). \end{aligned}$$

Recalling the definition of the electric field (1.18), and using this equation one can redefine Q, whose dimensions are those of an electric field, in terms of x that instead is the position of the harmonic oscillator:

$$Q(t) = \left(\frac{2\omega^2}{\epsilon_0 V}\right)^{1/2} x(t).$$

In conclusion, the equation of the electric field is

$$E_x(z,t) = \left(\frac{2\omega^2}{\epsilon_0 V}\right)^{1/2} x(t)\sin(kz).$$

To find the quantum operator associated to the electric field it is enough to replace x with the operator $\hat{x}$ in the previous equation. Remembering the definition of the ladder operators (1.11) and expressing $\hat{x}$ in function of a and $a^\dagger$, the quantum operator of the electric field takes the form

$$\hat{\mathbf{E}} = \left(\frac{\omega}{\epsilon_0 V}\right)^{1/2} \sin(kz)(a + a^\dagger)\mathbf{e}, \tag{1.21}$$

with $\mathbf{e}$ the polarization vector. Polarization describes the formation of dipoles in a material after the application of an electric field, the polarization vector is defined as the dipole momentum for unit of area. Replacing (1.21) in the equation (1.17), the interaction Hamiltonian operator is obtained

$$\hat{H}_{int} = \lambda \hat{d}(a + a^\dagger).$$

The terms $\hat{d} = \hat{\mathbf{d}} \cdot \mathbf{e}$ and $\lambda = -\sqrt{\omega/(\epsilon_0 V)}\sin(kz)$ are introduced.
The electric momentum operator $\hat{d}$ in the ON basis $\{|g\rangle, |e\rangle\}$ take the form

$$\hat{d} = \mathbb{I}\hat{d}\mathbb{I} = d_{ee}\,|e\rangle\langle e| + d_{gg}\,|g\rangle\langle g| + d_{eg}\,|e\rangle\langle g| + d_{ge}\,|g\rangle\langle e|$$

where

$$d_{ee} = \langle e|\,\hat{d}\,|e\rangle, \;\; d_{gg} = \langle g|\,\hat{d}\,|g\rangle \;\text{ and }\; d_{eg} = \langle e|\,\hat{d}\,|g\rangle = d_{ge}^\dagger.$$

The interaction Hamiltonian $-e\mathbf{r}\cdot\mathbf{E}(t)$ is odd respect the spatial inversion $r \to -r$. The eigenfunctions of the two-level atom have a spatial symmetry that is either even or odd. In this case a spatial inversion that preserves the ground state $|g\rangle$ is considered. With

this choice $H_{int}\,|g\rangle$ is odd and therefore $\langle g|\,H_{int}\,|g\rangle$ is null. A similar argument holds for $\langle e|\,H_{int}\,|e\rangle$ but in this case $|e\rangle$ is odd and $H_{int}\,|e\rangle$ is even. It follows that $\langle e|\,\hat{d}\,|e\rangle = \langle g|\,\hat{d}\,|g\rangle = 0$. It is also possible to choose $d_{eg} = d_{ge} = d_{eg}^{\dagger}$ then $d_{eg} = d_{ge} = d \in \mathbb{R}$. In conclusion, remembering equations (1.8), the operator can be written as

$$\hat{d} = d(\hat{S}_+ + \hat{S}_-),$$

and, removing the hat symbol, the interaction Hamiltonian is

$$H_{int} = k(S_+ + S_-) \otimes (a^{\dagger} + a), \tag{1.22}$$

where

$$k = d\lambda$$

is the matter-radiation coupling constant. It describes the strength of interaction between the electromagnetic field and the two-level atom.

The Rabi Hamiltonian, that describes the interaction between one mode of the electromagnetic field and the two-level atom, is the sum of the free Hamiltonian and the interaction Hamiltonian. In units $\hbar = 1$ it is

$$H_R = \omega_0 S_3 \otimes \mathbb{I} + \omega \mathbb{I} \otimes \left(\hat{n} + \frac{1}{2}\right) + k(S_+ + S_-) \otimes (a^{\dagger} + a). \tag{1.23}$$

The symbol $\otimes$ is omitted in the successive sections and the following correspondences are used

$$\begin{gathered} |g\rangle\,|n\rangle \equiv |g\rangle \otimes |n\rangle\,, \qquad |e\rangle\,|n\rangle \equiv |e\rangle \otimes |n\rangle\,, \\ O_a O_f \equiv O_a \otimes O_f, \qquad O_a \equiv O_a \otimes \mathbb{I}_2, \qquad O_f \equiv \mathbb{I} \otimes O_f, \end{gathered} \tag{1.24}$$

where $O_a \in \mathcal{E}_A$ and $O_f \in \mathcal{E}_F$.

In literature one meets other expressions for the Rabi Hamiltonian. For example, in the article of Braak [5] the expression for H_R is

$$H_R = \omega a^{\dagger} a + g\sigma_x(a + a^{\dagger}) + \Delta\sigma_z. \tag{1.25}$$

In this expression the constant term $\omega/2$ is discarded because it does not interfere in the study of the model and it can be easily reintroduced at any point without difficulties.

The differences in notation between Eqs. (1.23) and (1.25) are summarized in table 1.1. In Eq. (1.25) the Pauli matrices are preferred to the spin operators, moreover they are indexed as σ_x, σ_y and σ_z while in this thesis they are presented in equation (1.7) with names σ_1, σ_2 and σ_3, respectively. In analogy at the spin operator, one defines operators

$$\sigma_+ = \sigma_x + i\sigma_y, \qquad \sigma_- = \sigma_+^\dagger$$

and follows that σ_x can be expressed in function of σ_+ and σ_- as

$$\sigma_x = \sigma_+ + \sigma_-.$$

Regarding the term Δ, in this thesis it indicates the detunign of the system, so the difference between the frequencies ω and ω_0, in the work of Braak it is defined as $\Delta = \omega_0/2$. Other notations, different from the two in table 1.1, can be found in literature but their use in the works listed in the bibliography is limited so they are not reported.

Braak	ω	2Δ	σ_z	$\sigma_x = \sigma_+ + \sigma_-$	g	$\omega - 2\Delta$
This thesis	ω	ω_0	$2S_3$	$2(S_+ + S_-)$	k	$\Delta = \omega - \omega_0$

Table 1.1. Notations in Braak [5] and in this thesis, Eq. (1.25) and Eq. (1.23), respectively.

Chapter 2

Simplified models for the Rabi Hamiltonian

2.1 Introduction

In chapter 1 each term of the Hamiltonian has been described and the quantum Rabi Hamiltonian H_R has been introduced. Dispensing with the tensor product notation and in units $\hbar = 1$,

$$H_R = \omega_0 S_3 + \omega \left(\hat{n} + \frac{1}{2} \right) + k(S_+ + S_-)(a^\dagger + a), \tag{2.1}$$

with ω the field frequency, ω_0 the difference in energy of the two atom states; k is the coupling interaction strength. Let us recall that at resonance $\omega_0 = \omega$.

A description of the interaction terms in (2.1) is useful to understand their physical effects on the system and it is given in section 2.2. In this section are presented also two approximations of the Rabi Hamiltonian with their regions of validity, the Jaynes-Cummings and anti Jaynes-Cummings models. Successively, in sections 2.3 and 2.4, the Hamiltonians of these models are diagonalized to obtain two explicit expressions for the JC and anti-JC eigenvalues. In section 2.2, the rotating-wave approximation is applied to the Rabi Hamiltonian to obtain the Jaynes-Cummings model, therefore this approximation is presented in details in section 2.5.

2.2 Interaction terms and their effects

A brief description of the interaction terms in (2.1) is useful to understand their physical effects on the system.

- S_+a: This term describes the process of (stimulated) absorption, see [2]. The incident field forces an atom in its ground state $|g\rangle$ to oscillate and to emit a second quantum of light (photon), with the energy of one photon, such that it is in anti-phase with the incident field. These two fields interfere destructively with each other and the energy is absorbed by the atom that switches to the excited state $|e\rangle$.
- $S_-a^\dagger$: This term describes the phenomena of the stimulated emission. In this process an incident electromagnetic field, with the energy of one photon, forces the atom in its excited state $|e\rangle$ to oscillate and to emit a second electromagnetic field that interfere constructively with the incident field, so that the energy is emitted by the atom that switches to the ground state $|g\rangle$.
- $S_+a^\dagger$: In the process described by this term, the atom is forced to jump from its ground state $|g\rangle$ to its excited state $|e\rangle$ emitting an electromagnetic field. The emitted field is in phase with the incident field, so that a photon is gained by the incident electromagnetic field.
- S_-a: This term describes a process where the electromagnetic field loses one photon due to the destructive interaction with the field emitted by the atom, while it switches from its excited state $|e\rangle$ to its ground state $|g\rangle$.

The first two processes are the most intuitive because in these processes the energy is conserved. In the first case the energy lost by the field is used by the atom to jump from the ground state to the excited state and in the second case the inverse transition is described. In this case, absorption and stimulated emission are opposite processes.
This statement is not true in general. As reported by Pollnau in his work [3], the truly inverse processes are those that satisfies the following conditions:

- An absorbing atom removes one photon from an incident field containing $\phi + 1$ photons, such that the resulting field embodies ϕ photons,

- an emitting atom adds one photon to an incident field containing ϕ photons, such that $\phi + 1$ photons are contained in the resulting field.

Both phenomena involve the same number of photon and they are inverse processes in the amplitude-phase diagram. To prove this statement the adsorption of one photon from an incident field containing only one photon process and spontaneous emission process are taken into account and the amplitudes of the fields involved in these phenomena are reported in figures 2.1 and 2.2. These are truly inverse processes because the incident field, in the absorption process, has the same amplitude of the resulting field in the spontaneous emission process. Similarly, there is not incident field in the spontaneous emission and the same holds for the resulting field in the absorption process.
These observations do not hold when the two phenomena involves different numbers of photons. As example, the absorption of one photon process where the incident field contains two photon is considered, figure 2.3. The incident field has a different amplitude respect to the resulting field in the process of spontaneous emission. Moreover, the emitted field is not in opposite of phase and it has a different amplitude respect that of the incident one. This leads to the conclusion that the resulting field is not nullified. This process is not the opposite phenomenon of the spontaneous emission process, the truly inverse one is the stimulated emission where the incident field contains one photon.

The last two terms described in the list, $S_+a^\dagger$ and S_-a, are less intuitive and they describes two apparently unphysical processes because the total energy is not conserved by the single term. However, their effect over the total dynamics cannot be neglected in some regimes. These terms are known as virtual or counter-rotating terms (CRTs). The motivation for this name is discussed in section 2.5.

Although it is possible to understand the effect of each term in the Rabi Hamiltonian (2.1) the study of the full Hamiltonian is difficult. A proof of the existence of an exact solution for this equation has not been given yet, though a sort of analytical solution was found by Braak [5]. In literature, the method developed in [5] was subsequently applied to various extensions of the Rabi model, considering the interaction between the electromagnetic field with more than one atom or considering two or more modes of the electromagnetic radiation [8, 6, 4, 7].

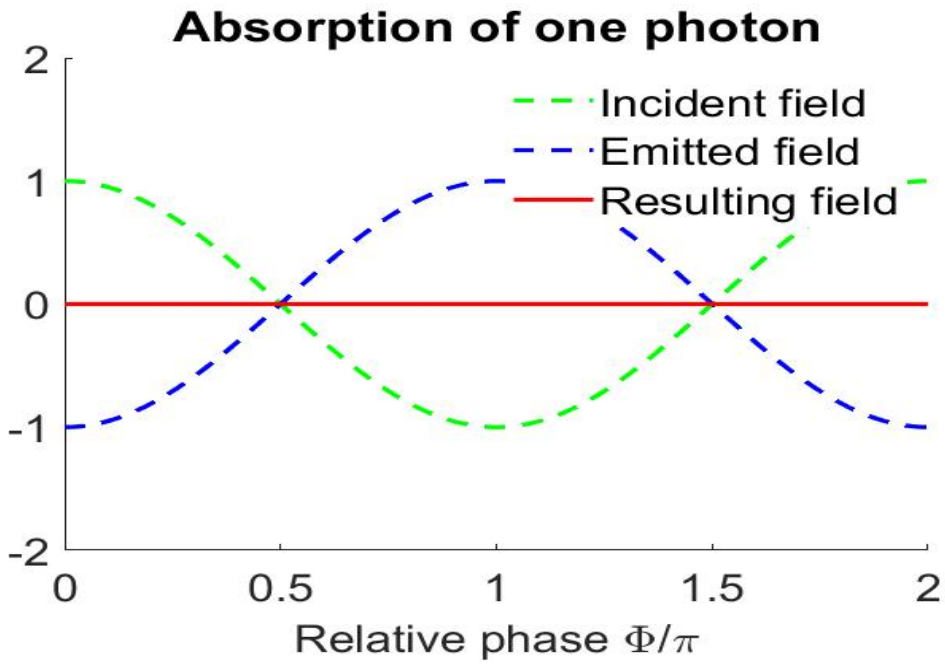

Figure 2.1. Absorption of one photon when the incident field contains only one photon. Incident field (green dashed line), field emitted by the atom (blue dashed line) and resulting field (red line). The incident and emitted fields destructively interfere each other, there is no resulting field. The behaviour of this process is the opposite of that described in figure 2.2, (see [3]).

Different dynamics for the Rabi Hamiltonian are shown for different regimes, which are defined by the relation among the three Hamiltonian parameter ω, ω_0 and k as presented by Xie [4]. Two approximations are of particular interest in this thesis, namely, the Jaynes-Cummings (JC) and the anti Jaynes-Cummings (anti-JC) models, whose regions of validity are, cf. [4]

- JC regime: $k \ll \omega$ and $|\omega - \omega_0| \ll |\omega + \omega_0|$,
- anti-JC regime: $k \ll \omega$ and $|\omega - \omega_0| \gg |\omega + \omega_0|$.

Jaynes-Cummings model was one of the first models employed to approximate the Rabi Hamiltonian in optics. It was proposed by Jaynes and Cummings and the rotating-wave approximation (RWA) was applied to the Rabi Hamiltonian. Briefly, this approximation consists in neglecting the so called counter rotating terms in (2.1), namely, the terms

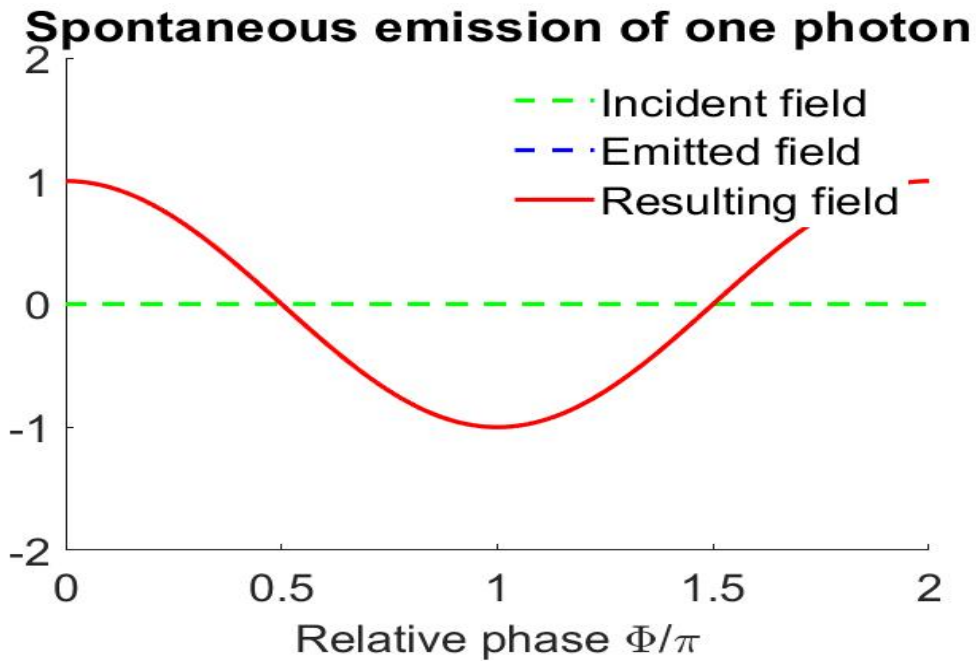

Figure 2.2. Spontaneous emission of one photon when there is no incident field. Incident field (green dashed line), field emitted by the atom (blue dashed line) and resulting field (red line). There is no incident field, the field emitted by the atom is also the resulting field. This is the opposite phenomenon of that reported in figure 2.1, (cf. [3]).

proportional to $S_+a^\dagger$ and S_-a. This approximation is valid only in regimes where the coupling strength is much weaker than the mode frequency, $k \ll \omega$. The second requested condition is $|\omega-\omega_0| \ll |\omega+\omega_0|$ and it is satisfied by a system close to resonance. The RWA turned out to be a valid approximation for the understanding of many experiments, then JC model is able to correctly describe most observed effects. The Hamiltonian achieved applying the RWA to (2.1) is the JC Hamiltonian

$$H_{JC} = \omega_0 S_3 + \omega(\hat{n} + \frac{1}{2}) + k(S_+a + S_-a^\dagger). \tag{2.2}$$

The JC model has a great relevance because for this model a closed form solution can be readily obtained, i.e., the Hamiltonian (2.2) can be analytically diagonalized.

In a quantum simulation one can go beyond conventional regimes and even reach unphysical situations [9], as when the atom and the mode have frequencies of opposite sign. In this case, $|\omega - \omega_0| \gg |\omega + \omega_0|$ and one can neglect the terms proportional to S_+a

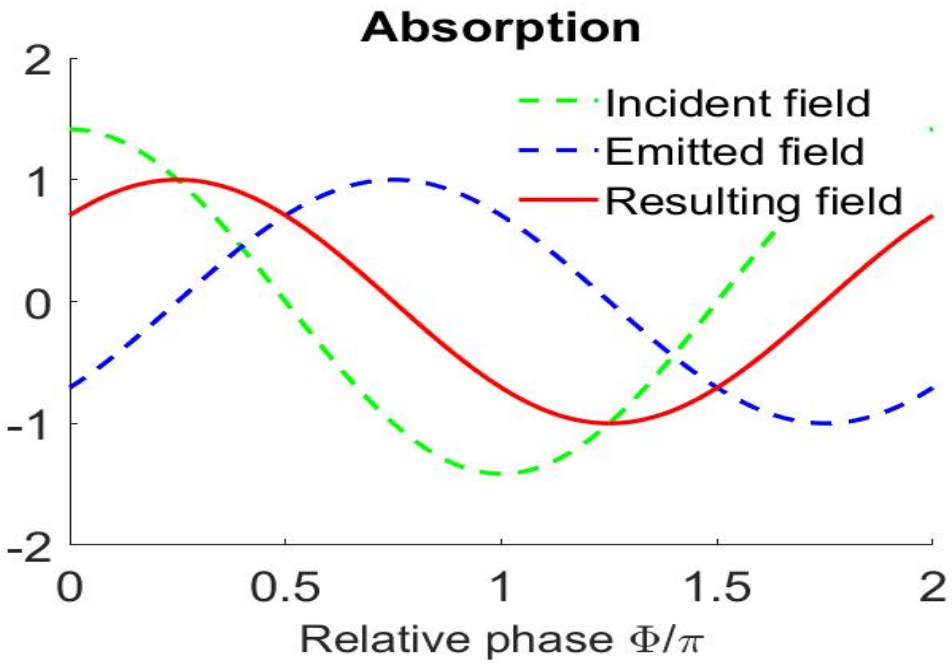

Figure 2.3. Amplitude of the incident field, green dashed line, field emitted by the atom of the system, blue dashed line, and resulting field, in red. The incident field contains two photon, (see [3]).

and $S_-a^\dagger$, called rotating terms. This approximation leads to the anti Jaynes-Cummings Hamiltonian

$$H_{AJC} = \omega_0 S_3 + \omega(\hat{n} + \frac{1}{2}) + k(S_+ a^\dagger + S_- a). \tag{2.3}$$

One recalls that both the Jaynes-Cummings and the anti Jaynes-Cummings models apply to a weak coupling regime. As the interaction grows in strength, both approximations can no longer be applied.

2.3 Diagonalization of H_{JC}

A closed form solution can be obtained for H_{JC} diagonalizing the equation (2.2) respect to states $\{|e\rangle|n\rangle\,;|g\rangle|n+1\rangle\},\ n \geq 0$. Different methods, other than the matrix representation, can be applied to obtain the spectrum of the Jaynes-Cummings Hamiltonian, e.g. the Unitary Transformation Method (UTM) [6]. In this thesis the matrix representation

method is preferred, carried out as in [10].

To study the matrix representation for the Jaynes-Cummings Hamiltonian, (2.2) is decomposed in

$$H_{JC} = H_0 + H_{int}$$

with

$$H_0 = \omega_0 S_3 + \omega\left(\hat{n} + \frac{1}{2}\right), \qquad H_{int} = k(S_+ a + S_- a^\dagger).$$

The model Hamiltonian (2.2) commute with the operator

$$N_{JC} = \hat{n} + S_3 \tag{2.4}$$

then N_{JC} is a constant of motion for H_{JC}. Physically, it represents the total excitation number, i.e. the sum of the field and the spin excitations is a conserved quantity for the system. Consequently, H_{JC} is endowed with a block-diagonal structure, whose nonzero elements correspond to product states for which N_{JC} assumes a constant value. This justifies the diagonalization with respect to the states $\{|e\rangle|n\rangle\,;|g\rangle|n+1\rangle\}$.

Recalling the action of the spin operators S_3, S_+, S_- on the states $|g\rangle$ and $|e\rangle$, cf. Eq. (1.8), and the action of the ladder operators $a, a^\dagger$ on the number states $|n\rangle$, see Eq. (1.16), one obtains

$$H_0|e\rangle|n\rangle = \frac{1}{2}\omega_0|e\rangle|n\rangle + \omega(n+\frac{1}{2})|e\rangle|n\rangle = \left[\omega(n+1) - \frac{\Delta}{2}\right]|e\rangle|n\rangle\,,$$
$$H_0|g\rangle|n+1\rangle = -\frac{1}{2}\omega_0|g\rangle|n+1\rangle + \omega(n+\frac{3}{2})|g\rangle|n+1\rangle = \left[\omega(n+1) + \frac{\Delta}{2}\right]|g\rangle|n+1\rangle\,,$$

$$H_{int}|e\rangle|n\rangle = k(S_+|e\rangle a|n\rangle + S_-|e\rangle a^\dagger|n\rangle) = k\sqrt{n+1}\,|g\rangle|n+1\rangle\,,$$
$$H_{int}|g\rangle|n+1\rangle = k(S_+|g\rangle a|n+1\rangle + S_-|g\rangle a^\dagger|n+1\rangle) = k\sqrt{n+1}\,|e\rangle|n\rangle\,,$$

where Δ is the detuning of the system

$$\Delta = \omega - \omega_0, \tag{2.5}$$

with $\Delta = 0$ at resonance.

To find the matrix representation of H_0 and H_{int} the following coefficients are calculated,

$$\langle n| \langle e| H_0 |e\rangle |n\rangle = \omega(n+1) - \frac{\Delta}{2},$$
$$\langle n+1| \langle g| H_0 |g\rangle |n+1\rangle = \omega(n+1) + \frac{\Delta}{2},$$
$$\langle n+1| \langle g| H_0 |e\rangle |n\rangle = 0 = \langle n| \langle e| H_0 |g\rangle |n+1\rangle ,$$

$$\langle n| \langle e| H_{int} |e\rangle |n\rangle = 0 = \langle n+1| \langle g| H_{int} |g\rangle |n+1\rangle ,$$
$$\langle n+1| \langle g| H_{int} |e\rangle |n\rangle = k\sqrt{n+1},$$
$$\langle n| \langle e| H_{int} |g\rangle |n+1\rangle = k\sqrt{n+1}.$$

With respects to the states $\{|e\rangle |n\rangle , |g\rangle |n+1\rangle\},\ n \geq 0$, one has

$$H_0 \equiv \begin{bmatrix} \omega(n+1) - \frac{\Delta}{2} & 0 \\ 0 & \omega(n+1) + \frac{\Delta}{2} \end{bmatrix}, \qquad H_{int} \equiv \begin{bmatrix} 0 & k\sqrt{n+1} \\ k\sqrt{n+1} & 0 \end{bmatrix},$$

and the general 2×2 matrix B_n of the model Hamiltonian H_{JC} is

$$B_n \equiv \begin{bmatrix} \omega(n+1) - \frac{\Delta}{2} & k\sqrt{n+1} \\ k\sqrt{n+1} & \omega(n+1) + \frac{\Delta}{2} \end{bmatrix}, \qquad n \geq 0.$$

B_n can be rewritten as

$$B_n = \omega(n+1)\mathbb{I}_2 + \widetilde{B}_n, \tag{2.6}$$

where

$$\widetilde{B}_n = \begin{bmatrix} -\frac{\Delta}{2} & k\sqrt{n+1} \\ k\sqrt{n+1} & \frac{\Delta}{2} \end{bmatrix}.$$

A careful observer can notice that the eigenvectors of B_n and $\widetilde{B}_n$ are the same, because they are not affected by the terms proportional to $\mathbb{I}_2$, and that the eigenvalues of B_n are given by the eigenvalues of $\widetilde{B}_n$ plus $\omega(n+1)$.

Moreover, $\operatorname{tr} \widetilde{B}_n = 0$, then the characteristic equation for $\widetilde{B}_n$ reduces to $\lambda^2 + \det \widetilde{B}_n = 0$ and the eigenvalues of $\widetilde{B}_n$ are readily provided. These observations give a justification for the decomposition (2.6) and thanks to it the study of the eigenvalues and eigenvectors of

H_{JC} is readily carried out.

The roots of the characteristic equation for $\widetilde{B}_n$,

$$\lambda^2 - \frac{\Delta^2}{4} + k^2(n+1) = 0$$

are

$$\lambda_{\pm,n} = \pm\frac{1}{2}\sqrt{\Delta^2 + 4k^2(n+1)} = \pm\frac{1}{2}R_n,$$

where R_n denotes the generalized Rabi frequency,

$$R_n = \sqrt{\Delta^2 + \Omega_n^2} \tag{2.7}$$

with $\Omega_n^2 = 4k^2(n+1)$. R_n describes the flopping frequency of the atom, hit by the incident electromagnetic field, between its two energy levels $|g\rangle$ and $|e\rangle$.

In conclusion, from (2.6), the energy eigenvalues $E_{\pm,n}$ of H_{JC} are

$$E_{\pm,n} = \omega(n+1) \pm \frac{1}{2}R_n. \tag{2.8}$$

These eigenvalues are expressed in natural units $\hbar = 1$, then their unit is frequency. To get the two values of the energy associated with the two eigenstates of the system, $\hbar$ must be reintroduced in (2.8).

The eigenvectors of B_n and $\widetilde{B}_n$ are the same, so the eigenvector for H_{JC} are computed with references to $\widetilde{B}_n$

$$\begin{bmatrix} -\frac{\Delta}{2} & k\sqrt{n+1} \\ k\sqrt{n+1} & \frac{\Delta}{2} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \pm\frac{1}{2}R_n \begin{bmatrix} x_n \\ y_n \end{bmatrix}.$$

Using the normalization condition $x_n^2 + y_n^2 = 1$, the eigenstates of H_{JC} are

$$|+,n\rangle = \sin\theta_n |e\rangle |n\rangle + \cos\theta_n |g\rangle |n+1\rangle \qquad \text{for } \lambda = +\frac{1}{2}R_n, \tag{2.9}$$

$$|-,n\rangle = \cos\theta_n |e\rangle |n\rangle - \sin\theta_n |g\rangle |n+1\rangle \qquad \text{for } \lambda = -\frac{1}{2}R_n, \tag{2.10}$$

where

$$\cos\theta_n = \frac{R_n - \Delta}{\sqrt{(R_n-\Delta)^2 + \Omega_n^2}}, \qquad \sin\theta_n = \frac{\Omega_n}{\sqrt{(R_n-\Delta)^2 + \Omega_n^2}}.$$

In literature the states (2.9), (2.10) are called "dressed states", whereas the states $|g\rangle|n+1\rangle$, $|e\rangle|n\rangle$ which are eigenstates of H_0, are known as "bare states".

This concludes the study of the eigenvalues and eigenvectors for the JC Hamiltonian through the matrix representation method. Nevertheless an eigenvalue is missing in this study and it is the eigenvalue associated to the state $|g\rangle|0\rangle$, this value is not found by the matrix representation. To calculate this coefficient, H_{int} and H_0 are applied to the state $|g\rangle|0\rangle$,

$$H_{int}|g\rangle|0\rangle = 0 \qquad H_0|g\rangle|0\rangle = \frac{\Delta}{2}|g\rangle|0\rangle$$

and so the coefficient is

$$\langle 0|\langle g|H_{JC}|g\rangle|0\rangle = \frac{\Delta}{2}.$$

It corresponds to a degenerate block i.e. one dimensional, because it reduces to a single coefficient. This degenerate eigenvalue is removed when the system is in resonance, $\Delta = 0$.

2.4 Diagonalization of H_{AJC}

To study the spectrum of the anti Jaynes-Cummings Hamiltonian one proceeds as for H_{JC}, therefore one derives the matrix representation of H_{AJC} and then finds its eigenvalues and eigenvectors. For H_{AJC}, (2.3), the constant of motion is

$$N_{AJC} = \hat{n} - S_3. \tag{2.11}$$

Accordingly, the matrix representation for H_{AJC} can be carried out respect to the product states $\{|g\rangle|n\rangle\,;|e\rangle|n+1\rangle\}$, $n \geq 0$, for which N_{AJC} assumes a constant value.

Comparing (2.11) with (2.4) one observes that they are very similar and they differ only by a sign. Physically, in the anti Jaynes-Cummings model the conserved quantity is the difference between the field and the spin excitations.

The matrix representation method is a natural way to obtain the spectrum of H_{AJC} but it is not the only and smartest one. An alternative way is to introduce a map between the JC and the anti-JC model and then use it to derive the spectrum of H_{AJC} from that of H_{JC}. To find this map an unitary operator is introduced, at this point is important to recall that an operator is unitary if it satisfies the condition $T^\dagger T = \mathbb{I}$ and that the

eigenvalues of an operator are preserved under the application of an unitary operator. It is therefore important to prove that the two Hamiltonian H_{JC} and H_{AJC} are not unitary equivalent. A more complete description of unitary operators and their properties is given in section 2.5.1.

Following [6], for any spin operator component S_j, $j = 1,2,3$ one has

$$e^{i\varphi S_j} = \cos\frac{\varphi}{2}\mathbb{I}_2 + 2i\sin\frac{\varphi}{2}S_j. \tag{2.12}$$

With $\varphi = \pi$, the unitary transformation T can be defined

$$T = \exp(i\pi S_1) = 2iS_1.$$

T can be applied to S_3 and $S_\pm$

$$e^{i\pi S_1}S_3e^{-i\pi S_1} = -S_3, \qquad e^{i\pi S_1}S_\pm e^{-i\pi S_1} = S_\mp,$$

so that

$$TH_{JC}T^\dagger = -\omega_0 S_3 + \omega\left(\hat{n} + \frac{1}{2}\right) + k(S_+a^\dagger + S_-a) = H_{AJC}^{(-\omega_0)}. \tag{2.13}$$

This result confirms that (2.2) and (2.3) are not unitary equivalent, as the atom transition frequency is modified by transformation T, $\omega_0 \to -\omega_0$. The non-unitary equivalence between the two Hamiltonians proves also that the two Hamiltonian have two different spectra and it is interesting to study the eigenvalue problem also for H_{AJC}. Moreover, due to the transformation T, a relation between the eigenvalue problems of the two models can be established,

$$H_{JC}\left|\Psi_{n,\pm}\right\rangle = E_{n,\pm}\left|\Psi_{n,\pm}\right\rangle, \tag{2.14}$$

$$H_{AJC}\left|\Phi_{n,\pm}\right\rangle = \mathcal{E}_{n,\pm}\left|\Phi_{n,\pm}\right\rangle. \tag{2.15}$$

Using the map (2.13) and $T^\dagger T = \mathbb{I}$ in Eq. (2.14), one has

$$H_{AJC}^{(-\omega_0)}T\left|\Psi_{n,\pm}\right\rangle = E_{n,\pm}T\left|\Psi_{n,\pm}\right\rangle \to H_{AJC}T\left|\Psi_{n,\pm}^{(-\omega_0)}\right\rangle = E_{n,\pm}^{(-\omega_0)}T\left|\Psi_{n,\pm}^{(-\omega_0)}\right\rangle. \tag{2.16}$$

Comparing (2.16) with (2.15), one finds that the eigenvalues and eigenvectors of H_{AJC} are linked with those of H_{JC} by the relations

$$\mathcal{E}_{n,\pm} = E_{n,\pm}^{(-\omega_0)}, \qquad \left|\Phi_{n,\pm}\right\rangle = T\left|\Psi_{n,\pm}^{(-\omega_0)}\right\rangle. \tag{2.17}$$

At this point, a set of eigenvectors which diagonalizes H_{JC} and a set of eigenvectors which diagonalizes H_{AJC} is found. Since the Rabi Hamiltonian can be written as a combination of H_{JC} and H_{AJC}

$$H_R = H_{JC} + H_{AJC} - H_0,$$

one thinks that exists also a set of eigenvectors which diagonalizes H_R, or alternatively, applying the UTM, exists an unitary transformation such that $UH_RU^\dagger$ is diagonal. Unluckily no one has proved that this transformation exists until now.

2.5 Rotating Wave Approximation (RWA)

The Rabi Hamiltonian, as presented in chapter 1, is the sum of two contributes, the free Hamiltonian H_0 and the interaction Hamiltonian H_{int},

$$H_0 = \omega_0 S_3 + \omega\left(\hat{n} + \frac{1}{2}\right), \qquad H_{int} = k(S_+ + S_-)(a^\dagger + a). \tag{2.18}$$

The terms proportional to $S_+a + S_-a^\dagger$ are known as rotating terms and those proportional to $S_+a^\dagger + S_-a$ are the virtual or counter-rotating terms (CRTs). CRTs are the terms neglected in the interaction Hamiltonian when the RWA is applied.

Until now, the dynamics of the Rabi model was not of interest then it was unnecessary to introduce the Schrödinger and the Heisenberg pictures because these are two points of view used to study the dynamics of systems in physics. However, to study the RWA is fundamental to introduce the time-dependence in the system then these pictures become of great importance. Between these two points of view there is a subtle difference. In the Schrödinger picture the time evolution of the dynamics is assigned to the eigenstates which are time-dependent, while the operators are time-independent. Conversely, in the Heisenberg picture the time-dependent elements are the operators and the state vectors are independent of time. Despite this difference, the results that they provide are equivalent and a relation exists between the two pictures, so a system can be studied indifferently according to one of them. Before describing the Heisenberg and Schrödinger picture, a fundamental element for their study is introduced, the unitary operators.

2.5.1 Unitary operators

In chapter 1 a brief description of the unitary matrices was given and in section 2.4 a first introduction to the unitary operators was presented. Nevertheless, the description of the unitary operators is still incomplete then in this section they are described in a more complete way, giving their definition and some of their properties, cf. [10]. In many cases a matrix representation for the unitary operators can be found, so no wonder that some of the properties listed here were already presented in chapter 1.
By definition an operator U is unitary if

$$U^\dagger U = UU^\dagger = \mathbb{I}.$$

Given an Hermitian operator A, $A^\dagger = A$, the corresponding unitary operator can be always obtained using the exponential map,

$$U = \exp(iA).$$

When an unitary operator U is applied to an arbitrary state vector $|\psi\rangle$, the latter is transformed into another vector $|\psi'\rangle$

$$|\psi'\rangle = U\,|\psi\rangle\,.$$

From this property it follows that the scalar product between two vector is conserved

$$\langle\psi_1'|\psi_2'\rangle = (U\,|\psi_1\rangle)^\dagger U\,|\psi_2\rangle = \langle\psi_1|\,U^\dagger U\,|\psi_2\rangle = \langle\psi_1|\psi_2\rangle\,,$$

and, in particular, this means that the unitary operator preserves the normalization of the original state vector

$$\langle\psi|\psi\rangle = 1 \rightarrow \langle\psi'|\psi'\rangle = \langle\psi|\,U^\dagger U\,|\psi\rangle = 1.$$

Unitary operators can be applied also to operators. The transformation rule for an operator A is found using the conservation property of the mean value

$$\langle\psi|\,A\,|\psi\rangle = \langle\psi'|\,A'\,|\psi'\rangle = \langle\psi|\,U^\dagger A' U\,|\psi\rangle\,,$$

so

$$A = U^\dagger A' U \rightarrow A' = UAU^\dagger.$$

It is important to state that unitary operators do not affect the eigenvalues of an operator A but only its eigenvectors. To prove this property an eigenvector $|\psi_a\rangle$ for the operator A, associated with the eigenvalue a is considered. The eigenvalue equation for A is

$$A\,|\psi_a\rangle = a\,|\psi_a\rangle\,. \tag{2.19}$$

Applying the operator U to both terms of the equation and recalling that $\mathbb{I} = U^\dagger U$, one has

$$UAU^\dagger U\,|\psi_a\rangle = aU\,|\psi_a\rangle\,.$$

Using the transformation property the eigenvalue equation for A' is obtained

$$A'\,|\psi'_a\rangle = a\,|\psi'_a\rangle\,. \tag{2.20}$$

Comparing the eigenvalue equations for A, (2.19), and A', (2.20), one observes that a is an eigenvalue for both the operators, associated to the eigenvectors $|\psi_a\rangle$ and $|\psi'_a\rangle$, respectively. This proves that the eigenvalues of an operator are preserved under the effects of an unitary operator. In light of this property, one senses the importance of the proof presented in section 2.4 of not unitary equivalence between H_{JC} and H_{AJC}, to prove that they have two different spectra.
The properties for the unitary operators listed in this section are used in this chapter, in section 2.5.2, to present and study the two pictures of Schrödinger and Heisenberg and the link between them.

2.5.2 Schrödinger picture and Heisenberg picture

The differences between the two pictures are highlighted considering a general system. In the Schrödinger picture a general system is described by the Hamiltonian H_S while its dynamic is embodied in the state vector, see [10],

$$|\Psi_S(t)\rangle = U(t)\,|\Psi_S(0)\rangle\,, \tag{2.21}$$

with $|\Psi_S(0)\rangle$ the initial state of the system and $U(t)$ the unitary transformation defined as, in units $\hbar = 1$,

$$U(t) = \exp(-iH_S t). \tag{2.22}$$

The definition (2.22) follows from the Schrödinger equation applied to the state vector

$$i\frac{\partial}{\partial t}\left|\Psi_S(t)\right\rangle = H_S\left|\Psi_S(t)\right\rangle.$$

Using (2.21) one gets

$$i\frac{\partial}{\partial t}U(t)\left|\Psi_S(0)\right\rangle = H_S U(t)\left|\Psi_S(0)\right\rangle.$$

$|\Psi(0)\rangle$ is constant and the equation must be true for any constant state vector then the unitary operator $U(t)$ obeys

$$i\frac{\partial}{\partial t}U(t) = H_S U(t)$$

and since H_S is time-independent its solution is exactly (2.22).

Conversely, in the Heisenberg picture, the state vector $|\Psi_H\rangle$ is time-independent and the time-dependence is absorbed by the operator $A_H(t)$. The operators and state vectors in the Heisenberg picture are derived from the corresponding operators and state vectors in the Schrödinger picture. Given A_S, an operator in the Schrödinger picture, and $A_H(t)$, the same operator in the Heisenberg picture, the expectation value of these operators must be the same in the two pictures. For A_S, its expectation value for a given state vector $|\Psi_S(t)\rangle$ is

$$\langle A_S\rangle_t = \langle\Psi_S(t)|\, A_S\left|\Psi_S(t)\right\rangle,$$

while the mean value for the operator $A_H(t)$ for a state $|\Psi_H\rangle$ is

$$\langle A_H\rangle_t = \langle\Psi_H|\, A_H(t)\left|\Psi_H\right\rangle.$$

Using (2.21) and the conservation property for the mean value, one has

$$\langle A_S\rangle_t = \langle A_H\rangle_t \rightarrow \langle\Psi_S(0)|\, U^\dagger(t) A_S U(t)\left|\Psi_S(t)\right\rangle = \langle\Psi_H|\, A_H(t)\left|\Psi_H\right\rangle.$$

In the Heisenberg picture the state vectors are constant and the time-dependence is carried out by the operators, then

$$\left|\Psi_H\right\rangle = \left|\Psi_S(0)\right\rangle = U^\dagger(t)\left|\Psi_S(t)\right\rangle, \qquad A_H(t) = U^\dagger(t) A_S U(t).$$

These results confirm that the two pictures are linked by the unitary transformation $U(t)$ defined as in (2.22).

Only in a particular case the operator is time-independent also in the Heisenberg picture. This happens when A_S is both time-independent and a constant of the motion, i.e. $[A_S, H_S] = 0 \rightarrow [A_S, U(t)] = 0$, in this case $A_S = A_H$. As an example, the Hamiltonian of the system H_S satisfies these conditions then $H_H = H_S$, the Hamiltonian is the same in both pictures.

2.5.3 Interaction picture and RWA

To understand the validity of the RWA, the interaction Hamiltonian H_{int} is not studied either in the Schrödinger or Heisenberg picture but from a further point of view, the interaction picture. In this representation both the state vectors and the operators exhibit time dependence. This picture can be seen as an intermediate representation between the Schrödinger and the Heisenberg picture.

The importance of this picture is due to the time dependence of the Hamiltonian, which is time-independent in the other two pictures. The time dependence enables to study the evolution of the Hamiltonian terms and consequently understand when their effects are negligible or not. To switch from the Schrödinger to the interaction picture a unitary transformation is applied to the state vectors and the operators of the system. The unitary transformation is linked only to a part of the system Hamiltonian. To be useful in simplifying the analysis of the problem, typically it is chosen the exactly solvable and well understood part of the Hamiltonian.

For the Rabi Hamiltonian (2.1) the well understood part is the free Hamiltonian H_0 and the unitary transformation is defined as

$$U(t) = \exp(-iH_0 t) = \exp[-i(\omega_0 S_3 + \omega \hat{n})t], \tag{2.23}$$

where the constant term $\omega/2$ in H_0 is neglected. The unitary transformation is applied to the interaction Hamiltonian H_{int} and not to the full Hamiltonian because the free Hamiltonian is invariant with respect the unitary transformation defined as in (2.23), so its study is of no particular interest. The invariance of H_0 with respect to $U(t)$ is easily

proved applying $U(t)$ to the terms of H_0,

$$\begin{aligned}
U(t)^\dagger \omega_0 S_3 U(t) &= e^{i(\omega_0 S_3+\omega\hat{n})t}\omega_0 S_3 e^{-i(\omega_0 S_3+\omega\hat{n})t} \\
&= \left[\cos\frac{\omega_0 t}{2}\mathbb{I}_2 + 2i\sin\frac{\omega_0 t}{2}S_3\right]\omega_0 S_3\left[\cos\frac{\omega_0 t}{2}\mathbb{I}_2 - 2i\sin\frac{\omega_0 t}{2}S_3\right] = \\
&= \omega_0 S_3\left[\cos^2\frac{\omega_0 t}{2} + \sin^2\frac{\omega_0 t}{2}\right] = \omega_0 S_3, \\
U(t)^\dagger \omega\hat{n} U(t) &= e^{i(\omega_0 S_3+\omega\hat{n})t}\omega\hat{n} e^{-i(\omega_0 S_3+\omega\hat{n})t} \\
&= e^{i\omega\hat{n}t}\omega a^\dagger a e^{-i\omega\hat{n}t} = \\
&= \omega a^\dagger e^{i\omega(\hat{n}+1)t}e^{-i\omega(\hat{n}+1)t}a = \omega\hat{n},
\end{aligned}$$

where at the first step the term $\exp[i\omega_0 S_3 t]$ is replaced with its expansion (2.12). Therefore, only the interaction Hamiltonian is studied in the interaction picture and its representation is obtained applying $U(t)$ to H_{int}. It is useful to study the effect of the unitary transformation over each term of H_{int}. In particular, it is interesting to study the effect of the term $\exp[i\omega\hat{n}t]$ when applied to the ladder operators $a,\ a^\dagger$ and the effects of the term $\exp[i\omega_0 t S_3]$ on the spin operators $S_+,\ S_-$. Only these cases are taken into account because $\exp[i\omega\hat{n}t]$ has no effects on $S_\pm$ and the same holds for $\exp[i\omega_0 t S_3]$ when applied to a and $a^\dagger$.

$$\begin{aligned}
e^{i\omega_0 t S_3}S_+ e^{-i\omega_0 t S_3} &= \left[\cos\frac{\omega_0 t}{2}\mathbb{I}_2 + 2i\sin\frac{\omega_0 t}{2}S_3\right]S_+\left[\cos\frac{\omega_0 t}{2}\mathbb{I}_2 - 2i\sin\frac{\omega_0 t}{2}S_3\right] = \\
&= \left[\cos\frac{\omega_0 t}{2}\mathbb{I}_2 + i\sin\frac{\omega_0 t}{2}\mathbb{I}_2\right]S_+\left[\cos\frac{\omega_0 t}{2}\mathbb{I}_2 - 2i\sin\frac{\omega_0 t}{2}S_3\right] = \\
&= S_+\left[\cos\frac{\omega_0 t}{2}\mathbb{I}_2 + i\sin\frac{\omega_0 t}{2}\mathbb{I}_2\right]^2 = S_+ e^{i\omega_0 t}, \\
e^{i\omega_0 t S_3}S_- e^{-i\omega_0 t S_3} &= \left[\cos\frac{\omega_0 t}{2}\mathbb{I}_2 + 2i\sin\frac{\omega_0 t}{2}S_3\right]S_-\left[\cos\frac{\omega_0 t}{2}\mathbb{I}_2 - 2i\sin\frac{\omega_0 t}{2}S_3\right] = \\
&= \left[\cos\frac{\omega_0 t}{2}\mathbb{I}_2 - i\sin\frac{\omega_0 t}{2}\mathbb{I}_2\right]S_-\left[\cos\frac{\omega_0 t}{2}\mathbb{I}_2 - 2i\sin\frac{\omega_0 t}{2}S_3\right] = \\
&= S_-\left[\cos\frac{\omega_0 t}{2}\mathbb{I}_2 - i\sin\frac{\omega_0 t}{2}\mathbb{I}_2\right]^2 = S_- e^{-i\omega_0 t}, \\
e^{i\omega\hat{n}t}a^\dagger e^{-i\omega\hat{n}t} &= a^\dagger e^{i\omega(\hat{n}+1)t}e^{-i\omega\hat{n}t} = a^\dagger e^{i\omega t}, \\
e^{i\omega\hat{n}t}a e^{-i\omega\hat{n}t} &= a e^{i\omega(\hat{n}-1)t}e^{-i\omega\hat{n}t} = a e^{-i\omega t}.
\end{aligned}$$

Combining these terms with each other, the interaction Hamiltonian in the interaction picture is obtained

$$\begin{aligned} H_{int}(t) &= U^{\dagger}(t) H_{int} U(t) \\ &= k\left(S_{-}ae^{-i(\omega+\omega_0)t} + S_{+}a^{\dagger}e^{i(\omega+\omega_0)t} + S_{+}ae^{-i(\omega-\omega_0)t} + S_{-}a^{\dagger}e^{i(\omega-\omega_0)t}\right). \end{aligned} \tag{2.24}$$

If the system is near resonance, the condition $|\omega - \omega_0| \ll |\omega + \omega_0|$ holds. Under this condition, terms oscillating with frequency $\omega - \omega_0 \simeq 0$ are nearly resonant (rotating terms) while terms oscillating with frequency $\omega + \omega_0 \simeq 2\omega$ are nearly anti-resonant (counter-rotating terms). Defining the detuning Δ of the system as in (2.5), $\Delta = \omega - \omega_0$, the time that it takes for the resonant terms to complete a full oscillation is $\tau = (2\pi)/\Delta$ and it is much greater than the period of the anti-resonant terms, $(2\pi)/(2\omega) \ll \tau$.

At the timescale over which the resonant behaviour is analyzed, the difference in the periods leads to a net effect of the quick oscillating anti-resonant terms which averages to zero, thus the anti-resonant terms can be neglected. This gives rise to the rotating-wave approximation, whose application to the Hamiltonian in the interaction picture retrieves

$$H_{int}^{RWA}(t) = k\left(S_{+}ae^{-i(\omega-\omega_0)t} + S_{-}a^{\dagger}e^{i(\omega-\omega_0)t}\right). \tag{2.25}$$

Transforming back into the Schrödinger picture, i.e., calculating $U(t)H_{int}(t)U^{\dagger}(t)$, gives

$$H_{int}^{RWA} = k(S_{+}a + S_{-}a^{\dagger}).$$

Considering the full Hamiltonian, $H_0 + H_{int}^{RWA}$, one obtains the Jaynes-Cummings Hamiltonian (2.2). H_0 is defined as in (2.18) because it is unaffected by this approximation. This study shows also the limitations of this approximation. If the system is far away from the resonance the effect of the anti-resonant terms is no longer negligible and they should be retained. Furthermore, when the frequencies of the atom and the mode are in opposite of sign, it becomes negligible the effect of the nearly resonant terms and one can ignore it. In this case, the interaction Hamiltonian in the interaction picture results to be

$$H_{int}^{CRT}(t) = k\left(S_{-}ae^{-i(\omega+\omega_0)t} + S_{+}a^{\dagger}e^{i(\omega+\omega_0)t}\right).$$

To find the anti Jaynes-Cummings model (2.3) is enough to transform back the Hamiltonian into the Schrödinger picture and to consider the full Hamiltonian $H_0 + H_{int}^{CRT}$, where

the free Hamiltonian is always defined as in (2.18).

At this point the near resonance condition for the validity of RWA is demonstrated, but the application of the RWA is linked also with a second condition, $k \ll \omega$. It was observed that the rotating wave approximation holds when the ratio between the coupling strength and the frequency of the electromagnetic field is no higher than $k/\omega \sim 0.1$. If this ratio grows, i.e., the coupling strength grows in intensity, the RWA breaks down because the effects of the counter-rotating terms are not negligible. Furthermore, the width of the validity range for the RWA decreases as one proceeds higher up the spectrum.

This physical range of coupling strength k in which RWA breaks down is known as "ultrastrong coupling regime", in contrast with the situation where RWA hold that is worldwide called "strong coupling regime". The objective of this thesis is to study the set of parameters in which the RWA is applicable and for what parameters it breaks down, so the focus is on the strong and ultrastrong coupling regimes, nevertheless it is interesting to report that other regimes exist. These regimes are achieved when the coupling strength k grows, they have lately been reached experimentally and they are the subject of recent studies. These new regimes are known as the "deep strong coupling regime", when $k/\omega \geq 1$, and the "extreme deep strong coupling regime", reached up when $k/\omega \geq 10$. The classification of the regimes is summarized in [7] and is reported in table 2.1.

Regimes	Interval of validity
strong coupling regime	$k/\omega \sim [0, 0.1]$
ultrastrong coupling regime	$k/\omega \sim [0.1, 1]$
deep strong coupling regime	$k/\omega \sim [1, 10]$
extreme strong coupling regime	$k/\omega \geq 10$

Table 2.1. Different regimes for different values of the ratio k/ω.

Another important consideration has to be made also for the term ω_0. This term is very important in the study of the validity for the RWA but it must be taken into account also to verify the correctness of the two-level approximation for the atom. In section 1.3.1, where the two-level approximation is introduced, it is reported that this approximation is valid only in conditions of near resonance. This means that when the system is far away

from resonance both approximations, two-level atom and RWA, break down. This effect is greater when the coupling strength grows, then for some systems in the ultrastrong coupling regimes also the two-level approximation cannot be applied. This means that in this case also the Rabi Hamiltonian (2.1) is no longer valid and some approximations have to be introduced in the Hamiltonian of the model. These situations are not uncommon in the experiment so it is important to mention their existence though these cases are not dealt with in this thesis.

Chapter 3

Numerical solution of the Rabi model

3.1 Introduction

In chapter 2 the eigenvalues of both the JC and anti-JC Hamiltonians are found. The JC and anti-JC eigenvalues approximate the Rabi eigenvalues in different regions of validity, which are enunciated in chapter 2 when the two simplified models are presented. In section 2.5.3, a justification for the validity of the RWA when the system is in the near-resonance condition is proposed and, in this way, a justification at the condition $|\omega - \omega_0| \ll |\omega + \omega_0|$ for the JC model is given. In the same section a proof of the anti-JC model validity is also given: it corresponds to the condition $|\omega - \omega_0| \gg |\omega + \omega_0|$. A further condition of validity commun to both JC and anti-JC models is $k \ll \omega$, the RWA holds when k/ω is not greater than 0.1, i.e. the system is in the strong coupling regime. If k/ω increases the system enters the ultrastrong coupling regime and the RWA breaks down. The regimes classification is presented in chapter 2 and it is summarized in table 2.1.

To find the exact interval of k in which the condition $k \ll \omega$ is satisfied a comparison between the eigenvalues of the Rabi Hamiltonian and its two approximation is done, so it

is important to study the eigenvalues of H_R (2.1) and then compare them with the closed-form eigenvalues of H_{JC} and H_{AJC} obtained in chapter 2. Since an exact solution for the Rabi Hamiltonian has not been found so far, the study is mainly of numerical nature. The problem is rewritten introducing an infinite size block matrix and the evaluation of the eigenvalues for the model is carried out numerically. Naturally, since the matrix has infinite size it must be truncated to solve the eigenvalue problem. This, in turn, introduces an error in the eigenvalues computation, the relative error analysis is therefore presented. This study is conducted in sections 3.2 and 3.4.

In the last two section 3.5 and 3.6 an alternative method for the computation of the Rabi eigenvalues, based on the recurrence relations, is developed. In section 3.5 the relations are obtained analytically. Successively, in section 3.6, the numerical study is conducted. The eigenvalues are computed numerically and the property of the successions are studied.

To verify the correctness of the equations obtained in section 3.2, they are used in section 3.3 to retrieve the results for the JC and anti-JC models obtained in chapter 2.

3.2 Eigenvalue problem of the Rabi Hamiltonian

The Rabi Hamiltonian (2.1) is here rewritten for convenience as:

$$H_R = \omega_0 S_3 + \omega \hat{n} + k(S_+ + S_-)(a^\dagger + a). \tag{3.1}$$

In Eq. (3.1) the constant term $\omega/2$ has been safely neglected because it does not modify the theory here developed: it can be reintroduced at the end of the analysis.

According to the Schrödinger picture, the dynamics of the system is embodied in the state vector $|\Psi(t)\rangle$, that is time-dependent. For example, in the JC model the state vectors are expressed in function of the bare states $\{|g\rangle|n+1\rangle, |e\rangle|n\rangle\}$. In the Rabi model (3.1) one can express the state vector $|\Psi(t)\rangle$ in terms of the states $|g\rangle|n\rangle$ and $|e\rangle|n\rangle$. This is possible because the set $\{|g\rangle|n\rangle, |e\rangle|n\rangle\}$, for $n = 0,1,2,\ldots$, is a complete, orthonormal basis for the system and the state vector can be expressed as a linear combination of the

elements of the base,

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} \left[g_n(t) \, |g\rangle \, |n\rangle + e_n(t) \, |e\rangle \, |n\rangle \right], \tag{3.2}$$

where $g_n(t)$ and $e_n(t)$ are coefficients to be obtained.

The time evolution of the state vector of the system is described by the Schrödinger equation

$$i \frac{d}{dt} |\Psi(t)\rangle = H_R \, |\Psi(t)\rangle \, . \tag{3.3}$$

To understand the behaviour of $|\Psi(t)\rangle$ it is fundamental to study the action of the Rabi Hamiltonian on the state vector. For the single terms of H_R one calculates:

$$\begin{aligned}
S_3 \, |\Psi(t)\rangle &= \sum_{n=0}^{\infty} \left(-\frac{1}{2} g_n(t) \, |g\rangle \, |n\rangle + \frac{1}{2} e_n(t) \, |e\rangle \, |n\rangle \right), \\
S_+ a \, |\Psi(t)\rangle &= \sum_{n=0}^{\infty} \left[g_n(t) \sqrt{n} \, |e\rangle \, |n-1\rangle + 0 \right] = \sum_{m=0}^{\infty} g_{m+1} \sqrt{m+1} \, |e\rangle \, |m\rangle \, , \\
S_- a \, |\Psi(t)\rangle &= \sum_{n=0}^{\infty} \left[0 + e_n(t) \sqrt{n} \, |g\rangle \, |n-1\rangle \right] = \sum_{m=0}^{\infty} e_{m+1} \sqrt{m+1} \, |g\rangle \, |m\rangle \, ,
\end{aligned}$$

where the index $m = n - 1$ is introduced. The summations start with $m = 0$ because for $m = -1$ the term $\sqrt{m+1}$ vanishes. It follows that the first two coefficients are $g_1(t)$ and $e_1(t)$.

$$S_+ a^\dagger \, |\Psi(t)\rangle = \sum_{n=0}^{\infty} \left[g_n(t) \sqrt{n+1} \, |e\rangle \, |n+1\rangle + 0 \right] = \sum_{p=1}^{\infty} g_{p-1} \sqrt{p} \, |e\rangle \, |p\rangle \, , \tag{3.4}$$

$$S_- a^\dagger \, |\Psi(t)\rangle = \sum_{n=0}^{\infty} \left[0 + e_n(t) \sqrt{n+1} \, |g\rangle \, |n+1\rangle \right] = \sum_{p=1}^{\infty} e_{p-1} \sqrt{p} \, |g\rangle \, |p\rangle \, , \tag{3.5}$$

where $p = n + 1$. The summations start with $p = 1$ because for $p = 0$ the term under the square root is zero. The first two non vanishing terms in the summations are $g_0(t)$, $e_0(t)$. Moreover, since m and p are dummy indexes, they can be both renamed as n. In conclusion, the r.h.s. of equation (3.3) is

$$\begin{aligned}
H_R \, |\Psi(t)\rangle = \sum_{n=0}^{\infty} \Bigg[& \left(\left(-\frac{\omega_0}{2} + \omega n \right) g_n + k \left(e_{n-1}(t) \sqrt{n} + e_{n+1}(t) \sqrt{n+1} \right) \right) |g\rangle \, |n\rangle + \\
& \left(\left(\frac{\omega_0}{2} + \omega n \right) e_n + k \left(g_{n+1}(t) \sqrt{n+1} + g_{n-1}(t) \sqrt{n} \right) \right) |e\rangle \, |n\rangle \Bigg].
\end{aligned} \tag{3.6}$$

From Eq. (3.2), the time-dependence of $|\Psi(t)\rangle$ is absorbed by the coefficients $g_n(t)$ and $e_n(t)$, so the l.h.s. of the Schrödinger equation (3.3) is explicitly

$$i\frac{d}{dt}|\Psi(t)\rangle = \sum_{n=0}^{\infty} i\left[\frac{dg_n}{dt}(t)\,|g\rangle\,|n\rangle + \frac{de_n}{dt}(t)\,|e\rangle\,|n\rangle\right]. \tag{3.7}$$

Using equations (3.7) and (3.6) in (3.3) and collecting the terms with $|g\rangle\,|n\rangle$ and $|e\rangle\,|n\rangle$ one obtains a system of infinite equations for $g_n(t)$ and $e_n(t)$, $n = 0,1,2,\ldots,$

$$i\frac{dg_n}{dt}(t) = \left(\omega n - \frac{\omega_0}{2}\right) g_n(t) + k\left(e_{n-1}(t)\sqrt{n} + e_{n+1}(t)\sqrt{n+1}\right), \tag{3.8}$$

$$i\frac{de_n}{dt}(t) = \left(\omega n + \frac{\omega_0}{2}\right) e_n(t) + k\left(g_{n+1}(t)\sqrt{n+1} + g_{n-1}(t)\sqrt{n}\right). \tag{3.9}$$

Solving this system means finding all the coefficient $e_n(t)$ and $g_n(t)$ of $|\Psi(t)\rangle$, which would solve the problem of the diagonalization of the Rabi Hamiltonian. Unluckily, due to the presence of the terms $e_{n-1}(t)$ and $e_{n+1}(t)$ in equation (3.8) and, correspondingly, $g_{n-1}(t)$ and $g_{n+1}(t)$ in equation (3.9), the equations are coupled and they cannot be solved explicitly.

Observing carefully the calculations leading to Eqs. (3.8), (3.9), one can notice that the terms $ke_{n-1}(t)\sqrt{n}$ and $kg_{n+1}(t)\sqrt{n+1}$ are given by applying the rotating terms S_+a and $S_-a^\dagger$ to the vector state $|\Psi(t)\rangle$. The terms $ke_{n+1}(t)\sqrt{n+1}$ and $kg_{n-1}(t)\sqrt{n}$ are the result of the application of the counter-rotating terms $S_+a^\dagger$ and S_-a to the state vector. It follows that the system of equations for the JC model is readily retrieved neglecting the terms linked with the counter-rotating terms in (3.8), (3.9),

$$\begin{aligned} i\frac{dg_n}{dt}(t) &= \left(\omega n - \frac{\omega_0}{2}\right) g_n(t) + ke_{n-1}(t)\sqrt{n}, \\ i\frac{de_n}{dt}(t) &= \left(\omega n + \frac{\omega_0}{2}\right) e_n(t) + kg_{n+1}(t)\sqrt{n+1}. \end{aligned} \qquad n = 0,1,2\ldots \tag{3.10}$$

On the other hand, if in (3.8) and (3.9) one ignores the terms linked with the rotating terms, the equations for the anti-JC model are obtained

$$\begin{aligned} i\frac{dg_n}{dt}(t) &= \left(\omega n - \frac{\omega_0}{2}\right) g_n(t) + ke_{n+1}(t)\sqrt{n+1}, \\ i\frac{de_n}{dt}(t) &= \left(\omega n + \frac{\omega_0}{2}\right) e_n(t) + kg_{n-1}(t)\sqrt{n}. \end{aligned} \qquad n = 0,1,2\ldots \tag{3.11}$$

One mentions that both systems (3.10) and (3.11) are utilized in section 3.3 in the framework of an alternative solution with respect to the one developed in chapter 2 for the JC and anti JC models.

Going back to the initial equations (3.8) and (3.9), where both the effects of the JC and anti-JC terms are considered, it is possible to simplify the equations normalizing with respect to ω, which means fixing $\omega = 1$. In this way the notation is simplified because of the following correspondence,

$$\frac{\omega_0}{2\omega} \to \frac{\omega_0}{2}, \qquad \frac{k}{\omega} \to k, \qquad \omega t \to t.$$

Note that since $\omega = 1$ the Rabi eigenvalues are also normalized with respect to ω ($\epsilon/\omega \to \epsilon$).

For convenience, the time dependence in $g_n(t)$ and $e_n(t)$ is omitted and the parameter $\Omega = \omega_0/2$ is introduced. With $n = 0,1,2,\ldots$, this leads to the normalized system of equations,

$$i\frac{dg_n}{dt} = (n - \Omega)g_n + k\left(e_{n-1}\sqrt{n} + e_{n+1}\sqrt{n+1}\right), \tag{3.12}$$

$$i\frac{de_n}{dt} = (n + \Omega)e_n + k\left(g_{n+1}\sqrt{n+1} + g_{n-1}\sqrt{n}\right). \tag{3.13}$$

For the numerical evaluation of the eigenvalues it is crucial to rewrite equations (3.12), (3.13) in a more convenient way, using the matrix and vector representation. One defines the vectors $\mathbf{g}$ and $\mathbf{e}$ which contain the unknown coefficients g_n and e_n, respectively,

$$\mathbf{g} = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \end{pmatrix}, \qquad \mathbf{e} = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ \vdots \end{pmatrix}.$$

Resorting to $\mathbf{g}$ and $\mathbf{e}$, Eqs. (3.12), (3.13) reduce to the following matrix equations,

$$i\frac{d\mathbf{g}}{dt} = (\mathbb{D} - \Omega\mathbb{I})\,\mathbf{g} + k\,(\mathbb{L} + \mathbb{U})\,\mathbf{e}, \tag{3.14}$$

$$i\frac{d\mathbf{e}}{dt} = (\mathbb{D} + \Omega\mathbb{I})\,\mathbf{e} + k\,(\mathbb{U} + \mathbb{L})\,\mathbf{g}, \tag{3.15}$$

where $\mathbb{D}$, $\mathbb{L}$ and $\mathbb{U}$ are three sparse, infinite-dimensional matrices. $\mathbb{D}$ is a diagonal matrix with coefficients $n = 0,1,2,\ldots$, $\mathbb{L}$ is a lower triangular matrix where the non-zero elements are only those in the first sub-diagonal and $\mathbb{U}$ is the transpose of $\mathbb{L}$ and is an upper

triangular matrix where the only non-zero elements are those on the first over-diagonal

$$\mathbb{D} = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 2 & \\ & & & \ddots \end{pmatrix}, \qquad \mathbb{L} = \begin{pmatrix} 0 & & & \\ \sqrt{1} & 0 & & \\ & \sqrt{2} & 0 & \\ & & \ddots & 0 \end{pmatrix}, \qquad \mathbb{U} = \mathbb{L}^{\dagger}.$$

In partitioned matrix form the coefficients of equations (3.14), (3.15) reduce the block matrix

$$\left(\begin{array}{c|c} \mathbb{D} - \Omega\mathbb{I} & k(\mathbb{L}+\mathbb{U}) \\ \hline k(\mathbb{U}+\mathbb{L}) & \mathbb{D} + \Omega\mathbb{I} \end{array}\right). \tag{3.16}$$

The resulting form of Eqs. (3.14), (3.15) is

$$i\frac{d}{dt}\begin{pmatrix} \mathbf{g} \\ \mathbf{e} \end{pmatrix} = \left(\begin{array}{c|c} \mathbb{D} - \Omega\mathbb{I} & k(\mathbb{L}+\mathbb{U}) \\ \hline k(\mathbb{U}+\mathbb{L}) & \mathbb{D} + \Omega\mathbb{I} \end{array}\right)\begin{pmatrix} \mathbf{g} \\ \mathbf{e} \end{pmatrix}. \tag{3.17}$$

Finally, it is possible to write the eigenvalue equation for this problem,

$$\epsilon\begin{pmatrix} \mathbf{g} \\ \mathbf{e} \end{pmatrix} = \left(\begin{array}{c|c} \mathbb{D} - \Omega\mathbb{I} & k(\mathbb{L}+\mathbb{U}) \\ \hline k(\mathbb{U}+\mathbb{L}) & \mathbb{D} + \Omega\mathbb{I} \end{array}\right)\begin{pmatrix} \mathbf{g} \\ \mathbf{e} \end{pmatrix}. \tag{3.18}$$

At this point the eigenvalue problem for the Rabi Hamiltonian entails the solution of the matrix equation (3.18). This is done numerically with a MATLAB program. Since the block matrix (3.16) and the coefficients vectors $\mathbf{g}$, $\mathbf{e}$ are infinite dimensional, for the numerical computation of the Rabi eigenvalues it is essential to truncate it.

3.3 Check for JC and anti-JC models

Before proceeding with the numerical computation of the Rabi eigenvalues, a check of correctness of the relations obtained in section 3.2 is presented.

In chapter 2 an analytical expression for the eigenvalues of the JC Hamiltonian is obtained diagonalizing H_{JC}, the eigenvalues of H_{AJC} are obtained observing the symmetry between H_{AJC} and H_{JC}. In this section the eigenvalue problems for the two models are obtained without considering the constants of motions for the models but only observing

the structures of the Eqs. (3.10) and (3.11).

Considering the system of equations (3.8) and (3.9) achieved in section 3.2 writing the Schrödinger equation (3.3) for the state vector $|\Psi\rangle$, one obtains the system for the JC model (3.10)

$$i\frac{dg_n}{dt}(t) = \left(\omega n - \frac{\omega_0}{2}\right) g_n(t) + k e_{n-1}(t)\sqrt{n}, \tag{3.19}$$

$$i\frac{de_n}{dt}(t) = \left(\omega n + \frac{\omega_0}{2}\right) e_n(t) + k g_{n+1}(t)\sqrt{n+1}, \tag{3.20}$$

where $n = 0,1,2,\ldots$, time-dependencies for the coefficients g_n and e_n are omitted and $\Omega = \omega_0/2$. One observes that the equation (3.19) depends from e_{n-1} while the equation (3.20) depends from g_{n+1}, for each n the equation for g_n is coupled with those of e_{n-1}. Moreover, imposing the condition $e_{-1} = 0$ the equation for g_0 is uncoupled. These observations are consistent with the procedure stated in chapter 2 where the JC Hamiltonian is diagonalized with respect the bare states $\{|g\rangle|n+1\rangle, |e\rangle|n\rangle\}$ and a degenerate block, i.e. one dimensional, is found for the state $|g\rangle|0\rangle$. Returning to the equations and applying the stated observations, one obtains the uncoupled equation

$$i\frac{dg_0}{dt} = -\Omega g_0, \tag{3.21}$$

and an infinite number of equations that couple g_n and e_{n-1}

$$\begin{aligned} i\frac{dg_n}{dt} &= (\omega n - \Omega) g_n + k e_{n-1}\sqrt{n}, \\ i\frac{de_{n-1}}{dt} &= (\omega n - \omega + \Omega) e_{n-1} + k g_n \sqrt{n}, \end{aligned} \qquad n = 1,2,3,\ldots. \tag{3.22}$$

For each n, the eigenvalue problem for the system is promptly written

$$\epsilon g_n = (\omega n - \Omega) g_n + k e_{n-1}\sqrt{n}, \tag{3.23}$$

$$\epsilon e_{n-1} = (\omega n - \omega + \Omega) e_{n-1} + k g_n \sqrt{n}, \tag{3.24}$$

while for the uncoupled equation (3.21), since $\epsilon g_0 = -\Omega g_0$, it is immediate that $\epsilon = -\Omega$. Using the matrix notation the equations (3.23) and (3.24) are recast

$$\begin{bmatrix} \omega n - \Omega - \epsilon & k\sqrt{n} \\ k\sqrt{n} & \omega n - \omega + \Omega - \epsilon \end{bmatrix} \begin{bmatrix} g_n \\ e_{n-1} \end{bmatrix} = 0, \qquad n = 1,2,3,\ldots$$

obtaining the matrix form for the eigenvalue problem. The equation for the eigenvalues of H_{JC} is obtained solving the determinant of the matrix and it is the same as the one obtained in chapter 2, cf. Eq. (2.8).

Similarly, if in the equations (3.8) and (3.9) those terms linked with the JC terms are disregarded the system (3.11) for the anti-JC Hamiltonian is obtained

$$i\frac{dg_n}{dt} = (\omega n - \Omega)g_n + ke_{n+1}\sqrt{n+1}, \tag{3.25}$$

$$i\frac{de_n}{dt} = (\omega n + \Omega)e_n + kg_{n-1}\sqrt{n}, \tag{3.26}$$

for $n = 0,1,2,\ldots$.

In the anti-JC system, the equation (3.26) depends from g_{n-1} and the equation (3.25) is dependent from e_{n+1}, that leads to have a system of two coupled equations for each n, g_n and e_{n+1}. Imposing the condition $g_{-1} = 0$ one finds a uncoupled equation for e_0. In chapter 2 it is observed that the anti-JC Hamiltonian is diagonalizable with respect the states $\{|e\rangle\,|n+1\rangle,\ |g\rangle\,|n\rangle\}$ and it is consistent with the results obtained in the equations (3.25) and (3.26).

Resuming, one has a uncoupled equation

$$i\frac{de_0}{dt} = \Omega e_0, \tag{3.27}$$

and an infinite number of equations that coupled g_n and e_{n+1}

$$\begin{aligned} i\frac{dg_n}{dt} &= (\omega n - \Omega)g_n + k\sqrt{n+1}e_{n+1}, \\ i\frac{de_{n+1}}{dt} &= (\omega n + \omega + \Omega)e_{n+1} + k\sqrt{n+1}g_n, \end{aligned} \qquad n = 0,1,2,\ldots \tag{3.28}$$

For the uncoupled equation (3.27) holds $\epsilon = \Omega$ and it follows from $\epsilon e_0 = \Omega e_0$. This is coherent with the observation presented in chapter 2, the anti-JC eigenvalues are linked with those of the JC Hamiltonian by the relation (2.17),

$$\mathcal{E}_{n,\pm} = E^{(-\omega_0)}_{n,\pm},$$

where $\mathcal{E}$ denotes the anti-JC eigenvalues and E denotes the eigenvalues of H_{JC}.

For the system (3.28) the eigenvalue problem is

$$\epsilon g_n = (\omega n - \Omega)g_n + k\sqrt{n+1}e_{n+1}, \tag{3.29}$$

$$\epsilon e_{n+1} = (\omega n + \omega + \Omega)e_{+1} + k\sqrt{n+1}g_n, \tag{3.30}$$

for $n = 0,1,2,\ldots$.

In matrix form it is

$$\begin{bmatrix} \omega n - \Omega - \epsilon & k\sqrt{n+1} \\ k\sqrt{n+1} & \omega n + \omega + \Omega - \epsilon \end{bmatrix} \begin{bmatrix} g_n \\ e_{n+1} \end{bmatrix} = 0, \qquad n = 1,2,3,\ldots$$

Computing the determinant of the matrix one obtains the expression for the eigenvalues of H_{AJC}.

One recalls that for the JC model the eigenvalues are given in Eq. (2.8) and the eigenvectors in equations (2.9) and (2.10), for the anti-JC model they are obtained using the relation (2.17) between the two models. This concludes that the equations obtained in section 3.2 are consistent, at least in the JC and anti-JC models, and they can be used to compute the Rabi eigenvalues.

3.4 Rabi eigenvalues and comparison with the JC and anti-JC eigenvalues

As observed in section 3.3 the equations (3.8) and (3.9) return the expected results when they are used to compute the eigenvalue problems for the JC and anti-JC cases. Thanks to this check one states the correctness of these equations.

In this section the eigenvalue problem for the Rabi model (3.18), obtained from the Eqs. (3.8) and (3.9), is employed to study numerically the eigenvalues of the model.
As reported in the section 3.2, the numerical computation of the eigenvalues needs a finite size matrix obtained truncating the block matrix (3.16). The truncation is a delicate step because it introduces an error in the computation of the eigenvalues. The evaluation of the error is therefore unavoidable. Specifically, it is studied how the truncation size affects the prediction of the eigenvalues. For this study reference eigenvalues are needed, they are computed with a matrix of dimension $N = 1000$ and at each step they are compared with those computed with a matrix of dimension n and the relative error between them is considered. With n varying between 1 to N, the definition of the relative error δ_r is

$$\delta_r = \frac{|\epsilon_n - E_N|}{|E_N|}, \tag{3.31}$$

where E_N is the reference eigenvalue computed with the matrix of dimension N and ϵ_n is the eigenvalue computed with a truncated matrix of dimension n. It is fixed $N = 1000$ because this huge size allows a reliable prediction for the eigenvalues. The relative error

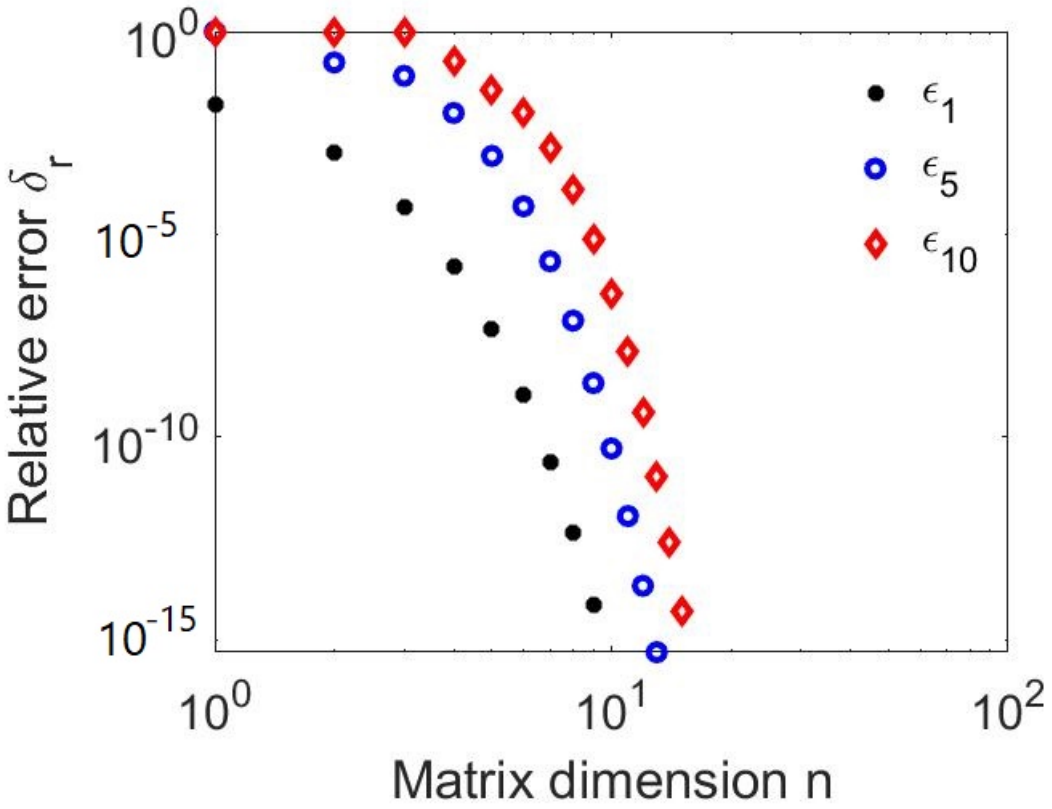

Figure 3.1. δ_r for the $1st$ (dots), $5th$ (circles) and $10th$ (diamond) eigenvalue. Each dot, circle or diamond indicates the relative error between the eigenvalue computed with a matrix of dimension n and the reference eigenvalues, reported in table 3.1 and computed with a matrix of size $N = 1000$. The parameters are $\Omega = 0.4$, $\omega = 1$ and coupling strength $k = 0.4$.

trend for the $1st$, $5th$ and $10th$ Rabi eigenvalue is reported in figure 3.1 and the reference values for the first ten eigenvalues are reported in table 3.1. In the figure the dots, circles and diamonds represent the relative error respect the reference values when the eigenvalues are computed with a matrix of size n. Dots are used for the $1st$ eigenvalue, circles report the error for the $5th$ eigenvalue and diamonds denote the error for the $10th$ eigenvalue. For all the eigenvalues, the error decreases to zero rapidly but one observes that proceeding

higher up in the spectrum a bigger size matrix is needed to have a good prediction of the Rabi eigenvalues. For the purpose of this thesis only the first eigenvalues are considered then it is enough to consider a small in size matrix, as example the relative error for the first ten eigenvalues computed with a matrix of dimension $n = 10$ is reported in table 3.1 and it is immediate to observe that the error is nearly zero. When the size of the matrix exceeds $n = 10$ the error is of the order of the machine accuracy. The block matrix (3.16) and its eigenvalues depend on the coupling strength k and on the parameter $\Omega = \omega_0/2$ that can be arbitrarily fixed, here $k = 0.4$, $\Omega = 0.4$ as in [5]. The value of ω is fixed at one.

	E_N	Relative error for a matrix of dimension $n = 10$
1^{st}	-0.4925	0
2^{nd}	0.0256	$1.0326\ 10^{-13}$
3^{rd}	0.7799	$1.2642\ 10^{-13}$
4^{th}	0.8862	$6.9478\ 10^{-13}$
5^{th}	1.7916	$5.2531\ 10^{-11}$
6^{th}	1.8971	$2.9321\ 10^{-11}$
7^{th}	2.7280	$3.4878\ 10^{-09}$
8^{th}	2.9635	$4.3641\ 10^{-09}$
9^{th}	3.6914	$1.6915\ 10^{-07}$
10^{th}	3.9948	$3.5545\ 10^{-07}$

Table 3.1. First ten eigenvalues for H_R computed with a matrix of size $N = 1000$ and used as reference eigenvalues in the error computation in figure 3.1. For each eigenvalue the relative error is computed with a matrix of dimension $n = 10$.

In the light of the claims stated observing figure 3.1 and table 3.1, the eigenvalues for the Rabi Hamiltonian and their variation as a function of the coupling strength is evaluated using a block matrix with size $n = 100$. Trends for the first ten eigenvalues are reported in figure 3.2 for the coupling strength range $0 \leq k \leq 0.8$, with parameters $\Omega = 0.4$ and $\omega = 1$. The results in figure 3.2 exactly reproduce the eigenvalues reported in the work of Braak, [5].

In figure, each eigenvalue is associated to a state vector of the system. The lowest line

reports the eigenvalue related with the eigenstate of the system $|-,0\rangle$ and the second line represents the eigenvalue associated with the state vector $|+,0\rangle$. Similarly, the second couple of eigenvalues is linked with the eigenstates $|-,1\rangle$, $|+,1\rangle$ and so on. The number $n = 0,1,2\ldots$, in the eigenstates is the number of the bosonic mode and for each mode is possible to find the system in the ground or excited state of the two level system, indicated with the symbols $-,+$. It follows that for each mode a couple of eigenvectors exists. This behaviour is a consequence of the two level system approximation of the model.

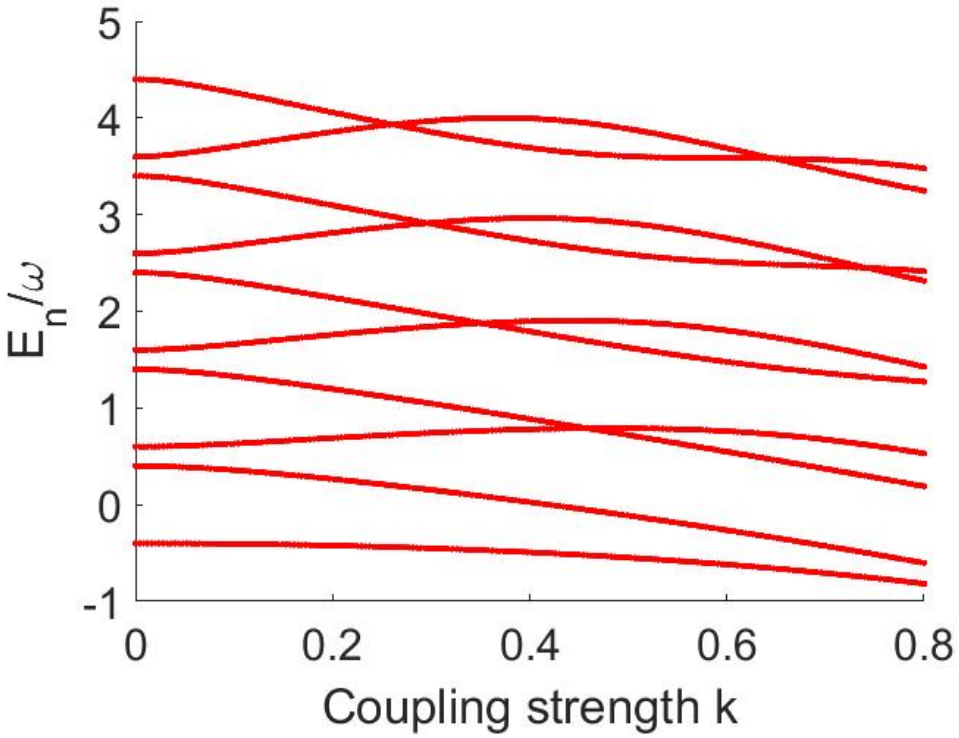

Figure 3.2. The first ten eigenvalues for the Rabi model are reported in function of the coupling strength. Matrix of size $n = 100$, $\omega = 1$ and $\Omega = 0.4$. The first couple of eigenvalues is associated to the first mode, respectively to the ground and excited state of the two level system, $|-,0\rangle$, $|+,0\rangle$. The second couple is related to the states $|\mp,1\rangle$ and so on. (see [5])

The eigenvalues of the Rabi Hamiltonian (3.1) plotted in figure 3.2 are compared with the JC eigenvalues analytically obtained in chapter 2 and defined by equation (2.8)

$$E_{\pm,n} = \omega(n+1) \pm \frac{1}{2}R_n.$$

It is important to observe that equation (2.8) is obtained considering the JC Hamiltonian (2.2)

$$H_{JC} = \omega_0 S_3 + \omega \left(\hat{n} + \frac{1}{2} \right) + k(S_+ a + S_- a^\dagger),$$

while the Rabi eigenvalues are calculated starting by the Hamiltonian (3.1)

$$H_R = \omega_0 S_3 + \omega \hat{n} + k(S_+ + S_-)(a^\dagger + a),$$

where the constant term $\omega/2$ is discarded. Therefore, this term must be removed also in the JC Hamiltonian for a consistent comparison, so that the JC eigenvalues become

$$E_{\pm,n} = \omega \left(n + \frac{1}{2} \right) \pm \frac{1}{2} R_n,$$

where R_n is the generalized Rabi frequency (2.7),

$$R_n = \sqrt{\Delta^2 + 4k^2(n+1)}.$$

The two sets of eigenvalues are represented in Fig. 3.3. The red lines represent the eigenvalues of the Rabi Hamiltonian, i.e. they refer to the same results reported in figure 3.2, while the eigenvalues of H_{JC} are reported with the blue, light blue and green lines. In green one sees the JC eigenvalue relative to the state $|g\rangle |0\rangle$, the blue and light blue lines are the eigenvalues of the JC Hamiltonian associated with the dressed states $|+, n\rangle$ (2.9) and $|-, n\rangle$ (2.10), respectively. The dashed line is drawn for $k = 0.1$. The parameters used in the computation of the two sets of eigenvalues are: $\Omega = \omega_0/2 = 0.4$, $\omega = 1$, block matrix size $n = 100$ and for the range of coupling strength $0 \leq k \leq 0.8$.

Observing figure 3.3 one notes that for $k \sim [0, 0.1]$ the eigenvalues obtained with the JC models perfectly approximate those obtained with the Rabi Hamiltonian. Accordingly, in this interval of coupling strength values the RWA can be applied to the Rabi model obtaining the JC model that closely approximates the behaviour of the system. Furthermore, in line with the considerations in section 2.5.3, proceeding higher up in the spectrum the range of RWA validity reduces. This effect is clearly visible in figure 3.3, for the lower eigenvalues the RWA is applicable for a larger interval of k and also for $k \sim [0, 0.2]$ the JC model returns a good prediction of the Rabi eigenvalues. For higher eigenvalues, when k exceeds 0.1, a slight deflection is observed and the JC model cannot

be used anymore. Out of these intervals of validity for the RWA, slight deviations are observed and they grow in intensity with the increasing of k.

Recalling the condition of validity for the RWA stated in chapter 2,

$$k \ll \omega, \qquad |\omega - \omega_0| \ll |\omega + \omega_0|,$$

and replacing the parameters $\omega = 1$ and $\omega_0 = 2\Omega = 0.8$ in the second condition, one notes that this condition is satisfied because $0.2 \ll 1.8$. One concludes that for this set of parameters the RWA is an admissible approximation.

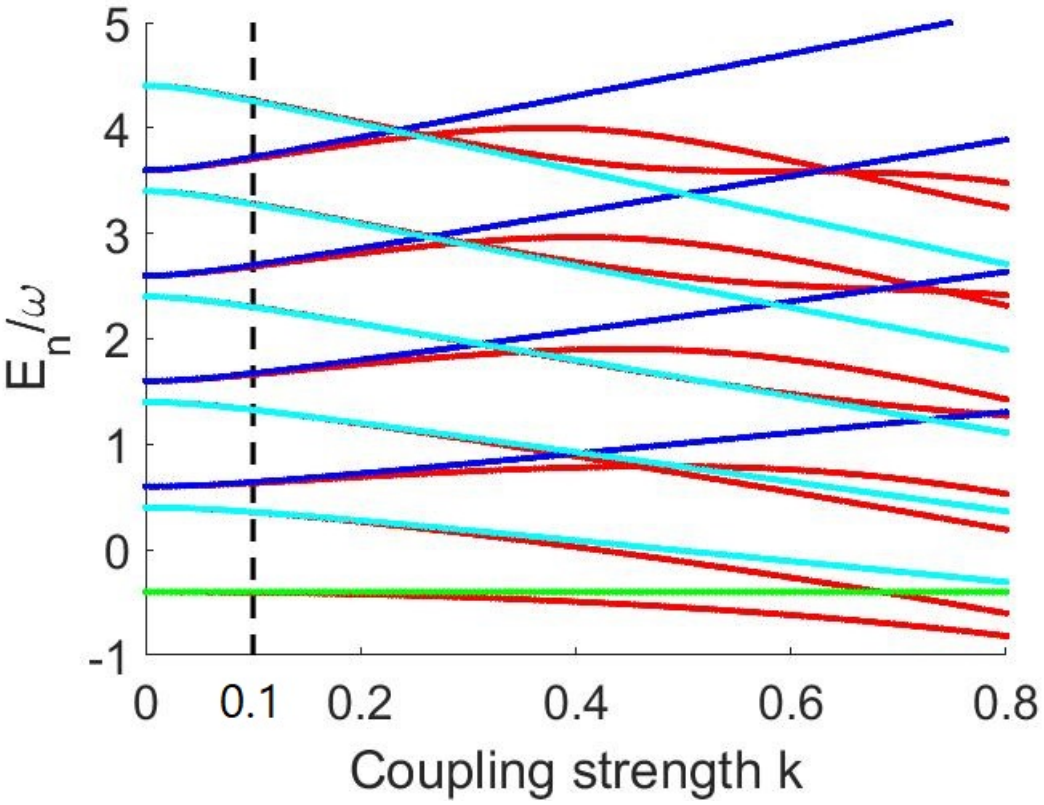

Figure 3.3. The red lines represent the Rabi eigenvalues. The blue and light blue lines are the JC eigenvalues with the dressed states $|+, n\rangle$ (2.9) and $|-, n\rangle$ (2.10), respectively. The green line represents the JC eigenvalue associated to the state $|g\rangle\,|0\rangle$. The black dashed line is drawn for $k = 0.1$. The parameters are $\Omega = \omega_0/2 = 0.4$, $\omega = 1$, $0 \leq k \leq 0.8$ and for the Rabi eigenvalues the block matrix has size 100

In figure 3.3 the influence of the coupling strength is studied for a given value of the detuning of the system $\Delta = \omega - \omega_0 = 0.2$. Conversely, in figure 3.4 the Rabi eigenvalues

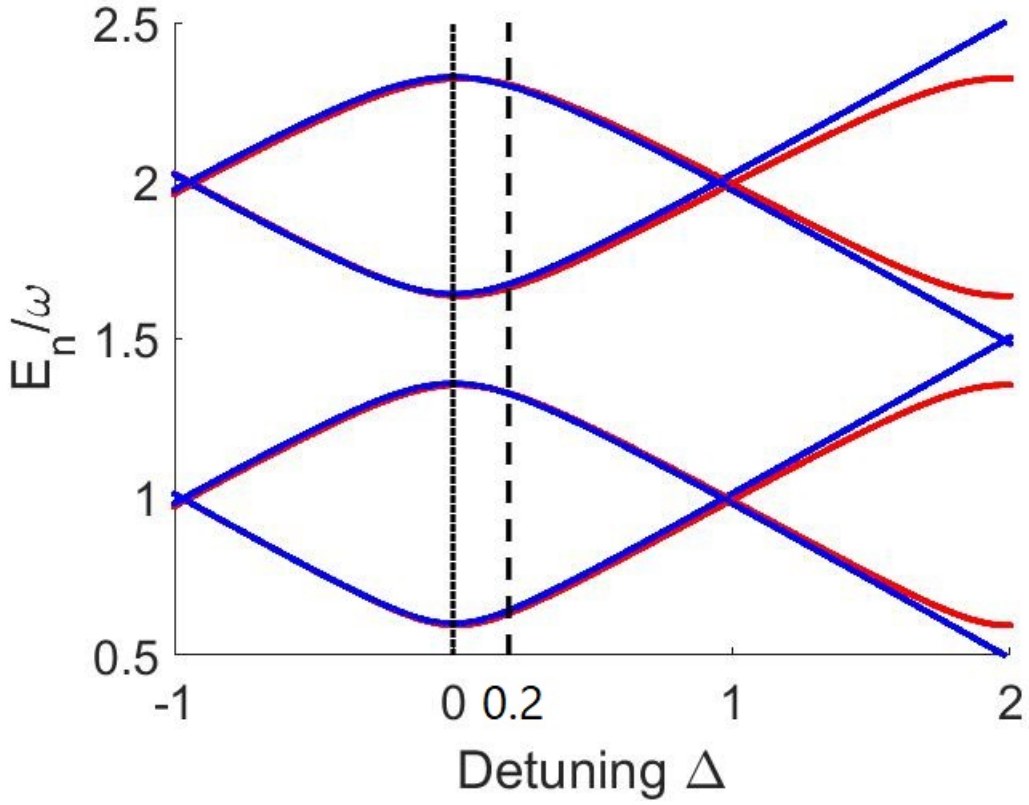

Figure 3.4. Eigenvalues of the Rabi model (red lines), eigenvalues of the JC model (blue lines). $\omega = 1$, $k = 0.1$, the Rabi block matrix has size 100, $\omega_0/2 = \Omega \in [-0.5, 1]$ and consequently $-1 \leq \Delta \leq 2$, $\Delta = \omega - \omega_0$. $3rd$, $4th$, $5th$ and $6th$ eigenvalues are reported. The dotted line is drawn for the resonance condition $\Delta = 0$ and the dashed line is drawn for $\Delta = 0.2$.

(red lines) and JC eigenvalues (blue lines) are plotted varying the detuning and for fixed value $k = 0.1$. In this figure, the second and third couple of eigenvalues, namely, the $3rd$, $4th$, $5th$ and $6th$ eigenvalue, are represented. The dotted line indicates the condition $\Delta = 0$ while the dashed line indicates the the condition $\Delta = 0.2$ used in figure 3.3. The parameter $\omega = 1$ is constant while Ω varies between -0.5 and 1, consequently $-1 \leq \Delta \leq 2$.

When $\Delta = 0$ and the system is in resonance the JC eigenvalues perfectly approximate those of the Rabi model, moving away from the resonance condition a slight deflection appears, which amplifies when the detuning increases, this effect is greater for higher eigenvalues. Observing carefully, $\Delta > 1$ when $\omega_0 = 2\Omega$ is negative because $\omega = 1$

is constant. This situation occurs when the atom and the electromagnetic field have frequencies of opposite sign. In this case the condition $|\omega - \omega_0| \ll |\omega + \omega_0|$ is not satisfied and the system is not in the JC regime. In this situation the system fulfills the condition $|\omega + \omega_0| \ll |\omega - \omega_0|$ and the Rabi Hamiltonian can be approximated by the anti-JC Hamiltonian.

This situation is illustrated in Fig. 3.5 where the eigenvalues of H_{AJC} are represented in blue and the Rabi eigenvalues are plotted in red. In this figure the frequency ω is always fixed at one, while $\Omega = -0.4$. The anti-JC eigenvalues reported in this figure are the translation of those found in chapter 2 and expressed by the relation (2.17). The eigenvalues, as for the JC case, are computed discarding the constant term $\omega/2$ for consistency in the anti-JC Hamiltonian

$$H_{AJC} = \omega_0 S_3 + \omega \hat{n} + k(S_+ a^\dagger + S_- a).$$

The results obtained in figure 3.5 are not surprising, the eigenvalues calculated for the anti-JC Hamiltonian closely approximate those for the Rabi Hamiltonian when the coupling strength k is the range [0,0.1] and an error appears when k grows. Similar to the JC case, for smaller eigenvalues the anti-JC model returns a good prediction for a larger interval of k. To facilitate the reading of the graph, for the condition $k = 0.1$, the dashed line is drawn.

3.5 Use of recursive relations to the Rabi model

In section 3.2 the computation of the eigenvalues of the Rabi Hamiltonian is carried out using the matrix representation, in this section an alternative method is proposed. This new method leads to the same results previously obtained through the matrix method but it is based on the use of recursive relations. This method is developed following the article [11, 12, 8].

Starting from the normalized equations (3.12) and (3.13),

$$\begin{aligned} i\frac{dg_n}{dt} &= (n - \Omega)g_n + k\left(e_{n-1}\sqrt{n} + e_{n+1}\sqrt{n+1}\right), \\ i\frac{de_n}{dt} &= (n + \Omega)e_n + k\left(g_{n+1}\sqrt{n+1} + g_{n-1}\sqrt{n}\right), \end{aligned} \qquad n = 0,1,2\ldots$$

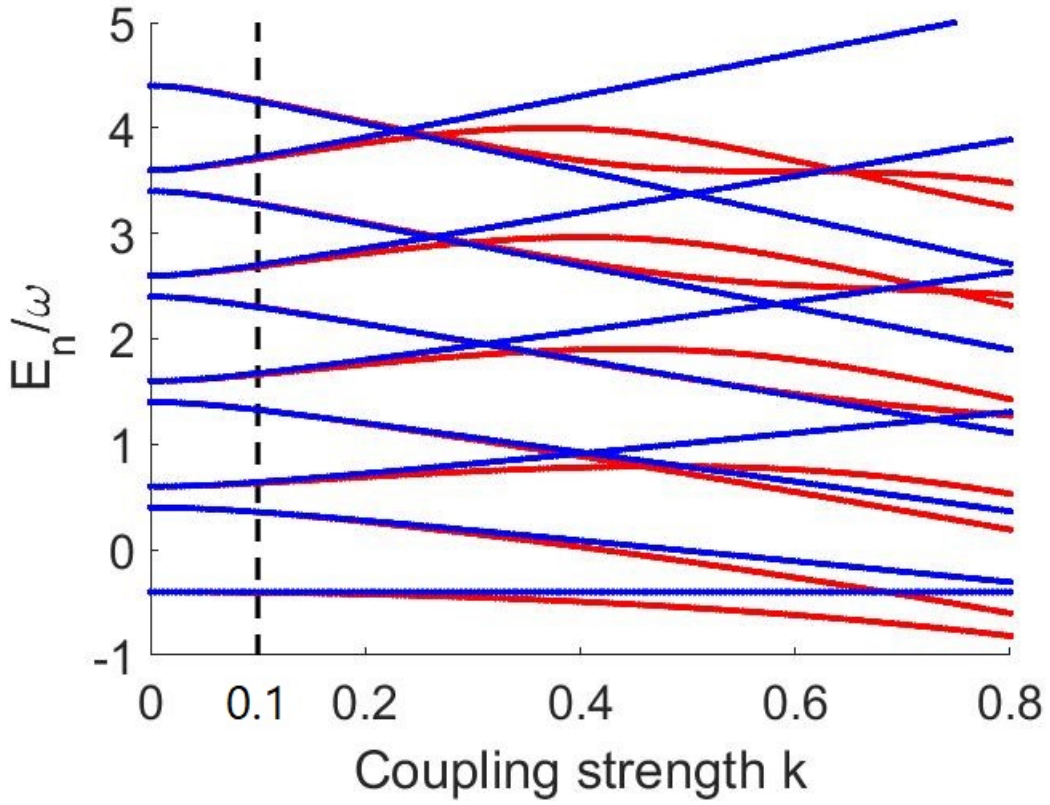

Figure 3.5. The red lines represent the eigenvalues of the Rabi Hamiltonian, the blue lines are the anti-JC eigenvalues. The parameters are $\Omega = \omega_0/2 = -0.4$, $\omega = 1$, $0 \leq k \leq 0.8$ and for the Rabi eigenvalues the block matrix has size 100. Dashed line is drawn for $k = 0.1$

the eigenvalue problem is easily written in the form

$$\epsilon g_n = (n - \Omega) g_n + k \left(e_{n-1}\sqrt{n} + e_{n+1}\sqrt{n+1} \right), \tag{3.32}$$

$$\epsilon e_n = (n + \Omega) e_n + k \left(g_{n+1}\sqrt{n+1} + g_{n-1}\sqrt{n} \right), \tag{3.33}$$

for $n = 0,1,2,\ldots$. One recalls that $\omega = 1,\ \Omega = \omega_0/2$.

This system of equations is equivalent to the eigenvalue problem (3.18). In this case the formulation has to be simplified and to express the equations in an simpler form one introduces the coefficients $\widetilde{g}_n$ and $\widetilde{e}_n$, see [8],

$$g_n = \sqrt{n!}\ \widetilde{g}_n, \qquad e_n = \sqrt{n!}\ \widetilde{e}_n. \tag{3.34}$$

Using definitions (3.34) in equations (3.32) and (3.33), one obtains the three terms relations for $\widetilde{g}_{n+1}$ and $\widetilde{e}_{n+1}$ in function of $\widetilde{e}_n$, $\widetilde{g}_n$, $\widetilde{e}_{n-1}$ and $\widetilde{g}_{n-1}$

$$\widetilde{g}_{n+1} = \frac{1}{n+1}\left[\frac{\epsilon-\Omega-n}{k}\widetilde{e}_n - \widetilde{g}_{n-1}\right], \tag{3.35}$$

$$\widetilde{e}_{n+1} = \frac{1}{n+1}\left[\frac{\epsilon+\Omega-n}{k}\widetilde{g}_n - \widetilde{e}_{n-1}\right], \tag{3.36}$$

for $n = 0,1,2,\ldots$.

For $n = 0$, the first computed terms are $\widetilde{g}_1$ and $\widetilde{e}_1$. Two initial conditions are needed for both the successions. In (3.35) are imposed $\widetilde{g}_{-1} = 0$ and $\widetilde{e}_0 = 1$. Similarly, in (3.36) the initial conditions are $\widetilde{e}_{-1} = 0$ and $\widetilde{g}_0 = 1$. The choice of $\widetilde{g}_0$ and $\widetilde{e}_0$ is arbitrary.

Observing the system it is immediate to notice the symmetry between (3.35) and (3.36). In particular it is important to observe that in the passage between the equation for $\widetilde{e}_{n+1}$ (3.36) and the equation for $\widetilde{g}_{n+1}$ (3.35) the sign of the atom frequency changes,

$$\widetilde{e}_n \to \widetilde{g}_n \implies \Omega \to -\Omega.$$

Equations (3.35) and (3.36) are coupled due to the presence of $\widetilde{e}_n$ in (3.35) and $\widetilde{g}_n$ in (3.36). From the remarks for equations (3.4) and (3.5), one knows that the first terms in the recursive relations are $\widetilde{g}_0$ and $\widetilde{e}_0$, so that the condition $\widetilde{e}_{-1} = \widetilde{g}_{-1} = 0$ follows. Combining this condition with the initial condition on $\widetilde{e}_0$, the equation (3.35) gives the value for $\widetilde{g}_1$. Then using $\widetilde{g}_1$ and $\widetilde{e}_0$ in the equation (3.36) one obtains $\widetilde{e}_2$. This process can be repeated indefinitely and in conclusion one has a succession where the values in the odd positions are computed using the equation for $\widetilde{e}_{n+1}$ (3.36) and those in the even positions are obtained with the equation for $\widetilde{g}_{n+1}$ (3.35)

$$\widetilde{g}_0 \to \widetilde{e}_1 \to \widetilde{g}_2 \to \widetilde{e}_3 \to \ldots \tag{3.37}$$

On the contrary, imposing the initial condition on $\widetilde{g}_0$ and using the equation (3.36) one obtains $\widetilde{e}_1$. The terms $\widetilde{g}_0$ and $\widetilde{e}_1$ allow to find $\widetilde{g}_2$ applying the equation (3.35), and so on. In this case one has the succession

$$\widetilde{e}_0 \to \widetilde{g}_1 \to \widetilde{e}_2 \to \widetilde{g}_3 \to \ldots \tag{3.38}$$

where the $\widetilde{e}$ elements in the even positions are computed with the equation (3.36) and the $\widetilde{g}$ elements in the odd positions with the equation (3.35).

Eqs. (3.37) and (3.38) are two independent successions for the eigenvalue problem and they must be recast to have two more compact relations. It is observed that in the odd positions for the succession (3.37) one finds the elements $\widetilde{e}$ and indexing them as $2m+1$, one has

$$\widetilde{e}_{2m+1} = \frac{1}{2m+1}\left[\frac{\epsilon-\Omega-2m}{k}\widetilde{g}_{2m} - \widetilde{e}_{2m-1}\right], \qquad m = 0,1,2,\ldots \tag{3.39}$$

In the even position, one finds the elements $\widetilde{g}$ indexed as $2m+2$

$$\widetilde{g}_{2m+2} = \frac{1}{2m+2}\left[\frac{\epsilon-\Omega-(2m+1)}{k}\widetilde{e}_{2m+1} - \widetilde{g}_{2m}\right], \qquad m = 0,1,2,\ldots \tag{3.40}$$

To find a single relation for the succession (3.37) one redefines $\widetilde{g}_0$ as X_0, $\widetilde{e}_1$ as X_1 and so on. In this way the equations (3.39) and (3.40) summarized into one single equation for X_n

$$X_{n+1} = \frac{1}{n+1}\left[C_n(\epsilon,\Omega)X_n - X_{n-1}\right], \qquad n = 0,1,2,\ldots \tag{3.41}$$

where

$$C_n(\epsilon,\Omega) = \frac{\epsilon+(-1)^n\Omega-n}{k}. \tag{3.42}$$

Similarly, for the succession (3.38) it is observed that the $\widetilde{g}$ elements in the odd positions are calculated by the equation (3.35) while the $\widetilde{e}$ elements in the even positions are found from equation (3.36). Redefining $\widetilde{e}_0 \equiv Y_0$, $\widetilde{g}_1 \equiv Y_1$, ..., the recursive relation for Y_n is obtained

$$Y_{n+1} = \frac{1}{n+1}\left[C_n(\epsilon,-\Omega)Y_n - Y_{n-1}\right], \qquad n = 0,1,2,\ldots \tag{3.43}$$

where $C_n(\epsilon,-\Omega)$ is defined as in (3.42).

To find the eigenvalues for the Rabi Hamiltonian the limit for $n\to\infty$ for the three terms recurrence relations (3.41) and (3.43) must be evaluated. The limit is studied fixing the value for Ω and varying ϵ. For each value of ϵ the two relations are applied recursively imposing for (3.41) the initial conditions $X_0 = 1$ and $X_{-1} = 0$ and for (3.43) $Y_0 = 1$ and $Y_{-1} = 0$

$$X_\infty(\epsilon) = \lim_{n\to\infty} X_n(\epsilon), \qquad X_0 = 1,\ X_{-1} = 0,$$
$$Y_\infty(\epsilon) = \lim_{n\to\infty} Y_n(\epsilon), \qquad Y_0 = 1,\ Y_{-1} = 0.$$

ϵ is an eigenvalue of the Rabi Hamiltonian if the limit of one of the two successions is zero when evaluated for this value, i.e., either $X_\infty(\epsilon)$ or $Y_\infty(\epsilon)$ vanishes. Furthermore, recalling the equation of the state vector (3.2) introduced in chapter 3

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} \left[g_n(t) \, |g\rangle \, |n\rangle + e_n(t) \, |e\rangle \, |n\rangle \right],$$

it is possible to find its coefficients.

If the eigenvalue ϵ makes X_n to vanish, then the eigenfunction relative to ϵ has coefficients $\widetilde{g}_{2m} = \widetilde{g}_0 X_{2m}(\epsilon)$ and $\widetilde{e}_{2m+1} = \widetilde{g}_0 X_{2m+1}(\epsilon)$, while the coefficients $\widetilde{g}_{2m+1}$ and $\widetilde{e}_{2m}$ are zero. This follows from the definition of X_n that arises from the succession (3.37) where the terms $\widetilde{g}$ appear in the even positions and the terms $\widetilde{e}$ occupy the odd positions.
Similarly, if Y_n vanishes when evaluated in ϵ, the relative eigenfunction has coefficients $\widetilde{g}_{2m+1} = \widetilde{e}_0 Y_{2m+1}(\epsilon)$ and $\widetilde{e}_{2m} = \widetilde{e}_0 Y_{2m}(\epsilon)$. The coefficients $\widetilde{g}_{2m}$ and $\widetilde{e}_{2m+1}$ vanish due the definition of Y_n that originates from the succession (3.38) where in the odd positions one finds the terms $\widetilde{g}$ and the terms $\widetilde{e}$ occupy the even positions.

3.6 Recurrence relations: Numerical solution

In section 3.5 the recursive relations (3.41) and (3.43) are obtained. In this section their limits are evaluated numerically for different values of ϵ and successively the property of the successions are discussed.
Since the theoretical recursion to infinity cannot be evaluated, for each ϵ the succession stops after a fixed number of iterations. In figure 3.6 is reported for each ϵ the normalized values of the successions X_n and Y_n at the last iteration $n = 25$. The normalized values for the successions are indicated with $\widetilde{X}_n$ and $\widetilde{Y}_n$ and they are defined

$$\widetilde{X}_n = \frac{X_n}{\sqrt{X_n^2 + Y_n^2}}, \qquad \widetilde{Y}_n = \frac{Y_n}{\sqrt{X_n^2 + Y_n^2}}.$$

In the same figure the eigenvalues of the Rabi Hamiltonian computed with the matrix method developed in section 3.2 are reported with the blue circles. In figure 3.6 one sees that when the recursion is zero at the last iteration, the respective value of ϵ is an eigenvalue. In figure 3.6, 25 iterations are considered but the doubt if this number of steps

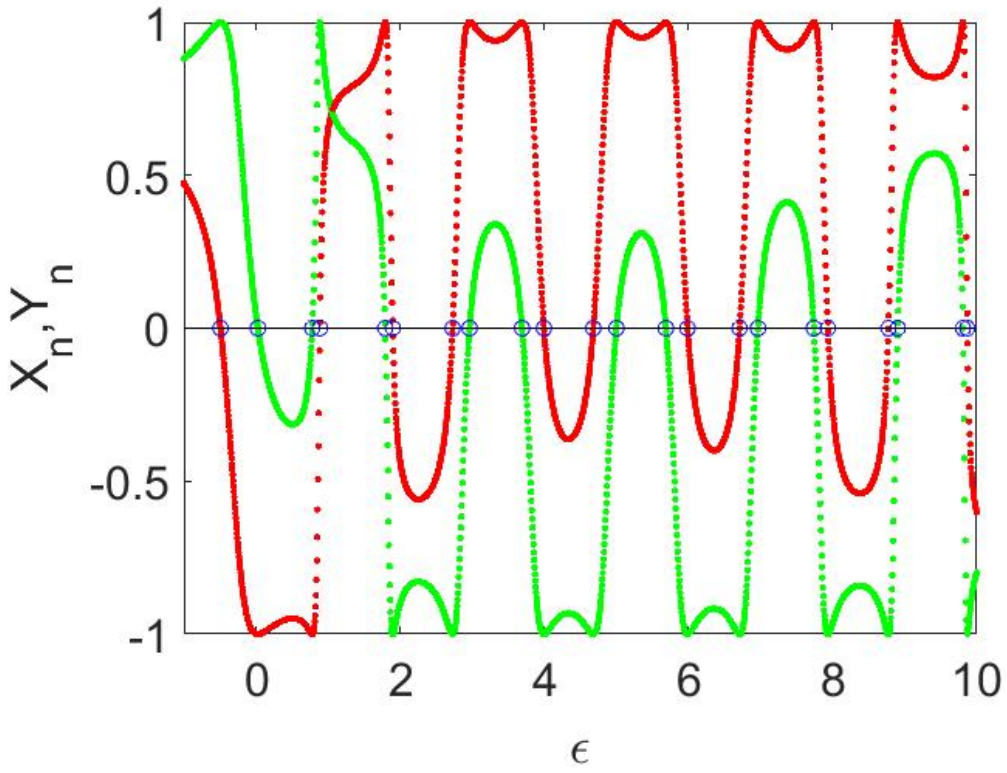

Figure 3.6. $\widetilde{X}_{25}$ (green line) and $\widetilde{Y}_{25}$ (red lines), i.e., the values for the succession X_n and Y_n at the 25^{th} iteration. Normalization $\sqrt{X_{25}^2 + Y_{25}^2} = 1$. The parameters are $\Omega = 0.4$, $k = 0.4$, $\omega = 1$, the size of the matrix is 50.

is enough to have a good prediction of the limit arises.

In order to clarify this doubt is interesting to observe the trend of the successions respect to n, so in figure 3.7 the successions X_n and Y_n are reported for $\epsilon = 0.78$. Successively, in Fig. 3.8 the successions for $\epsilon = 10$ are reported. For both the figure the other parameters are $\Omega = 0.4$, $k = 0.4$ and $\omega = 1$.

The choice of values of ϵ is not random. In figure 3.7 the value $\epsilon = 0.78$ is chosen because from table 3.1 one knows that it is an eigenvalue. In this figure one sees that both the successions oscillate around zero and after the first iterations the oscillations became almost constant. Moreover, the amplitude of the succession Y_n decreases to zero because of the chose of ϵ, which is an eigenvalue.

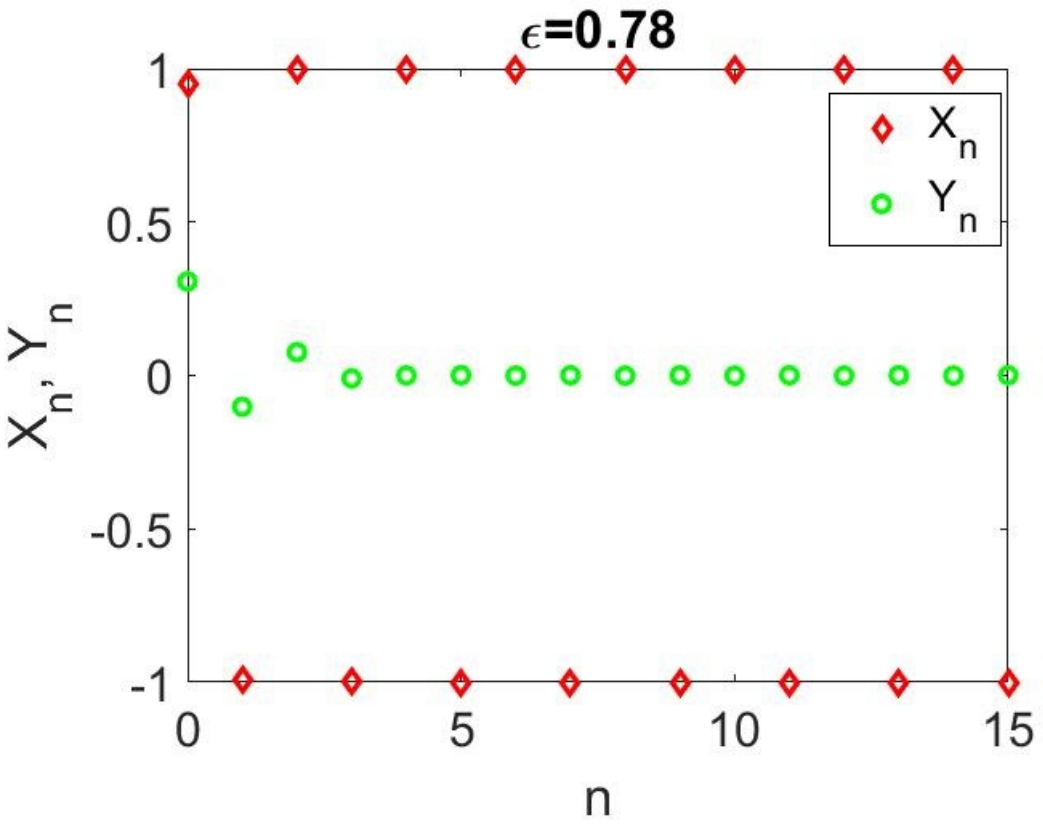

Figure 3.7. Trend of the values for the successions X_n and Y_n respect to the iteration step n, the parameters are $\Omega = 0.4,\ k = 0.4,\ \omega = 1$ and $\epsilon = 0.78$

In Fig. 3.8 the trend of X_n and Y_n is reported for $\epsilon = 10$. This chose is dictated by the figure 3.6 where ϵ varies between -1 and 10. Therefore, in figure 3.8 the worst case scenario is presented. In this case one sees that initially the succession oscillates around a value different form zero. In the first iterations the amplitude varies a lot but from the $15th$ iteration onward the oscillations of the successions became constant around the zero mean value. This behavior is the same as that observed in figure 3.7.

Comparing figures 3.7 and 3.8 one observes that for larger ϵ more iterations are required for stabilized results. This result is not surprising, in section 3.4 it is observed that proceeding higher up in the spectrum a bigger size matrix is necessary, in this case more iterations are requested.

In light of these observations one concludes that in figure 3.6, 25 iterations are sufficient

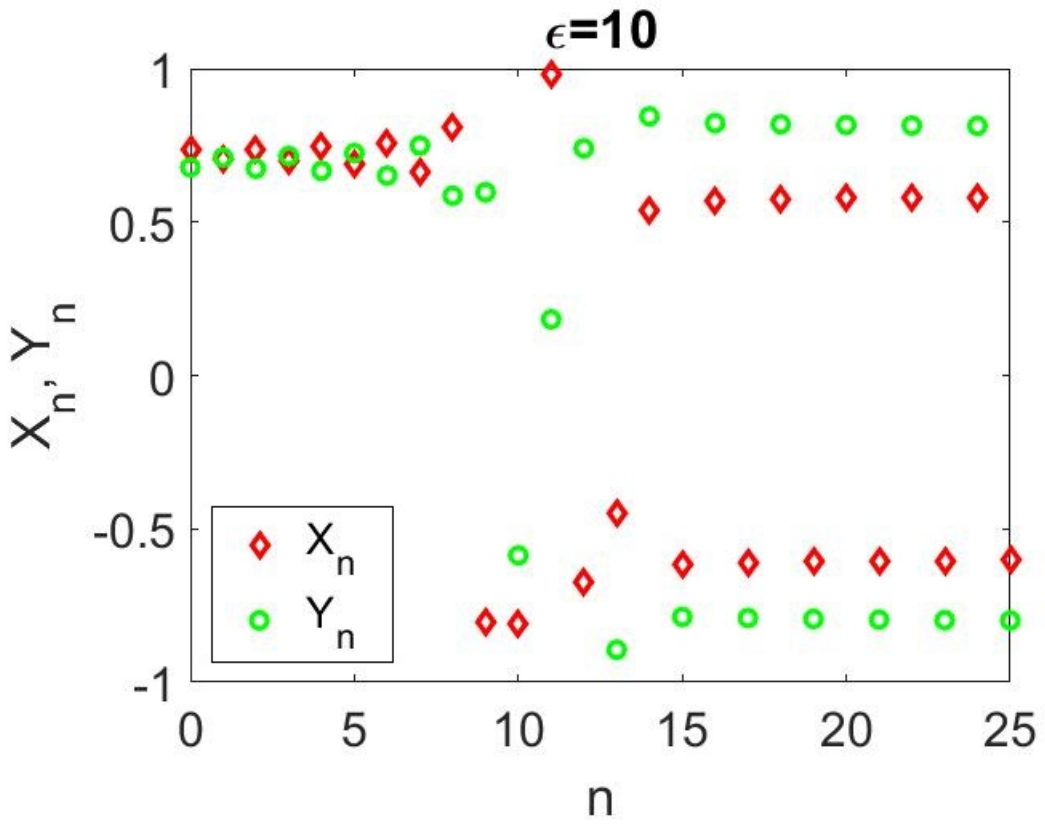

Figure 3.8. Trend of the values for the successions X_n and Y_n respect to the iteration step n, the parameters are $\Omega = 0.4,\ k = 0.4,\ \omega = 1$ and $\epsilon = 10$

to correctly evaluate the limit of the successions because ϵ varies in the fixed interval from -1 to 10. If larger values of ϵ are considered, the number of iterations for both successions must be clearly increased.

Chapter 4

Conclusions

The relevance of the JC and anti-JC models follows from their exact solvability, i.e., their eigenvalues and eigenvectors are exactly computed. Furthermore, these models return, in some regions of validity, excellent approximations at the Rabi model so they are two fundamental models in quantum optics.

Nevertheless, the usefulness of these models is restricted to a small interval of coupling strength and to the near resonance condition. The last requirement is valid only for the JC model and it is due to its nature, since it originates from the application of the rotating-wave approximation to the Rabi Hamiltonian. For the anti-JC model, this condition is replaced with the condition $|\omega - \omega_0| \gg |\omega + \omega_0|$.

The coupling strength interval of validity is investigated comparing the exact eigenvalues of the JC and anti-JC Hamiltonians with the numerical Rabi eigenvalues. One obtains that their validity is restricted to the interval $k \sim [0,0.1]$, which is slightly wider for the lower eigenvalues and reduces in the higher part of the spectrum. For this interval of coupling strength, the system is in the strong coupling regime.

The JC and anti-JC models are presented simultaneously because they share many features, they differs only in one important aspect. The Jaynes-Cummings model is applicable when the frequency of the atom and the frequency of the electromagnetic field have the same sign. When they have opposite sign the system is in the validity range of

the anti Jaynes-Cummings model.

When the coupling strength exceeds the strong coupling regime, the JC and anti-JC models cannot be applied anymore. In this situation it is unavoidable to use the full Rabi Hamiltonian. Since an explicit expression for its eigenvalues does not exist they are computed numerically. Two methods are presented in this thesis and both return an approximation of the exact eigenvalues.

The first method is based on the matrix representation: one writes the eigenvalue problem in matrix form and computes the eigenvalues numerically.
Truncating the matrix an error on the eigenvalues computation is introduced. The approximation error reduces increasing the dimension of the matrix and, for the same matrix, the error is greater for higher order eigenvalues.

The second method is based on two recursive relations. This method permits to compute the eigenvalues evaluating the limit of the recursions. The relations are evaluated for eigenvalues ϵ with a finite number of decimal digits, so only approximations of the eigenvalues can be computed. Furthermore, more iterations are required for stabilized relations when progressing higher up in the spectrum. This method returns also the state vector coefficients e_n and g_n. Computing the coefficients at each step an approximation of the state vector is obtained.

In conclusion, using a numerical approach, excellent approximations for eigenvalues and eigenvectors of the Rabi model can be computed. However, in the context of the present thesis, it is important to propose more studies on the Rabi Hamiltonian with the aim of:

- Improving the knowledge of the model and verify the existence of an explicit expression for the model eigenvalues and eigenvectors.

- Investigating different numerical methods to compute the eigenvalues and eigenvectors of the model with greater accuracy.

While in this thesis the standard Rabi model is considered, it is worth emphasizing that more general versions of this model are reported and investigated in literature. Such versions entail, for examples, the consideration of multi-photon interaction, multi-level atoms, non linear coupling strength, etc.

Bibliography

[1] R. Gilmore, *Lie groups, Lie algebras, and some of their applications*, Dover, 2005.

[2] M. Fox, *Quantum Optics : An Introduction*, Oxford University Press, 2006.

[3] M. Pollnau, Are absorption and spontaneous or stimulated emission inverse processes? The answer is subtle!, *App. Phys. B*, **125**, 25 (2019).

[4] Q. Xie et al, The quantum Rabi model: solution and dynamics, *J. Phys. A: Math. Theor.*, **50**, 113001 (2017).

[5] D. Braak, Integrability of the Rabi Model, *Phys. Rev. Lett.*, **107**, 100401 (2011).

[6] F. Raffa and M. Rasetti, Full spectrum of nonlinear Jaynes–Cummings and anti-Jaynes–Cummings models, *Int. J. Quant. Inf.*, **15**, 1740008 (2017).

[7] J. Larson and T. Mavrogordatos, *The Jaynes-Cummings model and its descendants*, Inst. of Physics Pub. Inc., 2011.

[8] Q. Chen et al, Exact solvability of the quantum Rabi model using Bogoliubov operators, *Phys. Rev. A*, **86**, 023822 (2012).

[9] J. S. Pedernales et al, Quantum Rabi model with trapped ions, *Sci. Rep.*, **5** (2015).

[10] F. Raffa, Introduction to the Hamiltonian formulation of classical and quantum systems, Draft of the lecture notes for the PhD in Physics, Politecnico di Torino, 2022.

[11] A. Moroz, On solvability and integrability of the Rabi model, *Annals of Physics*, **338**, 319-340 (2013).

[12] A. Moroz, Comment on "Integrability of the Rabi model", *arXiv preprint arXiv:1205.3139* (2012).

[13] C. Emary, On the Quantum Mechanics of Generalized Rabi Systems, PhD thesis, University of Manchester Institute of Science and Technology (UMIST), 2001