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Analytical characterization of the simplicial Kuramoto model

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Ai miei genitori

Summary

Synchronization is a ubiquitous phenomenon, universally found both in natural and human-engineered systems: from firing neurons to the twinkling of fireflies, from metronomes to power grids, from applauding audiences to the circadian rhythms of plants and animals. Patterns and order magically emerge from the disordered interplay of many interacting parts, with a complexity which seems to give no hope for simple models. The Kuramoto model is one of the hallmarks of complex systems' theory, with its simplicity, richness of behavior and surprising modeling power. It allows us to describe the onset of synchronization in systems of oscillators which interact in pairs according to a determined network topology.

A particular reformulation of the Kuramoto model naturally lends itself to a generalization which, on the recent wave of research on higher-order systems, allows one to go beyond pairwise interactions and consider oscillators influencing each other in groups. Such higher-order interactions can be easily described with simplicial complexes, discrete combinatorial objects which generalize graphs, from nodes and edges, to triangles, tetrahedra and so on. The rich theory of discrete exterior calculus, which brings classical calculus on manifolds to the discrete domains of simplicial complexes, provides us with a vast set of tools and ideas which can help us to study the simplicial Kuramoto model. Interesting relations between the topology of the complex and the synchronization properties of the model emerge.

In the thesis, after having introduced Hodge's decomposition theorem and discrete exterior calculus, we explore the properties of the simplicial Kuramoto model on weighted simplicial complexes, with a particular focus on its equilibrium properties. We study the peculiar character of simplicial interactions, generalize known results to find new notions of synchronization and phase-locking, together with necessary and sufficient conditions for their onset. We investigate the equilibrium phase transition which occurs when the strength of the interaction is increased, and its effect on the order parameter. Finally, we consider the case where an external frustration influences the interactions, study how it reshapes the equilibrium configurations of the system and how it can be used in a synchronization control perspective.

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Contents

"This is what makes the world, Ms. Lin. I believe this to be the fundamental dynamic. Transition. The point where one thing becomes another. It is what makes you, the city, the world, what they are. And that is the theme I'm interested in. The zone where the disparate become part of the whole."

[China Mieville, Perdido Street Station]

1 | Introduction

Complex networks theory is the study of phenomena which result from the interplay among many distinct parts interacting in a non-trivial way. Examples can be found everywhere: the spread of fake news on a social networks, the interaction network of proteins in an organism, the activation of neurons in the brain, climate networks, etc. The common modeling approach proceeds in a bottom-up fashion: first we identify the agents, then we decide which ones interact and, more importantly, *how* they interact. The hard part, then, is being able to say something about the emergent macroscopic phenomena directly from the microscopic evolution rules, which are usually quite simple. The idea that complex behaviors can *emerge* from simple rules is an old and well established one: from the physical world of disordered magnets subject to a magnetic field, to the computational world of cellular automata [27]. In some sense, we look for ways to "skip" the middle scales, where the agents interact and change in such complex and unpredictable ways that our exact analytical tools break down completely. In many real or ideal systems, in fact, there is a right scale, where this enormous complexity leaves space for the emergence of ordered phenomena which can be understood with simple mathematical models.

One of the most successful ways to give a mathematical description of these complex systems of interacting agents is the following:

- consider agents as nodes in a graph with state variables associated to them;
- represent interactions by edges connecting them;
- establish a discrete or continuous-time evolution rule for the nodes' states which depends, for each one, only on the states of its neighbors.

This is the mantra of complex networks theory, which stands as one of the major workhorses of modern scientific research. This simple recipe works surprisingly well and has been successfully applied in modeling a vast class of systems, from the spread of epidemics [20] and disinformation [26], to the regulation of genes in an organism [15]. Graph theory, moreover, is an old and extremely well studied subject, which provides precious tools to study the interplay between structure and dynamics.

Graphs, however, are only a part of the story. Recent years have seen the birth of a different paradigm which aims at going one step beyond [5]. The key observation is that nodes and edges can describe only *pairwise* interactions, i.e. the evolution of the

system comes from *pairs* of agents influencing each other. Group interactions are thus completely neglected or reduced to combinations of pairwise (or dyadic) ones. This is indeed a strong limitation, which can have harmful effects on the descriptive power of a model. It is not hard to see that if we want to describe, say, social dynamics, a group of friends is quite different from a collection of one-on-one friendships. The number of systems where higher-order interactions have been recognized to be present and relevant is rapidly growing: brain networks [28], biological communities [17], scientific research groups formation [19] are just a few examples.

The mathematical models which are able to describe higher-order interactions are multiple but, among them, the most common ones are *hypergraphs* and *simplicial complexes*. Simplicial complexes, in particular, allows us to go from nodes and edges, to triangles, tetrahedra and general higher order *simplices*. They are more structured than hypergraphs and, like graphs, possess a rich theory rooted in the mathematical field of topology. Their expressive power, moreover, is greatly increased by the possibility of naturally including *weights* [2]. Simplicial variants of known models [18, 6, 10] have thus emerged in the last years, displaying rich dynamical properties, from explosive phenomena [18] to chimera states [13].

Naturally, the process of "simplicialization" has reached also the study of synchronization. This particular field [21] deals with systems of oscillators, i.e. objects whose state follows a periodic evolution, which influence each other and sometimes become coherent. Think, for example, of frictionless pendula attached to the same support, or metronomes on a moving platform, or even a swarm of fireflies synchronously blinking in the night. Oscillatory materials, like the heart's muscle cells responsible for its periodic and synchronous beating, are object of this study as well. Among the different mathematical models which can describe oscillators' systems, the Kuramoto model [14] is the most renowned. With its simple evolution rule, written in the language of ordinary differential equations, it is able to characterize discrete systems of simple oscillators which, when left alone, evolve with constant angular speed. These oscillators are placed in the nodes of a network which regulate interactions, chosen in such a way that they naturally tend to minimize the phase difference between neighboring nodes. Thanks to its simplicity and descriptive power, the Kuramoto model has been the object of massive research which revealed interesting relations between the dynamical properties of the model, like its propensity to synchronize, and the topology of the network [11].

The simplicial Kuramoto [18, 1] is a natural generalization of the original model. With it, we are not constrained to considering the evolution of oscillators placed on nodes, but we can place them on simplices of any order. This change, which may seem completely arbitrary, actually allows us to consider higher order interactions: if an edge can connect only two nodes at a time, a triangle connects three edges, a tetrahedron four faces and so on. Simplicial oscillators of order k will interact through $(k+1)$-simplices, resulting in interactions of order $k+2$. In line with the guiding principles of higher-order network theory, the essential difference between agents and "carriers of interactions" fades away, leaving us with a much wider modeling freedom.

What we get is a remarkable model, whose behavior is peculiar. Its evolution equation, moreover, can be elegantly written by borrowing some of the concepts of *discrete exterior calculus* [8], the discrete analogous to differential geometry on manifolds. As we will see, this geometrical structure allows us to "open the black box" and get precious insights on the dynamical properties of the model and how they are related to the topological characteristics of the simplicial complex.

The thesis is divided in two parts:

- Chapters 2,3, and 4 describe the fundamentals required to build the simplicial Kuramoto model. In particular,
  - Chapter 2 introduces adjoint operators, the concept of Laplacian, homology and the Hodge decomposition theorem;
  - Chapter 3 is a brief review of discrete exterior calculus, aimed at defining simplicial complexes, the coboundary and codifferential operators acting on them and showing the relation between the Laplacian operator and the topology of the complex;
  - Chapter 4 describes the basis of the Kuramoto model for coupled oscillators, the original formulation and some of its variants. The chapter ends with the derivation of the Kuramoto model for weighted simplicial oscillators and the generalizations of the concepts of full synchronization and order parameter.
- Chapter 5, finally, contains a bottom-up analysis of the simplicial Kuramoto model, focused on the study of its equilibrium configurations. All the novel contributions of this work can be found there. In particular,
  - Section 5.1 studies the case of a single simplicial oscillator in isolation, highlighting how it differs from a standard Kuramoto oscillator;
  - in Section 5.2, Hodge's decomposition theorem is applied to find the main components of the dynamics which are found to evolve independently of each other. Separate equilibrium expressions for the curl-free and divergence-free components are found and are related to the equilibria of the dynamics projected onto adjacent simplices. A linear stability analysis of the equilibrium sets is also performed;
  - in Section 5.3, we investigate how the coupling strength influences the equilibrium properties of the system. Necessary and sufficient conditions for simplicial phase-locking are derived, generalizing known results for the standard graph Kuramoto to the weighted simplicial case;
  - in Section 5.4 we examine the order parameter of equilibrium configurations, giving upper and lower bounds for their value as functions of the interaction strength;
  - finally, in Section 5.5 we examine a variant of the Kuramoto model where the interactions are "frustrated" by the action of an external field. We discuss the

ways in which one can extend it to the simplicial case and study the case of a single, frustrated, simplicial oscillator.

Chapter 6 summarizes the work and the major contributions of the thesis.

2 | Hodge theory in a nutshell

We present here the fundamentals aspects of Hodge theory, a set of tools which will be extensively used in the rest of the thesis. As it is excellently done in [16], we choose here to give an abstract exposition in order to show the beauty and generality of this theory. Moreover, we choose to concentrate most of the proofs here, a sacrifice which will grant a more fluent exposition when all of this will be applied in later sections.

2.1 Fundamentals

In a few words, Hodge theory studies the property of sequences of linear maps which satisfy a particular "nesting" relation. We will see how, from a simple algebraic property, many useful concepts can be derived.

Consider three finite-dimensional Hilbert spaces (i.e. vector spaces with non-degenerate inner product) $(V, \langle\cdot,\cdot\rangle_V)$, $(W, \langle\cdot,\cdot\rangle_W)$, $(Z, \langle\cdot,\cdot\rangle_Z)$ and linear maps $f : V \to W$, $g : W \to Z$.

Property 1. *Hodge theory studies the properties of*

$$V \xrightarrow{f} W \xrightarrow{g} Z$$

when

$$g \circ f = 0,$$

i.e. when $Im(f) \subseteq ker(g)$ *(see Figure 2.1).*

If we choose bases for the (finite dimensional) vector spaces $\{v_i\}_i, \{w_i\}_i, \{z_i\}_i$ we can then represent f and g with matrices F,G. Property 1 then becomes $GF = 0$. In the same way, one can represent the inner products as symmetric positive definite matrices M_V, M_W, M_Z (called *Gram matrices*) whose elements are $(M_V)_{ij} = \langle v_i, v_j\rangle_V$, $(M_W)_{ij} = \langle w_i, w_j\rangle_W$ and $(M_Z)_{ij} = \langle z_i, z_j\rangle_Z$. The inner product between two vectors $x, y \in V$, for example, will be given by $x^\top M_V y$.

Definition 2.1.1 (Adjoint operator)**.** *If* $f : (V, \langle\cdot,\cdot\rangle_V) \to (W, \langle\cdot,\cdot\rangle_W)$ *is a linear map between Hilbert spaces, we define its* ***adjoint*** *as the linear map* $f^* : (W, \langle\cdot,\cdot\rangle_W) \to (V, \langle\cdot,\cdot\rangle_V)$ *such that* $\langle fx, y\rangle_W = \langle x, f^* y\rangle_V \ \forall x \in V, y \in W$.

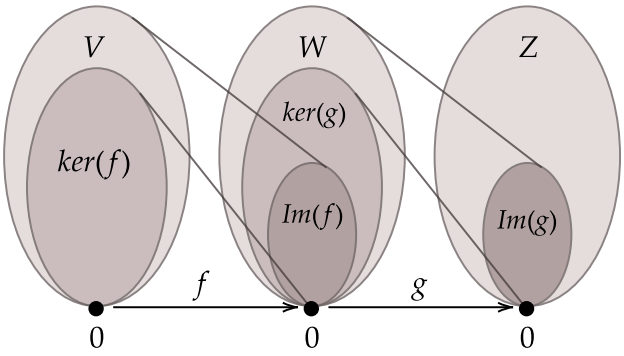


Figure 2.1: Representation of the property $g \circ f = 0$.

The adjoint operation is a way to invert the direction of a map without the strong requirements needed for an actual inverse.

Proposition 2.1.1. *The adjointness operation is idempotent, i.e.* $(f^*)^* = f$.

Proof.

$$\langle x, (f^*)^* y \rangle = \langle f^* x, y \rangle \underbrace{=}_{\text{symmetry}} \langle y, f^* x \rangle = \langle f y, x \rangle = \langle x, f y \rangle ,$$

hence $(f^*)^* = f$. □

It is easy to prove that the adjoint of an operator $f : V \to W$ represented by the matrix F is represented by $F^* = M_V^{-1} F^\top M_W$.

The analogous to Property 1 also holds for adjoints:

Proposition 2.1.2. *If* $g \circ f = 0$ *then the analogous holds for the adjoints i.e.* $f^* \circ g^* = 0$.

Proof. This is proven by considering

$$\langle f^* \circ g^* x, y \rangle_V = \langle g^* x, f y \rangle_W = \langle x, g \circ f y \rangle_Z = 0, \quad \forall x \in Z, y \in V,$$

meaning that $\langle Im(f^* \circ g^*), V \rangle_V = 0$, i.e. $f^* \circ g^* = 0$. □

We now state and prove some essential linear algebra properties which relate a map and its adjoint.

Theorem 2.1.1. *If $f : V \to W$ is a linear map between Hilbert spaces*

1. $ker(f^*) = Im(f)^\perp$,
2. $Im(f^*) = ker(f)^\perp$,
3. $V = ker(f) \oplus Im(f^*)$.

The $\perp$ and $\oplus$ reference orthogonality conditions with respect to the correct inner product.

Proof.
1. If $y \in ker(f^*)$ then, for every $x \in V$, $\langle y, fx\rangle_W = \langle f^*y, x\rangle_V = 0$, meaning that $y \in Im(f)^\perp$. If $y \in Im(f)^\perp$ then, for every $x \in V$ $\langle f^*y, x\rangle_V = \langle y, fx\rangle_V = 0$ i.e. $f^*y = 0$, $y \in ker(f^*)$.

2. $Im(f^*) = \big(Im(f^*)^\perp\big)^\perp = ker((f^*)^*)^\perp = ker(f)^\perp$. Because of the previous property and the fact that the adjointness operation is idempotent.

3. $V = ker(f) \oplus ker(f)^\perp = ker(f) \oplus Im(f^*)$.

□

One simple way to remember these properties is by noticing that they all come from the fact that one can swap the kernel and the image of a map by taking both the adjoint and the orthogonal complement.

Armed with these results, it is possible to introduce a fundamental objects which will be essential in later chapters.

Definition 2.1.2 (Discrete Hodge Laplacian)**.** *We define the* ***Hodge Laplacian*** *as the linear automorphism $\Delta : W \to W$ defined by*

$$\Delta = g^* \circ g + f \circ f^*.$$

We represent Δ with the square matrix $L = G^*G + FF^*$.

Proposition 2.1.3. *The Hodge Laplacian is a self-adjoint, positive semidefinite operator.*

Proof.
- Δ is self-adjoint because $\langle \Delta x, y\rangle_W = \langle g^*gx + ff^*x, y\rangle_W = \langle g^*gx, y\rangle_W + \langle ff^*x, y\rangle_W = \langle gx, gy\rangle_Z + \langle f^*x, f^*y\rangle_V = \langle x, g^*gy\rangle_W + \langle x, ff^*y\rangle_W = \langle x, \Delta y\rangle_W$;
- Δ is positive semidefinite because $\langle \Delta x, x\rangle_W = \langle g^*gx + ff^*x, x\rangle_W = \langle gx, gx\rangle_Z + \langle f^*x, f^*x\rangle_V = \|gx\|_Z^2 + \|f^*y\|_V^2 \geq 0$.

□

The matrix L directly inherits these two properties, being then Hermitian and positive semidefinite. It thus has real, non-negative eigenvalues.

If we define **harmonic vectors** as the ones belonging to $ker(g) \cap ker(f^*)$, we can see a first interesting property of the Hodge Laplacian.

Proposition 2.1.4.

$$ker(\Delta) = ker(g) \cap ker(f^*)$$

i.e. the kernel of Δ is the harmonic space.

Proof. A harmonic vector obviously belongs to $ker(\Delta)$ as, if $x \in \ker(g) \cap ker(f^*)$, then

$$\Delta(x) = g^* \circ g(x) + f \circ f^*(x) = 0.$$

The converse is also true:

$$\begin{aligned}
\Delta x = 0 &\iff g^*gx + ff^*x = 0 \iff g^*gx = -ff^*x \\
&\iff \langle g^*gx, y\rangle_W = -\langle ff^*x, y\rangle_W \ \ \forall y \in W \\
&\iff \langle gx, gy\rangle_Z = -\langle f^*x, f^*y\rangle_V \ \ \forall y \in W \\
&\implies \langle gx, gx\rangle_Z = -\langle f^*x, f^*x\rangle_V \\
\implies \|gx\|_Z^2 = -\|f^*x\|_V^2 &\implies gx = 0, f^*x = 0 \\
&\implies x \in ker(g) \cap ker(f^*),
\end{aligned}$$

where $\|x\|_W \triangleq \sqrt{\langle x, x\rangle_W}$ is the norm induced by the inner product of W. □

We are now ready to state and prove the core of Hodge theory: the decomposition theorem. This result allows one to decompose a vector space W in a way which reflects the structure of the maps f^*, g which act on it.

Theorem 2.1.2 (Hodge decomposition)**.** *The vector space W can be decomposed as the orthogonal sum*

$$W = Im(g^*) \oplus ker(\Delta) \oplus Im(f)$$

w.r.t the inner product on W.

Proof. According to property 3 applied to g, we have

$$W = Im(g^*) \oplus ker(g).$$

Let us focus on the second term:

$$ker(g) = W \cap ker(g) = (Im(f) \oplus ker(f^*)) \cap ker(g) = (Im(f) \cap ker(g)) \oplus (ker(f^*) \cap ker(g)).$$

Given that $g \circ f = 0$ we have that $Im(f) \subseteq ker(g)$ and thus

$$ker(g) = (Im(f) \cap ker(g)) \oplus (ker(f^*) \cap ker(g)) = Im(f) \oplus (ker(f^*) \cap ker(g)) = Im(f) \oplus ker(\Delta),$$

hence the thesis

$$W = Im(g^*) \oplus ker(\Delta) \oplus Im(f).$$

□

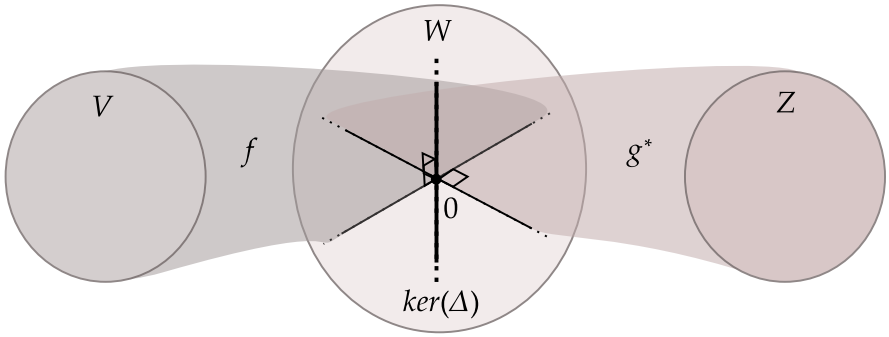


Figure 2.2: Graphical representation of Hodge's decomposition theorem 2.1.2.

2.2 Homology

The second important property of the Hodge Laplacian has to do with the concept of **homology**. Homology groups were first defined in the field of algebraic topology in order to have an algebraic tool which could count the holes of a topological space and be robust to deformations. Likewise, in differential geometry one has De Rham **co**homology which intuitively measures how a manifold fails to be globally covered by local charts. These two and many others, are aspects of the same underlying concept whose bare bones we briefly present here.

Definition 2.2.1 (Homology group)**.** *The* ***homology group*** *associated to the system* $V \xrightarrow{f} W \xrightarrow{g} Z$ *is defined as*

$$\mathcal{H} = ker(g)/Im(f).$$

The quotient is performed by considering the two vector spaces as Abelian groups under addition. In more detail, this means that we identify (consider **homologous**) two vectors x, y in the kernel of g if and only if they differ by a vector in $Im(f)$

$$x \sim y \iff \exists c \in Im(f) : \; x = y + c.$$

Intuitively, the homology group considers vectors which are mapped by g onto 0 while neglecting the ones which are images of vectors in V through f. $Im(f)$ which is contained

in $ker(g)$ is, in some sense, "contracted". This interpretation will be much clearer in part 3, when we will translate all of this to the world of simplicial complexes.

A natural question is whether one can find a reasonable way to pick representatives of the equivalence classes in $\mathcal{H}$. In particular, one would like to find a space which is easy to define and isomorphic to $\mathcal{H}$ which allow us to work on the homology without actually considering equivalence classes. It turns out that this is possible with the help of the Hodge Laplacian.

Theorem 2.2.1 (Harmonic representative)**.** *The homology group is isomorphic to the harmonic space*

$$\mathcal{H} \cong ker(\Delta).$$

Proof. Given an equivalence class $[z] \in \mathcal{H}$ we decide to pick, as a representative, the unique vector $x \in ker(g)$ which is orthogonal to every other vector in $Im(f)$. This means that $x \in Im(f)^{\perp}$.

- This vector exists. $x = z + c^*$, we ask the existence of a vector $c^* \in Im(f)$ such that $\langle z + c^*, c\rangle = 0 \ \forall c \in Im(f)$. If we write $c^* = f(y^*), c = f(y)$, then

$$\begin{aligned} \langle z + f(y^*), f(y)\rangle_W = 0 &\iff \langle f^*(z) + f^* \circ f(y^*), y\rangle_V = 0 \\ &\iff \langle f^*(z), y\rangle_V = \langle -f^* \circ f(y^*), y\rangle_V \\ &\iff f^*(z) = -f^* \circ f(y^*), \end{aligned}$$

  which admits solutions for y^*, as $f^*(z), -f^* \circ f(y^*) \in Im(f^*)$.

- This vector is unique as, if $x, y \in Im(f)^{\perp}$ and $x \sim y$, then $\exists c \in Im(f)$ such that $x - y = c$. This means that $c = x - y \in Im(f)^{\perp}$ and $c \in Im(f)$ and thus $c \in Im(f)^{\perp} \cap Im(f) = \{0\} \implies c = 0 \implies x = y$.

Given that $Im(f)^{\perp} = ker(f^*)$, we require that $x \in ker(g) \cap ker(f^*)$ meaning that x is indeed harmonic. The thesis follows from the property $ker(\Delta) = ker(g) \cap ker(f^*)$ (Proposition 2.1.4). □

We note here that, in the rest of this work, we will not work with homology but with **co**homology. For the sake of this thesis, the distinction is merely a matter of notation, as their properties turn out to be exactly the same. In other applications, however, homology and cohomology are distinct structures, dual to one another but nonetheless differently built and with different properties. We shall not concern with such issues here, but we refer the interested here to [9].

3 Introduction to discrete exterior calculus

In this first part of the thesis, we give an overview of the field of discrete exterior calculus, whose concepts and instruments will largely be used throughout all of this work. The exposition loosely follows the book by Grady [8] although with a different notation and a few differences in the mathematical constructions. The main goal will be to define and explore the properties of simplicial complexes, a particular class of combinatorial objects which extend the familiar notion of oriented graph to account for higher order connections. Simplicial complexes will take the role of geometrical domains, discrete in nature, upon which one can define many of the familiar concepts at the core of standard calculus such as differential forms, derivatives, integration and Laplacian operators.

It is important to underline that discrete calculus is not a mere *discretization* of differential calculus. It is, in truth, possible to discretize (more precisely triangulate) manifolds with simplicial complexes in such a way that the discrete objects converge to their continuous equivalents as the discretization becomes finer and finer. In this setting discrete calculus can be seen as a discretization of calculus, useful for performing computations regarding continuous objects, which, by their nature, cannot be fully described by a computer. Discrete calculus, however, can be applied to much more general situations and provides tools to work with other kind of geometries, such as networks, with no notion of an underlying continuous space.

3.1 Simplicial complexes

Differential geometry works with manifolds, topological spaces which are locally homeomorphic to Euclidean spaces. Continuous objects however, cannot be understood by a digital computer, which is discrete in nature. One therefore needs spaces with a discrete *combinatorial* structure which can be constructed with a finite set of "bricks". This role is covered by *simplicial complexes*, a particular family of mathematical objects which, in their purest form, contain only information regarding the *relation* between their parts and nothing about *position*. The bricks which make up a simplicial complex are called *simplices*, among which are points, segments, triangles, tetrahedra and so on.

3.1.1 Fundamentals

Definition 3.1.1 (Simplicial complex)**.** *Given a finite set $\mathcal{V} = \{v_0, \dots, v_{N-1}\}$ of N points, a **k-simplex** σ_i is a set $\{v_{i_0}, \dots, v_{i_k}\}$ of $k+1$ points in $\mathcal{V}$. A **face** of σ_i is any subset $\{v_{i_0}, \dots, v_{i_{j-1}}, v_{i_{j+1}}, \dots, v_{i_k}\}$ for some j. If τ_i is a face of σ_i we write $\tau_i \subset \sigma_i$. An **abstract simplicial complex** $\mathcal{X}$ is a finite collection of simplices closed under the inclusion of faces, i.e. if $\sigma_i \in \mathcal{X}$ then all of its faces belong to $\mathcal{X}$.*

We call n_k the number of kth order simplices in $\mathcal{X}$ and K the smallest integer for which $n_{K+1} = 0$, i.e. the maximum order of the complex. The kth order simplices together form the **k-skeleton** of the complex, denoted by $\mathcal{X}_k$. According to the definition above, an abstract simplicial complex is a purely combinatorial construct where no coordinates, and therefore no geometry, are involved. If we associate coordinates in some space $\mathbb{R}^N$ to the points in $\mathcal{V}$ then we can define a **geometric k-simplex** as the convex hull of $k+1$ affinely independent points (see Figure 3.2). A **geometric simplicial complex** is a finite collection of geometric simplices which is closed under inclusion and such that any intersection of simplices is itself a simplex belonging to the complex or the empty set. Every abstract simplicial complex can be given a geometric realization by gluing together the different simplices in a space with large enough dimension. When geometrized in this way, a simplicial complex can be endowed with a topology given by restriction of the ambient Euclidean topology.

A k-simplex $\{v_{i_0}, \dots, v_{i_N}\}$ can be **oriented** by fixing an order to its vertices $[v_{i_0}, \dots, v_{i_N}]$. Two orientations (orderings of the vertices) are equivalent if and only if they are related by an even permutation, i.e. if one is related to the other by an even number of swaps of two vertices. Every simplex can thus have only 2 orientations (see Figure 3.2). $[a, b, c]$ and $[c, a, b]$ are oriented in the same way as the second can be obtained by the first with two swaps: $[a, b, c] \to [a, c, b] \to [c, a, b]$. We say that an oriented simplex naturally **induces** an orientation on its faces by considering the order of each of the faces' vertices. For example, the 2-simplex $[a, c, b]$ induces the orientations $[a, c], [c, b], [b, a]$ of its faces.

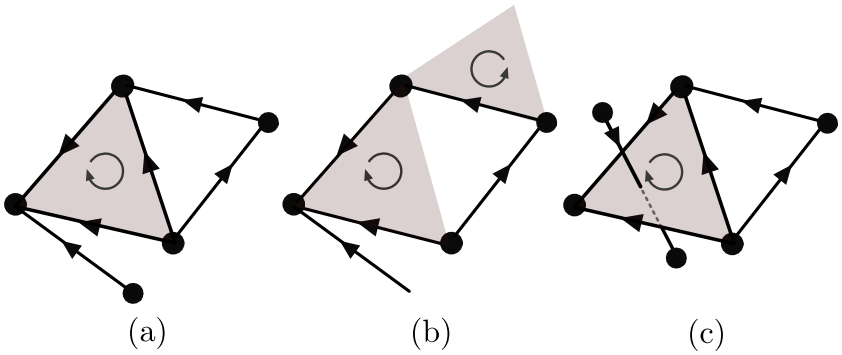


Figure 3.1: (a) is an allowed geometric simplicial complex. (b) is not a simplicial complex, as it is not closed under face inclusion. (c) is not a geometric simplicial complex because there is an intersection between simplices which is not a simplex.

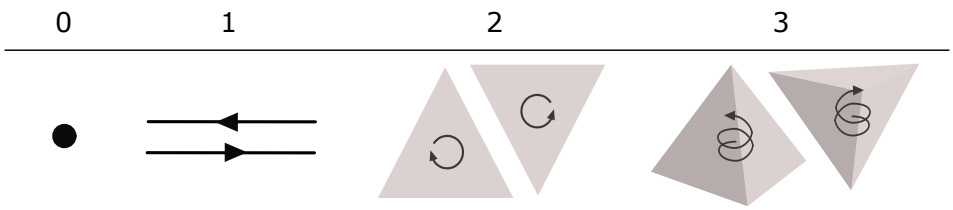


Figure 3.2: Oriented simplices by order.

Now that we have got an underlying space, we devote the rest of the section to the construction of a useful algebraic structure on it: the *cochain complex*.

Definition 3.1.2 (Chain space)**.** *We denote with $C_k(\mathcal{X};\mathbb{R})$ the set of ordered k-simplices with the structure of vector space over the field of real numbers*[a]*. We call it* ***chain space*** *of order k and its elements,* ***k-chains****, are formal linear combinations of the k-th order simplices. By definition, we change the orientation of a simplex in a chain by changing its sign.*

[a] One could choose coefficients in any field $\mathbb{F}$.

Consider for example the simplicial complex

$$\mathcal{X} = \{[a],[b],[c],[d],[a,b],[b,c],[c,d],[d,a],[a,c],[a,b,c]\}\,,$$

some examples of chains are

$$3[a]-[b] \in C_0(\mathcal{X};\mathbb{R}), \quad [a,b]+4[b,c]-\pi[a,c] \in C_1(\mathcal{X};\mathbb{R}), \quad -3[a,b,c] \in C_2(\mathcal{X};\mathbb{R}).$$

It also holds that $[a,b] = -[a,b]$ and $[a,b,c] = [c,a,b] = -[b,a,c]$. Note that, by construction, the set of k-th order simplices is a basis of $C_k(\mathcal{X};\mathbb{R})$ meaning that $dim(C_k(\mathcal{X};\mathbb{R})) = n_k$. We write a general k-chain as

$$x = x^i\tau_i,\ x_i \in \mathbb{R},\ \tau_i \in \mathcal{X}_k$$

using Einstein notation.

Definition 3.1.3 (Adjacency)**.** *Two k-simplices are* ***upper adjacent*** *in a complex $\mathcal{X}$ if they are both faces of a $(k+1)$-simplex in $\mathcal{X}_{k+1}$. They are* ***lower adjacent*** *if they have a common face of order $(k-1)$ in $\mathcal{X}_{k-1}$.*

For example, in a filled triangle, the edges are lower adjacent through the nodes and upper adjacent through the face. These relationships can be expressed by the action of a family of special linear operators acting on chain spaces.

Definition 3.1.4 (Boundary operator)**.** *We define the* ***boundary operator*** *of order k as the linear operator*

$$\partial_k : C_k(\mathcal{X};\mathbb{R}) \to C_{k-1}(\mathcal{X};\mathbb{R})$$

which acts on k-simplices in the following way

$$\partial_k[v_{i_0},\dots,v_{i_k}] \triangleq \sum_{j=0}^{k}(-1)^j[v_{i_0},\dots,v_{i_{j-1}},\not{v}_{i_j},v_{i_{j+1}},\dots,v_{i_k}].$$

The boundary of a k-simplex is thus the $(k-1)$-chain given by the alternating sum of its faces. All the lower adjacency structure of $\mathcal{X}_k$ is directly encoded in ∂_k.

A chain whose boundary is zero is called a **cycle**. Intuitively, a 1-cycle can be a closed loop of edges and a 2-cycle a triangulation of a closed surface like a sphere or a torus. A k-chain which is in the image of ∂_{k+1} is called a **boundary**.

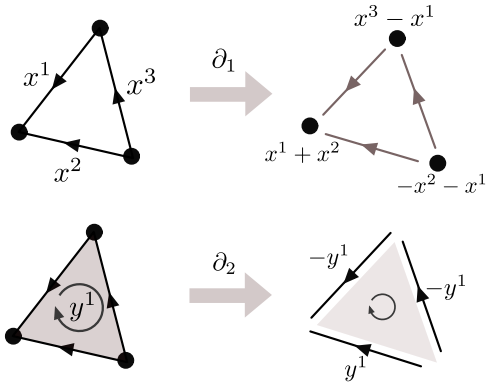


Figure 3.3: Action of the boundary operators on chains.

A fundamental property of chain spaces and boundary operators is that together they form a *chain complex*

$$C_K(\mathcal{X};\mathbb{R}) \xrightarrow{\partial_K} C_{K-1}(\mathcal{X};\mathbb{R}) \xrightarrow{\partial_{K-1}} \dots \xrightarrow{\partial_2} C_1(\mathcal{X},\mathbb{R}) \xrightarrow{\partial_1} C_0(\mathcal{X},\mathbb{R}) \xrightarrow{\partial_0} \{0\}$$

where we define $\partial_0 = 0$ and it holds that

Property 2.

$$\partial_{k-1}\partial_k = 0 \quad \forall k = 1,\dots,K. \tag{3.1}$$

This property, which is sometimes called *fundamental theorem of topology*, can be given geometrical sense by the intuitive notion that "a boundary has no boundary". We prove it for a single k-simplex.

$$\begin{aligned}\partial_{k-1}\partial_k[v_{i_0},\dots,v_{i_k}] &= \sum_{j=0}^{k}(-1)^j\partial_{k-1}[v_{i_0},\dots,v_{i_{j-1}},v_{i_{j+1}},\dots,v_{i_k}] = \\ &= \sum_{j<l}^{k}(-1)^j(-1)^l[v_{i_0},\dots,v_{i_{j-1}},v_{i_{j+1}},\dots,v_{i_{l-1}},v_{i_{l+1}},\dots,v_{i_k}] + \\ &+ \sum_{j>l}^{k}(-1)^j(-1)^l[v_{i_0},\dots,v_{i_{l-1}},v_{i_{l+1}},\dots,v_{i_{j-1}},v_{i_{j+1}},\dots,v_{i_k}] = 0\end{aligned}$$

as the terms of the two summands cancel in pairs. The result is extended to chains by linearity. Note that this is exactly the fundamental property required to apply the tools of Hodge theory (Property 1) and, in fact, we will do so after a few more concepts will be introduced.

When working in practice with these elements, we make use of the obvious isomorphism $C_k(\mathcal{X}, R) \cong \mathbb{R}^{n_k}$ and represent chains as n_k-dimensional real vectors. Accordingly, we represent the kth order boundary operator as a matrix with the so-called *incidence matrix*.

Definition 3.1.5 (Incidence matrix)**.** *We define the incidence matrix* $B_k \in \mathbb{R}^{n_{k-1} \times n_k}$ *as*

$$B_k(i,j) = \begin{cases} 0 & \text{if } \sigma_i \not\subset \sigma_j \\ 1 & \text{if } \sigma_i \subset \sigma_j \text{ and } \sigma_i \sim \sigma_j \\ -1 & \text{if } \sigma_i \subset \sigma_j \text{ and } \sigma_i \not\sim \sigma_j \end{cases}$$

where, when $\sigma_i \subset \sigma_j$*, we write* $\sigma_i \sim \sigma_j$ *if their orientations are coherent. For completeness, we also define* $B_0 = 0 \in \mathbb{R}^{1 \times n_0}$*.*

Property 2 is translated to matrix form to

$$B_{k-1} B_k = 0 \quad \forall k = 1, \dots, K.$$

In the same way as one defines a Riemannian metric on a manifold in order to introduce notions of length, area and volume, we can now define an inner product on each chain space giving it the structure of a Hilbert space

$$\langle x, y \rangle_{(k)} = W^{(k)}_{ij} x^i x^j$$

where $x, y \in C_k(\mathcal{X}; \mathbb{R})$ and $W^{(k)}$ is a symmetric positive definite **metric tensor** i.e. $W^{(k)} : C_k \times C_k \to \mathbb{R}$. One usually considers diagonal inner products $W^{(k)}_{ij} = W^{(k)}_{ii} \delta_{ji} > 0$. In this way the simplices form an orthogonal basis of the chain space and $W^{(k)}_{ii}$ can be interpreted as the **weight** of the k-simplex i. In matrix form, the inner product is simply a symmetric positive definite matrix $W^{(k)} \in \mathbb{R}^{n_k \times n_k}$.

Definition 3.1.6 (Norm)**.** *The inner product induces a* ***norm*** *on each chain space, defined as*

$$\|x\|_{(k)} \triangleq \sqrt{\langle x, x \rangle_{(k)}} \quad \forall x \in C_k(\mathcal{X}; \mathbb{R}).$$

When we write $\|x\|_k$ without the parentheses we will instead mean the standard k-norm of a vector i.e.

$$\|x\|_k = \left(\sum_{i=1}^{n} x_i^k \right)^{\frac{1}{k}}.$$

3.1.2 The cochain complex

Armed with a chain complex and an inner product, we can now build another important ingredient of discrete calculus: the *cochain complex*. If chains correspond to geometrical domains, cochains will correspond to the most natural objects which can be integrated on such domains, i.e. differential forms.

Definition 3.1.7 (Cochain space)**.** *We define the kth order* ***cochain space*** $C^k(\mathcal{X};\mathbb{R})$ *as the dual space to* $C_k(\mathcal{X};\mathbb{R})$*, i.e. the vector space of linear functionals* $C_k(\mathcal{X};\mathbb{R}) \to \mathbb{R}$ *endowed with an inner product given by the inverse of* $W^{(k)}$*, which we denote with* $W_{(k)}$*.*

In matrix form, the inner product $W_{(k)}$ is just the matrix inverse of $W^{(k)}$.

Every Hilbert space is isomorphic to its dual, therefore C_k and C^k have the same dimension, equal to the number of k-simplices n_k. A natural basis for $C^k(\mathcal{X};\mathbb{R})$ is the canonical basis $\{\tau^i\}$ whose elements are each supported on one single k-simplex

$$\tau^i(\tau_j) = \delta^i_j.$$

Cochains can thus be seen as real functions on simplices. The analogy with differential forms is clear if one looks explicitly at simplices as lists of vertices. Consider for example the complex $\mathcal{X} = \{\clubsuit, \diamondsuit, \spadesuit, [\clubsuit, \diamondsuit], [\diamondsuit, \spadesuit], [\clubsuit, \spadesuit], [\clubsuit, \diamondsuit, \spadesuit]\}$ and the basis 2-cochain $x = \tau$, where $\tau([\clubsuit, \diamondsuit, \spadesuit]) = 1$. If we think of τ as a real function of the vertices, then the property $\tau([\diamondsuit, \clubsuit, \spadesuit]) = \tau([\spadesuit, \diamondsuit, \clubsuit]) = -1$ is the same of asking the basis cochain τ to be an *antisymmetric* function of the vertices.

The role of the inner product, aside from introducing "geometry" to the simplicial complex, is in establishing a relation between chain and cochain space. This is done in two ways:

1. by inducing an isomorphism between chains and cochains;
2. by establishing a notion of *integration* of cochains onto chains, i.e. of differential forms onto domains.

First, one can canonically associate a cochain to a chain through the action of $W^{(k)}$. We do this with the **musical isomorphism operators** $\flat$ and $\sharp$.

$$\begin{aligned} \flat_k : C_k(\mathcal{X};\mathbb{R}) &\longrightarrow C^k(\mathcal{X};\mathbb{R}) \\ x = x^i\tau_i &\longmapsto x^{\flat_k} = W^{(k)}_{ij} x^j \tau^i \end{aligned}$$

$$\begin{aligned} \sharp_k : C^k(\mathcal{X};\mathbb{R}) &\longrightarrow C_k(\mathcal{X};\mathbb{R}) \\ x = x_i\tau^i &\longmapsto x^{\sharp_k} = W^{ij}_{(k)} x_j \tau_i \end{aligned}$$

Both $\flat_k$ and $\sharp_k$ are vector space isomorphisms because $W^{(k)}$ is square and invertible. Moreover, they are one the inverse of the other $\flat_k = \sharp_k^{-1}$ and they can be represented respectively with the inner product matrices $W^{(k)}$ and $W_{(k)}$.

Secondly, one can pair a cochain and a chain by integrating the first on the latter, in the same way as one would integrate a differential form on a domain.

Definition 3.1.8 (Integral)**.** *We define the* ***integral*** *of a k-cochain $y \in C^k(\mathcal{X};\mathbb{R})$ onto the k-chain $x \in C_k(\mathcal{X};\mathbb{R})$ as*

$$(x, y) \triangleq x^i y_i \in \mathbb{R} \quad \longleftrightarrow \quad \int_x y,$$

Note that this is the same as taking the inner product of x with the sharp of y.

Proposition 3.1.1. *It holds that*

$$(x, y) = \left\langle x, y^{\sharp_k} \right\rangle_{(k)}$$

Proof.

$$\left\langle x, y^{\sharp_k} \right\rangle_{(k)} = x^i W^{(k)}_{ij} W^{jl}_{(k)} y_l = x^i y_i,$$

as $W_{(k)}$ is the inverse of $W^{(k)}$. □

Having the concept of integration, we can define the "volume" of a k-chain as the integration of the **volume cochain** $w \triangleq W^{(k)}\mathbb{1}$ on x

$$(x, w) = W^{(k)}_{ij} x^i \mathbb{1}^j = \sum_i \sum_j W^{(k)}_{ij} x^i = \langle x, \mathbb{1} \rangle_{(k)},$$

i.e. the inner product of x with $\mathbb{1}$ in $C^k(\mathcal{X};\mathbb{R})$.

Let us return to the analogy with differential geometry. We have defined differential forms and the operation of integration, but something is clearly missing: the *differential*. The differential, which extends the notion of derivative, sends k-forms to $(k+1)$-forms in such a way that the differential of the differential always vanishes. To complete the construction of the cochain complex, we thus need to define a linear operator which sends k-cochains to $(k+1)$-cochains. The most natural way to do this is to, once again, employ the inner product.

Definition 3.1.9 (Coboundary operator)**.** *We define the* ***coboundary operator*** *$d^k : C^k(\mathcal{X};\mathbb{R}) \to C^{k+1}(\mathcal{X},\mathbb{R})$ as the dual operator to ∂_{k+1} w.r.t to the integration pairing*

$$(\partial_{k+1} x, y) = (x, d^k y) \tag{3.2}$$

for every $x \in C_{k+1}(\mathcal{X},\mathbb{R})$ and $y \in C^k(\mathcal{X};\mathbb{R})$.

If we look closely at equation 3.2 we can see a striking resemblance to Stoke's fundamental theorem of calculus as, formally,

$$\int_x dy \longleftrightarrow (x, d^k y) = (\partial_{k+1} x, y) \longleftrightarrow \int_{\partial x} y,$$

a surprising result which further motivates our efforts.

In analogy with the boundary operator, we call **cocycle** a cochain whose coboundary is 0 and **coboundary** a k-cochain in the image of d^{k-1}. Note that 3.2 implies that, if x is a cycle, $(x, d^k y) = 0$. This is, once again, in perfect analogy to the continuous case as it is the discrete version of the fact that conservative vector fields (coboundaries) vanish when integrated on closed curves.

From the definition, we directly prove that a dual property to 2 holds

Property 3.

$$d^{k+1} d^k = 0$$

i.e. the differential of a differential vanishes.

Proof. We prove it by showing that $\langle y, d_{k+1} d_k x \rangle_{(k+2)} = 0$ for every pair of cochains $y \in C^{k+2}, x \in C^k$.

$$\langle y, d_{k+1} d_k x \rangle_{(k+2)} = (y^{\sharp_{k+2}}, d_{k+1} d_k x) = (\partial_{k+2} y^{\sharp_{k+2}}, d_k x) = (\partial_{k+1} \partial_{k+2} y^{\sharp_{k+2}}, x) = (0, x) = 0.$$

□

We have built the *cochain complex*

$$0 \xrightarrow{d^0} C^1(\mathcal{X}, \mathbb{R}) \xrightarrow{d^1} C^2(\mathcal{X}, \mathbb{R}) \xrightarrow{d^2} \dots \xrightarrow{d^{K-2}} C^{K-1}(\mathcal{X}; \mathbb{R}) \xrightarrow{d^{K-1}} C^K(\mathcal{X}; \mathbb{R}),$$

where, contrarily to the chain complex, the action of the operator increases the order.

Proposition 3.1.2. *In terms of matrix representation, the coboundary operator is just the transpose of the incidence matrix* $D^k = B_{k+1}^\top \in \mathbb{R}^{n_{k+1} \times n_k}$.

Proof. To prove this, consider the component expressions of both the boundary and coboundary operators,

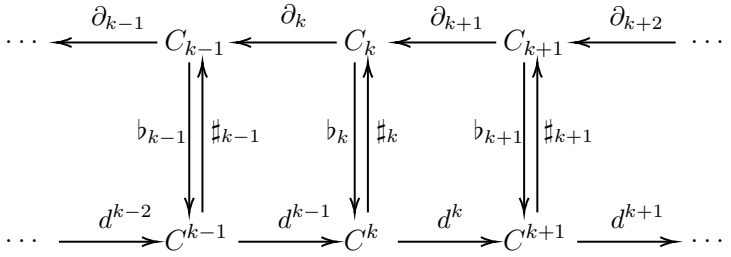
$$(\partial_k x)^i = B_j^i x^j, \quad (d^{k+1} y)_i = D_i^j y_j.$$

By definition one has

$$(\partial_{k+1} x, y) = (x, d^k y) \iff (\partial_{k+1} x)^i y_i = x^i (d^k y)_i \iff B_j^i x^j y_i = D_i^j x^i y_j \iff B_j^i = D_i^j,$$

hence the thesis. □

As a final step we can put together the chain and cochain complexes with boundary, coboundary operators and the musical isomorphisms relating them.



One needs not be constrained to the "ascending" order on cochain complexes given by d^k. In order to formulate the simplicial Kuramoto model (Section 4.1.5 of Chapter 4) we need to build an operator, analogous to ∂_k, which lowers the order of a cochain. The diagram above makes it an easy task as we can just go to the chain complex, perform the boundary there and then come back down to the cochain complex.

Definition 3.1.10 (Adjoint coboundary operator)**.** *Define the* ***adjoint coboundary*** *operator acting on cochains as*

$$\begin{aligned} \partial^k : \; & C^k(\mathcal{X};\mathbb{R}) \longrightarrow C^{K-1}(\mathcal{X};\mathbb{R}) \\ & x \longmapsto (\partial_k x^{\sharp_k})^{\flat_{k-1}} = W_{il}^{(k)} W_{(k)}^{lj} x_j \partial^k \tau^i \end{aligned}$$

represented by the matrix

$$B^k = W^{(k-1)} B_k W_{(k)}. \tag{3.3}$$

Note that, with the standard "Euclidean" inner product $W^{(k)} = I$ for every $k = 1, \dots, K$, the coboundary operator acting on cochains and the boundary acting on chains have the same matrix representation $B^k = B_k$. If the inner products are diagonal, then the adjoint coboundary differs from the boundary just by a rescaling in its components. For this reason, with a slight abuse of notation, we will sometimes call the *adjoint coboundary operator* just *boundary*.

A further important property emerges from this construction, motivating the "adjoint" part of the name.

Proposition 3.1.3. *The adjoint coboundary of order k ∂^k is the adjoint operator to the order $k-1$ coboundary d^{k-1} i.e.*

$$\left\langle \partial^k x, y \right\rangle_{(k-1)} = \left\langle x, d^{k-1} y \right\rangle_{(k)} \quad \forall x \in C^k, y \in C^{k-1}.$$

Proof.

$$\left\langle \partial^k x, y \right\rangle_{(k-1)} = \left\langle (\partial_k x^{\sharp_k})^{\flat_{k-1}}, y \right\rangle_{(k-1)} = (\partial_k x^{\sharp_k}, y) = (x^{\sharp_k}, d^{k-1} y) = \left\langle x, d^{k-1} y \right\rangle_{(k)}.$$

□

The adjoint coboundary operator satisfies Hodge's property as well.

Proposition 3.1.4. *It holds that*

$$\partial^k \partial^{k+1} x = 0 \quad \forall x \in C^{k+1}(\mathcal{X}; \mathbb{R}).$$

Proof. Because of Proposition 3.1.3, ∂^k is the adjoint of d^{k-1} for which it holds that $d^k d^{k-1} = 0$ (Proposition 3). The thesis follows from the direct application of Proposition 2.1.2 of Chapter 2. □

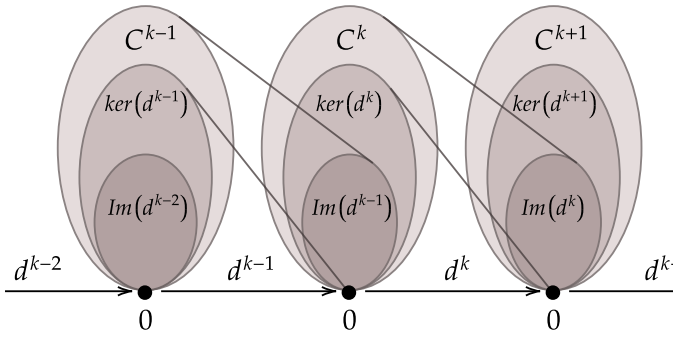


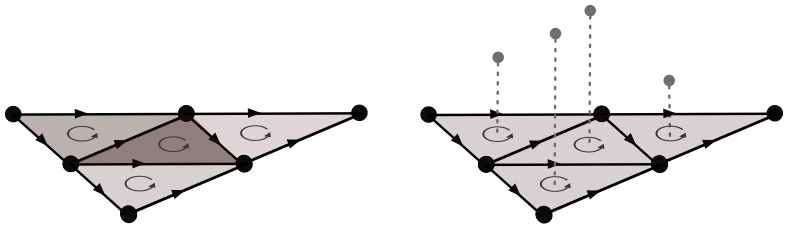
Figure 3.4: Representation of Property 3 of the coboundary operator. $d^k d^{k-1} = 0$ means that $Im(d^{k-1}) \subset ker(d^k)$. The analogous holds for the adjoint coboundary operators and their matrix representations.

Given the direct relation between dual coboundary and boundary, we call cochains in $ker(\partial^k)$ **cycles** and cochains in $Im(\partial^k)$ **boundaries**. If the inner product is not trivial, then the cycles of the dual coboundary correspond to *weighted* geometrical cycles in the complex.

Having now coboundary operators and their adjoints acting on cochains, in the rest of the thesis we will mainly restrict our attention to cochains, simply due to their direct analogy with differential forms. Notice that one could easily define the adjoint of the boundary operator on the chain complex and get a perfectly equivalent structure. One could have also started directly with cochains, but we think that this derivation is better able to show the geometric intuition involved. Furthermore, it is certainly more natural to weigh chains of simplices, in the same way as one establishes a Riemannian metric on a manifold, and induce the structure of the cochain space then directly weighing the cochains.

Because of 3, all of Hodge's theory, illustrated in Chapter 2, can now be applied to the cochain complex:

$$\cdots \underset{\partial^{k-1}}{\overset{d^{k-2}}{\rightleftarrows}} C^{k-1} \underset{\partial^{k}}{\overset{d^{k-1}}{\rightleftarrows}} C^{k} \underset{\partial^{k+1}}{\overset{d^{k}}{\rightleftarrows}} C^{k+1} \underset{\partial^{k+2}}{\overset{d^{k+1}}{\rightleftarrows}} \cdots$$



(a) (b)

Figure 3.5: Intuitive representation of the formal difference between simplicial chains and cochains. (a) represents a 2-chain where each 2-simplex is "weighted" with a scalar value. (b) represents a 2-cochain i.e. a scalar function on 2-simplices.

Of particular interest are the discrete Hodge Laplacian operators

$$\mathcal{L}^k = \underbrace{\partial^{k+1} d^k}_{\mathcal{L}^k_{up}} + \underbrace{d^{k-1} \partial^k}_{\mathcal{L}^k_{down}} \tag{3.4}$$

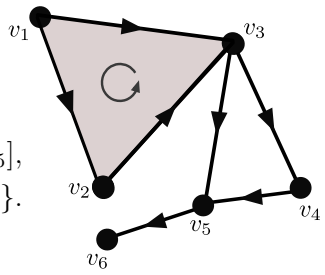
which provide a discrete generalization to the continuous Hodge Laplacians of differential geometry. The discrete Hodge Laplacians are represented by the square n_k-dimensional matrices

$$L^k = \underbrace{B^{k+1} D^k}_{L^k_{up}} + \underbrace{D^{k-1} B^k}_{L^k_{down}} = W^{(k)} B_{k+1} W_{(k+1)} B_k^\top + B_{k-1}^\top W^{(k-1)} B_k W_{(k)} \tag{3.5}$$

Remember that, as proven in Proposition 2.1.3 of Section 2, the $\mathcal{L}^k$ and L^k are both self-adjoint and positive semidefinite. We will study these operators and their properties in more detail in the next section.

3.1.3 Example

Before going forward, let us catch a breath and work on a simple example. Consider the simplicial complex $\mathcal{X}$ whose geometrical realization is shown on the right. Let us choose unitary weights for all simplices so that $W^{(k)} = I$ for every k. We have



$$\mathcal{X} = \{v_1, v_2, v_3, v_4, v_5, v_6, [v_1, v_2], [v_2, v_3], [v_1, v_3], [v_3, v_4], [v_3, v_5], [v_4, v_5], [v_5, v_6], [v_1, v_2, v_3]\}.$$

It follows that its skeletons are

$$\mathcal{X}_0 = \{v_1, v_2, v_3, v_4, v_5, v_6\}, \mathcal{X}_1 = \{[v_1, v_2], [v_2, v_3], [v_1, v_3], [v_3, v_4], [v_3, v_5], [v_4, v_5], [v_5, v_6]\}, \ \mathcal{X}_2 = \{[v_1, v_2, v_3]\}.$$

The boundary incidence matrices acting on cochains are

$$B^1 = B_1 = \underbrace{\begin{bmatrix} -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{edges}}, \ B^2 = B_2 = \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{faces}}$$

and the Hodge Laplacians

$$L^0 = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \ L^1 = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 3 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}, \ L^2 = 3.$$

3.2 Spectral Simplicial theory

In this section we complete the exposition began above and, in particular, we focus on the algebraic properties of the Hodge Laplacian operators and their striking relation to topology and signal analysis.

From now on, it will be convenient to put aside the coordinate free expressions and work with matrices and vector representations.

3.2.1 The Fourier basis as eigenvectors of the Laplacian

The goal of signal analysis is to study the characteristics of scalar functions, such as sounds (1-dimensional) or images (2-dimensional). The uncontested king of the tools in the belt of a signal analyst is without doubt the Fourier transform, which allows to study a signal not by its shape in space (or time), but by the fundamental frequencies it contains. To do so, one defines a new basis, called **Fourier basis**, of the signal space and describes the signal with its coefficients. What is often neglected is that the Fourier basis has an intimate relation with geometry as it is composed by the eigenfunctions of the Laplacian operator Δ on a precise geometrical domain, i.e. a circle for sounds and a torus for images. The natural generalization of the Laplacian, the Laplace-Beltrami operator, then allows one to define the Fourier basis even when the underlying space is a curved manifold of any dimension. Moreover, generalizing the Laplacian to work on k-forms, lets us build a Fourier basis even for vector fields. As we see here, the same holds for the discrete geometries of simplicial complexes.

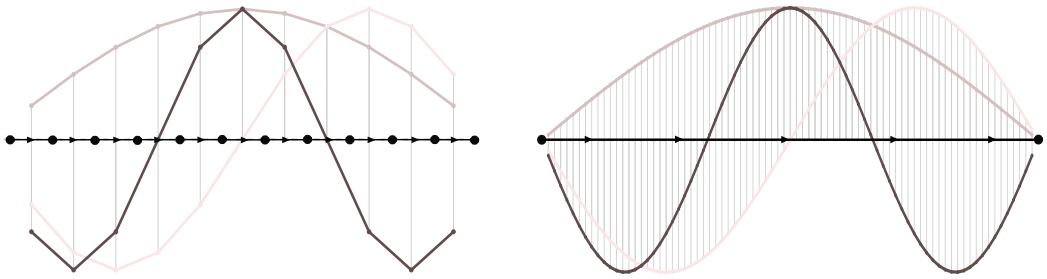


Figure 3.6: The first 3 eigenvector cochains of L^1 on a chain of 11 (left) and 79 (right) oriented edges. They clearly seem to resemble the continuous Fourier basis function on an interval.

We are thus interested in working with signals which are defined on the simplices of a simplicial complexes $\mathcal{X}$. A kth order signal is just a scalar function defined on the k-simplices of $\mathcal{X}$ and, as such, is a kth order cochain. The Hodge Laplacian of order k 3.4, in complete analogy with the Laplace-Beltrami operator, has the remarkable property that it may be used to build an alternative basis of the signal space $C^k(\mathcal{X};\mathbb{R})$ which is nonlocal and, as

it is extensively discussed in [3], is "topology-aware".

L^k is a self-adjoint matrix, and thus it is possible to find eigenvectors which form an orthonormal basis of $C^k(\mathcal{X};\mathbb{R}) \cong \mathbb{R}^{n_k}$

$$L^k = U^k \Lambda^k (U^k)^\top,$$

where $U^k \in \mathbb{R}^{n_k \times n_k}$ is an orthonormal matrix $(U^k)^\top W_{(k)} U^k = I$ and Λ^k is diagonal with non-negative elements. Each signal $x \in C^k \cong \mathbb{R}^{n_k}$ can be described with its coefficient in the Fourier basis given by the eigenvectors of L^k in U^k. We write $\hat{x} = U_k^T W_{(k)} s^k$ and call it **Fourier transform** of x. The matrix $W_{(k)}$ appears as we are considering a non-trivial inner product on $\mathbb{R}^{n_k}$.

The consistence with the standard Fourier basis can be qualitatively seen in Figure 3.6 where the first eigenvectors of L^1 on a line of 1-simplices are shown. The Fourier basis for periodic signals on an interval would be exactly obtained in the limit if one considered a closed line of 1-simplices i.e. the discretization of a circle.

3.2.2 Topological relevance of the harmonic space

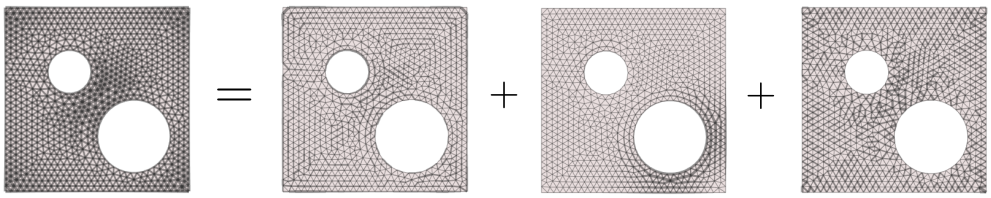
Of particular importance is the space of signals belonging to $ker(\mathcal{L}^k)$, i.e. the harmonic cochains. This space has the remarkable property that it is one of the three components of the Hodge decomposition of the signal cochain space. Recalling theorem 2.1.2 of Chapter 2 we have that every k-cochain can be uniquely written as the sum of three orthogonal[1] components:

$$C^k(\mathcal{X};\mathbb{R}) = Im(\partial^{k+1}) \oplus ker(\mathcal{L}^k) \oplus Im(d^{k-1}) \tag{3.6}$$

or, in matrix form,

$$\mathbb{R}^{n_k} = Im(B^{k+1}) \oplus ker(L^k) \oplus Im(D^{k-1}).$$

To make things clearer and more intuitive let us look at an example.



original signal $C^1(\mathcal{X},\mathbb{R})$ | divergence-free $Im(\partial^2)$ | harmonic $ker(\mathcal{L}^1)$ | curl-free $Im(d^0)$

Figure 3.7: Hodge decomposition of an edge signal. The edges' width is proportional to the absolute value of the signal components.

[1] w.r.t the inner product induced by the inner product $W^{(k)}$.

Consider the complex $\mathcal{X}$ obtained by triangulating a square in $\mathbb{R}^2$ with 2 holes and generate the edge signal $x \in C^1(\mathcal{X}; \mathbb{R})$ shown in the leftmost panel of Figure 3.7.If we reason by analogy with differential geometry and think of an edge signal (1-cochain) as a vector field, we can give interesting interpretations of the three component. The first one, which belongs to $Im(\partial^2)$, looks like a field which wraps around the domain and the holes without any sources. This comes from the fact that $Im(\partial^2) \subset ker(\partial^1)$ i.e. this component is actually a cycle which, in some sense, has to "close". We call this first component *divergence-free*. In the third component, belonging to $Im(d^0)$, the divergence aspect seems to be strong as the field "radiates" from the holes and the boundaries. This behavior is strongly reminiscent of a curl-free vector field, hence the name *curl-free*. Finally, it appears that the harmonic component is concentrated around one of the two holes. This last behavior is not a coincidence as a numerical computation shows that the dimension of the harmonic space is 2, the same as the number of holes of the domain, and the relative Laplacian eigenvectors are both supported around the two holes (see Figure 3.8).

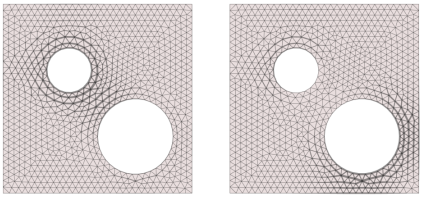


Figure 3.8: Fourier basis signals of $ker(\mathcal{L}^1)$.

The most beautiful property of the Hodge Laplacians is, in fact, their intimate connection to topology. Indeed, the dimensions of the harmonic subspaces of cochains $ker(\mathcal{L}^k)$ correspond to the **Betti numbers** of the geometric complex

$$dim(ker(\mathcal{L}^k)) = \beta_k.$$

The Betti numbers $\{\beta_k\}_{k\in\mathbb{N}}$ are well known topological invariants which encode the number of k-dimensional holes in the space. β_0 is the number of connected components of $\mathcal{X}$, β_1 is the number of loops, β_2 of closed voids, and so on. A torus, for example, would have $\beta_0 = 1$, $\beta_1 = 2$, $\beta_2 = 1$ and the disjoint union of two spheres $\beta_0 = 2$, $\beta_1 = 0$, $\beta_2 = 2$. This property comes from the fact, proved in general in Theorem 2.2.1 of Chapter 2, that the harmonic space of cochains is isomorphic to the **cohomology group** of the complex

$$\mathcal{H}^k(\mathcal{X}) = ker(d^k)/Im(d^{k-1}),$$

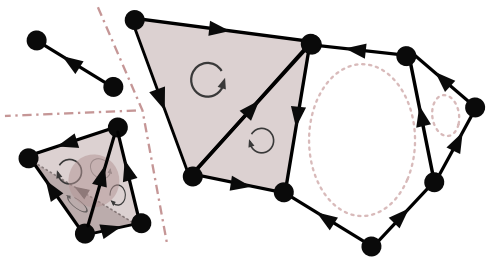
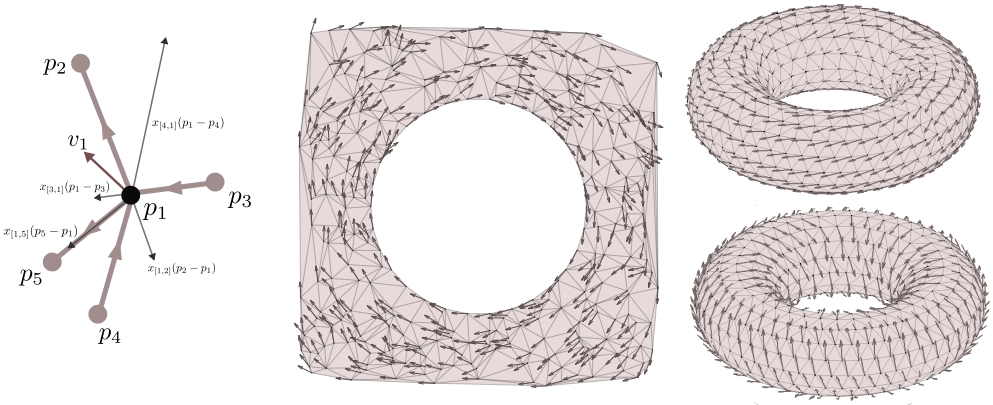


Figure 3.9: $\beta_0 = 3$ because the complex has 3 connected components. $\beta_1 = 2$ because there are 2 loops and $\beta_2 = 1$ because there is one empty void.

whose dimension is by definition the kth Betti number. For the sake of this thesis, the prefix *co* here is used simply because we are working with *co*chains instead of chains. It can be proven that $dim(\mathcal{H}^k(\mathcal{X})) = dim(\mathcal{H}_k(\mathcal{X}))$, where the second is the **homology group** of the complex

$$\mathcal{H}_k(\mathcal{X}; \mathbb{R}) = ker(\partial_k)/Im(\partial_{k+1}).$$



(a) (b) (c)

Figure 3.10: (a) Representation of the way one can associate a vector field on the nodes to an edge cochain with Equation 3.7. (b) Harmonic vector field (normalized) on a triangulated domain with a hole. (c) The two vector fields (normalized) spanning the harmonic space $ker(\mathcal{L}^1)$ of a triangulated torus. In both these cases, the vector field "wraps" around the two 1-dimensional holes of the Torus.

Intuition-wise, it is clearer to look at the homology group. The elements of the homology group are in fact equivalence classes of *cycles* (elements of $ker(\partial_k)$) which are identified when they differ by a *boundary* (element of $Im(\partial_{k+1})$). More intuitively, one counts k-holes with the homology group by considering all cycles, which correspond to both actual k-dimensional holes and boundaries of filled $(k+1)$-dimensional regions, and simply removes the last ones, which are contained in $Im(\partial_{k+1})$, through the quotient.

We can make the analogy *1-cochains - vector fields* more explicit and intuitive in the following way. Let $\mathcal{X}$ be a *geometric* simplicial complex embedded in $\mathbb{R}^n$ such that each vertex i has position $p_i \in \mathbb{R}^n$. If $x \in C^1(\mathcal{X}; \mathbb{R})$ is an edge signal, we can associate to each node a vector which is the sum of the *oriented* vectors connecting it to its neighbors, weighted by the components of x. More precisely we associate to vertex i a vector v_i

$$v_i = \underbrace{\sum_{[i,j] \in \mathcal{X}_1} x_{[i,j]}(p_j - p_i)}_{\text{edges leaving } i} + \underbrace{\sum_{[j,i] \in \mathcal{X}_1} x_{[j,i]}(p_i - p_j)}_{\text{edges entering } i} . \tag{3.7}$$

This association is graphically represented in Figure 3.10 (a). If we compute the associated vector fields to harmonic 1-cochains in a simplicial complex, we will find that they will wrap around 1-dimensional holes, just like it is shown in Figure 3.10 (b, c).

4 The Kuramoto model

Synchronization is a ubiquitous phenomenon, universally found both in natural and human-engineered systems. From firing neurons to the twinkling of fireflies, from metronomes to power grids, from applauding audiences to the circadian rhythms of plants and animals. All of these systems display similar synchronization behaviors. Patterns and order magically emerge from the disordered interplay of many interacting parts, with a complexity which seems to give no hope for simple models. As it often happens in complex systems, however, their obvious differences fade away when the phenomenon is examined at the right scale and with the right level of abstraction. The way synchronization occurs seems to be independent of whether the agents are singing crickets, heart cells or metronomes. The mathematical power of capturing *structure* regardless of *form* can thus be exploited to formulate powerful models which surprise with both their simplicity and explanatory power.

The father of these models is, no doubt, the one proposed by Yoshiki Kuramoto (蔵本由紀) in his seminal work of 1975 [14]. Its multiple variants have been extensively studied in the decades since its creation and, although many questions are still unanswered, thanks to it, we now have a much firmer grasp on the behavior of synchronizing systems.

4.1 The model

4.1.1 What is an oscillator?

The exposition of this first section loosely follows the book by Pikovsky [21].

We start our formal treatment by discussing the fundamental issue of describing oscillator systems. In our modeling effort this must necessarily be a simplifying step which has to capture the essential characteristics which relate, for example, a neuron to a firefly. We will describe oscillators which belong to the family of **self-sustained oscillators**. With this term, we denote a system in which an internal source of energy makes it oscillate with a given periodical pattern for an indefinite amount of time. When its state is slightly perturbed, the periodical pattern is restored after a transient. Requiring the existence of this internal source of energy lets us work with oscillators whose behavior is **autonomous**, i.e. time independent. We do not have to worry, for example, about the effect of air friction, which would quickly dissipate all of a pendulum's initial energy.

Simple examples of self-sustained oscillators are clocks and isolated fireflies. Of course, in nature there are no infinite sources of energy which can indefinitely power the oscillations. The clock's spring will eventually loosen and the firefly will unfortunately die after a couple of months. For modeling purposes, we assume that this source is big enough to be considered infinite w.r.t the timescale we want to consider (hours for the clock, seconds for the firefly).

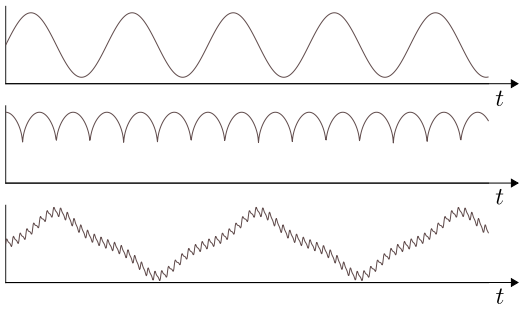


Figure 4.1: Example of oscillatory behaviors.

Some examples of oscillatory behaviors are represented in Figure 4.1 as real periodic functions of time $x(t)$. An important observation is that, if one has to describe the evolution of such systems, one variable is not enough. Knowing the position of a pendulum at a given time, for example, does not provide a complete description of its state as its velocity could both be pointing up or towards the equilibrium. A state description which also takes into consideration angular velocity is necessary. The state of the system is then represented by a point in the multidimensional **phase space**. Let us now focus on oscillators whose phase space is 2-dimensional, although analogous arguments hold for any dimension.

If we describe the state of the system in the phase space with a variable $x \in \mathbb{R}^2$, in order for $x(t)$ to describe the evolution of a self-sustained oscillator, we require that $x(t)$ is T-periodic, i.e. $\mathcal{C} \triangleq \{x(t) : t \in \mathbb{R}\}$ is a closed, non-intersecting curve in $\mathbb{R}^2$ which is periodically traveled by the system every T units of time. This curve is called the **limit cycle** of the oscillator.

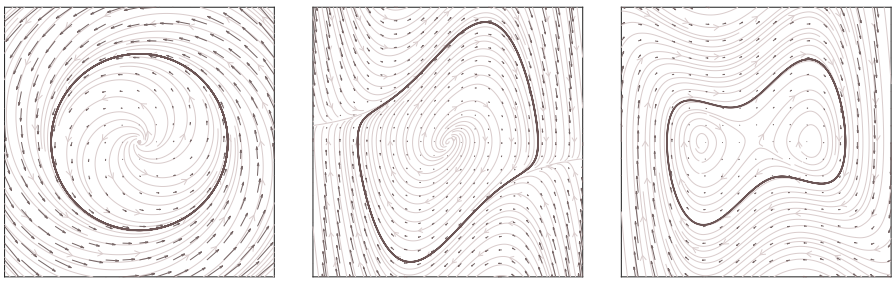


Figure 4.2: Phase space portrait of some oscillatory behaviors. The vector field representing the autonomous dynamics is shown together by some integral curves and its limit cycle.

Remember that in the definition of self-sustained oscillator we also require that its evolution is stable w.r.t small perturbations. This means that the dynamics of the system

is such that $\mathcal{C}$ is a **stable attractor**, i.e. trajectories which are close to $\mathcal{C}$ will converge to $\mathcal{C}$. Examples of oscillatory dynamics of this type, which are given by autonomous systems of differential equations, are shown in Figure 4.2.

If we parametrize the limit cycle $\mathcal{C}$, we call its scalar parameter **phase** and denote it by θ. It follows that $\theta(t)$ is a T-periodic function of time. The stability condition implies that a perturbation which is transversal to the limit cycle will go to 0 with time.

Of course, a general description of the evolution of such a vast class of oscillators is impossible. A restriction is surely necessary. Kuramoto's model makes two essential simplifications:

1. constrain the oscillator's state to lie on the unit circle (no transversal motion possible);

2. consider oscillators which travel the unit circle with constant angular speed ω i.e. $x(t) = (\cos(\omega t + \theta_0), \sin(\omega t + \theta_0))$.

The phase parameter θ can then be chosen to be the angle which the state makes with the x axis. With this in mind, the second simplification can be translated in asking for $\theta(t) = \omega t + \theta_0$ i.e. constant angular speed. This speed is called the **natural frequency** of the oscillator, as it is the angular frequency it will have when left unperturbed. If we have n different *isolated* oscillators, each with its own natural frequency $\omega_i \in \mathbb{R}$, we can describe the evolution of the system with the ordinary system of differential equations

$$\dot{\theta} = \omega. \tag{4.1}$$

We stress the word "isolated" because, up to this point, no interaction is considered. The oscillators are each evolving as by their nature, with no external disruptions (see Figure 4.3 for an example). It is also important to underline that, unlike the dynamics portrayed in Figure 4.2, here the motion is constrained to the limit cycle and thus the phase is enough to fully describe the evolution of the oscillator.

4.1.2 All-to-all interactions

We now come to the central point of explaining how it is possible for a system of oscillators to synchronize. Why do metronomes on a moving platform, each set with its own tempo (natural frequency), synchronize after a few beats? The reason is to be found in interaction. In each of the synchronizing systems mentioned above, one finds that the oscillators are always coupled, in the sense that they exchange information about their phase. This could mean different things depending on the specific system: the metronomes are coupled through the slight movements and vibrations of the moving platform, the fireflies can see each other's light, and the applauding crowd can feel the movement of their seats induced by the applause itself. No matter how weak, interactions disrupt the natural behavior of oscillators, allowing for the emergence of a variety of interesting collective phenomena, of which synchronization is just one.

Figure 4.3: Example trajectories of two isolated Kuramoto oscillators with different natural frequencies.

Let us consider a system of n Kuramoto oscillators with natural frequencies gathered in the vector $\omega \in \mathbb{R}^n$. We want to modify model 4.1 for isolated oscillators in order to account for physically meaningful interactions. Following Kuramoto's work, we do this in the following way:

$$\dot{\theta}_i = \omega_i - \sigma \sum_{j=1}^{n} \sin(\theta_i - \theta_j). \tag{4.2}$$

The newly added term tells us that the evolution of the phase of oscillator i is disrupted by the interaction between i and every other oscillator j, modulated by the **coupling** or **interaction strength** $\sigma > 0$. First notice that, by the properties of the sine function, the interaction between oscillators i and j will be 0 when $\theta_i - \theta_j = k\pi$ i.e. when θ_i and θ_j represent the same angle (**in-phase synchronization**) or opposite angles (**anti-phase synchronization**), modulus 2π. The interaction, instead, is strongest when the two phases differ by an odd multiple of $\frac{\pi}{2}$ i.e. when the two states, as seen on the unit circle, are orthogonal. This last facts suggests us to formulate a graphical and more intuitive interpretation of the interaction term (see Figure 4.4).

Consider a pair of oscillators as two points on the unit circle with position $x_i = (\cos(\theta_i), \sin(\theta_i))$ $i = 1,2$ such that they are pulled together by a force whose direction is the line connecting x_i to x_j and its intensity is proportional to their distance. The oscillators' states however need to be constrained to the circle and so the pull they actually experience is an orthogonal projection of $x_2 - x_1$ onto the tangent line to the circle, spanned by the unit vector $v(\theta_i) = (-\sin(\theta_i), \cos(\theta_i))^\top$. More formally

$$\dot{x}_i = \langle v(\theta_1), x_2 - x_1 \rangle \, v(\theta_1)$$
$$= (-\sin(\theta_i), \cos(\theta_i))(x_2 - x_1)(-\sin(\theta_i), \cos(\theta_i))^\top,$$

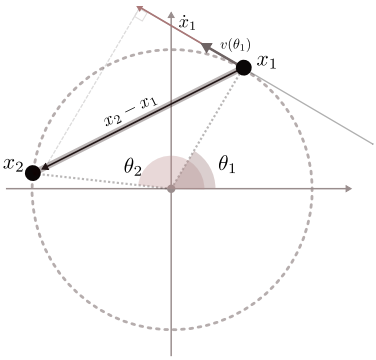


Figure 4.4: Geometrical interpretation of the pairwise interaction term.

which, after some algebra and trigonometry, becomes, for oscillator 1,

$$\dot{x}_1 = -\sin(\theta_1 - \theta_2)\begin{pmatrix} -\sin(\theta_i) \\ \cos(\theta_1) \end{pmatrix}.$$

The evolution of the phase θ_1 will then be

$$\begin{aligned}\dot{\theta}_1 &= \frac{d}{dt}\arctan\left(\frac{x_1^2}{x_1^1}\right) = \frac{1}{1+\left(\frac{x_1^2}{x_1^1}\right)^2}\frac{\dot{x}_1^2 x_1^1 - x_1^2 \dot{x}_1^1}{(x_1^1)^2} = \frac{1}{(x_1^1)^2 + (x_1^2)^2}(\dot{x}_1^2 x_1^1 - x_1^2 \dot{x}_1^1) \\ &= \frac{1}{\cos(\theta_1)^2 + \sin(\theta_1)^2}(-\cos(\theta_1)\sin(\theta_1 - \theta_2)\cos(\theta_1) - \sin(\theta_1)\sin(\theta_1 - \theta_2)\sin(\theta_1)) \\ &= -\sin(\theta_1 - \theta_2)(\sin(\theta_1)^2 + \cos(\theta_1)^2) = -\sin(\theta_1 - \theta_2).\end{aligned}$$

In the same way one finds

$$\dot{\theta}_2 = -\sin(\theta_2 - \theta_1),$$

which, if multiplied by the coupling strength σ, is precisely the interaction term of the Kuramoto model 4.2 for a single pair of oscillators. This intuition can be extended to any number of oscillators: each oscillator will be simultaneously pulled, as by springs, by all the others and will move according to their total (See Figure 4.5).

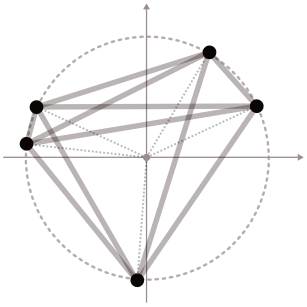


Figure 4.5: Representation of a system of 5 interacting oscillators.

Adding interaction terms together with natural frequencies will give us behaviors like the one shown in Figure 4.6. We can clearly see how the two oscillators, each with its own frequency, are pulled together by the interaction until their phases difference "locks" to a fixed amount, and they start oscillating at the same frequency. We call this configuration *phase-locked*. For now, we must be satisfied with an intuitive understanding, as we will formally define it in later sections.

Many questions can be asked about the behavior of the Kuramoto model 4.2. For example, one could investigate the shapes of the possible equilibrium configurations $\dot{\theta} = 0$ (when the interaction terms are balanced out by the natural frequencies) or look at the conditions on the coupling strength under which phase locking occurs. We will examine these kinds of questions in later sections, right after we will make a couple more generalizations of the model.

4.1.3 Network interactions

The first obvious generalization naturally comes with the observation that model 4.2 describes a system in which *every oscillator interacts with every other oscillator in the same way*. This, especially for large populations of oscillators, becomes an unrealistic assumption.

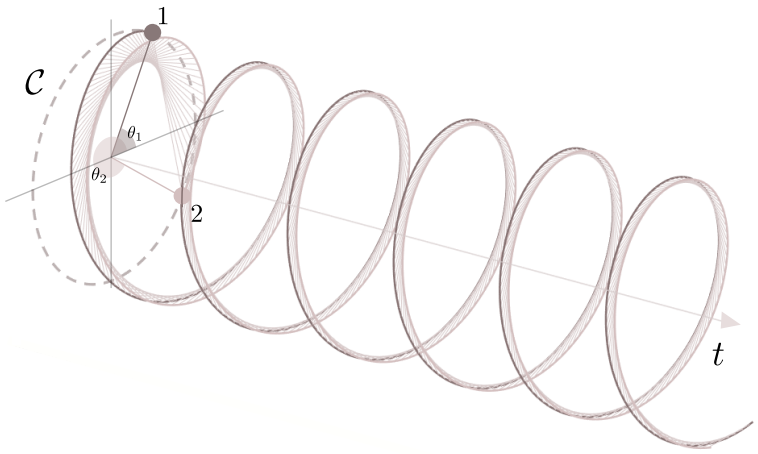


Figure 4.6: Example of phase locking in a system of two interacting oscillators with different natural frequencies.

In a swarm of thousands of fireflies, each insect will surely not be influenced by all the others, but only by the ones nearby. In other examples, the "all-to-all" assumption is even more harmful to the modeling purpose. Think of a country's power grid, where the power plants are not directly connected to one another, but are instead structured in a network fashion. The first generalization, therefore, makes it possible for the model to describe **network-like** interactions.

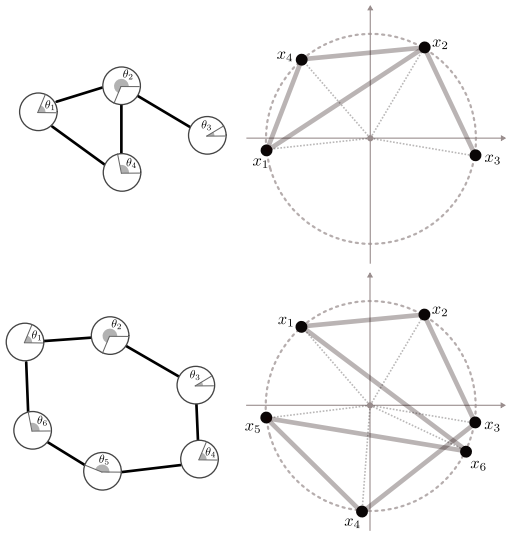


Figure 4.7: Examples of Kuramoto systems with network interactions.

Let G be an undirected graph with n nodes, no self-loops, and adjacency matrix A. Each node is an oscillator and each edge is a pairwise interaction between oscillators.

$$\dot{\theta}_i = \omega_i - \sigma \sum_{j=1}^{n} A_{ij} \sin(\theta_i - \theta_j). \tag{4.3}$$

Having network interactions lets us describe a much bigger class of systems, while also raising interesting questions regarding the relation between the synchronization properties of the system and its network topology. It is immediate to see how the all-to-all model 4.2 is just the network model 4.3 on the complete graph.

Notice that, when the natural frequencies coincide for all oscillators, $\omega_i = \omega \ \forall i$, it is

possible to perform a change of variable $\gamma_i = \theta_i - \omega t$ which results in 4.3 becoming

$$\dot{\gamma}_i = -\sigma \sum_{j=1}^{n} A_{ij} \sin(\gamma_i - \gamma_j). \tag{4.4}$$

This means that, if we put ourselves in a reference frame which rotates at frequency ω, we can describe the system as if the natural frequencies were 0. When the natural frequencies are not the same and the graph is connected, it is also common to change to a frame moving at the average frequency $\langle\omega\rangle = \frac{1}{n}\sum_i \omega_i$ i.e. $\gamma_i = \theta_i - \langle\omega\rangle t$, which leads to

$$\dot{\gamma}_i = (\omega_i - \langle\omega\rangle) - \sigma \sum_{j=1}^{n} A_{ij} \sin(\gamma_i - \gamma_j).$$

This change of variable, which for now has little meaning, is actually rooted in topological reasons and will naturally come up in Section 5.2.

A further natural extension of model 4.3 is to substitute the global coupling σ with a different strength for each pairwise interaction. On the line of the considerations which motivated the introduction of network interactions, this extension is useful for considering systems of oscillators in which the interaction depends on *spatial* factors. It is more realistic to say that a firefly will be more strongly influenced by its closer neighbors than by those further away. We do this by simply considering G as a *positively* weighted graph with adjacency matrix $A_{ij} = A_{ji} \geq 0$.

4.1.4 Polyadic network interactions

Recent years have seen a new wave of research in the thriving field of complex systems. Its origin can be traced to the realization that many complex phenomena cannot be captured, as it was always done, by network models [5]. Describing a system through the *pairwise* interactions between its agents is a powerful tool which dominated the scene because of its simplicity and its roots in graph theory, which goes as far as the 18th century. It should be understood, however, that the choice to consider pairwise interaction is indeed a choice and not a universal principle in nature. Interactions among multiple agents at once, also called *higher-order interactions*, are present in many cases, and they do, in fact, have an impact in the emergence of complex phenomena: for example, in social systems group interactions are crucial to cooperation and decision-making; in neuronal networks, interneurons mediate interactions between multiple other neurons; indirect competition for resources generates interactions among multiple species in ecological systems. It is natural therefore, at least for the sake of curiosity, to bring higher order interactions into the Kuramoto model as well. The reader who would like to learn more about higher order systems is referred to the "manifesto" [5] and the book [4].

A natural way to do this is to extend the underlying structure of interaction to something more general than a graph. We can imagine oscillators as nodes which are connected by edges (pairwise interactions), triangles and higher order structures. This construction

surely makes simplicial complexes come to mind but, we can see, is actually more general as it does not need to satisfy the inclusion relation of Definition 3.1.1. We can, in fact, have a group interaction between three oscillators (a, b, c), without having the "included" pairwise interactions (a, b), (b, c) or (a, c).

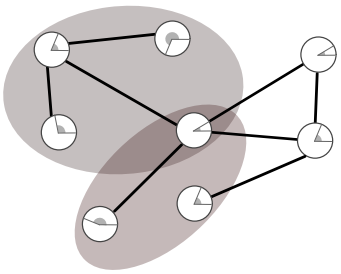


Figure 4.8: Graphical representation of the interactions of a higher-order Kuramoto system.

The mathematical object which describes best this situation is called a **hypergraph**, whose structure is encoded in the adjacency tensors $A^{(k)}$, $k \in \mathbb{N}$, such that $A^{(k)}_{i_1 \dots i_{k+1}} = 1$ if and only if there is a $(k+1)$th order interaction among oscillators $i_1, \dots, i_{k+1}$. A possible higher-order Kuramoto model on a hypergraph structure of order k is, as proposed in different forms in [24], [13] and [25],

$$\begin{aligned}\dot{\theta}_i = \omega_i &- \sigma_1 \sum_{j_1=1}^{n} A^{(1)}_{ij_1} \sin(\theta_i - \theta_{j_1}) \\ &- \sigma_2 \sum_{j_1,j_2=1}^{n} A^{(2)}_{ij_1j_2} \sin(2\theta_i - \theta_{j_1} - \theta_{j_2}) - \dots \\ &- \sigma_k \sum_{j_1,\dots,j_k=1}^{n} A^{(k)}_{ij_1\dots j_k} \sin(k\theta_i - \theta_{j_1} - \dots - \theta_{j_k}) \end{aligned} \tag{4.5}$$

Introducing higher-order interactions into the Kuramoto model makes it possible for interesting behaviors to appear, from chimera states [13] to chaotic dynamics [7]. Being this model very general, it is quite hard to study it analytically except for very simple cases.

In this work, we will abandon this model and consider an alternative way of introducing higher-order interactions, based on simplicial complexes.

4.1.5 Simplicial interactions

The main references for the simplicial Kuramoto model are [18] and [1].

The fundamental step which naturally leads to the simplicial formulation comes from a particular rewriting of the Kuramoto model on an unweighted graph 4.3.

Let $G = (\mathcal{V}, \mathcal{E})$ be an undirected graph, where $\mathcal{V}$ is the set of nodes and $\mathcal{E}$ its edges. Let us give an orientation[1] to each edge, coded in the action of two functions $\boldsymbol{h}, \boldsymbol{t} : \mathcal{E} \to \mathcal{V}$ which, respectively, return the head and tail node of a given edge. Let $B \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ be the node-link incidence matrix associated to the graph, i.e.

$$B_{i\epsilon} = \begin{cases} 1 \text{ if } i = \boldsymbol{h}(\epsilon) \\ -1 \text{ if } i = \boldsymbol{t}(\epsilon) \\ 0 \text{ otherwise} \end{cases} \qquad i \in \mathcal{V}, \epsilon \in \mathcal{E}.$$

[1] "Orientation" is different from *direction* as it does not stop the flow of information.

Then it holds that

$$\sum_{k=1}^{n} A_{ik} \sin(\theta_i - \theta_k) = B \sin(B^\top \theta).$$

Let us prove this fact. First, compute the action of $B^\top$ on the phases vector

$$(B^\top \theta)_\epsilon = \sum_{i \in \mathcal{V}} B^\top_{\epsilon i} \theta_i = \sum_{i \in \mathcal{V}} B_{i\epsilon} \theta_i = \theta_{\boldsymbol{h}(\epsilon)} - \theta_{\boldsymbol{t}(\epsilon)}$$

for every $\epsilon \in \mathcal{E}$. It follows that, for $i \in \mathcal{V}$,

$$\begin{aligned}
\Big[B \sin(B^\top \theta)\Big]_i &= \sum_{\epsilon \in \mathcal{E}} B_{ij} \sin(B^\top \theta)_\epsilon = \sum_{\epsilon \in \mathcal{E}} B_{i\epsilon} \sin(\theta_{\boldsymbol{h}(\epsilon)} - \theta_{\boldsymbol{t}(\epsilon)}) \\
&= \sum_{\epsilon : \boldsymbol{h}(\epsilon) = i} \sin(\theta_i - \theta_{\boldsymbol{t}(\epsilon)}) - \sum_{\epsilon : \boldsymbol{t}(\epsilon) = i} \sin(\theta_{\boldsymbol{h}(\epsilon)} - \theta_i) \\
&= \sum_{\epsilon : \boldsymbol{h}(\epsilon) = i} \sin(\theta_i - \theta_{\boldsymbol{t}(\epsilon)}) + \sum_{\epsilon : \boldsymbol{t}(\epsilon) = i} \sin(\theta_i - \theta_{\boldsymbol{h}(\epsilon)}) \\
&= \sum_{k \in \mathcal{N}(i)} \sin(\theta_i - \theta_k) = \sum_{k=1}^{n} A_{ik} \sin(\theta_i - \theta_k).
\end{aligned}$$

We then find that the Kuramoto dynamics 4.3 can be elegantly written in matrix form as

$$\dot{\theta} = \omega - \sigma B \sin(B^\top \theta). \tag{4.6}$$

Notice that this works independently of the orientation chosen for the edges.

We are now ready to make the mathematical leap necessary for our final generalization of Kuramoto's model. The trick is just to consider θ as a 0-cochain on the 0-simplices of the simplicial complex given by the oriented graph. With this in mind, we can finally take advantage of the theory developed in Chapter 3 to give new meaning to the terms in Equation 4.6. We immediately notice that $B^\top = D^0$ and B^1 are, respectively, the matrices representing the operators d^0 and ∂^1 defined in Section 3.1.2. Equation 4.6 is then written as

$$\dot{\theta} = \omega - \sigma B^1 \sin(D^0 \theta), \tag{4.7}$$

where ω too represents a 0-cochain. This last formulation is actually more powerful than 4.6 as it is capable of describing weighted interactions. This is easily done by defining a diagonal inner product on the 1-chain space $W^{(1)}$ such that $W^{(1)}_{ii}$ is the weight of the interaction mediated by edge i, and remembering that, according to 3.3, it induces a modification of the boundary (actually adjoint coboundary 3.1.10) operator $B^1 = B_1^\top W_{(1)}$.

Equation 4.7 moreover has components which possess indexes. In such a situation, any respectable mathematician must then ask him or herself what happens when these indexes are varied. This simple curiosity leads us to a first formulation of a simplicial version of the Kuramoto model:

$$\dot{\theta} = \omega - \sigma B^{k+1} \sin(D^k \theta). \tag{4.8}$$

In a general oriented simplicial complex $\mathcal{X}$ (see Section 3.1.1), Equation 4.8 describes the evolution of the phases of oscillators which are general k-simplices interacting through $(k+1)$-simplices. If 4.7 describes oscillators which interact in a pairwise fashion, 4.8 then describes oscillators which are involved in $(k+2)$th order interactions, as every $(k+1)$-simplex has $k+2$ faces. Accordingly, both the phase vector θ and the natural frequencies ω, represent kth order cochains in $C^k(\mathcal{X};\mathbb{R})$. In the left panel of Figure 4.9, for example, the oscillators are the edges, and they interact in triplets through the triangle faces. We effectively introduced higher order interactions in the Kuramoto model.

One further observation is necessary. kth order simplices can be connected to each other in two different ways (Definition 3.1.3): from "above", when they are faces of the same $(k+1)$-simplex, and from "below", when they have a common $(k-1)$th order face. It is easy to see that Equation 4.8 describes only the first kind of interaction, as both D^k and B^{k+1} depend only on the k and $(k+1)$-skeletons of the complex. This makes sense as it was obtained by directly extending the equation describing node oscillators, which are only connected from "above" by edges. When working with higher order simplices it is then worth considering this new other kind of interaction. We do this as it was proposed in [18] for unweighted complexes and in [1] for weighted complexes, and define the **simplicial Kuramoto model** of order k

$$\dot{\theta} = \omega - \sigma^d D^{k-1} \sin(B^k \theta) - \sigma^u B^{k+1} \sin(D^k \theta), \tag{4.9}$$

where each interaction term is regulated by a different coupling strength σ^d, σ^u[2]. One interesting thing to notice is that interactions from "above" always involve $k+2$ oscillators, as a $k+1$ simplex will always have $k+2$ faces. Interactions from "below", do not have this constraint because there can be an arbitrary number of k-simplices which share the same face.

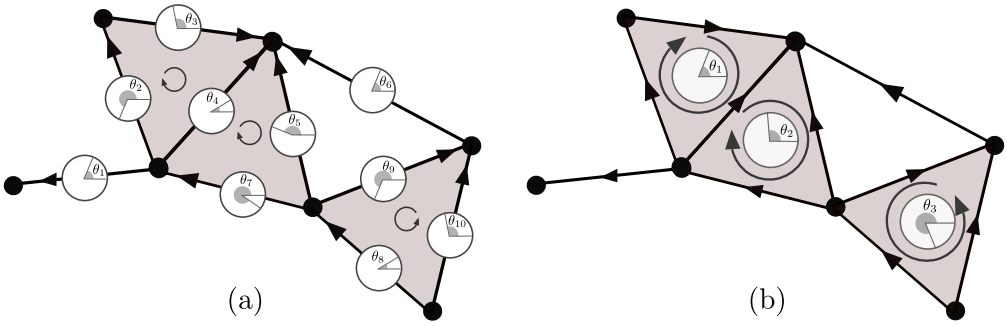


Figure 4.9: The same simplicial complex describes the interactions between 1st order oscillators (a) and 2nd order oscillators (b).

Returning again to the left panel of Figure 4.9, Equation 4.9 prescribes that the edge oscillators interact both in pairs through nodes and in triplets through triangles.

[2] d stands for "down" and u for "up".

It must be noted that the "rigid" structure of simplicial complex, which may seem like a great modeling weakness, is actually its greatest strength. The algebraic properties of boundary, coboundary and Laplacian operators give us powerful tools to investigate and understand the dynamics from a topological point of view. Moreover, as it is discussed in [2], the freedom in choosing weights makes simplicial complexes just as expressive as hypergraphs.

4.1.6 Synchronization and order parameter

Until now, we relied on an intuitive notion of "synchronization", without a precise mathematical definition. We will see that asking for the phases to be all equal is not a satisfying definition in the general simplicial case. In this section, we take care of this problem and discuss a notion of *simplicial* synchronization with firm topological roots.

Let us consider a population of node oscillators regulated by the graph Kuramoto model of Equation 4.6 on a *connected* graph. As intuition suggests, we say that the network of oscillators is **synchronized** when the phases are all equal $\theta \propto \mathbb{1}$. This does not mean that the system must be static, it just means that the phases change together with time, i.e. the phases vector does not leave the vector space spanned by $\mathbb{1}$. The key ingredient for understanding this comes from noticing that, because of connectedness, the graph Laplacian $L = BB^\top$ has eigenvalue with multiplicity 1 and its kernel is spanned by the vectors of $\mathbb{1}$. More precisely, this means that

$$ker(L) = ker(BB^\top) = ker(B^\top) = span\{\mathbb{1}\}.$$

A synchronized phases vector θ then is such that the interaction term $B\sin(B^\top\theta)$ vanishes and, in this case, this intuitive notion is equivalent to harmonicity.

With this in mind, it makes sense to define synchronization for the simplicial case in the same way.

Definition 4.1.1 (Complete synchronization)**.** *We say that a phase cochain $\theta \in C^k(\mathcal{X};\mathbb{R})$ on a simplicial complex $\mathcal{X}$ is **completely synchronized** w.r.t the simplicial Kuramoto model 4.9 if θ is harmonic i.e. $\theta \in ker(L^k)$.*

Just like the node case, harmonicity implies the vanishing of the interaction term. Disregarding the coupling strengths σ^u, σ^d,

$$D^{k-1}\sin(B^k\theta) + B^{k+1}\sin(D^k\theta) = 0$$

because

$$\theta \in ker(L^k) = ker(B^{k+1}D^k + D^{k-1}B^k) = ker(D^k) \cap ker(B^k).$$

Interestingly, with this definition, a configuration in which the phases of the simplicial oscillators are all equal is generally not synchronized. In fact, it often holds that $\mathbb{1} \notin \ker(L^k)$.

Having understood what it means for a configuration to be completely synchronized, we now move on to quantifying synchronization. In his original work in 1975 [14], Kuramoto introduced the **order parameter**, a scalar function of the phases, whose value quantifies the amount of synchronization of the system. Let us come back to the model with all-to-all interaction 4.2, the order parameter is defined as

$$R(\theta) = \|\bar{x}\| = \frac{1}{n}\left\|\sum_{i=1}^{n} x_i(\theta_i)\right\|_2 \geq 0,$$

i.e. the distance of the oscillators' centroid $\bar{x}$ from the origin. If the oscillators' phases have similar values then $\bar{x}$ will be necessarily close to the circle and when synchronization happens $\theta \propto \mathbb{1}\ x_i = x\ \forall i$ R reaches its maximum

$$R(\theta) = \frac{1}{n}\|nx\|_2 = \|x\| = \sqrt{\cos(\theta)^2 + \sin(\theta)^2} = 1.$$

Interestingly, the square order parameter acts as a sort of "potential" for the interaction

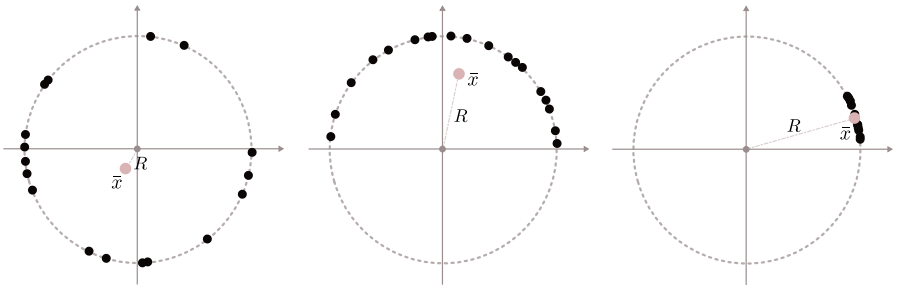


Figure 4.10: Representation of the order parameter as the distance of the oscillators' centroid from the origin.

term of the Kuramoto model. We see this by noticing that

$$\begin{aligned}
\frac{\partial}{\partial\theta_i}R^2(\theta) &= \frac{1}{n^2}\frac{\partial}{\partial\theta_i}\left[\left(\sum_{j=1}^{n}\cos(\theta_j)\right)^2 + \left(\sum_{j=1}^{n}\sin(\theta_j)\right)^2\right] \\
&= \frac{2}{n^2}\left[-\sin(\theta_i)\left(\sum_{j=1}^{n}\cos(\theta_j)\right) + \cos(\theta_i)\left(\sum_{j=1}^{n}\sin(\theta_j)\right)\right] \\
&= \frac{2}{n^2}\sum_{j\neq i}(\sin(\theta_j)\cos(\theta_i) - \cos(\theta_j)\sin(\theta_i)) = \frac{2}{n^2}\sum_{j\neq i}\sin(\theta_j - \theta_i),
\end{aligned}$$

meaning that we can rewrite 4.2 as

$$\dot{\theta} = \omega + \frac{n^2\sigma}{2}\nabla_\theta R^2(\theta) = \nabla_\theta\left(\omega^\top\theta + \frac{n^2\sigma}{2}R^2(\theta)\right).$$

Dynamics of this type are called **gradient systems**.

The order parameter for the graph Kuramoto model 4.6 is built in such a way as to be the potential of the interaction function $B\sin(B^\top\theta)$. Define

$$R^2(\theta) \triangleq \frac{1}{|\mathcal{E}|}\mathbb{1}^\top\cos(B^\top\theta). \tag{4.10}$$

First, if θ is harmonic $\theta \propto \mathbb{1}$, then $R^2(\theta) = 1$ meaning that this parameter actually measures synchronization in the sense of Definition 4.1.1. Second, it is easy to show that

$$\nabla_\theta R^2(\theta) = \frac{1}{|\mathcal{E}|} = -\frac{1}{|\mathcal{E}|}B^\top\sin(B^\top\theta)$$

and thus

$$\dot{\theta} = \omega - \sigma B\sin(B^\top\theta) = \nabla_\theta\left(\omega^\top\theta + \sigma\,|\mathcal{E}|\,R^2(\theta)\right).$$

If we modify Equation 4.10 in order to take into account interactions from "below" we can find an order parameter which is able to measure synchronization under the simplicial Kuramoto dynamics. We then define the **simplicial order parameter** (SOP), first proposed in [18].

Definition 4.1.2 (Simplicial order parameter)**.** *Define the (square) simplicial order parameter of order k as*

$$R_k^2(\theta) = \frac{1}{C_k}\left(w_{(k-1)}^\top\cos(B^k\theta) + w_{(k+1)}^\top\cos(D^k\theta)\right), \tag{4.11}$$

where $C_k = \mathbb{1}^\top w_{(k-1)} + \mathbb{1}^\top w_{(k+1)}$ is a normalization constant and $w_{(k)} = W_{(k)}\mathbb{1}$ is the inverse volume cochain of order k.

If θ is harmonic $\theta \in \ker(L^k)$, then $R_k^2(\theta) = 1$ as $\ker(L^k) = \ker(D^k)\cap\ker(B^k)$. Moreover, if the complex is unweighted $W^{(j)} = I\ \forall j$, then the simplicial Kuramoto model with $\sigma^u = \sigma^d = \sigma$ is a gradient system

$$\dot{\theta} = \nabla_\theta U_k(\theta) = \nabla_\theta\left(\omega^\top\theta + \sigma C_k R_k^2(\theta)\right). \tag{4.12}$$

What Equation 4.12 tells us is that, much like the well-known algorithm of gradient descent, the simplicial Kuramoto dynamics will always follow the direction of steepest ascent in the landscape of $U_k(\theta)$. This means that every local maximum of $U_k(\theta)$ will be a stable equilibrium point which, if reached by the dynamics, will correspond to a configuration of phases static in time.

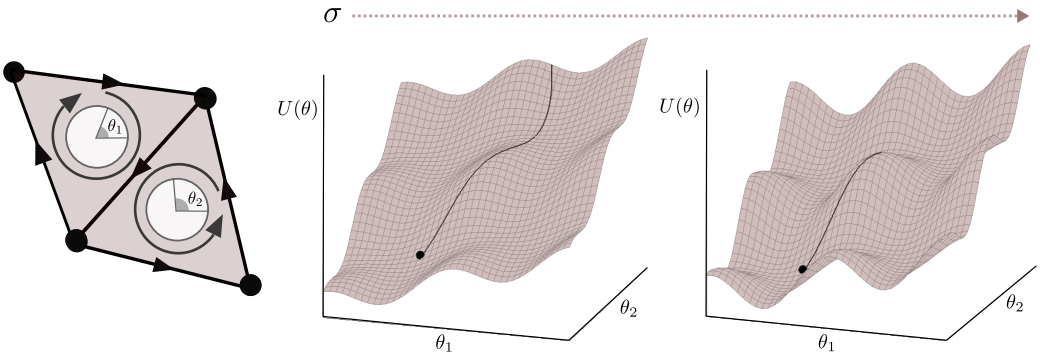


Figure 4.11: The function $U_2(\theta)$ is plotted for the 2nd order simplicial Kuramoto on the complex on the left for $\omega = (1,2)^\top$, together with a trajectory of the dynamics (dark line). The two plots were made for two different values of σ. When σ is high enough (plot on the right) local maxima appear in the landscape and the gradient system stops at a stable phase configuration.

5 | Analysis of the simplicial Kuramoto model

This last chapter is entirely devoted to the analysis of the simplicial Kuramoto model introduced and motivated in Chapter 4. We will take a bottom-up approach in trying to explore the behavior of this beautiful model, deriving all the results without omitting the proofs.

Our main focus will be exploring the equilibrium properties of this system. Are there configurations of phases which do not change in time? How are they made? Under what conditions do they exist? These are some of the questions which have been extensively asked about the Kuramoto model and which we will now ask for its simplicial variant.

5.1 The simplicial oscillator

In this first section, we take a look at one of the most peculiar properties of the simplicial Kuramoto model, which has to do with the fundamental behavior of its oscillators. Recall that in Section 4.1.5, the introduction of the model was motivated purely by formal (4.8) and symmetry (4.9) arguments and was not obtained in a bottom-up fashion like the original model 4.2. Here, therefore, we take a step back and look at the behavior of a single simplicial oscillator under the simplicial Kuramoto dynamics.

5.1.1 Interactions and self-interactions

Let us consider an unweighted simplicial complex $\mathcal{X}$ made by a single kth order simplex together with its faces, the faces of its faces and so on. Let also $\omega \in \mathbb{R}$ be its natural frequency. As there are no $(k+1)$-simplices, the "above" part of the dynamics 4.9 will not be present, thus leaving us with the scalar equation

$$\dot{\theta} = \omega - \sigma^d D^{k-1} \sin(B^k \theta), \tag{5.1}$$

where B^k will be a $(k+1) \times 1$ matrix whose elements are either 1 or -1 depending on the orientation of the $(k-1)$-faces w.r.t to the simplex. We can write $B^k = B_k = \xi$

with $\xi \in \{-1,1\}^{k+1}$. ξ_i will be the relative orientation of face i w.r.t the simplex. Since the complex is unweighted, 3.3 tells us that $D^{k-1} = B_k^\top$. Equation 5.1 becomes

$$\dot{\theta} = \omega - \sigma^d \xi^\top \sin(\xi\theta) = \omega - \sigma^d \xi^\top \xi \odot \sin(\theta \mathbb{1}) = \omega - \sigma^d \mathbb{1}^\top \sin(\theta \mathbb{1}),$$

as the sine is an odd function. Expanding the scalar product, we finally get the scalar equation

$$\dot{\theta} = \omega - \sigma(k+1)\sin(\theta), \tag{5.2}$$

where we write for simplicity $\sigma = \sigma^d$. This simple result clearly tells us a surprising and quite disturbing fact, i.e. that

simplicial oscillators are not Kuramoto oscillators

in the sense of Section 4.1.1 as they do not follow Equation 4.1.

The reason for this lies in the fact that simplicial oscillators can never actually be free of interactions. A simplicial complex is *by definition* (3.1.1) closed under the inclusion of faces, meaning that a single kth order oscillator will always have $k+1$ faces. The interaction term from "below" will never disappear and will result in the extra term of Equation 5.2. For these reasons, we name this extra term **self-interaction**.

The first thing that we notice is that the self-interaction term $-\sigma(k+1)\sin(\theta)$ is proportional both to the order (plus one) and to the coupling strength. This suggests that it comes from a sum of contributions for all the faces, each with strength σ. To understand the origin of this phenomenon, it is worth looking at the case of two k-simplices adjacent from below. If $k = 1$, for example, we are talking about two edges connected by a node. We can describe such a system with $B^1 = B_1 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$, and if $k = 2$ $B^2 = B_2 = \begin{pmatrix} -1 & 0 \\ -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$. By direct generalization, we see that a system of two adjacent oscillating k-simplices can be described with the boundary matrix

$$B^k = B_k = \begin{pmatrix} \xi_1 & \underline{0} \\ \nu_1 & \nu_2 \\ \underline{0} & \xi_2 \end{pmatrix},$$

where $\xi_i \in \{-1,1\}^k$ ($i \in \{1,2\}$) contains the orientations of the k faces of simplex i which are not faces of the other and ν_i is the orientation of the common face w.r.t the simplex i. By carrying out the simple computations, we find that the interaction term from "below" is

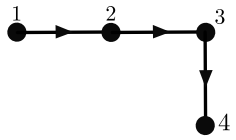
$$B_k^\top \sin(B_k\theta) = \begin{pmatrix} \xi_1^\top \sin(\xi_1\theta_1) + \nu_1 \sin(\nu_1\theta_1 + \nu_2\theta_2) \\ \nu_2 \sin(\nu_1\theta_1 + \nu_2\theta_2) + \xi_2^\top \sin(\xi_2\theta_2) \end{pmatrix} = \begin{pmatrix} k\sin(\theta_1) + \sin(\theta_1 + \nu_1\nu_2\theta_2) \\ k\sin(\theta_2) + \sin(\nu_1\nu_2\theta_1 + \theta_2) \end{pmatrix}.$$

The simplicial Kuramoto dynamics regulating the system will then be

$$\dot{\theta} = \omega - \sigma k \sin(\theta) - \sigma \begin{pmatrix} \sin(\theta_1 + \nu_1\nu_2\theta_2) \\ \sin(\nu_1\nu_2\theta_1 + \theta_2) \end{pmatrix}.$$

Notice how the self-interaction term is still present but, this time, is multiplied by k instead of $k+1$. The "missing" unit has become an interaction term which depends on the difference (or sum, depending on the relative orientation $\nu_1\nu_2$) of the phases of the two simplices. It is reasonable to deduce that a simplex interacting with two others will have a self-interaction term of strength $k-1$ and so on. Consider for example 3 oscillating edges in sequence with boundary matrix:

$$B_1 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$



We get

$$B_1^\top \sin(B_1\theta) = \begin{pmatrix} \sin(\theta_1) & +\sin(\theta_1 - \theta_2) & \\ & \sin(\theta_2 - \theta_3) & +\sin(\theta_2 - \theta_1) \\ \underbrace{\sin(\theta_3)}_{\text{Self-interaction}} & \underbrace{\qquad\qquad}_{\text{Head interaction}} & \underbrace{+\sin(\theta_3 - \theta_2)}_{\text{Tail interaction}} \end{pmatrix}$$

where we see that the middle simplex experiences no self-interaction term.

We are now ready to give an interpretation of this phenomenon and, as a result, of the behavior of the whole simplicial Kuramoto model. The interaction term from "below" $B_k^\top \sin(B_k\theta)$ means that each oscillating simplex "sends a message" through each of its faces. If there is a simplex of the same order on the other side an interaction bond is formed, otherwise the signal "bounces" back and causes a self-interaction. Figure 5.1 gives a graphical explanation of this fact.

This of course does not happen for nodes, as there are no (-1)-simplices.

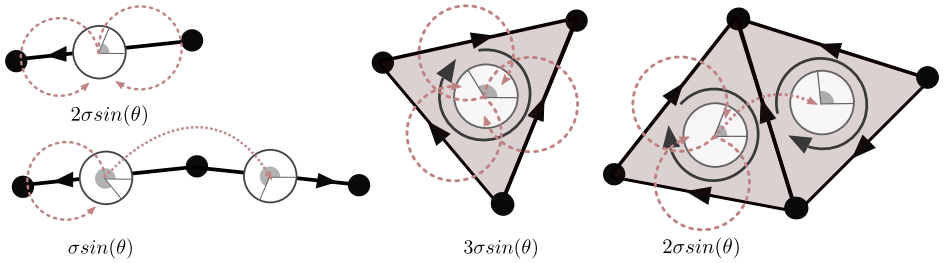


Figure 5.1: Self-interactions and interactions in the case of the edge and triangle dynamics.

5.1.2 Equilibrium properties of the simplicial oscillator

Having now given an interpretation of the self-interaction term, let us come back to the single oscillating k-simplex of Equation 5.2. Let us fix k, ω and a starting phase, and integrate the ODE for different values of σ, in order to get a sense of its behavior. The results, shown in Figure 5.2, suggest us at least three observations:

1. the oscillations, which of course correspond to Kuramoto oscillations for $\sigma = 0$, are increasingly deformed as σ grows;
2. the oscillations' period increases with σ;
3. after a certain value of σ, which we call σ_*, the oscillator is no longer *self-sustaining* and quickly stabilizes into an equilibrium position.

When σ passes σ_* the systems experiences a *phase transition*, from a self-sustaining oscillator to a dissipative system which stops moving after a few instants.

Finding the particular value of σ, which we call **critical coupling**, is actually an easy task. An equilibrium configuration of Equation 5.2 will have to satisfy

$$\sigma(k+1)\sin(\theta) = \omega \iff \sin(\theta) = \frac{\omega}{\sigma(k+1)},$$

which, given the bounded image of the sine function, will have solutions if and only if

$$\left|\frac{\omega}{\sigma(k+1)}\right| \leq 1.$$

It thus trivially follows that:

Theorem 5.1.1. *The simplicial oscillator admits equilibria if and only if*

$$\left|\frac{\omega}{\sigma(k+1)}\right| \leq 1.$$

If ω and k are fixed, we can easily turn this into a condition on σ and find

$$\sigma \geq \sigma_* \triangleq \frac{|\omega|}{k+1}. \tag{5.3}$$

Moreover, if 5.3 holds, we can even find an explicit expression of all the equilibrium configurations

$$\theta = (-1)^s \arcsin\left(\frac{\omega}{\sigma(k+1)}\right) + s\pi + 2m\pi \quad s \in \{0,1\}, \quad m \in \mathbb{Z}.$$

Modulo periodicity, then the equilibria are just two:

$$\theta_s = \arcsin\left(\frac{\omega}{\sigma(k+1)}\right), \quad \theta_u = -\arcsin\left(\frac{\omega}{\sigma(k+1)}\right) + \pi.$$

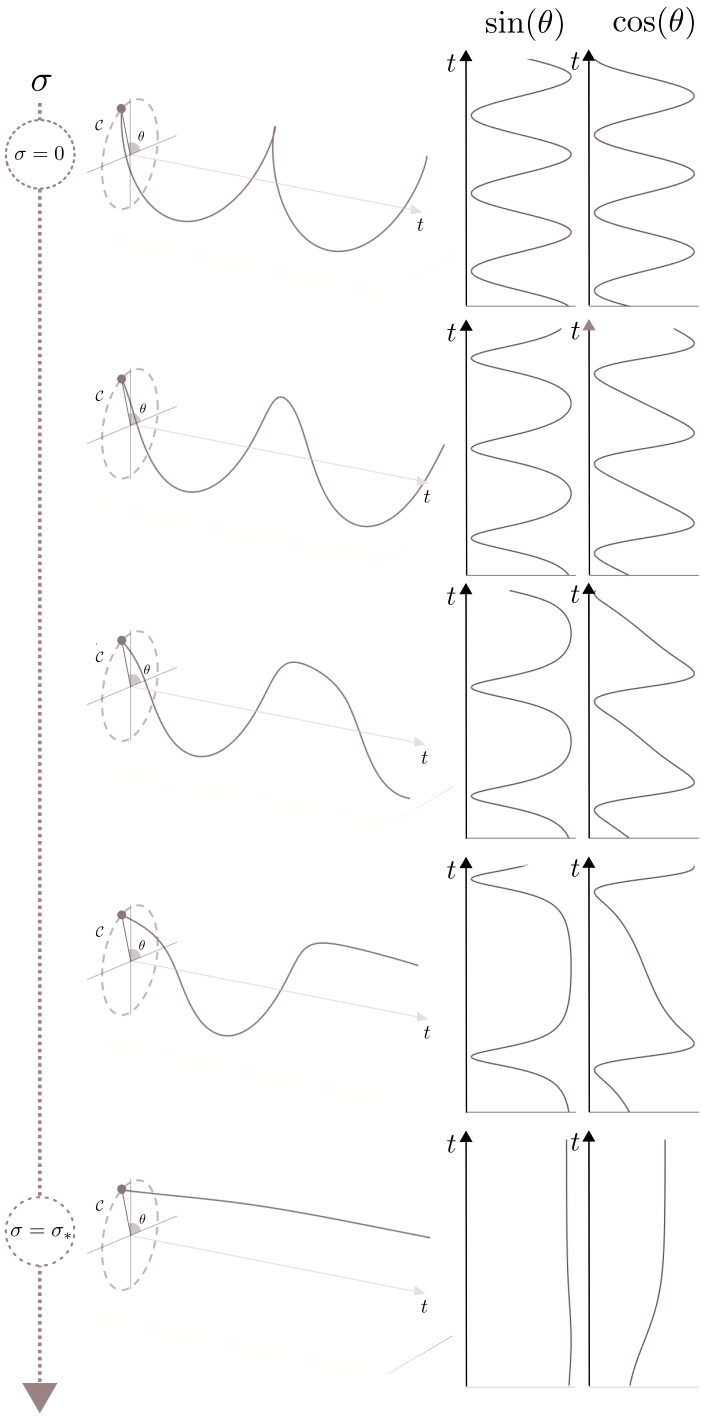


Figure 5.2: Dynamics of a kth order simplicial oscillator for different values of σ.

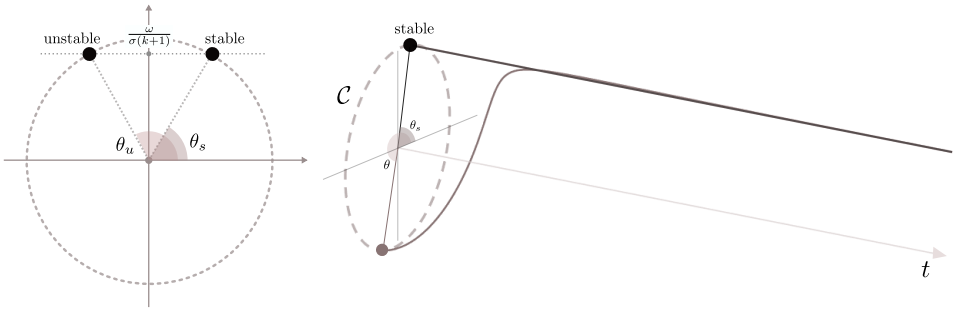


Figure 5.3: On the left, the two equilibria of a single oscillating simplex are shown. On the right, a simulation of the model shows convergence to θ_s.

Their stability can be computed through linear stability analysis by looking at the sign of the Jacobian of 5.2.

$$J(\theta) = \frac{\partial}{\partial\theta}\dot{\theta} = -\sigma(k+1)\cos(\theta).$$

If $\sigma > \sigma_*$

$$J(\theta_s) = -\sigma(k+1)\cos\left(\arcsin\left(\frac{\omega}{\sigma(k+1)}\right)\right) = -\sigma(k+1)\sqrt{1-\left(\frac{\omega}{\sigma(k+1)}\right)^2} < 0$$

and thus θ_s is asymptotically stable. Instead,

$$J(\theta_u) = -\sigma(k+1)\cos\left(-\arcsin\left(\frac{\omega}{\sigma(k+1)}\right)+\pi\right) = \sigma(k+1)\sqrt{1-\left(\frac{\omega}{\sigma(k+1)}\right)^2} > 0$$

meaning that θ_u is unstable. The two equilibrium configurations are symmetric w.r.t the y-axis, as shown in Figure 5.3.

Let us give a more geometrical interpretation of this phase transition. Remember how in Section 4.1.6 we defined the order parameter and rewrote the unweighted simplicial Kuramoto as a gradient system (Equation 4.12). If we do the same for a single simplicial oscillator, we can see that the dynamics 5.2 becomes

$$\dot{\theta} = \frac{\partial}{\partial\theta}U_k(\theta) = \frac{\partial}{\partial\theta}\left(\omega\theta + \sigma(k+1)\cos(\theta)\right).$$

It is then a gradient system whose potential landscape is given as the sum of two contributions:

- a line with slope ω;

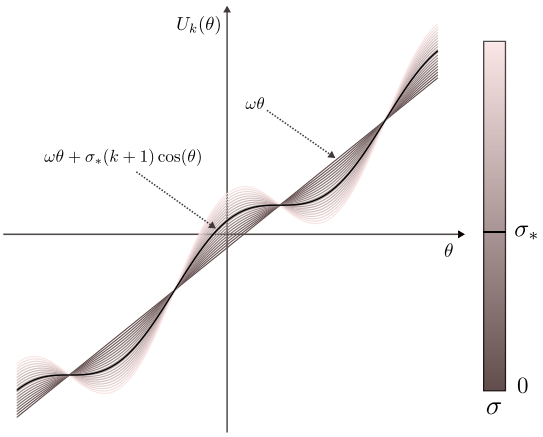


Figure 5.4: Plot of the potential $U_k(\theta)$ for different values of σ. The black line shows $U_k(\theta)$ when $\sigma = \sigma_*$.

- a cosine wave with amplitude $\sigma(k+1)$.

When $\sigma \geq \sigma_*$ the potential will have local maxima (corresponding to $\theta_s + 2m\pi, m \in \mathbb{Z}$) and thus the system will reach one of them and stop there (unless it starts in a local minimum corresponding to an unstable equilibrium $\theta_u + 2m\pi, m \in \mathbb{Z}$).

5.2 Equilibria of the simplicial Kuramoto model

Having gained a valuable insight on the way the simplicial Kuramoto model works in the most simple case, we can now go on studying more complex situations. The main question we address in this section has to do with the equilibrium properties of the model. Are there configurations of phases which are left unchanged by the dynamics? If so, how are they made? On which factors does their existence depend?

5.2.1 Decomposing the equilibrium problem

Let us consider a simplicial complex $\mathcal{X}$, weighted or unweighted, which describes the interactions between kth order oscillators. Remember that the equation regulating the dynamics (Equation 4.9) is

$$\dot{\theta} = \omega - \sigma^d D^{k-1} \sin(B^k \theta) - \sigma^u B^{k+1} \sin(D^k \theta). \tag{5.4}$$

A first important observation is that the addition of a harmonic cochain $x \in ker(L^k)$ to the phases has no effect on the dynamics. This comes from the fact that $\ker(L^k) = \ker(B^k) \cap \ker(D^k)$ (Proved in general in Proposition 2.1.4 of Chapter 2). Any change of variable $\gamma = \theta + x$, which corresponds to describing oscillator i in a reference system rotating at speed x_i, will thus leave Equation 5.4 formally unchanged. In this sense we can say that the harmonic space is the *gauge* of the simplicial Kuramoto. Note here the difference with respect to the node Kuramoto on a graph in which the trivial constant phase shift is the gauge transformation that leaves the dynamics unchanged.

To find the equilibrium conditions, a rewriting of the model is necessary. In fact, the dynamics can be better understood and greatly simplified if one resorts to the simplicial Hodge decomposition theorem (Equation 3.6) which, we recall, tells us that

$$C^k(\mathcal{X}; \mathbb{R}) = Im(\partial^{k+1}) \oplus ker(\mathcal{L}^k) \oplus Im(d^{k-1})$$

or, in matrix form,

$$\mathbb{R}^{n_k} = Im(B^{k+1}) \oplus ker(L^k) \oplus Im(D^{k-1}).$$

We use the theorem to decompose both the phases cochain θ and the natural frequencies ω:

$$\theta = \theta_{cf} + \theta_H + \theta_{df}, \ \omega = \omega_{cf} + \omega_H + \omega_{df}$$

(cf means *curl-free* and df means *divergence-free*) where

$$\theta_{cf} = B^{k+1}\phi, \ \theta_{df} = D^{k-1}\psi, \ \omega_{cf} = B^{k+1}\lambda, \ \omega_{df} = D^{k-1}\mu.$$

Equation 5.4 becomes

$$\begin{aligned}
\dot{\theta}_{cf} + \dot{\theta}_H + \dot{\theta}_{df} &= \omega_{cf} + \omega_H + \omega_{df} - \sigma^d D^{k-1} \sin(B^k(\theta_{cf} + \theta_H + \theta_{df})) \\
&\quad - \sigma^u B^{k+1} \sin(D^k(\theta_{cf} + \theta_H + \theta_{df})) \\
&= \omega_{cf} + \omega_H + \omega_{df} - \sigma^d D^{k-1} \sin(B^k \theta_{df}) - \sigma^u B^{k+1} \sin(D^k \theta_{cf}),
\end{aligned}$$

because $B^k B^{k+1} = 0$, $D^k D^{k-1} = 0$ and $B^k \theta_h = D^k \theta_H = 0$. Given the orthogonality[1] of the three components ensured by the theorem, the equation can be decomposed into the system of ODEs

$$\begin{cases} \dot{\theta}_{cf} = \omega_{cf} - \sigma^u B^{k+1} \sin(D^k \theta_{cf}) \\ \dot{\theta}_H = \omega_H \\ \dot{\theta}_{df} = \omega_{df} - \sigma^d D^{k-1} \sin(B^k \theta_{df}) \end{cases} . \tag{5.5}$$

The three equations in 5.5, which are equivalent to 5.4, are of crucial importance. They tell us that under the simplicial Kuramoto dynamics: i) the curl-free, the harmonic and the divergence-free components evolve independently of one another; and, ii) the harmonic component is not affected by the interaction terms (see Figure 5.5). Notice also that the interaction from "above" affects only the curl-free component, while the one from "below" affects only the divergence-free component.

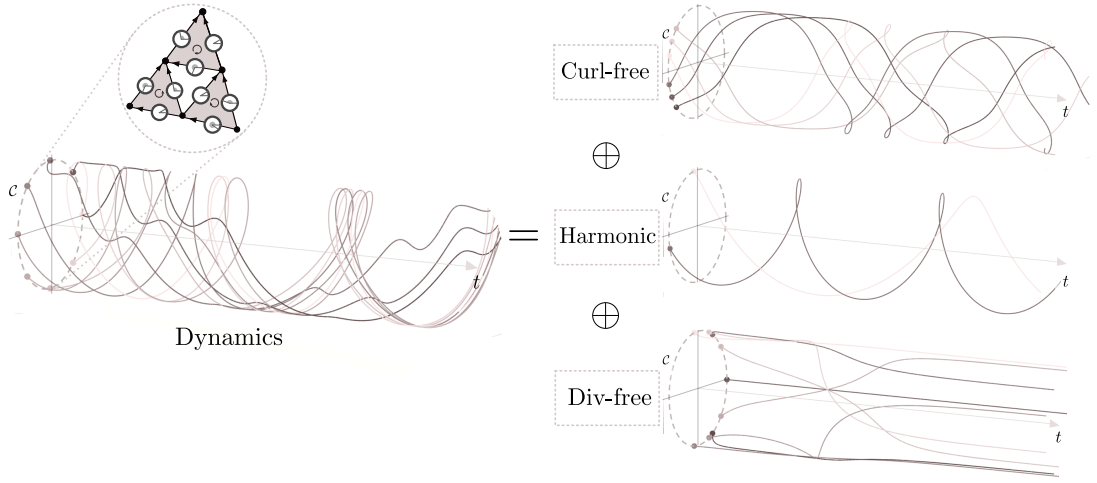


Figure 5.5: Hodge decomposition (Equation 5.5) of the edge simplicial Kuramoto dynamics on a small simplicial complex. The harmonic component divides the edges in two groups, the ones on the boundary of the hole, and the others. The two groups evolve with constant angular speed. Notice also how, in this case, the divergence-free component reaches an equilibrium configuration.

The independence of the harmonic term from the interaction $\dot{\theta}_H = \omega_H$ directly implies that *the harmonic component of the phases acts as a system of isolated Kuramoto oscillators with natural frequencies given by the harmonic component of the natural frequencies.* Moreover, if $\omega_H \neq 0$, there can be no equilibrium of the system as θ_H will always evolve with a fixed angular speed. In looser but more intuitive terms, we could state that

[1] w.r.t the inner product defined on $C^k(\mathcal{X}; \mathbb{R})$.

Observation 1. *the presence of topological k-holes (non-trivial harmonic space) is responsible for the existence of an underlying harmonic dynamics which wraps around them and evolves with constant angular speeds, one for each k-simplex.*

It is therefore convenient to change coordinates and pass to a system rotating at ω_H $\gamma = \theta - \omega_H$, where $\dot{\gamma}_H = 0$ and equilibrium is possible. This leads us to a new system

$$\begin{cases} \dot{\gamma}_{cf} = \omega_{cf} - \sigma^u B^{k+1} \sin(D^k \gamma_{cf}) \\ \dot{\gamma}_H = 0 \\ \dot{\gamma}_{df} = \omega_{df} - \sigma^d D^{k-1} \sin(B^k \gamma_{df}) \end{cases} \tag{5.6}$$

where only the curl-free and divergence-free components evolve. Notice that this is a direct generalization of what it is usually done in the literature when moving to the corotating frame, that is, asking for ω to have 0 mean (see Section 4.1.3). In fact, if the complex is connected and unweighted, $\ker(L^k) = span\{\mathbb{1}\}$ the projection of ω onto the harmonic space is

$$\omega_H = \left(\frac{1}{\sqrt{n}}\mathbb{1}\right)\left(\frac{1}{\sqrt{n}}\mathbb{1}\right)^\top \omega = \frac{1}{n}\mathbb{1}\mathbb{1}^\top\omega = \bar{\omega}\mathbb{1}.$$

Proposition 5.2.1. *Without loss of generality, we can always move to the harmonic co-rotating frame, that is, assume that the natural frequencies cochain has no harmonic component $\omega_H = 0$.*

Under this last assumption, studying the equilibrium properties of the simplicial Kuramoto model is equivalent to studying the equilibria of the curl-free and divergence-free components. If these two converge to an equilibrium configuration, then the complete system will converge to a configuration evolving with constant angular speed, given by ω_H.

We have that $\dot{\theta} = 0$ if and only if

$$\begin{cases} B^{k+1} \sin(D^k \theta_{cf}) = \frac{\omega_{cf}}{\sigma^u} \\ D^{k-1} \sin(B^k \theta_{df}) = \frac{\omega_{df}}{\sigma^d} \end{cases} . \tag{5.7}$$

These two equations need not be satisfied simultaneously: one could perfectly have a curl-free component in equilibrium and an evolving divergence-free component. To solve them, we employ the *Moore-Penrose pseudoinverse*, whose main properties are listed in Appendix B. Since $\omega_{cf} \in Im(B^{k+1})$ and $\omega_{df} \in Im(D^{k-1})$, the linear systems admits solutions which can be written as[2]

$$\begin{cases} \sin(D^k \theta_{cf}) = (B^{k+1})^\dagger \frac{\omega_{cf}}{\sigma^u} + x \\ \sin(B^k \theta_{df}) = (\ D^{k-1})^\dagger \frac{\omega_{df}}{\sigma^d} + y \end{cases} , \tag{5.8}$$

[2]This is done by virtue of Theorem B.0.1 of Appendix B.

for any $x \in ker(B^{k+1})$ and $y \in ker(D^{k-1})$. As they will come up surprisingly often, it is convenient to give names to the vector quantities above.

Definition 5.2.1 (Structure potentials)**.** *Given a simplicial complex $\mathcal{X}$ and a frequency k-cochain $\omega \in \mathbb{R}^{n_k}$, we call **structure potentials** of order k the quantities*

$$\beta^{(+)} \triangleq (B^{k+1})^\dagger \omega_{cf} \in \mathbb{R}^{n_{k+1}}, \ \beta^{(-)} \triangleq (D^{k-1})^\dagger \omega_{df} \in \mathbb{R}^{n_{k-1}}.$$

The adjective "structure" is chosen because they encode the structural properties of the system through the boundary matrix and the natural frequencies.

Proposition 5.2.2. *By the properties of the Moore-Penrose pseudoinverse we have that*

$$\beta^{(+)} \in Im((B^{k+1})^*) = \ker(B^{k+1})^\perp, \ \beta^{(-)} \in Im((D^{k-1})^*) = \ker(D^{k-1})^\perp.$$

We can immediately prove an interesting fact about the structure signals which relates them to the Hodge decomposition of ω and justifies the name "potentials".

Proposition 5.2.3. *$\beta^{(+)} = \lambda, \beta^{(-)} = \mu$, where $\omega_{cf} = B^{k+1}\lambda$ and $\omega_{df} = D^{k-1}\mu$*

Proof.

$$B^{k+1}\beta^{(+)} = B^{k+1}(B^{k+1})^\dagger \omega_{cf}$$

is the orthogonal projection of ω_{cf} onto $Im(B^{k+1})$ (See Property 9 of Appendix B). ω_{cf}, however, belongs to $Im(B^{k+1})$ and thus

$$B^{k+1}\beta^{(+)} = \omega_{cf}$$

hence the thesis. An analogous argument holds for $\beta^{(-)}$. □

Proposition 5.2.3 tells us that the structure signals correspond to the higher and lower order signals which make up the Hodge decomposition's components of the natural frequencies vector ω:

$$\omega = B^{k+1}\beta^{(+)} + \omega_H + D^{k-1}\beta^{(-)}.$$

Let us come back to the equilibrium conditions of Equation 5.8. We see that a *necessary* condition for the existence of a solution is

$$\left\| \frac{\beta^{(+)}}{\sigma^u} + x \right\|_\infty \leq 1, \ \left\| \frac{\beta^{(-)}}{\sigma^d} + y \right\|_\infty \leq 1,$$

because the image of the sine function is the closed interval $[-1,1]$. The kernel vectors x,y associated to equilibrium configurations are thus constrained to lie in a particular set which we now define.

Definition 5.2.2 (Admissible cycles)**.** *We call a cycle $x \in ker(B^{k+1})$* ***admissible*** *if*

$$\left\| \frac{\beta^{(+)}}{\sigma^u} + x \right\|_\infty \leq 1.$$

We call a cocycle $y \in ker(D^{k-1})$ ***admissible*** *if*

$$\left\| \frac{\beta^{(-)}}{\sigma^u} + y \right\|_\infty \leq 1.$$

With a slight abuse of notation, we call them both admissible cycles, and we name their sets $\mathcal{A}^{(+)}$ and $\mathcal{A}^{(-)}$.

The name "admissible cycles" is chosen to highlight an interesting observation.

Observation 2. *At least for the (+) component, the equilibrium properties of the model are related to the existence of cycles in the complex which are, in some sense, compatible with the natural frequencies.*

It is easy to see that $\mathcal{A}^{(\pm)}$ is the intersection of a vector space with an ∞-norm ball centered in $-\beta^{(\pm)}/\sigma$ with radius 1. $\mathcal{A}^{(\pm)}$ can thus be described with a set of linear inequalities, meaning that it is a *closed*, *convex*, *bounded polytope* whose vertices cannot be explicitly found in general.

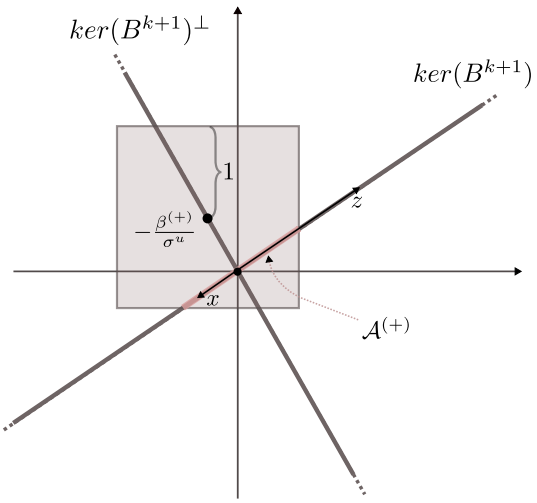


Figure 5.6: x is admissible as it belongs to the intersection of $\ker(B^{k+1})$ and the unit ∞-ball centered in $-\beta^{(+)}/\sigma^u$. z is not admissible.

When x and y are admissible, each component of the left-hand side of 5.8 belongs to $[-1,1]$. We can invert the sine, and find

$$\begin{cases} D^k\theta_{cf} = (-1)^{s_+} \odot \arcsin\left(\frac{\beta^{(+)}}{\sigma^u} + x\right) + s_+\pi + 2\pi m_+ \\ B^k\theta_{df} = (-1)^{s_-} \odot \arcsin\left(\frac{\beta^{(-)}}{\sigma^d} + y\right) + s_-\pi + 2\pi m_- \end{cases}, \tag{5.9}$$

where

$$s_+ \in \{0,1\}^{n_{k+1}},\ s_- \in \{0,1\}^{n_{k-1}},\ m_+ \in \mathbb{Z}^{n_{k+1}},\ m_- \in \mathbb{Z}^{n_{k-1}}.$$

Let us take a moment to recap what we did in the previous steps. We started with the simplicial Kuramoto equation 5.4 and found out that, through Hodge decomposition,

we could find independent equations 5.5 for the evolution of the curl-free, harmonic and divergence-free components. When considering a reference system with $\omega_H = 0$ (Proposition 5.2.1), we can state that the phases will be in equilibrium if and only if the curl-free and divergence-free components are in equilibrium. With a few algebraic steps, we finally found two independent equations 5.9 which give conditions for their equilibrium.

Notice, however, that the Equations 5.9 are not explicit but are cast in the form of linear systems. Matrices D^k and B^k are in general non-square and non-invertible, and they cannot be brought to the right side through the pseudoinverse as we have no guarantees that the system has solutions.

Let us take a closer look at the right-hand sides of the equations.

Definition 5.2.3 (Equilibrium sets)**.** *We define the* ***equilibrium sets*** *of the dynamics as*

$$\mathcal{E}^{(+)} \triangleq \left\{ (-1)^{s_+} \odot \arcsin\left(\frac{\beta^{(+)}}{\sigma^u} + x\right) + s_+\pi + 2\pi m_+ : x \in \mathcal{A}^{(+)}, \right. \\ \left. s_+ \in \{0,1\}^{n_{k+1}}, m_+ \in \mathbb{Z}^{n_{k+1}} \right\}$$

$$\mathcal{E}^{(-)} \triangleq \left\{ (-1)^{s_-} \odot \arcsin\left(\frac{\beta^{(-)}}{\sigma^d} + y\right) + s_-\pi + 2\pi m_- : y \in \mathcal{A}^{(-)}, \right. \\ \left. s_- \in \{0,1\}^{n_{k-1}}, m_- \in \mathbb{Z}^{n_{k-1}} \right\}.$$

The equilibrium sets, whose name will be justified in the next pages, are the sets of all possible right-hand sides of 5.9. A simple fact follows immediately:

Proposition 5.2.4. *If $\mathcal{E}^{(\pm)} = \emptyset$ then the curl-free/divergence-free component admits no equilibria. Moreover,*

$$\mathcal{E}^{(\pm)} = \emptyset \iff \mathcal{A}^{(\pm)} = \emptyset$$

i.e. there are no equilibrium vectors if and only if there are no admissible cycles.

Proof. The first statement comes from the fact that, if $\mathcal{E}^{(\pm)} = \emptyset$, then there are no possible right-hand sides of Equation 5.9, which cannot be solved and, therefore, gives no equilibria.

The second statement comes directly from the definition of equilibrium set 5.2.3. □

Moreover, we can see that there will be equilibria exactly when we can solve the equations in 5.9 i.e. when some vector in the right-hand side belongs to the image of the matrix on the left-hand side.

Proposition 5.2.5. *The curl-free/divergence-free component admits equilibria if and only if, respectively,*

$$\mathcal{E}^{(+)} \cap Im(D^k) \neq \emptyset, \ \mathcal{E}^{(-)} \cap Im(B^k) \neq \emptyset.$$

5.2.2 Simplicial phase-locking and types of equilibrium

Let us now focus on the $(k+1)$-cochain $D^k\theta_{cf}$. First notice that, by Hodge decomposition,

$$\theta^{(+)} \triangleq D^k\theta = D^k(\theta_{cf} + \theta_H + \theta_{df}) = D^k\theta_{cf}.$$

Let us consider the node dynamics of the standard Kuramoto model 4.6 on a simplicial complex $\mathcal{X}$ of order 1 (a graph with oriented edges) and look at the value of $\theta^{(+)}$. $\theta^{(+)}$ is a signal defined on the edges and its components are

$$\theta^{(+)}_{[i,j]} = (D^0\theta)_{[i,j]} = (B_1^\top\theta)_{[i,j]} = \theta_j - \theta_i,$$

when $[i,j] \in \mathcal{X}_1$. $\theta^{(+)}$ then contains the phase difference along each edge. It follows that, when $\theta^{(+)}_{[i,j]}$ does not change in time, the phases of oscillators i and j evolve maintaining a constant distance from each other. The equilibrium of the whole vector $\theta^{(+)}$ will correspond to a *phase-locked* configuration: the oscillators move at the same angular speed, but do not have the same phase. With this in mind, we can easily generalize this concept to Kuramoto dynamics of any order and define phase locking as the onset of a particular fixed relation between the oscillators' phases.

Definition 5.2.4 (Phase-locking)**.** *A configuration θ is said to be **phase-locked** from **above** w.r.t the kth order Kuramoto dynamics if $\dot{\theta}^{(+)} = 0$. Analogously, it is called **phase-locked** from **below** when $\dot{\theta}^{(-)} = 0$ with $\theta^{(-)} \triangleq B^k\theta$.*

Phase locking is an important phenomenon which relaxes the notion of complete synchronization to contain also configurations which are not harmonic but still evolve in an "ordered" manner. It is therefore worth studying the evolution and equilibrium properties of $\theta^{(-)}$ and $\theta^{(+)}$. This paradigm shift was first proposed in the original paper [18], where the phase transition of $\theta^{(\pm)}$ is investigated.

The interesting thing about $\theta^{(\pm)}$, which we can interpret as "projections" of the phases onto upper and lower simplices, respectively, is that one can explicitly find the equilibrium configurations of their dynamics. First, their evolution equations are readily obtained by multiplying Equations 5.4 by D^k and B^k

$$\begin{cases} D^k\dot{\theta} = D^k\omega - \sigma^u D^k B^{k+1}\sin(D^k\theta) \\ B^k\dot{\theta} = B^k\omega - \sigma^d B^k D^{k-1}\sin(B^k\theta) \end{cases} \implies \begin{cases} \dot{\theta}^{(+)} = \omega^{(+)} - \sigma^u L^{k+1}_{down}\sin(\theta^{(+)}) \\ \dot{\theta}^{(-)} = \omega^{(-)} - \sigma^d L^{k-1}_{up}\sin(\theta^{(-)}) \end{cases} \tag{5.10}$$

where $\omega^{(+)} \triangleq D^k\omega$, $\omega^{(-)} \triangleq B^k\omega$ and $L^{k+1}_{down}, L^{k-1}_{up}$ are the half Laplacian matrices defined in 3.5. The following algebraic properties of the Laplacians should be kept in mind:

$$Im(L^{k+1}_{down}) = Im(D^k), \quad Im(L^{k-1}_{up} = Im(B^k).$$

We can now solve the equilibrium equations. For example, $\theta^{(-)}$ will be in equilibrium if and only if

$$L^{k-1}_{up}\sin(\theta^{(-)}) = \frac{\omega^{(-)}}{\sigma^d} \underbrace{\iff}_{\omega^{(-)} \in Im(L^{k-1}_{up})} \sin(\theta^{(-)}) = (L^{k-1}_{up})^\dagger \frac{\omega^{(-)}}{\sigma^d} + x \quad \forall x \in ker(L^{k-1}_{up}) = ker(D^{k-1}).$$

This equation closely resembles Equation 5.8 in its form and in the fact that, to solve it, the vector x still needs to satisfy an *admissibility* condition (Definition 5.2.2). The admissibility condition, moreover, is exactly the same. It turns out that we can push this similarity further:

Proposition 5.2.6. *It holds that*

$$(L^{k-1}_{up})^\dagger \omega^{(-)} = \beta^{(-)}, \ (L^{k+1}_{down})^\dagger \omega^{(+)} = \beta^{(+)}.$$

Proof. Notice first that $L^{k-1}_{up} = B^k D^{k-1}$ and, by the construction of the adjoint coboundary operator, ∂^k and d^{k-1} are adjoint operators, $D^{k-1} = (B^k)^*$ which mean that, by Property 5 of the pseudoinverse,

$$(L^{k-1}_{up})^\dagger = ((B^k)^*)^\dagger (B^k)^\dagger.$$

Moreover

$$(L^{k-1}_{up})^\dagger \omega^{(-)} = (L^{k-1}_{up})^\dagger B^k \omega = ((B^k)^*)^\dagger (B^k)^\dagger B^k \omega$$

which, by Property 6 of the pseudoinverse applied to $((B^k)^*)^\dagger$, gives us the thesis

$$(L^{k-1}_{up})^\dagger \omega^{(-)} = ((B^k)^*)^\dagger \omega = (D^{k-1})^\dagger \omega = \beta^{(-)}.$$

The same steps can be repeated *mutatis mutandis* to obtain the thesis for the $(+)$ component. □

This result directly implies that a projected configuration $\theta^{(\pm)}$ will be in equilibrium if and only if it belongs to the equilibrium set $\mathcal{E}^{(\pm)}$ of Definition 5.2.3!

Proposition 5.2.7. $\dot{\theta}^{(\pm)} = 0$ *if and only if* $\theta^{(\pm)} \in \mathcal{E}^{(\pm)}$.

One must however be careful when looking at this last result. Notice that the dynamics for the $(-)$ component (the same holds for $(+)$) 5.10 states that the time derivative of $\theta^{(-)}$ will be the vector $\omega^{(-)} - \sigma^d L^{k-1}_{up}\sin(\theta^{(-)})$, which always belongs to $Im(B^k)$. This means that, if the initial configuration is $\theta^{(-)}_0$, the trajectories of the dynamics will live

in the affine space given by $Im(B^k) + \theta_0^{(-)}$ (See Figure 5.7). Only the equilibria in $\mathcal{E}^{(-)}$ which are also in $Im(B^k) + \theta_0^{(-)}$ are *actual* equilibria of the dynamics. Moreover, given the nature of its definition, $\theta_0^{(-)} = B^k\theta_0$ i.e. the projection onto lower simplices of the initial phase configuration. Thus $\theta_0^{(-)} \in Im(B^k)$ and the dynamics will live in the *vector space* $Im(B^k) + \theta_0^{(-)} = Im(B^k)$. So, practically, the equilibria which have a real effect on the dynamics are the ones in $\mathcal{E}^{(-)} \cap Im(B^k)$. For this reason, we call them *reachable*.

Definition 5.2.5 (Reachable equilibrium)**.** *We call an equilibrium configuration for the $(-)$ dynamics $\theta_{eq}^{(-)} \in \mathcal{E}^{(-)}$* ***reachable*** *if*

$$\theta_{eq}^{(-)} \in Im(B^k).$$

Analogously $\theta_{eq}^{(+)} \in \mathcal{E}^{(+)}$ is reachable if

$$\theta_{eq}^{(+)} \in Im(D^k).$$

We denote the sets of reachable equilibria $\mathcal{R}^{(\pm)} \subseteq \mathcal{E}^{(\pm)}$.

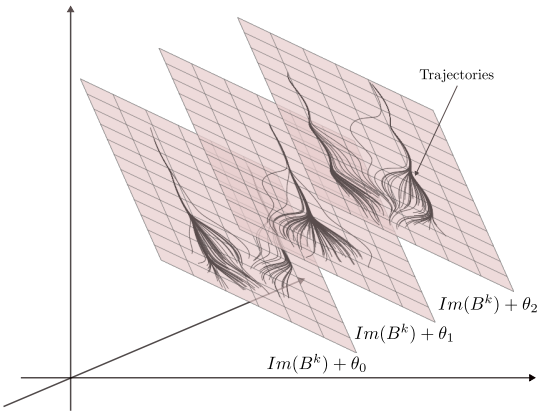


Figure 5.7: The trajectories of the $(-)$ component 5.10 live on the affine space $Im(B^k)$ centered in the starting phase configuration $\theta_i^{(\pm)}$.

To complete the picture, we need to make one last observation. Return to the decomposed equilibrium conditions for the curl-free and divergence-free components of Equations 5.9

$$\begin{cases} D^k\theta_{cf} = (-1)^{s_+} \odot \arcsin\left(\frac{\beta^{(+)}}{\sigma^u} + x\right) + s_+\pi + 2\pi m_+ \\ B^k\theta_{df} = (-1)^{s_-} \odot \arcsin\left(\frac{\beta^{(-)}}{\sigma^d} + y\right) + s_-\pi + 2\pi m_- \end{cases} \implies \begin{cases} D^k\theta_{cf} = \theta_{eq}^{(+)} \in \mathcal{E}^{(+)} \\ B^k\theta_{df} = \theta_{eq}^{(-)} \in \mathcal{E}^{(-)} \end{cases}.$$

If $\theta_{eq}^{(-)}$ is a reachable equilibrium $\theta_{eq}^{(-)} \in \mathcal{R}^{(-)} = \mathcal{E}^{(-)} \cap Im(D^k)$ then the first linear equation has a unique solution in $Im(D^k)$ and so the curl-free component will admit equilibria. Let us better formulate this fact in a proposition.

Proposition 5.2.8. *The curl-free (divergence-free) component admits equilibria if and only if $\theta^{(+)}$ ($\theta^{(-)}$) admits reachable equilibria. Moreover, the equilibria of the curl-free (divergence-free) component are in one-to-one correspondence with the reachable ones in $\mathcal{R}^{(+)}$ ($\mathcal{R}^{(-)}$) through the action of the matrix $(D^k)^\dagger$ ($(B^k)^\dagger$).*

It is useful now to focus on a simple case in order to clarify these concepts. We already discussed the case when $\omega_H = 0$. What happens when ω_{cf} or ω_{df} are 0?

First notice that, by Proposition 5.2.3, $\omega_{cf} = B^{k+1}\beta^{(+)}$ and, being $\beta^{(+)} \in ker(B^{k+1})^{\perp}$, we have that $\omega_{cf} = 0 \iff \beta^{(+)} = 0$. The same holds for ω_{df} and $\beta^{(-)}$.

Proposition 5.2.9. *If $\omega_{cf} = 0$ (respectively $\omega_{df} = 0$) then the null cochain $\theta^{(\pm)}_{eq} = 0$ is a reachable equilibrium for the $(\pm)$ component.*

Proof. We prove it for the $(+)$ component. First notice that the cycle $x = 0$ is admissible as

$$\left\| \frac{\beta^{(+)}}{\sigma^u} + x \right\|_\infty = \|x\|_\infty = 0 \leq 1.$$

By Definition 5.2.3 we have that to $x = 0$ corresponds a set of equilibria given by

$$(-1)^{s_+} \odot \arcsin(x) + s_+\pi + 2\pi m_+ = s_+\pi + 2\pi m_+.$$

The null vector belongs to this set and, given that every vector space contains it, it is also reachable. □

It follows that, when $\omega_{cf} = \omega_{df} = 0$, there are configurations when θ_{cf} and θ_{df} do not evolve and they are equal to 0. In this case, the only component of the phases which evolves is the harmonic one and thus $\theta(t) \in ker(L^k)\ \forall t$. This means, by Definition 4.1.1, that this is a case where *complete synchronization is possible*.

5.2.3 The shape of equilibria

Now that the heavy burden of the previous section has been lifted off our shoulders, let us build some intuition about these equilibria through some pictures and basic observations. Note that here we examine only the $(-)$ component but, for now, the situation is completely analogous for the $(+)$ component. If we plot the points of $\mathcal{E}^{(-)}$ in some particular region (Figure 5.8) we see that, depending on the complex, the value of σ^d and of $\omega^{(-)}$, peculiar geometrical structures emerge.

Remember the general equilibrium expression for the $(-)$ component in Definition 5.2.3:

$$\theta^{(-)}_{eq} = (-1)^s \odot \arcsin\left(\frac{\beta^{(-)}}{\sigma^d} + y\right) + s\pi + 2m\pi,\ y \in \mathcal{A}^{(-)},\ s \in \{0,1\}^{n_{k-1}},\ m \in \mathbb{Z}^{n_{k-1}}.$$

This expression deserves some further comments. Let us fix the integer vectors s and m and consider the map $f^{(-)}_{(s,m)} : \mathbb{R}^{n_{k-1}} \to \mathbb{R}^{n_{k-1}}$

$$f^{(-)}_{s,m}(x) = (-1)^{\underline{s}} \odot \arcsin\left(\frac{\beta^{(-)}}{\sigma} + x\right) + \underline{s}\pi + 2\underline{m}\pi.$$

$f^{(-)}_{s,m}$ is a C^∞ invertible map with C^∞ inverse, i.e. a C^∞-diffeomorphism.

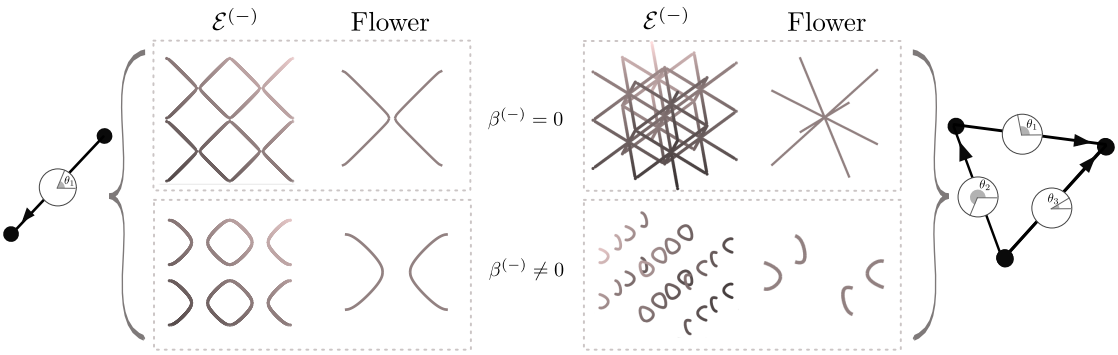


Figure 5.8: Examples of equilibrium sets $\mathcal{E}^{(-)}$ together with the flowers (Definition 5.2.7) which make them up. Small complexes with at most 3 nodes were chosen in order to make $\mathcal{E}^{(-)}$ representable.

Definition 5.2.6 (Petal). *We call* ***petal*** *the subset of* $\mathcal{E}^{(-)}$ *defined by* $\mathcal{E}^{(-)}(s,m) = \left\{ f^{(-)}_{s,m}(x) : x \in \mathcal{A}^{(-)} \right\}$, *having fixed* s *and* m.

A petal is thus a compact, connected, $\dim(\ker(D^{k-1}))$-dimensional manifold with boundary, homeomorphic to the bounded convex polytope $\mathcal{A}^{(-)}$. It follows that $\mathcal{E}^{(-)}$ is given by the union of petals, which differ only by rotation ($(-1)^s\odot$) and by position.

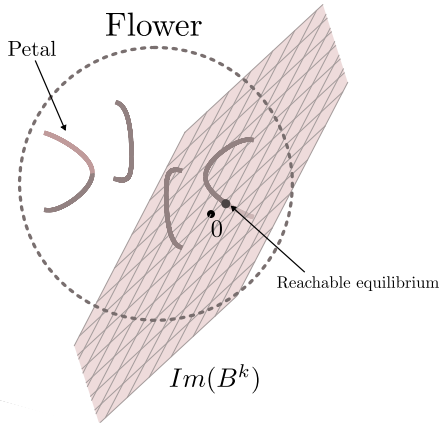


Figure 5.9: Example of a single flower for the $(-)$ component $B^1\theta$ of the edge dynamics on a single unweighted triangle complex.

If we fix $\underline{m}$ and take the union of the petals associated to all possible values of $\underline{s}$, we get a set which is the "fundamental atom" of the equilibrium set, in the sense that $\mathcal{E}^{(-)}$ is just a 2π-periodic tessellation of $\mathbb{R}^{n_{k-1}}$ with them. We call them *flowers*.

Definition 5.2.7 (Flower). *We call* ***flower*** *of the equilibrium set* $\mathcal{E}^{(-)}$ *the set*

$$\mathcal{E}^{(-)}_* = \bigcup_{s\in\{0,1\}^{n_{k-1}}} \mathcal{E}^{(-)}(s,m).$$

As it is shown in Figure 5.8 it often happens that the flowers intersect in some points, sometimes forming loops It is easy to see that every flower has a number of petals equal to $2^{n_{k-1}}$.

Remember that the equilibria we actually care about are the *reachable* ones (Definition 5.2.5) resulting from the intersection of $\mathcal{E}^{(-)}$ with $Im(B^k)$. It is impossible to give a general expression of such an intersection, but we can, through some numerical experiments, get

a sense of their shape. The examples of Figure 5.10 seem to suggest that $\mathcal{R}^{(-)}$ will be composed by a discrete set of points arranged in some lattice structure which periodically repeats itself in space.

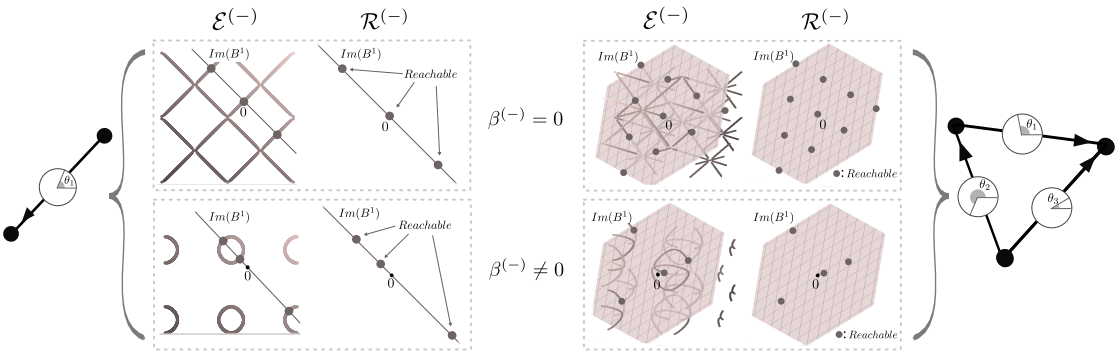


Figure 5.10: Example of reachable equilibria for $\theta^{(-)}$ for two unweighted small simplicial complexes. On the left $\theta^{(-)} \in \mathbb{R}^2$ as there are 2 nodes. On the right $\theta^{(-)} \in \mathbb{R}^3$ as there are 3. When $\beta^{(-)} = 0$ the equilibrium structures look more regular.

5.2.4 Linear stability analysis of the equilibrium sets

Now that we know the explicit expression of the equilibria of the projections onto upper and lower adjacent simplices, it is natural to ask whether they are stable or unstable. If $\theta^{(+)}$, for example, is close to a reachable equilibrium, will it converge?

Consider a reachable equilibrium point in $\mathcal{R}^{(\pm)}$ given by

$$\theta_{eq}^{(\pm)} = (-1)^s \odot \arcsin\left(\frac{\beta^{(\pm)}}{\sigma} + x\right) + s\pi + 2m\pi \tag{5.11}$$

for some integer m, s and admissible $x \in \mathcal{A}^{(\pm)}$. We can formulate a sufficient condition for asymptotic stability and individuate a *stable petal* in the fundamental flower of $\mathcal{E}^{(\pm)}$.

Theorem 5.2.1. *A reachable equilibrium point for the* $(\pm)$ *dynamics given by 5.11 is asymptotically stable if* $s = 0 \in \mathbb{R}^{n_{k\pm1}}$ *and* $\left\|\beta^{(\pm)}/\sigma + x\right\|_\infty < 1$.

Proof. We prove this for weighted simplicial complexes, i.e. $W_{(k)} = diag(w_{(k)})\ \forall k$.

To study the stability of a reachable equilibrium, we need to get rid of the redundancy contained in the dynamics and restrict our analysis to the reachable subspace. We do that by putting ourselves in a new basis and considering the evolution of the coefficients of $\theta^{(\pm)}$ w.r.t it. Let us adopt the following useful notation

$$L^{(-)} = L_{up}^{k-1}, \quad L^{(+)} = L_{down}^{k+1}.$$

Because of Proposition 2.1.3 in Chapter 2, $L^{(\pm)}$ is a Hermitian positive semidefinite matrix which, because of Theorem A.0.1 in Appendix A, admits an eigendecomposition $L^{(\pm)} = V\Lambda V^*$. Being $L^{(\pm)}$ in general not full rank, we can pass to the reduced decomposition,

$$L^{(\pm)} = \tilde{V}\tilde{\Lambda}\tilde{V}^*,$$

where

$$\tilde{V}^* = \tilde{W}^{(k\pm1)}\tilde{V}^\top W_{(k\pm1)}.$$

It also holds that the columns of $\tilde{V}$ are a basis of the reachable subspace $Im(L^{(\pm)})$. We can now write the following differential equation which describes the evolution of the coefficients c of $\theta^{(\pm)} = \tilde{V}c$.

$$\dot{\theta}^{(\pm)} = \omega^{(\pm)} - \sigma L^{(\pm)} \sin(\theta^{(\pm)}) \tag{5.12}$$

$$\implies \frac{d}{dt}(\tilde{V}c) = \omega^{(\pm)} - \sigma\tilde{V}\tilde{\Lambda}\tilde{V}^* \sin(\tilde{V}c) \tag{5.13}$$

$$\implies \tilde{V}^*\tilde{V}\dot{c} = \tilde{V}^*\omega^{(\pm)} - \sigma\tilde{V}^*\tilde{V}\tilde{\Lambda}\tilde{V}^* \sin(\tilde{V}c) \tag{5.14}$$

$$\implies \dot{c} = \tilde{V}^*\omega^{(\pm)} - \sigma\tilde{\Lambda}\tilde{V}^* \sin(\tilde{V}c). \tag{5.15}$$

Adopting the terminology of [11], where the node Kuramoto is studied with a perspective analogous to ours, we call this last equation **grounded dynamics**.

The associated Jacobian is then

$$J^{(\pm)}(c) = -\sigma\tilde{\Lambda}\tilde{V}^* diag(\cos(\tilde{V}c))\tilde{V}. \tag{5.16}$$

If the matrix calculated in the coefficient of $\theta_{eq}^{(\pm)}$ w.r.t to V has all strictly negative eigenvalues, then the equilibrium will be asymptotically stable. Given that $\theta_{eq}^{(\pm)}$ is reachable then $\theta_{eq}^{(\pm)} \in Im(L^{(\pm)}) = Im(\tilde{V})$ and thus we get

$$J_{eq}^{(\pm)} = -\sigma\tilde{\Lambda}\tilde{V}^* diag(\cos(\theta_{eq}^{(\pm)}))\tilde{V}.$$

If we replace $\theta^{(\pm)}$ with its expression 5.11 we get

$$\begin{aligned} J_{eq}^{(\pm)} &= -\sigma\tilde{\Lambda}\tilde{V}^* diag\left(\cos\left((-1)^{\underline{s}} \odot \arcsin\left(\frac{\beta^{(\pm)}}{\sigma} + x\right) + \underline{s}\pi + 2\underline{m}\pi\right)\right)\tilde{V} \\ &= -\sigma\tilde{\Lambda}\tilde{V}^* diag\left((-1)^{\underline{s}} \odot \sqrt{\mathbb{1} - \left(\frac{\beta^{(\pm)}}{\sigma} + x\right)^2}\right)\tilde{V}. \end{aligned}$$

$\tilde{\Lambda} = \tilde{\Lambda}^{\frac{1}{2}}\tilde{\Lambda}^{\frac{1}{2}}$ because it is a symmetric positive definite square matrix. With this in mind we find that $J_{eq}^{(\pm)}$ has the same eigenvalues as the following matrix

$$\tilde{J}_{eq}^{(\pm)} = -\sigma\tilde{\Lambda}^{\frac{1}{2}}\tilde{V}^* diag\left((-1)^{\underline{s}} \odot \sqrt{\mathbb{1} - \left(\frac{\beta^{(\pm)}}{\sigma} + x\right)^2}\right)\tilde{V}\tilde{\Lambda}^{\frac{1}{2}} \triangleq -\sigma\tilde{\Lambda}^{\frac{1}{2}}\tilde{V}^* S\tilde{V}\tilde{\Lambda}^{\frac{1}{2}},$$

as they are similar $\tilde{J}_{eq}^{(\pm)} = \tilde{\Lambda}^{\frac{1}{2}} J_{eq}^{(\pm)} \tilde{\Lambda}^{-\frac{1}{2}}$. It is immediate to see that

- $s = 0$ implies that S is positive semidefinite as its diagonal elements are non-negative (given that x is admissible and the square root returns non-negative numbers);
- $\left\|\beta^{(\pm)}/\sigma + x\right\|_\infty < 1$ implies that the diagonal elements of S are nonzero and S is invertible.

The combination of these two facts results in $\tilde{J}_{eq}^{(\pm)}$ having all negative eigenvalues. We prove this by showing that $\tilde{J}_{eq}^{(\pm)}$ is a negative semidefinite operator on the space of coefficient w.r.t the inner product $\tilde{W}_{(k\pm1)}$.

$$\left\langle \tilde{J}_{eq}^{(\pm)} c, c \right\rangle_{\tilde{W}_{(k)}} = -\sigma c^\top (\tilde{J}_{eq}^{(\pm)})^\top \tilde{W}_{(k\pm1)} c = -\sigma c^\top \tilde{\Lambda}^{\frac{1}{2}} \tilde{V}^\top S W_{(k\pm1)} \tilde{V} \tilde{W}^{(k\pm1)} \tilde{\Lambda}^{\frac{1}{2}} \tilde{W}_{(k\pm1)} c.$$

$\Lambda^{\frac{1}{2}}$ and $\tilde{W}_{(k\pm1)}$ are diagonal, and thus they commute.

$$\left\langle \tilde{J}_{eq}^{(\pm)} c, c \right\rangle_{\tilde{W}_{(k\pm1)}} = -\sigma c^\top \tilde{\Lambda}^{\frac{1}{2}} \tilde{V}^\top S W_{(k\pm1)} \tilde{V} \tilde{\Lambda}^{\frac{1}{2}} c = -\sigma \left\| (S W_{(k\pm1)})^{\frac{1}{2}} \tilde{V} \tilde{\Lambda}^{\frac{1}{2}} c \right\| \leq 0.$$

Finally, being V a basis, it holds that $Vc = 0 \iff c = 0$ and thus $\tilde{J}_{eq}^{(\pm)}$ is actually negative definite and has all strictly negative eigenvalues. □

This result can be conveniently restated in the following way: *the intersection of any equilibrium petal characterized by $s = 0$ with $Im(B^k)$ is asymptotically stable.* Figure 5.11 shows this fact in two simple cases.

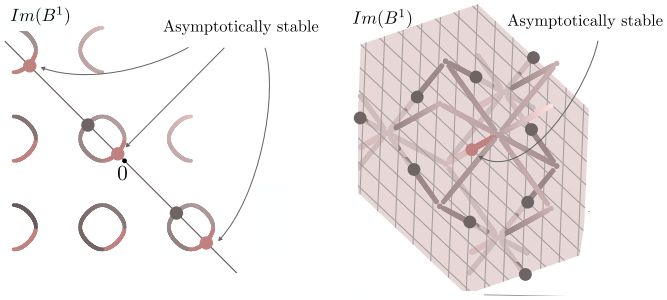


Figure 5.11: In red are shown examples of the stable petals of two equilibrium sets for $\theta^{(-)}$. Their intersections with $Im(B^1)$ are asymptotically stable reachable equilibria.

5.3 The role of the coupling strength

In this section, we investigate the relation between the equilibrium properties of the simplicial Kuramoto model and the value of the coupling strength. It is natural to think that having a stronger interaction would make it easier for the system to reach a phase-locked or synchronized configuration as the intrinsic differences among the oscillators, encoded by their natural frequencies, become secondary. We already encountered this fact in Section

5.1 for a single simplicial oscillator, where a proper *phase transition* occurs when tuning the value of σ, making the system go from an equilibrium-free potential landscape to a situation where the energy is "dissipated" and the oscillator stops moving. Does the same occur when more oscillators are combined in a simplicial complex?

An important point must be stressed. In the next pages we will focus on the relation between σ and the equilibria of the projected dynamics $\theta^{(\pm)}$ which, by Proposition 5.2.8, when reachable are perfectly equivalent to the equilibria of the curl-free and divergence-free components. Contrary to the case of the single simplicial oscillator, when we say "equilibrium" we mean a *phase-locked* configuration (Definition 5.2.4), although, by Equation 5.5, when ω has no harmonic component, phase-locking from below and from above imply actual equilibrium of the phases.

Finally, in this section we will always consider complexes which are weighted, i.e. they have diagonal inner product $W_{(k)} = diag(w_{(k)})$.

5.3.1 Simple bounds for phase-locking

Here we derive a first family of bounds on σ, the first of which is a sufficient condition for the *non*-existence of equilibria (and thus a necessary condition for their existence). The way we can find them is simply by taking advantage of the relation *admissible cycles - equilibria* of Proposition 5.2.4. Remember that, by Definition 5.2.2, a coclosed cochain $x \in ker(D^{k-1})$ is admissible for the $(-)$ component if

$$\left\| \frac{\beta^{(-)}}{\sigma} + x \right\|_\infty \leq 1.$$

Surely there will be no equilibria if the ∞-ball centered in $\beta^{(-)}/\sigma$ with radius 1 does not intersect the subspace $x \in ker(D^{k-1})$.

Proposition 5.3.1. *If*

$$\sigma < \frac{\left\|\beta^{(\pm)}\right\|_{(k\pm1)}}{\sqrt{\mathbb{1}^\top w_{(k\pm1)}}}$$

then the $(\pm)$ *component admits no equilibria* $\mathcal{E}^{(\pm)} = \emptyset$.

Proof. First recall that all norms are equivalent in a finite dimensional space and, in particular, we can bound a strong norm with a weaker one

$$\|v\|_{(k\pm1)} = \sqrt{\sum_{i=1}^{n_{k\pm1}} w_i v_i^2} \leq \sqrt{\left(\sum_i w_i\right)\left(\max_i (v_i)^2\right)} \leq \sqrt{\mathbb{1}^\top w_{(k\pm1)}}\, \|v\|_\infty .$$

With this in mind, we can write

$$\left\| \frac{\beta^{(-)}}{\sigma} + x \right\|_\infty \geq \frac{1}{\sqrt{\mathbb{1}^\top w_{(k\pm1)}}} \left\| \frac{\beta^{(-)}}{\sigma} + x \right\|_{(k\pm1)}.$$

The two addenda in the norm are orthogonal w.r.t. the inner product $W_{(k\pm1)}$ because, for the two cases,

$$\begin{aligned} x \in ker(D^{k-1}) \qquad & \text{and } \beta^{(-)} \in Im((D^{k-1})^\dagger) = Im((D^{k-1})^*) = ker(D^{k-1})^\perp \\ x \in ker(B^{k+1}) \qquad & \text{and } \beta^{(+)} \in Im((B^{k+1})^\dagger) = Im((B^{k+1})^*) = ker(B^{k+1})^\perp. \end{aligned}$$

Thus

$$\frac{1}{\sqrt{\mathbb{1}^\top w_{(k\pm1)}}} \left\| \frac{\beta^{(\pm)}}{\sigma} + x \right\|_{(k\pm1)} = \frac{1}{\sqrt{\mathbb{1}^\top w_{(k\pm1)}}} \sqrt{\left\| \frac{\beta^{(\pm)}}{\sigma} \right\|^2_{(k\pm1)} + \|x\|^2_{(k\pm1)}} \geq \frac{1}{\sqrt{\mathbb{1}^\top w_{(k\pm1)}}} \left\| \frac{\beta^{(\pm)}}{\sigma} \right\|_{(k\pm1)}.$$

If this last term is strictly greater than 1 then there will surely be no admissible cycles and, therefore, no equilibria.

What we did in this proof can be interpreted as bounding the ∞-ball from above with the smallest 2-ball bigger than it. This 2-ball will intersect the subspace if and only if its center is close enough to the origin (See Figure 5.12). □

Notice that, when the complex is unweighted, it holds that

$$\mathbb{1}^\top w_{(k\pm1)} = n_{k\pm1}.$$

This last bound is a necessary condition for the existence of *reachable* equilibria too as, naturally,

$$\mathcal{R}^{(\pm)} \subseteq \mathcal{E}^{(\pm)}.$$

So, if $\sigma^d < \left\|\beta^{(-)}\right\| / \sqrt{n_{k-1}}$, there is no configuration which is phase locked from below.

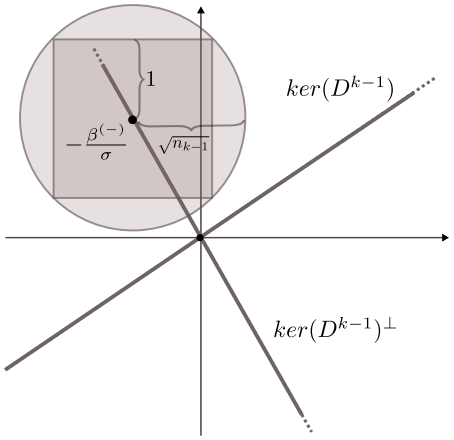


Figure 5.12: Representation of the idea behind Proposition 5.3.1.

A lower bound on σ which is sufficient for existence of equilibria (not necessarily reachable) can be easily obtained as well by finding conditions for the existence of a particular admissible cycle.

Proposition 5.3.2. *If* $\sigma \geq \sigma_\infty \triangleq \left\|\beta^{(\pm)}\right\|_\infty$ *then the* $(\pm)$ *component admits equilibrium points* $\mathcal{E}^{(\pm)} \neq \emptyset$.

Proof. It is straightforward to see that if $\sigma \geq \left\|\beta^{(\pm)}\right\|_\infty$ then the null cycle 0 is admissible.

The family of equilibria associated to this vector will be

$$\mathcal{E}^{(\pm)}(0) = \left\{ (-1)^s \odot \arcsin\left(\frac{\beta^{(\pm)}}{\sigma}\right) + s\pi + 2m\pi : m \in \mathbb{Z}^{n_{k\pm1}},\ s \in \{0,1\}^{n_{k\pm1}} \right\}.$$

□

5.3.2 The critical coupling

As it is shown by Proposition 5.2.4 the $(-)$ dynamics admits equilibrium points (not necessarily reachable) if and only if there exists an admissible cycle $x \in ker(D^{k-1})$ such that

$$\left\|\frac{\beta^{(-)}}{\sigma} + x\right\|_\infty \leq 1,$$

or, equivalently, if

$$\min_{x \in ker(D^{k-1})} \left\|\frac{\beta^{(\pm)}}{\sigma} + x\right\|_\infty \leq 1.$$

It is natural now to ask if it is possible to find the minimum value σ_* of the coupling parameter for which there are equilibrium solutions.

Definition 5.3.1 (Critical coupling)**.** *We call* ***critical coupling*** $\sigma_*^{(\pm)}$ *for the* $(\pm)$ *component the minimum value of* σ *such that there are admissible cycles in* $\mathcal{A}^{(\pm)}$*.*

Lemma 1. *The critical couplings* $\sigma_*^{(\pm)}$ *will satisfy*

$$\min_{x \in ker(D^{k-1})} \left\|\frac{\beta^{(-)}}{\sigma_*^{(-)}} + x\right\|_\infty = 1, \quad \min_{y \in ker(B^{k+1})} \left\|\frac{\beta^{(+)}}{\sigma_*^{(+)}} + y\right\|_\infty = 1.$$

Proof. If the statement were false and

$$\min_{x \in ker(D^{k-1})} \left\|\frac{\beta^{(-)}}{\sigma_*^{(-)}} + x\right\|_\infty = a$$

with $0 < a < 1$, then we could divide both sides by a and get

$$\min_{x \in ker(D^{k-1})} \left\|\frac{\beta^{(-)}}{a\sigma_*^{(-)}} + \frac{1}{a}x\right\|_\infty = 1,$$

which means that for $\sigma = a\sigma_*^{(-)} < \sigma_*$ there is an admissible cycle $x = \frac{x}{a}$. This is not possible because we assumed that σ_* is the smallest coupling with that property. □

Armed with this characterizing property, we can get the following important result.

Theorem 5.3.1. *The critical coupling $\theta_*^{(\pm)}$ can be found in the solution of a linear optimization problem*

$$\sigma_*^{(+)} = \min_{y \in ker(B^{k+1})} \left\| \beta^{(+)} + y \right\|_\infty, \quad \sigma_*^{(-)} = \min_{x \in ker(D^{k-1})} \left\| \beta^{(-)} + x \right\|_\infty \tag{5.17}$$

Proof. Let us take the expression of Lemma 1 and multiply both terms by $\sigma_*^{(-)}$

$$\min_{x \in ker(D^{k-1})} \left\| \beta^{(-)} + \sigma_*^{(-)} x \right\|_\infty = \sigma_*^{(-)}.$$

It is now possible to perform a linear change of variable in the optimization problem $\sigma_*^{(-)} x \to \tilde{x}$ which will change the optimal solution position but not the optimum itself! This means that $\sigma_*^{(-)}$ disappears from the left-hand side and $\sigma_*^{(-)}$ is found as the solution of an optimization problem.

$$\sigma_*^{(-)} = \min_{\tilde{x} \in ker(D^{k-1})} \left\| \beta^{(-)} + \tilde{x} \right\|_\infty.$$

□

Moreover, this problem can be easily recast into a linear program (LP)

$$\begin{aligned} \sigma_*^{(-)} = \min_{s \in \mathbb{R}, x \in \mathbb{R}^{n_{k-1}}} \quad & s \\ \text{s.t.} \quad & -s\mathbb{1} \leq \beta^{(-)} + x \leq s\mathbb{1} \\ & D^{k-1} x = 0 \end{aligned} \tag{5.18}$$

which can be efficiently solved with the simplex algorithm or other analogous mehods [12].

Let us take a look at the geometrical meaning of $\sigma_*^{(\pm)}$ in terms of the relative equilibrium set $\mathcal{E}^{(\pm)}$. In Figure 5.13 the equilibrium set $\mathcal{E}^{(-)}$ of a one-edge complex is shown for different values of σ. We can clearly see that, as expected, there are no equilibria for $\sigma < \sigma_*^{(-)}$ and that for $\sigma = \sigma_*^{(-)}$ the equilibrium petals (Definition 5.2.6) first appear as single points which "expand" into continuous structures for $\sigma > \sigma_*^{(-)}$. This is of course true because in the optimization problem 5.17 the optimal value $\sigma_*^{(-)}$ will be associated to an optimal solution x^* which will be the only admissible cycle of the complex. This means that

$$\mathcal{E}^{(-)} = \bigcup_{s,m} \mathcal{E}_{s,m}^{(-)}(x^*),$$

which is a union of $2^{n_{k-1}}$ discrete lattices, one for each value of s.

Notice that in Figure 5.13 some of the equilibria which appear for $\sigma = \sigma_*^{(-)}$ are reachable as they are on the line $Im(B^1)$. This is by no means true in general: most of the

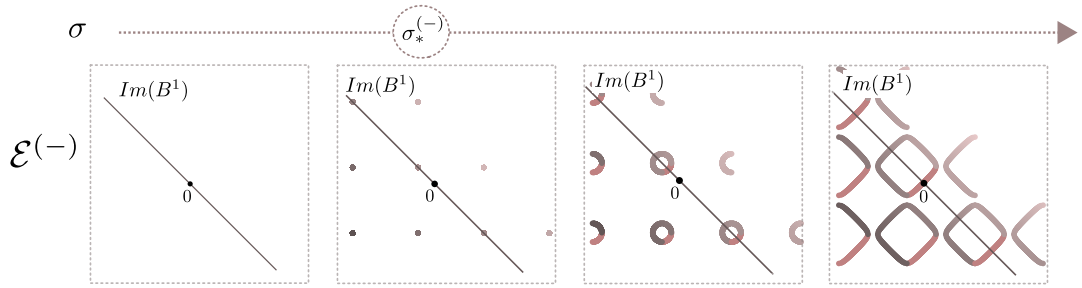


Figure 5.13: The meaning of the critical coupling in the shape of the equilibrium sets. For $\sigma = \sigma_*^{(-)}$ equilibrium flowers first appear as discrete sets of points

time, for $\sigma = \sigma_*^{(-)}$ there will be no reachable equilibria at all. This doesn't mean however that the critical coupling is a useless concept, in fact it holds that $\sigma_*^{(\pm)}$ *is the sharpest possible necessary condition for the existence of reachable equilibria.*

Proposition 5.3.3.

$$\mathcal{R}^{(\pm)} \neq \emptyset \implies \sigma \geq \sigma_*^{(\pm)}.$$

It trivially holds that the two values of σ discussed in Propositions 5.3.1 and 5.3.2 bound the critical coupling from below and from above.

Proposition 5.3.4.

$$\frac{\left\|\beta^{(\pm)}\right\|_{(k\pm1)}}{\sqrt{\mathbb{1}^\top w_{(k\pm1)}}} \leq \sigma_*^{(\pm)} \leq \sigma_\infty^{(\pm)}.$$

In the special case of the $(-)$ component of the edge dynamics on an unweighted connected complex, the set of admissible vectors and the critical coupling can both be found explicitly, giving a useful necessary condition for the equilibrium of the curl-free component.

Theorem 5.3.2. *For the $(-)$ component of the edge dynamics on a connected simplicial complex it holds that*

$$x \in \mathcal{A}^{(-)} \iff -\min\left(\frac{\beta^{(-)}}{\sigma}\right) - 1 \leq x \leq -\max\left(\frac{\beta^{(-)}}{\sigma}\right) + 1$$

and

$$\sigma_*^{(-)} = \frac{\max(\beta^{(-)}) - \min(\beta^{(-)})}{2}.$$

Proof. If the complex is connected we have that D^0 has a 1-dimensional kernel given by $span\{\mathbb{1}\}$. This means that there are admissible vectors if and only if

$$\left\| \frac{\beta^{(-)}}{\sigma} + x\mathbb{1} \right\|_\infty \leq 1 \iff -1 \leq \frac{\beta_i^{(-)}}{\sigma} + x \leq 1 \ \forall i = 1, \dots, n_0.$$

It is easy to see that this holds if and only if

$$\max\left(-\frac{\beta^{(-)}}{\sigma}\right) - 1 \leq x \leq \min\left(-\frac{\beta^{(-)}}{\sigma}\right) + 1$$

which is nonempty when

$$\begin{aligned} \max\left(-\frac{\beta^{(-)}}{\sigma}\right) - 1 \leq \min\left(-\frac{\beta^{(-)}}{\sigma}\right) + 1 &\iff \frac{1}{\sigma}\left(\max(-\beta^{(-)}) - \min(-\beta^{(-)})\right) \leq 2 \\ &\iff \sigma \geq \frac{\max(\beta^{(-)}) - \min(\beta^{(-)})}{2}. \end{aligned}$$

□

It is now worth noting that the properties of the projected components $\theta^{(+)} = D^k\theta$ and $\theta^{(-)} = B^k\theta$ are not entirely symmetrical. Let us consider the kth order dynamics on a simplicial complex $\mathcal{X}$ which, therefore, has a nonempty k-skeleton. Remember that the existence of equilibria for both of them depends on the presence or absence of admissible cycles (Definition 5.2.2) which, respectively, must belong to $ker(B^{k+1})$ and $ker(D^{k-1})$. The asymmetry stems from the fact that D^{k-1} cannot have a trivial kernel because

$$ker(D^{k-1}) = Im(B^k)^\perp \underbrace{=}_{\text{Hodge decomp.}} Im(D^{k-2}) \oplus ker(L^{k-1}) \tag{5.19}$$

and thus

- if $k = 1$ then $ker(D^0) = ker(L^0)$, which is nontrivial as there is at least one connected component;
- if $k > 1$ then $dim(ker(D^{k-1})) \geq dim(Im(D^{k-2}))$ which is nonzero because, by inclusion, there is a nonzero number of $(k-2)$-simplices.

The same cannot be said for B^{k+1} as, in general, there is no restriction on the number of $(k+1)$-cycles. In fact, on the same line of Equation 5.19,

$$ker(B^{k+1}) = Im(D^k)^\perp = Im(B^{k+2}) \oplus \ker(L^{k+1}),$$

which is empty when there are no $k+2$ simplices and no $(k+1)$-holes. Therefore, the case when $ker(B^{k+1}) = \{0\}$ deserves a special treatment.

Theorem 5.3.3. *If there are no $(k+1)$-cycles $ker(B^{k+1}) = \{0\}$ then the following properties hold:*

1. *if $\mathcal{A}^{(+)} \neq \emptyset$ then $\mathcal{A}^{(+)} = \{0\}$;*
2. *if $\mathcal{A}^{(+)} \neq \emptyset$ then the equilibrium set is a discrete set of point given by*

$$\mathcal{E}^{(+)}_{s,m} = (-1)^s \odot \arcsin\left(\frac{\beta^{(+)}}{\sigma}\right) + s\pi + 2m\pi;$$

3. $\sigma^{(+)}_* = \sigma^{(+)}_\infty = \left\|\beta^{(+)}\right\|_\infty$;
4. *All equilibria are reachable $\mathcal{E}^{(+)} = \mathcal{R}^{(+)}$.*

Proof. 1. It is trivial because 0 is the only vector in $ker(B^{k+1})$.

2. Directly follows from Definition 5.2.3 with $x = 0$.

3. 0 is the only vector in $ker(D^k)$ so it will be admissible if and only if

$$\left\|\frac{\beta^{(+)}}{\sigma}\right\|_\infty \leq 1.$$

The smallest value of σ for which this holds is $\sigma = \left\|\beta^{(+)}\right\|_\infty = \sigma^{(+)}_\infty$.

4. According to Definition 5.2.5, an equilibrium is reachable for the $(+)$-component if it belongs to $Im(D^k)$. In this case,

$$Im(D^k) = ker((D^k)^*)^\perp = ker(B^{k+1})^\perp = \{0\}^\perp = \mathbb{R}^{n_{k+1}},$$

hence the thesis.

□

In this case then the critical coupling $\sigma^{(+)}_*$ has a stronger meaning as it is also the transition value for reachable equilibria and, by virtue of Proposition 5.2.8, for the equilibria of the divergence-free component as well. This concept of *reachability transition* is of paramount importance and will be discussed in general in the next section.

An interesting result can be obtained without effort from Theorem 5.3.3.

Theorem 5.3.4. *Consider the node Kuramoto model on a connected tree graph (i.e. with no 1-cycles) with natural frequencies ω such that $\sum_i \omega_i = 0$. The critical coupling for which the node Kuramoto with natural frequencies ω admits equilibria is*

$$\sigma_* = \left\|\beta^{(+)}\right\|_\infty = \left\|(B^1)^\dagger \omega\right\|_\infty.$$

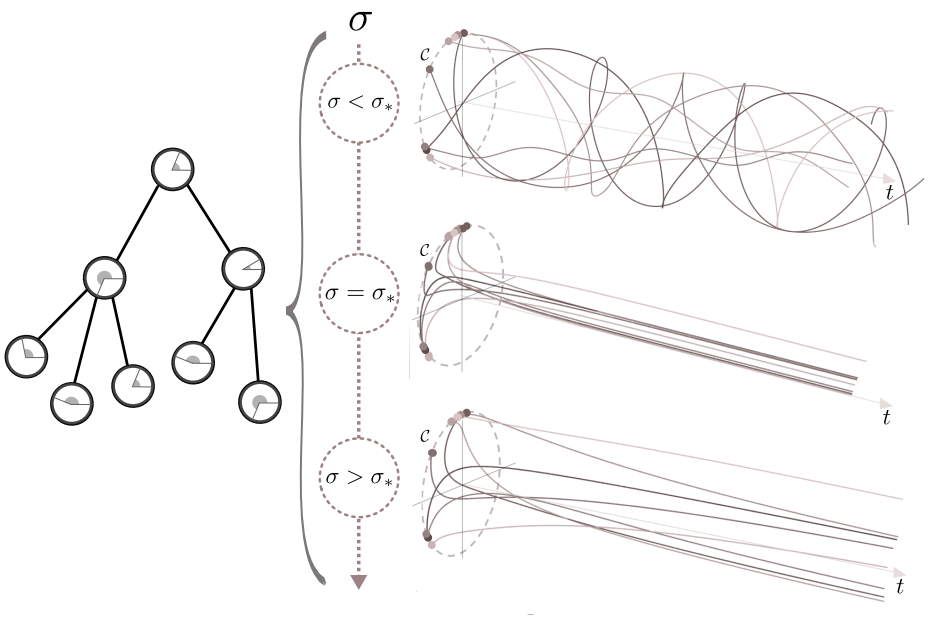


Figure 5.14: Equilibrium phase transition occurring in the tree graph on the left with random natural frequencies (with 0 sum). The critical coupling σ_* was computed exactly using Theorem 5.3.4.

Proof. If $\sum_i \omega_i = 0$ then $\omega \in \{\mathbb{1}\}^\perp$, meaning that ω has no harmonic component. So, by Equation 5.5, the system will admit equilibria if and only if its divergence-free component admits equilibria (there is no curl-free component). In turn, by Proposition 5.2.8, the divergence-free component will admit equilibria if and only if the $(+)$ component $\theta^{(+)} = D^0\theta$ will admit reachable equilibria. Given the fact that a tree graph has no cycles, we have that $\ker(B^1) = \{0\}$ and thus, by Theorem 5.3.3, its critical coupling will be

$$\sigma_* = \sigma_*^{(+)} = \sigma_\infty^{(+)} = \left\|\beta^{(+)}\right\|_\infty = \left\|(B^1)^\dagger \omega\right\|_\infty .$$

□

Note that this result, which is also derived in [11], guarantees that for $\sigma \geq \sigma_*$ the system will have actual equilibria i.e. configurations where the phases do not evolve anymore and are not just phase-locked.

5.3.3 The reachability problem

Let us quickly review the content of the last sections. We found out that the equilibrium properties of the curl-free and divergence-free components are related to the equilibria of the dynamics projected onto upper $\theta^{(+)}$ and lower $\theta^{(-)}$ simplices. Not all the equilibria of their ODE are actually "compatible" with the dynamics, as they live in a lower dimensional subspace (as it is shown in Figure 5.7). One therefore establishes the concept of *reachable equilibria* to talk about the special ones which also belong to this subspace.

Reachable equilibria are evidently more important because, as the name suggests, they can be actually be "reached" by the dynamics. It turns out that reachable equilibria are in one-to-one correspondence with the equilibria of the curl-free and divergence-free components, which we called *phase-locked* configurations (Definition 5.2.4). In the last section we proved bounds on σ which guarantee the existence of equilibria. The question rises naturally: *is it possible to find bounds which also guarantee reachability?* Or, more fundamentally, *does increasing the coupling strength always generate phase-locked configurations?*

Definition 5.3.2 (Reachability coupling)**.** *We call reachability coupling* $\sigma_r^{(\pm)}$ *the smallest* σ *for which reachable equilibria exist for the* $(\pm)$ *dynamics. If* $\mathcal{R}^{(\pm)} = \emptyset$ *for all* $\sigma > 0$*, we say that* $\sigma_r^{(\pm)} = \infty$*.*

Notice that, although this value is well-defined, it is not clear whether it is actually phase transitions. In other words, while for $\sigma = \sigma_r$ there surely will be reachable equilibria by definition, nothing is known a priori when $\sigma > \sigma_r$.

Finding the exact value of σ_r seems to be a quite hard task. There are indeed particular cases, like the one of Theorem 5.3.3 where $\sigma_r = \sigma_*$, where it can be done but, in general, we should lower our expectations and settle for something less precise. We would therefore be happy enough to find a *bound* on the value of σ_r which is "strict enough" and easy to compute.

It turns out that an elegant bound that satisfies these requirements (and even more) can be found by generalizing one of the result proved in [11] for the node Kuramoto.

Theorem 5.3.5 (Bound for the existence of stable reachable equilibria)**.** *If*

$$\sigma \geq \sigma_{fp}^{(\pm)} := \frac{\left\|\beta^{(\pm)}\right\|_{(k\pm1)}}{\sqrt{\min(w_{(k\pm1)})}}, \tag{5.20}$$

there exists an asymptotically stable reachable equilibrium for the $(\pm)$ *dynamics in the ball*

$$\left\|\theta^{(\pm)}\right\|_{(k\pm1)} \leq \frac{\pi}{2}\sqrt{\min w_{(k\pm1)}}.$$

This directly means that $\sigma_{fp}^{(\pm)} \geq \sigma_r^{(\pm)}$*.*

Proof. The idea of the proof, which directly follows the constructions in [11], is to rewrite the equilibrium equation for the reduced dynamics 5.15 as a fixed point equation and find σ such that it is a continuous function from a convex compact set to itself. Brouwer's fixed point theorem will then provide the existence of the fixed point (a reachable equilibrium) in this set.

We return to the grounded dynamics introduced in the proof of Theorem 5.2.1:

$$\dot{c} = \tilde{V}^* \omega^{(\pm)} - \sigma \tilde{\Lambda} \tilde{V}^* \sin(\tilde{V}c),$$

describing the evolution of the coefficients of $\theta^{(\pm)}$ w.r.t the basis given by the columns of $\tilde{V}$, which span the reachable subspace. The coefficients c are associated to a reachable equilibrium configuration if and only if

$$\tilde{V}^* \frac{\omega^{(\pm)}}{\sigma} = \tilde{\Lambda} \tilde{V}^* \sin(\tilde{V}c).$$

We want to reduce this equation to a fixed point equation, i.e. of the form $f(c) = c$.

$$\tilde{V}^* \frac{\omega^{(\pm)}}{\sigma} = \tilde{\Lambda} \tilde{V}^* \sin(\tilde{V}c) \iff \tilde{\Lambda}^{-1} \tilde{V}^* \frac{\omega^{(\pm)}}{\sigma} = \tilde{V}^* diag(\text{sinc}(\tilde{V}c)) \tilde{V}c$$

$$c = (\tilde{V}^* S(c) \tilde{V})^{-1} \tilde{\Lambda}^{-1} \tilde{V}^* \frac{\omega^{(\pm)}}{\sigma} \triangleq f(c), \tag{5.21}$$

where we defined $S(c) := diag(\text{sinc}(\tilde{V}c))$. In order to do this, we need to be able to invert the matrix $\tilde{V}^* S(c) \tilde{V}$. Let us therefore look for conditions for its invertibility.

We claim that, if $S(c)$ has strictly positive elements, then $\tilde{V}^* S(c) \tilde{V}$ is invertible. First, notice that S is diagonal which, together with the inner product matrix being diagonal, means that $S(c)$ is a Hermitian matrix ($S(c)^* = S(c)$), and so is its square root. We get the following

$$\tilde{V}^* S(c) \tilde{V} = (S^{\frac{1}{2}}(c) \tilde{V})^* (S^{\frac{1}{2}}(c) \tilde{V}) \triangleq A^* A.$$

Given that $ker(A^*) = Im(A)^\perp$, we deduce that A^*A is invertible if and only if $A = S^{\frac{1}{2}}(c)\tilde{V}$ has trivial kernel. Given that the columns of $\tilde{V}$ are a basis, we have that $\tilde{V}c = 0 \iff c = 0$. If $S^{\frac{1}{2}}(c)$ is invertible, then, its kernel will be trivial and, by extension, the same will hold for A.

If we give a condition for $S(c)$ to be positive definite, then we also have the invertibility of $\tilde{V}^* S(c) \tilde{V}$. For this purpose, we restrict ourselves to the ball

$$\left\| \theta^{(k\pm1)} \right\|_{(k\pm1)} = \left\| \tilde{V}c \right\|_{(k\pm1)} \leq \sqrt{\min(w_{(k\pm1)})} \frac{\pi}{2}.$$

We choose this because, under such an assumption,

$$\left\| \tilde{V}c \right\|_\infty \leq \left\| \tilde{V}c \right\|_2 = \sqrt{\sum_i (\tilde{V}c)_i^2} = \sqrt{\sum \frac{1}{(w_{(k\pm1)})_i} (w^{(k\pm1)})_i (\tilde{V}c)_i^2} \leq \frac{1}{\sqrt{\min(w_{(k\pm1)})}} \left\| \tilde{V}c \right\|_{(k\pm1)} \leq \frac{\pi}{2},$$

meaning that every component of $\theta^{(\pm)} = \tilde{V}c$ will belong to the interval $[-\pi/2, \pi/2]$. The sinc function, which is applied component-wise to $\tilde{V}c$, is strictly positive in $[-\pi/2, \pi/2]$,

hence the positive definiteness of $S(c)$.

We now want to prove that the left-hand side of the equilibrium fixed point equation 5.21 is a contraction from the set $\mathcal{B} = \left\{ c : \left\| \tilde{V} c \right\|_{(k\pm1)} \leq \sqrt{\min(w_{(k\pm1)})} \frac{\pi}{2} \right\}$ to itself.

First, we prove conditions for $f(c)$ to belong to $\mathcal{B}$, when $c \in \mathcal{B}$.

$$\|f(c)\|_{(k\pm1)} = \left\| (\tilde{V}^* S(c) \tilde{V})^{-1} \tilde{\Lambda}^{-1} \tilde{V}^* \frac{\omega^{(\pm)}}{\sigma} \right\|_{(k\pm1)} \leq \frac{1}{\sigma} \left\| (\tilde{V}^* S(c) \tilde{V})^{-1} \right\|_{(k\pm1)} \left\| \tilde{\Lambda}^{-1} \tilde{V}^* \omega^{(\pm)} \right\|_{(k\pm1)} . \tag{5.22}$$

Let us look at the two terms of Equation 5.22 separately, starting from the right one.

$$\left\| \tilde{\Lambda}^{-1} \tilde{V}^* \omega^{(\pm)} \right\|_{(k\pm1)} = \left\| \tilde{V}^* \tilde{V} \tilde{\Lambda}^{-1} \tilde{V}^* \omega^{(\pm)} \right\|_{(k\pm1)} = \left\| \tilde{V}^* (L^{(\pm)})^\dagger \omega^{(\pm)} \right\|_{(k\pm1)} = \left\| \tilde{V}^* \beta^{(\pm)} \right\|_{(k\pm1)} .$$

Moreover

$$\left\| \tilde{V}^* \beta^{(\pm)} \right\|_{(k\pm1)} = \sqrt{\left\langle \tilde{V}^* \beta^{(\pm)}, \tilde{V}^* \beta^{(\pm)} \right\rangle_{(k\pm1)}} = \sqrt{\left\langle \tilde{V} \tilde{V}^* \beta^{(\pm)}, \beta^{(\pm)} \right\rangle_{(k\pm1)}}.$$

According to Proposition A.0.4 of Appendix A, $\tilde{V}\tilde{V}^*$ is the orthogonal projection operator onto $Im(L^{(\pm)})$. Being $\beta^{(\pm)} \in Im(L^{(\pm)})$, then, we see that the projection will leave it unchanged.

$$\left\| \tilde{V}^* \beta^{(\pm)} \right\|_{(k\pm1)} = \sqrt{\left\langle \tilde{V} \tilde{V}^* \beta^{(\pm)}, \beta^{(\pm)} \right\rangle_{(k\pm1)}} = \left\| \beta^{(\pm)} \right\|_{(k\pm1)} .$$

Let us now analyze the first term of Equation 5.22 and try to bound it from above. When $c \in \mathcal{B}$ then $S(c)$ is positive definite and therefore we can write

$$\left\| (\tilde{V}^* S(c) \tilde{V})^{-1} \right\|_{(k\pm1)} = \left\| (A^* A)^{-1} \right\|_{(k\pm1)} ,$$

for which it holds that

$$\frac{1}{\|(A^*A)^{-1}\|_{(k\pm1)}} = \min_{\|c\|_{(k\pm1)}=1} \|A^* A c\|_{(k\pm1)} . \tag{5.23}$$

We apply here the Cauchy-Schwarz inequality

$$|\langle A^* A c, c \rangle| \leq \|A^* A c\| \, \|c\|$$

and find that 5.23 can be bounded from below

$$\begin{aligned}
\min_{\|c\|_{(k\pm1)}=1} \|A^*Ac\|_{(k\pm1)} &\geq \min_{\|c\|_{(k\pm1)}=1} \left|\langle A^*Ac, c\rangle_{(k\pm1)}\right| = \min_{\|c\|_{(k\pm1)}=1} \left|\langle Ac, Ac\rangle_{(k\pm1)}\right| \\
&= \min_{\|c\|_{(k\pm1)}=1} \|Ac\|^2_{(k\pm1)} = \left(\min_{\|c\|_{(k\pm1)}=1} \left\|S^{\frac{1}{2}}(c)\tilde{V}c\right\|_{(k\pm1)}\right)^2 \\
&= \left(\min_{\|\tilde{V}c\|_{(k\pm1)}=1} \left\|S^{\frac{1}{2}}(c)\tilde{V}c\right\|_{(k\pm1)}\right)^2 \\
&= \left(\min_{\|d\|_{(k\pm1)}=1, d\in Im(L^{(\pm)})} \left\|S^{\frac{1}{2}}(c)d\right\|_{(k\pm1)}\right)^2 \\
&\geq \left(\min_{\|d\|_{(k\pm1)}=1} \left\|S^{\frac{1}{2}}(c)d\right\|_{(k\pm1)}\right)^2 = \left(\left\|S^{-\frac{1}{2}}(c)\right\|^2_{(k\pm1)}\right)^{-1}.
\end{aligned}$$

We have proven that

$$\frac{1}{\|(A^*A)^{-1}\|_{(k\pm1)}} \geq \left(\left\|S^{-\frac{1}{2}}(c)\right\|^2_{(k\pm1)}\right)^{-1},$$

or, equivalently,

$$\left\|(A^*A)^{-1}\right\|_{(k\pm1)} \leq \left\|S^{-\frac{1}{2}}(c)\right\|^2_{(k\pm1)}. \tag{5.24}$$

According to Proposition A.0.2 of Appendix A, this term can be rewritten as

$$\left\|S^{-\frac{1}{2}}(c)\right\|^2_{(k\pm1)} = \left\|(W_{(k\pm1)})^{-\frac{1}{2}} S^{-\frac{1}{2}}(c)(W_{(k\pm1)})^{\frac{1}{2}}\right\|^2_2 = \left\|S^{-\frac{1}{2}}(c)\right\|^2_2 = \frac{1}{\min_i \operatorname{sinc}(\tilde{V}c)_i}$$

because $S(c)$ is diagonal with positive diagonal elements.

We now remove the dependence on c by taking a maximum over $\mathcal{B}$

$$\begin{aligned}
\left\|S^{-\frac{1}{2}}(c)\right\|^2_{(k\pm1)} &\leq \max_{c\in\mathcal{B}} \frac{1}{\min_i \operatorname{sinc}((\tilde{V}c)_i)} = \frac{1}{\min_{c\in\mathcal{B}} \min_i \operatorname{sinc}((\tilde{V}c)_i)} \\
&\leq \frac{1}{\min_{x\in[-\pi/2,\pi/2]} \operatorname{sinc}(x)} = \frac{1}{2/\pi} = \frac{\pi}{2}.
\end{aligned}$$

Putting all of this together in Equation 5.22, we finally find,

$$\|f(c)\|_{(k\pm1)} \leq \frac{1}{\sigma}\frac{\pi}{2}\left\|\beta^{(\pm)}\right\|_{(k\pm1)},$$

which means that $f(c) \in \mathcal{B}$ if and only if

$$\frac{1}{\sigma}\frac{\pi}{2}\left\|\beta^{(\pm)}\right\|_{(k\pm1)} \leq \sqrt{\min(w_{(k\pm1)})}\frac{\pi}{2} \iff \sigma \geq \frac{\left\|\beta^{(\pm)}\right\|_{(k\pm1)}}{\sqrt{\min(w_{(k\pm1)})}} \triangleq \sigma^{(\pm)}_{fp}.$$

This proves that, under the condition of the theorem, $f(c)$ maps the closed ball $\mathcal{B}$ to itself and so Brouwer's theorem ensures the existence of a fixed point (i.e. a reachable equilibrium) in $\mathcal{B}$.

The asymptotic stability of the equilibrium directly comes from the fact that

$$c \in \mathcal{B} \implies \left\|\theta^{(\pm)}\right\|_{(k\pm1)} \leq \sqrt{\min(w_{(k\pm1)})}\frac{\pi}{2} \implies \left\|\theta^{(\pm)}\right\|_{\infty} \leq \frac{\pi}{2} \implies \cos\left(\theta^{(\pm)}\right) \geq 0$$

and thus the Jacobian of Equation 5.16 is negative definite. □

Two important observations should be highlighted from this result:

Observation 3.
- *it is always possible to tune the coupling strength in order for the curl-free or divergence-free component to reach equilibrium;*
- *after a certain value of the coupling strength, these equilibrium configurations always exist.*

This result also makes intuitive sense as it states that the curl-free (divergence-free) component will have equilibria when the coupling is stronger then the curl-free (divergence-free) component natural frequencies.

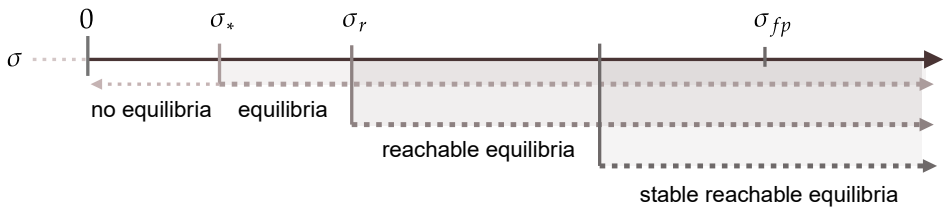


Figure 5.15: Ordering of the couplings values discussed in Section 5.3

The value of σ_{fp}, where fp stands for "fixed point", is quite significant as it often comes up when working with the simplicial Kuramoto model. We will see this later when we will concentrate on the order parameter.

It is thus worth to spend a few moments to directly look at its value. Let us consider the $(-)$ dynamics:

$$\sigma_{fp} = \left\|\beta^{(-)}\right\|_{(k-1)} = \left\|(D^{k-1})^{\dagger}\omega\right\|_{(k-1)}.$$

According to B.0.2, this can be seen as the $(k-1)$-norm of the weighted least squares solution to the linear system $D^{k-1}x = \omega$, i.e. the problem of finding the best $(k-1)$-th order signal whose coboundary is close to ω. The least squares problem is very common in engineering applications and therefore there are many efficient algorithms to solve it, with full or sparse matrices and even without the explicit computation of the pseudoinverse.

5.4 Bounding the simplicial order parameter in an equilibrium

Recall from Definition 4.1.2 that the kth order simplicial order parameter (SOP) is a scalar function of the phase cochain defined as

$$R_k^2(\theta) = \frac{1}{C_k}\left(w_{(k+1)}^\top \cos(D^k\theta) + w_{(k-1)}^\top \cos(B^k\theta)\right)$$

where $C_k = \mathbb{1}^\top w_{k+1} + \mathbb{1}^\top w_{k-1}$. Thanks to the work we did in the previous sections, it should be now clear that the two terms inside the cosines are just the projections onto upper and lower simplices Given $\theta^{(+)} = D^k\theta$ and $\theta^{(-)} = B^k\theta$. It follows that

$$R_k^2(\theta) = \frac{1}{C_k}\left(w_{(k+1)}^\top \cos\left(\theta^{(+)}\right) + w_{(k-1)}^\top \cos\left(\theta^{(-)}\right)\right).$$

Observation 4. *The simplicial order parameter R_k^2 depends only on the projections of the phases cochain onto upper and lower adjacent simplices.*

This means that the order parameter does not depend on the harmonic component of θ and is decomposed, modulo normalization constants, as the sum of two independent terms (See [18]), measuring respectively the order of the curl-free and divergence-free components.

Definition 5.4.1. *We define*

$$R_k^{(+)}(\theta) = \frac{1}{\mathbb{1}^\top w_{(k+1)}} w_{(k+1)}^\top \cos\left(\theta^{(+)}\right), \quad R_k^{(-)}(\theta) = \frac{1}{\mathbb{1}^\top w_{(k-1)}} w_{(k-1)}^\top \cos\left(\theta^{(-)}\right)$$

and call them, respectively, ***upper*** *and* ***lower order parameter****.*

Two questions naturally arises. Given an equilibrium configuration for the curl-free (or divergence-free) component, how ordered is it? Do all the equilibria have the same order? We can easily answer to both of them.

Let $\theta_{eq}^{(-)}$ be an equilibrium for the $(-)$ component $\theta_{eq}^{(-)} \in \mathcal{E}^{(-)}$. According to Definition 5.2.3 there exist $s \in \{0,1\}^{n_{k-1}}$, $m \in \mathbb{Z}^{n_{k-1}}$ and an admissible cycle $x \in \mathcal{A}^{(-)}$ such that

$$\theta_{eq}^{(-)} = (-1)^s \odot \arcsin\left(\frac{\beta^{(-)}}{\sigma^d} + y\right) + s\pi + 2\pi m.$$

If we plug this value into the expression of the lower order parameter $R_k^{(-)}$ we get

$$R_k^{(-)}\left(\theta_{eq}^{(-)}\right) = \frac{1}{\mathbb{1}^\top w_{(k-1)}} w_{(k-1)}^\top \cos\left((-1)^s \odot \arcsin\left(\frac{\beta^{(-)}}{\sigma^d} + y\right) + s\pi + 2\pi m\right) \tag{5.25}$$

$$= \frac{1}{\mathbb{1}^\top w_{(k-1)}} w_{(k-1)}^\top \left((-1)^s \odot \sqrt{\mathbb{1} - \left(\frac{\beta^{(-)}}{\sigma^d} + y\right)^2}\right), \tag{5.26}$$

where both the square and the square root act on vector in a component-wise manner.

A first observation is that the value of the order does not depend on m i.e. by which flower we are considering. This was to be expected given the 2π-periodicity of the phases. It depends, however, on the petal (Definition 5.2.6) it belongs to, established by the value of s. For example, we see that, if $\theta_{eq}^{(-)}$ belongs to the stable petal (Theorem 5.2.1) then

$$R_k^{(-)}\left(\theta_{eq}^{(-)}\right) = \frac{1}{\mathbb{1}^\top w_{(k-1)}} w_{(k-1)}^\top \sqrt{\mathbb{1} - \left(\frac{\beta^{(-)}}{\sigma^d} + y\right)^2},$$

which, as we see, depends on the admissible cycle y. Therefore, to answer the second question, *not* all equilibria have the same order and, even among the stable ones, we find variability. Moreover, we can easily prove that, for a stable equilibrium,

$$0 \leq R_k^{(-)}(\theta_{eq}^{(-)}) \leq 1$$

and

$$R_k^{(-)}(\theta_{eq}^{(-)}) = 1 \iff \frac{\beta^{(-)}}{\sigma} + y = 0$$

which, being $\beta^{(-)} \perp y$, is possible only when $\beta^{(-)} = 0$.

Just like above, the $(+)$ component on a complex with no $(k+1)$-cycles stands apart from the general case and allows us and find nice, exact results.

Proposition 5.4.1. *If $ker(B^{k+1}) = \{0\}$ then, when $\sigma^u \geq \sigma_\infty^{(+)}$, all the equilibria belonging to a petal have the same order*

$$R_k^{(+)}(\theta_{eq}^{(+)}) = \frac{1}{\mathbb{1}^\top w_{(k+1)}} \omega_{(k+1)}^\top (-1)^s \odot \sqrt{\mathbb{1} - \left(\frac{\beta^{(+)}}{\sigma^u}\right)^2}.$$

If $\theta_{eq}^{(+)}$ belongs to the stable petal ($s = 0$) therefore,

$$R_k^{(+)}(\theta_{eq}^{(+)}) = \frac{1}{\mathbb{1}^\top w_{(k+1)}} \omega_{(k+1)}^\top \sqrt{\mathbb{1} - \left(\frac{\beta^{(+)}}{\sigma^u}\right)^2}.$$

Notice how this means that, when $\beta^{(+)} = 0$, the upper order parameter of the stable configurations is strictly increasing in σ^u, with an asymptote in 1. The maximum order cannot be obtained when the natural frequencies have a nonzero divergence-free component.

Theorem 5.4.1. *If $\sigma \geq \sigma_*^{(\pm)}$, and thus $\mathcal{E}^{(\pm)} \neq \emptyset$, then we can bound the upper and lower order parameters in an equilibrium with*

$$R_k^{(\pm)}\left(\theta_{eq}^{(\pm)}\right) \leq 1 - \frac{\min(w_{(k\pm1)})}{2\mathbb{1}^\top w_{(k\pm1)}}\left(\frac{\sigma_{fp}^{(\pm)}}{\sigma}\right)^2,$$

where $\sigma_{fp}^{(\pm)}$ is defined in Theorem 5.3.5.

Proof. Let us consider an equilibrium $\theta_{eq}^{(\pm)} \in \mathcal{E}^{(\pm)}$. According to Equation 5.25 its order is

$$R_k^{(\pm)}\left(\theta_{eq}^{(\pm)}\right) = \frac{1}{\mathbb{1}^\top w_{(k\pm1)}} w_{(k\pm1)}^\top \left((-1)^s \odot \sqrt{\mathbb{1} - \left(\frac{\beta^{(\pm)}}{\sigma} + x\right)^2}\right).$$

Given that $w_{(k\pm1)} > 0$ and the square root is non-negative, we can bound $R_k^{(\pm)}$ with its value in a *stable* equilibrium

$$R_k^{(\pm)}\left(\theta^{(\pm)_{eq}}\right) \leq \frac{1}{\mathbb{1}^\top w_{(k\pm1)}} w_{(k\pm1)}^\top \sqrt{\mathbb{1} - \left(\frac{\beta^{(\pm)}}{\sigma} + x\right)^2}.$$

To obtain the bound we make use of the following inequality

$$\sqrt{1-x^2} \leq 1 - \frac{x^2}{2},$$

which is also the 2nd order Taylor expansion of $\sqrt{1-x^2}$ around 0. We get

$$\begin{aligned} R_k^{(\pm)}\left(\theta_{eq}^{(\pm)}\right) &\leq \frac{1}{\mathbb{1}^\top w_{(k\pm1)}} w_{(k\pm1)}^\top \left(\mathbb{1} - \frac{1}{2}\left(\frac{\beta^{(\pm)}}{\sigma} + x\right)^2\right) \\ &= \frac{\mathbb{1}^\top w_{(k\pm1)}}{\mathbb{1}^\top w_{(k\pm1)}} - \frac{1}{2\mathbb{1}^\top w_{(k\pm1)}} w_{(k\pm1)}^\top \left(\frac{\beta^{(\pm)}}{\sigma} + x\right)^2 \\ &= 1 - \frac{1}{2\mathbb{1}^\top w_{(k\pm1)}} \left\| \frac{\beta^{(\pm)}}{\sigma} + x \right\|_{(k\pm1)}^2 \end{aligned}$$

which, thanks to the Pythagorean theorem (being the two terms in the norm orthogonal), becomes

$$\begin{aligned} R_k^{(\pm)}\left(\theta_{eq}^{(\pm)}\right) &\leq 1 - \frac{1}{2\mathbb{1}^\top w_{(k\pm1)}} \left\| \frac{\beta^{(\pm)}}{\sigma} \right\|_{(k\pm1)}^2 - \frac{1}{2\mathbb{1}^\top w_{(k\pm1)}} \|x\|_{(k\pm1)}^2 \\ &\leq 1 - \frac{1}{2\mathbb{1}^\top w_{(k\pm1)}} \left\| \frac{\beta^{(\pm)}}{\sigma} \right\|_{(k\pm1)}^2 = 1 - \frac{\min(w_{(k\pm1)})}{2\mathbb{1}^\top w_{(k\pm1)}} \left(\frac{\sigma_{fp}^{(\pm)}}{\sigma}\right)^2. \end{aligned}$$

□

This upper bound is a continuous function of σ which converges to 1 from below as $\sigma \to +\infty$. This means that there can be no equilibria with maximum order for a finite value of σ unless $\sigma_{fp}^{(\pm)} = 0$.

It would now be useful to find a lower bound on the SOP, which would tell us the "minimum amount of synchronization" that the system has when reaching an equilibrium. Unfortunately, naive bounding approaches will only give the trivial bound for the equilibria of the stable petal $R_k^{(\pm)}(\theta_{eq}^{(\pm)}) \geq 0$. We can, however, restrict our attention to *some* equilibria, in particular the reachable, stable ones which are close to the origin, whose existence is guaranteed by Theorem 5.3.5.

Theorem 5.4.2. *If $\sigma^{(\pm)} \geq \sigma_{fp}^{(\pm)}$ then we can bound the order of the equilibria belonging to the set*

$$\left\|\theta_{eq}^{(\pm)}\right\|_{(k\pm1)} \leq \frac{\pi}{2}\sqrt{\min(w_{(k\pm1)})}$$

as

$$R_k^{(\pm)}(\theta_{eq}^{(\pm)}) \geq \cos\left(\frac{\pi}{2}\frac{\sigma_{fp}^{(\pm)}}{\sigma}\right).$$

Proof. If $\sigma \geq \sigma_{fp}^{(\pm)}$ then, according to Theorem 5.3.5, there is a stable reachable equilibrium $\theta_{eq}^{(\pm)}$ with

$$\left\|\theta_{eq}^{(\pm)}\right\|_{(k\pm1)} \leq \frac{\pi}{2}\sqrt{\min(w_{(k\pm1)})}.$$

Consider now the proof of Theorem 5.3.5 where we prove that

$$\|f(c)\|_{(k\pm1)} \leq \frac{\pi}{2\sigma}\left\|\beta^{(\pm)}\right\|_{(k\pm1)} = \frac{\pi}{2\sigma}\sigma_{fp}^{(\pm)}\sqrt{\min(w_{(k\pm1)})}.$$

It is easy to see that, if we take $\sigma > \sigma_{fp}^{(\pm)}$, we can ask for f to be contracting onto a ball with a smaller radius. That is, if $0 < a \leq \frac{\pi}{2}$,

$$\sigma \geq \frac{\pi}{2a}\sigma_{fp}^{(\pm)} \implies \|f(c)\|_{(k\pm1)} \leq a\sqrt{\min(w_{(k\pm1)})}.$$

This means that under this condition the reachable equilibria will be in the ball with center 0 and radius a. So it holds that

$$\sigma = \frac{\pi}{2a}\sigma_{fp}^{(\pm)} \implies \left\|\theta^{(\pm)}\right\|_{(k\pm1)} \leq a\sqrt{\min(w_{(k\pm1)})} \implies \left\|\theta^{(\pm)}\right\|_{\infty} \leq a \implies \cos\left(\theta_i^{(\pm)}\right) \geq \cos(a) \geq 0,$$

as $0 < a < \frac{\pi}{2}$. Finally

$$w_{(k\pm1)}^{\top}\cos\left(\theta^{(\pm)}\right) \geq \sum_i (w_{(k\pm1)})_i \cos(a) = \mathbb{1}^{\top} w_{(k\pm1)}\cos(a) = (\mathbb{1}^{\top} w_{(k\pm1)})\cos\left(\frac{\pi}{2}\frac{\sigma_{fp}^{(\pm)}}{\sigma}\right),$$

hence the thesis. □

In Figure 5.16, we see this bound plotted as a function of σ. Under the condition required by the theorem, $\sigma \geq \sigma_{fp}^{(\pm)}$, the bound is a strictly increasing function whose value is ≥ 0. Moreover it converges asymptotically to 1 from below, for $\sigma \to +\infty$. This means that there will always be a reachable stable equilibrium which is both close to the origin and whose order goes to the maximum possible value (which is 1) when σ is increased.

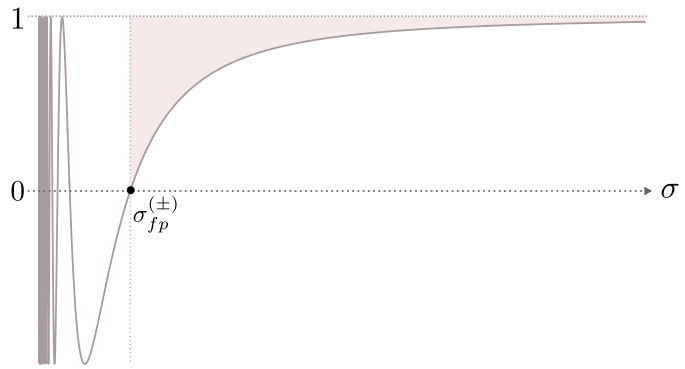


Figure 5.16: The lower bound on the order parameter of Theorem 5.4.2. The shaded region indicates where the bound holds.

Putting this together with Theorem 5.4.1, we find a nice bounding region which becomes indefinitely smaller with σ.

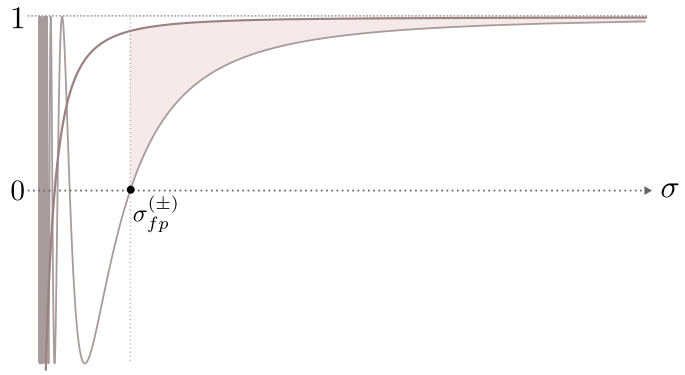


Figure 5.17: The lower bound of Theorem 5.4.2 together with the upper bound of Theorem 5.4.1.

5.5 Simplicial Sakaguchi-Kuramoto model

A particular variant of the Kuramoto model deserves particular attention. First proposed in [22], the *Sakaguchi-Kuramoto model* introduces a *phase-lag parameter* (or *frustration*) in the interactions which disrupts the natural synchronizing behavior of the Kuramoto

model. On a graph G with adjacency matrix A the model can be written as

$$\dot{\theta}_i = \omega_i - \sigma \sum_{k=1}^{n} A_{ik} \sin(\theta_i - \theta_k + \alpha_{ij}), \tag{5.27}$$

where α_{ij} is the frustration vector, defined on the edges. Intuitively, we can of the frustration as an external field changing the natural behavior of the oscillators, from striving to synchronize to searching for configurations where their phase difference is a particular value, given by α_{ij}.

It turns out that it is not possible to introduce frustrations with the incidence matrix in a trivial way, i.e. $B \sin(B^\top \theta + \alpha)$. To see this, it is enough to look at a single edge coupling two node oscillators. The incidence matrix will be $B = [1, -1]^\top$ and the resulting interaction term

$$B \sin(B^\top \theta + \alpha) = \begin{pmatrix} \sin(\theta_1 - \theta_2 + \alpha) \\ \sin(\theta_2 - \theta_1 - \alpha) \end{pmatrix},$$

meaning that the frustration acts differently for each node. We need a formulation for which the relative sign difference between the phases and the frustration is independent w.r.t the edge orientation.

This problem has been solved in [1] with a particular reformulation of the model, which also works for the simplicial case. Before analyzing it, we first explore the consequences of introducing frustrations in the "trivial" way.

5.5.1 Orientation-dependent frustration

It is a simple task to modify the simplicial Kuramoto model in order to add a frustration which disrupts the synchronization process without being orientation-independent. Given two frustration cochains $\omega_{k-1} \in C^{k-1}$, $\omega_{k+1} \in C^{k+1}$, respectively applied on $(k-1)$ and $(k+1)$-simplices we can write

$$\dot{\theta} = \omega - \sigma^d D^{k-1} \sin(B^k \theta + \omega_{k-1}) - \sigma^u B^{k+1} \sin(D^k \theta + \omega_{k+1}). \tag{5.28}$$

The equilibrium properties of this model are not so different from the one without frustration. Again, it is possible to resort to the Hodge decomposition of the dynamics

$$\begin{cases} \dot{\theta}_{cf} = \omega_{cf} - \sigma^u B^{k+1} \sin(D^k \theta_{cf} + \omega_{k+1}) \\ \dot{\theta}_H = \omega_H \\ \dot{\theta}_{df} = \omega_{df} - \sigma^d D^{k-1} \sin(B^k \theta_{df} + \omega_{k-1}) \end{cases},$$

and, equivalently, consider the projections onto upper and lower adjacent simplices

$$\theta^{(+)} = D^k \theta, \quad \theta^{(-)} = B^k \theta.$$

The dynamics of these two components are easily obtained

$$\dot{\theta}^{(\pm)} = \omega^{(\pm)} - \sigma L^{(\pm)} \sin(\theta^{(\pm)} + \omega_{k\pm1}). \tag{5.29}$$

It is clear from this last equation that the frustration will account only for a shift of the equilibrium set! In fact, following the same steps in Section 5.2, we find that every equilibrium can be written as

$$\theta^{(\pm)} = (-1)^s \odot \arcsin\left(\frac{\beta^{(\pm)}}{\sigma} + x\right) + s\pi + 2m\pi - \omega_{k\pm1}$$

for admissible $x \in \mathcal{A}^{(\pm)}$, $s \in \{0,1\}^{n_{k\pm1}}$ $m \in \mathbb{Z}^{n_{k\pm1}}$.

The situation, however, is not entirely analogous to the non-frustrated case. Looking at the dynamics of the projection 5.29 we see that, by construction, $\dot{\theta}^{(\pm)} \in Im(L^{(\pm)})$ and, given that $\theta^{(\pm)}(0) \in Im(L^{(\pm)})$, we have that $\theta^{(\pm)}(t) \in Im(L^{(\pm)})$ for all times t. This means that, while the equilibrium set $\mathcal{E}^{(\pm)}$ is translated by a vector $-\omega_{k\pm1}$, the reachable subspace stays the same. By direct consequence, the set of reachable equilibria $\mathcal{R}^{(\pm)}$ will have, in general, a different shape. See Figure 5.18 for an example.

Proposition 5.5.1. *The sets of reachable equilibria for the frustrated model 5.28 are*

$$\mathcal{R}^{(-)} = \left(\mathcal{E}^{(-)} - \omega_{k-1}\right) \cap Im(B^k), \quad \mathcal{R}^{(+)} = \left(\mathcal{E}^{(+)} - \omega_{k+1}\right) \cap Im(D^k).$$

Notice also that, if the frustration cochain belongs to the reachable subspace, $\omega_{k-1} \in Im(B^k)$ or $\omega_{k+1} \in Im(D^k)$, then the sets of reachable equilibria will just be translated. This amounts to moving $\theta_{eq}^{(\pm)}$ to the origin and then shifting it to $\theta_*^{(\pm)}$.

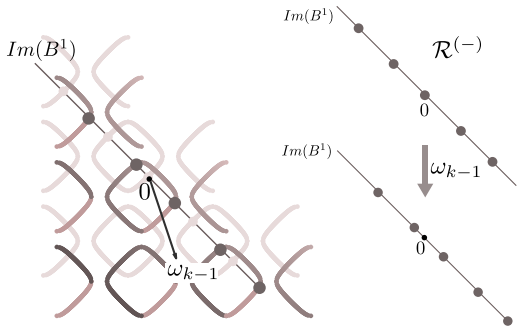


Figure 5.18: The orientation-dependent frustration acts on the equilibrium set $\mathcal{E}^{(-)}$ by translating it in $\mathbb{R}^{n_{k-1}}$. The intersection of the shifted set with $Im(B^1)$ gives the reachable equilibria.

An interesting property of such a frustration is that, by tuning it appropriately, one can turn every equilibrium into a reachable one.

Proposition 5.5.2. *If $\theta_{eq}^{(\pm)}$ is an equilibrium solution of the $(\pm)$ dynamics without frustration $\theta_{eq}^{(\pm)} \in \mathcal{E}^{(\pm)}$, then by choosing $\omega_{k\pm1} = \theta_{eq}^{(\pm)}$ one will have a frustrated system with a reachable equilibrium in the origin.*

Full synchronization (in the classical sense) is then achieved in a system which would otherwise not admit it as an equilibrium. An exemplification of this can be seen in Figure 5.19.

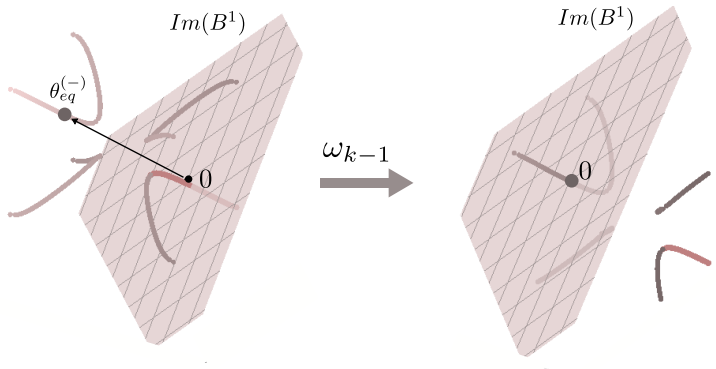


Figure 5.19: Any equilibrium $\theta_{eq}^{(\pm)} \in \mathcal{E}^{(\pm)}$ can be made reachable by choosing $\omega_{k\pm1} = \theta_{eq}^{(\pm)}$.

If $\sigma \geq \sigma_{\infty}^{(\pm)}$ then $x = 0$ is an admissible vector (Proposition 5.3.2), and thus a natural choice for $\theta_{eq}^{(\pm)}$ is

$$\theta_{eq}^{(\pm)} = \arcsin\left(\frac{\beta^{(\pm)}}{\sigma}\right) \in \mathcal{E}^{(\pm)},$$

which, by virtue of 5.2.1, also belongs to the stable petal. If we choose

$$\omega_{k\pm1} = \arcsin\left(\frac{\beta^{(\pm)}}{\sigma}\right)$$

we will have a system in which the configuration $\theta^{(\pm)} = 0$ is asymptotically stable.

In general, it is possible to make every reachable configuration an equilibrium.

Proposition 5.5.3. *Let $\theta_*^{(\pm)} \in Im(L^{(\pm)})$ be a reachable configuration and $\theta_{eq}^{(\pm)} \in \mathcal{E}^{(\pm)}$ an equilibrium. If we apply the following frustration*

$$\omega_{k\pm1} = \theta_{eq}^{(\pm)} + \theta_*^{(\pm)}$$

then $\theta_*^{(\pm)} \in \mathcal{R}^{(\pm)}$.

We conclude this section with an interesting reformulation of the trivial frustration which will come helpful in the next section. The idea is to apply the simple trigonometric identity for the sine of a sum to the interaction terms. For the "above" interaction we have:

$$\begin{aligned}
&\sigma^u B^{k+1} \sin(\theta^{(+)} + \omega_{k+1}) \\
&= \sigma^u B^{k+1} \left(\cos(\omega_{k+1}) \odot \sin(\theta^{(+)})\right) + \sigma^u B^{k+1} \left(\sin(\omega_{k+1}) \odot \cos(\theta^{(+)})\right) \\
&= \sigma^u B^{k+1} \Omega^{\cos}_{k+1} \sin(\theta^{(+)}) + \sigma^u B^{k+1} \Omega^{\sin}_{k+1} \cos(\theta^{(+)}),
\end{aligned} \tag{5.30}$$

where we defined $\Omega^{\sin}_{k+1} \triangleq diag(\sin(\omega_{k+1}))$ and $\Omega^{\cos}_{k+1} \triangleq diag(\cos(\omega_{k+1}))$. In the same way

$$\sigma^d D^{k-1} \sin(\theta^{(-)} + \omega_{k-1}) = \sigma^d D^{k-1} \Omega^{\cos}_{k-1} \sin(\theta^{(-)}) + \sigma^d D^{k-1} \Omega^{\sin}_{k-1} \cos(\theta^{(-)}). \tag{5.31}$$

Observation 5. *The trivial frustration splits the interaction term into two components:*

- *the first is the usual simplicial Kuramoto interaction, weighted*[3] *by* $\Omega^{\cos}_{k\pm1}$*;*
- *the second one has the same form, weighted by* $\Omega^{\sin}_{k\pm1}$*, except for the cosine acting on* $\theta^{(\pm)}$ *instead of the sine. Intuitively, if the sine tends to push the phases together, the cosine tends to make them as distant as possible, hence we may call it an anti-synchronization force.*

The dependence of the model on the orientation of $(k \pm 1)$-simplices is clear from Equations 5.30 and 5.31. If we change, for example, the orientation of a $(k+1)$-simplex indexed by i we will have that the boundary and coboundary operators change in the following way

$$\tilde{B}^{k+1} = B^{k+1} P, \quad \tilde{D}^k = P D^k,$$

where P is a diagonal matrix with -1 in position i and 1 everywhere else. We thus see that

$$\begin{aligned}
&\sigma^u \tilde{B}^{k+1} \Omega^{\cos}_{k+1} \sin(\tilde{\theta}^{(+)}) + \sigma^u \tilde{B}^{k+1} \Omega^{\sin}_{k+1} \cos(\tilde{\theta}^{(+)}) \\
&= \sigma^u B^{k+1} P \Omega^{\cos}_{k+1} \sin(P\theta^{(+)}) + \sigma^u B^{k+1} P \Omega^{\sin}_{k+1} \cos(P\theta^{(+)}) \\
&= \sigma^u B^{k+1} P \Omega^{\cos}_{k+1} P \sin(\theta^{(+)}) + \sigma^u B^{k+1} P \Omega^{\sin}_{k+1} \cos(\theta^{(+)}) \\
&= \sigma^u B^{k+1} P^2 \Omega^{\cos}_{k+1} \sin(\theta^{(+)}) + \sigma^u B^{k+1} P \Omega^{\sin}_{k+1} \cos(\theta^{(+)}) \\
&= \sigma^u B^{k+1} \Omega^{\cos}_{k+1} \sin(\theta^{(+)}) + \sigma^u B^{k+1} P \Omega^{\sin}_{k+1} \cos(\theta^{(+)}),
\end{aligned}$$

as P and $\Omega_{k\pm1}$, being diagonal, commute and $P^2 = I$. Notice that the first term, the "syncrhonization" force, has not changed and is therefore orientation-independent. In the second one, the "anti-syncrhonization" term, an extra P comes up, changing the effect of the frustration. The same holds for the interaction from below.

[3] *Note that they are not actual weights, as they can have negative or zero value. The intuition, however, still stands.*

5.5.2 Orientation-independent frustration

We now take a look at a smarter way of introducing frustration and see how it relates to the trivial one. Following the recipe proposed in [1], the idea is to double the simplices by mapping them to a space where they have all possible orientations, apply frustration there and then come back using a particular projection operator. To do this, we need to define a few ingredients.

Definition 5.5.1 (Lifted chain space)**.** *Let $\mathcal{X}$ be an oriented simplicial complex. We define the* ***lifted chain space*** *of order k $\mathcal{D}_k(\mathcal{X};\mathbb{R})$ as the real vector space of linear combinations of the simplices with both orientations.*

The lifted chain space contains chains of simplices with both orientations, without the identification of different oriented simplices by changing the sign, as its is done in chain spaces. For example, if $\mathcal{X} = \{a, b, c, d, [a, b], [a, c], [b, c], [b, d], [c, d], [a, b, c], [b, c, d]\}$, some lifted chains are

$$[a, b] - 2[b, a] + [b, d] + 3[d, b] \in \mathcal{D}_1(\mathcal{X};\mathbb{R}), \quad [a, b, c] + \pi[b, a, c] - [b, c, d] \in \mathcal{D}_2(\mathcal{X};\mathbb{R}).$$

It is easy to see that a natural basis is given by the simplices of both orientations and thus $dim(\mathcal{D}_k) = 2n_k$. In the same way we did in Chapter 3, we also define the **lifted cochain space** $\mathcal{D}^k(\mathcal{X};\mathbb{R})$ as the dual to $\mathcal{D}_k(\mathcal{X};\mathbb{R})$.

Given a cochain on a complex $C^k(\mathcal{X};\mathbb{R})$, we can associate a lifted cochain in a natural way, by simply associating to the simplices with different orientation the same coefficient with changed sign.

Definition 5.5.2 (Lift operator)**.** *We define the* ***lift operator*** *of order k $\mathcal{V}^k : C^k(\mathcal{X};\mathbb{R}) \to D^k(\mathcal{X};\mathbb{R})$ by its action on the basis cochains*

$$\mathcal{V}^k([v_{i_0}, \dots, v_{i_k}]) = [v_{i_0}, v_{i_1}, \dots, v_{i_k}] - [v_{i_1}, v_{i_0}, \dots, v_{i_k}].^a$$

[a] Note that, with a slight abuse of notation, we are denoting the basis cochains with the simplex they are supported on.

For example, for 1-cochains,

$$v_1(4[1,2] + [2,3]) = 4[1,2] - 4[2,1] + [2,3] - [3,2].$$

In the natural basis we have that $\mathcal{V}^k$ is represented by the following matrix

$$V^k = \begin{pmatrix} I \\ -I \end{pmatrix} \in \mathbb{R}^{2n_k \times n_k}.$$

For more details, we refer to [23], where this construction was first introduced in order to

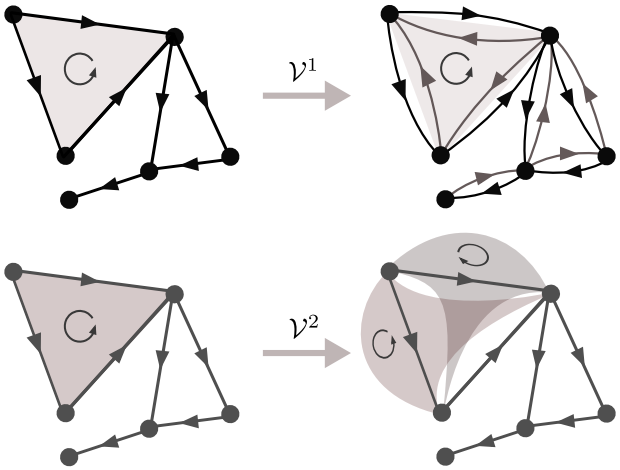


Figure 5.20: The lift operator (Definition 5.5.2) acts by doubling the simplices and changing the sign of the coefficient on the orientation-reversed simplices.

define random walks on simplicial complexes. Finally, we define the projection onto the positive/negative entries of a matrix as

$$A^{\pm} = \frac{1}{2}(A \pm |A|),$$

where $|\cdot|$ is applied element-wise.

Putting everything together, we write the **simplicial Sakaguchi-Kuramoto** model as

$$\begin{aligned}\dot{\theta} = \omega &- \sigma^u \left(B^{k+1}(V^{k+1})^\top\right)^- \sin(V^{k+1}D^k\theta + U^{k+1}\omega_{k+1}) \\ &- \sigma^d \left(D^{k-1}(V^{k-1})^\top\right)^- \sin(V^{k-1}B^k\theta + U^{k-1}\omega_{k-1}),\end{aligned} \tag{5.32}$$

where

- ω_{k-1}, ω_{k+1} are the frustration cochains respectively on interactions from below and from above;
- V^k is the lift matrix of Definition 5.5.2,
- U^k is the matrix representation of a lifting operator which, contrary to V^k, does not change the sign on the differently-oriented simplices, i.e.

$$U^k = \begin{pmatrix} I \\ I \end{pmatrix} \in \mathbb{R}^{2n_k \times n_k}.$$

Note that the formulation proposed is a slight generalization of the one in [1], as it accounts for frustration on lower order simplices as well.

Let us unpack the complicated expression of Equation 5.32. We can actually expand the lift operator and the projections onto negative elements, and find an expression which makes the behavior of the model clearer. Let us focus on the interaction term from above for simplicity.

$$\begin{aligned}
&\left(B^{k+1}(V^{k+1})^\top\right)^- \sin(V^{k+1}D^k\theta + U^{k+1}\omega_{k+1}) \\
&= \left(B^{k+1}(V^{k+1})^\top\right)^- \sin\left(\begin{pmatrix}\theta^{(+)}\\ -\theta^{(+)}\end{pmatrix} + \begin{pmatrix}\omega_{k+1}\\ \omega_{k+1}\end{pmatrix}\right) \\
&= \left((B^{k+1})^- \ (-B^{k+1})^-\right) \sin\begin{pmatrix}\theta^{(+)} + \omega_{k+1}\\ -\theta^{(+)} + \omega_{k+1}\end{pmatrix} \\
&= (B^{k+1})^- \sin(\theta^{(+)} + \omega_{k+1}) + (B^{k+1})^+ \sin(\theta^{(+)} - \omega_{k+1}),
\end{aligned}$$

because, trivially, $(-B^{k+1})^- = -(B^{k+1})^+$. If we consider the node Kuramoto, we can see that, basically, the last equation is applying the frustration ω_{k+1} on the heads of the edges and $-\omega_{k+1}$ on their tails. We can go a step further and replace the projections with their explicit forms and expand the sums inside the sine.

$$\begin{aligned}
&(B^{k+1})^- \sin(\theta^{(+)} + \omega_{k+1}) + (B^{k+1})^+ \sin(\theta^{(+)} - \omega_{k+1}) \\
&= \frac{1}{2}\left(B^{k+1} - \left|B^{k+1}\right|\right) \sin(\theta^{(+)} + \omega_{k+1}) + \frac{1}{2}\left(B^{k+1} + \left|B^{k+1}\right|\right) \sin(\theta^{(+)} - \omega_{k+1}) \\
&= \frac{1}{2}B^{k+1}\left(2\sin(\theta^{(+)}) \odot \cos(\omega_{k+1})\right) - \frac{1}{2}\left|B^{k+1}\right|\left(2\cos(\theta^{(+)}) \odot \sin(\omega_{k+1})\right) \\
&= B^{k+1}\Omega^{\cos}_{k+1}\sin(\theta^{(+)}) - \left|B^{k+1}\right|\Omega^{\sin}_{k+1}\cos(\theta^{(+)}).
\end{aligned} \tag{5.33}$$

This formulation of the interaction term looks extremely close to the one obtained with the trivial frustration 5.30. In fact, they are exactly the same, except for the fact that the orientation-dependence of the anti-synchronization term is "corrected" by taking (minus) the absolute value of the boundary matrix. A change in orientation of a $(k+1)$-simplex will not change $\cos(\theta^{(+)})$, as the cosine is even, nor will it change $\left|B^{k+1}\right|$. We can interpret this in the following way

Observation 6. *The frustration in Equation 5.32 acts as if it was on a different complex, oriented in such a way that all $(k+1)$-simplices are incoherent with their faces (hence $-\left|B^{k+1}\right|$ has all negative components). We could say that a reference system independent of the orientation of $(k+1)$-simplices is fixed for the frustration. The same is true for the frustration on lower simplices.*

If we change the projection $(\cdot)^-$ into $(\cdot)^+$ we find that it is equivalent to considering frustration on a complex where all $(k \pm 1)$-simplices are oriented *coherently* with their faces. Note that this complex may not actually exist: given an orientation to the faces of a k-simplex τ, there will almost always be no orientation for τ which is coherent with all of its faces.

5.5.3 Single oscillating simplex with frustration

Just like we did in Section 5.1, it is now worth taking a look at the simplest possible cases of orientation-independent frustration, i.e. a single oscillating simplex with frustration acting on its faces.

Let us consider an unweighted simplicial oscillator of order k with natural frequency ω and a frustration cochain $\omega_{k-1} \in \mathbb{R}^{k+1}$ on its $k+1$ faces. Let us explicitly write the simplicial Sakaguchi-Kuramoto dynamics of Equation 5.32, starting from the reformulation 5.33 (in its variant for the interactions from below).

$$\dot{\theta} = \omega - \sigma \left(D^{k-1} \Omega^{\cos}_{k-1} \sin(B^k \theta) - \left| D^{k-1} \right| \Omega^{\sin}_{k-1} \cos(B^k \theta) \right).$$

Given that we are considering unitary weights, we have that

$$B^k = \xi \in \{-1,1\}^{k+1}, \quad D^{k-1} = (B^k)^\top = \xi^\top \implies \left| D^{k-1} \right| = \mathbb{1}^\top,$$

where ξ_i is the relative orientation of face i w.r.t to the k-simplex. Substituting these into the equation gives us

$$\dot{\theta} = \omega - \sigma \left(\xi^\top diag(\cos(\omega_{k-1})) \sin(\xi\theta) - \mathbb{1}^\top diag(\sin(\omega_{k-1})) \cos(\xi\theta) \right).$$

We can take ξ out of the sine because it is an odd function and remove it in the cosine because it is even.

$$\dot{\theta} = \omega - \sigma \left(\xi^\top diag(\cos(\omega_{k-1})) \xi \sin(\theta) - \mathbb{1}^\top diag(\sin(\omega_{k-1})) \mathbb{1} \cos(\theta) \right).$$

Finally

$$\xi^\top diag(\cos(\omega_{k-1}))\xi = \sum_{i=1}^{k+1} \cos(\omega_{k-1})_i (\xi_i)^2 = \sum_{i=1}^{k-1} \cos(\omega_{k-1})_i \triangleq c,$$

$$\mathbb{1}^\top diag(\sin(\omega_{k-1}))\mathbb{1} = \sum_{i=1}^{k+1} \sin(\omega_{k-1})_i \triangleq s,$$

from which

$$\dot{\theta} = \omega - \sigma c \sin(\theta) + \sigma s \cos(\theta). \tag{5.34}$$

The first thing to notice is that the dynamics does not depend on the orientations of the faces. This is no surprise, as the simplicial Sakaguchi-Kuramoto model was explicitly built with this in mind. Moreover, the frustration cochain comes in the equation only through c and s, i.e. the sums respectively of the sines and cosines of its components. The order of the simplex k comes into play only as the number of summands. In this regard, this next proposition is particularly useful.

Proposition 5.5.4. *The constants c and s in Equation 5.34 can take all the values in the disk of radius $k+1$, i.e.*

$$c^2 + s^2 \leq (k+1)^2.$$

Proof. Let us define

$$\mathcal{C} = \left\{ \left(\sum_{i=1}^{k+1} \cos(\omega_{k-1})_i, \sum_{i=1}^{k+1} \sin(\omega_{k-1})_i \right) \in \mathbb{R}^2 : \ \omega_{k-1} \in \mathbb{R}^{k+1} \right\}$$

and

$$\mathcal{B} = \left\{ (c, s) \in \mathbb{R}^2 : \ c^2 + s^2 \leq (k+1)^2 \right\}$$

The problem can be restated in the following way: given $k+1$ points on the unit circle given by their phases $(\omega_{k-1})_i$, what are their possible vector sums? More intuitively, which points of the plane can we reach by taking exactly $k+1$ unit length steps from the origin? The claim is that *every* point in the disk of radius $k+1$ can be reached with exactly $k+1$ unit steps.

It is easy to see that $\mathcal{C} \subseteq \mathcal{B}$, as

$$\begin{aligned}
&\left(\sum_i \cos(\omega_{k-1})_i \right)^2 + \left(\sum_i \sin(\omega_{k-1})_i \right)^2 \\
&= \sum_i \left(\cos(\omega_{k-1})_i^2 + \sin(\omega_{k-1})_i^2 \right) + 2 \sum_{i<j} \left(\cos(\omega_{k-1})_i \cos(\omega_{k-1})_j + \sin(\omega_{k-1})_i \sin(\omega_{k-1})_j \right) \\
&\leq \sum_i 1 + 2 \sum_{i<j} (1+1) = (k+1) + 2 \frac{(k+1)^2 - (k+1)}{2} = (k-1)^2.
\end{aligned}$$

This means that the furthest we can go from the origin is the circle of radius $(k+1)$

Proving the converse is a bit harder. Let us choose a point in the disk $(x, y) \in \mathcal{B}$ and prove that we can reach it from the origin in $k+1$ unit steps. We write its polar coordinates (R, θ), which are, respectively, the distance from the origin and the angle. If $R = k+1$ then $(x, y) \in \partial\mathcal{B}$ and we can reach it by taking $k+1$ steps in the same direction. If $R < k+1$ we proceed in the following way (Figure 5.21):

- take $\lfloor R \rfloor - 1$ unit steps in the direction from the origin to the point;
- given that $R < k+1$ we have that $\lfloor R \rfloor - 1 \leq k-1$ and thus we have at least 2 steps left;
- take one step on the circle with radius 1 centered in (x, y);
- move on the circle until we have one step left, and then we reach the center, i.e. (x, y).

□

We can completely describe the frustrated simplicial oscillator, forgetting about $\omega_{k-1} \in \mathbb{R}^{k+1}$ and considering only the two values c and s in the disk of radius $k+1$. We reparametrized the frustration, going from $k+1$ parameters to two, without loss of information. It is natural now to ask what is the behavior of the oscillator for the different

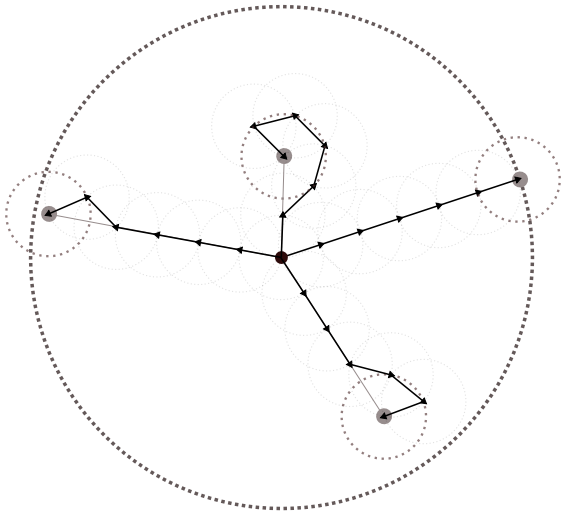


Figure 5.21: Construction of the proof of Proposition 5.5.4 for $k+1=6$.

possible values of c and s. For which ones, for example, equilibrium configurations exist?

Theorem 5.5.1. *The dynamics of the frustrated simplicial k-oscillator 5.34 admits equilibria if and only if $s^2+c^2 \geq \left(\frac{\omega}{\sigma}\right)^2$. Moreover, they are given by*

$$\theta_{eq}^{\pm} = 2\arctan\left(\frac{c \pm \sqrt{c^2+s^2-\left(\frac{\omega}{\sigma}\right)^2}}{\frac{\omega}{\sigma}-s}\right) + 2m\pi$$

if $\frac{\omega}{\sigma}-s \neq 0$, otherwise

$$\theta_{eq}^{0} = 2\arctan\left(\frac{\omega}{\sigma c}\right) + 2m\pi$$

with $m \in \mathbb{Z}$.

Proof. We need to solve the following trigonometric equation

$$\frac{\omega}{\sigma} = c\sin(\theta) - s\cos(\theta). \tag{5.35}$$

To do this, we perform the change of variable

$$\eta = \tan\left(\frac{\theta}{2}\right),$$

which allows us to write

$$\sin(\theta) = \frac{2\eta}{1+\eta^2}, \quad \cos(\theta) = \frac{1-\eta^2}{1+\eta^2}.$$

Equation 5.35, becomes

$$\begin{aligned}\frac{\omega}{\sigma} = c\frac{2\eta}{1+\eta^2} - s\frac{1-\eta^2}{1+\eta^2} &= \frac{2c\eta - s(1-\eta^2)}{1+\eta^2} \\ \frac{\omega}{\sigma}(1+\eta^2) &= 2c\eta - s(1-\eta^2) \\ \eta^2\left(\frac{\omega}{\sigma} - s\right) + \eta(-2c) + \left(\frac{\omega}{\sigma} + s\right) &= 0.\end{aligned}$$

If $\frac{\omega}{\sigma} - s = 0$ then we get a linear equation whose solution is

$$\eta^0 = \frac{\omega}{\sigma c}.$$

Otherwise, the solutions to the quadratic equation are

$$\eta^{\pm} = \frac{c \pm \sqrt{c^2 + s^2 - \left(\frac{\omega}{\sigma}\right)^2}}{\frac{\omega}{\sigma} - s},$$

defined when $c^2 + s^2 - \frac{\omega}{\sigma}^2$. We can now return to θ by taking the inverse tangent and directly find the thesis. □

This last result can be made clearer by plotting the values of θ^+_{eq}, θ^-_{eq} and θ^0_{eq} for different values of s, c such that $s^2 + c^2 \geq \left(\frac{\omega}{\sigma}\right)^2$. Figure 5.22 shows the resulting surfaces obtained for a single value of $m = 0$. We can clearly see how θ^+_{eq} and θ^-_{eq} are associated to two spiral surfaces wrapping around a central hole of radius $\left|\frac{\omega}{\sigma}\right|$ and periodically repeating as m varies. As the right panel shows, if we plot them together, we see that the curve θ^0_{eq} "stitches" the two surface together, perfectly fitting in the line where they are not defined. A single continuous surface is obtained in this way, describing, for each pair of frustration constants c, s, the position of the equilibrium phases.

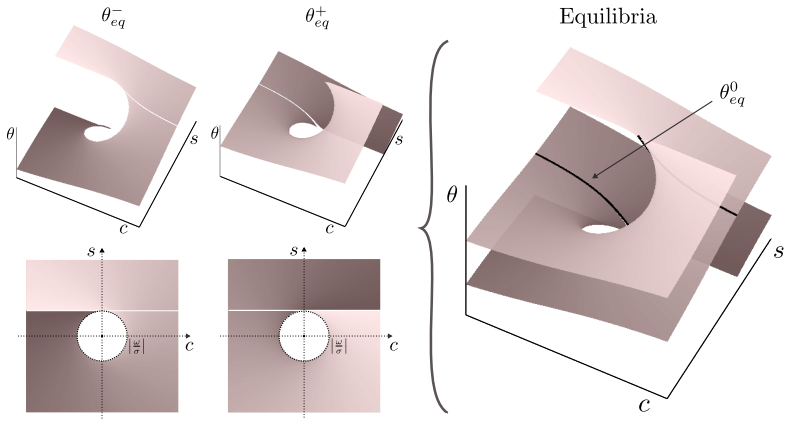


Figure 5.22: The values of θ^0_{eq} "stich" together the two sets of equilibria into a single spiral surface wrapping around the circle of radius $\left|\frac{\omega}{\sigma}\right|$.

We should not forget that, having fixed the order of the oscillator k, the possible values of c, s are constrained in the set $c^2 + s^2 \leq (k+1)^2$ (Proposition 5.5.4). We then have a situation just like the one shown in Figure 5.23, where the accessible regions for the frustration are shown for different values of k. It is easy to see that, graphically, there will be equilibria if and only if the disk with radius $k+1$ is bigger then the one with radius $\left|\frac{\omega}{\sigma}\right|$.

Proposition 5.5.5. *The frustrated simplicial oscillator admits equilibria if and only if*

$$\left|\frac{\omega}{\sigma(k+1)}\right| \leq 1.$$

Notice that this is the same condition we found for the non-frustrated single oscillator 5.1.1. The frustration, interestingly, has no effect on whether there are equilibria or not, it can only change their position. Moreover, Theorem 5.5.1 tells us that, when $c^2+s^2 = \left(\frac{\omega}{\sigma}\right)^2$, there will be a single equilibrium

$$\theta_{eq} = 2\arctan\left(\frac{c}{\frac{\omega}{\sigma} - s}\right).$$

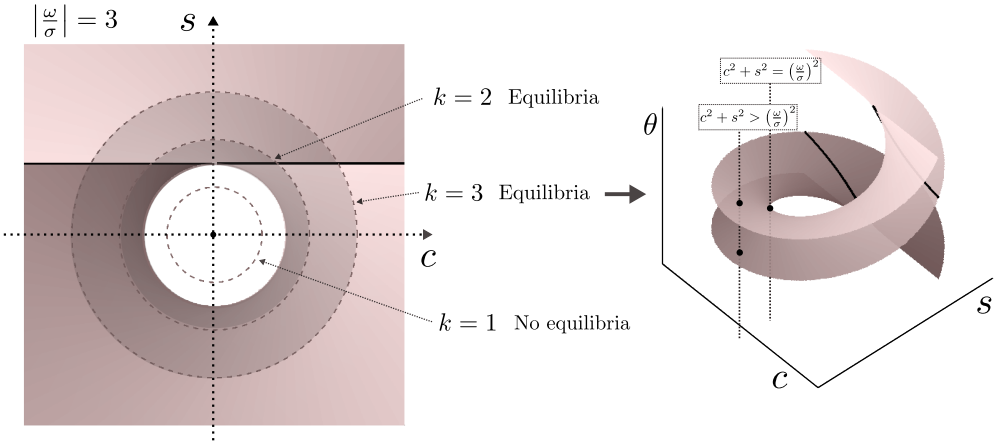


Figure 5.23: On the left: fixing the order of the oscillator constrains the possible values of c, s to lie within the disk of radius $k+1$. On the right: choosing c, s to lie exactly on the circle of radius $\left|\frac{\omega}{\sigma}\right|$ gives a single equilibrium configuration (modulo 2π). Two equilibria (modulo 2π) are associated to c, s chosen outside the circle.

We can ask, finally, whether these equilibria are stable or unstable.

Proposition 5.5.6. *If $c^2 + s^2 > \left(\frac{\omega}{\sigma}\right)^2$ then θ_{eq}^- is asymptotically stable and θ_{eq}^+ in unstable.*

Proof. To prove this we need to compute the Jacobian and look at its sign.

$$J(\theta) = \frac{\partial}{\partial\theta}(\omega - \sigma c\sin(\theta) + \sigma s\cos(\theta)) = -\sigma(c\cos(\theta) + s\sin(\theta)).$$

We now need to plug in $\theta_{eq}^{\pm}$ of Theorem 5.5.1 inside J and take advantage of the trigonometric identities

$$\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}, \quad \cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}.$$

After some long but simple algebraic steps we get

$$J(\theta_{eq}^{\pm}) = \pm\sigma\sqrt{c^2 + s^2 - \left(\frac{\omega}{\sigma}\right)^2},$$

which is strictly negative for θ_{eq}^{-} and positive for θ_{eq}^{+}. □

Putting this result together with the easily proven fact that θ_{eq}^{0} is asymptotically stable when $c > 0$ and unstable when $c < 0$, we get that the spiral surfaces on the left of Figure 5.22 are respectively stable and unstable. For any couple of frustration constants c, s which do not lie on the circle with radius $|\omega/\sigma|$, the system will have a pair of equilibria, one stable and one unstable, just like the non frustrated case.

6 | Conclusions

In this thesis, we gave a preliminary analytical study of the simplicial Kuramoto model. Focusing on its equilibrium properties, we were able to set up a theoretical framework which can help to understand it in detail. We recognized its difference from the standard Kuramoto in the behavior of a single simplicial oscillator, showing how, under certain conditions, it is not self-sustaining. The Hodge decomposition allowed us to reveal the three-fold nature of the dynamics, unveiling the beautiful geometry of the equilibrium structures of the projected dynamics. Looking at them explicitly allowed us both to formulate optimal necessary conditions for phase-locking (Theorem 5.17), give a geometrical interpretation to the reachability phase transition and understand the action of the orientation-dependent frustration (Proposition 5.5.1). In the same way, we were able to give upper and lower bounds for the order parameter in an equilibrium configuration. As it often happens in mathematics, the simplicial generalization sheds new light on the meaning and behavior of the standard Kuramoto. Well known results, like the sufficient bound for phase locking (Theorem 5.3.5), were generalized, while others, like the exact value of the phase transition for trees (Theorem 5.3.4), were proven in a different way. Finally, we extended the orientation-independent Sakaguchi-Kuramoto model [1] to include frustration on lower simplices and, by expanding its expression, gave an interpretation to its behavior.

There is, however, much left to do. Many aspects of the model, which have not been adequately discussed here, deserve further investigations in future works. Some ideas follow.

- First, the shape of the equilibrium structures needs to be examined in more detail, especially for higher-dimensional cases. The computations of persistent homology of the equilibrium structures could help in those situations in which visualization is not possible. On the same note, the equilibria for the simplicial Sakaguchi-Kuramoto model (Equation 5.32) have not yet been computed, mainly because, in that case, Hodge decomposition does not produce decoupled dynamics. If an analytical approach is not feasible, numerical experiments showing how the frustration reshapes the equilibrium sets could certainly be interesting.

- Numerical experiments show that reachable equilibria are present for values much lower than σ_{fp}. Almost always we see that σ_∞, whose value does not depend explicitly on the number of simplices, is enough to guarantee it. This suggests that

better sufficient bounds for reachability can be found, if not in general, at least for particular classes of complexes and natural frequency cochains.

- Finally, it is easy to see that the coupling strengths σ^u,σ^d do not need to be distinct elements of the model, as they could perfectly be incorporated as rescalings of the weights. We could therefore move on to a coupling-agnostic formulation and study, with more generality, the impact of weights on the properties of the model.

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A | Linear algebra with inner products

When working with finite dimensional Hilbert spaces, it is often useful to fix a basis and pass to a matrix representation. In this way, elements of the space become finite dimensional vectors and the inner product is represented by a positive definite matrix $\langle x, y\rangle = x^\top G y$. What one usually do is then change the basis in order to reshape the inner product into the standard Euclidean one. Sometimes, however, it is not possible or convenient to perform this change, and we must stick with a particular basis. In this case, we need to adopt a different kind of linear algebra, often neglected, which takes into consideration the presence of a different inner product.

Let us consider two finite-dimensional Hilbert spaces which can be represented by $\mathbb{R}^n$ and R^m together with inner products given by the symmetric, positive definite matrices W, G. If $A \in \mathbb{R}^{n\times m}$ is a matrix representing a linear operator between the two vector spaces $A : (\mathbb{R}^m, G) \to (\mathbb{R}^n, W)$, then we write its **adjoint** as

$$A^* = G^{-1}A^\top W \in \mathbb{R}^{m\times n}.$$

A is **self-adjoint** (or **Hermitian**) if $A^* = A$. The norm of a vector $x \in \mathbb{R}^m$ is defined as

$$\|x\|_G \triangleq \sqrt{x^\top G x}.$$

From this, we can immediately establish a relation between the G-norm and the familiar Euclidean norm

Proposition A.0.1.

$$\|x\|_G = \left\|G^{\frac{1}{2}}x\right\|_2$$

Proof.

$$\|x\|_G = \sqrt{x^\top G x} = \sqrt{x^\top G^{\frac{1}{2}} G^{\frac{1}{2}} x} = \sqrt{(G^{\frac{1}{2}}x)^\top G^{\frac{1}{2}}x} = \left\|G^{\frac{1}{2}}x\right\|_2.$$

□

A similar result can be found for the matrix norm.

Proposition A.0.2.

$$\|A\|_G = \left\|W^{\frac{1}{2}} A G^{-\frac{1}{2}}\right\|_2 .$$

Proof.

$$\begin{aligned}\|A\|_G &= \sup_{\|x\|_G=1} \|Ax\|_W = \sup_{\left\|G^{\frac{1}{2}}x\right\|_2=1} \left\|W^{\frac{1}{2}} A x\right\|_2 \qquad (y \triangleq G^{\frac{1}{2}}x)\\ &= \sup_{\|y\|_2=1} \left\|W^{\frac{1}{2}} A G^{-\frac{1}{2}} y\right\|_2 = \left\|W^{\frac{1}{2}} A G^{-\frac{1}{2}}\right\|_2\end{aligned}$$

□

Being $\|\cdot\|_G$ the operator norm, it also holds that

Proposition A.0.3.

$$\|Ax\|_W \leq \|A\|_W \|x\|_W \quad \forall x \in \mathbb{R}^m.$$

When performing the eigendecomposition of a matrix, it is important to take into consideration the inner product as well.

Theorem A.0.1. *Let $B \in \mathbb{R}^{m\times m}$ be a Hermitian matrix. There exists two matrices V, Λ such that*

$$B = V\Lambda V^* = V\Lambda G^{-1} V^\top G,$$

where Λ is a diagonal matrix containing the (real) eigenvalues of B and V is unitary $V^*V = VV^* = I$.

When the image of B does not coincide with the entire $\mathbb{R}^m$, we will have that $r \triangleq rank(B) < m$ and Λ will have only r nonzero elements in its diagonal. In this case, we can pass to a "reduced" eigendecomposition by considering only the nonzero eigenvalues.

$$\begin{aligned}B = V\Lambda V^* &= \begin{pmatrix} v_1 & \dots & v_m \end{pmatrix} \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} (v_1^*)^\top \\ \vdots \\ (v_m^*)^\top \end{pmatrix} \\ &= \begin{pmatrix} v_1 & \dots & v_r \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{pmatrix} \begin{pmatrix} (v_1^*)^\top \\ \vdots \\ (v_r^*)^\top \end{pmatrix} = \tilde{V}\tilde{\Lambda}\tilde{V}^*,\end{aligned}$$

where $\tilde{V} \in \mathbb{R}^{m \times r}, \Lambda \in \mathbb{R}^{r \times r}$. This rewriting is particularly useful because it holds that the columns of $\tilde{V}$ are a basis for $Im(B)$. Moreover, when G is a diagonal matrix, it is easy to prove that

$$\tilde{V}^* = \tilde{G}^{-1}\tilde{V}^\top G,$$

where $\tilde{G} \in \mathbb{R}^{r \times r}$ contains only the elements of G corresponding to nonzero eigenvalues. In this way, we can consider $\tilde{V}$ as an operator from the Hilbert space of coefficients $(\mathbb{R}^r, \tilde{G})$ to $(\mathbb{R}^m, G)$ and $\tilde{V}^*$ is its adjoint. The fact that V is unitary translates to

Proposition A.0.4.
- $\tilde{V}^*\tilde{V} = I$;
- $\tilde{V}\tilde{V}^*$ *is the* G*-orthogonal projection operator onto* $Im(B)$.

Proof. We prove the second using the characterizing property of an orthogonal projection operator: P is an orthogonal projection w.r.t an inner product if and only if $P^2 = I$ and $\langle x - Px, Px \rangle = 0 \ \forall x$. First

$$(\tilde{V}\tilde{V}^*)^2 = \tilde{V}\tilde{V}^*\tilde{V}\tilde{V}^* = \tilde{V}\tilde{V}^*.$$

And secondly

$$\left\langle x - \tilde{V}\tilde{V}^*x, \tilde{V}\tilde{V}^*x \right\rangle_G = \left\langle \tilde{V}^*x - \tilde{V}^*\tilde{V}\tilde{V}^*x, \tilde{V}^*x \right\rangle_{\tilde{G}} = \left\langle \tilde{V}\tilde{V}^*x - (\tilde{V}\tilde{V}^*)^2x, x \right\rangle_G = 0 \ \ \forall x,$$

by the adjointness relation of $\tilde{V}$ and $\tilde{V}^*$. □

B | Moore-Penrose pseudoinverse

In this second Appendix, we give a quick introduction to the concept of Moore-Penrose pseudoinverse and its fundamental properties without proofs. This is by no means a complete exposition, but just a reference to be used in the rest of the thesis.

The Moore-Penrose pseudoinverse is the most known generalization of the concept of inverse, which can be applied to any matrix, regardless of whether it is invertible or even square.

Let us consider two finite-dimensional Hilbert spaces which can be represented by $\mathbb{R}^n$ and R^m together with inner products given by the symmetric, positive definite matrices W, G (see Appendix A).

Definition B.0.1. *Given a matrix $A \in \mathbb{R}^{n\times m}$, we call the **Moore-Penrose pseudoinverse** of A a matrix $A^\dagger \in \mathbb{R}^{m\times n}$ such that it satisfies the following properties*

- $A^\dagger A A^\dagger = A^\dagger$,
- $A A^\dagger A = A$,
- $A A^\dagger$ *is Hermitian,*
- $A^\dagger A$ *is Hermitian.*

From this definition it is easy to see that, when $m = n$ and $G = W = I$ (meaning that the adjoint is the transpose), the pseudoinverse of an invertible matrix $A \in \mathbb{R}^{n\times n}$ is just its inverse $A^{-1} = A^\dagger$:

- $A^{-1} A A^{-1} = A^{-1} I = A^{-1}$,
- $A A^{-1} A = A I = A$,
- $A A^{-1} = I = I^\top$,
- $A^{-1} A = I = I^\top$.

When A is singular or even non-square, it can be proven that a pseudoinverse that satisfies the properties of the definition always *exists and is unique*.

We now give a list of important properties without proof.

1. $(A^\dagger)^\dagger = A$;
2. $(A^\top)^\dagger = (A^\dagger)^\top$;
3. $(A^*)^\dagger = (A^\dagger)^*$;
4. $(cA)^\dagger = \frac{1}{c}A^\dagger$ for any scalar $c \in \mathbb{R}$, $c \neq 0$;
5. $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$;
6. $A = AA^*(A^\dagger)^* = (A^\dagger)^* A^* A$;
7. $Im(A^\dagger) = Im(A^*)$;
8. $ker(A^\dagger) = ker(A^*)$;
9. $AA^\dagger$ is the orthogonal[1] projection matrix onto $Im(A)$;
10. $A^\dagger A$ is the orthogonal[2] projection matrix onto $Im(A^*)$;
11. $I - AA^\dagger$ is the orthogonal projection matrix onto $ker(A^*)$;
12. $I - A^\dagger A$ is the orthogonal projection matrix onto $ker(A)$.

The pseudoinverse can be used to give an explicit expression to all the solutions of a linear system.

Theorem B.0.1. *If the system $Ax = b$ admits solutions, then they can be written as*

$$x = A^\dagger b + (I - A^\dagger A)c$$

for every $c \in \mathbb{R}^m$.

It can be shown that the first term is the solution with minimum norm and the second term, by virtue of Property 12, captures the vector space of homogeneous solutions $Ax = 0$ i.e. $ker(A)$.

The pseudoinverse is also useful when the system admits no solutions.

[1] w.r.t the inner product given by W

[2] w.r.t the inner product given by G

Theorem B.0.2. *If the system $Ax = b$ admits no solutions then*

$$x = A^\dagger b$$

is its least squares solution, i.e.

$$A^\dagger b = \underset{x \in \mathbb{R}^n}{argmin} \|Ax - b\|_W .$$

This directly comes from the fact that $AA^\dagger b$, because of Property 9, is the orthogonal projection of b onto $Im(A)$.

Finally, we state a useful result which relates the pseudoinverse of a Hermitian, positive definite matrix to its eigendecomposition (Theorem A.0.1).

Theorem B.0.3. *Let $B \in \mathbb{R}^{m \times m}$ be a positive definite Hermitian matrix such that its reduced eigendecomposition is*

$$B = \tilde{V}\tilde{\Lambda}\tilde{V}^*.$$

Then its pseudoinverse can be written as

$$B^\dagger = \tilde{V}\tilde{\Lambda}^{-1}\tilde{V}^*.$$

Proof. By Definition B.0.1.

1. $B^\dagger B B^\dagger = \tilde{V}\tilde{\Lambda}^{-1}\cancelto{I}{\tilde{V}^*\tilde{V}}\tilde{\Lambda}\cancelto{I}{\tilde{V}^*\tilde{V}}\tilde{\Lambda}^{-1}\tilde{V}^* = \tilde{V}\tilde{\Lambda}^{-1}\tilde{\Lambda}\tilde{\Lambda}^{-1}\tilde{V}^* = \tilde{V}\tilde{\Lambda}^{-1}\tilde{V}^* = B^\dagger$;
2. $B B^\dagger B = \tilde{V}\tilde{\Lambda}\cancelto{I}{\tilde{V}^*\tilde{V}}\tilde{\Lambda}^{-1}\cancelto{I}{\tilde{V}^*\tilde{V}}\tilde{\Lambda}\tilde{V}^* = \tilde{V}\tilde{\Lambda}\tilde{\Lambda}^{-1}\tilde{\Lambda}\tilde{V}^* = \tilde{V}\tilde{\Lambda}\tilde{V}^* = B$;
3. $BB^\dagger = \tilde{V}\tilde{\Lambda}\cancelto{I}{\tilde{V}^*\tilde{V}}\tilde{\Lambda}^{-1}\tilde{V}^* = \tilde{V}\tilde{V}^*$ is Hermitian because $(\tilde{V}\tilde{V}^*)^* = (\tilde{V}^*)^*\tilde{V}^* = \tilde{V}\tilde{V}^*$;
4. $B^\dagger B = \tilde{V}\tilde{\Lambda}^{-1}\cancelto{I}{\tilde{V}^*\tilde{V}}\tilde{\Lambda}\tilde{V}^* = \tilde{V}\tilde{V}^*$ is Hermitian because $(\tilde{V}\tilde{V}^*)^* = (\tilde{V}^*)^*\tilde{V}^* = \tilde{V}\tilde{V}^*$.

□