POLITECNICO DI TORINO

**Master's Degree Course
in Mathematical Engineering**

Master's Degree Thesis

Mean-field limits for entropic multi-population dynamical systems

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Summary

We prove the well-posedness of a multi-population dynamical system with an entropy regularization, and compare the entropic solution to the entropy-free solution. The mean-field limit of such a system is performed in both the entropic and entropy free cases. We also address the case of different time scales between the agents' locations and labels dynamics, performing a fast reaction limit.

Contents

"Keep calm and carry on." Richard Ross

1 Introduction

The concept of mean-field interaction, originally used in statistical physics by Kac [18] and then by McKean [22] to describe the collisions between particles in a gas, has later proved to be a powerful tool to analyze the asymptotic behavior of systems of interacting agents. Models of multi-agent interactions have been recently used to describe phenomena in biology, sociology, and economics; for example cell aggregation and motility [19, 10], coordinated animal motion [4], synthetic agent behaviour and interactions, such as cooperative robots [11], and influence of key investors in the stock market [6, indroduction]. Two main mechanisms are considered in such models to define the dynamics.

The first takes inspiration from physical laws of motion and is based on pairwise forces encoding observed "second principles" of biological, social, or economical interactions, including repulsion/attraction, alignment, self-propulsion/friction. Typically, this leads to Newton-like first or second order equations with "social interaction" forces. In fact, the evolution of N agents with time-dependent locations, $x_t^1, \dots, x_t^N$ in $\mathbb{R}^d$ is described by the ODE system

$$\dot{x}_t^i = \frac{1}{N} \sum_{j=1}^{N} f(x_t^i, x_t^j) \quad \text{for } i = 1, \dots, N,\ t \in (0, T],$$

where f is a pre-determined pairwise interaction force between pairs of agents. First order models of this form are ubiquitous in the literature and have, for instance, been used to model multi-agent interactions in opinion formation [16], vehicular traffic flow [14], pedestrian motion [12], and synchronisation of chemical and biological oscillators in neuroscience [20].

In the second mechanism, the dynamics is driven by an evolutionary game where players simultaneously optimize their cost. Examples are game theoretic models of evolution [17] or mean-field games, introduced in [21] and independently under the name Nash Certainty Equivalence (NCE) in [9], later greatly popularized, *e.g.*, within consensus problems.

More recently, the notion of spatially inhomogeneous evolutionary games has been proposed [2] where the transport field for the agent population is directed by an evolutionary game on their available strategies (see also [1] for a proposal of a numerical scheme for spatially inhomogeneous evolutionary games). Unlike mean-field games, the optimal dynamics is not chosen via an underlying non-local optimal control problem, but by the agents' local (in time and space) decisions. This is implemented by the well-known replicator dynamics [17]. As above, agents may move in $\mathbb{R}^d$. We denote by U be the set of *pure* strategies. A pay-off function $J\colon (\mathbb{R}^d \times U)^2 \to \mathbb{R}$ is supposed to be known, so that $J(x, u, x', u')$ is the pay-off that player in position x gets playing pure strategy u against player in position x' with pure strategy u'. Each agent does not always play the same strategy, but picks instead different strategies according to a probability measure $\sigma \in \mathcal{P}(U)$, which is referred to as *mixed* strategy. The state of each agent is given by their position and mixed strategy, *e.g.*, (x, σ), so that

$$\int_U J(x, u, x', u') \,\mathrm{d}\sigma'(u')$$

is the pay-off that player in position x gets playing strategy u against player in position x' with mixed strategy σ'. Therefore, if we consider N agents, and denote by (x_t^i, σ_t^i), $i = 1, \dots, N$, their states, the pay-off that the i-th player gets playing strategy u against all the other players at time t is

$$\mathcal{J}(x_t^i, u) \coloneqq \sum_{j=1}^{N} \int_U J(x_t^i, u, x_t^j, u') \,\mathrm{d}\sigma_t^j(u').$$

Maximizing this pay-off is a consequence of playing the best strategy possible. In order to do so, the i-th player has to compare the pay-off obtained by a certain strategy with the mean pay-off over all possible strategies according to their mixed strategy σ_t^i. This leads us to the system of ODEs

$$\begin{cases} \dot{x}_t^i = v(x_t^i, \sigma_t^i) \\ \dot{\sigma}_t^i = \left(\mathcal{J}(x_t^i, u) - \displaystyle\int_U \mathcal{J}(x_t^i, v) \,\mathrm{d}\sigma_t^i(v) \right) \sigma_t^i \end{cases} \qquad \text{for } i = 1, \ldots, N, \ t \in (0, T].$$

A more general case has been analyzed in [23], where the velocity of the agents is also depending on the behavior of the other ones, and the replicator dynamics for the strategies has been replaced by a more general vector field $\mathcal{T}$, that is

$$\begin{cases} \dot{x}_t^i = v_{\Lambda_t^N}(x_t^i, \sigma_t^i) \\ \dot{\sigma}_t^i = \mathcal{T}_{\Lambda_t^N}(x_t^i, \sigma_t^i) \end{cases} \qquad i = 1, \ldots, N, \ t \in (0, T],$$

where $\Lambda_t^N = \sum_{j=1}^N \delta(x_t^j, \sigma_t^j) \in \mathcal{P}(\mathbb{R}^d \times \mathcal{P}(U))$ is a distribution of agents with strategies at time t. The interpretation, given in [23], of these types of systems differs from the one of game theory: the interacting agents are assumed to belong to a number of different species, or populations, and therefore we deal with *labels* instead of strategies. This point of view can be used to distinguish informed agents steering pedestrians, to highlight the influence of few key investors in the stock market, or to recognize leaders from followers in opinion formation models. Throughout this work, we will adopt this perspective.

In [5], the replicator equation is slightly modified adding an entropy regularization $\mathcal{H}$, see (11) below. However, it is very complicated to define such a functional on the space of probability measure $\mathcal{P}(U)$. Therefore, we fix a probability measure $\eta \in \mathcal{P}(U)$, and consider only probabilities densities with respect to η. The aim of this work is to study the system

$$\begin{cases} \dot{x}_t^i = v_{\Lambda_t^N}(x_t^i, \ell_t^i) \\ \dot{\ell}_t^i = \lambda \left[\mathcal{T}_{\Lambda_t^N}(x_t^i, \sigma_t^i) + \varepsilon \,\mathcal{H}(\ell_t^i) \right] \end{cases} \qquad i = 1, \ldots, N, \ t \in (0, T], \tag{I}$$

where ℓ_t^i denotes the label of the i-th agent, and therefore it is a probability density with respect to η; $\varepsilon > 0$ is a small parameter which modulates the intensity of the entropy functional; $\lambda \geq 1$ takes into account the possible time scale difference between the positions and labels dynamics. We are interested in the well-posedness of this system, and in comparing the entropic solution, *i.e.*, $\varepsilon > 0$, to the entropy-free solution, *i.e.*, $\varepsilon = 0$. Our intent is also to compute the mean-field limit of system (I). The underlying idea is that the behavior of this system, as the number of agents is large enough, *i.e.*, $N \to \infty$, can be efficiently treated by replacing the influence of all other individuals in the dynamics on a given agent by a single averaged effect. From a mathematical point of view, this amounts to passing from a particle description to a kinetic description, consisting of a limit PDE whose unknown is the particle density distribution in the state space. More precisely, if we denote by $b_{\Lambda_t^N}(x_t^i, \ell_t^i)$ the vector field which drives the state (x_t^i, ℓ_t^i) in system (I), the mean-field limit of such a system is probability measure on the state space, which solves the continuity equation for measures

$$\partial_t \Lambda_t + \mathrm{div}(b_{\Lambda_t} \Lambda_t) = 0.$$

The well-posedness of such models has therefore to be proved in spaces of measures (we refer to [3] for a monograph on this topic). Lastly, we are interested in the case of instantaneous changes for the labels dynamic, *i.e.*, the fast reaction limit $\lambda \to +\infty$. In the *undisclosed* setting [5], that is $\mathcal{T}_{\Lambda_t^N}(x_t^i, \ell_t^i) = \mathcal{T}(x_t^i, \ell_t^i, x_t^1, \dots, x_t^N)$, we prove the convergence of system (I) to a Newton-like system of the form

$$\dot{x}_t^i = v_{\Lambda_t^N}(x_t^i, \ell_t^{*\,i}(x_t^1, \dots, x_t^N)) \qquad \text{for } i = 1, \dots, N,\ t \in (0, T],$$

where $\ell_t^{*\,i}$ is a solution to a particular minimum problem. The mean-field limit of this last system is a byproduct of the mean-field limit of system (I).

In Sections 2 and 3, we present our notation, and recall some tools of functional analysis and measure theory. In Section 4 we outline the basic settings of the problem and make the general assumptions. In Section 5 we study the entropy functional focusing just on the dynamics for the labels. In section 6 we prove the well-posedness of system (I), and compare the entropic solution to the entropy free solution. In Section 7 we introduce the notions of Eulerian and Lagrangian solutions, which are used in Section 8 to obtain the mean-field limit of system (I). In Section 9, we obtain the fast reaction limit of system (I). We conclude this thesis with Section 10, where we indicate some possible lines of research for further development of these topics.

2 Basic notation

If $(\mathcal{X}, \mathsf{d}_{\mathcal{X}})$ is a metric space we denote by $\mathcal{P}(\mathcal{X})$ the space of probability measures on $\mathcal{X}$. The notation $\mathcal{P}_c(\mathcal{X})$ will be used for probability measures on $\mathcal{X}$ having compact support. We denote by $C_0(\mathcal{X})$ the space of continuous functions vanishing at the boundary of $\mathcal{X}$, and by $C_b(\mathcal{X})$ the space of bounded continuous functions. Whenever $\mathcal{X} = \mathbb{R}^d$, $d \geq 1$, it remains understood that is endowed with the Euclidean norm (and induced distance), which shall be simply denoted by $|\cdot|$. For a Lipschitz function $f \colon \mathcal{X} \to \mathbb{R}$ we denote by

$$\mathrm{Lip}(f) := \sup_{\substack{x, y \in \mathcal{X} \\ x \neq y}} \frac{|f(x) - f(y)|}{\mathsf{d}_{\mathcal{X}}(x, y)}$$

its Lipschitz constant. The notations $\mathrm{Lip}(\mathcal{X})$ and $\mathrm{Lip}_b(\mathcal{X})$ will be used for the spaces of Lipschitz and bounded Lipschitz function on $\mathcal{X}$, respectively. Both are normed spaces with the norm $\|f\| := \|f\|_\infty + \mathrm{Lip}(f)$, where $\|\cdot\|_\infty$ is the supremum norm. In a complete and separable metric space $(\mathcal{X}, \mathsf{d}_{\mathcal{X}})$, we shall use the Kantorovich-Rubinstein distance $\mathcal{W}_1$ in the class of

$\mathcal{P}(\mathcal{X})$, defined as

$$\mathcal{W}_1(\mu,\nu) := \sup\left\{ \int_{\mathcal{X}} \varphi(x)\,\mathrm{d}\mu(x) - \int_{\mathcal{X}} \varphi(x)\,\mathrm{d}\nu(x) : \varphi \in \mathrm{Lip}_b(\mathcal{X}),\ \mathrm{Lip}(\varphi) \le 1 \right\} \tag{1}$$

or, equivalently (thanks to the Kantorovich duality), as

$$\mathcal{W}_1(\mu,\nu) := \inf\left\{ \int_{\mathcal{X}\times\mathcal{X}} \mathsf{d}_{\mathcal{X}}(x,y)\,\mathrm{d}\Pi(x,y) : \Pi(A\times\mathcal{X}) = \mu(A),\ \Pi(\mathcal{X}\times B) = \nu(B) \right\},$$

involving couplings Π of μ and ν. It can be proved that the infimum is actually attained. Notice that $\mathcal{W}_1(\mu,\nu)$ is finite if μ and ν belong to the space

$$\mathcal{P}_1(\mathcal{X}) := \left\{ \mu \in \mathcal{P}(\mathcal{X}) \colon \int_{\mathcal{X}} \mathsf{d}_{\mathcal{X}}(x,\overline{x})\,\mathrm{d}\mu(x) < +\infty \text{ for some } \overline{x} \in \mathcal{X} \right\} \tag{2}$$

and that $(\mathcal{P}_1(\mathcal{X}),\mathcal{W}_1)$ is complete if $(\mathcal{X},\mathsf{d}_{\mathcal{X}})$ is complete. For a probability measure $\mu \in \mathcal{P}(\mathcal{X})$, if $\mathcal{X}$ is also a Banach space, we define the first moment $m_1(\mu)$ as

$$m_1(\mu) := \int_{\mathcal{X}} \|x\|_{\mathcal{X}}\ \mathrm{d}\mu(x).$$

So that, the finiteness of the integral above is equivalent to $\mu \in \mathcal{P}_1(\mathcal{X})$, whenever the distance $\mathsf{d}_{\mathcal{X}}$ is induced by the norm $\|\cdot\|_{\mathcal{X}}$.
Let $\mu \in \mathcal{P}(\mathcal{X})$ and $f \colon \mathcal{X} \to Z$ a μ-measurable function be given. The push-forward measure $f_{\#}\mu \in \mathcal{P}(Z)$ is defined by $f_{\#}\mu(B) = \mu(f^{-1}(B))$ for any Borel set $B \subset Z$. It also holds the change of variables formula

$$\int_Z g(z)\,\mathrm{d}f_{\#}\mu(z) = \int_{\mathcal{X}} g(f(x))\,\mathrm{d}\mu(x)$$

whenever either one of the integrals is well defined.

For E being a Banach space, the notation $C^1_b(E)$ will be used to denote the subspace $C_b(E)$ of functions having bounded continuous Fréchet differential at each point. The notation $D\phi(\cdot)$ will be used to denote the Fréchet differential. In the case of a function $\phi\colon [0,T]\times E \to \mathbb{R}$, the symbol ∂_t will be used to denote partial differentiation with respect to t, while D will only stand for the differentiation with respect to the variables in E.

3 Well-posedness of ODEs in Banach spaces

We recall a theorem by Brezis [7, section I.3, Theorem 1.4, Corollary 1.1] on the well-posedness of ODEs in Banach spaces.

Theorem 1. *Let $(E,\|\cdot\|_E)$ be a Banach space, let C be a closed convex subset of E, and, for $t \in [0,T]$, let $A(t,\cdot)\colon C \to E$ be a family of operators satisfying the following properties:*

(i) there exists a constant $L \ge 0$ such that for every $c_1, c_2 \in C$ and $t \in [0,T]$

$$\|A(t,c_1) - A(t,c_2)\|_E \le L\,\|c_1 - c_2\|_E\,;$$

(ii) *for every* $c \in C$ *the map* $t \mapsto A(t, c)$ *is continuous in* $[0, T]$*;*

(iii) *for every* $R > 0$ *there exists* $\theta_R > 0$ *such that for every* $c \in C \cap \{e \in E : \|e\|_E \leq R\}$

$$c + \theta_R A(t, c) \in C.$$

Then for every $\overline{c} \in C$ *there exists a unique curve* $c \colon [0, T] \to C$ *of class* C^1 *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} c_t = A(t, c_t) \quad \textit{in } [0, T], \quad c_0 = \overline{c}.$$

Moreover, if c^1 *and* c^2 *are the solutions starting from the initial data* $\overline{c}^1, \overline{c}^2 \in C$*, respectively, we have*

$$\left\| c^1_t - c^2_t \right\|_E \leq e^{Lt} \left\| \overline{c}^1 - \overline{c}^2 \right\|_E \qquad \textit{for every } t \in [0, T].$$

For our purposes, we need the following generalization where assumption (i) in Theorem 1 is weakened to a local Lipschitz continuity condition, provided we additionally assume at most linear growth of the operator A.

Corollary 1. *Let* $(E, \|\cdot\|_E)$ *be a Banach space, let* C *be a closed convex subset of* E*, and, for* $t \in [0, T]$*, let* $A(t, \cdot) \colon C \to E$ *be a family of operators satisfying the following properties:*

(i) *for every* $R > 0$ *there exists a constant* $L_R > 0$ *such that for every* $t \in [0, T]$ *and* c_1*,* $c_2 \in C \cap \{e \in E : \|e\|_E \leq R\}$

$$\|A(t, c_1) - A(t, c_2)\|_E \leq L_R \|c_1 - c_2\|_E;$$

(ii) *for every* $c \in C$ *the map* $t \mapsto A(t, c)$ *is continuous in* $[0, T]$*;*

(iii) *for every* $R > 0$ *there exists* $\theta_R > 0$ *such that for every* $c \in C \cap \{e \in E : \|e\|_E \leq R\}$

$$c + \theta_R A(t, c) \in C;$$

(iv) *there exists* $M > 0$ *such that for every* $c \in C$*, there holds*

$$\|A(t, c)\|_E \leq M(1 + \|c\|_E).$$

Then for every $\overline{c} \in C$ *there exists a unique curve* $c \colon [0, T] \to C$ *of class* C^1 *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} c_t = A(t, c_t) \quad \textit{in } [0, T], \quad c_0 = \overline{c}. \tag{3}$$

Moreover, if c^1 *and* c^2 *are the solutions starting from the initial data* $\overline{c}^1, \overline{c}^2 \in C \cap \{e \in E \colon \|e\|_E \leq R\}$*, respectively, there exists a constant* $L = L(M, R, T) > 0$ *such that*

$$\left\| c^1_t - c^2_t \right\|_E \leq e^{Lt} \left\| \overline{c}^1 - \overline{c}^2 \right\|_E \qquad \textit{for every } t \in [0, T]. \tag{4}$$

Proof. Let us fix the initial datum $\bar{c} \in C$ and let us choose $\overline{R} := (\|\bar{c}\|_E + MT)e^{MT}$. Consider a smooth function with compact support $\chi \colon \mathbb{R}^+ \to [0,1]$ such that $\chi(r) = 1$ for every $r \le \overline{R}$ and set $B(t,c) := \chi(\|c\|_E)A(t,c)$. Then one can see that B satisfies hypotheses (i) and (ii) of Theorem 1. To see that hypothesis (iii) of Theorem 1 is also satisfied, it is suffices to notice that, by convexity and since $0 \le \chi \le 1$, the point $c + \theta\chi(\|c\|_E)A(t,c)$ belongs to C whenever $c + \theta A(t,c) \in C$. Therefore there exists a unique solution $t \mapsto c(t)$ of class C^1 to

$$\frac{\mathrm{d}}{\mathrm{d}t} c_t = B(t, c_t) \quad \text{in } [0,T], \qquad c_0 = \bar{c}. \tag{5}$$

Using again that $0 \le \chi \le 1$ and assumption (iv), one can see that

$$\|c_t\|_E \le \|\bar{c}\|_E + MT + M \int_0^T \|c_s\|_E \,\mathrm{d}s,$$

hence, Grönwall's Lemma implies that $\|c_t\|_E \le \overline{R}$ for every $t \in [0,T]$. With this, c_t solves (3). A similar argument shows that any other solution $t \mapsto \hat{c}_t$ to (3) must satisfy $\|\hat{c}_t\|_E \le \overline{R}$ for every $t \in [0,T]$. Thus, uniqueness of solutions for (3) follows from the uniqueness of solutions to (5). A similar argument also yields (4). □

4 Functional setting

The space of labels (U, d) will be assumed to be a separable metric space. Consider the Borel σ-algebra $\mathfrak{B}$ on U induced by the metric d and let us fix a probability measure $\eta \in \mathcal{P}(U)$ which we can assume, without loss of generality, to have full support, *i.e.*, $\mathrm{spt}(\eta) = U$. Notice that the measure space $(U, \mathfrak{B}, \eta)$ is σ-finite and separable[1]. For $p \in [1, +\infty]$, we denote by E the Banach space $L^p(U, \eta)$ which we suppose to be separable[2]. Given two constants r and R such that $0 < r < 1 < R < +\infty$, we introduce the set of probability densities with respect to η, having lower bound r and upper bound R:

$$C_{r,R} := \left\{ \ell \in E : \int_U \ell(u) \,\mathrm{d}\eta(u) = 1 \text{ and } r \le \ell \le R \ \eta - a.e. \right\}. \tag{6}$$

For our purposes, both constants will depend on a small parameter $\varepsilon > 0$, *i.e.*, $r = r_\varepsilon$ and $R = R_\varepsilon$, so that we shall use the simpler notation C_ε instead of $C_{r_\varepsilon, R_\varepsilon}$. Such dependence will be discussed in detail in the Section 5. With an abuse of notation, we denote by $C_{0,\infty}$ the set of unbounded probability densities with respect to η, having L^p regularity:

$$C_{0,\infty} := \left\{ \ell \in E : \int_U \ell(u) \,\mathrm{d}\eta(u) = 1 \text{ and } \ell \ge 0 \ \eta - a.e. \right\}. \tag{7}$$

[1] A measure space $(\mathcal{X}, \mathcal{A}, \mu)$ is said to be separable if there exists a countable family of elements in $\mathcal{A}$, for which the σ-algebra generated by this family coincides with $\mathcal{A}$, see [8, Definition on page 98].

[2] For $p \in [1, +\infty)$, such a space is actually separable because of [8, Theorem 4.13]. For $p = +\infty$, a sufficient condition would be that the elements of U are finite.

Since $\eta(U) = 1$, the inclusion $L^p(U,\eta) \subset L^1(U,\eta)$ holds for all $p \in [1,+\infty]$ and therefore C_ε and $C_{0,\infty}$ are both closed sets with respect to the p-norm. Thus, when equipped with the p-norm, both C_ε and $C_{0,\infty}$ are separable[3] Banach spaces. Notice that C_ε and $C_{0,\infty}$ are also convex and their interiors are empty.

The state variable of the our system is $y := (x,\ell) \in \mathbb{R}^d \times C_\varepsilon =: Y_\varepsilon$. The component $x \in \mathbb{R}^d$ describes the location of an agent in space, whereas the component $\ell \in C_\varepsilon$ describes the distribution of labels (or pure strategies in the case of a replicator dynamics) of the agent. A probability distribution $\Psi \in \mathcal{P}(Y_\varepsilon)$ denotes a distribution of agents with labels. To outline the functional setting for the dynamics, we define $\overline{Y} := \mathbb{R}^d \times E$ and the norm $\|\cdot\|_{\overline{Y}}$ by

$$\|y\|_{\overline{Y}} = \|(x,\ell)\|_{\overline{Y}} := |x| + \|\ell\|_{L^p(U,\eta)}\,. \tag{8}$$

Since $Y_\varepsilon \subset \overline{Y}$, we equip Y_ε with the $\|\cdot\|_{\overline{Y}}$ norm. For a given $R > 0$, we denote by B_R the closed ball of radius R in $\mathbb{R}^d$ and by $B_R^{Y_\varepsilon}$ the closed ball of radius R in Y_ε, namely, $B_R^{Y_\varepsilon} = \{y \in Y_\varepsilon \colon \|y\|_{\overline{Y}} \le R\}$. The Banach space structure of $\overline{Y}$ allows us to define the first moment $m_1(\Psi)$ for a probability measure $\Psi \in \mathcal{P}(Y_\varepsilon)$ as

$$m_1(\Psi) := \int_{Y_\varepsilon} \|y\|_{\overline{Y}}\, \mathrm{d}\Psi(y),$$

so that the space $\mathcal{P}_1(Y_\varepsilon)$, see (5), can be equivalently characterized as

$$\mathcal{P}_1(Y_\varepsilon) = \{\Psi \in \mathcal{P}(Y_\varepsilon) \colon m_1(\Psi) < +\infty\}.$$

Given the arbitrariness of ε and since we will be also interested in the limit as $\varepsilon \to 0$, we define the space $Y := \mathbb{R}^d \times C_{0,\infty}$. Similarly, we equip Y with the norm $\|\cdot\|_{\overline{Y}}$ and we denote by B_R^Y the closed ball of radius R in Y. Notice that $B_R^{Y_\varepsilon} \subset B_R^Y$ for every $\varepsilon > 0$.

5 The decoupled problem

In this section, we focus our attention just on the dynamics for a single distribution of labels. Such dynamics is described by a vector field $\mathcal{T}$ which is regularized by the entropy functional $\mathcal{H}$ defined below. The intensity of the entropy functional is modulated by a small parameter $\varepsilon > 0$, so that the overall dynamics, for a distribution of labels $\ell \in C_\varepsilon$, is determined by the vector field $\mathcal{R}^\varepsilon(\ell) = \mathcal{T}(\ell) + \varepsilon\,\mathcal{H}(\ell)$. In order to state the regularity assumptions that we make on $\mathcal{R}^\varepsilon$, we will discuss first the assumptions on $\mathcal{T}$ and then define the entropy regularization functional $\mathcal{H}$. To this aim, we recall definitions (6) and (7), and that we have set $E = L^p(U,\eta)$.

We assume that the operator

$$\mathcal{T} \colon C_{0,\infty} \to E \tag{9}$$

satisfies the following properties:

(d$\mathcal{T}$1) $\mathcal{T}(\ell)$ has zero mean for every $\ell \in C_{0,\infty}$:

$$\int_U \mathcal{T}(\ell)(u)\, \mathrm{d}\eta(u) = 0\,;$$

[3] A subset of a separable metric space is also separable, [13].

(d$\mathcal{T}$2) for every $R > 0$ there exists a positive constant $L_{\mathcal{T},R}$ such that for every $\ell_1, \ell_2 \in C_{0,\infty}$ with $\|\ell_1\|_E, \|\ell_2\|_E \le R$

$$\|\mathcal{T}(\ell_1) - \mathcal{T}(\ell_2)\|_E \le L_{\mathcal{T},R} \, \|\ell_1 - \ell_2\|_E\,; \tag{10}$$

(d$\mathcal{T}$3) there exist a monotone increasing function $\omega\colon [0,+\infty) \to [0,+\infty)$, for which

$$\exists \limsup_{s\to 0^+} \frac{\omega(s)}{s} =: \underline{\omega} \in [0,+\infty) \qquad \text{and} \qquad \exists \limsup_{s\to +\infty} \frac{\omega(s)}{s} =: \overline{\omega} \in [0,+\infty),$$

and a positive constant $C_{\mathcal{T}}$ such that for every $\ell \in C_{0,\infty}$ and for η-a.e. $u \in U$

$$|\mathcal{T}(\ell)(u)| \le C_{\mathcal{T}}\, \omega(\ell(u)).$$

The entropy functional that we consider is

$$\begin{aligned} \mathcal{H}\colon C_\varepsilon &\to E \\ \ell &\mapsto \ell\, [I(\ell) - \log \circ\, \ell]\,, \end{aligned} \tag{11}$$

where $I(\ell)$ is the negative entropy of the probability density ℓ, namely

$$I(\ell) := \int_U \ell(u)\, \log(\ell(u))\, \mathrm{d}\eta(u).$$

In Lemma 1 below, we show that the entropy functional $\mathcal{H}$ is well defined for every choice of $0 < r_\varepsilon < 1 < R_\varepsilon < +\infty$. Notice that $\mathcal{H}$ can be written as

$$\mathcal{H}(\ell) = -\ell\, \log \circ\, \ell - (-I(\ell))\, \ell,$$

where, in the term $I(\ell)\ell$, the presence of the (negative) entropy $I(\ell)$ has the effect that $\mathcal{H}(\ell)$ has zero mean, while the presence of ℓ guarantees that $\mathcal{H}(\ell)$ is compatible with a replicator dynamics, see, *e.g.*, [5].

Lemma 1. *Let $\mathcal{T}$ be a functional defined as in (9) satisfying properties (d$\mathcal{T}$1), (d$\mathcal{T}$2), and (d$\mathcal{T}$3). Then, for every $\varepsilon > 0$ there exist two constants r_ε and R_ε, such that $0 < r_\varepsilon < 1 < R_\varepsilon < +\infty$, and the functional*

$$\begin{aligned} \mathcal{R}^\varepsilon\colon C_\varepsilon &\to E \\ \ell &\mapsto \mathcal{T}(\ell) + \varepsilon\, \mathcal{H}(\ell) \end{aligned} \tag{12}$$

satisfies the following properties:

($\mathcal{R}$1) *$\mathcal{R}^\varepsilon(\ell)$ has zero mean for every $\ell \in C_\varepsilon$:*

$$\int_U \mathcal{R}^\varepsilon(\ell)(u)\, \mathrm{d}\eta(u) = 0\,;$$

($\mathcal{R}$2) *$\mathcal{R}^\varepsilon$ is Lipschitz continuous with respect to the p-norm: there exists $L_\varepsilon > 0$ such that for every $\ell_1, \ell_2 \in C_\varepsilon$*

$$\|\mathcal{R}^\varepsilon(\ell_1) - \mathcal{R}^\varepsilon(\ell_2)\|_E \le L_\varepsilon\, \|\ell_1 - \ell_2\|_E\,;$$

($\mathcal{R}3$) *there exists a positive constant θ_ε such that for every $\ell \in C_\varepsilon$*

$$\ell + \theta_\varepsilon \mathcal{R}^\varepsilon(\ell) \in C_\varepsilon;$$

where $C_\varepsilon := C_{r_\varepsilon, R_\varepsilon}$ is defined as in (6), and $\mathcal{H}$ is the entropy regularization functional defined in (11).

Proof. We first check that $\mathcal{R}^\varepsilon$ is well defined for any choice of $R_\varepsilon > 1 > r_\varepsilon > 0$. To this end, it is sufficient to show that $\mathcal{H}(C_\varepsilon) \subset L^\infty(U, \eta)$, because $L^\infty(U, \eta) \subset E$ and $C_\varepsilon \subset C_{0,\infty}$. Fixed $\ell \in C_\varepsilon$, for every $u \in U$ we have that $\ell(u) = r_\varepsilon\, \zeta(u) + (1 - \zeta(u))\, R_\varepsilon$, with $0 \leq \zeta(u) \leq 1$. Thus, using the convexity of the function $x \mapsto x \log(x)$ we get

$$I(\ell) \leq r_\varepsilon \,\log(r_\varepsilon) \int_U \zeta(u)\, \mathrm{d}\eta(u) + R_\varepsilon \,\log(R_\varepsilon) \int_U (1 - \zeta(u))\, \mathrm{d}\eta(u).$$

Since ℓ is a probability density it is straightforward to check that

$$\int_U \zeta(u)\, \mathrm{d}\eta(u) = \frac{R_\varepsilon - 1}{R_\varepsilon - r_\varepsilon},$$

and therefore

$$I(\ell) \leq \frac{R_\varepsilon - 1}{R_\varepsilon - r_\varepsilon}\, r_\varepsilon \,\log(r_\varepsilon) + \left(1 - \frac{R_\varepsilon - 1}{R_\varepsilon - r_\varepsilon}\right) R_\varepsilon \,\log(R_\varepsilon).$$

For the sake of simplicity we define

$$\alpha_\varepsilon := \frac{(R_\varepsilon - 1)\, r_\varepsilon}{R_\varepsilon - r_\varepsilon} \in (0{,}1), \tag{13}$$

and the previous inequality takes the simpler form

$$I(l) \leq \alpha_\varepsilon \,\log(r_\varepsilon) + (1 - \alpha_\varepsilon)\,\log(R_\varepsilon) =: k_\varepsilon. \tag{14}$$

Moreover, by Jensen's inequality

$$I(\ell) \geq \int_U \ell(u)\, \mathrm{d}\eta(u)\, \log\left(\int_U \ell(u)\, \mathrm{d}\eta(u)\right) = 0. \tag{15}$$

From the bounds on ℓ and $I(\ell)$ it is easy to check that

$$-R_\varepsilon \,\log(R_\varepsilon) \leq \mathcal{H}(\ell) \leq R_\varepsilon\, k_\varepsilon + \frac{1}{e}\,,$$

so that $\mathcal{H}(\ell) \in L^\infty(U, \eta)$. Since $\mathcal{H}(\ell)$ has zero mean and ($\mathrm{d}\mathcal{T}1$) holds, ($\mathcal{R}1$) also holds. To prove ($\mathcal{R}2$) it is sufficient to show that $\mathcal{H}$ is Lipschitz on C_ε because (10) holds with $L_{\mathcal{T}, R_\varepsilon}$ for every $\ell_1, \ell_2 \in C_\varepsilon$. First, we observe that since $x \mapsto x\log(x)$ is Lipschitz on $[r_\varepsilon, R_\varepsilon]$ whenever $r_\varepsilon > 0$

$$\begin{aligned}\sup_{u \in U} \frac{|\ell_1(u)\log(\ell_1(u)) - \ell_2(u)\log(\ell_2(u))|}{|\ell_1(u) - \ell_2(u)|} &\leq \sup_{x_1, x_2 \in [r_\varepsilon, R_\varepsilon]} \frac{|x_1 \log(x_1) - x_2 \log(x_2)|}{|x_1 - x_2|} \\ &\leq \sup_{x \in [r_\varepsilon, R_\varepsilon]} |\log(x) + 1| =: \wp_\varepsilon\end{aligned}$$

and therefore

$$
\begin{aligned}
|\mathcal{H}(\ell_1)(u)-\mathcal{H}(\ell_2)(u)| &\le |I(\ell_1)\ell_1(u)-I(\ell_2)\ell_2(u)| + |\ell_1(u)\log(\ell_1(u))-\ell_2(u)\log(\ell_2(u))| \\
&\le |I(\ell_1)-I(\ell_2)|\,|\ell_1(u)| + |I(\ell_2)|\,|\ell_1(u)-\ell_2(u)| + \wp_\varepsilon\,|\ell_1(u)-\ell_2(u)| \\
&\le R_\varepsilon\,|I(\ell_1)-I(\ell_2)| + k_\varepsilon\,|\ell_1(u)-\ell_2(u)| + \wp_\varepsilon\,|\ell_1(u)-\ell_2(u)| \\
&\le R_\varepsilon \int_U |\ell_1(u)\log(\ell_1(u))-\ell_2(u)\log(\ell_2(u))|\,\mathrm{d}\eta(u) + (k_\varepsilon+\wp_\varepsilon)\,|\ell_1(u)-\ell_2(u)| \\
&\le R_\varepsilon\,\wp_\varepsilon \int_U |\ell_1(u)-\ell_2(u)|\,\mathrm{d}\eta(u) + (k_\varepsilon+\wp_\varepsilon)\,|\ell_1(u)-\ell_2(u)|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|\mathcal{H}(\ell_1)-\mathcal{H}(\ell_2)\|_E &\le R_\varepsilon\,\wp_\varepsilon\,\|\ell_1(u)-\ell_2(u)\|_{L^1(U,\eta)} + (k_\varepsilon+\wp_\varepsilon)\;\|\ell_1-\ell_2\|_E \\
&\le R_\varepsilon\,\wp_\varepsilon\,\|\ell_1-\ell_2\|_E + (k_\varepsilon+\wp_\varepsilon)\;\|\ell_1-\ell_2\|_E \\
&= (R_\varepsilon\,\wp_\varepsilon + k_\varepsilon + \wp_\varepsilon)\,\|\ell_1-\ell_2\|_E\,,
\end{aligned}
$$

where we have used the inclusion $L^p(U,\eta)\subset L^1(U,\eta)$. Regardless of p, the Lipschitz constant of $\mathcal{R}^\varepsilon$ in not greater than $L_\varepsilon = L_{\mathcal{T},R_\varepsilon} + R_\varepsilon\,\wp_\varepsilon + k_\varepsilon + \wp_\varepsilon$. We have proved so far that for any choice of $R_\varepsilon > 1 > r_\varepsilon > 0$ the functional $\mathcal{R}^\varepsilon$ is well defined, and both $(\mathcal{R}1)$ and $(\mathcal{R}2)$ hold. Now we want to show that for some particular choices of r_ε and R_ε also $(\mathcal{R}3)$ holds. To this end, r_ε must satisfy the following inequality

$$
\varepsilon\,\log\left(\frac{3}{4\,r_\varepsilon}\right) \ge C_{\mathcal{T}}\,\frac{\omega(\frac{4}{3}r_\varepsilon)}{r_\varepsilon}. \tag{16}
$$

Observe that there exists at least one $r_\varepsilon\in(0,1)$ which satisfies such inequality because

$$
\limsup_{r_\varepsilon\to 0^+} \varepsilon\log\left(\frac{3}{4\,r_\varepsilon}\right) = +\infty \qquad \text{and} \qquad \limsup_{r_\varepsilon\to 0^+} C_{\mathcal{T}}\,\frac{\omega(\frac{4}{3}r_\varepsilon)}{r_\varepsilon} = \frac{4}{3}\,C_{\mathcal{T}}\,\underline{\omega}.
$$

Instead, R_ε must satisfy the following inequality

$$
\alpha_\varepsilon\,\log\left(\frac{R_\varepsilon}{r_\varepsilon}\right) \ge \frac{2\,C_{\mathcal{T}}\,\omega(R_\varepsilon)}{\varepsilon\,R_\varepsilon}. \tag{17}
$$

Observe that there exists at least one $R_\varepsilon > 1$ which satisfies such inequality because

$$
\limsup_{R_\varepsilon\to+\infty} \alpha_\varepsilon\,\log\left(\frac{R_\varepsilon}{r_\varepsilon}\right) = +\infty \qquad \text{and} \qquad \limsup_{R_\varepsilon\to+\infty} \frac{2\,C_{\mathcal{T}}\,\omega(R_\varepsilon)}{\varepsilon\,R_\varepsilon} = \frac{2\,C_{\mathcal{T}}\,\overline{\omega}}{\varepsilon}\,.
$$

Let us show now that we can choose θ_ε such that for η-a.e. $u\in U$

$$
\ell(u) + \theta_\varepsilon\,\mathcal{R}^\varepsilon(\ell)(u) \le R_\varepsilon\,.
$$

In fact, using $(\mathrm{d}\mathcal{T}3)$ and (14) we get

$$
\begin{aligned}
\ell(u) + \theta_\varepsilon\;[T(\ell)(u) + \varepsilon\,\mathcal{H}(\ell)(u)] &\le \ell(u) + \theta_\varepsilon\;[C_{\mathcal{T}}\,\omega(\ell(u)) + \varepsilon\,\mathcal{H}(\ell)(u)] \\
&\le \ell(u) + \theta_\varepsilon\;[C_{\mathcal{T}}\,\omega(R_\varepsilon) + \varepsilon\,\ell(u)\;(I(\ell) - \log(\ell(u)))] \\
&\le \ell(u) + \theta_\varepsilon\;[C_{\mathcal{T}}\,\omega(R_\varepsilon) + \varepsilon\,\ell(u)\,(\alpha_\varepsilon\;\log(r_\varepsilon) + (1-\alpha_\varepsilon)\log(R_\varepsilon) - \log(\ell(u)))]\,.
\end{aligned} \tag{18}
$$

Because of (17)

$$\lim_{\ell(u)\to R_\varepsilon^-} [C_{\mathcal{T}}\,\omega(R_\varepsilon) + \varepsilon\,\ell(u)\,(\alpha_\varepsilon\,\log(r_\varepsilon) + (1-\alpha_\varepsilon)\log(R_\varepsilon) - \log(\ell(u)))] =$$

$$= C_{\mathcal{T}}\,\omega(R_\varepsilon) - \varepsilon\,\alpha_\varepsilon\,R_\varepsilon\,\log\left(\frac{R_\varepsilon}{r_\varepsilon}\right) \le -C_{\mathcal{T}}\,\omega(R_\varepsilon) < 0,$$

so that there exists $R'_\varepsilon < R_\varepsilon$ for which the right-hand side of (18) is not greater than R_ε whenever $\ell(u) \in [R'_\varepsilon, R_\varepsilon]$. Otherwise, if $\ell(u) \le R'_\varepsilon$

$$\ell(u) + \theta_\varepsilon\,\mathcal{R}^\varepsilon(\ell)(u) \le R'_\varepsilon + \theta_\varepsilon\,\left[C_{\mathcal{T}}\,\omega(R_\varepsilon) + \varepsilon\,R_\varepsilon\,k_\varepsilon + \frac{\varepsilon}{e}\right],$$

and there exists at least one positive value, that we shall denote θ_ε^1, for which the right-hand side of the previous inequality is not greater than R_ε for every positive $\theta_\varepsilon \le \theta_\varepsilon^1$. Similarly, we show that we can choose θ_ε such that for η-a.e. $u \in U$

$$\ell(u) + \theta_\varepsilon\,\mathcal{R}^\varepsilon(\ell)(u) \ge r_\varepsilon\,.$$

In fact, using (d$\mathcal{T}$3) and (15)

$$\ell(u) + \theta_\varepsilon\,[\mathcal{T}(\ell)(u) + \varepsilon\,\mathcal{H}(\ell)(u)] \ge \ell(u) + \theta_\varepsilon\,[-C_{\mathcal{T}}\,\omega(\ell(u)) - \varepsilon\,\ell(u)\log(\ell(u))].$$

If $\ell(u) > \frac{4}{3}\,r_\varepsilon$

$$\ell(u) + \theta_\varepsilon\,[\mathcal{T}(\ell)(u) + \varepsilon\,\mathcal{H}(\ell)(u)] \ge \frac{4}{3}\,r_\varepsilon + \theta_\varepsilon\,[-C_{\mathcal{T}}\,\omega(R_\varepsilon) - \varepsilon\,R_\varepsilon\,\log(R_\varepsilon)]\,,$$

and there exists at least one positive value, that we shall denote θ_ε^2, for which the right-hand side of the previous inequality is not less than r_ε for every positive $\theta_\varepsilon \le \theta_\varepsilon^2$. Otherwise, if $\ell(u) \le \frac{4}{3}\,r_\varepsilon$

$$\begin{aligned}\ell(u) + \theta_\varepsilon\,[\mathcal{T}(\ell)(u) + \varepsilon\,\mathcal{H}(\ell)(u)] &\ge \ell(u) + \theta_\varepsilon\,\ell(u)\left[-C_{\mathcal{T}}\,\frac{\omega(\ell(u))}{\ell(u)} - \varepsilon\,\log(\ell(u))\right]\\ &\ge \ell(u) + \theta_\varepsilon\,\ell(u)\left[-C_{\mathcal{T}}\frac{\omega\left(\frac{4}{3}\,r_\varepsilon\right)}{r_\varepsilon} + \varepsilon\,\log\left(\frac{3}{4\,r_\varepsilon}\right)\right]\\ &\ge \ell(u),\end{aligned}$$

where the last inequality follows from (16). Finally, taking $0 < \theta_\varepsilon \le \min\{\theta_\varepsilon^1, \theta_\varepsilon^2\}$ we get

$$r_\varepsilon \le \ell(u) + \theta_\varepsilon\,\mathcal{R}^\varepsilon(\ell(u)) \le R_\varepsilon$$

for η-a.e. $u \in U$. □

Remark 1. The Lipschitz continuity of $\mathcal{R}^\varepsilon$ is a consequence of the boundedness of C_ε with respect to the p-norm, in fact $\mathcal{T}$ is Lipschitz, and not just locally Lipschitz, on C_ε.

Remark 2. With a similar line of reasoning, it can be proved that for any choice of $0 \le a < 1 < b < +\infty$ there exists $\theta > 0$ such that for every $\ell \in C_{a,b}$ we have that $\ell + \theta\,\varepsilon\,\mathcal{H}(\ell) \in C_{a,b}\,.$

We now have a well-posedness theorem for the labels dynamics.

Theorem 2. *For every $\varepsilon > 0$, there exist two constants $r_\varepsilon \in (0,1)$, and $R_\varepsilon \in (1, +\infty)$ such that for every $T > 0$, and every $\overline{\ell} \in C_\varepsilon$ there exists a unique curve $\ell \colon [0,T] \to C_\varepsilon$ of class C^1 satisfying*

$$\frac{\mathrm{d}}{\mathrm{d}t}\ell_t = \mathcal{R}^\varepsilon(\ell_t) \quad \text{in } [0,T], \quad \ell_0 = \overline{\ell},$$

where $\mathcal{R}^\varepsilon$ is defined as in (12).

Proof. For any choice of $0 < r_\varepsilon < 1 < R_\varepsilon < +\infty$ the space C_ε is a convex and closed subset of the Banach space E. The map $[0,T] \times C_\varepsilon \ni (t, \ell) \mapsto R^\varepsilon(\ell) \in E$ is independent of t, and therefore it satisfies hypothesis (ii) of Theorem 1. For Lemma 1 hypothesis (i) is also satisfied with $L = L_{\mathcal{T},R_\varepsilon} + \varepsilon\,(R_\varepsilon\,\wp_\varepsilon + k_\varepsilon + \wp_\varepsilon)$. Again, for Lemma 1 there exists a particular choice of r_ε and R_ε for which such a map also satisfies hypothesis (iii) of Theorem 1. □

We conclude this section with few remarks.

Remark 3. The modelling choice of setting the probability densities in C_ε instead of the larger space $C_{0,\infty}$ is due to two main reasons. First, the lower bound r_ε guarantees that the entropy functional is Lipschitz continuous. Second, the upper bound R_ε guarantees the non-degeneracy of the term $\ell \log \circ\, \ell$. In fact, given the vector space structure of E, for any choice of θ, if $\ell \in E$ then $\ell + \theta\,[\mathcal{T}(\ell) + \varepsilon\,\mathcal{H}(\ell)] \in E$ if and only if $\mathcal{H}(\ell) \in E$. However, if $\ell \in E$ it's not necessarily true that $\ell \log \circ\, \ell \in E$, unless ℓ is bounded.

Remark 4. The choice of r_ε and R_ε is made once ε is fixed. Therefore, for $\varepsilon \to 0^+$ from (16) and (17) we get that $r_\varepsilon \to 0^+$ and $R_\varepsilon \to +\infty$. More precisely, for every monotone decreasing sequence $\varepsilon_n \to 0^+$ there exist a monotone decreasing sequence $r_{\varepsilon_n} \to 0^+$ and a monotone increasing sequence $R_{\varepsilon_n} \to +\infty$. For this reason

$$C_{0,\infty} = \overline{\bigcup_{n=1}^{\infty} C_{\varepsilon_n}}^{\|\cdot\|_E} \tag{19}$$

which, given the encapsulation of the sets C_{ε_n}, we shall denote it simply by $C_\varepsilon \to C_{0,\infty}$ as $\varepsilon \to 0^+$. To check why this is true, let

$$\begin{aligned} \chi_\varepsilon \colon \mathbb{R} &\to \{0,1\} \\ x &\mapsto \chi_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in [r_\varepsilon, R_\varepsilon], \\ 0 & \text{otherwise}, \end{cases} \end{aligned}$$

and for every $\ell \in C_{0,\infty}$ consider the function

$$\mathrm{Avg}(\chi_\varepsilon(\ell)\,\ell) := \left(\int_U \chi_\varepsilon(\ell(u))\,\ell(u)\,\mathrm{d}\eta(u) \right)^{-1} \chi_\varepsilon(\ell)\,\ell\,.$$

Clearly $\mathrm{Avg}(\chi_\varepsilon(\ell)\,\ell)$ is a probability density and it's greater than r_ε for η-a.e. $u \in U$, however it's not necessarily smaller than R_ε for η-a.e. $u \in U$. Nevertheless, fixed $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $m > n$, and $\mathrm{Avg}(\chi_{\varepsilon_n}(\ell)\,\ell) \in C_{\varepsilon_m}$. For $n \to +\infty$ we have that $||\mathrm{Avg}(\chi_{\varepsilon_n}(\ell)\ell) - \ell||_E \to 0$.

Remark 5. The entropy regularization functional acts similarly to the Laplacian operator on the space of labels U. Indeed, if we consider a dynamics of the type $\dot{\ell}_t = \mathcal{H}(\ell_t)$, for $\Delta t \ll 1$ we can approximate $\ell_{t+\Delta t} \approx \ell_t + \Delta_t \mathcal{H}(\ell_t)$. For $u \in U$ such that $\ell(u) \leq e^{I(\ell_t)}$ we have that $\mathcal{H}(\ell_t)(u) \geq 0$, otherwise $\mathcal{H}(\ell_t)(u) \leq 0$. Iterating, ℓ_t gets closer to $u \mapsto 1$ since $I(\ell_t)$ gets closer to zero. This is shown in detail in [5] taking $J(x, u, x'u') = 0$ for the replicator dynamics.

6 The coupled problem

Recall that the state of our system is described by pairs $y = (x, \ell) \in Y_\varepsilon = \mathbb{R}^d \times C_\varepsilon$. The element $x \in \mathbb{R}^d$ denotes the position of an agent, whereas the element $\ell \in C_\varepsilon$ denotes a distribution of labels, see (6). The space Y_ε is equipped with the norm $\|y\|_{\overline{Y}} = |x| + \|\ell\|_E$, where $\overline{Y} = \mathbb{R}^d \times E$, see (8). A distribution of states will be described by an element $\Lambda^\varepsilon \in \mathcal{P}(Y_\varepsilon)$. We will be concerned with the evolution of Λ^ε, given an initial $\overline{\Lambda}^\varepsilon$, determined by the laws of evolution of x and ℓ, which are going to be discussed below.
For $y = (x, \ell) \in Y_\varepsilon$ and $\Psi \in \mathcal{P}_1(Y_\varepsilon)$, see (2), we define the entropic vector field $b^\varepsilon_\Psi : Y_\varepsilon \to \overline{Y}$ through

$$b^\varepsilon_\Psi(y) := \begin{pmatrix} v_\Psi(y) \\ \mathcal{R}^\varepsilon_\Psi(y) \end{pmatrix}. \tag{20}$$

The first component of b^ε_Ψ is a velocity field in $\mathbb{R}^d$ determined by the global state of the system Ψ; the second component is expressed in terms of an entropy perturbed vector filed $\mathcal{T}_\Psi(y)$ for the labels dynamics: $\mathcal{R}^\varepsilon_\Psi(y) = \mathcal{T}_\Psi(y) + \varepsilon\, \mathcal{H}(\ell)$, where $\mathcal{H}$ is defined as in (12). For $y \in Y$ and $\Psi \in \mathcal{P}_1(Y)$, the entropy free vector field $b_\Psi : Y \to \overline{Y}$ is defined as

$$b_\Psi(y) := \begin{pmatrix} v_\Psi(y) \\ \mathcal{T}_\Psi(y) \end{pmatrix}, \tag{21}$$

where $Y = \mathbb{R}^d \times C_{0,\infty}$, see (7). In order to state the regularity assumptions that we make on b^ε_Ψ and b_Ψ, we will discuss separately the assumptions that we make on v_Ψ and on $\mathcal{T}_\Psi$. Given the arbitrariness of ε and since we will be also interested in the limit as $\varepsilon \to 0$, we state our regularity assumptions on the larger space Y instead of Y_ε. In doing so, we'll be interested in comparing the entropic and the entropy free solutions, upon knowing that $Y_\varepsilon \to Y$ in the sense of (19). We recall that $B^{Y_\varepsilon}_R \subset B^Y_R$ and we'll use the notation $\mathcal{P}(K)$ to denote a probability measure with support contained on a given bounded subset $K \subset Y$, for example $K = B^Y_R$ or $K = B^{Y_\varepsilon}_R$. Notice that trivially we have $\mathcal{P}(K) \subset \mathcal{P}_1(Y)$.
We assume that the velocity field $v_\Psi : Y \to \mathbb{R}^d$ satisfies the following conditions:

(v1) for every $R > 0$, for every $\Psi \in \mathcal{P}(B^Y_R)$, $v_\Psi \in \mathrm{Lip}(B^Y_R; \mathbb{R}^d)$ uniformly with respect to Ψ, namely there exists $L_{v,R} > 0$ such that

$$|v_\Psi(y^1) - v_\Psi(y^2)| \leq L_{v,R}\, ||y^1 - y^2||_{\overline{Y}};$$

(v2) for every $R > 0$, there exists $L_{v,R} > 0$ such that for every $y \in B^Y_R$, and for every Ψ^1, $\Psi^2 \in \mathcal{P}(B^Y_R)$

$$|v_{\Psi^1}(y) - v_{\Psi^2}(y)| \leq L_{v,R}\, \mathcal{W}_1(\Psi^1, \Psi^2);$$

(v3) there exists $M_v > 0$ such that for every $y \in Y$, and for every $\Psi \in \mathcal{P}_1(Y)$ there holds

$$|v_\Psi(y)| \le M_v \,(1 + \|y\|_{\overline{Y}} + m_1(\Psi)).$$

We now describe the assumptions on $\mathcal{T}$. For every $\Psi \in \mathcal{P}_1(Y)$, let $\mathcal{T}_\Psi \colon Y \to E$ be an operator such that

(T0) $\mathcal{T}_\Psi(y)$ has zero mean for every $(y, \Psi) \in Y \times \mathcal{P}_1(Y)$:

$$\int_U \mathcal{T}_\Psi(y)(u)\,\mathrm{d}\eta(u) = 0\,;$$

(T1) for every $(y, \Psi) \in Y \times \mathcal{P}_1(Y)$, there exists $M_{\mathcal{T}} > 0$ such that

$$\|\mathcal{T}_\Psi(y)\|_E \le M_{\mathcal{T}}\,(1 + \|y\|_E + m_1(\Psi));$$

(T2) for every $R > 0$ there exists $L_{\mathcal{T},R} > 0$ such that for every (y^1, Ψ^1), $(y^2, \Psi^2) \in B_R^Y \times \mathcal{P}(B_R^Y)$

$$||\mathcal{T}_{\Psi^1}(y^1) - \mathcal{T}_{\Psi^2}(y^2)||_E \le L_{\mathcal{T},R}\,\Big(||y^1 - y^2||_{\overline{Y}} + \mathcal{W}_1(\Psi^1, \Psi^2)\Big)\,;$$

(T3) there exist a monotone increasing function $\omega\colon\, [0, +\infty) \to [0, +\infty)$, for which

$$\exists \limsup_{s\to 0^+} \frac{\omega(s)}{s} =: \underline{\omega} \in [0, +\infty) \qquad \text{and} \qquad \exists \limsup_{s\to\infty} \frac{\omega(s)}{s} =: \overline{\omega} \in [0, +\infty),$$

and a constant $C_{\mathcal{T}} > 0$ such that for every $(y, \Psi) \in Y \times \mathcal{P}_1(Y)$

$$|\mathcal{T}_\Psi(y)(u)| \le C_{\mathcal{T}}\,\omega(\ell(u))$$

for η-almost every $u \in U$.

Since we're interested in the limit as ε goes to zero, we wish that the entropy free solution also exists. To this end, we require that

(T4) for every $R > 0$ there exists $\delta_R > 0$ such that for every $(y, \Psi) \in B_R^Y \times \mathcal{P}(B_R^Y)$ we have

$$\ell(u) + \delta_R\,\mathcal{T}_\Psi(y)(u) \ge 0$$

for η-almost every $u \in U$.

We now have the following regularity properties for the entropy free vector field b_Ψ.

Proposition 1. *For $y \in Y$ and $\Psi \in \mathcal{P}_1(Y)$, define $b_\Psi(y)$ as in (21). Assume that $v_\Psi \colon Y \to \mathbb{R}^d$ satisfies properties* (v1)-(v3)*, and $\mathcal{T}_\Psi \colon Y \to E$ satisfies properties* (T0)-(T4)*. Then*

(i) for every $R > 0$, there exists $L_R > 0$ such that for every $\Psi \in \mathcal{P}(B_R^Y)$, and for every y^1, $y^2 \in B_R^Y$

$$||b_\Psi(y^1) - b_\Psi(y^2)||_{\overline{Y}} \le L_R\,||y^1 - y^2||_{\overline{Y}}\,;$$

(ii) *for every* $R > 0$*, there exists* $L_R > 0$ *such that for every* Ψ^1, $\Psi^2 \in \mathcal{P}(B_R^Y)$*, and for every* $y \in B_R^Y$

$$||b_{\Psi^1}(y) - b_{\Psi^2}(y)||_{\overline{Y}} \leq L_R \mathcal{W}_1(\Psi^1, \Psi^2);$$

(iii) *for every* $R > 0$*, there exists* $\theta_R > 0$ *such that for every* $y \in B_R^Y$*, and for every* $\Psi \in \mathcal{P}(B_R^Y)$

$$y + \theta_R b_\Psi(y) \in Y;$$

(iv) *there exists* $M > 0$ *such that for every* $y \in Y$ *and for every* $\Psi \in \mathcal{P}_1(Y)$ *there holds*

$$||b_\Psi(y)||_{\overline{Y}} \leq M(1 + \|y\|_{\overline{Y}} + m_1(\Psi)).$$

Proof. Property (i) is a consequence of (v1) and (T2); property (ii) follows from (v2) and (T2); property (iv) is a direct consequence of (v3) and (T1). Lastly, because of the vector space structure of $\mathbb{R}^d$, property (iii) follows from (T0) and (T4). Notice that (T0) and (T4) can be combined together because for our notation $\Psi \in \mathcal{P}(B_R^Y) \subset \mathcal{P}_1(Y)$. Observe also that we haven't use (T3). □

We now have the following regularity properties for the entropic vector field b^ε_Ψ.

Proposition 2. *Assume that* $v_\Psi : Y \to \mathbb{R}^d$ *satisfies* (v1)-(v3) *and* $\mathcal{T}_\Psi : Y \to E$ *satisfies* (T0)-(T4)*. Then for every* $\varepsilon > 0$ *there exist* $r_\varepsilon \in (0,1)$ *and* $R_\varepsilon \in (1, +\infty)$ *such that for every* $(y, \Psi) \in Y_\varepsilon \times \mathcal{P}_1(Y_\varepsilon)$*, with* $\mathrm{spt}(\Psi) \subset Y_\varepsilon$ *, the vector field* $b^\varepsilon_\Psi(y)$ *defined as in* (20) *satisfies the following properties:*

(i) *for every* $R > 0$*, there exists* $L_{\varepsilon,R} > 0$ *such that for every* $\Psi \in \mathcal{P}(B_R^{Y_\varepsilon})$*, and for every* y^1, $y^2 \in B_R^{Y_\varepsilon}$

$$||b^\varepsilon_\Psi(y^1) - b^\varepsilon_\Psi(y^2)||_{\overline{Y}} \leq L_{\varepsilon,R} \, ||y^1 - y^2||_{\overline{Y}}\,; \tag{22}$$

(ii) *for every* $R > 0$*, there exists* $L_R > 0$ *such that for every* Ψ^1, $\Psi^2 \in \mathcal{P}(B_R^{Y_\varepsilon})$*, and for every* $y \in B_R^{Y_\varepsilon}$

$$||b^\varepsilon_{\Psi^1}(y) - b^\varepsilon_{\Psi^2}(y)||_{\overline{Y}} \leq L_R \mathcal{W}_1(\Psi^1, \Psi^2); \tag{23}$$

(iii) *for every* $R > 0$*, there exists* $\theta_R > 0$ *such that for every* $y \in B_R^{Y_\varepsilon}$ *and for every* $\Psi \in \mathcal{P}(B_R^{Y_\varepsilon})$

$$y + \theta_R b^\varepsilon_\Psi(y) \in Y_\varepsilon\,; \tag{24}$$

(iv) *there exists* $M_\varepsilon > 0$ *such that for every* $y \in Y_\varepsilon$ *and for every* $\Psi \in \mathcal{P}_1(Y_\varepsilon)$ *with* $\mathrm{spt}(\Psi) \subset Y_\varepsilon$ *there holds*

$$||b^\varepsilon_\Psi(y)||_{\overline{Y}} \leq M_\varepsilon \,(1 + \|y\|_{\overline{Y}} + m_1(\Psi)). \tag{25}$$

Proof. We start noticing that $B_R^{Y_\varepsilon} \subset B_R^Y$ and therefore for our notation $\mathcal{P}(B_R^{Y_\varepsilon}) \subset \mathcal{P}(B_R^Y)$. Moreover, $\mathcal{H}$ is Lipschitz on C_ε, with respect to the p-norm, for every choice of $r_\varepsilon \in (0,1)$ and $R_\varepsilon \in (1, +\infty)$. Then, properties (i) and (ii) follow from Proposition 1. Observe that the Lipchitz constant in (23) does not depend on ε because $\mathcal{H}$ does not depend on Ψ, and

$L_{\varepsilon,R} = L_R + \varepsilon\,\mathrm{Lip}(\mathcal{H})$. Similarly, property (iv) follows from Proposition 1 because for a measure $\Psi \in \mathcal{P}_1(Y_\varepsilon)$, which support is at most Y_ε , it holds that $\Psi \in \mathcal{P}_1(Y)$, and because

$$\begin{aligned} ||\mathcal{H}(\ell)||_E &\leq (I(\ell) + (-\log(r_\varepsilon) \vee \log(R_\varepsilon)))\,\|\ell\|_E \\ &\leq (\log(R_\varepsilon) + (-\log(r_\varepsilon) \vee \log(R_\varepsilon)))\,\|\ell\|_E =: h_\varepsilon\,\|\ell\|_E\,, \end{aligned} \tag{26}$$

having in the end $M_\varepsilon = M + \varepsilon\,h_\varepsilon$. To prove (iii), because of the vector space structure of $\mathbb{R}^d$, we simply have to show that for every $R > 0$, there exists $\theta_R > 0$ such that for every $y = (x, \ell) \in B_R^{Y_\varepsilon}$ and $\Psi \in \mathcal{P}(B_R^{Y_\varepsilon})$

$$\ell + \theta_R\,[\mathcal{T}_\Psi(y) + \varepsilon\,\mathcal{H}(\ell)] \in C_\varepsilon. \tag{27}$$

Chosen r_ε and R_ε satisfying (16) and (17) respectively, the proof is identical to the one of Lemma 1. □

Remark 6. It is important to notice that since the choice of r_ε and R_ε has to be independent of the pair (y, Ψ) also the constant $C_\mathcal{T}$ has to be independent of (y, Ψ). For the same reason we can't allow $C_\mathcal{T}$ to be dependent on the constant R of formula (27).

Remark 7. In the proof of Proposition 2 assumption (T4) has not been used. Nevertheless, it can be used to show that for every $R > 0$ there exists $\theta_R > 0$ such that for every $(y, \Psi) \in B_R^{Y_\varepsilon} \times \mathcal{P}(B_R^{Y_\varepsilon})$ and for η-a.e. $u \in U$

$$\ell + \theta_R\,[\mathcal{T}_\Psi(y) + \varepsilon\,\mathcal{H}(\ell)] \geq r_\varepsilon,$$

making inequality (16) redundant. Thus, one can pick $r_\varepsilon = \varepsilon$ as long as (T4) holds.

Remark 8. Manipulating inequality (17) it is easy to see that $\varepsilon\,\log R_\varepsilon \not\to 0$ whenever $\overline{\omega} \neq 0$. Indeed, by (17), and recalling (13), we have that

$$\frac{2\,C_\mathcal{T}\,w(R_\varepsilon)}{R_\varepsilon} + \alpha_\varepsilon\,\varepsilon\,\log(r_\varepsilon) \leq \varepsilon\,\alpha_\varepsilon\log(R_\varepsilon) \leq \varepsilon\,\log(R_\varepsilon),$$

taking the limit as $\varepsilon \to 0^+$

$$\lim_{\varepsilon\to 0^+} \varepsilon\,\log(R_\varepsilon) \geq 2\,C_\mathcal{T}\,\overline{\omega}\,.$$

For every $\varepsilon > 0$, we now consider a particle system of N agents evolving according to

$$\begin{cases} \dot{x}_t^{\varepsilon,i} = v_{\Lambda_t^N}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}), \\ \dot{\ell}_t^{\varepsilon,i} = \mathcal{T}_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}) + \varepsilon\,\mathcal{H}(\ell_t^{\varepsilon,i}) \end{cases} \qquad \text{for } i = 1, \ldots, N,\ t \in [0, T] \tag{28}$$

where $x_t^{\varepsilon,i} \in \mathbb{R}^d$, $\ell_t^{\varepsilon,i} \in C_\varepsilon$ for each $i \in \{1, \ldots, N\}$, and

$$\Lambda_t^{\varepsilon,N} := \frac{1}{N}\sum_{i=1}^N \delta_{(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i})} \tag{29}$$

is the empirical measure associated with the system. Recalling that definition of b^ε in (20), the evolution (28) can be written in compact form as

$$\dot{y}_t^{\varepsilon,i} = b^\varepsilon_{\Lambda_t^{\varepsilon,N}}(y_t^{\varepsilon,i}) \qquad \text{for } i = 1, \ldots, N,\ t \in [0, T]. \tag{30}$$

We discuss the well-posedness of system (30) for every choice of an initial datum $\overline{y}^{\varepsilon,i} = (\overline{x}^{\varepsilon,i}, \overline{\ell}^{\varepsilon,i})$, for $i = 1, \ldots, N$.

Theorem 3. *Assume that* $v_\Psi\colon Y \to \mathbb{R}^d$ *satisfies* (v1)-(v3) *and* $\mathcal{T}_\Psi\colon Y \to E$ *satisfies* (T0)-(T4). *Then, for every* $\varepsilon > 0$ *there exist* $r_\varepsilon \in (0,1)$ *and* $R_\varepsilon \in (1,+\infty)$ *such that for every choice of* $\overline{y}^{\varepsilon,i} \in Y_\varepsilon$ *,* $i = 1,\dots,N$*, the system* (30) *has a unique solution in* $[0,T]$*. Moreover, for* $i = 1,\dots,N$

$$||y_t^{\varepsilon,i}||_{\overline{Y}} \le \left(\sum_{i=1}^N ||\overline{y}^{\varepsilon,i}||_{\overline{Y}} + NMT\right) e^{(2M+\varepsilon\, h_\varepsilon)T}. \tag{31}$$

Proof. We introduce the vector-valued variable

$$\boldsymbol{y}^\varepsilon := (y^{\varepsilon,1},\dots,y^{\varepsilon,N}) \in Y_\varepsilon^N \subset \overline{Y}^N,$$

which we endow with the norm

$$\|\boldsymbol{y}^\varepsilon\|_{\overline{Y}^N} := \frac{1}{N}\sum_{i=1}^N ||y^{\varepsilon,i}||_{\overline{Y}},$$

and the associated empirical measure

$$\Lambda^{\varepsilon,N} := \frac{1}{N}\sum_{i=1}^N \delta_{y^{\varepsilon,i}}\,,$$

which belongs to $\mathcal{P}(B_R^{Y_\varepsilon})$ whenever $\boldsymbol{y} \in (B_R^{Y_\varepsilon})^N$. Consider the map $\boldsymbol{b}^{\varepsilon,N}\colon Y_\varepsilon^N \to \overline{Y}^N$ whose components are defined through

$$b_i^{\varepsilon,N}(\boldsymbol{y}^\varepsilon) := b^\varepsilon_{\Lambda^N}(y^{\varepsilon,i}).$$

Then the Cauchy problem associated with (30) can be written as

$$\begin{cases} \dot{\boldsymbol{y}}_t^\varepsilon = \boldsymbol{b}^{\varepsilon,N}(\boldsymbol{y}_t^\varepsilon),\\ \boldsymbol{y}_0^\varepsilon = \overline{\boldsymbol{y}}^\varepsilon. \end{cases}$$

In order to apply Corollary 1 to the system above, we first notice that assumption (ii) is automatically satisfied since the system is autonomous. To check the other assumptions, we fix a ball $B_R^{Y_\varepsilon^N}$ and notice that $B_R^{Y_\varepsilon^N} \subset (B_{NR}^{Y_\varepsilon})^N$. Applying (24) with $\Psi = \Lambda^{\varepsilon,N}$ to each component $y^{\varepsilon,i}$ of $\boldsymbol{y}^\varepsilon$, we get that assumption (iii) of Corollary 1 is satisfied with $\theta = \theta_{RN}$. We now show that assumption (i) holds. Fix $\boldsymbol{y}_1^\varepsilon$, $\boldsymbol{y}_2^\varepsilon \in B_R^{Y_\varepsilon^N} \subset (B_{NR}^{Y_\varepsilon})^N$, and let $\Lambda_1^{\varepsilon,N}$, $\Lambda_2^{\varepsilon,N}$ be the associated empirical measures. Recalling (1), we notice that

$$\mathcal{W}_1(\Lambda_1^{\varepsilon,N},\Lambda_2^{\varepsilon,N}) \le \frac{1}{N}\sum_{i=1}^N ||y_1^{\varepsilon,i} - y_2^{\varepsilon,i}||_{\overline{Y}} = ||\boldsymbol{y}_1^\varepsilon - \boldsymbol{y}_2^\varepsilon||_{\overline{Y}^N},$$

an therefore by the triangle inequality, (22), and (23)

$$\begin{aligned} ||\boldsymbol{b}^{\varepsilon,N}(\boldsymbol{y}_1^\varepsilon) - \boldsymbol{b}^{\varepsilon,N}(\boldsymbol{y}_2^\varepsilon)||_{\overline{Y}^N} &= \frac{1}{N}\sum_{i=1}^N ||b^\varepsilon_{\Lambda_1^N}(y_1^{\varepsilon,i}) - b^\varepsilon_{\Lambda_2^N}(y_2^{\varepsilon,i})||_{\overline{Y}} \\ &\le L_{NR}\,\mathcal{W}_1(\Lambda_1^{\varepsilon,N},\Lambda_2^{\varepsilon,N}) + \frac{L_{\varepsilon,NR}}{N}\sum_{i=1}^N ||y_1^{\varepsilon,i} - y_2^{\varepsilon,i}||_{\overline{Y}} \\ &\le 2\,L_{\varepsilon,NR}\,||\boldsymbol{y}_1^\varepsilon - \boldsymbol{y}_2^\varepsilon||_{\overline{Y}^N} \end{aligned}$$

for every $\boldsymbol{y}_1^\varepsilon, \boldsymbol{y}_2^\varepsilon \in B_R^{Y_\varepsilon^N}$. To see that also assumption (iv) holds, we apply (25), upon noticing that $m_1(\Lambda^{\varepsilon,N}) = \|\boldsymbol{y}^\varepsilon\|_{\overline{Y}^N}$,

$$
\begin{aligned}
||\boldsymbol{b}^{\varepsilon,N}(\boldsymbol{y}^\varepsilon)||_{\overline{Y}^N} &= \frac{1}{N}\sum_{i=1}^N ||b^\varepsilon_{\Lambda^{\varepsilon,N}}(y^{\varepsilon,i})||_{\overline{Y}} \\
&\leq \frac{M_\varepsilon}{N}\sum_{i=1}^N (1 + ||y^{\varepsilon,i}||_{\overline{Y}} + m_1(\Lambda^{\varepsilon,N})) = M_\varepsilon\,(1 + 2||\boldsymbol{y}^\varepsilon||_{\overline{Y}^N}).
\end{aligned}
$$

Therefore we can apply Corollary 1, which proves the existence and uniqueness of the solution. Finally, because of (25), and (26), we have that

$$
\begin{aligned}
||\boldsymbol{y}_t^\varepsilon||_{\overline{Y}^N} &\leq ||\overline{\boldsymbol{y}}^\varepsilon||_{\overline{Y}^N} + \frac{1}{N}\sum_{i=1}^N \int_0^T ||b^\varepsilon_{\Lambda_s^{\varepsilon,N}}(y_s^{\varepsilon,i})||_{\overline{Y}}\,\mathrm{d}s \\
&\leq ||\overline{\boldsymbol{y}}^\varepsilon||_{\overline{Y}^N} + \frac{1}{N}\sum_{i=1}^N \int_0^T \Big(||b_{\Lambda_s^{\varepsilon,N}}(y_s^{\varepsilon,i})||_{\overline{Y}} + \varepsilon||\mathcal{H}(\ell_s^{\varepsilon,i})||_E\Big)\,\mathrm{d}s \\
&\leq ||\overline{\boldsymbol{y}}^\varepsilon||_{\overline{Y}^N} + \frac{1}{N}\sum_{i=1}^N \int_0^T \Big[M(1 + ||y_s^{\varepsilon,i}||_{\overline{Y}} + m_1(\Lambda_s^{\varepsilon,N})) + \varepsilon\,h_\varepsilon\,||\ell_s^{\varepsilon,i}||_E\Big]\,\mathrm{d}s \\
&\leq ||\overline{\boldsymbol{y}}^\varepsilon||_{\overline{Y}^N} + \frac{1}{N}\sum_{i=1}^N \int_0^T \Big[M(1 + ||y_s^{\varepsilon,i}||_{\overline{Y}} + ||\boldsymbol{y}_s^\varepsilon||_{\overline{Y}^N}) + \varepsilon\,h_\varepsilon\,||y_s^{\varepsilon,i}||_{\overline{Y}}\Big]\,\mathrm{d}s \\
&= ||\overline{\boldsymbol{y}}^\varepsilon||_{\overline{Y}^N} + \int_0^T \Big[M(1 + 2\,||\boldsymbol{y}_s^\varepsilon||_{\overline{Y}^N}) + \varepsilon\,h_\varepsilon\,||\boldsymbol{y}_s^\varepsilon||_{\overline{Y}^N}\Big]\,\mathrm{d}s \\
&= ||\overline{\boldsymbol{y}}^\varepsilon||_{\overline{Y}^N} + MT + (2M + \varepsilon\,h_\varepsilon)\int_0^T ||\boldsymbol{y}_s^\varepsilon||_{\overline{Y}^N}\,\mathrm{d}s\,,
\end{aligned}
$$

and therefore by Grönwall's Lemma

$$
||\boldsymbol{y}_t^\varepsilon||_{\overline{Y}^N} \leq (||\overline{\boldsymbol{y}}^\varepsilon||_{\overline{Y}^N} + MT)\,e^{(2M+\varepsilon\,h_\varepsilon)T}.
$$

Noticing that $||y^{\varepsilon,i}||_{\overline{Y}} \leq N||\boldsymbol{y}^\varepsilon||_{\overline{Y}^N}$ gives us (31). □

Since we are also interested in the entropy free case, we consider also a particle system of N agents evolving according to

$$
\begin{cases} \dot{x}_t^i = v_{\Lambda_t^N}(x_t^i, \ell_t^i), \\ \dot{\ell}_t^i = \mathcal{T}_{\Lambda_t^N}(x_t^i, \ell_t^i) \end{cases} \qquad \text{for } i = 1, \ldots, N,\ \ t \in [0,T] \tag{32}
$$

where $x_t^i \in \mathbb{R}^d$, $\ell_t^i \in C_{0,\infty}$ for each $i \in \{1, \ldots, N\}$, and

$$
\Lambda_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{(x_t^i,\ell_t^i)} \tag{33}
$$

is the empirical measure associated with the system. Recalling that definition of b in (21), the evolution (32) can be written in compact form as

$$
\dot{y}_t^i = b_{\Lambda_t^N}(y_t^i) \qquad \text{for } i = 1, \ldots, N,\ \ t \in [0,T]. \tag{34}
$$

We discuss the well-posedness of system (34) for every choice of an initial datum $\overline{y}^i = (\overline{x}^i, \overline{\ell}^i)$, for $i = 1, \dots, N$.

Theorem 4. *Assume that* $v_\Psi : Y \to \mathbb{R}^d$ *satisfies* (v1)-(v3) *and* $\mathcal{T}_\Psi : Y \to E$ *satisfies* (T0)-(T4). *Then, for every choice of* $\overline{y}^i \in Y$, $i = 1, \dots, N$, *the system* (34) *has a unique solution in* $[0, T]$. *Moreover*

$$||y_t^i||_{\overline{Y}} \leq \left(\sum_{i=1}^N ||\overline{y}^i||_{\overline{Y}} + NMT \right) e^{2MT}. \tag{35}$$

Proof. Since Proposition 1 holds, with a similar line of reasoning of Theorem 3 one can prove the statement. □

We are now interested in comparing the entropic and the entropy free solutions, that is the solutions of (30) and (34) respectively. Fixed the initial data $\overline{\boldsymbol{y}}^\varepsilon \in Y_\varepsilon^N$ and $\overline{\boldsymbol{y}} \in Y^N$, we shall denote by $\boldsymbol{y}^\varepsilon \in Y_\varepsilon^N$ and by $\boldsymbol{y} \in Y^N$ the entropic and entropy free solutions respectively. We shall also denote by $\Lambda^{\varepsilon,N}$ and Λ^N the associated empirical measures. Then, we have that

$$\begin{aligned} \|\boldsymbol{y}_t - \boldsymbol{y}_t^\varepsilon\|_{\overline{Y}^N} &= \frac{1}{N} \sum_{i=1}^N \left\| y_t^i - y_t^{\varepsilon,i} \right\|_{\overline{Y}} \\ &\leq \frac{1}{N} \sum_{i=1}^N \|\overline{y} - \overline{y}^\varepsilon\|_{\overline{Y}} + \frac{1}{N} \sum_{i=1}^N \int_0^t \left\| b_{\Lambda_s^N}(y_s^i) - b^\varepsilon_{\Lambda_s^{\varepsilon,N}}(y_s^{\varepsilon,i}) \right\|_{\overline{Y}} \mathrm{d}s. \end{aligned} \tag{36}$$

We focus our attention on the integrand of (36): by the triangle inequality, (22), (23), and (26)

$$\begin{aligned} \left\| b_{\Lambda_s^N}(y_s^i) - b^\varepsilon_{\Lambda_s^{\varepsilon,N}}(y_s^{\varepsilon,i}) \right\|_{\overline{Y}} &\leq \left\| b_{\Lambda_s^N}(y_s^i) - b_{\Lambda_s^N}(y_s^{\varepsilon,i}) \right\|_{\overline{Y}} + \left\| b_{\Lambda_s^N}(y_s^{\varepsilon,i}) - b^\varepsilon_{\Lambda_s^{\varepsilon,N}}(y_s^{\varepsilon,i}) \right\|_{\overline{Y}} \\ &\leq L_\Re \left\| y_s^i - y_s^{\varepsilon,i} \right\|_{\overline{Y}} + \left\| b_{\Lambda_s^N}(y_s^{\varepsilon,i}) - b_{\Lambda_s^{\varepsilon,N}}(y_s^{i,\varepsilon}) \right\|_{\overline{Y}} + \varepsilon \left\| \mathcal{H}(\ell_s^{\varepsilon,i}) \right\|_E \\ &\leq L_\Re \left\| y_s^i - y_s^{\varepsilon,i} \right\|_{\overline{Y}} + L_\Re \mathcal{W}_1(\Lambda_s^N, \Lambda_s^{\varepsilon,N}) + \varepsilon \, h_\varepsilon \left\| \ell_s^{\varepsilon,i} \right\|_E \\ &\leq L_\Re \left\| y_s^i - y_s^{\varepsilon,i} \right\|_{\overline{Y}} + L_\Re \|\boldsymbol{y}_s - \boldsymbol{y}_s^\varepsilon\|_{\overline{Y}^N} + \varepsilon \, h_\varepsilon \, \Re, \end{aligned} \tag{37}$$

where

$$\Re := \sum_{i=1}^N \left(||\overline{y}^i||_{\overline{Y}} + ||\overline{y}^{\varepsilon,i}||_{\overline{Y}} + NMT \right) e^{(2M+\varepsilon h_\varepsilon)T} \geq \sup_{t\in[0,T]} \max_{i=1,\dots,N} \left(||y_t^i||_{\overline{Y}} \vee ||y_t^{\varepsilon,i}||_{\overline{Y}} \right)$$

because of (31) and (35). Combining (36), and (37), we get

$$\|\boldsymbol{y}_t - \boldsymbol{y}_t^\varepsilon\|_{\overline{Y}^N} \leq \|\overline{\boldsymbol{y}} - \overline{\boldsymbol{y}}^\varepsilon\|_{\overline{Y}^N} + \varepsilon \, h_\varepsilon \, \Re T + 2 \, L_\Re \int_0^T \|\boldsymbol{y}_s - \boldsymbol{y}_s^\varepsilon\|_{\overline{Y}^N} \, \mathrm{d}s \, .$$

and by Grönwall's lemma

$$\|\boldsymbol{y}_t - \boldsymbol{y}_t^\varepsilon\|_{\overline{Y}^N} \leq \left(\|\overline{\boldsymbol{y}} - \overline{\boldsymbol{y}}^\varepsilon\|_{\overline{Y}^N} + \Re \, \varepsilon \, h_\varepsilon \, T \right) e^{2 \, L_\Re \, T}. \tag{38}$$

Even if $\overline{\boldsymbol{y}}^\varepsilon \to \overline{\boldsymbol{y}}$ as $\varepsilon \to 0^+$, estimate (38) does not guarantee convergence of the entropic solution to the entropy free solution because of *Remark* 8. To avoid this obstacle, one has to avoid using inequality (17) by making stronger assumptions on $\mathcal{T}$. For example, instead of (T3) we could require that

(T3') for every $R > 0$, there exists $\theta_R > 0$ such that for every $\varepsilon > 0$ and for every $(y, \Psi) \in (B_R^{Y_\varepsilon}, \mathcal{P}(B_R^{Y_\varepsilon}))$

$$\ell + \theta\, \mathcal{T}_\Psi(y) \in C_\varepsilon$$

as long as $r_\varepsilon \in (0,1)$ and $R_\varepsilon \in (1, +\infty)$.

Notice that if (T3') holds, recalling *Remark* 2, and using the convexity of C_ε, one can easily check that for every $R > 0$, there exists $\theta_R > 0$ such that for every $\varepsilon > 0$, and for every $(y, \Psi) \in (B_R^{Y_\varepsilon}, \mathcal{P}(B_R^{Y_\varepsilon}))$

$$\ell + \theta\, [\mathcal{T}_\Psi(y) + \varepsilon\, \mathcal{H}(\ell)] \in C_\varepsilon$$

for any choice of $r_\varepsilon \in (0,1)$ and $R_\varepsilon \in (1, +\infty)$. Therefore we could pick $r_\varepsilon = \varepsilon$, because of *Remark* 7, and $R_\varepsilon = 1 + \varepsilon^{-1}$, so that $\varepsilon\, h_\varepsilon \to 0$ as $\varepsilon \to 0^+$, having convergence of the entropic solution to the free entropy one. However, assumption (T3') is too stringent, in fact it is not a priori satisfied by the replicator dynamics.

7 Eulerian and Lagrangian Solutions

We are interested in the mean-field limit as $N \to \infty$ of the solutions $\boldsymbol{y}_t^{\varepsilon,N}$ to (30) or, equivalently, the limiting behaviour of the associated empirical measure $\Lambda_t^{\varepsilon,N}$ defined in (29). In order to do so, we first need to recall the concept of Eulerian solution to the continuity equation.

Definition 1 (Eulerian solution, [3]). *Let $\Lambda^\varepsilon \in C^0([0,T]; (\mathcal{P}_1(Y_\varepsilon), \mathcal{W}_1))$ and let $\overline{\Lambda}^\varepsilon \in \mathcal{P}_c(Y_\varepsilon)$ be a given initial datum. We say that Λ^ε is an Eulerian solution to the initial value problem for the continuity equation*

$$\partial_t \Lambda_t^\varepsilon + \operatorname{div}(b_{\Lambda_t^\varepsilon}^\varepsilon\, \Lambda_t^\varepsilon) = 0 \tag{39}$$

starting from $\overline{\Lambda}^\varepsilon$ if and only if $\Lambda_0^\varepsilon = \overline{\Lambda}^\varepsilon$ and, for every $\phi \in C_b^1([0,T] \times \overline{Y})$,

$$\int_{Y_\varepsilon} \phi(t,y)\, \mathrm{d}\Lambda_t^\varepsilon(y) - \int_{Y_\varepsilon} \phi(0,y)\, \mathrm{d}\Lambda_0^\varepsilon(y) = \int_0^t \int_{Y_\varepsilon} (\partial_t \phi(s,y) + D\phi(s,y) \cdot b_{\Lambda_s^\varepsilon}^\varepsilon(y))\, \mathrm{d}\Lambda_s^\varepsilon(y)\, \mathrm{d}s, \tag{40}$$

where $D\phi(s,y)$ is the Fréchet differential of ϕ in the y variable.

The main result of this work is the following theorem, stating the existence of a unique Eulerian solution to (39) and its characterization as the mean-field limit of solutions to the discrete problem (28).

Theorem 7. Let $r > 0$, and $\overline{\Lambda}^\varepsilon \in \mathcal{P}(B_r^{Y_\varepsilon})$ be a given initial datum. Then

(i) there exists a unique Eulerian solution $t \mapsto \Lambda_t^\varepsilon$ to (39) starting from $\overline{\Lambda}^\varepsilon$;

(ii) if $\overline{\Lambda}^{\varepsilon,N} = \frac{1}{N}\sum_{i=1}^N \delta_{\overline{y}_{i,N}^\varepsilon}$ is a sequence of atomic measures in $\mathcal{P}(B_r^{Y_\varepsilon})$ such that

$$\lim_{N\to\infty} \mathcal{W}_1(\overline{\Lambda}^\varepsilon, \overline{\Lambda}^{\varepsilon,N}) = 0$$

and, for fixed N, $\Lambda_t^{\varepsilon,N}$ are the empirical measures associated with the unique solution to (28) with initial datum $\overline{y}_{i,N}^\varepsilon$, we have

$$\lim_{N\to\infty} \mathcal{W}_1(\Lambda_t^\varepsilon, \Lambda_t^{\varepsilon,N}) = 0 \qquad \text{uniformly with respect to } t \in [0,T].$$

The proof of Theorem 7 will be based on a fixed point argument and on the notion of Lagrangian solution, which are going to be introduced below.
We start by proving an auxiliary well-posedness result for an ODE in Y_ε of the form

$$\dot{y}_t^\varepsilon = b^\varepsilon_{\Psi_t^\varepsilon}(y_t^\varepsilon), \qquad y_0^\varepsilon = \overline{y}^\varepsilon \tag{41}$$

where $[0,T] \ni t \mapsto \Psi_t^\varepsilon \in \mathcal{P}_1(Y_\varepsilon)$ is a continuous curve and $\overline{y}^\varepsilon \in Y_\varepsilon$.

Proposition 3. *Assume that for every $y \in Y$ and $\Psi \in \mathcal{P}_1(Y)$ the velocity $v_\Psi\colon Y \to \mathbb{R}^d$ satisfies* (v1)-(v3) *and the operator $\mathcal{T}_\Psi\colon Y \to E$ satisfies* (T0)-(T3)*. Let $\Psi^\varepsilon \in C^0([0,T];(\mathcal{P}_1(Y_\varepsilon),W_1))$ and assume that there exists $R>0$ such that $\Psi_t \in \mathcal{P}(B_R^{Y_\varepsilon})$ for all $t \in [0,T]$. Then, for every choice of $\overline{y}^\varepsilon \in Y_\varepsilon$, the ODE* (41) *has a unique solution.*

Proof. We set $A(t,y) := b^\varepsilon_{\Psi_t^\varepsilon}(y)$. Since $t \mapsto \Psi_t^\varepsilon$ is continuous, using (23) we get that, for any fixed $y \in Y_\varepsilon$, $A(\cdot,y)$ is continuous. Moreover, $A(t,\cdot)$ is locally Lipschitz because of (22) with the Lipschitz constant independent of t because $\Psi_t \in \mathcal{P}(B_R^{Y_\varepsilon})$ for all $t \in [0,T]$. For the same reason, by (25), we have the sub-linear growth of A:

$$||A(t,y)||_{\overline{Y}} \le M_\varepsilon\,(1 + ||y||_{\overline{Y}} + m_1(\Psi_t)) \le M_\varepsilon\,(1 + ||y||_{\overline{Y}} + R).$$

Similarly, from (24), we have that condition (iii) of Corollary 1 is satisfied, so that existence and uniqueness of the solution follow directly from Corollary 1. □

In view of the previous result the following definition is justified.

Definition 2 (Transition map)**.** *The transition map $\mathbf{Y}^\varepsilon_\Psi(t,s,\overline{y}^\varepsilon)$ associated with the ODE* (41)*, replacing the initial condition by $y_s^\varepsilon = \overline{y}^\varepsilon$ is defined through*

$$\mathbf{Y}^\varepsilon_\Psi(t,s,\overline{y}^\varepsilon) = y_t^\varepsilon\,, \tag{42}$$

where $t \mapsto y_t^\varepsilon$ is the unique solution to (41).

We can now proceed to defining the notion of Lagrangian solution to (39).

Definition 3 (Lagrangian solutions)**.** *Let $\Lambda^\varepsilon \in C^0([0,T];(\mathcal{P}_1(Y_\varepsilon),\mathcal{W}_1))$ and let $\overline{\Lambda}^\varepsilon \in \mathcal{P}_c(Y_\varepsilon)$ be a given initial datum. We say that Λ^ε is a Lagrangian solution to the initial value problem for* (39) *starting from $\overline{\Lambda}^\varepsilon$ if and only if it satisfies the fixed point condition*

$$\Lambda_t^\varepsilon = \mathbf{Y}^\varepsilon_{\Lambda^\varepsilon}(t,0,\cdot)_\#\overline{\Lambda}^\varepsilon \qquad \textit{for every } 0 \le t \le T, \tag{43}$$

where $\mathbf{Y}^\varepsilon_{\Lambda^\varepsilon}(t,s,\overline{y}^\varepsilon)$ are the transition maps associated with the ODE (41).

Remark 9. Recalling the definition of push-forward measure, it can be easily proven that Lagrangian solutions are also Eulerian solutions. Indeed, for every $\phi \in C_b^1([0,T]\times\overline{Y})$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{Y_\varepsilon}\phi(t,y)\,\mathrm{d}\Lambda_t^\varepsilon(y) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{Y_\varepsilon}\phi(t,\mathbf{Y}^\varepsilon_{\Lambda^\varepsilon}(t,0,\overline{y}))\,\mathrm{d}\overline{\Lambda}^\varepsilon(\overline{y})\\
&= \int_{Y_\varepsilon}\Big[\partial_t\phi(t,\mathbf{Y}^\varepsilon_{\Lambda^\varepsilon}(t,0,\overline{y})) + D\phi(t,\mathbf{Y}^\varepsilon_{\Lambda^\varepsilon}(t,0,\overline{y}))\cdot b^\varepsilon_{\Lambda_t^\varepsilon}(\mathbf{Y}^\varepsilon_{\Lambda^\varepsilon}(t,0,\overline{y}))\Big]\,\mathrm{d}\overline{\Lambda}^\varepsilon(\overline{y})\\
&= \int_{Y_\varepsilon}\Big(\partial_t\phi(t,y) + D\phi(t,y)\cdot b^\varepsilon_{\Lambda_t^\varepsilon}(y)\Big)\,\mathrm{d}\Lambda_t^\varepsilon(y)
\end{aligned} \tag{44}$$

which, upon integrating, coincides with (40).

Remark 10. For a fixed $N \in \mathbb{N}$, let $\Lambda_t^{\varepsilon,N}$ be the empirical measures associated with the unique solution to (30) with initial datum $\overline{y}^{\varepsilon,i}$, $i = 1, \ldots, N$. If we now set $\overline{\Lambda}^{\varepsilon,N} = \frac{1}{N}\sum_{i=1}^N \delta_{\overline{y}^{\varepsilon,i}}$, by Definition 2 there holds

$$\Lambda_t^{\varepsilon,N} = \mathbf{Y}^{\varepsilon}_{\Lambda^{\varepsilon,N}}(t,0,\cdot)_{\#}\overline{\Lambda}^{\varepsilon} \qquad \text{for every } 0 \le t \le T. \tag{45}$$

Indeed, $\Lambda^{\varepsilon,N}$ is continuous, since $t \mapsto y_t^{\varepsilon,i}$ are continuous, and it has compact support because of (31).

We now want to show that an infinite-dimensional converse of Proposition 3 holds, proving that indeed, in our case, every Eulerian solution is also a Lagrangian solution. This stems out of a general abstract principle known as the *superposition principle* in the version introduced in [2]. In the statement below, the *evaluation map* ev_t is defined, at a given $t \in [0,T]$, by

$$\mathrm{ev}_t(\gamma) := \gamma(t) \qquad \text{for all } \gamma \in C([0,T];E);$$

we also use the notion of cylindrical functions, which are defined in the following way. We say that $\phi \in C_b^1(E)$ is a cylindrical function if there exists a function $\varphi \in C_b^1(\mathbb{R}^N)$ and $z_1', \ldots, z_N' \in E'$ such that

$$\Phi(y) = \varphi(\langle z_1', y\rangle, \ldots, \langle z_1', y\rangle),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality map between E' and E.

Theorem 5 (Superposition principle). *Let $(E, \|\cdot\|)$ be a separable Banach space, let $b\colon (0,T)\times E \to E$ be a Borel vector field, and let $[0,T] \ni t \mapsto \mu_t \in \mathcal{P}(E)$ be a continuous curve with*

$$\int_0^T \int_E \|b_t\|_E \, \mathrm{d}\mu_t \, \mathrm{d}t < +\infty\,.$$

If

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t + \mathrm{div}(b_t\,\mu_t) = 0$$

in duality with cylindrical functions $\phi \in C_b^1(E)$[4]*, then there exists $\boldsymbol{\eta} \in \mathcal{P}(C([0,T];E))$ concentrated on absolutely continuous solutions to the ODE $\dot{y} = b_t(y)$ and with $(\mathrm{ev}_t)_{\#}\boldsymbol{\eta} = \mu_t$ for all $t \in [0,T]$.*

Proof. See [2, Theorem 5.2] for the proof. □

Combining the *Superposition Principle* with the uniqueness granted from Proposition 3, we can prove the announced equivalence result. Notice that the proof has an intermediate step, since in order to apply Proposition 3 we first must ensure that an Eulerian solution Λ_t^{ε} has equi-compact support for all t. We are able to deduce this from the *Superposition Principle* and the assumption that the initial datum $\overline{\Lambda}^{\varepsilon} \in \mathcal{P}_c(Y_\varepsilon)$.

[4] By this we mean that the definition of weak solutions to the continuity equation has only to be satisfied for test functions that are cylindrical.

Theorem 6. *Let $\Lambda^\varepsilon \in C^0([0,T];(\mathcal{P}_1(Y_\varepsilon),\mathcal{W}_1))$ and let $\overline{\Lambda}^\varepsilon \in \mathcal{P}_c(Y_\varepsilon)$ be a given initial datum. Assume that Λ^ε is an Eulerian solution to the initial value problem for $\partial_t \Lambda^\varepsilon_t + \operatorname{div}(b^\varepsilon_{\Lambda^\varepsilon_t}\Lambda^\varepsilon_t) = 0$ starting from $\overline{\Lambda}^\varepsilon$, in the sense of (39), then there exists $R > 0$ such that $\Lambda^\varepsilon_t \in \mathcal{P}(B^{Y_\varepsilon}_R)$ for all $t \in [0,T]$, and*

$$\Lambda^\varepsilon_t = \mathbf{Y}^\varepsilon_{\Lambda^\varepsilon}(t,0,\cdot)_\# \overline{\Lambda}^\varepsilon \qquad \text{for every } 0 \le t \le T, \tag{46}$$

where $\mathbf{Y}^\varepsilon_{\Lambda^\varepsilon}(t,s,\overline{y}^\varepsilon)$ are the transition maps associated with the ODE (41).

Proof. Since $\Lambda^\varepsilon \in C^0([0,T];(\mathcal{P}_1(Y_\varepsilon),\mathcal{W}_1))$, the map

$$t \mapsto m_1(\Lambda^\varepsilon_t) = \int_{Y_\varepsilon} \|y\|_{\overline{Y}} \,\mathrm{d}\Lambda^\varepsilon_t(y)$$

is continuous and, hence, bounded in $[0,T]$. We set $b_t(y) := b^\varepsilon_{\Lambda^\varepsilon_t}(y)$ for $y \in Y_\varepsilon$, and extend it to zero on $\overline{Y} \setminus Y_\varepsilon$. Using (25), we have

$$\begin{aligned}
\int_0^T \int_{\overline{Y}} ||b^\varepsilon_t(y)||_{\overline{Y}} \,\mathrm{d}\Lambda^\varepsilon_t(y)\,\mathrm{d}t &= \int_0^T \int_{Y_\varepsilon} ||b^\varepsilon_t(y)||_{\overline{Y}} \,\mathrm{d}\Lambda^\varepsilon_t(y)\,\mathrm{d}t \\
&\le \int_0^T \int_{Y_\varepsilon} M_\varepsilon\,(1 + \|y\|_{\overline{Y}} + m_1(\Lambda^\varepsilon_t))\,\mathrm{d}\Lambda^\varepsilon_t(y)\,\mathrm{d}t \\
&\le T M_\varepsilon \left(1 + 2 \max_{t\in[0,T]} m_1(\Lambda^\varepsilon_t)\right) < +\infty.
\end{aligned}$$

Hence we can apply Theorem 5 with $E = \overline{Y}$ and $\mu_t = \Lambda^\varepsilon_t$, obtaining that $\Lambda^\varepsilon_t = (\mathrm{ev}_t)_\#\boldsymbol{\eta}$ for a suitable $\boldsymbol{\eta} \in \mathcal{P}(C([0,T];\overline{Y}))$ concentrated on absolutely continuous solutions to the ODE

$$\dot{y}^\varepsilon = b^\varepsilon_{\Lambda^\varepsilon_t}(y^\varepsilon) \quad \text{in } [0,T], \text{ for every initial datum } \overline{y}^\varepsilon \in Y_\varepsilon. \tag{47}$$

Now, using (25) again, we have

$$||b^\varepsilon_{\Lambda^\varepsilon_t}(y)||_{\overline{Y}} \le M_\varepsilon(1 + \|y\|_{\overline{Y}} + m_1(\Lambda^\varepsilon_t)) \le M_{\Lambda^\varepsilon}(1 + \|y\|_{\overline{Y}}) \tag{48}$$

where we set $M_{\Lambda^\varepsilon} := M_\varepsilon(1 + \max_{t\in[0,T]} m_1(\Lambda^\varepsilon_t))$. The equality $\overline{\Lambda}^\varepsilon = \Lambda^\varepsilon_0 = (\mathrm{ev}_0)_\#\boldsymbol{\eta}$, which reads

$$\int_{C([0,T];\overline{Y})} \phi(\gamma(0))\,\mathrm{d}\boldsymbol{\eta}(\gamma) = \int_{Y_\varepsilon} \phi(y)\,\mathrm{d}\overline{\Lambda}^\varepsilon(y)$$

for each $\phi \in C_b(Y_\varepsilon)$, implies that $\boldsymbol{\eta}$ is concentrated on the set of solutions to (47) satisfying $y^\varepsilon(0) \in B^{Y_\varepsilon}_r$, where r is such that $\operatorname{spt}(\overline{\Lambda}^\varepsilon) \subset B^{Y_\varepsilon}_r$. With (48) and the Grönwall's Lemma, each of these solutions must satisfy $y^\varepsilon(t) \in B^{Y_\varepsilon}_R$, where R is explicitly given by

$$R := R_{r,M_\varepsilon,\Lambda^\varepsilon,T} = (r + M_{\Lambda^\varepsilon}T)e^{M_{\Lambda^\varepsilon}T}.$$

From the equality $\Lambda^\varepsilon_t = (\mathrm{ev}_t)_\#\boldsymbol{\eta}$ we deduce that $\Lambda_t \in \mathcal{P}_1(B^{Y_\varepsilon}_R)$ for all $t \in [0,T]$. We can therefore apply Proposition 3 and exploit uniqueness of the solution to the Cauchy problem to deduce the representation

$$\gamma(t) = \mathbf{Y}^\varepsilon_{\Lambda^\varepsilon}(t,0,\gamma(0))$$

with $\gamma(0) \in B_r^{Y_\varepsilon}$, for each continuous path $\gamma \in \mathrm{spt}(\boldsymbol{\eta})$. With the equality $\Lambda_t^\varepsilon = (\mathrm{ev}_t)_\# \boldsymbol{\eta}$, this gives

$$\int_{Y_\varepsilon} \phi(y) \,\mathrm{d}\Lambda_t^\varepsilon(y) = \int_{C([0,T];\overline{Y})} \phi(\mathbf{Y}^\varepsilon_{\Lambda^\varepsilon}(t,0,\gamma(0))) \,\mathrm{d}\boldsymbol{\eta}(\gamma) = \int_{Y_\varepsilon} \phi(\mathbf{Y}^\varepsilon_{\Lambda^\varepsilon}(t,0,y)) \,\mathrm{d}\overline{\Lambda}^\varepsilon(y)$$

for each $\phi \in C_b(Y_\varepsilon)$, which implies the conclusion. □

Remark 11. The same notions of Eulerian and Lagrangian solutions, with the proper adjustments, hold for the entropy free vector field (21) in the larger space Y. With the same line of reasoning the equivalence between the two can indeed be proved.

8 Mean-field limit

As mentioned in the previous section, we are interested in the mean-field limit as $N \to \infty$ of the solutions $\boldsymbol{y}_t^{\varepsilon,N}$ to (30) or, equivalently, the limiting behaviour of the associated empirical measure $\Lambda_t^{\varepsilon,N}$, defined in (29). Such behaviour is described in Theorem 7, stating the existence of a unique Eulerian solution to (39) and its characterization as the mean-field limit of solutions to the discrete problem (28). In this section, we prove Theorem 7 and compare the entropic mean-field limit to the entropy free mean-field limit. As a preliminary step towards the proof, we need the following lemma, ensuring that the size of the support of a Lagrangian solution in the sense of Definition 3 can be a priori estimated from the data of the problem.

Lemma 2. *Assume that for every $y \in Y$ and $\Psi \in \mathcal{P}_1(Y)$ the velocity $v_\Psi : Y \to \mathbb{R}^d$ satisfies* (v1)-(v3) *and the operator $\mathcal{T}_\Psi : Y \to E$ satisfies* (T0)-(T4)*. Let $\Lambda^\varepsilon \in C^0([0,T];(\mathcal{P}_1(Y_\varepsilon),\mathcal{W}_1))$ and let $\overline{\Lambda}^\varepsilon \in \mathcal{P}_c(Y_\varepsilon)$ be a given initial datum. Fix $r > 0$ such that $\overline{\Lambda}^\varepsilon$ has support in $B_r^{Y_\varepsilon}$, and let M_ε be the constant given by* (25)*. Assume that Λ^ε is the a Lagrangian solution to the initial value problem for* (39) *starting from $\overline{\Lambda}^\varepsilon$ in the sense of* (44)*. Then, for $R = (r + M_\varepsilon T)\, e^{2M_\varepsilon T}$ we have*

$$\Lambda_t^\varepsilon \in \mathcal{P}(B_R^{Y_\varepsilon}) \qquad \textit{for all } t \in [0,T].$$

Proof. For r, R as in statement, it suffices to show that

$$\max_{y \in B_r^{Y_\varepsilon}} \|\mathbf{Y}^\varepsilon_{\Lambda^\varepsilon}(t,0,y)\|_{\overline{Y}}$$

for all $t \in [0,T]$. Indeed, if this holds the statement follows immediately from (44) and elementary properties of the push-forward measure, taking into account that $\overline{\Lambda}^\varepsilon$ has support in $B_r^{Y_\varepsilon}$. To prove the above claim, we first observe that by definition of Lagrangian solutions and the fact that $\overline{\Lambda}^\varepsilon \in \mathcal{P}(B_r^{Y_\varepsilon})$ we immediately have

$$m_1(\Lambda_t^\varepsilon) \leq \max_{y \in B_r^{Y_\varepsilon}} \|\mathbf{Y}^\varepsilon_{\Lambda^\varepsilon}(t,0,y)\|_{\overline{Y}} \tag{49}$$

for all $t \in [0,T]$. We now set $f(s) = \max_{y \in B_r^{Y_\varepsilon}} \|\mathbf{Y}^\varepsilon_{\Lambda^\varepsilon}(s,0,y)\|_{\overline{Y}}$. Then, we have by definition of the transition map, (25) and (49) that for every choice of $y \in B_r^{Y_\varepsilon}$

$$\begin{aligned} \|\mathbf{Y}_{\Lambda^\varepsilon}(t,0,y)\|_{\overline{Y}} &\leq r + M_\varepsilon \int_0^T (1 + \|\mathbf{Y}_{\Lambda^\varepsilon}(s,0,y)\|_{\overline{Y}} + m_1(\Lambda_s^\varepsilon)) \,\mathrm{d}s \\ &\leq r + M_\varepsilon \int_0^T (1 + 2\, f(s)) \,\mathrm{d}s\,, \end{aligned}$$

which implies by Grönwall inequality that $f(t) \leq R$ for all $t \in [0,T]$. $\square$

We can now prove Theorem 7.

Theorem 7. *Let $r > 0$ and $\overline{\Lambda}^{\varepsilon} \in \mathcal{P}(B_r^{Y_\varepsilon})$ be a given initial datum. Then*

(i) there exists a unique Eulerian solution $t \mapsto \Lambda_t^\varepsilon$ to (39) starting from $\overline{\Lambda}^{\varepsilon}$;

(ii) if $\overline{\Lambda}^{\varepsilon,N} = \frac{1}{N}\sum_{i=1}^{N} \delta_{\overline{y}_{i,N}^{\varepsilon}}$ is a sequence of atomic measures in $\mathcal{P}(B_r^{Y_\varepsilon})$ such that

$$\lim_{N\to\infty} \mathcal{W}_1(\overline{\Lambda}^{\varepsilon}, \overline{\Lambda}^{\varepsilon,N}) = 0$$

and, for fixed N, $\Lambda_t^{\varepsilon,N}$ are the empirical measures associated with the unique solution to (28) with initial datum $\overline{y}_{i,N}^{\varepsilon}$, we have

$$\lim_{N\to\infty} \mathcal{W}_1(\Lambda_t^{\varepsilon}, \Lambda_t^{\varepsilon,N}) = 0 \qquad \textit{uniformly with respect to } t \in [0,T].$$

Proof. The proof goes through a finite-dimensional approximation and involves three steps. *Step 1: Stability of Lagrangian solutions.* We fix $r > 0$, two initial data $\overline{\Lambda}^{\varepsilon,1}, \overline{\Lambda}^{\varepsilon,2} \in \mathcal{P}(B_r^{Y_\varepsilon})$, and assume that two Lagrangian solutions $\Lambda_t^{\varepsilon,1}$, $\Lambda_t^{\varepsilon,1}$ starting from $\overline{\Lambda}^{\varepsilon,1}$ and $\overline{\Lambda}^{\varepsilon,2}$, respectively, exist. We fix $R = (r + M_\varepsilon T)e^{2M_\varepsilon T}$ and the corresponding constant $L_{\varepsilon,R}$ provided by (22). Notice that such Lipschitz constant is greater than the one of (23). We claim that

$$\mathcal{W}_1(\Lambda_t^{\varepsilon,1}, \Lambda_t^{\varepsilon,2}) \leq e^{L_{\varepsilon,R}t + L_R e^{L_{\varepsilon,R}t}} \mathcal{W}_1(\overline{\Lambda}^{\varepsilon,1}, \overline{\Lambda}^{\varepsilon,2}) \qquad \text{for all } t \in [0,T]. \tag{50}$$

To prove this claim, we fix $\overline{y}^1$ and $\overline{y}^2 \in B_r^{Y_\varepsilon}$ and observe that by the previous Lemma

$$||\mathbf{Y}^{\varepsilon}_{\Lambda^{\varepsilon,i}}(t,0,\overline{y}^i)||_{\overline{Y}} \leq R \tag{51}$$

for all $t \in [0,T]$ and $i = 1, 2$. With (51), (22) and (23), the solutions y_t^1 and y_t^2 to the ODEs $\dot{y}^i = b^{\varepsilon}_{\Lambda_t^{\varepsilon,i}}(y^i)$ with initial data $\overline{y}^1$ and $\overline{y}^2$ respectively satisfy

$$\begin{aligned} ||y_t^1 - y_t^2||_{\overline{Y}} &\leq ||\overline{y}^1 - \overline{y}^2||_{\overline{Y}} + \int_0^t \left(||b^{\varepsilon}_{\Lambda_s^{\varepsilon,1}}(y_s^1) - b^{\varepsilon}_{\Lambda_s^{\varepsilon,1}}(y_s^2)||_{\overline{Y}} + ||b^{\varepsilon}_{\Lambda_s^{\varepsilon,1}}(y_s^2) - b^{\varepsilon}_{\Lambda_s^{\varepsilon,2}}(y_s^2)||_{\overline{Y}} \right) \mathrm{d}s \\ &\leq ||\overline{y}^1 - \overline{y}^2||_{\overline{Y}} + L_R \int_0^t \mathcal{W}_1(\Lambda_s^{\varepsilon,1}, \Lambda_s^{\varepsilon,2}) \mathrm{d}s + \int_0^t L_{\varepsilon,R}\, ||y_s^1 - y_s^2||_{\overline{Y}}\, \mathrm{d}s\,. \end{aligned}$$

This gives, by Grönwall's Lemma, that

$$||y_t^1 - y_t^2||_{\overline{Y}} \leq \left(||\overline{y}^1 - \overline{y}^2||_{\overline{Y}} + L_R \int_0^t \mathcal{W}_1(\Lambda_s^{\varepsilon,1}, \Lambda_s^{\varepsilon,2}) \mathrm{d}s \right) e^{L_{\varepsilon,R}t}$$

equivalently,

$$||\mathbf{Y}^{\varepsilon}_{\Lambda^{\varepsilon,1}}(t,0,\overline{y}^1) - \mathbf{Y}^{\varepsilon}_{\Lambda^{\varepsilon,2}}(t,0,\overline{y}^2)||_{\overline{Y}} \leq \left(||\overline{y}^1 - \overline{y}^2||_{\overline{Y}} + L_R \int_0^t \mathcal{W}_1(\Lambda_s^{\varepsilon,1}, \Lambda_s^{\varepsilon,2}) \mathrm{d}s \right) e^{L_{\varepsilon,R}t} \tag{52}$$

for all $t \in [0,T]$ and $\overline{y}^1, \overline{y}^2 \in B_r^{Y_\varepsilon}$.
Now, let Π be an optimal coupling between $\overline{\Lambda}^{\varepsilon,1}$ and $\overline{\Lambda}^{\varepsilon,2}$. Then clearly, by the definition of Lagrangian solutions, $(\mathbf{Y}^\varepsilon_{\Lambda^{\varepsilon,1}}(t,0,\cdot), \mathbf{Y}^\varepsilon_{\Lambda^{\varepsilon,2}}(t,0,\cdot))_{\#}\Pi$ is a coupling between $\Lambda_t^{\varepsilon,1}$ and $\Lambda_t^{\varepsilon,2}$. Therefore

$$\begin{aligned}\mathcal{W}_1(\Lambda_t^{\varepsilon,1}, \Lambda_t^{\varepsilon,2}) &\le \int_{Y_\varepsilon \times Y_\varepsilon} ||\mathbf{Y}^\varepsilon_{\Lambda^{\varepsilon,1}}(t,0,y^1) - \mathbf{Y}^\varepsilon_{\Lambda^{\varepsilon,2}}(t,0,y^2)||_{\overline{Y}}\, \mathrm{d}\Pi(y^1,y^2)\\ &= \int_{B_r^{Y_\varepsilon} \times B_r^{Y_\varepsilon}} ||\mathbf{Y}^\varepsilon_{\Lambda^{\varepsilon,1}}(t,0,y^1) - \mathbf{Y}^\varepsilon_{\Lambda^{\varepsilon,2}}(t,0,y^2)||_{\overline{Y}}\, \mathrm{d}\Pi(y^1,y^2),\end{aligned}$$

where we also used that $\overline{\Lambda}^{\varepsilon,1}, \overline{\Lambda}^{\varepsilon,2} \in \mathcal{P}(B_r^{Y_\varepsilon})$. Hence, using (52) we get

$$\begin{aligned}\mathcal{W}_1(\Lambda_t^1, \Lambda_t^2) &\le e^{L_{\varepsilon,R}t} \int_{B_r^{Y_\varepsilon} \times B_r^{Y_\varepsilon}} ||y^1 - y^2||_{\overline{Y}}\, \mathrm{d}\Pi(y^1,y^2) + L_R\, e^{L_{\varepsilon,R}t} \int_0^t \mathcal{W}_1(\Lambda_s^{\varepsilon,1}, \Lambda_s^{\varepsilon,2}) \mathrm{d}s\\ &= e^{L_{\varepsilon,R}t}\, \mathcal{W}_1(\overline{\Lambda}^{\varepsilon,1}, \overline{\Lambda}^{\varepsilon,2}) + L_R\, e^{L_{\varepsilon,R}t} \int_0^t \mathcal{W}_1(\Lambda_s^{\varepsilon,1}, \Lambda_s^{\varepsilon,2}) \mathrm{d}s\,.\end{aligned}$$

With this and the Grönwall Lemma, we get (50).
Step 2: Existence and approximation of Lagrangian solutions. We start by fixing a sequence of atomic measures $\overline{\Lambda}^{\varepsilon,N} \in \mathcal{P}(B_r^Y)$ such that

$$\lim_{N\to\infty} \mathcal{W}_1(\overline{\Lambda}^{\varepsilon,N}, \overline{\Lambda}^{\varepsilon}) = 0. \tag{53}$$

Such a sequence can be for instance constructed as follows: choose $\overline{y}^i(z) \in Y_\varepsilon$ independent and identically distributed, with law $\overline{\Lambda}^\varepsilon$, so that the random measures $\overline{\Lambda}^{\varepsilon,N} := \frac{1}{N}\sum_{i=1}^N \delta_{\overline{y}^i(z)}$ almost surely converge in $\mathcal{P}_1(Y_\varepsilon)$ to $\overline{\Lambda}^\varepsilon$, and choose a realization z such that this convergence takes place. Now let $\Lambda_t^{\varepsilon,N}$ be the empirical measures associated with the unique solution to (28) with initial datum $\overline{y}^i$, $i = 1, \dots, N$. As noticed in (45), $\Lambda_t^{\varepsilon,N}$ are Lagrangian solutions to (39) starting from $\overline{\Lambda}^{\varepsilon,N}$. Hence, (50) provides a constant $C := C(\varepsilon, M_\varepsilon, r, T)$ such that

$$\mathcal{W}_1(\Lambda_t^{\varepsilon,N}, \Lambda_t^{\varepsilon,M}) \le C\, \mathcal{W}_1(\overline{\Lambda}^{\varepsilon,N}, \overline{\Lambda}^{\varepsilon,M})$$

for all $t \in [0,T]$ and $N, M \in \mathbb{N}$. If follows that $\Lambda_t^{\varepsilon,N} \in C([0,T]; (\mathcal{P}_1(B_R^{Y_\varepsilon}), \mathcal{W}_1))$ is a Cauchy sequence. Let then $\Lambda_t^\varepsilon \in C([0,T]; (\mathcal{P}_1(B_R^{Y_\varepsilon}), \mathcal{W}_1))$ be the limit of the sequence $\Lambda_t^{\varepsilon,N}$. For a given $\overline{y}^\varepsilon \in B_r^{Y_\varepsilon}$, consider now the solution $y_t^{\varepsilon,N}$ and y_t^ε to the ODEs $\dot{y}^{\varepsilon,N} = b^\varepsilon_{\Lambda_t^{\varepsilon,N}}(y^{\varepsilon,N})$ and $\dot{y}^\varepsilon = b^\varepsilon_{\Lambda_t^\varepsilon}(y^\varepsilon)$, respectively, with initial datum $\overline{y}^\varepsilon$. Let $R' \ge R$ be an upper bound[5] for $\max_{t\in[0,T]} ||y_t^\varepsilon||_{\overline{Y}}$, which can be taken independent form $\overline{y}^\varepsilon \in B_r^{Y_\varepsilon}$. With (22) and (23) we obtain again that

$$||\mathbf{Y}^\varepsilon_{\Lambda^\varepsilon}(t,0,\overline{y}^\varepsilon) - \mathbf{Y}^\varepsilon_{\Lambda^{\varepsilon,N}}(t,0,\overline{y}^\varepsilon)||_{\overline{Y}} \le L_R\, e^{L_{\varepsilon,R'}t} \int_0^t \mathcal{W}_1(\Lambda_s^\varepsilon, \Lambda_s^{\varepsilon,N}) \mathrm{d}s,$$

[5] Observe that at this point of the proof we cannot a priori exclude that $R' > R$, since we still do not know that Λ^ε is a Lagrangian solution, hence we cannot apply (51) (which holds instead for $\Lambda^{\varepsilon,N}$).

which entails the uniform convergence of $\mathbf{Y}^{\varepsilon}_{\Lambda^{\varepsilon,N}}(\cdot,t,\cdot)$ to $\mathbf{Y}^{\varepsilon}_{\Lambda^N}(\cdot,t,\cdot)$ in $[0,T]\times B^{Y_\varepsilon}_r$. For each $t\in[0,T]$ this implies together with (53) and the fact that $\mathbf{Y}^{\varepsilon}_{\Lambda^\varepsilon}(t,0,\cdot)$ is a Lipschitz map on $B^{Y_\varepsilon}_r$, that

$$\Lambda^{\varepsilon,N}_t=\mathbf{Y}^{\varepsilon}_{\Lambda^{\varepsilon,N}}(t,0,\cdot)_{\#}\overline{\Lambda}^{\varepsilon,N}\to\mathbf{Y}^{\varepsilon}_{\Lambda^\varepsilon}(t,0,\cdot)_{\#}\overline{\Lambda}^{\varepsilon}$$

in $\mathcal{P}_1(Y_\varepsilon)$, which gives $\Lambda^\varepsilon_t=\mathbf{Y}^{\varepsilon}_{\Lambda^\varepsilon}(t,0,\cdot)_{\#}\overline{\Lambda}^{\varepsilon}$.
Step 3: Uniqueness and conclusion. Uniqueness of Lagrangian solutions, given the initial datum, follows now from (50). Existence and uniqueness of Eulerian solutions is now a consequence of *Remark* 9 and Theorem 6, respectively. □

Remark 12. The same theorem, with the proper adjustment holds for the entropy free vector field (21) in the larger space Y.

We now want to compare the entropic solution, to the continuity equation (39), to entropy free solution for the correspondent continuity equation that is

$$\partial_t\Lambda_t+\operatorname{div}(b_{\Lambda_t}\,\Lambda_t)=0, \tag{54}$$

where now $\Lambda\in C^0([0,T];(\mathcal{P}_1(Y),\mathcal{W}_1))$ and $\overline{\Lambda}\in\mathcal{P}(Y)$.
Consider two initial data $\overline{\Lambda}\in\mathcal{P}(B^Y_r)$ and $\overline{\Lambda}^\varepsilon\in\mathcal{P}(B^{Y_\varepsilon}_r)$, so that there exist two unique solutions Λ and Λ^ε to the continuity equations (39) and (54), starting from $\overline{\Lambda}$ and $\overline{\Lambda}^\varepsilon$ respectively. Let Λ^N_t and $\Lambda^{\varepsilon,N}_t$ be the relative empirical measures in the sense of *Step 2* of Theorem 7. We notice that all these measures have support contained in the balls B^Y_R, with $R=(r+MT)e^{2MT}$, for the entropy free case, and in the balls $B^Y_{R'}$, with $R'=(r+M_\varepsilon T)e^{2M_\varepsilon T}$, for the entropic case. Moreover $R'\geq R$, since $M_\varepsilon=M+\varepsilon\,h_\varepsilon$ (see the proof of Proposition 2). By the triangle inequality

$$\mathcal{W}_1(\Lambda^\varepsilon_t,\Lambda_t)\leq\mathcal{W}_1(\Lambda^\varepsilon_t,\Lambda^{\varepsilon,N}_t)+\mathcal{W}_1(\Lambda^{\varepsilon,N}_t,\Lambda^N_t)+\mathcal{W}_1(\Lambda^N_t,\Lambda_t). \tag{55}$$

With a similar reasoning to the one that led us to (50) one has

$$\begin{aligned}\mathcal{W}_1(\Lambda^{\varepsilon,N}_t,\Lambda^N_t)&\leq e^{L_{R'}\,t\,(1+e^{L_{R'}t})}\,\mathcal{W}_1(\overline{\Lambda}^{\varepsilon,N},\overline{\Lambda}^N)+\varepsilon\,h_\varepsilon\,R'\,t\,e^{L_{R'}\,t\,(1+e^{L_{R'}t})}\\&\leq e^{L_{R'}\,t\,(1+e^{L_{R'}t})}\,[\mathcal{W}_1(\overline{\Lambda}^{\varepsilon,N},\overline{\Lambda}^\varepsilon)+\mathcal{W}_1(\overline{\Lambda}^\varepsilon,\overline{\Lambda})+\mathcal{W}_1(\overline{\Lambda},\overline{\Lambda}^N)]\\&\quad+\varepsilon\,h_\varepsilon\,R'\,t\,e^{L_{R'}\,t\,(1+e^{L_{R'}t})}\end{aligned} \tag{56}$$

Combining (55) with (56), and taking the limit as $N\to+\infty$, we get

$$\mathcal{W}_1(\Lambda^\varepsilon_t,\Lambda_t)\leq e^{L_{R'}\,t\,(1+e^{L_{R'}t})}\,\mathcal{W}_1(\overline{\Lambda}^\varepsilon,\overline{\Lambda})+\varepsilon\,h_\varepsilon\,R'\,t\,e^{L_{R'}\,t\,(1+e^{L_{R'}t})}.$$

If $\mathcal{T}_\Psi$ satisfies (T3') instead of (T3) then $\varepsilon\,h_\varepsilon\to 0$, so that $R'\to R$. Using this and assuming that the initial entropic data converge to the entropy free ones, that is $\overline{\Lambda}^\varepsilon\to\overline{\Lambda}$, we have the desired result

$$\lim_{\varepsilon\to 0^+}\mathcal{W}_1(\Lambda^\varepsilon_t,\Lambda_t)=0$$

uniformly with respect to $t\in[0,T]$.

9 Fast Reaction Limit

The aim of this section is to address the case in which the dynamics for the labels runs at a much faster time scale than the dynamics for the agents' positions. In this case, introducing the fast time scale $\tau = \lambda t$, with $\lambda \gg 1$, system (28) takes the form

$$\begin{cases} \dot{x}_t^{\varepsilon,i} = v_{\Lambda_t^N}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}), \\ \dot{\ell}_t^{\varepsilon,i} = \lambda[\mathcal{T}_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}) + \varepsilon\,\mathcal{H}(\ell_t^{\varepsilon,i})] \end{cases} \qquad \text{for } i = 1, \dots, N,\ \ t \in [0,T]. \tag{57}$$

Note that the well-posedness of (57) is still guaranteed by Theorem 3. Our attention is focused on the behaviour of system (57) as $\lambda \to +\infty$, thus we are interested in the case of instantaneous adjustment for the labels (or the strategies if a replicator dynamics is being considered). To this end, we suppose that the dynamics for the labels can be written as follows

$$\dot{\ell}_t^{\varepsilon,i} = \lambda\left(\int_U \partial_\xi F_{\pi_\#\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}(u), u)\, \ell_t^{\varepsilon,i}(u)\,\mathrm{d}\eta(u) - \partial_\xi F_{\pi_\#\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}, \cdot)\right)\ell_t^{\varepsilon,i}, \tag{58}$$

where

$$\begin{aligned} \pi\colon Y_\varepsilon &\to \mathbb{R}^d \\ y &\mapsto x \end{aligned}$$

is the position projection, and

$$\begin{aligned} F\colon \mathcal{P}_1(\mathbb{R}^d) \times \mathbb{R}^d \times [r_\varepsilon, R_\varepsilon] \times U &\to [-\infty, +\infty] \\ (\Sigma, x, \xi, u) &\mapsto F_\Sigma(x, \xi, u) \end{aligned} \tag{59}$$

satisfies the following properties:

(FI) for any $R > 0$, $\Sigma \in \mathcal{P}(B_R)$, $x \in B_R$, and $\ell \in C_\varepsilon$ the map $u \in U \mapsto F_\Sigma(x, \ell(u), u)$ is η-Lebesgue integrable;

(FII) for any $R > 0$, $\Sigma \in \mathcal{P}(B_R)$, $x \in B_R$, and for η-a.e. $u \in U$ the map

$$\begin{aligned} g\colon [r_\varepsilon, R_\varepsilon] &\to \mathbb{R} \\ \xi &\mapsto F_\Sigma(x, \xi, u) \end{aligned}$$

is twice differentiable with second derivative strictly positive and bounded uniformly with respect to u, that is $2\alpha < g''(\xi) < M$ for every $\xi \in [r_\varepsilon, R_\varepsilon]$ and for some $M, \alpha > 0$ independent of u;

(FIII) for any $R > 0$ there exists $L_R > 0$ such that for every $\Sigma_1, \Sigma_2 \in \mathcal{P}(B_R)$, $x_1, x_2 \in B_R$, $\xi \in [r_\varepsilon, R_\varepsilon]$, and for η-a.e. $u \in U$

$$|F_{\Sigma_1}(x_1, \xi, u) - F_{\Sigma_2}(x_2, \xi, u)| \le L_R\,(|x_1 - x_2| + W_1(\Sigma_1, \Sigma_2)),$$

and

$$|\partial_\xi F_{\Sigma_1}(x_1, \xi, u) - \partial_\xi F_{\Sigma_2}(x_2, \xi, u)| \le L_R\,(|x_1 - x_2| + W_1(\Sigma_1, \Sigma_2)).$$

Note that by Rademacher's Theorem, the map $x \mapsto F_\Sigma(x, \xi, u)$ is differentiable for almost every $x \in \mathrm{int}(B_R)$ and $|\partial_x F_\Sigma(x, \xi, u)| \le L_R$.

Remark 13. Assumptions (FI), (FII), and (FIII) do hold for the replicator dynamics in a particular case, which is referred to as *undisclosed setting* [5]. The pay-off that player in position x gets playing strategy u against all the other players according to a distribution Ψ of players with mixed strategies is

$$\mathcal{J}_\Psi(x,u) = \int_{Y_\varepsilon}\int_U J(x,u,x',u')\,\ell'(u')\,\mathrm{d}\eta(u')\mathrm{d}\Psi(x',\ell').$$

If we suppose that opponents' strategies are *undisclosed*, that is $J(x,u,x',u') = J(x,u,x')$, we have that

$$\mathcal{J}_\Psi(x,u) = \int_{Y_\varepsilon} J(x,u,x')\,\mathrm{d}\Psi(x',\ell') = \int_{\mathbb{R}^d} J(x,u,x')\,\mathrm{d}\pi_\#\Psi(x') =: \mathcal{J}_{\pi_\#\Psi}(x,u),$$

and therefore

$$\mathcal{T}_\Psi(x,\ell) = \Big(\mathcal{J}_{\pi_\#\Psi}(x,\cdot) - \int_U \mathcal{J}_{\pi_\#\Psi}(x,u)\,\ell(u)\,\mathrm{d}\eta(u)\Big)\,\ell.$$

This leads us to

$$\mathcal{T}_\Psi(x,\ell) + \varepsilon\mathcal{H}(\ell) = \Big(\int_U [\varepsilon\log(\ell(u)) - \mathcal{J}_{\pi_\#\Psi}(x,u)]\,\ell(u)\,\mathrm{d}\eta(u) - [\varepsilon\log(\ell) - \mathcal{J}_{\pi_\#\Psi}(x,\cdot)]\Big)\,\ell,$$

and finally

$$F_{\pi_\#\Psi}(x,\xi,u) = -\mathcal{J}_{\pi_\#\Psi}(x,u)\,\xi + \varepsilon\,[\xi\log(\xi) - \xi].$$

It easy to check that such a function satisfies properties (FI), (FII), and (FIII) whenever $J \in \mathrm{Lip}_b(\mathbb{R}^d \times U \times \mathbb{R}^d \times U)$.

The following proposition provides a set of conditions under which the assumptions stated above are satisfied for $F_\Sigma(x,\xi,u)$ being an integral function.

Proposition 4. *Let F be a function as in (59) defined as follows*

$$F_\Sigma(x,\xi,u) = \int_{\mathbb{R}^d} f(x,\xi,u,x')\,\mathrm{d}\Sigma(x'),$$

where $f\colon \mathbb{R}^d \times [r_\varepsilon, R_\varepsilon] \times U \times \mathbb{R}^d \times C_\varepsilon \to [-\infty,+\infty]$ satisfies the following properties

(fI) *for every $R > 0$, $\Sigma \in \mathcal{P}(B_R)$, $x \in B_R$, $\ell \in C_\varepsilon$ the map*

$$u \mapsto \int_{\mathbb{R}^d} f(x,\ell(u),u,x')\,\mathrm{d}\Sigma(x')$$

is η-Lebesgue integrable;

(fII) *for every $R > 0$, $x \in B_R$, $x' \in B_R$, and for η-a.e. $u \in U$ the map $\xi \mapsto f(x,\xi,u,x')$ is twice differentiable with second derivative bounded from above and below by two positive constants independent of u and x';*

(fIII) *for every $R > 0$, $x, x' \in B_R$, $\xi \in [r_\varepsilon, R_\varepsilon]$, and for η-a.e. $u \in U$ the function $f(x,\xi,u,\cdot) \in \mathrm{Lip}_b(\mathbb{R}^d)$, and the function $f(\cdot,\xi,u,x') \in \mathrm{Lip}(\mathbb{R}^d)$, with Lipschitz constants dependent only on R;*

(fIV) *for every* $R > 0, x, x' \in B_R$, $\xi \in [r_\varepsilon, R_\varepsilon]$, *and for* η*-a.e.* $u \in U$ *the function* $\partial_\xi f(x, \xi, u, \cdot) \in \mathrm{Lip}_b(\mathbb{R}^d)$, *and the function* $\partial_\xi f(\cdot, \xi, u, x') \in \mathrm{Lip}(\mathbb{R}^d)$, *with Lipschitz constants dependent only on* R.

Then F *satisfies properties* (FI), (FII), *and* (FIII).

Proof. Assumption (FI) coincides with (fI). Assumption (FII) follows from (fII), (fIII), and (fIV) by applying Leibniz integral rule. Assumption (FIII) is a direct consequence of (fIII) and (fIV). □

Our goal is to prove the convergence, as $\lambda \to +\infty$, of system (57) to a suitable system of agents with labels, where such labels are defined as minima of some particular functionals. In Proposition 5 we introduce the prototype for these functionals and present some of its properties. Before, stating Proposition 5, we recall the definition of Fréchet-differentiability.

Definition 4 (Fréchet-differentiability). *A functional* $\mathcal{F}\colon L^p(U, \eta) \to \mathbb{R}$ *is said to be continuos Fréchet-differentiable, if for every* $\ell \in L^p(U; \eta)$ *there exists a unique linear application* $D\mathcal{F}(\ell) \in \mathcal{L}(L^p(U, \eta); \mathbb{R})$ *such that*

$$\lim_{\tilde{\ell} \xrightarrow{L^p} \ell} \frac{|\mathcal{F}(\tilde{\ell}) - \mathcal{F}(\ell) - D\mathcal{F}(\ell)(\tilde{\ell} - \ell)|}{||\tilde{\ell} - \ell||_{L^p(U,\eta)}} = 0,$$

and the map $L^p(U, \eta) \ni \ell \to D\mathcal{F}(\ell) \in \mathcal{L}(L^p(U, \eta); \mathbb{R})$ *is continuous. A functional* $\mathcal{F}\colon C_\varepsilon \to \mathbb{R}$ *is continuous Fréchet-differentiable if it is the restriction of a continuous Frechèt-differentiable functional over* $L^p(U, \eta)$.

Proposition 5. *Let* F *be a function defined as in* (59) *which satisfies properties* (FI), (FII) *and* (FIII). *Then for any* $R > 0$, $\Psi \in \mathcal{P}(B_R^{Y_\varepsilon})$, *and* $x \in B_R$ *the functional*

$$\begin{aligned} G_\Psi(x, \cdot)\colon C_\varepsilon &\to \mathbb{R} \\ \ell \mapsto G_\Psi(x, \ell) &:= \int_U F_{\pi_\# \Psi}(x, \ell(u), u) \, \mathrm{d}\eta(u) \end{aligned} \tag{60}$$

is well defined, Fréchet differentiable if $p > 1$, *strongly convex if* $1 \le p \le 2$ *and uniformly convex if* $2 < p < +\infty$. *Moreover, for every* $\ell \in C_\varepsilon$, *the map*

$$B_R \times \mathcal{P}(B_R) \ni (x, \pi_\# \Psi) \mapsto G_\Psi(x, \ell) \tag{61}$$

is Lipschitz continuous, with Lipschitz constant dependent only on R.

Proof. The functional $G_\Psi(x, \cdot)$ defined in (60) is well defined as a consequence of property (FI). Furthermore, it is Fréchet-differentiable in $\ell_1 \in C_\varepsilon$ with differential

$$\begin{aligned} DG_\Psi(x, \ell_1)\colon C_\varepsilon &\to \mathbb{R} \\ \ell_2 &\mapsto \int_U \partial_\xi F_{\pi_\# \Psi}(x, \ell_1(u), u) \, \ell_2(u) \, \mathrm{d}\eta(u). \end{aligned}$$

Indeed, because of (FII) there exists $M > 0$, independent of u, such that $|g''(\xi)| < M$ for every $\xi \in [r_\varepsilon, R_\varepsilon]$ and therefore

$$
\begin{aligned}
&|G_\Psi(x,\ell_2) - G_\Psi(x,\ell_1) - DG_\Psi(x,\ell_1)(\ell_2-\ell_1)| = \\
&= \left| \int_U [F_{\pi_\# \Psi}(x,\ell_2(u),u) - F_{\pi_\# \Psi}(x,\ell_1(u),u) - \partial_\xi F_{\pi_\# \Psi}(x,\ell_1(u),u)(\ell_2-\ell_1)(u)]\,\mathrm{d}\eta(u) \right| \\
&= \int_U \frac{1}{2}\,\partial_{\xi\xi} F_{\pi_\# \Psi}(x,\ell_{1,2}(u),u)(\ell_2-\ell_1)^2(u)\,\mathrm{d}\eta(u) \\
&\le \frac{M}{2}\int_U (\ell_2-\ell_1)^2(u)\,\mathrm{d}\eta(u) \\
&\le \frac{M}{2}||\ell_2-\ell_1||_{L^p(U,\eta)}||\ell_2-\ell_1||_{L^q(U,\eta)},
\end{aligned}
$$

where $\ell_{1,2}(u)$ is the value provided by the Mean Value Theorem and q is the conjugate exponent of p. This leads us to

$$
\lim_{\ell_2 \xrightarrow{L^p} \ell_1} \frac{|G_\Psi(x,\ell_2) - G_\Psi(x,\ell_1) - DG_\Psi(x,\ell_1)(\ell_2-\ell_1)|}{\|\ell_2-\ell_1\|_{L^p(U,\eta)}} \le \frac{M}{2} \lim_{\ell_2 \xrightarrow{L^p} \ell_1} \|\ell_1-\ell_2\|_{L^q(U,\eta)}\,.
$$

If $p \ge q$, then $\|\ell_2-\ell_1\|_{L^q(U,\eta)} \le \|\ell_2-\ell_1\|_{L^p(U,\eta)}$. Otherwise, if $q \ge p$

$$
\|\ell_2-\ell_2\|^q_{L^q(u,\eta)} \le (R_\varepsilon - r_\varepsilon)^{q-p}\int_U |\ell_2-\ell_1|^p(u)\mathrm{d}\eta(u) = (R_\varepsilon - r_\varepsilon)^{q-p}\,\|\ell_2-\ell_1\|^p_{L^p(U,\eta)}\,.
$$

In either case, $\ell_2 \xrightarrow{L^p} \ell_1$ implies that $\ell_2 \xrightarrow{L^q} \ell_1$, proving the Fréchet differentiability for all $p \ge 1$. We now prove the strong convexity for $1 \le p \le 2$. Because of property (FII), the map $\xi \mapsto F_\Sigma(x,\xi,u)$ is strongly convex and therefore

$$
\begin{aligned}
G_\Psi(x,\ell_1) - G_\Psi(x,\ell_2) &= \int_Y [F_{\pi_\# \Psi}(x,\ell_1(u),u) - F_{\pi_\# \Psi}(x,\ell_2(u),u)]\mathrm{d}\eta(u) \\
&\ge \int_U \left[\partial_\xi F_{\pi_\# \Psi}(x,\ell_2(u),u)(\ell_1(u)-\ell_2(u)) + \alpha\,(\ell_1(u)-\ell_2(u))^2\right]\mathrm{d}\eta(u) \\
&= \int_U \partial_\xi F_{\pi_\# \Psi}(x,\ell_2(u),u)(\ell_1-\ell_2)(u)\,\mathrm{d}\eta(u) + \alpha\,\|\ell_1-\ell_2\|^2_{L^2(U,\eta)} \\
&= DG_\Psi(x,\ell_2)(\ell_1-\ell_2) + \alpha\,\|\ell_1-\ell_2\|^2_{L^2(U,\eta)} \\
&\ge DG_\Psi(x,\ell_2)(\ell_1-\ell_2) + \alpha\,\|\ell_1-\ell_2\|^2_{L^p(U,\eta)}\,.
\end{aligned}
\tag{62}
$$

Otherwise, if $2 < p < +\infty$

$$
\|\ell_1-\ell_2\|^p_{L^p(U,\eta)} \le (R_\varepsilon - r_\varepsilon)^{p-2}\,\|\ell_1-\ell_2\|^2_{L^2(U,\eta)} \tag{63}
$$

which, together with (62), proves that $G_\Psi(x,\cdot)$ is uniformly convex. Finally, the Lipschitz continuity is a direct consequence of property (FIII), indeed

$$
\begin{aligned}
|G_{\Psi_2}(x_2,\ell) - G_{\Psi_1}(x_1,\ell)| &\le \int_U |F_{\pi_\# \Psi_2}(x_2,\ell(u),u) - F_{\pi_\# \Psi_1}(x_1,\ell(u),u)|\,\mathrm{d}\eta(u) \\
&\le L_R\,(|x_2-x_1| + W_1(\pi_\# \Psi_2, \pi_\# \Psi_1)),
\end{aligned}
$$

which concludes the proof. □

As a consequence of Proposition 5 we have the following corollary.

Corollary 2. *Let F be a function defined as in (59) which satisfies properties* (FI), (FII), (FIII), *and let G be defined as in (60). Then for every $R > 0$, $\Psi \in \mathcal{P}(B_R^{Y_\varepsilon})$, $x \in B_R$, and $1 \leq p < +\infty$ there exists a unique solution ℓ^* to $\min\limits_{\ell \in C_\varepsilon} G_\Psi(x, \ell)$, and for every $\ell \in C_\varepsilon$*

$$G_\Psi(x, \ell) - G_\Psi(x, \ell^*) \geq \alpha \, ||\ell - \ell^*||^2_{L^2(U,\eta)}. \tag{64}$$

Moreover, the map

$$B_R \times \mathcal{P}(B_R) \ni (x, \pi_\# \Psi) \ \mapsto G_\Psi(x, \ell^*), \tag{65}$$

is Lipschitz continuous for every $1 \leq p < \infty$, while the map

$$B_R \times \mathcal{P}(B_R) \ni (x, \pi_\# \Psi) \mapsto \ell^* \tag{66}$$

is Lipschitz continuous for $1 \leq p \leq 2$ and Hölder continuous for $2 < p < +\infty$.

Proof. The existence and uniqueness to the minimum problem is a direct consequence of the strong and uniform convexity of $G_\Psi(x, \cdot)$ together with the convexity of C_ε. Then, by (62) and by the minimality of ℓ^*

$$G_\Psi(x, \ell) - G_\Psi(x, \ell^*) \geq \underbrace{DG_\Psi(\ell^*)(\ell - \ell^*)}_{\geq 0} + \alpha ||\ell - \ell^*||^2_{L^2(U,\eta)} \geq \alpha ||\ell - \ell^*||^2_{L^2(U,\eta)}.$$

Let $x_1, x_2 \in B_R$, $\Psi_1, \Psi_2 \in \mathcal{P}_1(B_R^{Y_\varepsilon})$, and let ℓ_1^* and ℓ_2^* be the solutions to $\min_{\ell \in C_\varepsilon} G_{\Psi_1}(x_1, \ell)$ and $\min_{\ell \in C_\varepsilon} G_{\Psi_2}(x_2, \ell)$ respectively. If $G_{\Psi_2}(x_2, \ell_2^*) \geq G_{\Psi_1}(x_1, \ell_1^*)$, using the minimality of ℓ_2^* and the Lipschitz continuity of (61)

$$\begin{aligned} |G_{\Psi_2}(x_2, \ell_2^*) - G_{\Psi_1}(x_1, \ell_1^*)| &= G_{\Psi_2}(x_2, \ell_2^*) - G_{\Psi_2}(x_2, \ell_1^*) + G_{\Psi_2}(x_2, \ell_1^*) - G_{\Psi_1}(x_1, \ell_1^*) \\ &\leq G_{\Psi_2}(x_2, \ell_1^*) - G_{\Psi_1}(x_1, \ell_1^*) \\ &\leq L_R \left(|x_2 - x_1| + W_1(\pi_\# \Psi_2, \pi_\# \Psi_1)\right). \end{aligned}$$

Otherwise, if $G_{\Psi_2}(x_2, \ell_2^*) \leq G_{\Psi_1}(x_1, \ell_1^*)$, with a similar line of reasoning, we reach the same result:

$$\begin{aligned} |G_{\Psi_2}(x_2, \ell_2^*) - G_{\Psi_1}(x_1, \ell_1^*)| &= G_{\Psi_1}(x_1, \ell_1^*) - G_{\Psi_1}(x_1, \ell_2^*) + G_{\Psi_1}(x_1, \ell_2^*) - G_{\Psi_2}(x_2, \ell_2^*) \\ &\leq G_{\Psi_1}(x_1, \ell_2^*) - G_{\Psi_2}(x_2, \ell_2^*) \\ &\leq L_R \left(|x_2 - x_1| + W_1(\pi_\# \Psi_2, \pi_\# \Psi_1)\right), \end{aligned}$$

which gives us the Lipshitz conitnuity of (65). Since $G_\Psi(x, \cdot)$ is strongly convex for $p = 2$ we have that

$$[DG_{\Psi_2}(x_2, \ell_2^*) - DG_{\Psi_2}(x_2, \ell_1^*)](\ell_2^* - \ell_1^*) \geq \alpha \, ||\ell_2^* - \ell_1^*||^2_{L^2(U,\eta)}.$$

By minimaility

$$DG_{\Psi_2}(x_2, \ell_2^*)(\ell_2^* - \ell_1^*) \leq 0 \leq DG_{\Psi_1}(x_1, \ell_1^*)(\ell_2^* - \ell_1^*),$$

and therefore using property (FIII)

$$\begin{aligned}\alpha\,||\ell_2^* - \ell_1^*||^2_{L^2(U,\eta)} &\leq [DG_{\Psi_1}(x_1,\ell_1^*) - DG_{\Psi_2}(x_2,\ell_1^*)](\ell_2^* - \ell_1^*) \\ &= \int_U [\partial_\xi F_{\pi_\#\Psi_1}(x_1,\ell_1^*(u),u) - \partial_\xi F_{\pi_\#\Psi_2}(x_2,\ell_1^*(u),u)](\ell_2^* - \ell_1^*)(u)\,\mathrm{d}\eta(u) \\ &\leq L_R\,(|x_2 - x_1| + W_1(\pi_\#\Psi_1,\pi_\#\Psi_2)) \int_U |\ell_2^* - \ell_1^*|(u)\,\mathrm{d}\eta(u) \\ &\leq L_R\,(|x_2 - x_1| + W_1(\pi_\#\Psi_1,\pi_\#\Psi_2))\,||\ell_2^* - \ell_1^*||_{L^2(U,\eta)}.\end{aligned}$$

Finally, if $p \leq 2$ then

$$||\ell_2^* - \ell_1^*||_{L^p(U,\eta)} \leq ||\ell_2^* - \ell_1^*||_{L^2(U,\eta)} \leq \frac{L_R}{\alpha}\,(|x_2 - x_1| + W_1(\pi_\#\Psi_1,\pi_\#\Psi_2)),$$

otherwise remembering (63)

$$||\ell_2^* - \ell_1^*||^{p-1}_{L^p(U,\eta)} \leq \frac{L_R}{\alpha}\,(R_\varepsilon - r_\varepsilon)^{p-2}\,(|x_2 - x_1| + W_1(\pi_\#\Psi_1,\pi_\#\Psi_2)).$$ □

As intermediate step towards the main result of this section we have the following Lemma.

Lemma 3. *Let F be a function defined as in (59) which satisfies properties (FI), (FII), (FIII), and let G be defined as in (60). Consider system (57) and suppose that the dynamics for the labels can be written as in (58). Then, for any $1 \leq p < +\infty$, for every $i = 1,\dots,N$, and for almost every $t \in (0,T]$*

$$\lim_{\lambda\to\infty} ||\ell_t^{\varepsilon,i} - \ell_t^{*\,\varepsilon,i}||_E = 0, \tag{67}$$

where $\ell_t^{*\,\varepsilon,i}$ is the unique solution to $\min_{\ell\in C_\varepsilon} G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell)$.

Proof. The proof consists of two steps. In the first step, we obtain some useful estimates and properties of system (57), which we'll use in the second step to prove (67).
Step 1. First, we have a bound for every agent's location. In fact, using (v3), remembering that $m_1(\Lambda_t^{\varepsilon,N}) \leq ||\boldsymbol{y}_t^\varepsilon||_{\overline{Y}^N}$, and noticing that $||\ell_t^{\varepsilon,i}||_E < R_\varepsilon$ for any $p < +\infty$, we have that

$$\begin{aligned}|x_t^{\varepsilon,i}| &\leq |x_0^{\varepsilon,i}| + \int_0^T |v_{\Lambda_s^{\varepsilon,N}}(x_s^{\varepsilon,i},\ell_s^{\varepsilon,i})|\,\mathrm{d}s \\ &\leq |x_0^{\varepsilon,i}| + \int_0^T M_v(1 + |x_s^{\varepsilon,i}| + ||\ell_s^{\varepsilon,i}||_E + m_1(\Lambda_s^{\varepsilon,N}))\,\mathrm{d}s \\ &< |x_0^{\varepsilon,i}| + M_v\,(1 + R_\varepsilon)\,T + \int_0^T M_v\,(|x_s^{\varepsilon,i}| + ||\boldsymbol{y}_s^{\boldsymbol{\varepsilon}}||_{\overline{Y}^N})\,\mathrm{d}s \\ &< |x_0^{\varepsilon,i}| + M_v\,(1 + 2R_\varepsilon)\,T + \int_0^T M_v\,(|x_s^{\varepsilon,i}| + ||\boldsymbol{x}_s^{\boldsymbol{\varepsilon}}||_{\overline{Y}^N})\,\mathrm{d}s.\end{aligned}$$

Therefore, a straightforward computation leads us to

$$||\boldsymbol{x}_t^\varepsilon||_{\overline{Y}^N} < ||\boldsymbol{x}_0^\varepsilon||_{\overline{Y}^N} + M_v\,(1 + 2R_\varepsilon)\,T + 2M_v \int_0^T ||\boldsymbol{x}_s^\varepsilon||_{\overline{Y}^N}\,\mathrm{d}s,$$

and applying Grönwall's Lemma

$$||\boldsymbol{x}_t^\varepsilon||_{\overline{Y}^N} < (||\boldsymbol{x}_0^\varepsilon||_{\overline{Y}^N} + M_v\,(1+2R_\varepsilon)\,T)\,e^{2M_v\,T}.$$

Then for each agent's location

$$|x_t^{\varepsilon,i}| < N\,(||\boldsymbol{x}_0^\varepsilon||_{\overline{Y}^N} + M_v\,(1+2R_\varepsilon)\,T)\,e^{2M_v\,T} =: R. \tag{68}$$

Second, the map $t \mapsto x_t^{\varepsilon,i}$ is Lipschitz continuous:

$$\begin{aligned} |x_{t_2}^{\varepsilon,i} - x_{t_1}^{\varepsilon,i}| &\le \int_{t_1}^{t_2} |v_{\Lambda_s^{\varepsilon,N}}(x_s^{\varepsilon,i}, \ell_s^{\varepsilon,i})|\,\mathrm{d}s \\ &< M_v\,(1+R+2R_\varepsilon+R/N)\,|t_2-t_1| =: A\,|t_2-t_1|, \end{aligned} \tag{69}$$

and therefore $t \mapsto \pi_\#\Lambda_t^{\varepsilon,N}$ is also Lipschitz continuous:

$$W_1(\pi_\#\Lambda_{t_1}^{\varepsilon,N}, \pi_\#\Lambda_{t_2}^{\varepsilon,N}) \le ||\boldsymbol{x}_{t_2}^\varepsilon - \boldsymbol{x}_{t_1}^\varepsilon||_{\overline{Y}^N} \le A\,|t_2-t_1|. \tag{70}$$

Step 2. Using the convexity of $G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\cdot)$ we have that

$$\begin{aligned} &G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}) - G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{*\,\varepsilon,i}) \le DG_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}(u), u)(\ell_t^{\varepsilon,i} - \ell_t^{*\,\varepsilon,i}) \\ &= \int_U \partial_\xi F_{\pi_\#\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}(u), u)(\ell_t^{\varepsilon,i} - \ell_t^{*\,\varepsilon,i})(u)\mathrm{d}\eta(u) \\ &= \int_U [\partial_\xi F_{\pi_\#\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}(u), u) - \int_U F_{\pi_\#\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}(v), v)\ell_t^{\varepsilon,i}(v)\mathrm{d}\eta(v)](\ell_t^{\varepsilon,i} - \ell_t^{*\,\varepsilon,i})(u)\mathrm{d}\eta(u) \\ &\le \left\|\partial_\xi F_{\pi_\#\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}, \cdot) - \int_U F_{\pi_\#\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}(u), u)\ell_t^{\varepsilon,i}(u)\mathrm{d}\eta(u)\right\|_{L^2(U,\eta)} ||\ell_t^{\varepsilon,i} - \ell_t^{*\,\varepsilon,i}||_{L^2(U,\eta)} \end{aligned}$$

which together with (64) lead us to

$$\begin{aligned} &\alpha\,[G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}) - G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{*\,\varepsilon,i})] \le \\ &\le \left\|\partial_\xi F_{\pi_\#\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}, \cdot) - \int_U F_{\pi_\#\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}(u), u)\,\ell_t^{\varepsilon,i}(u)\,\mathrm{d}\eta(u)\right\|_{L^2(U,\eta)}^2. \end{aligned} \tag{71}$$

For almost every $t \in (0,T]$

$$\begin{aligned} &\frac{d}{dt}\,[G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}) - G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{*\,\varepsilon,i})] \\ &\quad = \underbrace{\int_U \partial_\xi F_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}(u), u)\,\dot{\ell}_t^{\varepsilon,i}(u)\,\mathrm{d}\eta(u)}_{\mathrm{I}} + \underbrace{\partial_x G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}) \cdot \dot{x}_t^{\varepsilon,i}}_{\mathrm{II}} + \\ &\quad + \underbrace{\frac{d}{d\tau} G_{\Lambda_\tau^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i})\Big|_{\tau=t}}_{\mathrm{III}} \underbrace{- \frac{d}{dt} G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{*\,\varepsilon,i})}_{\mathrm{IV}} \end{aligned} \tag{72}$$

We now show that the terms II, III, and IV are well defined and uniformly bounded with respect to λ. Because of property (FIII), (v3), and bound (68) we get

$$
\begin{aligned}
\mathrm{II} &= \partial_x G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})\cdot \dot{x}_t^{\varepsilon,i}\\
&= \int_U \partial_x F_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})\cdot v_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})\,d\eta(u)\\
&\le \int_U |\partial_x F_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})|\,|v_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})|\,\mathrm{d}\eta(u)\\
&\le \int_U L_{R+R_\varepsilon}\,M_v(1+\|y_t^{\varepsilon,i}\|_{\overline{Y}}+m_1(\Lambda_t^{\varepsilon,N}))\,\mathrm{d}\eta(u)\\
&\le L_{R+R_\varepsilon}\,M_v\,(1+R+2R_\varepsilon+R/N)=L_{R+R_\varepsilon}\,A.
\end{aligned}
\tag{73}
$$

Given the Lipschitz continuity of map (61), and because of bounds (68) and (70), the map $\tau\mapsto G_{\Lambda_\tau^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})$ is Lipschitz continuous:

$$
|G_{\Lambda_{\tau_2}^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})-G_{\Lambda_{\tau_1}^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})|\le L_{R+R_\varepsilon}W_1(\pi_\#\Lambda_{\tau_1}^{\varepsilon,N},\pi_\#\Lambda_{\tau_2}^{\varepsilon,N})\le L_{R+R_\varepsilon}\,A\,|\tau_2-\tau_1|,
$$

and therefore differentiable by Rademacher's Theorem for almost every $t\in(0,T]$, with derivative bounded by its Lipschitz constant, that is

$$
\mathrm{III} = \frac{d}{d\tau}G_{\Lambda_\tau^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})\Big|_{\tau=t}\le L_{R+R_\varepsilon}\,A. \tag{74}
$$

Because of bound (68), the Lipschitz continuity of (65), (69), (70), and Rademacher's Theorem, we have that

$$
\mathrm{IV}\le\left|\frac{d}{dt}G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{*\,\varepsilon,i})\right|\le 2\,L_{R+R_\varepsilon}\,A. \tag{75}
$$

Regarding I instead, using (58) and (71) we have that

$$
\begin{aligned}
\mathrm{I} &= \int_U \partial_\xi F_{\pi_\#\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})\,\dot{\ell}_t^{\varepsilon,i}(u)\,\mathrm{d}\eta(u)\\
&= \int_U\left(\partial_\xi F_{\pi_\#\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})-\int_U \partial_\xi F_{\pi_\#\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i}(v),v)\,\ell_t^{\varepsilon,i}(v)\,\mathrm{d}\eta(v)\right)\dot{\ell}_t^{\varepsilon,i}(u)\,\mathrm{d}\eta(u)\\
&= -\lambda\int_U\left(\partial_\xi F_{\pi_\#\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})-\int_U \partial_\xi F_{\pi_\#\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i}(v),v)\,\ell_t^{\varepsilon,i}(v)\,\mathrm{d}\eta(v)\right)^2\ell_t^{\varepsilon,i}(u)\,\mathrm{d}\eta(u)\\
&\le -\lambda\,r_\varepsilon\int_U\left(\partial_\xi F_{\pi_\#\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})-\int_U \partial_\xi F_{\pi_\#\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i}(v),v)\,\ell_t^{\varepsilon,i}(v)\,\mathrm{d}\eta(v)\right)^2\mathrm{d}\eta(u)\\
&\le -\lambda\,r_\varepsilon\,\alpha\,[G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})-G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{*\,\varepsilon,i})]\,.
\end{aligned}
\tag{76}
$$

Using (73), (74), (75), and (76), from (72) we get the following inequality

$$
\begin{aligned}
\frac{d}{dt}&[G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})-G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{*\,\varepsilon,i})]\le\\
&\le -\lambda\,r_\varepsilon\,\alpha\,[G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})-G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{*\,\varepsilon,i})]+4\,L_{R+R_\varepsilon}\,A,
\end{aligned}
$$

or equivalently

$$\begin{aligned}&\frac{d}{dt}\left[G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})-G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{*\,\varepsilon,i})-\frac{4\,A}{\lambda\,\alpha\,r_\varepsilon}\,L_{R+R_\varepsilon}\right]\leq\\&\leq-\lambda\,r_\varepsilon\,\alpha\left[G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})-G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{*\,\varepsilon,i})-\frac{4\,A}{\lambda\,\alpha\,r_\varepsilon}\,L_{R+R_\varepsilon}\right].\end{aligned}$$

Therefore, by Grönwall's Lemma

$$\begin{aligned}&\left[G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i})-G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{*\,\varepsilon,i})-\frac{4\,A}{\lambda\,\alpha\,r_\varepsilon}\,L_{R+R_\varepsilon}\right]\leq\\&\leq\left[G_{\Lambda_0^{\varepsilon,N}}(x_0^{\varepsilon,i},\ell_0^{\varepsilon,i})-G_{\Lambda_0^{\varepsilon,N}}(x_0^{\varepsilon,i},\ell_0^{*\,\varepsilon,i})-\frac{4\,A}{\lambda\,\alpha\,r_\varepsilon}\,L_{R+R_\varepsilon}\right]e^{-\lambda\,r_\varepsilon\,\alpha\,T},\end{aligned}$$

and applying (64)

$$\begin{aligned}&\alpha\,||\ell_t^{\varepsilon,i}-\ell_t^{*\,\varepsilon,i}||^2_{L^2(U,\eta)}\leq\frac{4\,A}{\lambda\,\alpha\,r_\varepsilon}\,L_{R+R_\varepsilon}+\\&+\left[G_{\Lambda_0^{\varepsilon,N}}(x_0^{\varepsilon,i},\ell_0^{\varepsilon,i})-G_{\Lambda_0^{\varepsilon,N}}(x_0^{\varepsilon,i},\ell_0^{*\,\varepsilon,i})-\frac{4\,A}{\lambda\,\alpha\,r_\varepsilon}\,L_{R+R_\varepsilon}\right]e^{-\lambda\,r_\varepsilon\,\alpha\,T}.\end{aligned}$$

Taking the limit as $\lambda\to\infty$ gives the desired result. □

Remark 14. Since $\ell_0^{\varepsilon,i}$ is independent of λ so is $\ell_0^{*\,\varepsilon,i}$, and therefore it is obvious that

$$\lim_{\lambda\to\infty}||\ell_0^{\varepsilon,i}-\ell_0^{*\,\varepsilon,i}||_E=||\ell_0^{\varepsilon,i}-\ell_0^{*\,\varepsilon,i}||_E\,.$$

We are now ready to state and prove our main result for the *undisclosed setting*.

Theorem 8. *Let F be a function defined as in (59) which satisfies properties (FI), (FII), (FIII), and let G be defined as in (60). Let $(\overline{\boldsymbol{x}}^\varepsilon,\overline{\boldsymbol{\ell}}^\varepsilon)\in Y_\varepsilon^N$. Consider system (57) with initial datum $(\overline{\boldsymbol{x}}^\varepsilon,\overline{\boldsymbol{\ell}}^\varepsilon)$, suppose that the dynamics for the labels can be written as in (58), and denote by $t\mapsto\boldsymbol{y}_t^\varepsilon$ its unique solution. Consider the system*

$$\begin{cases}\dot{\hat{x}}_t^{\varepsilon,i}=v_{\hat{\Lambda}_t^{\varepsilon,N}}(\hat{x}_t^{\varepsilon,i},\hat{\ell}_t^{\varepsilon,i})\\ \hat{\ell}_t^{\varepsilon,i}=\arg\min\limits_{\ell\in C_\varepsilon}G_{\hat{\Lambda}_t^{\varepsilon,N}}(\hat{x}_t^{\varepsilon,i},\ell)\\ \hat{x}_0^{\varepsilon,i}=\overline{x}^{\varepsilon,i}\end{cases}\qquad\textit{for } i=1,\dots,N,\ t\in(0,T]\,,\tag{77}$$

where

$$\hat{\Lambda}_t^{\varepsilon,N}=\frac{1}{N}\sum_{i=1}^N\delta_{(\hat{x}_t^{\varepsilon,i},\hat{\ell}_t^{\varepsilon,i})}\,.\tag{78}$$

Then, for any $1\leq p\leq 2$ and for almost every $t\in(0,T]$, problem (77) admits a unique solution $t\mapsto\hat{\boldsymbol{y}}_t^\varepsilon$. Moreover, for almost every $t\in(0,T]$, the solution $\boldsymbol{y}_t^\varepsilon$ converges to $\hat{\boldsymbol{y}}_t^\varepsilon$ as $\lambda\to+\infty$, namely

$$\lim_{\lambda\to+\infty}||\boldsymbol{y}_t^\varepsilon-\hat{\boldsymbol{y}}_t^\varepsilon||_{\overline{Y}^N}=0.$$

Proof. Existence and uniqueness of the solution to system (77) follows form the Lipschitz continuity of the map (66), by standard results on ODE theory [15]. For the same reason, we have that

$$\begin{aligned}
||\boldsymbol{\ell}_t^\varepsilon - \hat{\boldsymbol{\ell}}_t^\varepsilon||_{E^N} &= \frac{1}{N}\sum_{i=1}^N ||\ell_t^{\varepsilon,i} - \hat{\ell}_t^{\varepsilon,i}||_E \\
&\leq \frac{1}{N}\sum_{i=1}^N ||\ell_t^{\varepsilon,i} - \ell_t^{*\,\varepsilon,i}||_E + \frac{1}{N}\sum_{i=1}^N ||\ell_t^{*\,\varepsilon,i} - \hat{\ell}_t^{\varepsilon,i}||_E \\
&\leq ||\boldsymbol{\ell}_t^\varepsilon - \boldsymbol{\ell}_t^{*\,\varepsilon}||_{E^N} + L_{R+R_\varepsilon}(||\boldsymbol{x}_t^\varepsilon - \hat{\boldsymbol{x}}_t^\varepsilon||_{(\mathbb{R}^d)^N} + W_1(\pi_\#\Lambda_t^{\varepsilon,N}, \pi_\#\hat{\Lambda}_t^{\varepsilon,N})) \\
&\leq ||\boldsymbol{\ell}_t^\varepsilon - \boldsymbol{\ell}_t^{*\,\varepsilon}||_{E^N} + 2\,L_{R+R_\varepsilon}\,||\boldsymbol{x}_t^\varepsilon - \hat{\boldsymbol{x}}_t^\varepsilon||_{(\mathbb{R}^d)^N},
\end{aligned}$$

where R is the bound defined in (68). From Lemma 3, there exists $C(\lambda) > 0$ such that $C(\lambda) \sim 1/\lambda$ as $\lambda \to +\infty$, and $||\boldsymbol{\ell}_t^\varepsilon - \boldsymbol{\ell}_t^{*\,\varepsilon}||_{E^N} \leq C(\lambda)$. Thus

$$||\boldsymbol{\ell}_t^\varepsilon - \hat{\boldsymbol{\ell}}_t^\varepsilon||_{E^N} \leq C(\lambda) + 2\,L_{R+R_\varepsilon}\,||\boldsymbol{x}_t^\varepsilon - \hat{\boldsymbol{x}}_t^\varepsilon||_{(\mathbb{R}^d)^N}.$$

Using the Lipschitz continuity of the vector filed v, and the previous inequality, we can estimate

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}||\boldsymbol{x}_t^\varepsilon - \hat{\boldsymbol{x}}_t^\varepsilon||_{(\mathbb{R}^d)^N} &\leq \frac{1}{N}\sum_{i=1}^N |\dot{x}_t^{\varepsilon,i} - \dot{\hat{x}}_t^{\varepsilon,i}| \\
&= \frac{1}{N}\sum_{i=1}^N |v_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}) - v_{\hat{\Lambda}_t^{\varepsilon,N}}(\hat{x}_t^{\varepsilon,i}, \hat{\ell}_t^{\varepsilon,i})| \\
&\leq \frac{1}{N}\sum_{i=1}^N L_{R+R_\varepsilon}(|x_t^{\varepsilon,i} - \hat{x}_t^{\varepsilon,i}| + ||\ell_t^{\varepsilon,i} - \hat{\ell}_t^{\varepsilon,i}||_E + W_1(\Lambda_t^{\varepsilon,N}, \hat{\Lambda}_t^{\varepsilon,N})) \\
&\leq 2\,L_{R+R_\varepsilon}\,(||\boldsymbol{x}_t^\varepsilon - \hat{\boldsymbol{x}}_t^\varepsilon||_{(\mathbb{R}^d)^N} + ||\boldsymbol{\ell}_t^\varepsilon - \hat{\boldsymbol{\ell}}_t^\varepsilon||_{E^N}) \\
&\leq 2\,L_{R+R_\varepsilon}C(\lambda) + 4\,L_{R+R_\varepsilon}\,||\boldsymbol{x}_t^\varepsilon - \hat{\boldsymbol{x}}_t^\varepsilon||_{(\mathbb{R}^d)^N} \\
&\leq 4\,L_{R+R_\varepsilon}\,(C(\lambda) + ||\boldsymbol{x}_t^\varepsilon - \hat{\boldsymbol{x}}_t^\varepsilon||_{(\mathbb{R}^d)^N})
\end{aligned}$$

or equivantely

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(C(\lambda) + ||\boldsymbol{x}_t^\varepsilon - \hat{\boldsymbol{x}}_t^\varepsilon||_{(\mathbb{R}^d)^N}\right) \leq 4\,L_{R+R_\varepsilon}\left(C(\lambda) + ||\boldsymbol{x}_t^\varepsilon - \hat{\boldsymbol{x}}_t^\varepsilon||_{(\mathbb{R}^d)^N}\right).$$

Therefore, by applying Grönwall's Lemma, for any $\tau > 0$ we obtain

$$C(\lambda) + ||\boldsymbol{x}_t^\varepsilon - \hat{\boldsymbol{x}}_t^\varepsilon||_{(\mathbb{R}^d)^N} \leq (C(\lambda) + ||\boldsymbol{x}_\tau^\varepsilon - \hat{\boldsymbol{x}}_\tau^\varepsilon||_{(\mathbb{R}^d)^N})\,e^{4L_{R+R_\varepsilon}(t-\tau)},$$

which leads us to

$$||\boldsymbol{x}_t^\varepsilon - \hat{\boldsymbol{x}}_t^\varepsilon||_{(\mathbb{R}^d)^N} \leq (C(\lambda) + ||\boldsymbol{x}_\tau^\varepsilon - \hat{\boldsymbol{x}}_\tau^\varepsilon||_{(\mathbb{R}^d)^N})\,e^{4L_{R+R_\varepsilon}T}.$$

Since the same initial data are being considered, we have that

$$\begin{aligned}
||\boldsymbol{x}_\tau^\varepsilon - \hat{\boldsymbol{x}}_\tau^\varepsilon||_{(\mathbb{R}^d)^N} &\leq \frac{1}{N}\sum_{i=1}^N \int_0^\tau |v_{\Lambda_t^{\varepsilon,N}}(x_s^{\varepsilon,i}, \ell_s^{\varepsilon,i}) - v_{\hat{\Lambda}_s^{\varepsilon,N}}(\hat{x}_s^{\varepsilon,i}, \hat{\ell}_s^{\varepsilon,i})|\,\mathrm{d}s \\
&\leq L_{R+R_\varepsilon}\frac{1}{N}\sum_{i=1}^N \int_0^\tau (|x_s^{\varepsilon,i} - \hat{x}_s^{\varepsilon,i}| + ||\ell_s^{\varepsilon,i} - \hat{\ell}_s^{\varepsilon,i}||_E + ||\boldsymbol{x}_s^\varepsilon - \hat{\boldsymbol{x}}_s^\varepsilon||_{(\mathbb{R}^d)^N})\,\mathrm{d}s \\
&\leq L_{R+R_\varepsilon}(1 + 4\,R + 4\,R_\varepsilon)\,\tau\,.
\end{aligned}$$

Finally, choosing $\tau = 1/\lambda$

$$||\boldsymbol{x}_t^\varepsilon - \hat{\boldsymbol{x}}_t^\varepsilon||_{(\mathbb{R}^d)^N} \leq \left(C(\lambda) + L_{R+R_\varepsilon}(1 + 4\,R + 4\,R_\varepsilon)\,\frac{1}{\lambda}\right)\, e^{4L_{R+R_\varepsilon}T}.$$

Taking the limit as $\lambda \to +\infty$ completes the proof. □

Remark 15. With the same line of reasoning of Section 8, it can be proved that the mean-field limit of system (77) is a probability measure $\Sigma_t \in C^0([0,T];(\mathcal{P}_1(\mathbb{R}^d),\mathcal{W}_1))$ which satisfies the continuity equation

$$\partial_t \Sigma_t^\varepsilon + \operatorname{div}\left(v_{\hat{\Lambda}_t^\varepsilon}(\cdot, \ell^*_{\Sigma_t^\varepsilon})\,\Sigma_t^\varepsilon\right) = 0,$$

where

$$\ell^*_{\Sigma_t^\varepsilon}(x) := \operatorname*{arg\,min}_{\ell\in C_\varepsilon} \int_U F_{\Sigma_t^\varepsilon}(x,\ell(u),u)\,\mathrm{d}\eta(u) \qquad \text{and} \qquad \hat{\Lambda}_t^\varepsilon := (id, \ell^*_{\Sigma_t^\varepsilon})_\# \Sigma_t^\varepsilon\,,$$

that is

$$\int_{\mathbb{R}^d} \phi(t,x)\mathrm{d}\Sigma_t^\varepsilon(x) - \int_{\mathbb{R}^d} \phi(0,x)\mathrm{d}\Sigma_0^\varepsilon(x) = \int_0^t \int_{\mathbb{R}^d} (\partial_t\phi(s,x) + \nabla\phi(s,x)\cdot v_{\hat{\Lambda}_s^\varepsilon}(x, \ell^*_{\Sigma_s^\varepsilon}(x)))\mathrm{d}\Sigma_s^\varepsilon(x)\mathrm{d}s$$

for every $\phi \in C_b^1([0,T]\times\mathbb{R}^d)$. As a consequence of Theorem 8, we have the following result

$$\lim_{\lambda\to+\infty} \mathcal{W}_1(\Sigma_t^\varepsilon, \Lambda_t^\varepsilon) = 0,$$

where Λ_t^ε is the mean-field limit of system (57). Indeed, by the triangle inequality we have that

$$\mathcal{W}_1(\hat{\Lambda}_t^\varepsilon, \Lambda_t^\varepsilon) \leq \mathcal{W}_1(\hat{\Lambda}_t^\varepsilon, \hat{\Lambda}_t^{\varepsilon,N}) + \mathcal{W}_1(\hat{\Lambda}_t^{\varepsilon,N}, \Lambda_t^{\varepsilon,N}) + \mathcal{W}_1(\Lambda_t^{\varepsilon,N}, \Lambda_t^\varepsilon),$$

and taking the limit as $\lambda \to +\infty$ we obtain

$$\mathcal{W}_1(\hat{\Lambda}_t^\varepsilon, \Lambda_t^\varepsilon) \leq \mathcal{W}_1(\hat{\Lambda}_t^\varepsilon, \hat{\Lambda}_t^{\varepsilon,N}) + \mathcal{W}_1(\Lambda_t^{\varepsilon,N}, \Lambda_t^\varepsilon).$$

Finally, recalling (1) we have that

$$\begin{aligned}
\mathcal{W}_1(\hat{\Lambda}_t^\varepsilon, \hat{\Lambda}_t^{\varepsilon,N}) &= \sup_{\substack{\varphi\in \mathrm{Lip}_b(Y_\varepsilon)\\ \mathrm{Lip}(\varphi)\leq 1}} \int_{Y_\varepsilon} \varphi(x,\ell)\,\mathrm{d}\hat{\Lambda}_t^\varepsilon(x,\ell) - \int_{Y_\varepsilon} \varphi(x,\ell)\,\mathrm{d}\hat{\Lambda}_t^{\varepsilon,N}(x,\ell)\\
&= \sup_{\substack{\varphi\in \mathrm{Lip}_b(Y_\varepsilon)\\ \mathrm{Lip}(\varphi)\leq 1}} \int_{\mathbb{R}^d} \varphi(x,\ell^*_{\Sigma_t^\varepsilon}(x))\,\mathrm{d}\Sigma_t^\varepsilon(x) - \frac{1}{N}\sum_{i=1}^N \varphi(\hat{x}_t^{\varepsilon,i}, \hat{\ell}_t^{\varepsilon,i})\\
&= \sup_{\substack{\varphi\in \mathrm{Lip}_b(Y_\varepsilon)\\ \mathrm{Lip}(\varphi)\leq 1}} \int_{\mathbb{R}^d} \varphi(x,\ell^*_{\Sigma_t^\varepsilon}(x))\,\mathrm{d}\Sigma_t^\varepsilon(x) - \int_{\mathbb{R}^d} \varphi(x,\ell^*_{\pi_\#\hat{\Lambda}_t^{\varepsilon,N}}(x))\,\mathrm{d}\pi_\#\hat{\Lambda}_t^{N,\varepsilon}(x),
\end{aligned}$$

by the triangle inequality

$$\begin{aligned}
\mathcal{W}_1(\hat{\Lambda}_t^\varepsilon, \hat{\Lambda}_t^{\varepsilon,N}) = &\sup_{\substack{\varphi\in \mathrm{Lip}_b(Y_\varepsilon)\\ \mathrm{Lip}(\varphi)\leq 1}} \int_{\mathbb{R}^d} \varphi(x,\ell^*_{\Sigma_t^\varepsilon}(x))\,\mathrm{d}\Sigma_t^\varepsilon(x) - \int_{\mathbb{R}^d} \varphi(x,\ell^*_{\Sigma_t^\varepsilon}(x))\,\mathrm{d}\pi_\#\hat{\Lambda}_t^{N,\varepsilon}(x)\\
&+ \sup_{\substack{\varphi\in \mathrm{Lip}_b(Y_\varepsilon)\\ \mathrm{Lip}(\varphi)\leq 1}} \int_{\mathbb{R}^d} [\varphi(x,\ell^*_{\Sigma_t^\varepsilon}(x)) - \varphi(x,\ell^*_{\pi_\#\hat{\Lambda}_t^{\varepsilon,N}}(x))]\,\mathrm{d}\pi_\#\hat{\Lambda}_t^{N,\varepsilon}(x).
\end{aligned}$$

Since there exists $R > 0$ such that Σ_t^ε, $\pi_\# \hat{\Lambda}_t^{\varepsilon,N} \in \mathcal{P}(B_R)$ for all $t \in [0, T]$ and $N \geq \overline{N}$, and because of (66), we have that

$$\mathcal{W}_1(\hat{\Lambda}_t^\varepsilon, \hat{\Lambda}_t^{\varepsilon,N}) \leq (1 + L_R)\mathcal{W}_1(\Sigma_t^\varepsilon, \pi_\# \hat{\Lambda}_t^{\varepsilon,N}) + L_R\, \mathcal{W}_1(\Sigma_t^\varepsilon, \pi_\# \hat{\Lambda}_t^{\varepsilon,N}).$$

In the end, we have

$$\mathcal{W}_1(\hat{\Lambda}_t^\varepsilon, \Lambda_t^\varepsilon) \leq (1 + 2\, L_R)\mathcal{W}_1(\Sigma_t^\varepsilon, \pi_\# \hat{\Lambda}_t^{\varepsilon,N}) + \mathcal{W}_1(\Lambda_t^{\varepsilon,N}, \Lambda_t^\varepsilon),$$

which gives the desired result taking the limit as $N \to \infty$.

10 Possible Improvements

In Section 9, we have analyzed just the *undisclosed* case. In order to address the general case, the equations for the labels (58) suggest us to suppose, for the functional F, a full dependence on the state of the system and not just on the agents' locations. Thus, we shall consider $F_{\Lambda_t^N}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}(u), u)$, instead of $F_{\pi_\# \Lambda_t^N}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}(u), u)$. By doing so, the functional G takes the following form

$$G_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}) = \int_U F_{\Lambda_t^N}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}(u), u)\, \mathrm{d}\eta(u),$$

instead of (60), which in turn leads us to a problematic expression for the labels equations of system (77):

$$\hat{\ell}_t^{*\,i} = \operatorname*{arg\,min}_{\ell \in C_\varepsilon} \int_U F_{\hat{\Lambda}_t^N}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}(u), u)\, \mathrm{d}\eta(u) \qquad i = 1, \dots, N.$$

The expression above does not make any sense because on the right-hand side the other solutions to the minimum problem appear in the term $\hat{\Lambda}_t^N$. While it may initially resemble a fixed point problem, a closer inspection reveals that it is not, due to the vector nature of the problem. For this reason, instead of (58), we are lead to consider

$$\dot{\ell}_t^{\varepsilon,i} = \lambda \left(\int_U DF_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}(u), u)\, \ell_t^{\varepsilon,i}(u)\, \mathrm{d}\eta(u)\, \Pi - DF_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i}, \ell_t^{\varepsilon,i}, \cdot) \right) (\ell_t^{\varepsilon,i}, \boldsymbol{\ell}_t^\varepsilon), \quad (79)$$

where now

$$\begin{aligned} \Pi \colon C_\varepsilon \times C_\varepsilon^N &\to C_\varepsilon \\ (\ell, \boldsymbol{\ell}) &\mapsto \ell \end{aligned}$$

and DF is the Fréchet-differential, in the sense of Definition 4, of

$$\begin{aligned} F_{\frac{1}{N}\sum_{j=1}^N \delta_{(x^j, \cdot)}}(x^i, \cdot, u) \colon [r_\varepsilon, R_\varepsilon] \times C_\varepsilon^N &\to [-\infty, +\infty] \\ (\xi, \boldsymbol{\ell}) &\mapsto F_{\frac{1}{N}\sum_{j=1}^N \delta_{(x^j, \ell^j)}}(x^i, \xi, u)\ . \end{aligned}$$

We notice that since (79) has to be compatible with (57) in order to use Definition 4, we would have to extend smoothly the entropy functional $\mathcal{H}$ to zero. We also observe that in the undisclosed setting, the expression in (79) gives us back that in (58). To avoid dealing with derivatives in the space of measures, we deal with integral functions only, that is

$$F_{\frac{1}{N}\sum_{j=1}^{N}\delta_{(x^j,\ell^j)}}(x^i,\xi,u) = \frac{1}{N}\sum_{j=1}^{N} f(x^i,\xi,u,x^j,\ell^j). \tag{80}$$

In such cases, to guarantee that similar assumptions to (FI), (FII), and (FIII) are satisfied, it is sufficient to make similar hypothesis on f to the ones of Proposition 4. At this point we can define

$$G(\boldsymbol{x},\boldsymbol{\ell}) := \frac{1}{N}\sum_{i=1}^{N}\int_U F_{\Lambda^N}(x^i,\ell^i(u),u)\,\mathrm{d}\eta(u), \qquad \text{with } \Lambda^N = \frac{1}{N}\sum_{j=1}^{N}\delta_{(x^j,\ell^j)}\,,$$

and have an analogous results to Proposition 5 and Corollary 2; in particular one can prove the existence and uniqueness of the solution $\boldsymbol{\ell}^*$ to the minimum problem $\min_{\boldsymbol{\ell}\in C_\varepsilon^N} G(\boldsymbol{x},\boldsymbol{\ell})$. To prove a result similar to Lemma 3, we require an additional property:

$$\begin{aligned} &m_t^i(u)\bigg(m_t^i(u)(\ell_t^{\varepsilon,i},\boldsymbol{\ell}_t^\varepsilon),\Big(m_t^j(u)(\ell_t^{\varepsilon,j},\boldsymbol{\ell}_t^\varepsilon)\Big)_{j=1}^{N}\bigg) \\ &\geq C\,||m_t^i(u)||^2_{\mathcal{L}([r_\varepsilon,R_\varepsilon]\times C_\varepsilon^N;\mathbb{R})}\,||(\ell_t^{\varepsilon,i}(u),\boldsymbol{\ell}_t^\varepsilon)||_{[r_\varepsilon,R_\varepsilon]\times C_\varepsilon^N}\,, \end{aligned}$$

where

$$m_t^i(u) = -\left(\int_U DF_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i}(u),u)\,\ell_t^{\varepsilon,i}(u)\,\mathrm{d}\eta(u)\,\Pi - DF_{\Lambda_t^{\varepsilon,N}}(x_t^{\varepsilon,i},\ell_t^{\varepsilon,i}(u),u)\right).$$

Unfortunately, the replicator dynamics does not satisfies this last hypothesis, and satisfies the other ones only for $J(x,u,x',u') = -J(x',u',x,u)$, so that a cooperative behavior is not allowed. Therefore, we have to think about how to weaken the assumptions on f to still guarantee the desired results. Another problem arise when we want to perform the mean-field limit. We have defined our functional F over $[r_\varepsilon,R_\varepsilon]\times C_\varepsilon^N$ in the form (80), but taking the limit as $N\to\infty$ to obtain a mean-field description is an ill-posed operation since the spaces change with N. Therefore, the particular form (80) does not solve our issue and it seems that considering measures derivatives can be a possible way to deal with the general problem.

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