POLITECNICO DI TORINO

Master Degree course in
Physics of Complex Systems

Master Degree Thesis

Non-equilibrium statistical mechanics of endogenous fluctuations in systems of heterogeneous interacting agents

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A mio padre

Summary

In this work it is discussed a model of heterogeneous agents inspired by the El Farol Bar problem and the Minority Game which reproduce the main stylized facts of anomalous collective fluctuations in finance.

The rules are simple: a finite number of players have to choose between two sides and the quantity they want to bet; whoever ends up in the minority side is a winner. By definition all players cannot be winners, and therefore they should not agree with each other as to what is the best strategy. Thus the Minority Game punishes herding and rewards diversity; the players make do with only a limited information processing capacity. Although this game is a generic model of competition between adaptive agents, it has attracted much attention as a model of financial markets, bringing appealing insights into market dynamics.

As in financial markets, the agents interact each other through a collective variable - that play the role of price - such that its value depends on the choice of the players at each time. Stock market prices are characterized by anomalous fluctuations, known as stylized facts, these are:

- the global efficiency, that measure the coordination among the agents
- the predictability, that measure the symmetry of the game

It is first introduced the Poisson Minority Game, whose dynamics is described by a set of deterministic equation and it is studied the role of different types of agents: producers and speculators.
Producers are traders who use the market for exchanging goods, their decision originate from outside and not on the market dynamics.
Speculators trade to gain from market fluctuations.

Compared with the classical Minority Game, here it is discussed a version in which the agents not only choose to “buy” or “sell”, but also the number of shares they bid, that it is assumed to be drawn from a Poisson distribution. A variable ϵ, that models the incentives of agents for trading in the market, is also added.

Then the stochastic differential equations for continuum time limit of the dynamics are derived. It is shown that this dynamics admits a Lyapunov function, that is a function which is minimized along the trajectories of the dynamics of the system. So the problem of studying the stationary state of a stochastic dynamical system turn into the characterization of the local minima of a function, considered as an Hamiltonian, the stationary state of the system corresponds to the ground state of the Hamiltonian, which can be computed analytically in the relevant thermodynamic limit.

Using the assumption of Poisson distribution, it has been derived, in the stationary state, a relation between the collective behavior. Numerical results gives evidence of the accuracy of the association.

The minimization of the Hamiltonian of the system has been done by using the Replica Method and it is shown that the analytical solution and the result found by numerical simulation coincide in the stationary state. In particular in the limit $\epsilon \to 0$, the minimization of the Hamiltonian reveals that the collective behaviour of the Minority Game features a phase transition as a function of the number N of agents. When there are less agents than the critical number, the price evolution seems predictable to an external agent (but not to those already playing), whereas when the number of agents is beyond the critical number, the market becomes unpredictable.

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Non cercare di diventare
un uomo di successo,
ma piuttosto un uomo di valore.
[A. Einstein]

Part I

Prima Parte

Chapter 1

Introduction

Minority game can be regarded as an Ising Model for system of heterogeneous adaptive agents, who interact through a global mechanism which involves competition for limited resources as in financial markets. For this reason it qualifies as a complex systems.

Markets are institutions which allow agents to exchange goods. Through these exchanges, traders can reallocate their resources in order to increase their profit. Since trading itself does not create wealth, the market must be a zero sum game. Taking out transaction costs and other frictions which are needed to reward the market maker for the service he is providing, the game becomes a Minority Game, i.e. a setting where only a minority of agents can win

Comparing real financial market with the Minority Game it is obvious that the latter cannot describe the complexity of the former in all its conditions and regimes, but the Minority Game can be considered a faithful picture of a financial market. The key observation is that the Minority Game is tailored to study fluctuation phenomena and their statistical properties. Statistical physics suggests that the collective behaviour of a system of many interacting units, is qualitatively rather indifferent to microscopic details.

Stock market prices are characterized by anomalous collective fluctuations which are strongly reminiscent of critical phenomena.The connection with critical phenomena is natural, because financial markets are indeed complex systems of many interacting degrees of freedom, the traders.
Statistical mechanics of disordered systems provides analytical and numerical tools for the description of complex systems, in particular analytic approaches provide exact result for the limit of infinitely many agents.

Minority Games provide a natural microscopic explanation for the volatility correlations found in real markets. Hence the Minority Game offers a broad picture of how financial markets operate, consistent with empirical data.

Chapter 2

Poisson Minority Game

2.1 Model

In the market described by the Minority Game a large number N of agents have to make one of two opposite actions such as e.g. "buy" or "sell" in every period $t = 1, 2, ...$ and only those agents who choose the minority action are rewarded.

In order to model this situation, let $\mathcal{N} = (1, ..., N)$ be the set of agents and let $\mathcal{A} = (-1, +1)$ be the set of the two possible actions.
The actions of agents depend on the value $\mu(t)$ of a public information variable which can be in one of P states: $\mu \in \mathcal{P} = (1, ..., P)$.

The variable μ encodes all possible information on the state of the environment where agents live, so sometimes μ is called "information". Here it is supposed that μ is drawn from a uniform distribution on $\mathcal{P}$.

We assume that P is large and of the same order of N such that:

$$\alpha = \frac{P}{N}$$

is finite in the limit $N \to \infty$
Let's define the $\mathcal{A}^{\mathcal{P}}$ the set of all strategies: an element of $\mathcal{A}^{\mathcal{P}}$ is a function:

$$a : \mu \in \mathcal{P} \longrightarrow a^{\mu} \in \mathcal{A}$$

that is P dimensional vector with coordinates a^μ, $\forall \mu \in \mathcal{P}$. There are $|\mathcal{A}^\mathcal{P}| = 2^P$ possible function. The quantity $a_i \in \mathcal{A}^\mathcal{P}$ is called a possible pure strategy for agent $i \in \mathcal{N}$, with elements $a_i^\mu \in \mathcal{A}$ for all $\mu \in \mathcal{P}$.

Here it is assumed that the strategies a_i^μ of traders are randomly and independently drawn (with replacement) from $\mathcal{A}^\mathcal{P}$. More precisely:

$$P(a_i^\mu = +1) = P(a_i^\mu = -1) = \frac{1}{2}, \quad \forall i \in \mathcal{N}, \mu \in \mathcal{P} \tag{2.1}$$

Note that independence of $\vec{a_i}$ across agents is reasonable because μ is a sunspot and no pre-play communication is possible (agents are assigned their $\vec{a_i}$ before the game starts).

If $a_i^{\mu(t)} \in \mathcal{A}$ is the action of agent $i \in \mathcal{N}$, the payoff of agent i is given by:

$$U_i(t+1) = U_i(t) - a_i^{\mu(t)} \frac{A(t)}{P} - \epsilon \tag{2.2}$$

Where

$$A(t) = \sum_i a_i^{\mu(t)} n_i(t) \tag{2.3}$$

this is a global or aggregate quantity which is produced by all players. This type of interaction is typical of market systems and it is similar to the long-range interaction assumed in mean-field models of statistical physics.

The factor n_i describes the number of shares taken by agent i.

The threshold ϵ is such that the agents will gain if their payoff is larger then it. This quantity can be seen as a benchmark, it models the incentives of agents for trading in the market.

Let's define the following variable:

$$\lambda_i(t) = \max(0\ ;\ \Gamma U_i(t)) \tag{2.4}$$

Where $\Gamma > 0$ represents the percentage of payoff that agents bid.
According to this definition, here it is supposed that:

$$P(n_i(t) = k) = \frac{\lambda_i(t)^k}{k!} exp(-\lambda_i(t)), \quad \forall i \in \mathcal{N} \tag{2.5}$$

that is, at each time step, the quantity played by each agents is drawn from a Poisson distribution with parameter $\lambda_i(t)$

In order to see that the game rewards the minority group notice that the MG interaction is described by the logical XOR function:

$\text{sign}(a_i(t))$	$\text{sign}(A(t))$	$\text{sign}(\Delta U_i + \epsilon)$
-	-	-
-	+	+
+	-	+
+	+	-

Table 2.1: The Poisson Minority Game interaction.

The agents from the minority (who took the action $a(t) = -\text{sign}(A(t))$) are rewarded with a gain $|A(t)|$, and those from the majority (who took the action $a(t) = \text{sign}(A(t))$) are punished by a loss $-|A(t)|$.

2.2 Speculators and Producers

Real markets are not zero sum games D. Challet [2001]. In real markets the participants can be divided into two groups: speculators and producers. Producers can be characterized by those using the market for purposes other than speculation, their trading strategy originate from outside opportunities related to the economic activity and not on the market dynamics itself. They need market for hedging, financing, or any ordinary business. They thus pay less or no attention to "timing the market".
Speculators, on the other hand, join the market with the aim of exploiting the marginal profit pockets, their aim is to gain from market fluctuations.

The producers inject information into the market prices, and the speculators make a living exploiting this information. The reason why producers let themselves be taken advantage of is that they have other business in mind. To conduct their business, they need the market, and their expertises and talents in other areas give them still better games to play.
Speculators make do exploiting the "meager margin" left in the competitive market.

In this version of MG, producers will be limited in choice, their activities

outside the game are not represented. Thus, the they have a fixed pattern in their market behavior and put a measurable amount of information into the market, which is exploited by the speculators.
In this thesis, following D. Challet [2000], it will be considered a population of N speculators and $\theta^2 N$ heterogeneous producers, so that θ^2 is the fraction of producers per speculator. The actions producers can do is just "buy" or "sell" and the quantity they bid is restricted to 1. The outcome is then:

$$A(t) = A(t)_{spec} + A_{prod}^{\mu(t)} = \sum_{i}^{N} a_i^{\mu(t)} n_i(t) + A_{prod}^{\mu(t)}$$

where $A_{prod}^{\mu(t)} \sim G(0, \theta\sqrt{N})$ that is a Gaussian random variable with zero mean and variance $\theta^2 N$, that it is rewritten as:

$$A_{prod}^{\mu(t)} = \sqrt{N} A_0^{\mu(t)}, \quad \text{with} \quad A_0^{\mu(t)} \sim G(0, \theta)$$

Producers always benefit from the presence of speculators, and reversely: both types of agents live in symbiosis. Indeed, the producers introduce systematic biases into the market, and without speculators, their losses would be proportional to these biases. The speculators precisely try to remove this kind of bias, reducing also systematic fluctuations in the market, thus reducing the losses of the producers and their own losses.

2.3 Characterization of collective behavior

Before entering in the analysis of the game, here is presented the notation that it will be used in the next sections.
First of all the average of a measure O over the information μ is given by:

$$\bar{O} = \frac{1}{P} \sum_{\mu=1}^{P} O^{\mu} \tag{2.6}$$

The average over the Poisson distribution:

$$\langle O(t) \rangle = \sum_{k=0}^{\infty} P(O(t) = k) k \tag{2.7}$$

where $O(t)$ is distributed according to (2.5).

As a preliminary it is useful to introduce the key quantities which describe the collective behavior, full description in D. Challet [2005]

Symmetry arguments suggest that none of the two groups $\text{sign}(A(t)) = -1$ or $\text{sign}(A(t)) = 1$ will be systematically the minority one.
This means that $A(t)$ will fluctuate around zero and $\langle A \rangle = 0$.
The size of fluctuations of $A(t)$, instead, displays a remarkable non-trivial behaviour. The variance

$$\sigma^2 = \overline{\langle A^2 \rangle} = \frac{1}{P} \sum_{\mu=1}^{P} \left\langle \left(\sum_j a_j^\mu n_j(t) + \sqrt{N} A_0^\mu \right)^2 \right\rangle \tag{2.8}$$

of $A(t)$ in the stationary state is a measure of how effective the system is at distributing resources. The smaller σ^2 is, the larger a typical minority group is, that correspond to an efficient coordination among agents. In other words σ^2 is a reciprocal measure of the global efficiency of the system, that measures market's fluctuation.
Numerical simulations of the Global efficiency for the Poisson Minority game are shown below:

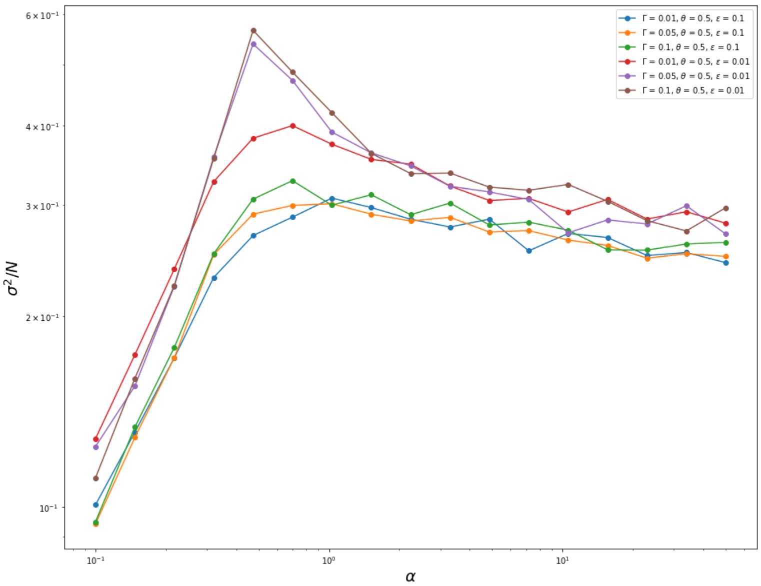

Figure 2.1: Global efficiency σ^2/N as a function of $P/N = \alpha$ for $\Gamma = 0.01$, $0.05, 0.1$; $\theta = 0.5$; $\epsilon = 0.01, 0.1$; averaged over 70 realizations of the game.

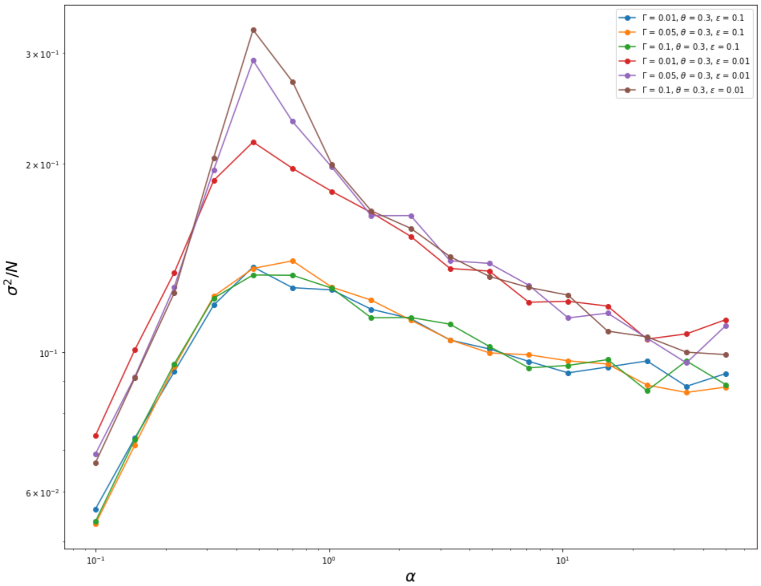

Figure 2.2: Global efficiency σ^2/N as a function of $P/N = \alpha$ for $\Gamma = 0.01$, $0.05, 0.1$; $\theta = 0.5$; $\epsilon = 0.01, 0.1$; averaged over 70 realizations of the game.

In both values of $\theta = 0.5, 0.3$ it can be seen an increasing trend in the value of the global efficiency when the threshold ϵ decrease. This is due to the fact that there exist a critical value for α, that it will be calculated later that is $\alpha_c = 1/2$ in which the system undergo trough a phase transition.

In the asymmetric phase, $\langle A|\mu\rangle \neq 0$ for at least one μ. Hence knowing the history $\mu(t)$ at time t, makes the sign$(A(t))$ statistically predictable. As a measure of the asymmetry, it is introduced the predictability:

$$H = \overline{\langle A \rangle^2} = \frac{1}{P} \sum_{\mu=1}^{P} \left(\sum_j a_j^\mu \langle n_j(t) \rangle_j + \sqrt{N} A_0^\mu \right)^2 \tag{2.9}$$

In the symmetric phase $\langle A|\mu\rangle = 0$ for all μ and hence $H = 0$, that correpsond to the situation in which the market is unpredictable or informationally efficient.

For the same values of the parameters before, numerical simulations of the Predictability for the Poisson Minority game are shown below:

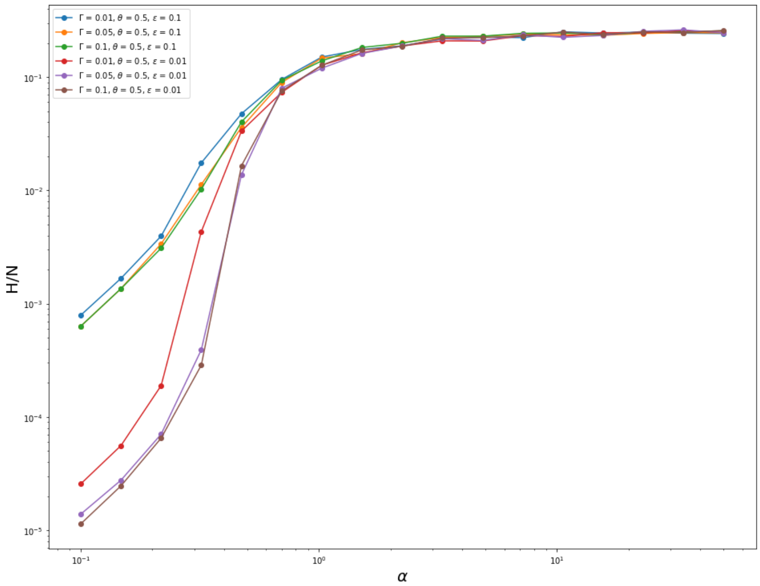

Figure 2.3: Predictability H/N as a function of $P/N = \alpha$ for $\Gamma = 0.01$, $0.05, 0.1$; $\theta = 0.5$; $\epsilon = 0.01, 0.1$; averaged over 70 realizations of the game.

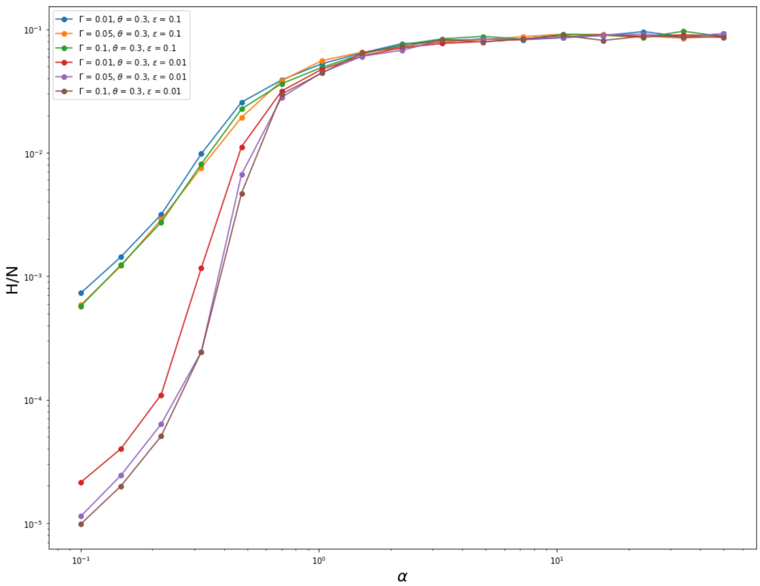

Figure 2.4: Predictability H/N as a function of $P/N = \alpha$ for $\Gamma = 0.01$, $0.05, 0.1$; $\theta = 0.5$; $\epsilon = 0.01, 0.1$; averaged over 70 realizations of the game.

In this case, instead, numerical simulation reveal that the Predictability decrease when ϵ goes down. In particular what it will proven is that for $\epsilon \to 0$ and values of $\alpha < \alpha_c = 1/2$ the quantity $H/N = 0$ that is the market becomes unpredictable.

Both the quantities σ^2 and H are extensive, that is, are proportional to N for $N \to \infty$, so the case of interested is in the finite quantities $\frac{\sigma^2}{N}$ and $\frac{H}{N}$.

Part II

Seconda Parte

Chapter 3

Continuum time limit

3.1 Continuum time limit and the dynamics in the long run

In this chapter, it is first derived a continuum time dynamics for (2.2) which captures the long run behavior of the system.

Then it will be shown that the collective behaviour of agents, within this continuum time dynamics, admits a Lyapunov function, that is a function which is minimized along the trajectories of the dynamics of the system. The dynamics therefore converges to the minima of this function.
This is a quite important step, since it allows to turn the study of the stationary state of the dynamical model into the study of the local minima of the Lyapunov function. Therefore one can regard the Lyapunov function as the Hamiltonian of a system and resort to the powerful tools of statistical mechanics in order to study the statistical properties of its ground state (global minimum) and eventually of its meta-stable states (local minima). This shall be the subject of the next chapter

In order to study the stationary state properties of the system, it is considered the long time limit of the dynamics of scores.
The approach below, which follows that of M. Marsili [2001], is based on the following key observations:

- It is expected that $U_i(t)$ changes significantly and systematically only over time-scales of order $\Delta t \sim P$ (characteristic times of the dynamics are proportional to P).

- The scaling $\sigma^2 \sim N$, with fixed α, means that typically $A(t) \sim \sqrt{N}$.

- Time increments of $U_i(t)$ are small (i.e. of order $\frac{\sqrt{N}}{P} \sim \frac{1}{\sqrt{P}}$)

In the limit $P, N >> 1$, the system is studied on a fixed infinitesimal increment of time $d\tau$ such that $Pd\tau = \alpha N d\tau$. In doing so the continuum time limit $d\tau \to 0$ is taken after the thermodynamic limit $N \to \infty$.
In order to capture the long time dynamics of scores, set

$$U_i(t) = u_i(\tau), \quad \text{with } \tau = \frac{t}{P} \tag{3.1}$$

The dynamics for $Pd\tau$ time steps, from $t = P\tau$ to $t = P(\tau + d\tau)$ will be:

$$\begin{aligned} u_i(\tau + d\tau) - u_i(\tau) &= - \sum_{t=P\tau}^{P(\tau+d\tau)-1} \left(a_i^{\mu(t)} \frac{A(t)}{P} + \epsilon_i \right) \\ &= - \sum_{t=P\tau}^{P(\tau+d\tau)-1} a_i^{\mu(t)} \frac{A(t)}{P} - \epsilon_i P d\tau \end{aligned} \tag{3.2}$$

The goal is to obtain the following expression for the dynamics :

$$u_i(t+1) - u_i(t) = du_i(\tau) + dW_i(\tau) \tag{3.3}$$

where $du_i(\tau)$ is the deterministic part and $dW_i(\tau)$ a noise term.
In doing so the quantity $a_i^{\mu(t)} A(t)$ is averaged first with respect to the uniform distribution of $\mu(t)$ and the probability distribution of $n_i(t)$, then averaged with respect to time interval $P\tau \le t \le P(\tau + d\tau)$, that is:

$$\frac{1}{Pd\tau}\sum_{t=P\tau}^{P(\tau+d\tau)-1}\left[\overline{a_i\left(\sum_j a_j\langle n_j(t)\rangle_j+\sqrt{N}A_0^\mu\right)}\right]=$$
$$\frac{1}{Pd\tau}\sum_{t=P\tau}^{P(\tau+d\tau)-1}\left[\frac{1}{P}\sum_{\mu=1}^{P}a_i^\mu\left(\sum_j a_j^\mu\langle n_j(t)\rangle_j+\sqrt{N}A_0^\mu\right)\right]=$$
$$\frac{1}{Pd\tau}\sum_{t=P\tau}^{P(\tau+d\tau)-1}\left[\frac{1}{P}\sum_{\mu=1}^{P}a_i^\mu\left(\sum_j a_j^\mu\sum_{k=0}^{\infty}P(n_j(t)=k)k+\sqrt{N}A_0^\mu\right)\right]=$$
$$\frac{1}{P}\sum_{\mu=1}^{P}a_i^\mu\left(\sum_j a_j^\mu\sum_{k=0}^{\infty}\left[\underbrace{\frac{1}{Pd\tau}\sum_{t=P\tau}^{P(\tau+d\tau)-1}P(n_j(t)=k)}\,k\right]+\sqrt{N}A_0^\mu\right)=\overline{a_i\langle A\rangle_{\pi(\tau)}}$$

where for the time interval $P\tau \le t \le P(\tau+d\tau)$ it is defined the random variable for each agent i

$$\begin{aligned}\pi_{k,i}(\tau)=&\frac{1}{Pd\tau}\sum_{t=P\tau}^{P(\tau+d\tau)-1}P(n_i(t)=k)=\\=&\frac{1}{Pd\tau}\sum_{t=P\tau}^{P(\tau+d\tau)-1}\frac{\lambda_i(t)^k}{k!}exp(-\lambda_i(t))\end{aligned}$$

which is the frequency with which agent i takes k shares in the time interval $P\tau \le t \le P(\tau+d\tau)$.

Once the above random variable has been introduced, it is derived the average of the number of shares taken by agent i in the time interval considered, that is:

$$\begin{aligned}\lambda_i(\tau)=&\frac{1}{Pd\tau}\sum_{t=P\tau}^{P(\tau+d\tau)-1}P(n_i(t)=k)k\\=&\max(0\ ;\ \Gamma u_i(\tau))\end{aligned}$$

According to this definition, it is rewritten:

$$a_i^{\mu(t)}A(t)=\overline{a_i\langle A\rangle_{\pi(\tau)}}+X_i(t) \tag{3.4}$$

that is, the above quantity is split into an average over the distribution of information and frequency and a stochastic part that has zero mean.

Substituting into (3.2) it is obtained

$$
\begin{aligned}
u_i(\tau + d\tau) - u_i(\tau) &= du_i(\tau) + dW_i(\tau) = \\
&= -\frac{1}{P}\sum_{t=P\tau}^{P(\tau+d\tau)-1}\left(\overline{a_i\langle A\rangle_{\pi(\tau)}} + X_i(t)\right) - \epsilon_i P d\tau = \\
&= -\overline{a_i\langle A\rangle_{\pi(\tau)}}d\tau - \frac{1}{P}\sum_{t=P\tau}^{P(\tau+d\tau)-1} X_i(t) - \epsilon_i P d\tau
\end{aligned}
$$

Consider the term $\frac{1}{P}\sum_{t=P\tau}^{P(\tau+d\tau)-1} X_i(t)$, according to (3.4) it is a a sum of $Pd\tau$ random variables $X_i(t)$ with zero average.

By taking $d\tau$ fixed and N very large, so that $Pd\tau >> 1$, it is possible to use limit theorems. According to the model, $\mu(t)$ and $n_i(t)$ are drawn independently at each time that means that the variables $X_i(t)$ are independent from time to time and identically distributed , hence in the time interval $[P\tau, P(\tau + d\tau)]$ it is possible to use the Central limit theorem, that is, the terms dW_i can be approximated by a Gaussian variable that have zero mean and covariance matrix given by:

$$
\langle dW_i(\tau)dW_j(\tau')\rangle = \frac{1}{P^2}\sum_{t=P\tau}^{P(\tau+d\tau)-1}\sum_{t'=P\tau'}^{P(\tau'+d\tau)-1}\langle X_i(t)X_j(t')\rangle_{\pi(\tau)} \tag{3.5}
$$

because of independence of $X_i(t)$ for different time, (3.5) becomes:

$$
\begin{aligned}
\langle dW_i(\tau)dW_j(\tau')\rangle &= \frac{\delta(\tau-\tau')}{P^2}\sum_{t=P\tau}^{P(\tau+d\tau)-1}\overline{\langle X_i(t)X_j(t)\rangle_{\pi(\tau)}} = \\
&= \frac{\delta(\tau-\tau')}{P}d\tau\overline{\langle X_i(t)X_j(t)\rangle_{\pi(\tau)}}
\end{aligned}
$$

by expanding the product inside the sum:

$$
\frac{\overline{\langle X_i(t)X_j(t)\rangle_{\pi(\tau)}}}{P} = \frac{\overline{a_i a_j\langle A^2\rangle_{\pi(\tau)}}}{P} - \frac{\overline{a_i\langle A\rangle_{\pi(\tau)}}\;\overline{a_j\langle A\rangle_{\pi(\tau)}}}{P}
$$

The second term is of order N^0 so it vanishes in the limit $N \to \infty$. Indeed $a_i^\mu\langle A|\mu\rangle_{\pi(\tau)}$ is of order $\sqrt{N}$ but its sign fluctuates, when it is averaged over $P \sim N$ different μ, it is obtained a quantity of order N^0.
It is implicitly assumed that $\overline{\langle A\rangle} \simeq 0$

The first term, instead

$$\sigma^2(\tau) \equiv \langle A^2 \rangle_{\pi(\tau)} \approx \langle A^2 | \mu \rangle_{\pi(\tau)} \tag{3.6}$$

which corresponds to:

$$\begin{aligned} \langle A^2 | \mu \rangle_{\pi(\tau)} &= \sum_{i,j} a_i^\mu a_j^\mu \lambda_i(\tau) \lambda_j(\tau) = \\ &= \sum_i \lambda_i(\tau)^2 + \sum_{i \neq j} \sum_{k,l=0}^{\infty} a_i^\mu a_j^\mu \lambda_i(\tau) \lambda_j(\tau) \end{aligned}$$

that is of order N.

Within this approximation, the correlation, in the limit $N \to \infty$ becomes:

$$\langle dW_i(\tau) dW_j(\tau') \rangle \cong \frac{\sigma^2(\tau)}{N} \overline{a_i a_j} \delta(\tau - \tau') d\tau \tag{3.7}$$

Summarizing the dynamics of u_i is describes by a continuum time Langevin equation:

$$\frac{du_i(\tau)}{d\tau} = -\overline{a_i \langle A \rangle_{\pi(\tau)}} - \epsilon_i P + \eta_i(\tau) \tag{3.8}$$

$$\langle \eta_i(\tau) \rangle = 0 \tag{3.9}$$

$$\langle \eta_i(\tau) \eta_j(\tau') \rangle \cong \frac{\sigma^2(\tau)}{N} \overline{a_i a_j} \delta(\tau - \tau') \tag{3.10}$$

Hence (3.8) are complex non-linear stochastic differential equations with a time dependent noise term.

3.2 Stationary State

The stochastic differential equation for the payoff is:

$$du_i(\tau) = -\left(\frac{1}{P} \sum_{\mu=1}^{P} a_i^\mu \left(\sum_{j=1}^{N} a_j^\mu \lambda_j + \sqrt{N} A_0^\mu \right) + \epsilon P \right) d\tau + dW_i(\tau) \tag{3.11}$$

Notice that $\pi_i(\tau)$ is a stochastic variables, hence it is defined the average on the stationary state as:

$$\langle ... \rangle = \lim_{\tau_0, T \to \infty} \frac{1}{T} \int_{\tau_0}^{\tau_0 + T} d\tau \langle ... \rangle_{\pi(\tau)} \tag{3.12}$$

So that the average over the stationary state of the dynamics will be read:

$$\left\langle \frac{du_i(\tau)}{d\tau} \right\rangle = -\overline{a_i\langle A\rangle} - \epsilon P, \qquad \langle A|\mu\rangle = \sum_j a_j^\mu \lambda_j + \sqrt{N} A_0^\mu \tag{3.13}$$

From (2.2):

$$\begin{aligned} \left\langle \frac{du_i(\tau)}{d\tau} \right\rangle &= -\overline{a_i\langle A\rangle} - \epsilon P \\ &= \begin{cases} 0 & \text{if } \lambda_i > 0, \\ < 0 & \text{if } \lambda_i = 0. \end{cases} \end{aligned} \tag{3.14}$$

This can be understood with the following argument: suppose that $\langle du_i \rangle = 0$ that means the score stabilize around a positive value and λ_i is picked such that $\lambda = \max(0\ ;\ \Gamma u_i)$. On the other hand, if $du_i < 0$ the score associated to agent i, $u_i \to -\infty$, hence $\lambda_i = 0$.
Consider the right part of (3.14):

$$\begin{aligned} \overline{a_i\langle A\rangle} + \epsilon P &= \frac{1}{P}\left(\sum_{\mu=1}^{P} a_i^\mu \langle A|\mu\rangle \right) + \epsilon P \\ &= \frac{1}{P} \sum_{\mu=1}^{P} a_i^\mu \left(\sum_{j=1}^{N} a_j^\mu \lambda_j + \sqrt{N} A_0^\mu \right) + \epsilon P \end{aligned}$$

this can be seen as the minimization problem of a function $\tilde{H}$ defined as the following:

$$\tilde{H} = \frac{1}{2P} \sum_{\mu=1}^{P} \left(\sum_{j=1}^{N} a_j^\mu \lambda_j + \sqrt{N} A_0^\mu \right)^2 + \epsilon P \sum_{j=1}^{N} \lambda_j \tag{3.15}$$

Such that the following conditions should be satisfied

$$\frac{\partial \tilde{H}}{\partial \lambda_j} = \begin{cases} 0 & \text{if } \lambda_j > 0, \\ > 0 & \text{if } \lambda_j = 0. \end{cases} \tag{3.16}$$

that is to find:

$$\lambda_i = \arg\min_{\vec{z}} \tilde{H}(z_i) \tag{3.17}$$

3.3 Numerical result

In the stationary state the distribution λ_i does not depend on Γ, neither does the predictability H. In addition:

$$\langle n_i(t) n_j(t) \rangle = \langle n_i(t) \rangle \langle n_j(t) \rangle = \lambda_i \lambda_j, \quad \text{for } i \neq j$$

According to this, it is possible to relate the global efficiency and the predictability through the following:

$$\begin{aligned}
\sigma^2 =& \frac{1}{P} \sum_{\mu=1}^{P} \langle A^2 | \mu \rangle = \frac{1}{P} \sum_{\mu=1}^{P} \left\langle \left(\sum_j a_j^\mu n_j(t) + \sqrt{N} A_0^\mu \right)^2 \right\rangle = \\
=& \frac{1}{P} \sum_{\mu=1}^{P} \left(\sum_{i,j} a_i^\mu a_j^\mu \langle n_i(t) n_j(t) \rangle + 2 \sum_i a_i^\mu \langle n_i(t) \rangle \sqrt{N} A_0^\mu + N (A_0^\mu)^2 \right)^2 \\
=& H + \frac{1}{P} \sum_{\mu=1}^{P} \sum_{i}^{N} (a_i^\mu)^2 \left[\langle n_i^2 \rangle - \langle n_i \rangle^2 \right] = H + \sum_i^N \langle n_i \rangle
\end{aligned}$$

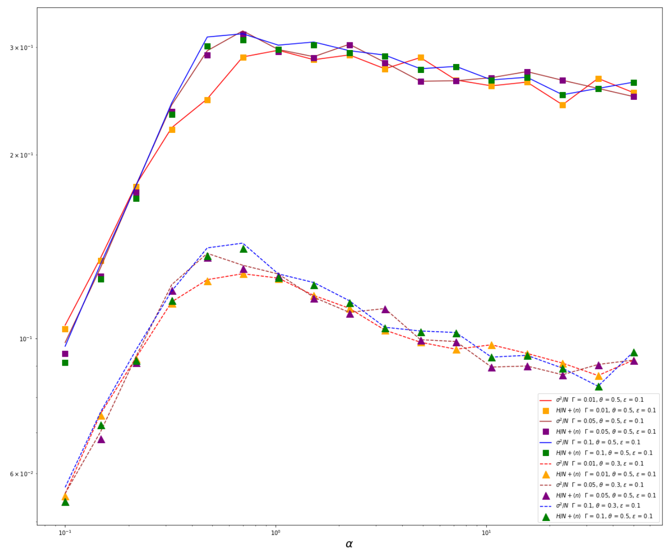

Figure 3.1: Relation σ^2/N and $H/N + \langle n \rangle$ as a function of $P/N = \alpha$ in the stationary state for $\theta = 0.3, 0.5; \epsilon = 0.01, 0.1$; averages were taken over 70 realizations of the game.

Chapter 4

Replica Method

4.1 Introduction

The goal of this section is to compute and characterize the minimum of the Hamiltonian $\tilde{H} = \frac{1}{2P}\sum_{\mu=1}^{P}\left(\sum_{j=1}^{N} a_j^\mu \lambda_j + \sqrt{N}A_0^\mu\right)^2 + \epsilon P \sum_{j=1}^{N}\lambda_j$. Considering $\tilde{H}$ as an Hamiltonian of a statistical mechanic's system, this can be done analyzing the zero temperature limit. First the partition function is built

$$Z(\beta) = \int_0^\infty e^{-\beta \tilde{H}(z)}\, dz \tag{4.1}$$

where β is the inverse temperature. This is nothing else than a generating function, from which all the statistical properties can be computed.

The quantity of interest is the minimum of $\tilde{H}$, taking the limit $\beta \to \infty$ at the end of the calculus.

$$\min_{\vec{z}} \tilde{H}(\vec{z}) = \lim_{\beta\to\infty} \frac{1}{\beta} \log Z(\beta) \tag{4.2}$$

This in principle depends on the specific realization a_i^μ of rules chosen by agents and $\sqrt{N}A_0^\mu$ the number of producers $\forall \mu \in \mathcal{P}$. In practice however, to leading order in N, all realizations of a_i^μ and $\sqrt{N}A_0^\mu$ yield the same limit, which then coincides with the average of $\min_{\vec{z}} \tilde{H}(\vec{z})$ over a_i^μ and $\sqrt{N}A_0^\mu$. In other words rather than focusing on the solution of this problem for a particular game, that is for a given realization of the structure of interactions, the interested is in the generic properties which hold for 'typical' realizations of the game in the limit $N \to \infty$. These properties are called self-averaging because they hold for almost all realizations. In other words, in this limit, all the realizations of the game are characterized by the same statistical behaviour, that is, the same values for all the relevant quantities.

The average of $\log Z$ over the $a's$, which we denote by $\langle ... \rangle_{a_i^\mu}$ and the one over the producers, which we denote by $\langle ... \rangle_{A_0^\mu}$ is reduced to that of moments of Z using the replica trick M. Mezard [1987]:

$$\langle \langle \log Z(\beta) \rangle_{a_i^\mu} \rangle_{A_0^\mu} = \lim_{n \to 0} \frac{\log \langle \langle Z(\beta)^n \rangle_{a_i^\mu} \rangle_{A_0^\mu}}{n} \tag{4.3}$$

With integer n the calculation of $\langle \langle Z^n \rangle_{a_i^\mu} \rangle_{A_0^\mu}$ amounts to study n identical copies of the original system all of them with the same realisation of a_i^μ and A_0^μ. A set of dynamical variables $z^a = \{z_i^a\}$ is introduced for each replica, which are labeled by the index $a = 1, ..., n$.

4.2 Partition Function

$$\langle\langle Z(\beta)^n\rangle_{a_i^\mu}\rangle_{A_0^\mu} = \left\langle\left\langle\left(\int_0^\infty e^{-\beta\tilde{H}(z)}\,dz\right)^n\right\rangle_{a_i^\mu}\right\rangle_{A_0^\mu}$$
$$= \int_0^\infty dz_1 ... \int_0^\infty dz_n \left\langle\left\langle e^{-\beta[\tilde{H}(z_1)+...+\tilde{H}(z_n)]}\right\rangle_{a_i^\mu}\right\rangle_{A_0^\mu} \tag{4.4}$$

The sum of the Hamiltonians over all the realizations is:

$$\sum_{a=1}^n \tilde{H}(z_a) = \sum_{a=1}^n \frac{1}{2P}\sum_{\mu=1}^P\left(\sum_{i=1}^N a_i^\mu z_i^a + \sqrt{\frac{P}{\alpha}}A_0^\mu\right)^2 + \epsilon P\sum_{a=1}^n\sum_{i=1}^N z_i^a \tag{4.5}$$

Substituting (4.5) in (4.4) it is obtained:

$$\int_0^\infty dz \left\langle\left\langle e^{-\beta\left[\sum_{a=1}^n \frac{1}{2P}\sum_{\mu=1}^P\left(\sum_{i=1}^N a_i^\mu z_i^a+\sqrt{\frac{P}{\alpha}}A_0^\mu\right)^2+\epsilon P\sum_{a=1}^n\sum_{i=1}^N z_i^a\right]}\right\rangle_{a_i^\mu}\right\rangle_{A_0^\mu} =$$
$$= \int_0^\infty dz \left\langle\left\langle e^{\sum_{a=1}^n\sum_{\mu=1}^P -\frac{\beta}{2P}\left(\sum_{i=1}^N a_i^\mu z_i^a+\sqrt{\frac{P}{\alpha}}A_0^\mu\right)^2 -\beta\epsilon P\sum_{a=1}^n\sum_{i=1}^N z_i^a}\right\rangle_{a_i^\mu}\right\rangle_{A_0^\mu} =$$
$$= \int_0^\infty dz \left\langle\left\langle \prod_{a=1}^n\prod_{\mu=1}^P e^{-\frac{\beta}{2P}\left(\sum_{i=1}^N a_i^\mu z_i^a+\sqrt{\frac{P}{\alpha}}A_0^\mu\right)^2}\right\rangle_{a_i^\mu}\right\rangle_{A_0^\mu} e^{-\beta\epsilon P\sum_{a=1}^n\sum_{i=1}^N z_i^a}$$

It is now used the so called Hubbard–Stratonovich transformation defined by the integral identity:

$$e^{-\frac{x^2}{2}} = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}+ixy}\,dy \tag{4.6}$$

In the following it will be denoted by $\langle ... \rangle_y$ the expectation over the Gaussian variable (zero mean and unit variance) y and it has been introduced the variable y_a^μ for each a and μ.

With this transformation, the Partition Function becomes:

$$\int_0^\infty dz \left\langle \prod_{a=1}^{n} \prod_{\mu=1}^{P} \left\langle \left\langle e^{i\sqrt{\frac{\beta}{P}}\left(\sum_{i=1}^{N} a_i^\mu z_i^a + \sqrt{\frac{P}{\alpha}} A_0^\mu\right) y_a^\mu} \right\rangle_{A_0^\mu} \right\rangle_y \right\rangle_{a_i^\mu} e^{-\beta\epsilon P \sum_{a=1}^{n}\sum_{i=1}^{N} z_i^a} =$$

$$= \int_0^\infty dz \left\langle \prod_{\mu=1}^{P} \left(\prod_{i=1}^{N} \left\langle e^{i\sqrt{\frac{\beta}{P}}\left(\sum_{a=1}^{n} z_i^a y_a^\mu\right) a_i^\mu} \right\rangle_{a_i^\mu} \right) \left\langle e^{i\sqrt{\frac{\beta}{\alpha}} \sum_{a=1}^{n} A_0^\mu y_a^\mu} \right\rangle_{A_0^\mu} \right\rangle_y e^{-\beta\epsilon P \sum_{a=1}^{n}\sum_{i=1}^{N} z_i^a}$$

The average over a_i^μ con be compute explicitly using the distribution:

$$P(a_i^\mu = +1) = P(a_i^\mu = -1) = \frac{1}{2}, \quad \forall i \in \mathcal{N}, \mu \in \mathcal{P} \tag{4.7}$$

that gives:

$$\left\langle e^{i\sqrt{\frac{\beta}{P}}\left(\sum_{a=1}^{n} z_i^a y_a^\mu\right) a_i^\mu} \right\rangle_{a_i^\mu} = \cos\left(\sqrt{\frac{\beta}{P}} \sum_{a=1}^{n} z_i^a y_a^\mu \right) = e^{-\frac{\beta}{2P}\left(\sum_{a=1}^{n} z_i^a y_a^\mu\right)^2}$$

Where in the last passage it is used the relation $\cos(x) \simeq 1 - \frac{x^2}{2} \simeq e^{-\frac{x^2}{2}}$ which is correct to order x^2 in a power expansion.

Notice also that:

$$\left\langle e^{i\sqrt{\frac{\beta}{\alpha}} \sum_{a=1}^{n} A_0^\mu y_a^\mu} \right\rangle_{A_0^\mu} = e^{-\frac{\beta}{2\alpha}\theta^2 \left(\sum_{a=1}^{n} y_a^\mu\right)^2}$$

that is the Characteristic function of a Gaussian distribution.
With this approximation it is obtained the following expression:

$$\int_0^\infty dz \left\langle \prod_{\mu=1}^{P} \left(\prod_{i=1}^{N} e^{-\frac{\beta}{2P}\left(\sum_{a=1}^{n} z_i^a y_a^\mu\right)^2} \right) e^{-\frac{\beta}{2\alpha}\theta^2 \left(\sum_{a=1}^{n} y_a^\mu\right)^2} \right\rangle_y e^{-\beta\epsilon P \sum_{a=1}^{n}\sum_{i=1}^{N} z_i^a} =$$

$$= \int_0^\infty dz \left\langle e^{-\frac{\beta}{2P} \sum_{\mu=1}^{P} \sum_{a,b=1}^{n} y_a^\mu y_b^\mu \sum_{i=1}^{N} z_i^a z_i^b - \frac{\beta}{2\alpha}\theta^2 \sum_{\mu=1}^{P} \left(\sum_{a=1}^{n} y_a^\mu\right)^2} \right\rangle_y e^{-\beta\epsilon P \sum_{a=1}^{n}\sum_{i=1}^{N} z_i^a}$$

Let's now introduce the matrix order parameters $\hat{Q} \equiv \{Q_{a,b},\ a, b = 1, ..., n\}$ that has elements:

$$Q_{a,b} = \frac{1}{N} \sum_{i=1}^{n} z_i^a z_i^b$$

such that the following identity holds:

$$e^{-\frac{\beta}{2P}\sum\limits_{\mu=1}^{P}\sum\limits_{a,b=1}^{n} y_a^\mu y_b^\mu \sum\limits_{i=1}^{N} z_i^a z_i^b} =$$

$$= \int d\hat{Q} \prod_{a\leq b} \delta\left(\sum_{i=1}^{n} z_i^a z_i^b - NQ_{a,b}\right) e^{-\frac{\beta}{2\alpha}\sum\limits_{\mu=1}^{P}\sum\limits_{a,b=1}^{n} y_a^\mu Q_{a,b} y_b^\mu}$$

Consider now the average over Gaussian variables, by definition:

$$\begin{aligned}\langle \ldots \rangle_y &= \prod_{\mu=1}^{P}\prod_{a=1}^{n}\int_{-\infty}^{\infty} dy_a^\mu \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(y_a^\mu\right)^2} \\ &= \prod_{\mu=1}^{P}\prod_{a=1}^{n}\int_{-\infty}^{\infty} dy_a^\mu \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\sum\limits_{b=1}^{n} y_a^\mu y_b^\mu \delta_{a,b}}\end{aligned}$$

such that it is rewritten:

$$\left\langle e^{-\frac{\beta}{2P}\sum\limits_{\mu=1}^{P}\sum\limits_{a,b=1}^{n} y_a^\mu y_b^\mu \sum\limits_{i=1}^{N} z_i^a z_i^b - \frac{\beta}{2\alpha}\theta^2 \sum\limits_{\mu=1}^{P}\left(\sum\limits_{a=1}^{n} y_a^\mu\right)^2} \right\rangle_y =$$

$$= \int d\hat{Q} \prod_{a\leq b} \delta\left(\sum_{i=1}^{n} z_i^a z_i^b - NQ_{a,b}\right) \left\langle e^{-\frac{\beta}{2\alpha}\sum\limits_{\mu=1}^{P}\sum\limits_{a,b=1}^{n} y_a^\mu Q_{a,b} y_b^\mu - \frac{\beta}{2\alpha}\theta^2 \sum\limits_{\mu=1}^{P}\left(\sum\limits_{a=1}^{n} y_a^\mu\right)^2} \right\rangle_y =$$

$$= \int d\hat{Q} \prod_{a\leq b} \delta\left(\sum_{i=1}^{n} z_i^a z_i^b - NQ_{a,b}\right) \left\langle e^{-\frac{\beta}{2\alpha}\sum\limits_{\mu=1}^{P}\left(\sum\limits_{a,b=1}^{n} y_a^\mu Q_{a,b} y_b^\mu + \theta^2\left(\sum\limits_{a=1}^{n} y_a^\mu\right)^2\right)} \right\rangle_y =$$

$$= \int d\hat{Q} \prod_{a\leq b} \delta\left(\sum_{i=1}^{n} z_i^a z_i^b - NQ_{a,b}\right) \left[\int dy e^{-\frac{1}{2}\sum\limits_{a,b=1}^{n} y_a\left(\delta_{a,b} + \frac{\beta}{\alpha}\left(Q_{a,b}+\theta^2\right)\right) y_b}\right]^P =$$

$$= \int d\hat{Q} \prod_{a\leq b} \delta\left(\sum_{i=1}^{n} z_i^a z_i^b - NQ_{a,b}\right) \left(\frac{1}{\sqrt{\det(\hat{M})}}\right)^P$$

Where it has been defined the matrix:

$$\hat{M} = \hat{I} + \frac{\beta}{\alpha}\left(\hat{Q} + \theta^2\right)$$

that is a matrix of the form:

$$M_{a,b} = m + M\delta_{a,b}$$

where $m = \frac{\beta}{\alpha}(q+\theta^2)$ and $M = 1 + \frac{\beta}{\alpha}(Q-q)$.

Last transformation that will be taken into account is the integral representation of Dirac's delta function, by introducing the matrix $\hat{R} \equiv \{R_{a,b},\ a,b = 1,...,n\}$, such that:

$$
\begin{aligned}
1 &= \int d\hat{Q} \prod_{a\le b} \delta\left(\sum_{i=1}^{n} z_i^a z_i^b - NQ_{a,b}\right) \\
&= \int\int d\hat{Q} d\hat{R} e^{\frac{\alpha\beta^2}{2}\sum\limits_{a\le b} R_{a,b}\left(\sum\limits_{i=1}^{n} z_i^a z_i^b - NQ_{a,b}\right)}
\end{aligned}
$$

With this the partition function becomes:

$$
\begin{aligned}
&\langle Z(\beta)^n \rangle = \\
&= \int\int d\hat{Q} d\hat{R} \left(\frac{1}{\sqrt{\det(\hat{M})}}\right)^P \int_0^\infty dz e^{\frac{\alpha\beta^2}{2}\sum\limits_{a\le b} R_{a,b}\left(\sum\limits_{i=1}^{n} z_i^a z_i^b - NQ_{a,b}\right)} e^{-\beta\epsilon P \sum\limits_{a=1}^{n}\sum\limits_{i=1}^{N} z_i^a} \\
&= \int\int d\hat{Q} d\hat{R} e^{-\beta n N F_\beta(\hat{Q},\hat{R})}
\end{aligned}
$$

where:

$$
\begin{aligned}
F_\beta(\hat{Q},\hat{R}) = &\frac{\alpha}{2n\beta} \log\det\left(\hat{M}\right) \quad (4.8) \\
&+\frac{\alpha\beta}{2n}\sum_{a\le b} R_{a,b}Q_{a,b} + \quad (4.9) \\
&-\frac{1}{n\beta}\log\int dz e^{\frac{\alpha\beta^2}{2}\sum\limits_{a\le b} R_{a,b} z^a z^b - \epsilon P\beta \sum\limits_{a} z^a} \quad (4.10)
\end{aligned}
$$

The first term arise from the expectation over y_a^μ, instead the second and the third terms arise from the integral representation of the delta functions.

The key point is that, in the limit $N \to \infty$ the integral over the matrices $\hat{Q}$ and $\hat{R}$ is dominated by their saddle point values, i.e. by the values of $Q_{a,b}$ and $R_{a,b}$ for which F attains its minimum value. One should then study the first order conditions $\frac{\partial F}{\partial Q_{a,b}} = 0$ and $\frac{\partial F}{\partial R_{a,b}} = 0$ for all a, b. Here it is assumed the replica symmetric approximation that is the matrices have the following form:

$$
Q_{a.b} = q + (Q-q)\delta_{a,b} \qquad R_{a.b} = r + (R-r)\delta_{a,b} \quad (4.11)
$$

Taking the limit $n \to 0$, following the calculation in Appendix A and putting together (A.1), (A.2) and (A.3), the expression of the free energy is:

$$\begin{aligned} F_\beta^{RS}(Q,q,R,r) = &\frac{\alpha}{2\beta}\log\left(1+\frac{\beta}{\alpha}(Q-q)\right)+\frac{1}{2}\frac{\alpha(q+\theta^2)}{\alpha+\beta(Q-q)}+ \\ &+\frac{\alpha\beta}{2}\left(RQ-\frac{rq}{2}\right)-\frac{1}{\beta}\left\langle\log\int dz e^{-\beta V_y(z)}\right\rangle_y \end{aligned} \tag{4.12}$$

where it has been introduced the potential:

$$V_y(z) = -\frac{\alpha\beta}{2}\left(R-\frac{r}{2}\right)z^2+\epsilon P z-\sqrt{\frac{\alpha r}{2}}zy \tag{4.13}$$

The last term of F_β^{RS} looks like the free energy of a particle with potential $V_y(z)$ where y plays the role of disorder.

These equation have to be studied in the limit $\beta \to \infty$, where the minimum of $\tilde{H}$ is recovered by (4.2), that is:

$$\lim_{N\to\infty}\min_{\vec{z}}\tilde{H}(z) = -\lim_{\beta,N\to\infty}\frac{1}{\beta N}\langle\log Z(\beta)\rangle = \lim_{\beta\to\infty}F_\beta^{RS}(Q,q,R,r)\bigg|_{sp} \tag{4.14}$$

In order to study the limit $\beta \to \infty$ it is convenient to define the response function, given by

$$\chi = \frac{\beta}{\alpha}(Q-q) \implies q = Q-\frac{\alpha}{\beta}\chi \tag{4.15}$$

$Q-q$ measures the distance between two different replicas of the system:

$$\begin{aligned} Q-q =&(z^a)^2-z^a z^b = z^a(z^a-z^b) = \\ =&\frac{1}{2}z^a(z^a-z^b)+\frac{1}{2}z^b(z^b-z^a) = \\ =&\frac{1}{2}(z^a-z^b)^2 \sim \frac{1}{\beta} \end{aligned}$$

A replica is nothing that a realization of the stochastic process with given initial conditions. A finite value of χ means simply that two processes with different initial conditions converge, in the stationary state, to the same point, that is in the limit $\beta \to \infty$, are searched solutions where $q \to Q$ and χ, which it is called susceptibility, remains finite.

The following variables are also defined:

$$\nu = -\alpha\beta\Big(R - \frac{r}{2}\Big), \qquad \xi = \sqrt{\frac{\alpha r}{2}}, \qquad \epsilon P \to \epsilon \tag{4.16}$$

With this change of variables, the potential (4.13) becomes:

$$V_y(z) = \frac{\nu}{2}z^2 + (\epsilon - \xi y)z$$

When $\beta \to \infty$, the last term in (4.12) is dominated by $z^*(y)$ which is the solution of

$$z^*(y) = \arg\min_z V_y(z)$$

that is it will be averaged over an exponential family distribution whose potential is picked around $z^*(y)$.
To find that value, the derivative of $V_y(z)$ is taken and made equal to 0

$$\begin{aligned} 0 = \frac{\partial V_y(z)}{\partial z} &= \nu z + \tilde{\epsilon} - y\xi \\ \implies z^*(y) &= \frac{1}{\nu}\Big(y\xi - \tilde{\epsilon}\Big) \end{aligned}$$

Clearly this equation makes sense only when $z^*(y) \geq 0$, that corresponds to the condition for the Gaussian variable

$$y \geq \frac{\epsilon}{\xi}$$

that is

$$z^*(y) = \begin{cases} \frac{1}{\nu}\Big(y\xi - \epsilon\Big) & \text{if } y \geq \frac{\epsilon}{\xi}, \\ 0 & \text{if } y < \frac{\epsilon}{\xi}. \end{cases}$$

So that in the limit $\beta \to \infty$ it is obtained:

$$\begin{aligned} F_\infty^{RS}(Q, \chi, \nu, \xi) =& \frac{1}{2}\frac{(Q + \theta^2)}{1 + \chi} + \\ & - \frac{1}{2}\nu Q + \frac{1}{2}\alpha\xi^2\chi + \\ & + \Big\langle \frac{\nu}{2}z^*(y)^2 + (\epsilon - \xi y)z^*(y) \Big\rangle_y \end{aligned} \tag{4.17}$$

The Free Energy F_∞^{RS} is computed at the saddle point values, so the parameters Q, χ, ν, ξ are fixed by the first order conditions

$$\frac{\partial F_\beta^{RS}}{\partial Q} = \frac{\partial F_\beta^{RS}}{\partial \chi} = \frac{\partial F_\beta^{RS}}{\partial \nu} = \frac{\partial F_\beta^{RS}}{\partial \xi} = 0$$

that is to solve the following system

$$\begin{cases} \dfrac{\partial F_\beta^{RS}}{\partial Q} = 0 \Longrightarrow \nu = \dfrac{1}{1+\chi} & (4.18a) \\ \dfrac{\partial F_\beta^{RS}}{\partial \chi} = 0 \Longrightarrow \xi = \nu\sqrt{\dfrac{Q+\theta^2}{\alpha}} & (4.18b) \\ \dfrac{\partial F_\beta^{RS}}{\partial \nu} = 0 \Longrightarrow Q = \left\langle z^*(y)^2 \right\rangle & (4.18c) \\ \dfrac{\partial F_\beta^{RS}}{\partial \xi} = 0 \Longrightarrow \alpha\chi\xi = \left\langle z^*(y)y \right\rangle & (4.18d) \end{cases}$$

Now let's evaluate the thermal average:

$$\left\langle z^*(y)y \right\rangle = \frac{1}{\sqrt{2\pi}} \int_{\epsilon/\xi}^{\infty} dy\; e^{-\frac{y^2}{2}} z^*(y)y =$$

$$= \frac{\xi}{\nu\sqrt{2\pi}} \int_{\epsilon/\xi}^{\infty} dy\; e^{-\frac{y^2}{2}} \left(y - \frac{\epsilon}{\xi}\right) y = \frac{\xi}{\nu} I_1\left(\frac{\epsilon}{\xi}\right)$$

in the same way let's evaluate the other thermal average:

$$\left\langle z^*(y)^2 \right\rangle = \frac{1}{\sqrt{2\pi}} \int_{\epsilon/\xi}^{\infty} dy\; e^{-\frac{y^2}{2}} z^*(y)^2 =$$

$$= \frac{\xi^2}{\nu^2\sqrt{2\pi}} \int_{\epsilon/\xi}^{\infty} dy\; e^{-\frac{y^2}{2}} \left(y - \frac{\epsilon}{\xi}\right)^2 = \frac{\xi^2}{\nu^2} I_2\left(\frac{\epsilon}{\xi}\right)$$

Where the two following quantity has been defined:

$$\frac{1}{\sqrt{2\pi}} \int_{\epsilon/\xi}^{\infty} dy\; e^{-\frac{y^2}{2}} \left(y - \frac{\epsilon}{\xi}\right) y = I_1\left(\frac{\epsilon}{\xi}\right), \quad \frac{1}{\sqrt{2\pi}} \int_{\epsilon/\xi}^{\infty} dy\; e^{-\frac{y^2}{2}} \left(y - \frac{\epsilon}{\xi}\right)^2 = I_2\left(\frac{\epsilon}{\xi}\right)$$

Once these expressions have been derived, it follows from the system of equation that:

$$\begin{cases} \nu = \frac{1}{1+\chi} \\ \alpha = \frac{\nu^2(Q+\theta^2)}{\xi^2} = I_2\left(\frac{\epsilon}{\xi}\right) + \frac{\nu^2\sigma^2}{\xi^2} \\ \frac{\nu^2 Q}{\xi^2} = I_2\left(\frac{\epsilon}{\xi}\right) \\ \alpha\chi\nu = I_1\left(\frac{\epsilon}{\xi}\right) \end{cases}$$

Combining the expressions above, the following relation is obtained:

$$\alpha\chi\nu = \alpha\frac{\chi+1-1}{\chi+1} = \alpha\left(1 - \frac{1}{\chi+1}\right) = \alpha(1-\nu) = I_1\left(\frac{\epsilon}{\xi}\right)$$

from which one gets:

$$\nu = 1 - \frac{1}{\alpha}I_1\left(\frac{\epsilon}{\xi}\right)$$

and substituting it in the expression for α, finally it gets:

$$\alpha = I_2\left(\frac{\epsilon}{\xi}\right) + \frac{\theta^2}{\epsilon^2}\left(\frac{\epsilon}{\xi}\right)^2\left[1 - \frac{1}{\alpha}I_1\left(\frac{\epsilon}{\xi}\right)\right]^2 \tag{4.19}$$

By solving equation (4.19) numerically as a function of parameters ϵ, ξ, θ and substitute the roots in the system above, to find the other variables, it is obtained a curve that describes the Free Energy F_β^{RS} in the limit $\beta \to \infty$ as a function of the order parameter α

The curves found analytically are compared with the numerical simulations in the stationary state, in particular, below is shown the Free Energy for values of $\theta = 0.5, 0.3$ and values of $\epsilon = 0.01, 0.1$

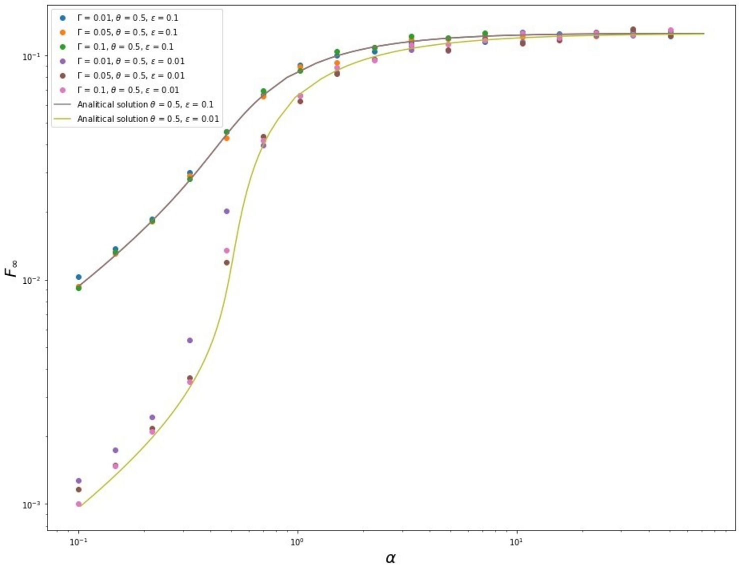

Figure 4.1: Free Energy F_{β}^{RS} as a function of $P/N = \alpha$ for $\theta = 0.5; \epsilon = 0.01, 0.1$; averages were taken over 70 realizations of the game.

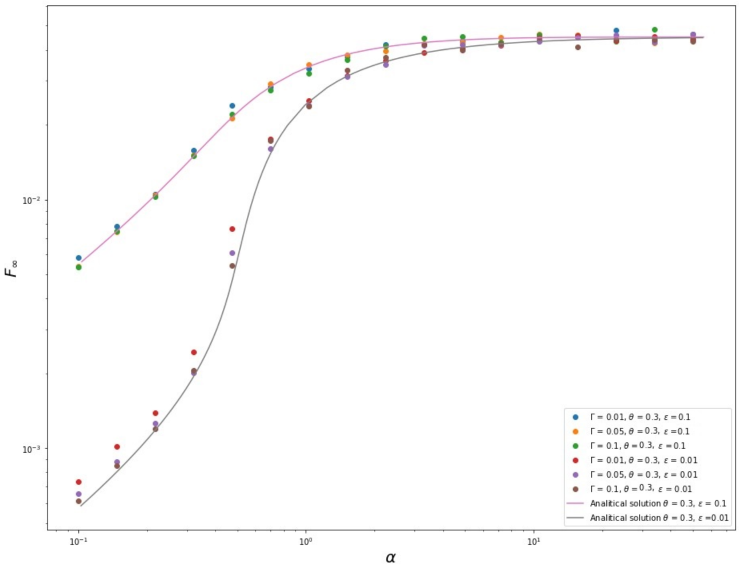

Figure 4.2: Free Energy F_{β}^{RS} as a function of $P/N = \alpha$ for $\theta = 0.3; \epsilon = 0.01, 0.1$; averages were taken over 70 realizations of the game.

4.3 limit $\epsilon \to 0$

In this section it will be analyzed the limit $\epsilon \to 0$ that will simplify the saddle point equations living only two equations to be solved for ξ and ν in terms of the parameters α, ϵ and σ. These equation are:

$$\begin{cases} \alpha(1-\nu) = I_1\left(\frac{\epsilon}{\xi}\right) \\ \left(\alpha - I_2\left(\frac{\epsilon}{\xi}\right)\right)\left(\frac{\xi}{\nu}\right)^2 = \theta^2 \end{cases}$$

In order to study this limit, assume that:

$$\frac{\epsilon}{\xi} = \omega$$

where as before it is denoted by

$$\begin{aligned}
&I_1(\omega) = \\
=&\frac{1}{\sqrt{2\pi}}\int_{\omega}^{\infty} dy\ e^{-\frac{y^2}{2}}\big(y-\omega\big)y = \frac{1}{\sqrt{2\pi}}\int_{\omega}^{\infty} dy\ e^{-\frac{y^2}{2}}y^2 - \frac{\omega}{\sqrt{2\pi}}\left(-e^{-\frac{y^2}{2}}\right)\Bigg|_{\omega}^{\infty} = \\
=&-\frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial\lambda}\int_{\omega}^{\infty} dy\ e^{-\lambda y^2}\Bigg|_{\lambda=\frac{1}{2}} - \frac{\omega}{\sqrt{2\pi}}e^{-\frac{\omega^2}{2}} = \\
=&-\frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial\lambda}\left(\frac{1}{\sqrt{\lambda}}\int_{\sqrt{\lambda}\omega}^{\infty} dz\ e^{-\omega^2}\right)\Bigg|_{\lambda=\frac{1}{2}} - \frac{\omega}{\sqrt{2\pi}}e^{-\frac{\omega^2}{2}} = \\
=&-\frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial\lambda}\left(\frac{1}{\sqrt{\lambda}}\frac{\sqrt{\pi}}{2}\Big(1-erf\big(\sqrt{\lambda}\omega\big)\Big)\right)\Bigg|_{\lambda=\frac{1}{2}} - \frac{\omega}{\sqrt{2\pi}}e^{-\frac{\omega^2}{2}} = \\
=&-\frac{1}{\sqrt{2\pi}}\frac{\sqrt{\pi}}{2}\left(-\frac{\omega}{\sqrt{\pi}}e^{-\lambda\omega^2} - \frac{1}{2\sqrt{\lambda}}\Big(1-erf\big(\sqrt{\lambda}\omega\big)\Big)\right)\frac{1}{\lambda}\Bigg|_{\lambda=\frac{1}{2}} - \frac{\omega}{\sqrt{2\pi}}e^{-\frac{\omega^2}{2}} = \\
=&\frac{1}{\sqrt{2\pi}}\frac{\omega}{2}e^{-\lambda\omega^2}\frac{1}{\lambda}\Bigg|_{\lambda=\frac{1}{2}} + \frac{1}{\sqrt{2\pi}}\frac{\sqrt{\pi}}{2}\frac{1}{2\sqrt{\lambda}}\Big(1-erf\big(\sqrt{\lambda}\omega\big)\Big)\frac{1}{\lambda}\Bigg|_{\lambda=\frac{1}{2}} - \frac{\omega}{\sqrt{2\pi}}e^{-\frac{\omega^2}{2}} = \\
=&\frac{\omega}{\sqrt{2\pi}}e^{-\frac{\omega^2}{2}} + \frac{1}{2}\left(1-erf\left(\frac{\omega}{\sqrt{2}}\right)\right) - \frac{\omega}{\sqrt{2\pi}}e^{-\frac{\omega^2}{2}} = \frac{1}{2}\left(1-erf\left(\frac{\omega}{\sqrt{2}}\right)\right)
\end{aligned}$$

In the same way let's evaluate the other thermal average:

$$I_2(\omega) = \frac{1}{\sqrt{2\pi}} \int\limits_{\omega}^{\infty} dy\; e^{-\frac{y^2}{2}} \Big(y-\omega\Big)^2 =$$

$$= \frac{1}{\sqrt{2\pi}} \int\limits_{\omega}^{\infty} dy\; e^{-\frac{y^2}{2}} y^2 - \frac{2\omega}{\nu^2\sqrt{2\pi}} \left(-e^{-\frac{y^2}{2}}\right)\Bigg|_{\omega}^{\infty} + \frac{\omega^2\sqrt{2}}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{2} \left(1 - erf\left(\frac{\omega}{\sqrt{2}}\right)\right) =$$

$$= -\frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial\lambda} \int\limits_{\omega}^{\infty} dy\; e^{-\lambda y^2}\Bigg|_{\lambda=\frac{1}{2}} - \frac{2\omega}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}} + \frac{\omega^2}{2} \left(1 - erf\left(\frac{\omega}{\sqrt{2}}\right)\right) =$$

$$= \frac{\omega}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}} + \frac{1}{2}\left(1 - erf\left(\frac{\omega}{\sqrt{2}}\right)\right) - \frac{2\omega}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}} + \frac{\omega^2}{2} \left(1 - erf\left(\frac{\omega}{\sqrt{2}}\right)\right) =$$

$$= \frac{\omega^2+1}{2}\left(1 - erf\left(\frac{\omega}{\sqrt{2}}\right)\right) - \frac{\omega}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}}$$

when $\epsilon \to 0$ there are two possible solutions, either ξ and ν tend to a finite value, or they both tend to zero.

4.3.1 Finite ξ, ν solution

If ξ attains a finite value, then $I_1, I_2 \to 1/2$ as $\epsilon \to 0$.
The system of equations is easily solved and gives

$$\begin{cases} \xi = \frac{\theta}{\alpha}\sqrt{\alpha - \frac{1}{2}} \\ \nu = \frac{1}{\alpha}\Big(\alpha - \frac{1}{2}\Big) \end{cases}$$

that is valid for $\alpha > \frac{1}{2}$.
By substituting this expression, from (4.18c) it is found:

$$Q = \Big\langle z^*(y)^2 \Big\rangle = \frac{\xi^2}{\nu^2} I_2(\omega) = \frac{\xi^2}{2\nu^2} = \frac{\theta^2}{2\alpha - 1} \tag{4.20}$$

finally in the limit considered:

$$\lim_{\epsilon\to 0} F_{\infty}^{RS} = \frac{\theta^2}{2\alpha}\Big(\alpha - \frac{1}{2}\Big) \tag{4.21}$$

and also, it is calculated:

$$\langle n_i \rangle = \Big\langle z^*(y) y \Big\rangle = \frac{\theta}{\sqrt{2\pi\Big(\alpha - \frac{1}{2}\Big)}} \tag{4.22}$$

4.3.2 $\xi, \nu \to 0$ solution

Let's set

$$\nu = \frac{\xi}{y}$$

from (4.2), according to this limit:

$$\alpha = I_1(\omega) \tag{4.23}$$

that yields

$$y = \frac{\theta}{\sqrt{\alpha - I_2(\omega)}} = \frac{\theta}{\sqrt{I_1(\omega) - I_2(\omega)}}$$

since by definition $\omega \geq 0$, this solution describes the region $\alpha \leq \frac{1}{2}$.
What is interested now is to study $\alpha \to \frac{1}{2}$, in this limit the solution has the leading behaviour:

$$\omega = \sqrt{2\pi}\left(\frac{1}{2} - \alpha\right) + O\left(\frac{1}{2} - \alpha\right)^2$$

that therefore implies

$$Q \simeq \frac{\theta^2 \alpha}{\frac{1}{2} - \alpha}$$

$$H = 0$$

As it is shown, in the limit $\epsilon \to 0$, the minimization of $\tilde{H}$ reveals that the collective behaviour of the Minority Game features a phase transition as a function of the number N of agents. When there are less agents than a critical number, the price evolution seems predictable to an external agent (but not to those already playing), whereas when the number of agents is beyond the critical number, the market becomes unpredictable. This suggests that, as long as there are few participants, the market will attract more and more agents, thus approaching the critical number where the market becomes unpredictable and hence unattractive.

Appendix A

Free Energy Calculation

A.1 First term

Consider the first term (4.8):

$$\frac{\alpha}{2n\beta}\log\det\left(\hat{M}\right)$$

In order to compute the determinant of this matrix, it can be easily find the trivial eigenvector $v^{(1)} = (1, ...1)$ of all ones with eigenvalue $\lambda^{(1)} = mn + M$. Notice also that the matrix M is symmetric, it means that eigenvector associated with different eigenvalues are orthogonal, that is all the others eigenvector should satisfy the following property:

$$0 = \sum_{a=1}^{n} v_a^{(k)} v_a^{(1)} = \sum_{a=1}^{n} v_a^{(k)} 1 = \sum_{i=1}^{n} v_a^{(k)} \qquad \forall k \in [2, ..., n]$$

so it can be written the following eigenvalue equation

$$\hat{M} \bullet v^{(k)} = \sum_{a=1}^{n} M_{a,b} v_a^{(k)} = \sum_{a=1}^{n} (m + M\delta_{a,b}) v_a^{(k)} = M v_b^{(k)}$$

so M is eigenvalue for all the others eigenvector $v^{(k)}$, $\forall k \in [2, ..., n]$ and M has degeneracy $n - 1$.
Finally it has been obtained that $\det\hat{M} = (mn + M)M^{n-1}$.

Now (4.8) can be rewritten as

$$\begin{aligned}
&\frac{\alpha}{2n\beta}\log\det\left(\hat{I}+\frac{\beta}{\alpha}\left(\hat{Q}+\theta^2\right)\right) = \\
=&\frac{\alpha}{2n\beta}\log\left[\left(n\frac{\beta}{\alpha}(q+\theta^2)+1+\frac{\beta}{\alpha}(Q-q)\right)\left(1+\frac{\beta}{\alpha}(Q-q)\right)^{n-1}\right] = \\
=&\frac{\alpha}{2n\beta}\left[n\log\left(1+\frac{\beta}{\alpha}(Q-q)\right)+\log\left[1+\frac{n\frac{\beta}{\alpha}(q+\theta^2)}{1+\frac{\beta}{\alpha}(Q-q)}\right]\right] = \\
=&\frac{\alpha}{2\beta}\log\left(1+\frac{\beta}{\alpha}(Q-q)\right)+\frac{\alpha}{2n\beta}\frac{n\frac{\beta}{\alpha}(q+\theta^2)}{1+\frac{\beta}{\alpha}(Q-q)}+O(n) = \\
=&\frac{\alpha}{2\beta}\log\left(1+\frac{\beta}{\alpha}(Q-q)\right)+\frac{1}{2}\frac{\alpha(q+\theta^2)}{\alpha+\beta(Q-q)}+O(n)
\end{aligned} \tag{A.1}$$

A.2 Second term

Let's now focus on (4.18b):

$$\frac{\alpha\beta}{2n}\sum_{a\leq b}R_{a,b}Q_{a,b}$$

the sum is split

$$\sum_{a\leq b}R_{a,b}Q_{a,b}=\sum_{a<b}R_{a,b}Q_{a,b}+\sum_{a=b}R_{a,b}Q_{a,b}$$

because in the first sum it has been put explicitly that $a \neq b$, referring to (4.11), there would be only the contribution coming from q and r, instead in the second sum there would be only the contribution coming from Q and R, in practice

$$\sum_{a<b}rq+\sum_{a=b}RQ=\frac{n(n-1)}{2}rq+nRQ$$

In total it is found

$$\frac{\alpha\beta}{2}\left[RQ+\frac{(n-1)}{2}rq\right]$$

Remembering that the limit $n \to 0$ is taken, (4.18b) would be

$$\frac{\alpha\beta}{2}\left[RQ-\frac{rq}{2}\right]+O(n) \tag{A.2}$$

A.3 Third term

In the end the third term is considered (4.18c):

$$\frac{1}{n\beta}\log\int dz e^{\frac{\alpha\beta^2}{2}\sum_{a\leq b}R_{a,b}z^az^b-\epsilon P\beta\sum_a z^a}$$

By using the (4.11) it is possible to rewrite it as:

$$\frac{1}{n\beta}\log\int dz e^{\frac{\alpha\beta^2}{2}\frac{r}{2}\left(\sum_a z^a\right)^2+\frac{\alpha\beta^2}{2}\left(R-\frac{r}{2}\right)\sum_a(z^a)^2-\epsilon P\beta\sum_a z^a}$$

Using again the Hubbard–Stratonovich transformation in (4.6), the above equation becomes

$$\begin{aligned}
&\frac{1}{n\beta}\log\int dz e^{\frac{\alpha\beta^2}{2}\left(R-\frac{r}{2}\right)\sum_a(z^a)^2-\epsilon P\beta\sum_a z^a}\left\langle e^{\beta\sqrt{\frac{\alpha r}{2}}\sum_a z^a y^a}\right\rangle_{y^a}\\
=&\frac{1}{n\beta}\log\left\langle\left[\int dz e^{\frac{\alpha\beta^2}{2}\left(R-\frac{r}{2}\right)z^2-\epsilon P\beta z+\beta\sqrt{\frac{\alpha r}{2}}zy}\right]^n\right\rangle_y\\
=&\frac{1}{n\beta}\log\left\langle\left[\int dz e^{-\beta V_y(z)}\right]^n\right\rangle_y\\
=&\frac{1}{n\beta}\log\left\langle\left[1+n\log\int dz e^{-\beta V_y(z)}+O(n)\right]\right\rangle_y\\
=&\frac{1}{n\beta}\log\left[1+n\left\langle\log\int dz e^{-\beta V_y(z)}\right\rangle_y+O(n)\right]\\
=&\frac{1}{n\beta}n\left\langle\log\int dz e^{-\beta V_y(z)}\right\rangle_y+O(n)\\
=&\frac{1}{\beta}\left\langle\log\int dz e^{-\beta V_y(z)}\right\rangle_y+O(n)
\end{aligned}\tag{A.3}$$

Where, as before, it has been averaged over the Gaussian variable (zero mean and unit variance) y and it has been introduced the potential

$$V_y(z)=-\frac{\alpha\beta}{2}\left(R-\frac{r}{2}\right)z^2+\epsilon Pz-\sqrt{\frac{\alpha r}{2}}zy$$

Appendix B

Source code

This appendix contains the source code to simulate the Poisson Minority Game, it measures $\frac{\sigma^2}{N}$, $\frac{H}{N}$ and $\langle n \rangle$ for random histories.
In order to measure the collective behaviour in the stationary state is has been introduced a time of equilibrium (TEQ) and then the data are collected through a certain number of iterations (NIT);
both this quantity are proportional to P.

Initially every player (speculator) starts with a payoff $U_i(0) = 0, \ \forall i \in \mathcal{N}$, instead the rate of producers is drawn by a Gaussian Distribution:

$$A^{\mu}_{prod} \sim G(0, \theta\sqrt{N}), \forall \mu \in \mathcal{P}$$

Before the simulation starts initialization of the variable is done:

- M is the length of the history
- α indicates the ratio $\frac{P}{N}$.
- Gamma is the percentage of payoff that agents bid
- epsilon is the incentive of agents for trading in the market.
- theta is the square root of the fraction of producers per speculator
- avgA is a vector containing for each position $\mu \in \mathcal{P}$ the value $\langle A|\mu \rangle$
- T is a vector containing the number of time a given μ is drawn

```
import numpy as np
import matplotlib.pyplot as plt
import math
### Parameters
M = 5 #length of history
NITP = 500 #number of iterations/P
Gamma = [0.01, 0.05, 0.1]
epsilon = 0.01/32
theta = 0.5
alphas = np.logspace(math.log10(0.1), 1.7, 17)

def Number_of_agents(p, alph):
    n = int(p/alph)
    if n%2 == 0:
        n = n + 1
    return n

def Minority_Game(M, NITP, alpha, Gamma, theta, eps):
    ### Parameters
    P = 2**M
    N = Number_of_agents(P, alpha)
    NIT = NITP*P #number of iterations
    TEQ = 300*P #equilibrium time
    NIT = NIT + TEQ
    ### Memory allocation
    number = np.zeros(N)
    a = np.zeros((N,P))
    avgA = np.zeros(P)
    T = np.zeros(P)
    A_0 = np.zeros(P)
    ### Initialisation of the players
    for j in range(P):
        for i in range(N):
            a[i,j] = 2*np.random.binomial(1,0.5,1) - 1
        A_0[j] = np.random.normal(0, np.sqrt(N)*theta)
    ### Beginning of the game
    U = np.zeros(N)
    mu = np.random.randint(0, P) # mu(t=0) is randomly drawn
    sigma2 = 0
    for it in range(int(NIT)):
        if it == TEQ:
            sigma2 = 0
            for nu in range(P):
                avgA[nu] = 0
                T[nu] = 0
```

```
            for i in range(N):
                number[i] = 0
        A = A_0[mu]
        for i in range(N):
            n = np.random.poisson(max(0,Gamma*U[i]))
            number[i] = number[i] + n
            A = A + a[i,mu]*n
        avgA[mu] = avgA[mu] + A # builds <A|mu>
        T[mu] = T[mu] + 1 #number of times mu appears
        sigma2 = sigma2 + A*A
        #update of the strategy scores
        for i in range(N):
            U[i] = U[i] - a[i,mu]*A/P - eps
        mu = np.random.randint(0, P)
    H = 0
    for mu in range(P):
        if T[mu] > 0:
            H = H + (avgA[mu]*avgA[mu])/T[mu]
    sum_i = 0
    for i in range(N):
        sum_i = sum_i + number[i]
    sum_i = sum_i/((NIT - TEQ)*N)
    H = H/((NIT - TEQ)*N)
    sigma2 = sigma2/((NIT - TEQ)*N)
    return sigma2, H, sum_i
```

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