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Conservation laws and temporal witnesses of non-classicality

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Summary

The problem of the universality of Quantum Mechanics has been debated for a long time, with some physicists claiming that there are no fundamental limits to its domain of applicability and others instead who consider Classical Physics to be more fundamental, to the extent that it is not necessary to quantise all physical systems. One of the most challenging topics emerging from this debate in contemporary physics is Quantum Gravity, whose goal is to unify the most fundamental theories known so far: General Relativity and Quantum Physics. Recently, promising laboratory-scale tests of quantum gravity effects have been proposed. They are based on the so-called "Bose-Marletto-Vedral effect", which can prove the non-classicality of gravity by using the latter to entangle two quantum masses. This effect is based on a general entanglement-based witness of non-classicality, which allows one to conclude that a system is non-classical if it can entangle any two quantum probes. Variants of this scheme include detection of gravity-induced non-Gaussianity (i.e., of states whose Wigner function is negative) in a single mass' dynamical evolution. Here we provide a radical generalisation of this single-mass argument, removing the underlying assumption of a specific interaction model: we show that if a system M (e.g. gravity) can induce the dynamical evolution of another system Q (e.g. a mass) from an eigenstate of one of its observables to an eigenstate of a different, non-commuting observable, then M must be non-classical provided that a global quantity of the total system $Q \oplus M$ is conserved. The argument qualifies as a general temporal equivalent of the entanglement-based witness of non-classicality that underlies the Bose-Marletto-Vedral effect using the formalism of quantum theory: assuming the conservation of a global quantity of Q and M, the non-trivial change of basis for the physical system Q will automatically rule out all possible classical models for M. The correspondence between the spatial and temporal witnesses sheds further light on the role of time in nature, expanding on the existing theories that suggest parallels between spatial and temporal quantum correlations. The result leads to new perspectives on the design of an experiment capable of assessing the non-classical nature of *whatever* physical systems, from gravity to biological entities, suggesting new answers to fundamental questions about the universality of Quantum Physics and its applicability to the

macroscopic domain.

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"Omne ignotum pro magnifico"

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Giuseppe.

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Acronyms

QG
: Quantum Gravity

CG
: Classical Gravity

BMV
: Bose-Marletto-Vedral

QIS
: Quantum Information Science

BEC
: Bose-Einstein Condensate

SNR
: Signal-to-noise ratio

MDDI
: Magnetic dipole-dipole interactions

CKW
: Coffman-Kundu-Wootters

CVQIS
: Continuous variable Quantum Information Science

H-P
: Holstein-Primakoff Transformation

Chapter 1

Introduction

Since the dawn of Quantum Mechanics, the question has been whether there are any fundamental limits to its *domain of applicability*. Although in principle it appears to be universal, there have been many arguments against this universality. Some physicists have argued that observers must ultimately be classical [1], which implies that Classical Physics is actually more fundamental than Quantum Physics [2]. Consequently, it may not even necessary to quantise all physical systems. For instance, General Relativity, our best available description of *gravity*, is a *classical* theory. It has been confirmed experimentally to a high accuracy so far and claims to be *universally applicable* in principle. This means that we should be able to either apply a quantisation procedure to the gravitational field ("quantising gravity") [3] as in many - yet untested- proposals for quantum gravity [4], or to "gravitise" quantum theory, maintaining the principles of general relativity [5, 6], or to build a *more general* theory out of which general relativity and quantum mechanics can be retrieved as particular cases, e.g. string theory [7].

The motivations for a theory of Quantum Gravity (QG) come from observed phenomena related to *black holes* and the presumed high-density initial state of the universe at the *big bang*. General relativity predicts for them *singularities* in which the equivalence principle is no longer valid, so that the theory itself becomes inapplicable for these points in spacetime, thus challenging its universality. A theory of QG, then, should explain what happens in those cases in which general relativity predicts singularities capturing the presumed quantum properties of the gravitational field and of dynamical spacetime. More broadly, it should explain in a consistent way how gravity and/or spacetime could be compatible with a quantum world consisting of quantum matter and quantum interactions.

Despite almost a century of work in this direction, it appears very difficult to find an uncontroversial proposal for QG. The reasons for these difficulties are:

- Conceptual incompatibility of the two theories: general relativity treats gravity as a *classical* dynamical field (in what we call *classical* gravity (CG)), but according to quantum mechanics dynamical fields have *quantum* properties, requiring then a quantisation of the gravitational field.
  Moreover, in general relativity the gravitational field is represented by the metric of spacetime; therefore, a quantisation of the gravitational field would correspond to a quantisation of the metric of spacetime. The quantum dynamics of the gravitational field would correspond to a dynamical quantum spacetime, but quantum field theories presuppose a fixed, non-dynamical background space for the description of the dynamics of quantum fields.
  Finally, in general relativity, time is dynamically involved in the interaction between matter/energy and the spacetime metric, while quantum mechanics treats it as a global background parameter, not even as a physical observable represented by a quantum operator;

- Lack of experimental evidence: the regimes at which quantum gravity is expected to be observable are traditionally considered far beyond the range of current experimental capabilities: to probe general relativity near a small length scale where quantum theory effects of spacetime become relevant, we would need to build a particle accelerator as big as Milky-Way [8], so that detecting a graviton is often considered practically impossible [9].

These two points have opened the debate on the actual need to consider gravity as a quantum entity [10] or a fundamental force [11], giving rise to alternative theories such as collapse models [10] which predict the breakdown of quantum mechanics above a certain scale, or models where gravity is treated as a classical agent with a stochastic noise [12, 13], or simply modifying the Einstein action in a way such that gravity becomes weaker at short distances and small timescales [14], or semi-classical theories where matter is quantised but gravity remains fundamentally classical [15].

In this work we want to address a completely new approach to testing Quantum Gravity based on *Quantum Information Science*: in this way we will be able to propose both a theoretically robust and experimentally feasible test of quantum-like features in gravity without relying on the gravitational coupling constant, which we know is about 43 orders of magnitude smaller than the fine structure constant. Moreover, this approach is not limited to gravity alone, but can be used to investigate the non-classicality of *whatever* physical system, from gravity to living systems like bacteria, satisfying some *general* principles. It therefore qualifies as an important ally in the investigation of the universality of Quantum Mechanics.

Two milestones in this direction are represented by the Bose-Marletto-Vedral

effect and by the creation of non-Gaussianity in the quantum field of matter.
The former can prove the non-classicality of gravity relying on a more general entanglement-based argument, which states that under the condition that all interactions but the one with a mediator M can be excluded, the creation of entanglement between two masses is a witness of non-classicality in the mediator M itself, while the latter considers the possibility that only a quantum field has, compared to its classical counterpart, of inducing non-Gaussianity in the quantum state of matter with which the field is coupled. The argument in support of the Bose-Marletto-Vedral effect is more general as it does not make any specific dynamical assumption. Instead, the non-Gaussianity witness assumes a specific dynamics, as it is based on discriminating a quantum model of matter-gravity interaction and its classical counterpart. However, it is interesting to notice how the latter can be seen as a temporal version of the former, although less general: while in the Bose-Marletto-Vedral effect we deal with the entanglement between two masses that are displaced in different locations, in the non-Gaussianity based witness of non-classicality we consider the time evolution of a system interacting with an external mediator.
It is natural then to ask whether a general version of the argument exists for the temporal case too: in this way we will achieve the goal of having a strong theorem easily implementable in an experimental protocol that has to deal with a single system and its time evolution to infer on the non-classicality of the mediator that is inducing the evolution, ideally gravity but actually whatever physical system we want to assess the nature of.

Here we conjecture this general argument using the formalism of quantum theory: we present a theorem that uses a *single probe* Q at *two times*, and a mediator M that *induces* the time evolution of Q, to *infer* on the non-classicality of the mediator itself.
This general argument needs another assumption, which is the conservation of a global quantity of the system $Q \oplus M$. We propose the conservation law as the temporal generalisation of the local-interaction assumption of the spatial version of the argument, so that the idea bodes well with the relation between temporal and spatial entanglement, shedding new light on the role of time in nature and on the meaning of *locality in time.*

In Sections 2 and 3 we will review in their details the two proposals introduced above: Bose-Marletto-Vedral effect, with the general theoretical argument supporting it, and non-Gaussianity based witness of non-classicality.
In Section 4.1 we comment on the possibility to join these two theories in a single, more general argument, to capture the advantages of both of them, in particular

the strong theoretical argument behind the Bose-Marletto-Vedral effect and the experimental feasibility of a protocol working with a single mass in its time evolution. In Sections 4, 5, 6 we will derive our original general temporal argument starting with two qubits, moving then to $N+1$ qubits and concluding with an harmonic oscillator and a spin-$\frac{1}{2}$ particle. Not only we will present the argument as a proof of principle, but we will also provide proofs of the necessity for the mediator to be quantum. The connections with the spatial argument will be emphasised and an interesting perspective on the role of time in nature will be discussed. Finally, in Section 6.3 an interesting generalisation in terms of mode entanglement will be analysed, in order to make a stronger connection between spatial and temporal arguments and to shed a light on the deep meaning of entanglement.

Chapter 2

Entanglement-based witnesses of non-classicality: the Bose-Marletto-Vedral effect

Here the Bose-Marletto-Vedral effect and its theoretical foundations are explained in details: the creation of *entanglement* between two *space-like separated* systems is a witness of quantum-like features in the mediator M, under certain conditions. An *experimental implementation* of the witness is presented. The *theoretical implications* regarding models to describe gravity are discussed.

The Bose-Marletto-Vedral (BMV) effect [16, 17] is one of the most promising proposals to witness quantum-like features in gravity at laboratory scale [18]. It adopts a totally different approach with respect to those described above as it takes advantage of *quantum information science* (QIS) techniques to testing quantum gravity: given that all other interactions can be excluded, the creation of *entanglement* between two masses in a superposition of two locations can be used as a witness of QG. The key is to witness two non-commuting observables of the gravitational field by setting up an experiment where gravity induces entanglement in two quantum probes.

2.1 Theoretical argument

The BMV effect is based on a general argument for a witness of non-classicality, which we now discuss: If a physical system M, e.g. the gravitational field, can mediate *locally* the generation of *entanglement* between two quantum systems Q_A and Q_B, e.g. two masses, then it must be itself *quantum* [16].
The notion of classicality we adopt here is *information theoretic*: "classical" means that all of its physical variables can be simultaneously measured to arbitrarily high accuracy by the *same* device, in a single-shot fashion. This means that we can label them collectively as a single observable T [19, 20]. A system is "non-classical" if it has at least one additional dynamical variable S, disjoint from T, which cannot be *simultaneously* measured jointly with T to arbitrarily high accuracy. In quantum theory, this means that T and S are represented by operators that do not commute. In the remainder of this section, I shall present the argument using the formalism of quantum theory. This definition is radically different from other existing ones - e.g. the system being in a coherent state or its being a decoherent channel [12] - and here lies the strength of this proposal: it relies on *information theoretic* tools to witness quantum-like feature in gravity, in a radically different approach to the topic.

An example with three qubits illustrates the logic: Q_A and Q_B are the two qubits to be entangled, Q_M is the mediator qubit[1]. We prepare the system $Q_A \oplus Q_M \oplus Q_B$ in a product $|0\rangle_A |0\rangle_M |0\rangle_B$ state (here $|0\rangle$ is the +1-eigenstate of the Z component of each qubit) and we let it evolve assuming Q_A and Q_B interacting each *locally* with Q_M.
A simple model describing the generation of entanglement between Q_A and Q_B is shown in Fig.2.1: the entangling gate acts on Q_A and Q_M, creating a Bell state; then, a SWAP gate between Q_M and Q_B allows the two qubits Q_A and Q_B to become entangled at the same degree as Q_M and Q_A were before (in our case maximally entangled). In order for this to be possible, Q_M must engage another variable which does not commute with its Z component: the X- or Y- component, for instance.

The argument will go by contradiction: we consider the physical system M to be *classical*, labelling its single observable T and we show that a contradiction arises if the entanglement between the quantum systems Q_A and Q_B is generated under the assumptions we will detail below.
For simplicity, we consider Q_A and Q_B to be two qubits and we introduce $\hat{q}^{(A)} \doteq$

[1] We assume the mediator to obey quantum theory. In the general argument this assumption will be relaxed.

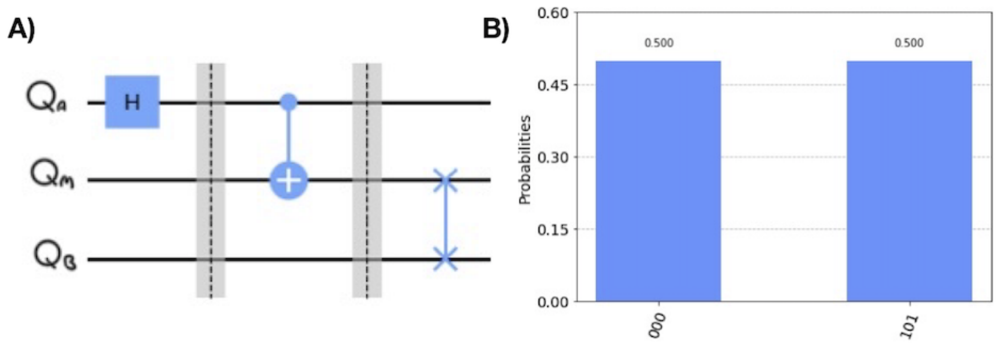

Figure 2.1: A) Circuit associated with the example in the main text. B) Outcome of the simulation: we see that the qubits Q_A and Q_B are entangled.

$(\sigma_x \otimes \mathcal{I}_{B,M}, \sigma_y \otimes \mathcal{I}_{B,M}, \sigma_z \otimes \mathcal{I}_{B,M})$ as the vector of generators $q_\alpha^{(A)}$ of the algebra of observables of the qubit Q_A, where σ_α, $\alpha = x, y, z$ are the Pauli operators and $\mathcal{I}_{B,M}$ is the unit on $Q_A \oplus M$. Similarly, $\hat{q}^{(B)} \doteq (\sigma_x \otimes \mathcal{I}_{A,M}, \sigma_y \otimes \mathcal{I}_{A,M}, \sigma_z \otimes \mathcal{I}_{A,M})$ can be defined for the qubit Q_B. The classical system M is considered to be a bit, so with T being a binary variable represented as an operator $q_z^{(M)} \doteq \mathcal{I}_{A,B} \otimes \sigma_z$, where $\mathcal{I}_{A,B}$ is the unit on Q_A and Q_B. We can think about T as the discretised version of one of the quadratures of the gravitational field, although the argument would apply to continuous systems too.
Assume that, initially, Q_A, Q_B and M are *disentangled* and independently prepared in an eigenstate of σ_z, as in the example above. We allow Q_A and Q_B to interact, separately, with M, but we forbid Q_A and Q_B to interact directly. This is crucial, as otherwise the entanglement generation may be due to such a direct interaction between Q_A and Q_B themselves. Suppose that after these interactions Q_A and Q_B are confirmed to be entangled. This is in *contradiction* with M being classical.
In fact, the most general form of a state of $Q_i \oplus M$, $i = A, B$, if M is classical is:

$$\rho = \frac{1}{4}\left(\mathcal{I}_{A,B,M} + \vec{r} \cdot \hat{q}^{(i)} + s_z q_z^{(M)} + \vec{t} \cdot \hat{q}^{(i)} q_z^{(M)}\right), \tag{2.1}$$

for some real-valued vectors $\vec{r}$, $\vec{t}$ and for some real coefficient s_z. Here $\mathcal{I}_{A,B,M}$ is the unit on $Q_A \oplus Q_B \oplus M$. If we interpret this state as a two-qubit state, we can easily check from the absence of terms proportional to $\hat{q}^{(A)} \cdot \hat{q}^{(B)}$ that it is *separable.*
Hence, the most general state of the system $Q_A \oplus Q_B \oplus M$ is separable too, if the three systems start globally disentangled and Q_A and Q_B cannot interact directly. In particular, the state of $Q_A \oplus Q_B$ will be separable. The contradiction arises when we consider M to be endowed with only one variable, T in our discussion,

i.e. being *classical*: the mediator M *must* have at least another *complementary* observable in addition to T. Thus, if Q_A and Q_B are found entangled and if that entanglement has been mediated by the interaction with M, then M must have at least two non commuting observables, i.e. it *must* be *non-classical* according to our earlier definition.

The argument we have discussed here is very general, compared for instance to the one proposed in [21]: it does not assume any *specific dynamics*. This means that, considering the mediator M to be the gravitational field, the proposal is independent of particular models of quantum gravity, representing an advantage given that there are many different proposed ones as summarised in the previous Section. Moreover, the fact that the argument does not rely on a specific dynamical model makes it applicable to prove the quantisation in different scenarios.
Another peculiar aspect of the argument is that it requires no quantum manipulation of the gravitational field itself: the entanglement between the two systems Q_A and Q_B can be confirmed by directly measuring observables on the two of them only, in a different basis, to implement a witness - but no measurements are ever performed on M (although we allow the possibility to measure it in the classical basis).

Finally, we see here that the argument still relies on quantum theory: if the mediator M is shown to be *non-classical* through this argument, then it *must* be *quantum*. It may be interesting to provide a generalisation of the theoretical argument the Bose-Marletto-Vedral effect is based on in a framework where the typical assumptions of quantum theory are *relaxed*. This is the framework of *Constructor Theory*, introduced in Appendix A, which relies on two, general principles, the Principle of Locality and the Principle of Interoperability. A cue for the possibility of this generalisation stays in the fact that all the interactions the argument assumes as legal must be *local* - namely there cannot be any action at a distance between Q_A and Q_B, and that Q_A interacts with M only, as well as Q_B does. This means that the argument obeys Principle of Locality. Rephrasing the proposed theoretical argument in the framework of Constructor Theory may allow for other possible theories to describe the mediator M, still being *non-classical* in the information-theoretic meaning given above, but different from Quantum Mechanics, e.g. post-Quantum theories. This generalisation is proposed in Appendix B, [22].

2.2 Experimental scheme

We discuss now an experiment based on the theoretical argument in the previous Section, to generate entanglement between the two masses Q_A and Q_B when the mediator M is the gravitational field, in the so-called *Bose-Marletto-Vedral* effect. In [16, 17] two different, but equivalent, experimental setups are proposed. They don't require any quantum control over gravity, just over the two probes. Moreover, they are within reach of current technologies:

- In [16], a general experimental scheme is discussed, which could be realised in a Mach-Zender interferometer or any other equivalent technology that allows one to out put two masses each in a superposition of two locations. In this scheme, the witness of quantum gravity is given by the entanglement between positional degrees of freedom of the masses, which could be massive molecules, two nanomechanical oscillators or two split Bose condensates;
- In [17] a Stern-Gerlach interferometer is used. Here the witness is given by spin entanglement and possible candidates as test masses are microdiamonds with an embedded nitrogen-vacancy center spin or Yb microcrystals with a single doped atomic two-level system in optical traps.

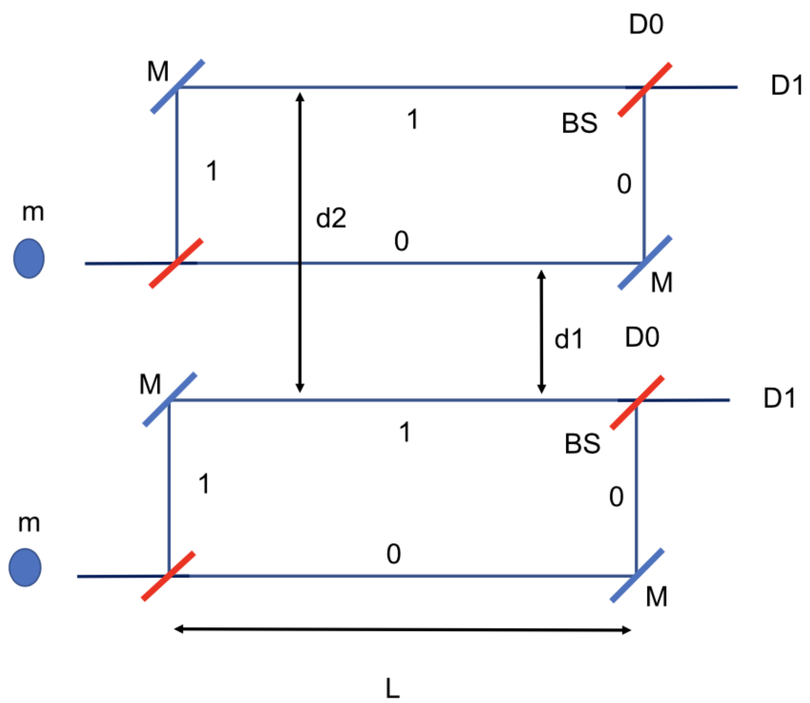

Figure 2.2: Entanglement-based witness of QG with two equal masses. In red we see the Beam Splitter (BS), in blue simple mirrors (M), and with D_i we refer to detectors on path $i = 0{,}1$.

We analyse the proposal in [16], referring to the Fig.2.2. Suppose to have two quantum systems Q_1 and Q_2 with equal mass m each in one of two Mach-Zender

interferometers. These interferometers are located in a way such that both masses experience the same Earth's gravitational field.
It is fundamental to notice here that the two masses *cannot interact directly*, in agreement with Principle of Locality expressed in previous Section. If this were the case, other theories including a classical gravitational field would be capable of predicting the entanglement generation between the two masses.
We label by $|0\rangle_i$ the state describing the particle i in the lower arm of the interferometer and with $|1\rangle_i$ the state describing it in the upper one. The first beam splitter, then, puts each mass in a superposition of the two states, so that:

$$|\Psi(t=0)\rangle_{12} = \frac{1}{\sqrt{2}}(|0\rangle_1 + |1\rangle_1)\frac{1}{\sqrt{2}}(|0\rangle_2 + |1\rangle_2). \tag{2.2}$$

The masses on different paths can interact via the gravitational field, which we assume can be treated within the Newtonian approximation, neglecting general-relativistic contributions. This means that the we are imposing a constraint condition in the Einstein's equation, mathematically represented by the Poisson equation $\nabla^2\Phi = -4\pi G\mu$, being $\mu(\mathbf{r})$ the mass density. This is the reason why, in [23], it has been argued that the gravity-induced entanglement would *not* verify quantum gravity because the relevant degrees of freedom involved in the experiments are *pure gauge*, without physical content. Anyway, as shown in [24], a quantum field without constraints on its Hilbert space and characterized only by dynamical degrees of freedom is *not* capable of inducing the entanglement we are looking for in these experiments. This means that even the Newtonian potential could demonstrate the BMV effect; this is quite remarkable as it is the first order perturbative QG, which means that *all* full QG theory predicts the effect to be real as they all agree in this regime.

Supposing the gravitational interaction of the masses on the two most distant arms to be negligible, the final state for the composite system *before* the last beam splitter is met will be:

$$|\Psi(t)\rangle = \frac{1}{2}|0\rangle\,[|0\rangle + e^{i\phi_1}|1\rangle] + \frac{1}{2}e^{i\phi_1}|1\rangle\,[|0\rangle + e^{i\Delta\phi}|1\rangle], \tag{2.3}$$

being ϕ_i the relative phases acquired by the masses due to the gravitational potential generated when they are, respectively, at distance d_i from one another. According to what we have said before, then:

$$\phi_i = \frac{m^2 G}{\hbar d_i}t = \left(\frac{m}{m_P}\right)^2 \alpha \tag{2.4}$$

with G the gravitational coupling constant and $t = \frac{L}{v}$ is the time spent by each mass in its arm of length L with velocity v. Notice that the presence of both G

and $\hbar$ in the above phase makes questionable its detectability. We can comment more on this introducing the Planck mass $m_P = \sqrt{\frac{\hbar c}{G}}$ and giving it a new meaning [25]: it is the mass scale at which quantum superposition of spacetimes curved by the masses themselves is detectable. Because of the control on quantum coherence, we have that the ratio $\frac{m}{m_P}$ is small: the small quantity that determines the physical effect. In order to make it measurable, we have to deal with $\alpha = \frac{ct}{d_i}$, which then has to be made as large as required by the reduction of the employed masses m with respect to the reference one m_P. Anyway, this nothing has to do with the entanglement between the masses and the field, which can even be so small to be undetectable in practice, just like a spontaneous emission of a graviton [9].

But there is something more behind this α: changing it (under the assumption the BMV effect is detectable) we can see that the state in $|\Psi(t)\rangle$ is entangled to a different degree. In each of the interferometers, the probabilities p_α for the mass to emerge on path $\alpha = 0{,}1$ *after* the second beam splitter are:

$$p_0 = \frac{1}{2}\left(\cos^2\frac{\phi_1}{2} + \cos^2\frac{\Delta\phi}{2}\right) \tag{2.5}$$

$$p_1 = \frac{1}{2}\left(\sin^2\frac{\phi_1}{2} + \sin^2\frac{\Delta\phi}{2}\right) \tag{2.6}$$

which shows two extreme regimes:

- The two masses are maximally entangled by the action of the gravitational field: $p_0 = p_1 = \frac{1}{2}$: it occurs when $\phi_1 = 2n\pi$ and $\Delta\phi = \pi$, with $n \in \mathbb{N}$;
- The two masses are not entangled at all; they simply undergo an ordinary interference experiment emerging both on path 0 of the interferometer: it occurs when $\phi_1 = \Delta\phi = 2n\pi$.

Of course all the other intermediate cases are possible as well, by correctly tuning the parameter α once having fixed the masses m. This entanglement, then, will be a witness that the task T_E in Eq.B.2 is possible, so of the quantum (non-classical)[2] nature of gravity.

There is still one point left open by these proposals: we actually would have to make sure that the entanglement is *really* generated by gravity, and not by other sources of interaction such as Van der Waals forces, Casimir-Podoler interaction or other electromagnetic interactions. These other interactions should be made weaker

[2] Actually to state the field is a proper quantum system we should prove something more, related to the two non-commuting variables: they must be both measurable.

than gravitational one, which can be way difficult: in [17], for instance, it is argued that $d_1 \approx 200\mu m$ in order for the Casimir-Podoler interaction to be negligible, but of course the distance between the interferometers cannot be arbitrarily large with this purpose because otherwise the gravitational interaction will become too weak. This is, together with the possible existence of direct interactions between the masses, an important issue of these entangled-based witnesses of QG.

2.3 Entanglement-based witness as a Bell's theorem for QG models

Suppose to be able to detect entanglement between the two masses when it is generated only through gravity, matching all the conditions required in Section 2.2: which one, among the proposed models for coupling matter with gravity, will be the correct one? Our argument *does not* provide an answer to this question, as it guarantees that the gravitational field must have at least two non commuting observables in an *indirect* way, i.e. without trying to detect directly quanta of the gravitational field.
As noted in Section 2.1, anyway, we may try to use this witness to *exclude* particular models of QG which will not be capable of predicting BMV effect in their formulation, as a Bell's Theorem for QG models. This is the content of [26], which we are going to discuss now.

As we already know, all those models that resort to a field that is classical, in the sense that it has no pair of non-commuting observables, will be ruled out observing the entanglement between the masses. Among this kind of models, we have the so-called "semi-classical" theories of gravity [27, 28]: the background spacetime is classical, but the back-action of the masses, prepared in some quantum state, on the field can be taken into account as an average of the energy momentum tensor in the quantum state of the masses. We can see this from a different perspective than in Section 2.1 referring directly to our experiment and considering an initial state for the masses as the one in Eq.2.2. Each mass would be affected by the average of the gravitational field generated by the other mass positioned at a distance which is the average of the position of the other mass in its quantum state, so that the state would become $\frac{1}{\sqrt{2}}(|0\rangle + exp(i\phi_m)|1\rangle)$ with $\phi_m = G\frac{m^2 t}{\hbar d_m}$ and $d_m = \frac{d_1+d_2}{2}$. Such a phase is a *local phase*, which cannot generate entanglement, therefore the final state of the two masses will always be a product state.

There is another approach which treats classical gravity as field induced by the quantum fluctuations of all other fields [11, 29]. In this way, the obtained gravitational field will anyway be semi-classical, so unable to take into account the entanglement generation we are talking about.

The last class of theories that will be ruled out are all those collapse models which predict the collapse of the wavefunction of each mass at the Planck mass m_P scale [10]. This is the way in which such a scale was understood before the introduction of the experiment we are discussing in Section 2.2. According to these models, the masses involved in the experiment undergo a transition to a state where the position is sharp, so that no entanglement could be generated via the gravitational interaction. Such a collapse is induced by the gravity itself and breaks the linearity of quantum theory. According to Penrose's collapse model, the decoherence time will be of order $t = \frac{\hbar d}{Gm^2} \approx 10^{-13}s$, which is below the $10^{-6}s$ required in our experiment with masses of $10^{-12}kg$ and distances of order of $10^{-4}m$.

This means that the only known models of QG that are compatible with the observed entanglement are *local linearised quantum gravity models*. This is a good result, as all the canonical approaches like loop quantum gravity [4] and string theory [7] reduce to linear quantum gravity.

Chapter 3

A proposal to probe quantum gravity with a single system: non-Gaussianity

Here gravity-induced *non-Gaussianity* generation in a single mass' dynamical evolution is used as a witness of non-classicality in gravity, given a *specific* interaction model. An *experimental implementation* of the witness is discussed. The *limitations* of a single mass witness, without further assumptions, are exposed.

One of the most important take-home messages from the Section 2 is that the link between QG and QIS leads to new theoretically strong and experimentally feasible way of testing QG. Anyway, as underlined in Section 2.2, there are some controversial points in the entanglement-based tests for QG: the need to delete all the direct interaction between the two masses and the difficulties in excluding all interactions but gravity in the experimental setup. These are the reasons why in [18] a new theoretical link between QG and QIS is presented together with an experimental scheme which works with a *single mass* in a way such that it is easy to let gravitational interaction be *the only one* acting on the system.

3.1 Theoretical argument

The argument relies on the impossibility for a classical theory of gravity to create *non-Gaussianity* in the quantum field state of matter: only its *quantum counterpart* will be capable of doing so. We need processes that are non-quadratic in quantum operators to create non-Gaussianity, and only QG, compared to CG, can contain such processes: as we see in Fig.3.1, assuming for simplicity to be in a weak field, perturbative gravitational interaction framework with a real scalar field representing matter ($\hat{a}$ and $\hat{a}^\dagger$ its annihilation and creation operators), the interaction is associated with three quantum operator, thus inducing non-Gaussianity, if the gravitational leg represents a graviton $\hat{h}_{\mu\nu}$.

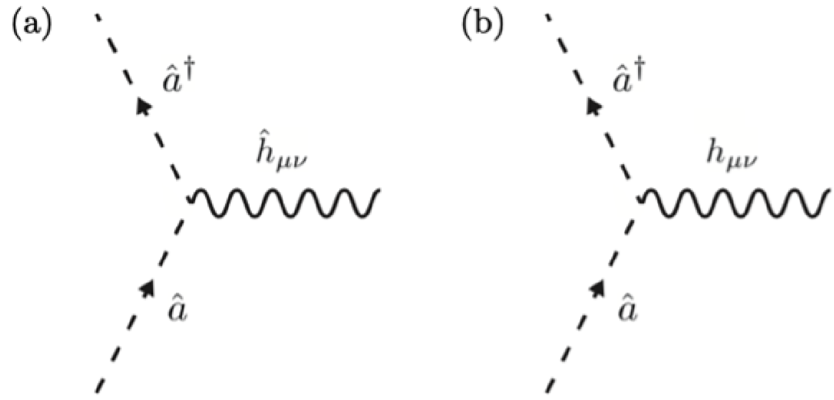

Figure 3.1: Feynman diagram for matter interacting with QG (a) and CG (b): in (a) matter emits a graviton $\hat{h}_{\mu\nu}$, in (b) a classical gravitational wave $h_{\mu\nu}$.

We notice here that the argument compares a *particular* QG model with its own *classical counterpart* in the ability to induce non-Gaussianity in the quantum state of matter, which means that we have something less general than what discussed in Section 2.1. We will comment further on this point in Section 3.3.

Let us introduce the quadrature operators associated to a field with a discrete, finite mode spectrum for simplicity, $\hat{x}_k = \hat{a}_k + \hat{a}_k^\dagger$ and $\hat{p}_k = i(\hat{a}_k^\dagger - \hat{a}_k)$ such that $\hat{x}_k \left|x\right\rangle_k = x_k \left|x\right\rangle_k$ and $\hat{p}_k \left|p\right\rangle_k = p_k \left|p\right\rangle_k$[1]. The most general form for a Hamiltonian

[1]The quadrature eigenvalues x_k and p_k are part of continuous eigenspectra that build a continuous phase space where we can encode our quantum information: we talk about continuous variable QIS (CVQIS) then, although the formulation using, for instance, qubits is valid as well.

that is at most quadratic in quadratures is:

$$\hat{H} = \sum_k \lambda_k(t)\hat{\mathbf{x}}_k + \sum_{k,l} \hat{\mathbf{x}}_k^T \mu_{kl}(t)\hat{\mathbf{x}}_l \tag{3.1}$$

where $\hat{\mathbf{x}}_k^T = (\hat{x}_k, \hat{p}_k)$, $\lambda_k(t)$ and $\mu_{kl}(t)$ are instead 2×1 and 2×2 real-valued arbitrary functions of time. Notice in particular that the Hamiltonian for a free field has this form as well. Consider now an entity $\mathcal{G}$ interacting with this quantum field and assuming there are no terms of higher order than quadratic in the quantum matter field (which will induce quantum self-interaction of matter also present in a flat space):

- If $\mathcal{G}$ is classical, it will be just absorbed in $\lambda_k(t)$ or in $\mu_{kl}(t)$, leaving the Hamiltonian in the form in Eq.3.1. This means that the Hamiltonian of the classical interaction preserves Gaussianity;
- If $\mathcal{G}$ is quantised, the resulting Hamiltonian can gain terms that are no more either linear nor quadratic in quadrature operators, thus inducing non-Gaussianity: the entity $\mathcal{G}$ will be associated with an operator, no longer preserving Gaussianity of Eq.3.1.

Due to the universal coupling of gravity, we can easily think to $\mathcal{G}$ as gravity, concluding our argument: creation of non-Gaussianity in the quantum state of matter would provide evidence for a quantum theory of gravity.

3.2 Experimental scheme

The proposed experiment that uses non-Gaussianity as a witness for QG is based on a *single* Bose-Einstein condensate (BEC) in a *single location* [18]. It can be described by a non-relativistic scalar quantum field $\hat{\Psi}(\mathbf{r})$ in a low temperature framework, such that the ground-state is macroscopically occupied and we can neglect the thermal component of the gas:

$$\hat{\Psi}(\mathbf{r}) \approx \psi(\mathbf{r})\hat{a} \tag{3.2}$$

being $\hat{a}$ the annihilation operator for the condensate and $\psi(\mathbf{r})$ the wavefunction of a condensed atom: we are considering then all the atoms in the same state, with the same wavefunction, equally delocalised across the BEC [30].
It is important here to underline that this test is based on a single mass, the BEC, as in this way we do not have to worry about possible direct interactions between the different parts of a multi-partite system anymore, solving the first problem left opened by the entanglement-based proposal in Section 2. Anyway, as we are going to comment more on in Section 3.3, using a single mass is *not* as strong as using

two (or more) masses on theoretical ground to witness quantum features in gravity.

Neglecting for the moment other possible kind of interactions, we let the atoms interact gravitationally with each other and we assume the system to be non-relativistic, so that the Newtonian approximation of gravity discussed in Section 2 can be used here as well. If gravity obeys classical theory, we will get

$$\hat{H}_{CG} = \lambda_{CG}[\Psi]\hat{a}^{\dagger}\hat{a} \tag{3.3}$$

where

$$\lambda_{CG}[\Psi](t) = Gm \int d^3\mathbf{r}|\psi(\mathbf{r})|^2\Phi[\Psi](t,\mathbf{r}) \tag{3.4}$$

and $\Phi[\Psi](t,\mathbf{r})$ the classical Newtonian potential, which is quadratic in quantum operators and so, for what discussed in Section 3.1, preserving Gaussianity in the state of the matter quantum field.

Looking for the *quantum counterpart* of it, we should quantise both $\rho(\mathbf{r})$, the mass density, and $\Phi(\mathbf{r})$, the Newtonian potential resulting from the solution of the scalar constraint on Einstein's equation mentioned in Section 2.2:

- For what concerns the mass density, in this Newtonian limit it will contain two copies of the matter field irrespectively of the spin of the field itself:

$$\hat{\rho}(\mathbf{r}) = m\hat{\Psi}^{\dagger}(\mathbf{r})\hat{\Psi}(\mathbf{r}); \tag{3.5}$$

- Regarding the Newtonian potential, we should solve the quantised version of Poisson's equation:

$$\hat{\Phi}(\mathbf{r}) = -Gm \int d^3\mathbf{r}' \frac{\hat{\Psi}^{\dagger}(\mathbf{r}')\hat{\Psi}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{3.6}$$

being m the mass of the atoms. This will result in:

$$\hat{H}_{QG} = \frac{1}{2}m \int d^3\mathbf{r} :\hat{\Psi}^{\dagger}(\mathbf{r})\hat{\Psi}(\mathbf{r})\hat{\Phi}(\mathbf{r}): \tag{3.7}$$

where :: refers to normal ordering, and using $\hat{\Psi}(\mathbf{r}) = \psi(\mathbf{r})\hat{a}$ we finally get:

$$\hat{H}_{QG} = \frac{1}{2}\lambda_{QG}\hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{a} \tag{3.8}$$

where:

$$\lambda_{QG} = -Gm^2 \int d^3\mathbf{r}d^3\mathbf{r}' \frac{|\psi(\mathbf{r}')|^2|\psi(\mathbf{r})|^2}{|\mathbf{r}-\mathbf{r}'|}. \tag{3.9}$$

It is important for the rest of the work to notice here that this Hamiltonian Eq.3.8 depends on *a single parameter* λ_{QG}, which contains the mass of the atom. We will come back on this observation in Section 4.1.

The main idea of this experimental scheme relies on the fact that for a gaussian distribution all cumulants higher than second order vanish, so measuring a non-zero value of such cumulants will mean that a non-Gaussianity has been created in the BEC, providing a witness for QG. Since the third order cumulant κ_3 can be zero for a non-Gaussian distribution as well if it is symmetric, it is preferable to concentrate on the fourth-order one κ_4:

$$\kappa_4 = \langle \hat{q}^4 \rangle - 4\langle \hat{q} \rangle \langle \hat{q}^3 \rangle - 3\langle \hat{q}^2 \rangle^2 + 12\langle \hat{q}^2 \rangle \langle \hat{q} \rangle^2 - 6\langle \hat{q}^4 \rangle \tag{3.10}$$

being $\hat{q}(\phi) = \hat{a}e^{-i\phi} + \hat{a}^\dagger e^{i\phi}$ a generalized quadrature. In our experimental case we deal with a finite sample to estimate κ_4, so an unbiased estimator for it is preferred: the k statistics $\langle k_4 \rangle = \kappa_4$ [31]. This allows us to define the signal-to-noise ratio (SNR) for the measurement considering as noise the standard deviation of k_4:

$$SNR = \frac{|\kappa_4|}{\sqrt{Var(k_4)}} \tag{3.11}$$

where, for a large number of measurements $\mathcal{M}$, we can apply the central limit theorem to get $Var(k_4) \propto \frac{1}{\mathcal{M}}$.
In order to maximise the SNR, the system is prepared in a Gaussian state as a squeezed one[2] [32]. The system is left evolve for a time t before the κ_4 is measured. To achieve this, we have two possible routes:

- To use a homodyne detection scheme for κ_4 [33], extending the techniques in [34][3]. This would requires a single-atom counting in a quantum gas with high efficiency on small length scales;

- To determine the Wigner function of the BEC. The only states that have negative Wigner functions are non-Gaussian states. To measure it, we can either use full state tomography with projective measurement [35], or weak measurements of the position quadrature and projective measurements of the momentum quadrature [36].

If κ_4 is measured to be different from zero at time t, then a non-Gaussianity has been induced on the state of the system, providing a witness for QG given that all

[2] A squeezed state is far less demanding to create with respect to a N00N one we have seen in Section 2.2, but with equal performances in quantum metrology.

[3] We could in principle work directly with κ_3, but it is predicted to be zero for a BEC initially prepared in a squeezed vacuum state.

the other possible quantum interactions can be excluded. The most relevant are the electromagnetic interactions between the atoms of the BEC due to Van der Waals and magnetic-dipole-dipole interactions (MDDIs), since the system is globally neutral. These interactions can be easily distinguished from gravitational effects in BEC thanks to the presence of optical and magnetic Feshbach resonances capable of controlling the strength of the electromagnetic interactions between the atoms by the application of an external optical or magnetic field [30]. Feshbach resonances occur when a magnetic or optical field is capable of sticking together two slow colliding atoms in an unstable compound with short life-time, so called resonance. Given the importance of this point in comparison with entanglement-based test of QG (see Section 2), we want to deepen it.
Let's consider a *weakly* interacting BEC of mass M in a spherical harmonic trap with frequency ω_0 at the low temperature at which BECs operate; we obtain the Gaussian wavefunction for BEC [30]:

$$\psi(\mathbf{r}) = \frac{1}{\pi^{3/4}R^{3/2}}e^{-r^2/(2R^2)} \tag{3.12}$$

with $R = \sqrt{\frac{\hbar}{m\omega_0}}$ effective radius of the spherical BEC and r the modulus of $\mathbf{r}$. The Hamiltonian for a BEC with electromagnetic interactions:

$$\begin{aligned}\hat{H} = \int d^3\mathbf{r}\Bigg[&-\frac{\hbar}{2m}\hat{\Psi}^\dagger(\mathbf{r})\nabla^2\hat{\Psi}(\mathbf{r}) + \frac{1}{2}m\omega_0^2 r^2\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r})\\ &+\frac{1}{2}\int d^3\mathbf{r}'\left[\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}^\dagger(\mathbf{r}')\hat{\Psi}(\mathbf{r})\hat{\Psi}(\mathbf{r}')\left(\frac{4\pi\hbar^2 a_s}{m}\delta^{(3)}(\mathbf{r}-\mathbf{r}') + \frac{\mu_0\mu^2}{4\pi}\frac{1-3\cos^2\theta}{|\mathbf{r}-\mathbf{r}'|^3}\right)\right]\Bigg]\end{aligned} \tag{3.13}$$

reduces to:

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\lambda_s\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} \tag{3.14}$$

where:

$$\hbar\omega = \hbar\omega_0 + \frac{3}{4}m\omega_0^2R^2 \tag{3.15}$$

$$\lambda_s = \frac{g}{2\sqrt{2}\pi^{\frac{3}{2}}R^3} = \sqrt{\frac{2}{\pi}}\frac{a_s\hbar^2}{mR^3}. \tag{3.16}$$

We notice that MDDIs are not relevant anymore in our discussion, because of the spherical symmetry of the BEC, so that only s-wave interactions have to be treated now. Applying a magnetic field of strength B to the BEC, the s-wave scattering length a_s becomes:

$$a_s(B) = a_s^{bg}\left[1 - \frac{\Delta}{B-B_0}\right] \tag{3.17}$$

being a_s^{bg} the background scattering length and B_0 and Δ the resonance position and width, respectively. If we set $B = B_0 + \Delta$, we see that $\lambda_s = 0$ and the electromagnetic interactions in the BEC are turned off.
So this non-Gaussianity-based proposal manages to overcome both the problems left opened by BMV proposal.

3.3 Is a single mass enough for our purposes?

One of the major drawbacks of this proposal is that it focuses on the comparison between the classical and quantised version of the same model for the interaction between gravity and matter. This means that we cannot repeat the same argument we have explained in Section 2.3 for this kind of test of QG. Why does this occur? The reason lies in the fact that we work here with a *single* mass. With a single mass experiment, we are not able to ensure that no other theories involving a purely *classical* gravitational field could be formulated in order to explain those effects we relate to an hypothetical quantised field.

This can be better explained looking at the thought experiment proposed by Feynman during the Chapel Hill conference on gravity [37]. It is the starting point for the argument we have deepened in Section 2: a *single* test mass is prepared in a superposition of two locations and interacts with a gravitational field. The mass and the field becomes entangled: in order for the field to be quantised the mass should interfere in a full interference experiment, meaning that the coupling to gravity has been reverted and, thus, the unitary dynamics in quantum theory confirmed.
This is not enough, anyway, to conclude that the gravitational field is quantum:

- It is not possible with this argument to exclude possible gravitational collapse theories as the one in [10] already mentioned or the one in [5], which interestingly works with a BEC as well; Feynman itself took into account these theories;

- In the argument the two spatial states of the mass acquire different phases, which can be induced by interaction with entirely classical gravitational field. Different experiments have been done in order to witness this, such as Collela-Overhauser-Werner and related ones [38].

Thus, a single mass is *not* enough to definitely prove the quantum nature of gravity: we should always accompany this kind of tests with something else based on a multi-partite system to have a stronger argument on this topic [5].

Chapter 4

Temporal witnesses of non-classicality: two-qubit toy model

Here the general temporal argument is discussed using *two qubits*. The crucial role of the *conservation law* is introduced. A proof of the necessary and sufficient conditions to have a *non-classical mediator* driving the evolution is given. The *connections with the entanglement-based witnesses* are disclosed and a new perspective on the role of *locality and entanglement in time* is explored.

4.1 What if we merge the two proposals?

The two arguments in Sections 2 and 3 seem to complete each other: the first one gives a strong theoretical argument for the mediator of the entanglement to be non-classical, working as a Bell's theorem for QG models when applied to gravity in the so-called BMV effect, while the latter manages to overcome those issues related to direct interactions between masses and deletion of all interactions but gravity throughout the experiment. The key to the solution, then, could be to show that the two proposals are *equivalent*, two different ways of looking at the same *QIS-based* test for QG capable of exploiting the advantages of both the proposals: proving the gravity to definitely be *non-classical* in a Bell's theorem like approach to gravity-matter interaction models with a single mass.

A cue for this possibility to be reasonable is that the QG signal has the same dependency on m^2 and t in both the proposals. Under the assumption that QG acts in the limit of small $\chi = \frac{|\lambda_{QG}|}{\hbar}$ and large number of atoms N in the BEC, the SNR can be of order $\chi N^2 t\sqrt{\mathcal{M}}$ being t the interaction time [18]. In the same setup leading to Eq.3.12, we can obtain [5]:

$$\chi t N^2 = \sqrt{\frac{2}{\pi}} \frac{GM^2 t}{\hbar R} \tag{4.1}$$

so that the signal-to-noise ratio becomes:

$$SNR \approx \chi t N^2 \sqrt{\mathcal{M}} = \sqrt{\frac{2}{\pi}} \frac{GM^2 t}{\hbar R} \sqrt{\mathcal{M}} \tag{4.2}$$

which, compared with Eq.2.4, matches the relative phase generated in the BMV effect upon the replacement of R with the smallest possible distance d between the two masses[1].

The idea, then, could be to associate a control system to the (only) parameter λ_{QG} on which Eq.3.8 depends and let it interact with the BEC: if the BEC can acquire a non-Gaussianity in its quantum state because of this interaction, then we can recover in the non-Gaussianity-based witness the two systems required by Locality and Interoperability Principles to be applicable, joining the two proposals as we claim. As a consequence, the argument proposed in Section 2.1 can be applied here as well, but with a fundamental difference: the entanglement is created between the BEC at time t and *itself* at time $t' > t$ through the auxiliary system behind λ_{QG}, giving rise to *entanglement in time*.
This is a very deep topic independently of QG: we know that two-fold correlations in space and time can be treated in a symmetrical way, but what about entanglement? Time and space are differently handled in quantum mechanics both at the conceptual and mathematical level: time enters as an external parameter in the dynamical evolution of the system, while spatial coordinates are considered as quantum mechanical observables, whose states live in an Hilbert space so that a composite quantum system can be in a state that is not separable regardless of the spatial separation of its components, being then *entangled in space*. Although we know that it is meaningful to talk about *entanglement in time* [39, 40], we don't know yet which the relationship between entanglement in space and in time is; and what the consequences of combining them might be: if we manage to prove

[1]There is more we can read here: since SNR depends on $\sqrt{\mathcal{M}}$, we can lower the total mass M required by increasing the number of measurements, which could be useful in order to devise the experiment.

that the non-Gaussianity-based witness of QG is the *temporal equivalent* of the BMV effect in the sense expressed above, then a deeper theory than quantum field theory in which the two need to be treated in a more equal foot may be required. Instead, if the task of driving the system to the non-Gaussian state through the interaction with the auxiliary system behind λ_{QG} is not possible, then it could be interesting to ask which one of the principles introduced in Section 2.1 fails and why. According to Constructor Theory of Information (see Appendix A), this can open a huge point of view on the role of space and time in nature, without any dynamical or scale assumptions.

Thus, in this and in the following Sections, we will derive this *temporal version* of the theoretical argument supporting the Bose-Marletto-Vedral effect working in a more general framework that is not specifically related to QG, so to have a tool to witness non-classicality in *whatever* physical system, being based on information-theoretic techniques. This means that we are going to replace the BEC introduced in Section 3 with a general system A, and the gravitational field with a general mediator B. The witnessing task introduced above can be formally stated as:

Witnessing Task. *The system A, prepared in an eigenstate of one of its observables at time t, evolves to an eigenstate of a different, non commuting observable at time $t' > t$, interacting with the mediator B only.*

It is interesting now to wonder whether this task is *possible* or not, under which assumptions and for what kind of interactions and initial states. We will discover that a crucial element in this witness is the requirement of a *conservation law* throughout the evolution, which prevents the spontaneous evolution of the system A and corresponds to the requirement of local interactions between the probes and the mediator in the entanglement-based witness introduced in section 2.1. Here lies the connection between space and time, with all the fascinating consequences it has on their symmetry.

4.2 Two-qubit toy model: an introduction

We start the investigation of the temporal non-classicality witness adopting two qubits Q_A and Q_B: the latter plays the same role as the auxiliary system associated with λ_{QG} in Section 3, while the former is the single probe involved in our proposal. Let us define the computational basis $\{|0\rangle, |1\rangle\}$ as that made of the eigenstates of the Z component of a qubit. Thus, the eigenstates of the X component will be $\{|+\rangle = 1/\sqrt{2}\,(|0\rangle + |1\rangle), |-\rangle = 1/\sqrt{2}\,(|0\rangle - |1\rangle)\}$.

The qubit Q_A is prepared in the state $|0\rangle$ at time t and we claim, according to the Witnessing Task, it can reach the eigenstate $|+\rangle$ at time $t' > t$ interacting with the qubit Q_B as shown in Fig.4.1. Intuitively, the initial state $|0\rangle$ should

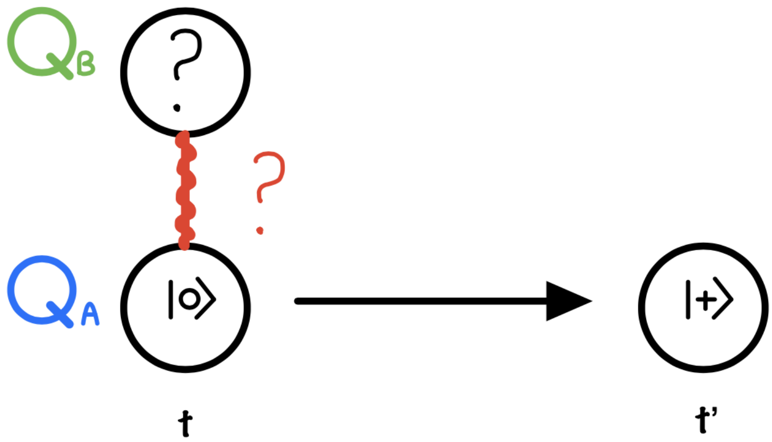

Figure 4.1: A pictorial representation of the Witnessing Task whose possibility we are assessing: what states of Q_B allow Q_A to achieve the transition from a state sharp in a given basis (e.g. Z) to another state sharp in a different basis (e.g. X), given a set of allowed interactions between the qubits?

represent the Gaussian state in which the BEC is prepared, while the final state $|+\rangle$ the non-Gaussian one induced by the interaction with the quantum field. This connection can be made more clear by looking at the computational role that Gaussian and non-Gaussian states have in quantum information: the evolution of Q_A $|0\rangle \rightarrow |+\rangle$ allows the qubit to reach states over all the Bloch sphere, so that it is capable of performing quantum computations and not only classical ones (as if it is constrained to move only along the z axis, having assumed that Z is the classical basis), exactly as only non-Gaussian states, compared to Gaussian ones, allows for quantum computations with pure states [41].

In order for this to be possible, which has to be the initial state of the qubit Q_B? We should expect it to be prepared in a superposition of eigenstates of X and Z as in this way it can have two non-commuting degrees of freedom, being consistent with the conditions for $\mathbf{M}$ in Section 2.1: the goal is to conclude that the only *non trivial* (see Section 4.3) way to let Q_A evolve as described above driven by Q_B is to have two non-commuting degrees of freedom of the qubit Q_B directly involved in the interaction.
The other important point concerns the *interactions* that make this controlled

evolution for Q_A possible. Our goal is to find the Hamiltonian pertaining to the gate we should implement: it could be interesting to look for the symmetries of this Hamiltonian, reflecting of course the symmetries of the interaction it describes. Gravitational interaction, which is the one we are interested in, is symmetric, so it is important to find a coherent result even with this simple qubit-based example.

This comment leads us to the final remarks on the perspectives of this work: how to generalise this to the BEC discussed in Section 3? Well, if we can show that the Witnessing Task with the two qubits discussed in this section is possible, then the system behind the parameter λ_{QG} must have two degrees of freedom that don't commute: in the continuous-variable quantum information limit we can think about a *mean field* involving these non-commuting degrees of freedom of the control system.

4.3 Two trivial examples for the Witnessing Task

We want to present here two simple examples to illustrate a very elementary way to achieve the Witnessing Task.

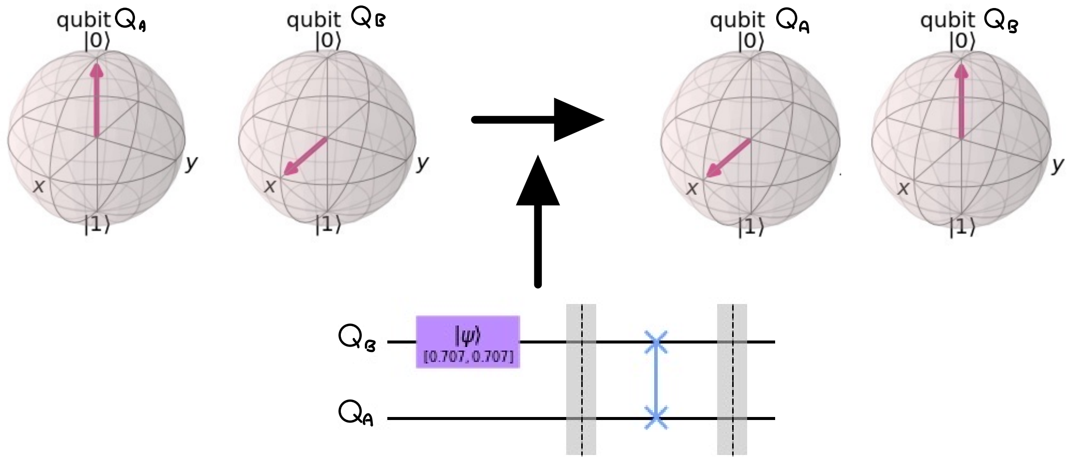

Figure 4.2: From left to right, we see the initial states of the two qubits, the swap gate discussed in the main text and the final state of the qubits.

The first example is the one shown in Fig.4.2: the qubit Q_B is *already* prepared in the state $|+\rangle$, with Q_A in $|0\rangle$ as always; we apply the swap gate and we check that, obviously, the qubit Q_A is now in the state $|+\rangle$ as claimed by the Witnessing Task. This is *not* what we want because only one observable of the qubit Q_A will be involved in the evolution of the system, so that we *cannot* say it to be non-classical.

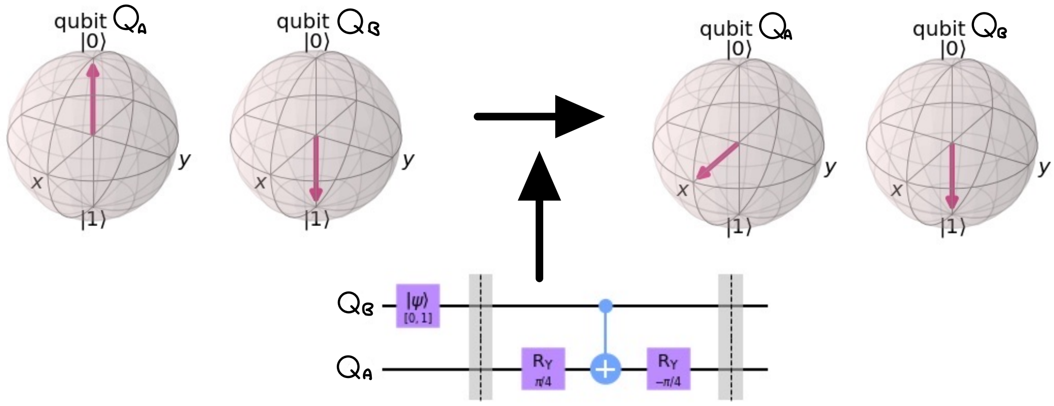

Figure 4.3: From left to right, we start with the qubit Q_B in $|1\rangle$ to match the controlling condition, the circuit discussed in the main text operates as a controlled-Hadamard gate to reach the final state where Q_A is in $|+\rangle$.

Another example is shown in Fig.4.3. Suppose the qubit Q_B is a *control program* acting in this way: if Q_B is in $|0\rangle$, then leave Q_A in $|0\rangle$; if Q_B is in $|1\rangle$, then evolve Q_A in $|+\rangle$. This is actually a *circular solution* of the problem for two reasons:

- Here we claim Q_B to be either in $|0\rangle$ or in $|1\rangle$, so sharp in Z, and for the same reason discussed in the previous example this is not enough to say it is non-classical;
- We are invoking here an already existing gate implementing a Hamiltonian capable of creating the state $|+\rangle$ in Q_A when the condition on Q_B is matched, that is exactly what we want to find in order to prove the possibility of the discussed task.

From these two examples we learn an important thing: the gate we are looking for should be built *starting from* Q_B in a way involving two non-commuting degrees of freedom, and not from Q_A because otherwise we end up with these *trivial* solutions to our problem.

4.4 The role of the conservation law

The “trivial” examples we have discussed before have in common the requirement that the qubit Q_A *cannot evolve spontaneously*: we need a *mediator*, here the qubit Q_B, which *interacts* with the qubit Q_A and induces an effective evolution to generate the state $|+\rangle$. This means that the Hamiltonian we are looking for must *conserve* a physical quantity of the composite system $Q_A \oplus Q_B$, as in this way the evolution *must* involve both the qubits. The requirement of a conservation law is perfectly in line with all the Hamiltonians of the different field theories, which do conserve a given physical quantity, and motivates us towards the research and understand of which this observable should be.

In order to give a cue on this conserved quantity for our two-qubit model, we can look back at the “trivial” example shown in Fig.4.3 and work in the Heisenberg picture: since we have a controlled Hadamard gate, we have to focus on the Hamiltonian describing the Hadamard gate: we should prevent it to be used. Let $\hat{q}^{(A)} := (\sigma_x \otimes I_B, \sigma_y \otimes I_B, \sigma_z \otimes I_B)$, where σ_α, $\alpha = x, y, z$ are the Pauli operators, be the vector of generators $q_\alpha^{(A)} = \sigma_\alpha \otimes I_\beta$ of the algebra of observables of the qubits Q_A. Let $\hat{q}^{(B)}$ be defined in a similar way [16]. The Hamiltonian for the Hadamard gate is

$$H_{Hadamard} = \frac{q_z^{(A)} + q_x^{(A)}}{\sqrt{2}} \tag{4.3}$$

and we see that requiring the conservation of $q_z^{(A)}$ will immediately forbid us to use it, as it allows the spontaneous evolution of Q_A.
It follows from this discussion that part of the conserved quantity we are looking for should involve the sum of the z-components of the generators for the qubits Q_A and Q_B, $q_z^{(A)} + q_z^{(B)}$. Anyway, this cannot be enough for our purposes: in order to let the qubits interact, we will for sure have to deal with *controlled* operations, e.g. Control-NOT gates. These kind of gates will perform computations on the qubit Q_A depending on the eigenvalue of the z-component of the descriptor $\hat{q}^{(B)}$, meaning that in the conserved quantity we should also take into account the product $q_z^{(A)} q_z^{(B)}$, so the physical observable we are looking for is:

$$q_z^{(A)} + q_z^{(B)} + q_z^{(A)} q_z^{(B)}. \tag{4.4}$$

Our goal now is to find a Hamiltonian that is capable to performing the task of changing of basis for the qubit Q_A that conserves this physical quantity.

4.5 A quantum circuit for the Witnessing Task

A quantum network performing this Witnessing Task is the one shown in Fig.4.4. First one applies a CNOT gate using Q_B as control qubit in order to let the two qubits interact; since Q_B could as well be initialised in $|0\rangle$, leaving the gate inoperative, one then applies a rotation of an angle $\theta = \frac{\pi}{2}$ around the Y-axis; then a CPH gate is applied, so that the qubits will become entangled irrespectively of

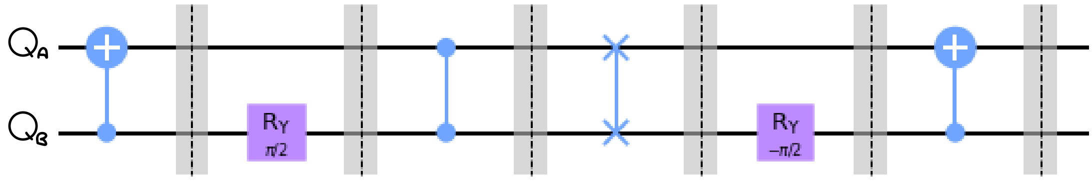

Figure 4.4: A possible symmetric quantum circuit capable of rotating the qubit Q_A from $|0\rangle$ to $|+\rangle$ conserving the observable $q_z^{(A)} + q_z^{(B)} + q_z^{(A)} q_z^{(B)}$.

the initial state of Q_B; a SWAP gate follows; finally, the same sequence of rotation ($\theta' = -\frac{\pi}{2}$) and CNOT gates is applied to the system in order to disentangle the qubits, making the circuit more symmetrical. The gates applied are represented as follows:

$$CNOT_{B,A}(t_i) = \frac{1}{2}\left(I + q_z^{(B)}(t_{i-1})\right) + \frac{1}{2}\left(I - q_z^{(B)}(t_{i-1})\right) q_x^{(A)}(t_{i-1}) \tag{4.5}$$

$$CPH_{B,A}(t_i) = \frac{1}{2}\left(I + q_z^{(B)}(t_{i-1})\right) + \frac{1}{2}\left(I - q_z^{(B)}(t_{i-1})\right) q_z^{(A)}(t_{i-1}) \tag{4.6}$$

$$R^{(B)}_{Y_{(\pm\frac{\pi}{2})}}(t_i) = \frac{\sqrt{2}}{2}\left(I \mp i q_y^{(B)}(t_{i-1})\right) \tag{4.7}$$

$$\begin{aligned} SWAP(t_i) = \frac{1}{2}\Big(I &+ q_x^{(A)}(t_{i-1}) q_x^{(B)}(t_{i-1}) \\ &+ q_y^{(A)}(t_{i-1}) q_y^{(B)}(t_{i-1}) + q_z^{(A)}(t_{i-1}) q_z^{(B)}(t_{i-1})\Big) \end{aligned} \tag{4.8}$$

and the final Hamiltonian for the circuit is:

$$H_{circ} = 2CNOT_{B,A} + R^{(B)}_{Y_{(\frac{\pi}{2})}} + R^{(B)}_{Y_{(-\frac{\pi}{2})}} + CPH_{B,A} + SWAP. \tag{4.9}$$

The evolution of the Heisenberg descriptors $\left\{q_x^{(A,B)}, q_y^{(A,B)}, q_z^{(A,B)}\right\}$ is shown in the Table 4.1.

Trying to simulate the circuit, we notice the role played by the rotation gates as anticipated above: the Witnessing Task is successfully performed even if the qubit Q_B is initialised in $|0\rangle$ or $|+\rangle$, meaning that we have a robust solution to the problem introduced in this section as it works with Q_B initialised in whatever state. This is shown in Fig.4.5.

Moreover, the proposed quantum circuit is such that the physical quantity introduced in Section 4.4 is actually conserved:

$$\left[H_{circ}, q_z^{(A)} + q_z^{(B)} + q_z^{(A)} q_z^{(B)}\right] = 0. \tag{4.10}$$

	Qubit A	Qubit B
t_0	$\left\{q_x^{(A)}, q_y^{(A)}, q_z^{(A)}\right\}$	$\left\{q_x^{(B)}, q_y^{(B)}, q_z^{(B)}\right\}$
t_1	$\left\{q_x^{(A)}, q_y^{(A)}q_z^{(B)}, q_z^{(A)}q_z^{(B)}\right\}$	$\left\{q_x^{(A)}q_x^{(B)}, q_x^{(A)}q_y^{(B)}, q_z^{(B)}\right\}$
t_2	$\left\{q_x^{(A)}, q_y^{(A)}q_z^{(B)}, q_z^{(A)}q_z^{(B)}\right\}$	$\left\{q_z^{(B)}, q_x^{(A)}q_y^{(B)}, -q_x^{(A)}q_x^{(B)}\right\}$
t_3	$\left\{-q_x^{(B)}, -q_z^{(A)}q_y^{(B)}, q_z^{(A)}q_z^{(B)}\right\}$	$\left\{q_z^{(A)}, q_y^{(A)}q_x^{(B)}, -q_x^{(A)}q_x^{(B)}\right\}$
t_4	$\left\{q_z^{(A)}, q_y^{(A)}q_x^{(B)}, -q_x^{(A)}q_x^{(B)}\right\}$	$\left\{-q_x^{(B)}, -q_z^{(A)}q_y^{(B)}, q_z^{(A)}q_z^{(B)}\right\}$
t_5	$\left\{q_z^{(A)}, q_y^{(A)}q_x^{(B)}, -q_x^{(A)}q_x^{(B)}\right\}$	$\left\{-q_z^{(A)}q_z^{(B)}, -q_z^{(A)}q_y^{(B)}, -q_x^{(B)}\right\}$
t_6	$\left\{q_z^{(A)}, -q_y^{(A)}, q_x^{(A)}\right\}$	$\left\{-q_z^{(B)}, -q_y^{(B)}, -q_x^{(B)}\right\}$

Table 4.1: Evolution of the system $Q_A \oplus Q_B$ in the Heisenberg Picture. All the components are expressed as function of the descriptors at time t_0.

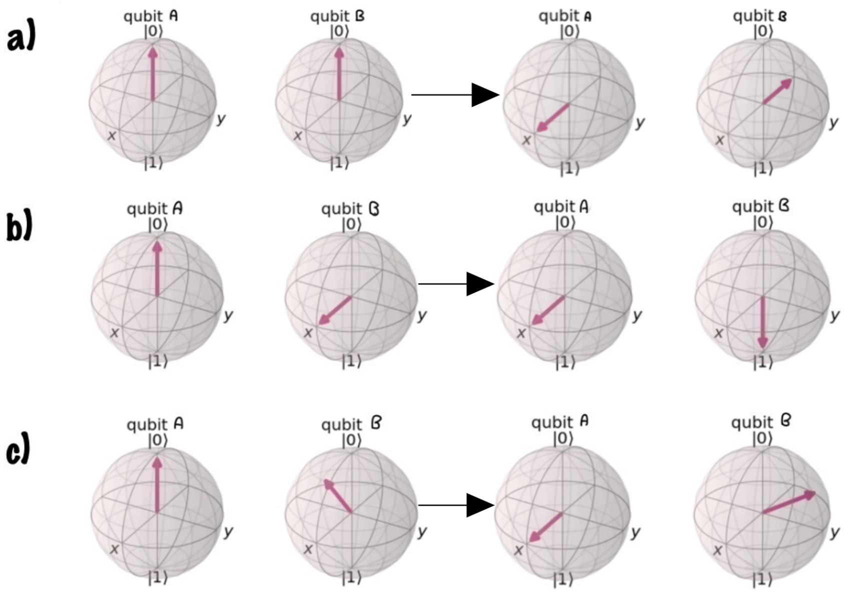

Figure 4.5: Initial and final state of the system $Q_A \oplus Q_B$ when a) Q_B is initialised in $|0\rangle$, b) Q_B is initialised in $|+\rangle$, c) Q_B is initialised in a random state, under the action of H_{circ}.

The last remark that we can do here concerns the degrees of freedom involved in the interaction between the qubits: we can see that we need at least two non-commuting degrees of freedom involved in the interaction because we make use of Controlled-operations (as CNOT and CPH) to drive the qubit A from $|0\rangle$ to $|+\rangle$.

4.6 A general proof

We have seen in the previous Section that if we assume the control system to be actually a quantum system, in our case a qubit, then it is possible to build a sequence of legal interactions capable of rotating the qubit Q_A from an eigenstate of the Z operator to an eigenstate of another variable not commuting with Z itself.

In this Section we want to show that the condition for the control system to be quantum is also *necessary*, by proving the following theorem (where, for generality purposes, we will call the two systems involved in the Witnessing Task A and B, respectively):

Theorem 1. *If the Witnessing Task is possible, then B must be non-classical itself.*

The proof will go by contradiction. Let's suppose that the qubit A is described by the descriptors $\hat{q}^{(A)} = (\sigma_x \otimes I_B, \sigma_y \otimes I_B, \sigma_z \otimes I_B) = \left(q_x^{(A)}, q_y^{(A)}, q_z^{(A)}\right)$ while the system B is a *classical* bit described by the z-component of its descriptors vector $q_z^{(B)} = I_A \otimes \sigma_z$.
The system A is, for the time being, initialised in $|0\rangle$ and we suppose that the physical quantity $q_z^{(A)} + q_z^{(B)} + q_z^{(A)} q_z^{(B)}$ is conserved, so that A cannot evolve if isolated, it can evolve only through its interaction with B. Finally, we suppose that at the end of the evolution the qubit A is found in $|+\rangle$, eigenstate of the X operator. Let's see how this leads to a contradiction working with the density operator for the system $A \oplus B$.
According to our assumptions, the initial density matrix will read:

$$\rho(t_0) \coloneqq \rho_0 = \frac{1}{2}\left(I + q_z^{(A)}\right) \otimes \frac{1}{2}\left(I + q_z^{(B)}\right) = \frac{1}{4}\left(I + q_z^{(A)} + q_z^{(B)} + q_z^{(A)} q_z^{(B)}\right) \tag{4.11}$$

and we let it evolve under the most general Hamiltonian describing the interaction between A and B:

$$H_{AB} = \alpha q_x^{(A)} + \beta q_y^{(A)} + \gamma q_z^{(A)} + a q_x^{(A)} q_z^{(B)} + b q_y^{(A)} q_z^{(B)} + c q_z^{(A)} q_z^{(B)}. \tag{4.12}$$

Before doing this, we have to enforce the conservation law required for A not to be capable of evolving spontaneously:

$$\left[H_{AB}, q_z^{(A)} + q_z^{(B)} + q_z^{(A)} q_z^{(B)}\right] = 0 \tag{4.13}$$

which implies:

$$\alpha = -a$$
$$\beta = -b$$

so that:

$$H'_{AB} = \alpha q_x^{(A)} + \beta q_y^{(A)} + \gamma q_z^{(A)} - \alpha q_x^{(A)} q_z^{(B)} - \beta q_y^{(A)} q_z^{(B)} + c q_z^{(A)} q_z^{(B)} \tag{4.14}$$

and we will label H'_{AB} as H_{AB} to lighten the notation. Now we can evolve the density operator ρ_0 under the Hamiltonian H_{AB}, but what we soon realise is that, because of the conservation law requirement, the effect will be to leave the initial state unaltered, so that:

$$\rho(t) = e^{-iH_{AB}} \rho_0 e^{iH_{AB}} = \rho_0 \tag{4.15}$$

that is not an eigenstate of X if interpreted as a single qubit state. Hence, the state of the system initialised in $|00\rangle$ and let evolved under the most general Hamiltonian describing the interactions between the qubit A and the bit B can never reach an eigenstate of the X operator for the qubit A. This is a contradiction, as we have supposed the qubit A to *have been found* in an eigenstate of X at the end of the evolution. The contradiction arises because of the assumption of B being classical, in the sense that it does not have two or more non-commuting degrees of freedom.

There is something more we can do in order to make this proof more robust: we want to relax the assumption of A being initialised in an eigenstate of the Z operator, so that it can be prepared in whatever state. In this case, the initial density matrix ρ_0 will become:

$$\rho_0 = \frac{1}{4}\Big(I_{A,B} + r_x q_x^{(A)} + r_y q_y^{(A)} + r_z q_z^{(A)} + s_z q_z^{(B)} \\ + t_x q_x^{(A)} q_z^{(B)} + t_y q_y^{(A)} q_z^{(B)} + t_z q_z^{(A)} q_z^{(B)}\Big) \tag{4.16}$$

being $\vec{r} = (r_x, r_y, r_z)$, $\vec{t} = (t_x, t_y, t_z)$ and s_z real valued coefficients; again we have to let it evolve under H_{AB} in Eq.4.14. We first of all notice that $\left[H_{AB}, q_z^{(B)}\right] = 0$, which means that:

$$q_z^{(B)}(t) = e^{-iH_{AB}} q_z^{(B)} e^{iH_{AB}} = q_z^{(B)} \tag{4.17}$$

i.e. B cannot evolve because of the interaction with A: if it is $|0\rangle$, it stays in $|0\rangle$. Hence, the effects of H_{AB} on A is to *rotate it* around the axis whose directors are:

$$\begin{cases} n_x = \alpha(1 - q_z^{(B)}) \\ n_y = \beta(1 - q_z^{(B)}) \\ n_z = \gamma + c q_z^{(B)}. \end{cases} \tag{4.18}$$

Our goal now is to find (n_x, n_y, n_z) such that it is possible to perform a rotation of the qubit Q_A mapping $q_z^{(A)} \to q_x^{(A)}$, $q_y^{(A)} \to -q_y^{(A)}$, $q_x^{(A)} \to q_z^{(A)}$ involving the bit B.

Looking at the directors in Eq.4.18, we notice that if the eigenvalue of $q_z^{(B)}$ is $+1$, then both n_x and n_y will vanish, meaning that if the bit B is initialised in $|0\rangle$, A can only rotate around the z-axis, which is not good if we want the mapping discussed above. Hence, B should be initialised in $|1\rangle$, so that:

$$\begin{cases} n_x = 2\alpha \\ n_y = 2\beta \\ n_z = \gamma - c. \end{cases} \tag{4.19}$$

Considering the unitary:

$$R_\theta = \cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)\left(n_x q_x^{(A)} + n_y q_y^{(A)} + n_z q_z^{(A)}\right) \tag{4.20}$$

with $\theta = \frac{\pi}{2}$, we can evolve component by component the descriptor for the qubit A according to Eq.4.20:

$$q_z^{(A)}(t) = R_\theta^\dagger q_z^{(A)} R_\theta \implies \begin{cases} -n_y + n_x n_z = 1 \\ n_x + n_y n_z = 0 \\ \frac{1}{2} + \frac{1}{2}\left(n_z^2 - n_x^2 - n_y^2\right) = 0 \end{cases} \tag{4.21}$$

which gives three roots out of which only one can be accepted because of the condition $n_x^2 + n_y^2 + n_z^2 = 1$: $(0, -1{,}0)$. Let's go on with the x component:

$$q_x^{(A)} = R_\theta^\dagger q_x^{(A)} R_\theta \implies \begin{cases} n_y + n_x n_z = 1 \\ -n_z + n_x n_y = 0 \\ \frac{1}{2} + \frac{1}{2}\left(n_x^2 - n_y^2 - n_z^2\right) = 0 \end{cases} \tag{4.22}$$

out of which we again extract three roots with a single one acceptable: (0,1,0), which is consistent with what found before. Finally, we evolve the y component:

$$q_y^{(A)} = R_\theta^\dagger q_y^{(A)} R_\theta \implies \begin{cases} -n_x + n_y n_z = 0 \\ n_z + n_x n_y = 0 \\ \frac{1}{2} + \frac{1}{2}\left(n_y^2 - n_x^2 - n_z^2\right) = 1 \end{cases} \tag{4.23}$$

which, instead, gives no acceptable roots: we cannot find a consistent axis around which the rotation we are looking for can be performed. The conclusion is that the task we claim is *not possible* if the system B has no two or more non-commuting degrees of freedom. Moreover, we could have a possible cue on the impossibility of this task with B considered as a bit from what said before: B initialised in $|1\rangle$ reminds us of the "trivial" case presented in Fig.4.3, where B controls the rotation of A with an already existing gate, that is actually what we want to find. We have shown now that such a gate cannot exist under the conditions required by our argument.

4.7 Connections with the entanglement-based witnesses and Bell's inequalities

It is interesting, given the theorem discussed in the previous Section, to compare it with the one in [16] and, remarkably, see how their mapping can be connected with the mapping between spatial and temporal Bell's inequalities. We restate here the two theorems we are going to compare:

Theorem 1. *If the Witnessing Task is possible, then B must be quantum itself.*

Theorem 2. *If two quantum systems become entangled through the interaction with a third system, than the third system must be quantum itself;*

Let us work with the assumptions that in both the theorems lead to the contradictions. Starting from Theorem 1:

$$\begin{cases} A' : The\ qubit\ A\ has\ evolved\ from\ |0\rangle\ to\ |+\rangle \\ B' : The\ qubit\ A\ cannot\ evolve\ spontaneously \\ C' : The\ mediator\ is\ classical \end{cases} \implies \textbf{Contradiction}$$

and continuing with Theorem 2:

$$\begin{cases} A : The\ quantum\ systems\ become\ entangled \\ B : The\ quantum\ systems\ cannot\ interact\ directly \\ C : The\ mediator\ is\ classical \end{cases} \implies \textbf{Contradiction}$$

We immediately notice the first analogy between the two arguments: the assumptions C and C', in both cases leading to the contradictions, are actually *the same*. The connection between A and A' follows naturally, as they are both related to the way in which we assess the occurrence of a contradiction: both the effects involved in these assumptions, in fact, could be verified with other experimental methods. The last connection we have is between the assumptions B and B'. This is interesting for two reasons:

- If the two quantum systems cannot interact directly, then the interactions involved in the proposal [16] are *local in space*. In our proposal we are considering a time evolution for the qubit A, in particular preventing it from a spontaneous time evolution: this could suggest the idea of a *locality in time* for the interactions between the two qubits;
- The requirement we have for the qubit A not to be capable of evolving spontaneously translates in a conservation law, involving the observable $q_z^{(A)} + q_z^{(B)} + q_z^{(A)} q_z^{(B)}$. This could mean that also the impossibility for the two quantum systems to interact directly is associated with a conserved quantity, and it might be interesting to wonder which it is.

Let's deepen them in the following Sections.

4.7.1 Connections with Bell's inequalities

The former point can be made more clear if we look at the interesting connection between the two theorems and the temporal and spatial Bell's inequalities. We know that Bell's inequalities are obtained from two hypothesis: Realism and Locality [39, 42]. The violation of the Bell's inequalities by Quantum Mechanics is the sign of a failure of the logical connection between these assumptions - in particular because of the Realism one which as a consequence rules out all those theories for QM involving hidden variables - and the emergence of *entanglement* as a purely quantum phenomenon, without a classical equivalent. Now, we see that in Theorem 2 one of the assumptions is the actual induction of *entanglement in space* between the two quantum systems and according to the connection between the proofs, we have that the assumption of the actual changing of basis for the qubit A is connected with *entanglement in time*. This means that the other two assumptions B and B', and C and C' should be connected with the two assumptions that made the discovery of entanglement possible: Realism and Locality.
The fact that the hypothesis C and C' are equal suggests that they can be mapped into the Realism hypothesis behind Bell's inequalities. This is reasonable if we think that the failure of the logical connections between the hypothesis in both Theorem 1 and Theorem 2 is due to the assumption for the mediator to be classical, in the same way as QM violates Bell's inequalities because of the Realism hypothesis in both spatial and temporal arguments.
As a consequence, B and B' must be connected with the remaining hypothesis of Locality: since B is clearly related to Locality in space, then B' has to be mapped into Locality in time, confirming what said before and shedding a light on the meaning of Locality in time: the interaction between two systems could be said *local in time* when their evolution occurs along the same temporal line, without the possibility for one or the other to evolve spontaneously and then to occupy a different temporal line.

4.7.2 A conserved quantity for the two-qubit case

The goal now is to relate the forbidding of direct interactions between the probes involved in Theorem 2 to the conservation law of a physical quantity, in the same way as the spontaneous evolution of the qubit A is prevented through a conservation law. Let us exploit the connection we have between the two systems to have an ansatz on such a quantity. We call the masses and the mediator in Theorem 2 respectively M_1, M_2 and C; from now on, we will consider them as qubits as well. From what said in the previous Sections, the subsystem $M_1 \oplus M_2$ corresponds to the qubit A of our proposal and the impossibility for M_1 and M_2 to interact directly, with the subsequent possibility of becoming entangled without the mediation of C, can be seen as *the impossibility for the subsystem $M_1 \oplus M_2$ to evolve spontaneously.* This mean that we have to enforce the conservation of a quantity that has the form:

$$q_z^{(M_1M_2)} + q_z^{(C)} + q_z^{(M_1M_2)} q_z^{(C)} \tag{4.24}$$

where $q_z^{(M_1M_2)}$ is a function of $q_z^{(M_1)}$ and $q_z^{(M_2)}$ defined on $M_1 \oplus M_2$. Who is $q_z^{(M_1M_2)}$, then?
In order to understand it, we recall the quantum circuit presented in [22] and shown in Fig.2.1:

1. Instead of using the Hadamard gate to rotate M_1, we can take advantage of the quantum circuit discussed in Section 4.5 that as we know will guarantee the conservation of the quantity $q_z^{(M_1)} + q_z^{(C)} + q_z^{(M_1)} q_z^{(C)}$;

2. Now the qubits M_1 and C are both in a superposition of $|0\rangle$ and $|1\rangle$. Since M_1 and M_2 cannot interact directly, we let C interact with the latter in order to create an entangled state;

3. We can finally apply a SWAP gate between the qubits M_1 and C in order for M_1 to become entangled with M_2 without having interacted directly with the latter. The degree of entanglement between the two of them will be the same as the one between M_2 and C.

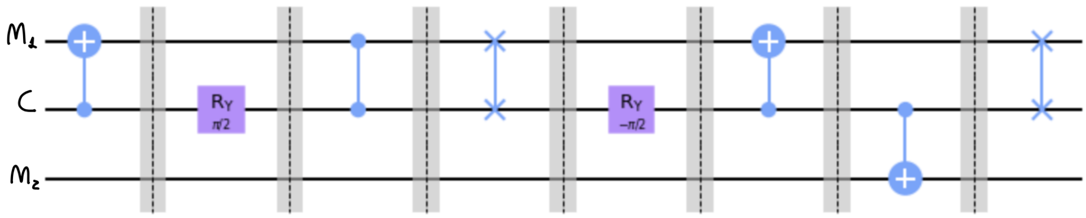

Figure 4.6: A circuit capable of inducing entanglement between M_1 and M_2 without direct interactions between themselves obtained with our original circuit, a $CNOT_{C,M_2}$ and a $SWAP_{M_1,C}$ gates.

The relevant circuit is represented in Fig.4.6 and a simulation of its working is shown in Fig.4.7 where the qubits are initialised in $|000\rangle$ for simplicity, although they can be initialised in whatever state giving eventually different degrees of entanglement.
The qubit M_2 comes into the game as the controlled qubit in the last but one operation and we can then use the results we have in Section 4.4 to ensure that

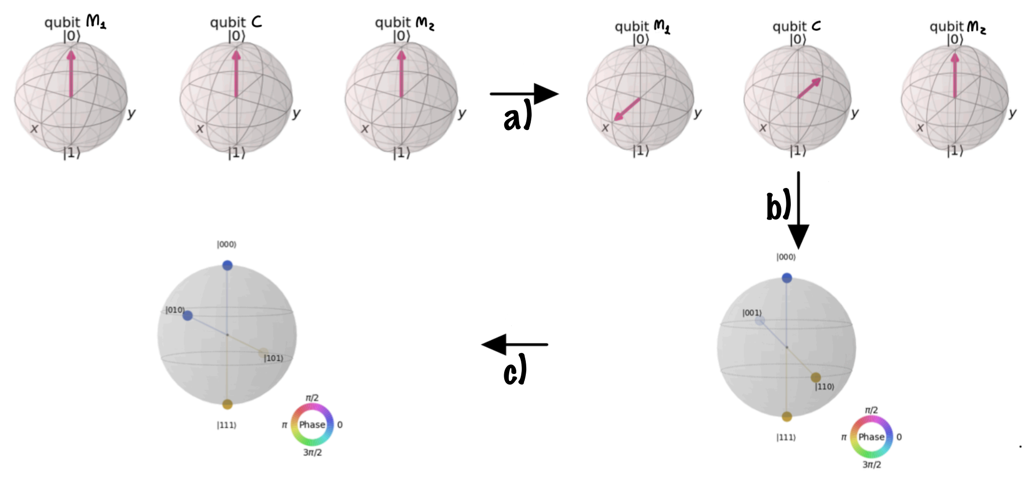

Figure 4.7: Simulation of the circuit in Fig.4.6 with the qubits initialised in $|000\rangle$: a) The circuit introduced in Section 4.5 allows the rotation of M_1 and C in $\pm|1\rangle$, b) The $CNOT_{C,M_2}$ induces entanglement between M_2 and C, c) The $SWAP_{M_1,C}$ gate transfers the entanglement state to the subsystem $M_1 \oplus M_2$.

$q_z^{(M_2)} + q_z^{(M_2)} q_z^{(C)}$ is conserved by this gate. This means that the conserved quantity we are looking for is:

$$q_z^{(M_1)} + q_z^{(M_2)} + q_z^{(C)} + q_z^{(M_1)} q_z^{(C)} + q_z^{(M_2)} q_z^{(C)} \tag{4.25}$$

so that $q_z^{(M_1 M_2)} = q_z^{(M_1)} + q_z^{(M_2)}$.

We have to check what happens to this observable when we allow M_1 and M_2 to interact, in which case we expect it to be non conserved. After having applied our circuit to the qubit M_1, we are again in the situation where both M_1 and C are in a superposition of $|0\rangle$ and $|1\rangle$, but now we can immediately apply a CNOT gate with M_1 as controller on M_2 and directly let them become entangled without the mediation of C. This is shown in Fig.4.8.
If we now try to compute $\left[CNOT_{M_1,M_2}, q_z^{(M_2)} + q_z^{(M_2)} q_z^{(C)}\right]$ we immediately see that it is not vanishing, so that the quantity in Eq.4.25 will *not* be conserved in the case M_1 and M_2 can interact directly, i.e., the subsystem $M_1 \oplus M_2$ can evolve spontaneously.

We see that the conserved quantity is the very simple generalisation of the observable introduced for the single qubit case in Section 4.4 to the case of two particles: we have just add to the latter the terms related to the second qubit that is now part of the system. This suggests the possibility for an even bigger generalisation of it to the case of N qubits: we have just to recursively iterate

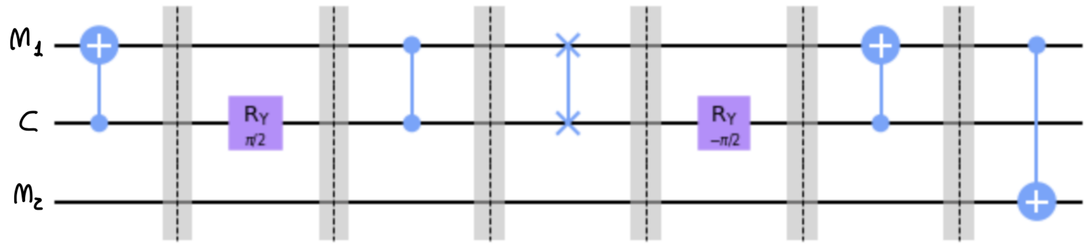

Figure 4.8: A circuit capable of inducing entanglement between M_1 and M_2 with direct interactions between themselves obtained with our original circuit ans a $CNOT_{M_1,M_2}$ gate.

the reasoning we have expressed here whenever a new qubit is add as part of the system, so that the quantity becomes:

$$\sum_{i=1}^{N} q_z^{(i)} + \sum_{j<l} q_z^{(j)} q_z^{(l)}. \tag{4.26}$$

This last observation is interesting also because the it gives another element to underline the symmetry between our original proposal and the one in [16], which can be add to the ones we have seen discussing the general theorems.

Chapter 5

Temporal witnesses of non-classicality: probing N qubits

Here the general temporal argument is discussed using a *Quantum Homogeniser* of *N qubits*. The homogenisation task is discussed with both a quantum and classical reservoir, showing that if and only if the homogeniser is *quantum* the evolution is *possible*. The study of the entanglement generation between the qubits involved in the system shed a light on how *information* flows during the task and makes a solid *connection* with the two-qubit case.

Once understood the problem working with two qubits, the next step in order to get closer to the continuous variable quantum information limit we aim is to let the quantum system Q_A interact with N *qubits*. This can be done exploiting the *Quantum Homogeniser* [43], a quantum device capable of "homogenising" the quantum state of the system with the state of the reservoir of N qubits to an arbitrary precision improving as we increase N.
Formally, by *homogenisation* we mean the task such that:

$$\forall N \geq N_\delta, D(\rho^N, \xi) \leq \delta; \tag{5.1}$$

$$\forall k, 1 \leq k \leq N, D(\xi'_k, \xi) \leq \delta, \tag{5.2}$$

where ρ^N is the state of the system after N interactions with the reservoir, ξ is the initial state of the reservoir, ξ'_k is the final state of the k-th reservoir qubit after the interaction with the system one, $D(\cdot, \cdot)$ denotes some distance in a certain metric and $\delta \geq 0$ is a (small) parameter representing the *degree of homogenisation*. This means that after N interactions the state of the reservoir qubits will change only little, while the system qubit's state becomes close to the initial state of the reservoir. Geometrically, the task is to continuously deform the Bloch sphere representing the initial state of the qubit into another sphere of radius δ centered at the point representing the state ξ on the Bloch sphere: we start with N reservoir qubits in the state ξ and the system qubit in an arbitrary state ρ and we end up with $N+1$ qubits within the sphere of radius δ introduced before, as shown in Fig.5.1 [43].

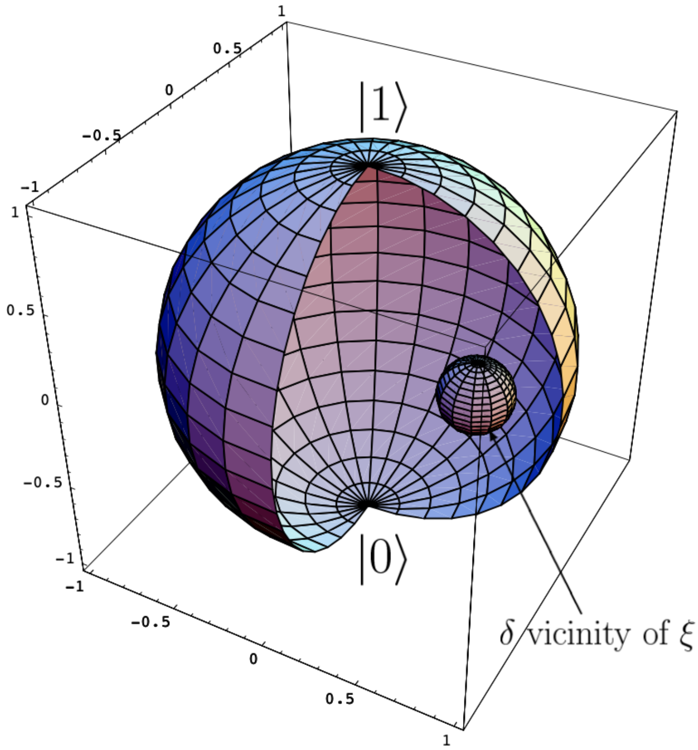

Figure 5.1: Geometrical description of the homogenisation task: in the end of the process all the $N+1$ qubits will be contained within the sphere of radius δ.

5.1 Quantum homogenisation of the system Q_A

Let us assume for simplicity that the quantum system Q_A involved in the homogenisation procedure is still a qubit that we want to evolve from $|0\rangle$ to $|+\rangle$ according to the the Witnessing Task introduced in Section 4.1. This means that the initial state for the system is $\rho_0 = \frac{1}{2}(I + q_z^{(A)})$, while we prepare the homogeniser, i.e. all the qubits in the reservoir, in the state $\xi = \frac{1}{2}(I + q_x^{(Q)})$, so that:

$$(\rho \otimes \xi^{\otimes N})_0 = \frac{1}{2}(I + q_z^{(A)}) \otimes \frac{1}{2}(I + q_x^{(Q)})^{\otimes N}. \tag{5.3}$$

The system qubit interacts with each of the qubits in the reservoir, one by one and only once, at each time step, as shown in Fig.5.2.

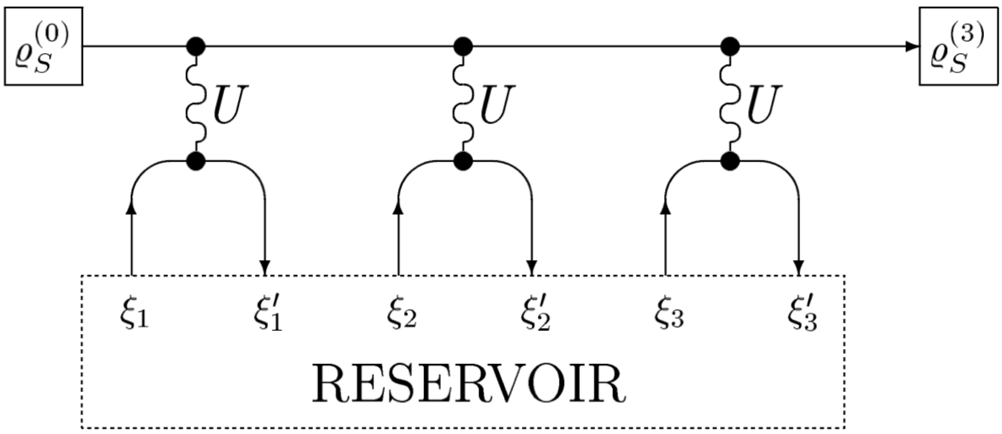

Figure 5.2: A scheme representing the homogenisation process.

The interaction is described by the unitary *partial SWAP* gate:

$$P(\eta) = \cos(\eta) + i\sin(\eta) S \tag{5.4}$$

where S is the SWAP gate in Eq.4.8 and η represents the strength of the homogenisation. It can be shown that this is the *unique* unitary operation capable of performing the homogenisation we are looking for [43]. Notice moreover that, given two qubits Q_A and Q:

$$\left[P(\eta), q_z^{(A)} + q_z^{(Q)} + q_z^{(A)} q_z^{(Q)}\right] = 0 \tag{5.5}$$

which means that the homogenisation task *conserves* the physical quantity we have introduced in Section 4.4 as well, providing a further motivation on the reason why we use a quantum homogeniser here.

After the first interaction, we get:

$$(\rho \otimes \xi)_1 = \cos^2(\eta)\rho \otimes \xi + \sin^2(\eta)\xi \otimes \rho + \frac{1}{4} i \sin(\eta)\cos(\eta)\left[q_z^{(A)} + q_x^{(Q)} + q_z^{(A)} q_x^{(Q)}, S\right] \tag{5.6}$$

and computing the commutator:

$$(\rho \otimes \xi)_1 = \cos^2(\eta)\rho\otimes\xi + \sin^2(\eta)\xi\otimes\rho + \\ + \frac{1}{4} i \sin(\eta)\cos(\eta)\left(iq_y^{(A)}q_x^{(Q)} - iq_x^{(A)}q_y^{(Q)} + iq_y^{(A)}q_z^{(Q)} - iq_z^{(A)}q_y^{(Q)}\right). \quad (5.7)$$

We can now firstly trace out the reservoir qubit to get:

$$\rho^{(1)} = \cos^2\rho + \sin^2\xi + i\cos\eta\sin\eta\,[\xi,\rho] \quad (5.8)$$

and then trace out the system qubit to have:

$$\xi_1' = \cos^2\xi + \sin^2\rho - i\cos\eta\sin\eta\,[\xi,\rho]. \quad (5.9)$$

In this way we can apply recursively the partial-swap gate following the rules of the homogenisation process to get, after n steps:

$$\rho^{(n)} = \cos^2\rho^{(n-1)} + \sin^2\xi + i\cos\eta\sin\eta\left[\xi,\rho^{(n-1)}\right]$$
$$\xi_n' = \cos^2\xi + \sin^2\rho^{(n-1)} - i\cos\eta\sin\eta\left[\xi,\rho^{(n-1)}\right]$$

which can be rewritten focusing on the terms proportional to ξ:

$$\rho^n = \sin^2\eta\sum_{k=0}^{n-1}\cos^{2k}\xi + \rho^n_{rest} = \left(1-\cos^{2n}\eta\right)\xi + \rho^n_{rest} \quad (5.10)$$

$$\xi_n' = \sin^2\eta\left(1-\cos^{2(n-1)}\eta\right)\xi + \xi_{n,rest}. \quad (5.11)$$

It can be shown (see Appendix C) that:

- As $n\to\infty$, ρ^n_{rest} converges monotonically to $\emptyset$, meaning that $\rho^{(N)}\to\xi$ and the condition in Eq.5.1 is fulfilled provided N large enough;
- Since $\left[\xi,\rho^{(n-1)}\right]\to 0$ as n increases, then $D(\xi_n',\xi)\le D(\xi_{n-1}',\xi)$, which means that ξ_n' becomes more similar to ξ. Notice moreover that what just written means that the condition in Eq.5.2 is satisfied $\forall k$ if and only if it is satisfied for $k=1$, i.e. if the unitary implementing the homogenisation is a *contractive map*.

This means that the homogenisation process is as much accurate as we increase the number of qubits N in the reservoir, so that the number of interactions n will increase as well. In particular, given a certain $\delta>0$ and provided $\sin\delta = \sqrt{\frac{\delta}{2}}$, we require

$$N \ge N_\delta = \frac{\ln\frac{\delta}{2}}{\ln\left(1-\frac{\delta}{2}\right)} \quad (5.12)$$

in order to achieve the homogenization with a required *fidelity* [43]. Notice moreover that both N and η are determined by δ.

5.2 Is it possible to use a classical homogeniser for the Witnessing Task?

We have seen that a reservoir made of *quantum systems* is capable of performing the homogenisation task described in the previous Section, but it is interesting, as we have done in the two-qubit case, to wonder if such a condition is also *necessary* for the task to be possible. Here, we will show that it is so.

Let us suppose that the reservoir is made of *bits*, so that it is initialised in the state:

$$\xi = |0\rangle\langle 0|^{\otimes N} = \frac{1}{2}\left(I + q_z^{(Q)}\right)^{\otimes N}, \tag{5.13}$$

since the computational basis is assumed to be $\{|0\rangle, |1\rangle\}$. This means that the system qubit Q_A will be initialised in:

$$\rho = |+\rangle\langle +| = \frac{1}{2}\left(I + q_x^{(A)}\right) \tag{5.14}$$

and the task we are looking for is an homogenisation of ρ to ξ, i.e. a rotation for the qubit Q_A from the state $|+\rangle$ to the state $|0\rangle$.
We assume, as in the quantum homogeniser, that the system qubit is allowed to interact with one bit per time step and only once. The most general unitary describing the interactions will be:

$$U(\eta) = \cos(\eta) + i\sin(\eta)H_{AQ} \tag{5.15}$$

being H_{AQ} the same introduced in Eq.4.12, with B replaced by Q. As said above, the partial SWAP does conserve the physical quantity $q_z^{(A)} + q_z^{(Q)} + q_z^{(A)}q_z^{(Q)}$; this means that we should enforce again the conservation of such quantity in our Hamiltonian. Recalling the calculations performed in Section 4.6, we get:

$$H_{AQ} = \alpha q_x^{(A)} + \beta q_y^{(A)} + \gamma q_z^{(A)} - \alpha q_x^{(A)}q_z^{(Q)} - \beta q_y^{(A)}q_z^{(Q)} + c q_z^{(A)}q_z^{(Q)}. \tag{5.16}$$

Now we are ready to let the system qubit interact with the first *bit* in the reservoir:

$$\begin{aligned}(\rho\otimes\xi)^1 &= U^\dagger(\eta)\,(\rho\otimes\xi)\,U(\eta)\\ &= \frac{1}{4}\cos^2(\eta)\left(I + q_x^{(A)} + q_z^{(Q)} + q_x^{(A)}q_z^{(Q)}\right)\\ &+ \frac{1}{4}\sin^2(\eta)\left(I + H_{AQ}^\dagger q_x^{(A)}H_{AQ} + q_z^{(B)} + H_{AQ}^\dagger q_x^{(A)}H_{AQ}q_z^{(Q)}\right)\\ &+ i\sin(\eta)\cos(\eta)\left[q_x^{(A)}, H_{AQ}\right] + i\sin(\eta)\cos(\eta)\left[q_x^{(A)}, H_{AQ}\right]q_z^{(Q)}.\end{aligned} \tag{5.17}$$

Let us focus on the term proportional to $\sin^2\eta$: as we know from Eq.5.6, it is necessary for it to be proportional to $\xi \otimes \rho$ in order for the fixed point of the contractive map to be reached at the end of the homogenisation procedure. Of course, we cannot expect the bit to rotate in an eigenstate of the X operator, so what we are actually looking for is a proportionality with $\xi \otimes \xi$. We have:

$$I + H_{AQ}^{\dagger} q_x^{(A)} H_{AQ} + q_z^{(Q)} + H_{AQ}^{\dagger} q_x^{(A)} H_{AQ} q_z^{(Q)} \tag{5.18}$$

that should match:

$$I + q_z^{(A)} + q_z^{(Q)} + q_z^{(A)} q_z^{(Q)}, \tag{5.19}$$

but in order for this to be possible, H_{AQ} should be capable of rotating the qubit Q_A around a given axis such that $|+\rangle \rightarrow |0\rangle$ and we have shown in Section 4.6 that it is *not* possible to define consistently such a rotation axis.
This means that we will *never* have a term like $\sin^2(\eta)\xi \otimes \xi$ in Eq.5.17, independently on the other terms that may appear in it, so that it is *not possible* to perform the Witnessing Task we are looking for using a *classical* reservoir.

5.3 A more in depth analysis of the entanglement between the qubits

Once understood that the reservoir in our system *must* be quantum, as well as the "control" system in the two-qubit case, in order for the Witnessing Task to be possible, it is interesting to look at the entanglement between the qubits involved in this $N+1$ qubits system in order to have a better insight on the connections between the two cases discussed in this Section and Section 4.
From now on, we will label $c = \cos\eta$, $s = \sin\eta$.

Let us define the bi-partite concurrence [44, 45] between two qubits in the state ρ_{jk} as:

$$C_{jk} = C(\rho_{jk}) := max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \tag{5.20}$$

being λ_is the eingenvalues of the matrix $R = \rho_{ij}(\sigma_y \otimes \sigma_y)\rho_{ij}^*(\sigma_y \otimes \sigma_y)$[1] listed in decreasing orders. Our actual entanglement measure will be the *square* of the bi-partite concurrence, known as *tangle*:

$$\tau_{ij} = [C_{ij}]^2 . \tag{5.21}$$

[1]Here σ_y is the usual Pauli matrix written in the computational basis.

5.3.1 Entanglement between the system qubit and one qubit of the reservoir

Let us start discussing the entanglement induction between the system qubit Q_A and the k-th qubit in the reservoir Q_k.
First of all, we're going to derive the general bi-partite concurrence between the two qubits [43], and then we will set it in our particular case.

The reservoir is initialised in the state $|\xi\rangle$, while the system qubit in the state $|\rho_A^{(0)}\rangle$ described by the density matrix $\rho_A^{(0)}$. Following the homogenisation scheme, the qubit Q_A will interact with Q_k at the k-th time step[2], so after $k-1$ steps its state is $\rho_A^{(k-1)}$, which we can express in the basis $\left\{|\xi\rangle, |\xi^{\perp}\rangle\right\}$ as:

$$\rho_A^{(k-1)} = a_{k-1}\,|\xi\rangle\langle\xi| + (1-a_{k-1})\,|\xi^{\perp}\rangle\langle\xi^{\perp}| \\ + b_{k-1}\,|\xi\rangle\langle\xi^{\perp}| + b_{k-1}^{*}\,|\xi^{\perp}\rangle\langle\xi|\,. \quad (5.22)$$

Now we can apply the k-th partial swap operation between Q_A and Q_k: $\rho_{Ak}^{(k)} = P^{\dagger}(\eta)\rho_{Ak-1}P(\eta)$. This will give the bi-partite density matrix:

$$\rho_{Ak}^{(k)} = \begin{pmatrix} a_{k-1} & cb_{k-1} & isbk-1 & 0 \\ b_{k-1}^{*} & (1-a_{k-1})c^2 & isc(1-a_{k-1}) & 0 \\ -isb_{k-1}^{*} & -isc(1-a_{k-1}) & s^2(1-a_{k-1}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5.23)$$

Building the matrix R as explained above and extracting its eigenvalues, we find that the only non vanishing one is $4c^2s^2(1-a_{k-1})^2$. This means that the concurrence is:

$$C_{Ak}^{(k)} = 2sc^{2k-1}(1-a_0) \quad (5.24)$$

being a_0 the coefficient through which we decompose $\rho_A^{(0)}$ in the basis $\left\{\{|\xi\rangle, |\xi^{\perp}\rangle\right\}$.

We can now compute it, then, in our specific case, where $|\xi\rangle = |+\rangle$ and $|\rho_A^{(0)}\rangle = |0\rangle$:

$$a_0 \coloneqq \langle +|\,0\,|+\rangle = \frac{1}{2} \quad (5.25)$$

so that:

$$C_{Ak}^{(k)} = 2sc^{2k-1}\left(1-\frac{1}{2}\right) = sc^{2k-1}. \quad (5.26)$$

[2]This means that *before* the interaction $n = k$, the bi-partite concurrence $C_{Ak}^{(n)} = 0$.

We conclude from this expression that the system qubit *is entangled* with the qubit Q_k of the reservoir after the k-th interaction.

The result we have got can be easily generalised looking at the entanglement between Q_A and Q_k at the n-th interaction, with $n = 1, ..., N$:

$$C_{Ak}^{(n)} = \begin{cases} 0, & \text{if } n < k \leq N \\ sc^{n+k-1}, & \text{if } k \leq n \leq N \end{cases} \tag{5.27}$$

from which it follows that the entanglement between the system qubit and any of the reservoir qubits is induced only after their interaction and, interestingly, it tends to zero as the number of interactions with the reservoir n increases; this means that when $n \to \infty$ the system qubit will completely get disentangled from the reservoir's one, exactly as we have seen in the two-qubit example discussed in Section 4.

5.3.2 Entanglement between two reservoir qubits

We want now to repeat the same analysis focusing our attention to two qubits of the reservoir, say Q_j and Q_k, setting $j < k$ without loss of generality.
In this case it is not so easy to assess what happens in the case $j < n < k$, since the reservoir qubits are *not* allowed to *interact directly*, but only locally with the system qubit. We focus then to the case $j < n < k$ first.

Since $n > j$, the system qubit has interacted with the qubit Q_j in the reservoir, but as $n < k$ it has not interacted with Q_k yet, so that:

$$\rho_{jk}^{(n)} = \xi_j' \otimes |0\rangle\langle 0| \tag{5.28}$$

and if we build the matrix R in order to extract the needed eigenvalues for the bi-partite concurrence, we find out that they are all vanishing. This means that *before* the qubit Q_A has interacted with *both* Q_j and Q_k, we cannot find entanglement between the two reservoir qubits.

Let us investigate what happens, instead, when $j < k \leq n$, i.e. when the system qubit has interacted with Q_k as well. In this case:

$$\begin{aligned} \rho_{jk}^{(n)} &= \xi_j' \otimes \xi_k' \\ &= \left\{ s^2 \rho_A^{(j-1)} + c^2 \xi + ics \left[\rho_A^{(j-1)}, \xi \right] \right\} \otimes \left\{ \rho_A^{(k-1)} + c^2 \xi + ics \left[\rho_A^{(k-1)}, \xi \right] \right\} \end{aligned} \tag{5.29}$$

from which we get, building the $R_{jk}^{(n)}$ matrix, extracting the eigenvalues and recalling that $a_0 = \frac{1}{2}$:

$$eig\left(R_{jk}^{(n)}\right) = \left\{ s^4 c^{2(j+k-2)}, 0, 0, 0 \right\} \tag{5.30}$$

that is:

$$C_{jk}^{(n)} = \begin{cases} 0, & \text{if } n < k \leq N \\ s^2 c^{j+k-2}, & \text{if } k \leq n \leq N. \end{cases} \tag{5.31}$$

We can conclude that the two reservoir qubits Q_j and Q_k gets entangled only after having *both* interacted with the reservoir qubits: the system qubit Q_A behaves as a *mediator* of entanglement between the reservoir qubits, which provides an evidence of the entanglement-based witness of non-classicality behind the BMV effect[3] discussed in [16, 17]. Moreover, notice that their entanglement degree will stay constant independently of the number of interactions n the qubit Q_A will perform, but later the interaction between Q_j and Q_k with Q_A, the smaller the degree of their mutual entanglement.

5.3.3 Entanglement between the system qubit and the whole reservoir

It is interesting now to wonder what happens when we consider the entanglement between the system qubit Q_A and the *whole* reservoir, as in this way we can start to understand how the information flows within the interacting systems.
Referring to [46], we can define a measure of entanglement between the single qubit and the rest of the system through the determinant of its density matrix:

$$\tau_A^{(n)} = 4 \det \rho_A^{(n)}. \tag{5.32}$$

In our case, we have:

$$\tau_A^{(n)} = 4 \det \left\{ \left(1 - c^{(2n)}\right) |+\rangle \langle +| + c^{(2n)} |0\rangle \langle 0| \right\} \tag{5.33}$$

from which it follows:

$$\tau_A^{(n)} = 2c^{(2n)} \left(1 - c^{(2n)}\right). \tag{5.34}$$

We can check here, coherently with what seen for the entanglement between the system qubit and one reservoir qubit in Section 5.3.1, that $\tau_A^{(n)}$ is monotonically decreasing with the number of interactions n and in the end converges to zero as $n \to \infty$. This is an even more relevant evidence of the procedure of entanglement and disentanglement that the system requires with its controller in order for the task we are looking for to be performed, and it is interesting that this occurs *exactly* when $n \to \infty$, which reflects the continuous variable limit we will investigate in the subsequent Sections.

[3]Recall here that the reservoir qubits cannot interact directly with each other

5.3.4 Entanglement between one reservoir qubit and the rest of the system

As the information stored between the system qubit and the whole reservoir diminishes as n increases, it is interesting to ask the same question of the previous Section can be asked regarding one reservoir qubit and the rest of the system, in order to understand where the information shared in the interactions is *actually* stored.

Let us deal with the j-th qubit of the reservoir and compute the tangle between itself and the rest of the system:

$$\tau_j^{(n)} = 4 \det \rho_j^{(n)}. \tag{5.35}$$

For what we have discussed in Section 5.3.2, the entanglement between two reservoir qubits is induced only once they *both* have interacted with the system qubit, so that if we are considering the entanglement of Q_j with the rest of the system we have firstly to wait for it to interact with Q_A:

$$\tau_j^{(n)} = \begin{cases} 0, & \text{if } n < j \leq N \\ 4 \det \xi_j', & \text{if } j \leq n \leq N. \end{cases} \tag{5.36}$$

In our case, $\xi_j' = \left(1 - s^2 c^{2(j-1)}\right) |+\rangle \langle +| + s^2 c^{2(j-1)} |0\rangle \langle 0|$, so that:

$$\tau_j^{(n)} = \begin{cases} 0, & \text{if } n < j \leq N \\ 2 s^2 c^{2(j-1)} \left(1 - s^2 c^{2(j-1)}\right), & \text{if } j \leq n \leq N \end{cases} \tag{5.37}$$

meaning that the reservoir qubit Q_j gets entangled with the rest of the system once having interacted with Q_A, but then the induced entanglement degrees *stays constant* with the number of interaction n. This is a cue about the possibility that the information on the initial state of the qubit Q_A gets actually stored in the entanglement between all the reservoir qubits it has interacted with. We will come back in more details on this sentence in a while.

5.3.5 Relationship between bi-partite concurrences and entanglement with the rest of the system

Once introduced separately the entanglement degrees of two qubits and of a qubit with the rest of the system, it is interesting to connect the two of them in order to understand to what extent we can connect the induction of the entanglement between one of the qubits and the whole system with the contributions given by the bi-partite entanglement between them. This problem goes into the more general

one of the *multi-particle entanglement* [47, 48].
In this field, a recent study by Coffman et al. [46] have conjectured[4] that, preparing N qubits in a pure state, the sum of the bi-partite entanglement degree between one qubit and all the others is lower than or equal to the entanglement between itself and the rest of the system:

$$S_j(n) \coloneqq \sum_{k=1,k\neq j}^{N} \left[C_{jk}^{(n)}\right] \leq \left[C_{j,\bar{j}}^{(n)}\right]^2 = \tau_j^{(n)} \tag{5.38}$$

where $\bar{j}$ denotes all the element in the whole system but j. This is known as Coffman-Kundu-Wootters (CKW) inequality.

We can check, following the lines in [43] applied to our case, that the homogenisation process *saturates* the CKW inequalities:

- Starting with the entanglement of the system qubit with the reservoir, we can directly compare Eq.5.34, representing the right-hand side of Eq.5.38 with the sum of the bi-partite concurrences in Eq.5.27 up to the n-th interaction to have:

$$S_A^{(n)} = \sum_{k=1}^{n} \left[C_{Ak}^{(n)}\right]^2 = 2c^{(2n)}\left(1 - c^{(2n)}\right) = \tau_A^{(n)} \tag{5.39}$$

  as claimed.

- Moving now to the reservoir qubits, we consider the j-th one and we distinguish between two cases, referring to Eq.5.31 and Eq.5.37:

  - If $j < n$, then the CKW inequality is obviously saturated as we get zero both at the left and right-hand side;
  - If $j \geq n$, we have:

$$\begin{aligned} S_j^{(n)} &= \left[C_{Aj}^{(n)}\right]^2 + \sum_{k=1}^{j-1} \left[C_{jk}^{(n)}\right]^2 + \sum_{k=j+1}^{n} \left[C_{jk}^{(n)}\right]^2 \\ &= 2s^2 c^{2(n+j-1)} + 2s^4 c^{2(j-2)} \left(\sum_{k=1}^{j-1} c^{2k} + \sum_{k=j+1}^{n} c^{2k}\right) \\ &= 2s^2 c^{2(n+j-1)} + 2s^4 c^{2(j-2)} \left(\sum_{k=1}^{n} c^{2k} - 1 - c^{2j}\right) \\ &= 2s^2 c^{2(j-1)} \left(1 - s^2 c^{2(j-1)}\right) = \tau_j^{(n)} \end{aligned} \tag{5.40}$$

[4] And proved it for N=3.

So all in all:

$$S_j^{(n)} = \tau_j^{(n)} = \begin{cases} 0, & \text{if } n \leq N \\ 2s^2 c^{2(j-1)} \left(1 - s^2 c^{2(j-1)}\right), & \text{if } j \leq n \leq N \end{cases} \tag{5.41}$$

as claimed.

From this we can read that the tangle is linear in the case of homogenisation process: the entanglement between a qubit and the rest of the system, independently on which qubit it is, is given by the sum of the bi-partite entanglement degrees between itself and each other single qubit in the system.

We can finally notice that:

$$\lim_{N \to \infty} \frac{S_A(N)}{S_k(N)} = 1 \quad \forall k = 1...N \tag{5.42}$$

which means that:

- $S_0(N)$ and $S_k(N)$ decrease at the same speed when N diverges. This can be understood as a consequence of the saturation of CKW inequality we have discussed above;
- The entanglement of Q_A with the reservoir is the same as the entanglement of an arbitrary reservoir qubit with the rest of the homogenised system. This means that not only states of the individual qubits are the same, but also the amount of entanglement between each of the qubits and the rest of the system are equal. This gives an interesting cue on the *symmetry* of the problem we are facing: the role of mediator can be assumed by every qubit involved in the interaction in a perfectly symmetrical way, reflecting the symmetric behaviour of gravitational interaction. Hence, it is legitimate to expect not only that the mediator effects the two qubits inducing the entanglement between them, but also that the two qubits effect the mediator itself in a symmetrical way.

5.4 An information theoretical connection with the two-qubit case

We have introduced in this Section the $N + 1$ qubits Witnessing Task as the natural step forward with respect to the two-qubit case discussed in Section 4, but is legitimate to ask to what extent the equivalence between the control qubit Q_B and the quantum homogeniser can be pushed. In this section, we will show that they are perfectly equivalent in the sense that the amount of information the qubit Q_A exchanges with the control qubit Q_B in the two-qubit case is *the same* of the amount of information the system qubit Q_A exchanges with the quantum homogeniser.

Let us start with the two-qubit case: the amount of information provided by Q_A on Q_B[5] is described by the *mutual information* $I(A:B)$:

$$I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \tag{5.43}$$

being $S(\rho) = -\mathrm{Tr}\rho \log \rho$ the Von Neumann entropy.
We want to compute it at the end of the interaction, following the steps explicited in Table.4.1:

$$\rho_{AB} = \frac{1}{4}\left(I - q_z^{(A)} q_z^{(B)} + q_y^{(A)} q_y^{(B)} - q_x^{(A)} q_x^{(B)}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{5.44}$$

and from this we get easily, after a proper diagonalisation of ρ_{AB} :

$$S(\rho_{AB}) = -\sum_i \lambda_i \log \lambda_i = 0. \tag{5.45}$$

being λ_i the eigenvalues of the density matrix. Moving now to ρ_A, we can trace out the qubit Q_B from ρ_{AB} to get the partial density matrix we are looking for:

$$\rho_A = \mathrm{Tr}_B \rho_{AB} = \frac{1}{2}\left(I + q_z^{(A)}\right) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{5.46}$$

which gives instead:

$$S(\rho_A) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = 1. \tag{5.47}$$

Of course the same argument can be repeated for the qubit Q_B, tracing out the system Q_A. Since ρ_{AB} is symmetric in Q_A and Q_B, we can immediately write:

$$S(\rho_B) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = 1. \tag{5.48}$$

All in all, the information provided by the system Q_A on the system Q_B is:

$$I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) = 2. \tag{5.49}$$

[5]And obviously by Q_B on Q_A, by symmetry.

Let us now move to the quantum homogeniser. This is a more a tricky argument, because one may think that since the bi-partite concurrences between the system qubit Q_A and the reservoir qubits go to zero as N increases, the information about the initial state of Q_A gets lost in the end. What we show instead is that such an information is "hidden" in the mutual correlations between the qubits of the homogenised system and that the amount of information they store is exactly the same exchanged by Q_A and Q_B in the two-qubit case. This can be seen summing up all the bi-partite concurrences between all qubits:

$$\begin{aligned}\lim_{N\to\infty}\sum_{j<k}^{N}\left[C_{jk}^{N}\right]^2 &= \lim_{N\to\infty}\frac{1}{2}\sum_{j=0}^{N}S_j(N)\\ &= c^{(2n)}\left(1-c^{(2n)}\right)+\sum_{j=1}^{(n)}s^2c^{2(j-1)}\left(1-s^2c^{2(j-1)}\right)=2\end{aligned}\tag{5.50}$$

which shows again the 2 we have found for the mutual information $I(A:B)$. This means that the amount of information exchanged between the two qubits Q_A and Q_B and between the system qubit Q_A and the homogeniser is the same, as claimed above, which gives a stronger correspondence between the controllers that the two cases involve.

There is one final remark that is worth noting here: such an information theoretical correspondence, as well as other results before, is exact in the limit of infinitely many qubits in the reservoir. Such a limit can be discussed in the following way in the framework of the quantum homogeniser [43]: we first let the system qubit A interact with a quantum homogeniser made of N qubits, with N kept finite, establishing in this way all the needed concurrences according to the bounding introduced in Section 5.1, and then we let N go to infinity, paying attention to get the best possible homogenisation according to the aforementioned bounds.
This $N\to\infty$ behaviour gives an interesting cue on the extensibility of these results to the case of a control system characterised by *continuous degrees of freedom*. This is an interesting point worth studying, as we know that our goal in the end is to deal with the continuous variable limit because we recall that the non-Gaussianity based witness of non-classicality introduced in Section 3 applied to gravity - that we claim to be *equivalent* to the BMV effect via our original argument - is based on CVQIS.

Chapter 6

Temporal witnesses of non-classicality: probing a harmonic oscillator

Here the general temporal argument is discussed in the framework of continuous variable with an *harmonic oscillator*. Using the Holstein-Primakoff transformation, we identify in the creation, annihilation, and number operators the *non-classical degrees of freedom* involved in the evolution. An *experimental implementation* of the temporal witness is proposed. The general temporal argument is then rephrased in the framework of *second quantisation*, discussing the so-called *single particle entanglement* and *identifying* the two probes in the spatial argument with the *quantum modes* of the single system in our temporal argument

We are now ready to face the continuous variable limit for the control system and the best way to deal with it is using an *harmonic oscillator* interacting with our usual qubit system Q_A. The way in which we quantise the fields within Quantum Field Theory is through harmonic oscillators, so that discussing the Witnessing Task with one of them would mean discussing it referring to one of the modes of a given field, in our case the gravitational one.

6.1 Derivation of the Hamiltonian

The task we want to complete is always the Witnessing Task: we want to allow the qubit to change basis only through the interaction with another system, the latter being *non classical.* Thus, we can build the quantised model for this framework starting from the Hamiltonian in Eq.4.9, introduced in the two-qubit scenario, and applying the *Holstein-Primakoff transformation* [49].
The Holstein-Primakoff transformation is a mapping between spin operators $\hat{S} = (S_x, S_y, S_z)$ such that $[S_\alpha, S_\beta] = i\hbar\epsilon_{\alpha\beta\gamma}S_\gamma$, and creation (annihilation) operator $a^\dagger$ (a) that can be expressed as follows:

$$\begin{cases} S_z = \hbar\left(s - a^\dagger a\right) \\ S^+ = \hbar\sqrt{2s}\sqrt{1 - \frac{a^\dagger a}{2s}}a \\ S^- = \hbar\sqrt{2s}a^\dagger\sqrt{1 - \frac{a^\dagger a}{2s}} \end{cases} \tag{6.1}$$

being s the particle's spin and $S^\pm = S_x \pm iS_y$ the raising and lowering operators. We can describe the qubit as a spin $s = \frac{1}{2}$ particle so that, reintroducing the usual descriptors $\hat{q} = (q_x, q_y, q_z)$ as in Section 4, we get ($\hbar = 1$ from now on):

$$\begin{cases} q_x = \frac{\sqrt{1-a^\dagger a}a + a^\dagger\sqrt{1-a^\dagger a}}{2} \\ q_y = \frac{\sqrt{1-a^\dagger a}a - a^\dagger\sqrt{1-a^\dagger a}}{2i} \\ q_z = \frac{1}{2} - a^\dagger a. \end{cases} \tag{6.2}$$

This is the transformation we want to apply to Eq.4.9 in order to obtain the microscopic model describing the evolution of a spin-$\frac{1}{2}$ particle interacting with an harmonic oscillator. Let us see how it goes.

Recalling that:

$$H_{circ} = 2CNOT_{B,A} + R^{(B)}_{Y_{(\frac{\pi}{2})}} + R^{(B)}_{Y_{(-\frac{\pi}{2})}} + CPH_{B,A} + SWAP, \tag{6.3}$$

we can apply the transformation to each gate in order to build the one for the full Hamiltonian.
First of all, we associate the creation and annihilation operators $a^\dagger, a$, $b^\dagger, b$ to the qubits Q_A and Q_B respectively, so that:

$$\begin{aligned} CNOT_{B,A} &= \frac{1}{2}\left(I + q_z^{(B)}\right) + \frac{1}{2}\left(I - q_z^{(B)}\right)q_x^{(A)} \\ &\overset{\text{H-P}}{=} \frac{1}{2}\left[I + \frac{1}{2} - b^\dagger b + \frac{\sqrt{1-a^\dagger a}a + a^\dagger\sqrt{1-a^\dagger a}}{2}\right. \\ &\left. -\frac{\sqrt{1-a^\dagger a}a + a^\dagger\sqrt{1-a^\dagger a}}{2}\left(\frac{1}{2} - b^\dagger b\right)\right] \end{aligned} \tag{6.4}$$

$$\begin{aligned}CPH_{B,A} &= \frac{1}{2}\left(I+q_z^{(B)}\right)+\frac{1}{2}\left(I-q_z^{(B)}\right)q_z^{(A)} \\ &\overset{\text{H-P}}{=}\frac{1}{2}\left[I+\frac{1}{2}-b^\dagger b+\frac{1}{2}-a^\dagger a-\left(\frac{1}{2}-b^\dagger b\right)\left(\frac{1}{2}-a^\dagger a\right)\right] \end{aligned} \tag{6.5}$$

$$\begin{aligned}SWAP &= \frac{1}{2}\left(I+q_x^{(A)}q_x^{(B)}+q_y^{(A)}q_y^{(B)}+q_z^{(A)}q_z^{(B)}\right) \\ &\overset{\text{H-P}}{=}\frac{1}{2}\left[I+\frac{\sqrt{1-a^\dagger a}a+a^\dagger\sqrt{1-a^\dagger a}}{2}\frac{\sqrt{1-b^\dagger b}b+b^\dagger\sqrt{1-b^\dagger b}}{2}\right. \\ &+\frac{\sqrt{1-a^\dagger a}a-a^\dagger\sqrt{1-a^\dagger a}}{2i}\frac{\sqrt{1-b^\dagger b}b-b^\dagger\sqrt{1-b^\dagger b}}{2i} \\ &\left.+\left(\frac{1}{2}-a^\dagger a\right)\left(\frac{1}{2}-b^\dagger b\right)\right] \end{aligned} \tag{6.6}$$

$$R^{(B)}_{Y_{\left(\pm\frac{\pi}{2}\right)}} = \frac{\sqrt{2}}{2}\left(I\mp iq_y^{(B)}\right)\overset{\text{H-P}}{=}\frac{\sqrt{2}}{2}\left[I\mp i\frac{\sqrt{1-b^\dagger b}b-b^\dagger\sqrt{1-b^\dagger b}}{2i}\right]. \tag{6.7}$$

Now we can build the full Hamiltonian:

$$\begin{aligned}H = &\left[I+\frac{1}{2}-b^\dagger b+\frac{\sqrt{1-a^\dagger a}a+a^\dagger\sqrt{1-a^\dagger a}}{2}\right. \\ &\left.-\frac{\sqrt{1-a^\dagger a}a+a^\dagger\sqrt{1-a^\dagger a}}{2}\left(\frac{1}{2}-b^\dagger b\right)\right] \\ &+\frac{\sqrt{2}}{2}\left[I-i\frac{\sqrt{1-b^\dagger b}b-b^\dagger\sqrt{1-b^\dagger b}}{2i}\right]+\frac{\sqrt{2}}{2}\left[I+i\frac{\sqrt{1-b^\dagger b}b-b^\dagger\sqrt{1-b^\dagger b}}{2i}\right] \\ &+\frac{1}{2}\left[I+\frac{1}{2}-b^\dagger b+\frac{1}{2}-a^\dagger a-\left(\frac{1}{2}-b^\dagger b\right)\left(\frac{1}{2}-a^\dagger a\right)\right] \\ &+\frac{1}{2}\left[I+\frac{\sqrt{1-a^\dagger a}a+a^\dagger\sqrt{1-a^\dagger a}}{2}\frac{\sqrt{1-b^\dagger b}b+b^\dagger\sqrt{1-b^\dagger b}}{2}\right. \\ &\left.+\frac{\sqrt{1-a^\dagger a}a-a^\dagger\sqrt{1-a^\dagger a}}{2i}\frac{\sqrt{1-b^\dagger b}b-b^\dagger\sqrt{1-b^\dagger b}}{2i}+\left(\frac{1}{2}-a^\dagger a\right)\left(\frac{1}{2}-b^\dagger b\right)\right]\end{aligned}$$

which in the end reads:

$$\begin{aligned}H = &\frac{3}{2}(I-b^\dagger b)+\frac{1}{2}(I-a^\dagger a)+\frac{\sqrt{1-a^\dagger a}a+a^\dagger\sqrt{1-a^\dagger a}}{2}\left(\frac{1}{2}+b^\dagger b\right) \\ &+\frac{1}{4}\left[b^\dagger\sqrt{1-b^\dagger b}\sqrt{1-a^\dagger a}a+a^\dagger\sqrt{1-a^\dagger a}\sqrt{1-b^\dagger b}b\right]\end{aligned} \tag{6.8}$$

where we have neglected the constant terms.

It can be easily checked in Eq.6.8 that we have free terms of the form $\hbar\omega_a a^\dagger a$ and $\hbar\omega_b b^\dagger b$ describing the free evolution of our subsystems and three interaction terms suggesting again the behaviour of the harmonic oscillator as a controller for the spin-$\frac{1}{2}$ particle.
Moreover, as we can assess looking at the Holstein-Primakoff transformation in Eq.6.2, here the non commuting degrees of freedom of the mediator are the number operator $b^\dagger b$ and the creation and annihilation operators $b^\dagger$ and b. All the three are involved in the interaction that makes the Witnessing Task possible, which means that the mediator must be non classical also in the framework of continuous variable Quantum Information Science.

6.2 A possible experimental implementation of the Witnessing Task

It could be interesting to understand how the controlling of the harmonic oscillator on the qubit is expressed. This not only will better qualify the interaction terms we have in Eq.6.8, but also could give us an answer concerning the necessary quantum nature of the harmonic oscillator itself: it is not so trivial, in fact, that what we have proved for the discrete degrees of freedom framework is extended to the continuous one. We will check that even in this scenario the quantum nature of the controlling system, i.e. the harmonic oscillator, is necessary and sufficient for the Witnessing Task to be performed.

Consider an harmonic oscillator whose creation and annihilation operators are $a^\dagger$ and a and let it interact with a qubit that we call A, whose descriptors are $\hat{q}^{(A)} = \left(q_x^{(A)}, q_y^{(A)}, q_z^{(A)}\right)$.
We prepare the qubit in one of the eigenstates of its Z component, $|0\rangle$ in our case, and we allow it to interact with the harmonic oscillator prepared in a *coherent state* $|\alpha\rangle_F$, so that:

$$|\psi(t_0)\rangle = |0\rangle |\alpha\rangle_F . \tag{6.9}$$

The qubit is a *non linear* medium the harmonic oscillator is in contact with. This means that waiting for an amount of time $t = \frac{\pi}{2\xi_r}$, being ξ_r the anharmonicity of the resonator, we expect a Yurke-Stoler coherent state to be created [50]. This phenomenon can be described by an evolution operator:

$$U_1(t) = e^{-i\xi_r \left(a^\dagger a\right)^2 t} \tag{6.10}$$

so that:

$$|\psi(t_1)\rangle = U_1(t)|\psi(t_0)\rangle = |0\rangle \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \left(|\alpha\rangle_F + e^{i\frac{\pi}{2}} |-\alpha\rangle_F\right). \tag{6.11}$$

The idea now is to take advantage of the superposition of coherent state we have created to induce a superposition in the qubit state as well. This can be done through a proper control of A by the harmonic oscillator.
First of all, we don't want the two coherent states to have the same modulus, so we displace the cat state of an amount α:

$$|\psi(t_2)\rangle = D(\alpha)\psi(t_1) = e^{\alpha a^\dagger - \alpha^* a}\psi(t_1) = |0\rangle \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \left(|2\alpha\rangle_F + e^{i\frac{\pi}{2}} |0\rangle_F \right). \quad (6.12)$$

Now we can create the superposition for the qubit building a CNOT gate where the qubit is controlled by the harmonic oscillator:

- In the branch where the harmonic oscillator is in the coherent state $|2\alpha\rangle$, we flip the qubit in the state $|1\rangle$;
- In the branch where the harmonic oscillator is in the vacuum state, we do nothing on the qubit.

This can be formalised through the following operator:

$$CNOT_{2\alpha,A} = e^{-q_x^{(A)} a^\dagger a} \quad (6.13)$$

that acting on $|\psi(t_2)\rangle$ gives:

$$|\psi(t_3)\rangle = CNOT_{2\alpha,A} |\psi(t_2)\rangle = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \left(e^{-4|\alpha|^2} |1\rangle |2\alpha\rangle_F + e^{i\frac{\pi}{2}} |0\rangle |0\rangle_F \right). \quad (6.14)$$

Notice that the superposition has been created for the qubit as well, as we claimed. Our goal now is to disentangle the two subsystems, which we expect to be possible following the conclusions we have derived in Section 5.3.3. In order to do this, we can now use the qubit A as a controller on the harmonic oscillator:

- Displace the harmonic oscillator in the state $|2\alpha\rangle_F$ when the qubit finds in $|1\rangle$;
- Leave the harmonic oscillator unchanged when the qubit is in $|0\rangle$.

Inspired by the map introduced in [51], we can formalise this operation with the following evolution operator:

$$CD_{A,2\alpha} = D\left(-\alpha e^{i\xi_{qr} T_{wait}}\right) e^{i\xi_{qr} T_{wait} a^\dagger a \frac{1}{2}\left(I - q_z^{(A)}\right)} D(\alpha) \quad (6.15)$$

where $D(\cdot)$ is the displacement operator we have introduced in Eq.6.12 and ξ_{qr} is the qubit-resonator coupling. Its action can be described in three steps:

1. An unconditional displacement of an amount α is applied to both the branches of the superposition;

2. Waiting for an amount of time $T_{wait} = \frac{\pi}{\xi_{qr}}$, we flip the coherent state in the branch where the qubit is in the state $|1\rangle$, leaving the one in the other branch unchanged;

3. We apply again an unconditional displacement of an amount $-\alpha e^{i\xi_{qr}T_{wait}} = +\alpha$

and we can check that is the qubit to control which of the coherent states in the cat state has to evolve and which has to be left unchanged.
Applying it to our system, we get:

$$|\psi(t_4)\rangle = CD_{A,2\alpha}|\psi(t_4)\rangle = \frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\left(e^{-4|\alpha|^2}|1\rangle|2\alpha\rangle_F + e^{i\frac{\pi}{2}}|0\rangle|2\alpha\rangle_F\right) \tag{6.16}$$

which can be written also as:

$$|\psi(t_4)\rangle = \frac{1}{\sqrt{2}}e^{i\frac{3\pi}{4}}\left(e^{-4|\alpha|^2}e^{-i\frac{\pi}{2}}|1\rangle + |0\rangle\right)|2\alpha\rangle_F. \tag{6.17}$$

We see that the qubit and the harmonic oscillator are now *exactly disentangled* as we expected commenting on the homogenisation task in Section 5.
There's something interesting we can see here: if $|\alpha|^2 \to \infty$, i.e. if we consider a *classical* harmonic oscillator, the superposition for the qubit *cannot* be obtained via this Hamiltonian, meaning that the Witnessing Task would be *impossible* if the mediator is a *classical system.* This is perfectly in line with what shown in the previous Sections.
If, on the other hand, we consider the *microscopic limit* $|\alpha|^2 \to 0$, we find:

$$|\psi(t_5)\rangle = \frac{1}{\sqrt{2}}e^{i\frac{3\pi}{4}}(-i|1\rangle + |0\rangle)|2\alpha\rangle_F = e^{i\frac{3\pi}{4}}|-i\rangle|2\alpha\rangle_F \tag{6.18}$$

meaning that the Witnessing Task has actually being *successfully performed* considering a *non classical* mediator: the qubit is now found in an eigenstate of its Y component, which does not commute with the Z component. We can generalise this saying that, at the end of the Witnessing Task, the qubit A is found in an eigenstate of its Y component with a precision dependent on $|\alpha|^2$.

As a final remark, we can build a correspondence with what we have seen in this Section and the interaction terms in Eq.6.8:

- The summand
$$\frac{\sqrt{1-a^\dagger a}a + a^\dagger\sqrt{1-a^\dagger a}}{2}\left(\frac{1}{2} + b^\dagger b\right)$$
describes the CNOT in Eq.6.13. This is as well supported by the fact that it originates from a CNOT in the source Hamiltonian Eq.4.9 after the Holstein-Primakoff transformation;

- The second and third summands,

$$\frac{1}{4}\left[b^\dagger\sqrt{1-b^\dagger b}\sqrt{1-a^\dagger a}a + a^\dagger\sqrt{1-a^\dagger a}\sqrt{1-b^\dagger b}b\right],$$

  describes instead the controlled displacement in Eq.6.15.

This concludes our final generalisation to continuous variable limit for the control system: whichever the dimensionality of the control system's Hilbert space, if it is capable of inducing a rotation on the qubit, then it *must* be quantum.

6.3 A generalisation with mode entanglement

In Section 4.7 we have addressed a first interesting connection between the spatial form of our argument [16] and the temporal form of it, comparing the two theorems and focusing on the role of the conserved quantity as the equivalent of the local interaction of the masses M_1 and M_2 with the mediator C. Here we want to make this connection even deeper introducing the idea firstly proposed in [52] on the possibility for a single particle to show entanglement in its delocalised state.

One of the most important points to be understood when talking about entanglement is that it is not an *absolute* property of quantum states, but a property of a quantum state *relative* to a given set of subsystems. An example to understand this important point can be the following. Suppose to have two harmonic oscillators that are allowed to interact with each other. If we transform to normal modes, then the eigenstates of this system will be the tensor product of the eigenstates of the normal modes, meaning that the system is not entangled. If, instead, we consider the description of the original oscillators modes, we see that:

$$\psi(x_1, x_2; t) = \mathcal{N}(t)exp\left[-a_1(t)x_1^2 - a_2(t)x_2^2 + a_{12}(t)x_1x_2\right] \tag{6.19}$$

being x_1 and x_2 the coordinates of the two oscillators. As we see, as soon as $a_{12}(t) \neq 0$, we have entanglement *between the two subsystems*.
This is the heart point of the discussion: entanglement occurs between subsystems, so that every discussion concerning it must clearly state which subsystems are being considered. Thus, we should work in terms of *fields*, of which particles are only a manifestation, in order to provide a satisfactory argumentation about entanglement. Let's see what are the consequences of this discussion on our work.

Consider the final state of our system qubit A after the interaction with the mediator that now we call C: as explained in Section 4, it is an eigenstate of the X component of the qubit itself, namely

$$|\psi\rangle_F = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right), \tag{6.20}$$

which describes a superposition of a particle being in the ground and excited state. Addressing it in terms of fields, i.e. in occupation number notation, we would get:

$$|\psi\rangle_F = \frac{1}{\sqrt{2}} \left(|10\rangle + |01\rangle \right) \tag{6.21}$$

being the first label relative to the ground state and the second to the excited one. The state in Eq.6.21 is nothing but a Bell state, so a non-factorizable one showing some entanglement between two subsystems. Which are the two of them? They are the two *modes* of the field.
This means that the mediator C has been capable of mediating entanglement between the two modes of the field it is interacting with, because of the conservation law introduced in Section 4.4. By the entanglement-based witness, then, we can assess the non classicality of the mediator C.
Here lies the symmetry between the spatial and temporal forms of the argument: the two subsystems in [16], M_1 and M_2, are nothing but the two modes of our system A, while C is the mediator of the entanglement between both of them.

We have considered in the previous discussion the case in which the qubit rotates up to the eigenstate of X, i.e. the two modes of it get *maximally entangled.* Anyway, as we have understood in Section 4.6, the non-classicality claiming for the control system is achieved as well just considering a small rotation out of z-axis in the Bloch sphere for the qubit A. What does this mean in the framework we have here introduced?
This means that we will still get an entanglement between the two modes, but with a *different degree* depending on the asymmetry between Z and the observable the obtained state is an eigenstate of. In general, we will get:

$$|\psi\rangle_F = \alpha \, |10\rangle + \beta \, |01\rangle \tag{6.22}$$

where α and $\beta \in \left.\right) 0, \frac{1}{\sqrt{2}} \left.\right]$ depends on the *asymmetry degree* between the aforementioned final observable and Z. The result perfectly matches the possibility we have to explore all the possible entanglement degrees between M_1 and M_2 in [16], which makes the symmetry between spatial and temporal forms of the argument stronger. In particular, α^2 and β^2 are the temporal form substitutes to the probabilities in Eq.2.6 we get at the output of the Mach-Zender interferometer in the spatial argument.

In this discussion, again, the conserved quantity plays a crucial role: without ensuring its conservation, we are not able to implement the gate entangling the two modes, as well as with direct interactions between the two subsystems in [16] we are not able to assess the non-classical nature of the mediator of the entanglement.

This is another cue on the the conservation law can be used as a substitute to the requirement of local interactions, as addressed in Section 4.4. Moreover, this convinces us more about the fact that a straightforward generalisation of the conserved law introduced in this text, involving the three systems required by the entanglement-based witness, can be defined and expected as well in the spatial version of our argument.

Chapter 7

Conclusions

In summary, inspired by the necessity to have a quantum information science based tool to witness non-classical features in gravity not relying on the small gravitational coupling constant, which has always characterized gravity as the hardest fundamental force to test at a scale lower than Planck mass m_P, we have derived a new temporal proposal working with two physical systems and the evolution of one of them controlled by the other, under the conservation law of a global quantity of the total system.

First of all we presented the Bose-Marletto-Vedral effect, based on the entanglement between two masses, each in a superposition of two locations, interacting only locally with a gravitational field and in no way directly with each other. The possibility of this task can prove, within the framework of Constructor Theory of Information, the *non-classical* features of the gravitational field without any assumption on specific dynamics nor scale as shown in Section 2.1 and in Appendix B. An experimental setup as the one in Fig.2.2 can be devised in order to provide a witness of the spatial argument and test the BMV effect (i.e., prove the possibility of the task T_E in Eq.B.2): first, each mass becomes entangled with the field interacting with one of its observables; then, because of the interaction with the *other* observable of the field, the phases ϕ_i in Eq.2.4 are induced in each of the configurations of the masses built by the two Mach-Zender interferometers; finally, emerging from the interferometer, each mass becomes disentangled from the field, being now entangled with the other at a different degree depending on the employed masses m and the parameter $\alpha = \frac{ct}{d_i}$.
This argument is so strong from a theoretical point of view that it behaves as a sort of *Bell's theorem* for Quantum Gravity, leaving *local linearised quantum gravity models* the only known one capable of describing the interactions between gravitational field and matter. On the other side, the experimental scheme leaves

two important open problems: we should first of all ensure that no direct interactions between the masses are present, as otherwise the Locality Principle would be violated invalidating our argument, and then we should be able to remove all other interactions different from gravity, without hindering the latter.

In order to overcome these two problems, another connection between QG and QIS has been found in non-Gaussianity induction in the quantum state of matter because of the interaction with a gravitational field: as summarised in Fig.3.1, only a quantised version of the chosen gravity-matter interaction model would be capable of explaining such a generation. According to this argument, an experimental setup using a single mass in a single location has been devised in which, measuring the fourth-order cumulant κ_4, it is possible to determine whether the non-Gaussianity has been induced in the sequeezed initial state of matter, witnessing the quantised nature of the field.
Since it is based on a single mass, this experiment will solve the problem of the direct interactions between the two masses involved in the BMV effect-based experiment; moreover, working with a BEC in a spherical harmonic potential, it is capable as well of removing all the interactions but gravity between the atoms: magnetic dipole-dipole interactions will be removed because of the spherical symmetry of the system, while the electromagnetic interactions can be turned off with the application of a proper magnetic field and exploiting Feshbach resonances. On the other hand, this proposal tests only a specific model of matter-gravity coupling, comparing its classical and quantum version according to the theoretical argument related to the generation of non-Gaussianity in the quantum state of the system; but most importantly, the fact that the experiment works with a single mass is not enough to ensure that no other theories involving a purely classical gravitational field will be capable of explaining those effects we claim to be witness of quantum gravity.

Motivated by the correspondence between the relative phase ϕ_i involved in the BMV effect-based test and the signal-to-noise ratio in Eq.3.11 according to which we determine κ_4, we have been looking for a general argument qualifying as a *temporal* version of the entanglement-based witness supporting the BMV effect.
We started addressing the problem with two qubits in Section 4: once established that it is *necessary* to have a qubit as mediator in order for the Witnessing Task to be performed, we have underlined the importance of a *conserved quantity* in this temporal witness of non-classicality to prevent the spontaneous evolution of the system Q and we have connected it to the role that the requirement of local interactions between the probes and the mediator plays in the spatial argument. This means that forbidding direct interactions between the two probes of the spatial argument, i.e. assuming *locality in space*, is equivalent to forbidding spontaneous evolution of the single mass in the temporal argument, which is then connected by

symmetry to what we may call *locality in time*.
Such a symmetry extends to the actual witnesses of non-classicalities: in the spatial version, we have the induction of entanglement *in space* between the two masses; in the temporal argument this is replaced by the time evolution of the single system Q, which can then be interpreted as an instance of entanglement *in time*. This argument is made even stronger by the discussion of our original proposal in the framework of second quantisation provided in Section 6.3: the temporal task can be seen as the induction of the entanglement between the quantum modes of the physical system Q.
The last framework has also the merit of confirming the overlapping between the spatial and temporal arguments, as the quantum modes of the single system in the temporal version can be seen as the two probes involved in the spatial version. The equivalence is also confirmed by the fact that the requirement for local interactions between the probes and the mediator in the spatial argument can be cast in terms of a conservation law of a quantity that has the same mathematical form of the one introduced in our two-qubit discussion.
The result has a deeper implication: the temporal argument is more robust to the external noise than the spatial version because even if a third system should be involved in the evolution of the qubit Q, we could still keep track of it in the conservation law and, if already established its quantum nature, we can still apply the argument on the actual mediator M.

We then extended our temporal argument to the case where the mediator M is made of N qubits: working with the *homogenisation* process, we have been able to show that only a *quantum* homogeniser is capable of performing the Witnessing Task we are looking for under the constraint of the conservation law, confirming again the validity of our argument.
Interestingly, we have deepened the way in which the information is spread among the $N+1$ qubits the system is made of, noticing in particular that the entanglement between the system qubit A and one of the reservoir qubits Q_k is established once the two of them interact and decreases as the process of homogenisation goes on; instead, the entanglement between two qubits of the reservoir Q_j and Q_k is created once the system qubit interacts with both of them *singularly*, which means that A behaves as the mediator of the entanglement induction between Q_k and Q_j: it is an evidence of the spatial argument!
Moreover, the entanglement induced between the reservoir qubits stays constant throughout the homogenisation process, meaning that the information exchanged between A and the quantum homogeniser is stored there. Notably, the information exchanged between A and B in the two qubits example is the same as the one exchanged between A and the homogeniser.

Finally, we discussed our argument in the case of continuous variable limit for the control system with an harmonic oscillator. Again, we showed how its being quantum is a necessary condition for the Witnessing Task to be possible and we have derived the Hamiltonian describing the interactions with a spin-$\frac{1}{2}$ particle through the Holstein-Primakoff transformation. A possible experimental implementation of the Witnessing Task was then proposed, matching the operations with the terms in our Hamiltonian.

We can then express a definitive version of our new general temporal argument as follows:

Theorem 3. *If a system M can induce the dynamical evolution of another system Q from a state where one of its observables is sharp to another state where a different, non-commuting observable is sharp, then M must be non-classical, provided that a global observable of the system $Q \oplus M$ is conserved.*

The generality of the theorem stated here relies on the absence of any assumption on *specific model Hamiltonian* to couple the system with the mediator, meaning that once established the time evolution of the system Q under the conservation law of the global observable of the system $Q \oplus M$ we are able to rule out all possible *classical* model to describe the mediator M. This represents a step forward compared to the non-Gaussianity based witness we introduced in Section 3.

Three main perspectives follows this result:

- We could devise an experimentally feasible protocol based on the temporal argument to witness non-classicality in gravity. This because we can infer the non-classicality of the mediator M just assuming the full quantum control on the quantum system Q (despite the fact that the mediator M could be measured in its own classical basis) and the conservation law;
- Since the mediator can be *whatever* physical system, we could apply the temporal argument to Quantum Biology: using a living system (e.g., a bacterium) to control the evolution of the qubit, investigating the compatibility of Quantum Mechanics with life, an open question since the early days of Quantum Physics [53];
- In order to reach the same level of generality of the entanglement-based witness, we could provide a further generalisation of the argument which does not assume the validity of Quantum Mechanics, working in the framework of Constructor Theory of Information (see Appendix A): this will allow us to consider other possible model different from quantum theory to describe the mediator M, still being non-classical.

We leave the exploration of these interesting avenues to future research.

Appendix A

A brief introduction to Constructor Theory of Information

Constructor theory of information [54] supplements the traditional physics viewpoint based on trajectories, dynamical laws and initial conditions at a given scale with statements on what *tasks* are *possible* or *impossible* on a given *substrate*, i.e. the physical system on which transformations specified by the task can be performed. These tasks are specified in terms of ordered pairs of input/output *attributes*, the latter being the sets of all states where the substrate has a given property. A set of disjoint attributes of a substrate will then define a *variable*, which should be carefully distinguished by an *observable*, as explained below. Whenever a substrate is in a state with attribute $\mathbf{v} \in V$, being V a variable, we say that V is *sharp* with value v. A task is *impossible* if the laws of physics impose a limit to how accurately it can be performed. Otherwise we say it to be *possible* and we can build an arbitrarily good approximation to a substrate that never fails in delivering any of the input attributes of the task to the correct output attributes and, most importantly, is capable of performing again the task: the *constructor*.

The cardinal principle is the **principle of locality**:

Principle 1 (Principle of Locality). *The state of a substrate is a description of it that satisfies two properties:*

- *Any attribute of a substrate is a fixed function of the substrate's state at any given time t;*
- *Any state of a composite substrate* $\mathbf{S_1} \oplus \mathbf{S_2}$ *is an ordered pair* (s_1, s_2) *of* $\mathbf{S_1}$

and $\mathbf{S_2}$*, with the property that if a task is performed on* $\mathbf{S_1}$ *only, then the state of the substrate* $\mathbf{S_2}$ *is not changed thereby.*

The Locality expressed in this principle is satisfied by all known theories, included non-relativistic Quantum Mechanics [55] and Quantum Field Theory [56] which are of interest for our goals, making it extremely general.

Here we will characterize the mediator **M** as an *information medium*, i.e. a substrate with a set of attributes $\mathcal{X}$, called *information variable*, with the property that the task of cloning the attributes of the first replica of the substrate onto the second substrate

$$\bigcup_{x\in X}\{(\mathbf{x},\mathbf{x_0})\to(\mathbf{x},\mathbf{x})\} \tag{A.1}$$

and the logically reversible computation

$$\bigcup_{x\in X}\{\mathbf{x}\to\Pi(\mathbf{x})\} \tag{A.2}$$

for a given Π, are possible. In this way, we characterize the information medium as a substrate that can be used for *classical* information processing: the fact that the cloning task in Eq. A.1 is possible is guaranteed by the **principle of interoperability**:

Principle 2 (Principle of Interoperability)**.** *If* $\mathbf{S_1}$ *and* $\mathbf{S_2}$ *are information media, respectively with information variable* $\mathcal{X}_1$ *and* $\mathcal{X}_2$*, their composite system* $\mathbf{S_1}\oplus\mathbf{S_2}$ *is an information medium with information variable* $\mathcal{X}_1\times\mathcal{X}_2$*, where* $\times$ *denotes the Cartesian product of the set.*

This principle requires the possibility of performing computations on one of the information media without simultaneously affecting the other, as otherwise the logically reversible computation in Eq.A.2 will not be a possible task for any Π anymore. This is guaranteed by the principle of locality, which will then be required in order for the interoperability principle to hold.

In this way we have been able to provide a purely constructor-theoretic notion of classical information, and the same can be done with concepts such as measuring and distinguishing.
We say that a variable $\mathcal{X}$ of a substrate is distinguishable if the task

$$\bigcup_{x\in X}\{(\mathbf{x})\to(\mathbf{q_x})\} \tag{A.3}$$

is possible, being $\mathbf{q_x}$ an information variable. In particular, if we can store the result of the distinction process in a second replica of the substrate in a way such that the original substrate is not affected by this

$$\bigcup_{x\in X}\{(\mathbf{x},\mathbf{x_0})\rightarrow(\mathbf{x},\mathbf{p_x})\} \tag{A.4}$$

we obtain the *perfect measurement* task, which generalises in purely constructor-theoretic terms the idea of measurements. This task is always possible for any information media, according to the interoperability principle.
Coming back to the *distinguishability* task Eq.A.3, we can extent it to attributes as well saying that if a variable $\{\mathbf{x_0},\mathbf{x_1}\}$ is distinguishable, than the attribute $\mathbf{x_0}$ is distinguishable from the attribute $\mathbf{x_1}$. This allows us to define, for any attribute $\mathbf{x}$, the attribute $\bar{\mathbf{x}}$ as the union of all attributes that are distinguishable from $\mathbf{x}$; notice that this is the way in which we generalise the orthogonal complement of a vector space. Extending this definition to the variable $\mathcal{X}=\bigcup_{x\in\mathcal{X}}\mathbf{x}$, we get $\bar{\mathcal{X}}=\overline{\bigcup_{x\in\mathcal{X}}\mathbf{x}}$[1]; if $\bar{\mathcal{X}}$ is empty, than $\mathcal{X}$ is a *maximal* variable.
We can build in the same way $\bar{\bar{\mathbf{x}}}$ which leads us to the definition of *observable* as an information variable whose attributes have the property that $\bar{\bar{\mathbf{x}}}=\mathbf{x}$. The idea of *observable* we have in Constructor Theory of Information is that of a variable, say $\mathcal{Z}$, with the property that whenever a measurer of $\mathcal{Z}$ produces a sharp output z the input substrate really has that attribute z and we can show that what we have written above is a necessary and sufficient condition for a variable to be an observable.

Interestingly, the notion of observable generalises in constructor-theoretic terms that of a quantum observable. We are now ready to include in our discussion quantum systems as well. This is done through the notion of *superinformation medium*.
A superinformation medium is an information medium with at least two information observables, say $\mathcal{X}$ and $\mathcal{Z}$, such that their union is not an information observable. This means that a measurer of one of these observables must perturb the substrate where the other is sharp, so we call them *incompatible* as in quantum theory. Qubits, which we are going to employ in our argument, are examples of superinformation media: X and Z can be thought as two non-commuting observables that cannot be copied by the same cloner.
We will not require, anyway, **M** to be a superinformation medium, but a **non-classical** one: an information medium, with maximal information observable $\mathcal{T}$, that has a variable $\mathcal{V}$, disjoint from $\mathcal{T}$ and with the same cardinality, with the following properties:

[1]This is possible thanks to a constructor-theoretic principle which relies on the existence of regularity among observable phenomena in a substrate, since they are related by a unifying explanation: if every pair of attributes in a variable $\mathcal{X}$ is distinguishable, than so is $\mathcal{X}$.

- There exist a superinformation medium $\mathbf{S_1}$ and a distinguishable variable $\mathcal{E} = \{\mathbf{e_j}\}$ of the joint substrate $\mathbf{S_1} \oplus \mathbf{M}$, whose attributes $\mathbf{e_j} = \{(s_j, v_j)\}$ are sets of ordered pairs of states, where v_j is a state belonging to some attribute in $\mathcal{V}$ and s_j is a state of $\mathbf{S_1}$;
- $\mathcal{V} \cup \mathcal{T}$ is not a distinguishable variable;
- The task of distinguishing the variable $\mathcal{E}$ is possible by measuring incompatible observables of a *composite superinformation medium* including $\mathbf{S_1}$, but impossible by measuring observables of $\mathbf{S_1}$ only.

We see that non-classicality is weaker than superinformation since the medium may not have the full information power as a quantum system, as underlined by the fact that the variable $\mathcal{V}$ may or may not be an information observable, but at the same its existence requires $\mathcal{M}$ to enable non-classical tasks on superinformation medium, such as establishing entanglement.

Appendix B

Rephrasing the entanglement-based witnesses of non-classicality via Constructor Theory

Here we discuss the theoretical argument behind the BMV effect in the more general framework of Constructor Theory of Information [54] (see Appendix A). The argument is more general and it is does not assume the formalism of quantum theory. It is based on the principle of interoperability and the principle of locality *only* (see Appendix A for a formal definition), defining generalisation of concepts as *non-classicality*, defined in Section 2.1, and *observable* that are compatible with quantum theory and general relativity, but not assuming either of those. We will assume that:

- The mediator $\mathbf{M}$ is an information medium with a maximal observable T;
- The probes $\mathbf{Q_A}$ and $\mathbf{Q_B}$ are superinformation media having at least two disjoint maximal information observables whose union is not an information observable. For the purpose of this work, we will consider them to be qubits - which are specific superinformation media - and the maximal information observables to be their X and Z components.
- For simplicity, all the information observables are binary: $T = \{\mathbf{t_0}, \mathbf{t_1}\}$, $Z = \{\mathbf{z_0}, \mathbf{z_1}\}$ and $X = \{\mathbf{x_+}, \mathbf{x_-}\}$.
- Following the main hypothesis of this argument, we assume that by coupling M locally with each of the qubits via the same interaction, it is possible to

prepare them in one of two orthogonal maximally entangled states.

The interoperability principle guarantees that the task of copying any of the observables of one of the qubits, Q_α with $\alpha \in \{1,2\}$, onto the observable T of the mediator $\mathbf{M}$ via some interaction is possible, so the task of measuring the observable X of one of the qubits, using the mediator $\mathbf{M}$ as the target, must be possible:

$$T_M = \{(\mathbf{z_0}, \mathbf{t_0}) \to (\mathbf{z_0}, \mathbf{t_0}), (\mathbf{z_1}, \mathbf{t_0}) \to (\mathbf{z_1}, \mathbf{t_1})\} \quad \text{(B.1)}$$

How can we interpret this? In the *limit of weak field* [16, 17] $\mathbf{z_0}$ and $\mathbf{z_1}$ can be seen as two distinct locations of a mass, while $\mathbf{t_0}$ and $\mathbf{t_1}$ as two distinguishable configurations of the gravitational field induced by the two mass locations. It is interesting as well to think of $\mathbf{t_0}$ and $\mathbf{t_1}$ as two distinguishable spacetime geometries, solutions of Einstein's equations for the two different mass distributions, as prescribed in general relativity, [25]. Notice that a measurer of Z is capable of distinguishing X as well because the attributes $\mathbf{z} \in Z$ generalise quantum superposition of the eigenstates of X [20].

Suppose now that $\mathbf{Q_A}$ and $\mathbf{Q_B}$ are successfully entangled: this means that the task

$$T_E = \{(\mathbf{x}_+, \mathbf{t_0}, \mathbf{x}_+) \to \mathbf{e}_{++}, (\mathbf{x}_-, \mathbf{t_0}, \mathbf{x}_+) \to \mathbf{e}_{-+}\} \quad \text{(B.2)}$$

must be also possible, with $B = \{\mathbf{e}_{++}, \mathbf{e}_{-+}\}$ an information variable of $\mathbf{Q_A} \oplus \mathbf{M} \oplus \mathbf{Q_B}$ whose attributes correspond to the aforementioned two orthogonal, maximally entangled states of the qubits. If we assume that the constructor for the task T_E is the same as the one for the task T_M (in our case it is the gravitational interaction between a mass and the field, initially prepared in $\mathbf{t_0}$) and that T_E is performed without direct interactions between $\mathbf{Q_A}$ and $\mathbf{Q_B}$, then the Principle of Locality will require it to be performed in (at least) two steps[1], separately involving one qubit and the mediator:

- First step, performed on $\mathbf{Q_A} \oplus \mathbf{M}$:

$$T_1 = \{(\mathbf{x}_+, \mathbf{t_0}, \mathbf{x}_+) \to (\mathbf{s}_{+\mathbf{0}}, \mathbf{x}_+), (\mathbf{x}_-, \mathbf{t_0}, \mathbf{x}_+) \to (\mathbf{s}_{-\mathbf{0}}, \mathbf{x}_+)\} \quad \text{(B.3)}$$

  Since this task is possible according to the possibility of T_M, we see that the substrate $\mathbf{Q_A} \oplus \mathbf{M}$ has a variable $\mathcal{E} = \{\mathbf{s}_{+\mathbf{0}}, \mathbf{s}_{-\mathbf{0}}\}$, matching the first condition for $\mathbf{M}$ to be a non-classical mediator. Of course our discussion will proceed commenting on the properties of this variable, in order to check the other conditions for the non-classicality as discussed in Appendix A;

[1] We will assume that entanglement is obtained just with the following two steps, as it is straightforward to generalise to the case where more repetitions are required.

- Second step, performed on $\mathbf{M} \oplus \mathbf{Q_B}$:

  $$T_2 = \{(\mathbf{s_{+0}}, \mathbf{x}_+) \to \mathbf{e}_{++}, (\mathbf{s_{-0}}, \mathbf{x}_+) \to \mathbf{e}_{-+}\} \tag{B.4}$$

  Notice that the attributes in B can be distinguished by measuring the observables of $\mathbf{Q_A}$ and $\mathbf{Q_B}$ only. Moreover, the task T_2 allows us to build a bijective mapping between $\mathcal{E}$ and two *distinguishable* attributes of the qubits, $\mathbf{e}_{\alpha\beta}$, which makes $\mathcal{E}$ a *distinguishable variable.*

So now we want to discuss the properties of this variable $\mathcal{E}$, starting with the observation that, by the Principle of Locality, $\mathcal{E} = \{\mathbf{s_{+0}}, \mathbf{s_{-0}}\} = \{\mathbf{s_{\alpha 0}}\}$ (being $\alpha \in \{+, -\}$ now) has attributes which are fixed function of $(\hat{q}_A^{\alpha 0}, m^{\alpha 0})$, being $\hat{q}_A^{\alpha 0}$ a vector of q-numbers representing the three components of the qubit $\mathbf{Q_A}$ and $m^{\alpha 0}$ some state describing $\mathbf{M}$. We would like then to establish the properties of this state, introducing the set $V = \{\{m^{\alpha 0}\}\}$.

- First of all, we should prove that V and T have the same cardinality, i.e. $V = m^{+0}, m^{-0}$ is a binary variable.
  To show this, we first have to observe that, by the Principle of Locality, the states $\mathbf{e}_{++}$ and $\mathbf{e}_{-+}$ must be fixed functions of the states describing $\mathbf{Q_B}$ and $\mathbf{M}$ *after* performing T_2. In particular, let's focus on these states of Q_B: $\hat{q}_B^{\alpha +}$ is the state obtained when the overall attribute is $\mathbf{e}_{\alpha +}$, with $\alpha \in \{+, -\}$. So, by the Principle of Locality, $\hat{q}_B^{\alpha +} = H(\hat{q}_B, m^{\alpha 0})$ with H some function of $\hat{q}_B$, which is the state describing $\mathbf{Q_B}$ *before* the task T_2 is performed, when it is in its initial attribute $\mathbf{x}_+$ with X sharp with value x_+. We know that $\mathbf{e}_{++}$ is distinguishable from $\mathbf{e}_{-+}$ only by measuring observables of *both* qubits, but since in the attributes $(\mathbf{s_{+0}}, \mathbf{x}_+)$ and $(\mathbf{s_{-0}}, \mathbf{x}_+)$ $\mathbf{Q_B}$ is still in the same initial state $\hat{q}_B$ where the observable X is sharp with value x_+, we cannot distinguish them by the same measurements as above. This means that the state m^{+0} must be *different* from the state m^{-0}, as the dependence on $m^{\alpha 0}$ makes each of the $\{\hat{q}_B^{\alpha +}\}$ different from $\hat{q}_B$;

- Next, we should prove that the attributes in V are not distinguishable from, and do not overlap with, those in T.
  This is a direct consequence of the fact that the task $T_2 \cup T_M$ is possible, being the constructor that performs both T_2 and T_M the same. If each attribute $\{m^{\alpha 0}\}$ were distinguishable from $\mathbf{t_0}$ or $\mathbf{t_1}$, then the attributes $\mathbf{x}_+$ and $\mathbf{x}_-$ of $\mathbf{Q_A}$ would be distinguishable from either $\mathbf{z_0}$ or $\mathbf{z_1}$, in contradiction with the assumption that $\mathbf{Q_A}$ is a superinformation medium. For the same reason, $m^{\alpha 0} \notin \mathbf{t_0}$ and $m^{\alpha 0} \notin \mathbf{t_1}$;

- Finally we notice that the variable V can be distinguished only by measuring simultaneously the complementary observables X_A and Z_A and X_B and Z_B on the superinformation medium $\mathbf{Q_A} \oplus \mathbf{Q_B}$.

This proves that all the conditions for $\mathbf{M}$ to be *non-classical* are met and our claim is reached in very general way. The argument does not commit to any particular formalism to describe $\mathbf{M}$ and its interaction with $\mathbf{Q_A}$ and $\mathbf{Q_B}$. Moreover, the two principles are scale-independent and widely applicable: the interoperability principle holds in any physical theory that allows for observables; the principle of locality is satisfied by both quantum theory and general relativity. This ensures that, if the entanglement is observed, then *all* classical models for gravity obeying interoperability and locality principles are ruled out. This is why the theorem is akin to *Bell's theorem* (see Section 2.3).

Appendix C

Homogenisation is a contractive map

In this Appendix we show in full details what claimed in the main text concerning the homogenisation process: we prove that the partial SWAP map is capable of evolving the system's qubit initial state ρ_0 to the homogeniser's initial state ξ, whatever homogenisation strength η.
In order to do this, we take advantage of the *Banach theorem* [57], which states that each *contractive map* T has a *unique* fixed point ξ and the iteration of this map converges to it, i.e. $T^N[\rho] \to \xi$ for each $\rho \in \mathcal{S}$, being $\mathcal{S}$ the set of physical state for our system. Here by *contractive map* we mean a transformation T such that, given $D(\rho, \xi)$ a distance function between elements of $\mathcal{S}$, it fulfills the inequality $D(T[\rho], T[\xi]) \leq kD(\rho, T\xi)$ with $0 \leq k < 1$ for all $\rho, \xi \in \mathcal{S}$; a *fixed point* of the transformation T is instead an element of $\mathcal{S}$ for which $T[\xi] = \xi$.
We will divide our prove in two steps: first, we will show that the partial SWAP has a fixed point in ξ, the initial state of the reservoir, and then we will prove that the partial SWAP is a contractive map, so that the Banach theorem can straightforwardly be applied concluding the main result of the Section 5.

To show that ξ is a fixed point of the map in Eq. 5.4, we prepare the system qubit Q in an arbitrary state, described by the most general density operator:

$$\rho_0 = \frac{1}{2}\mathcal{I} + \vec{\omega} \cdot \vec{\sigma} \tag{C.1}$$

being $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$ with $|\vec{\omega}| \leq \frac{1}{2}$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ the usual Pauli operators; the reservoir qubits are instead prepared in the state:

$$\xi = \frac{1}{2}\mathcal{I} + \vec{t} \cdot \vec{\sigma} \tag{C.2}$$

with $\vec{t} = (t_x, t_y, t_z)$ such that $|\vec{t}| \leq \frac{1}{2}$. Working in the space of trace-like operator $\mathcal{T}(\mathcal{H})$ in the basis made of the operators $\frac{1}{2}\mathcal{I}, \sigma_x, \sigma_y, \sigma_z$ since $\mathcal{H}$ is the Hilbert space of a qubit, we can write our states as $\rho_0 = (1, \omega_x, \omega_y, \omega_z)$ and $\xi = (1, t_x, t_y, t_z)$. Now we apply the partial SWAP between the system qubit and the first reservoir qubit. As we know from Eq.5.8, the state of the system becomes (here $c = \cos\eta, s = \sin\eta$:

$$\begin{aligned}
\rho_0 \to \rho_1 &= c^2\rho_0 + s^2\xi + ics\,[\xi, \rho_0] \\
&= \frac{1}{2}\mathcal{I} + \left(s^2\vec{t} + c^2\vec{\omega}\right)\cdot\vec{\sigma} + ics\left[\vec{t}\cdot\vec{\sigma}, \vec{\omega}\cdot\vec{\sigma}\right] \\
&= \frac{1}{2}\mathcal{I} + \left[s^2\vec{t} + c^2\vec{\omega} - 2cs\left(\vec{t}\times\vec{\omega}\right)\right]\cdot\vec{\sigma} \\
&= \frac{1}{2}\mathcal{I} + \vec{\omega}'\cdot\vec{\sigma}
\end{aligned} \tag{C.3}$$

where $\omega'_j = s^2 t_j + \left(c^2\delta_{jl} - 2cs\epsilon_{jkl}t_k\right)\omega_l$ with $j = x, y, z$ and the identity $\sigma_k\sigma_l = \delta_{kl}\mathcal{I} + i\epsilon_{jkl}\sigma_j$. The transformation $\rho_0 \to \rho_1$ can be written in vectorial form as:

$$\begin{pmatrix} 1 \\ \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ s^2t_x & c^2 & 2cst_z & -2cst_y \\ s^2t_y & -2cst_z & c^2 & 2cst_x \\ s^2t_z & 2cst_y & -2cst_x & c^2 \end{pmatrix}\begin{pmatrix} 1 \\ \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}, \tag{C.4}$$

which means $\rho_1 = T\rho_0$, where:

$$T = \begin{pmatrix} 1 & \vec{0}^T \\ s^2\vec{t} & \mathbf{T} \end{pmatrix} \tag{C.5}$$

is the matrix representing the super-operator acting on the linear space $\mathcal{T}(\mathcal{H})$. It is easy to check that in our case $\mathbf{T}\vec{t} = c^2\vec{t}$, with $|c^2| \leq 1$, which means that the state ξ is a *fixed* point of the map under consideration:

$$T\xi = \xi. \tag{C.6}$$

Now we can prove that, given a certain homogenisation strength η, the iterative application of the partial SWAP map makes $\rho_0 \to \xi$. In order to do this, we can apply the transformation n times, so that the state of the system reads, following Eq.5.10:

$$\begin{aligned}
\rho_n &= \frac{1}{2}\mathcal{I} + \left[\sum_{j=0}^{n-1} s^2\mathbf{T}^j\vec{t} + \mathbf{T}^n\vec{\omega}\right]\cdot\vec{\sigma} \\
&= \frac{1}{2}\mathcal{I} + \left[s^2\sum_{j=0}^{n-1} c^{2j}\vec{t} + \mathbf{T}^n\vec{\omega}\right]\cdot\vec{\sigma} \\
&= \frac{1}{2}\mathcal{I} + \left[\left(1 - c^{2n}\right)\vec{t} + \mathbf{T}^n\vec{\omega}\right]\cdot\vec{\sigma},
\end{aligned} \tag{C.7}$$

where in the last step we summed the geometric sum $\sum_{j=0}^{n-1}(c^2)^j = \frac{(1-c^{2n}}{1-c^2}$. Let's discuss the two summands involved in the product with $\vec{\sigma}$:

- Unless $c = \cos\eta = 1$, we have easily that $c^{2n} \to 0$;
- We can numerically check that $\mathbf{T}^n \to \mathbf{O}$, being $\mathbf{O}$ the zero operator.

This means that, as $n \to \infty$:

$$\rho_n \to \frac{1}{2}\mathcal{I} + \vec{t}\cdot\vec{\sigma} = \xi \tag{C.8}$$

as we wanted.

It is important now, to formally conclude our proof, to show that what we saw above is true *for all* η. We will thus be able to conclude that the partial SWAP is a *contractive map*.
First of all, we have to define a distance function $D(\cdot,\cdot)$ on $\mathcal{S}$, our set of physical states. We introduce the *trace distance* $D(\rho,\omega) = Tr|\rho-\omega|$ and we look for the *contraction parameter* k, as we know by definition that a map T is contractive if it fulfils the inequality $D(T[\rho], T[\omega]) \leq kD(\rho,\omega)$ with $0 \leq k < 1$.
Let us introduce the vectors $\vec{v} = (1, v_x, v_y, v_z)$ and $\vec{r} = \vec{\omega} - \vec{v}$. For a qubit, this means:

$$D(\rho, v) = Tr|(\vec{\omega}-\vec{v})\cdot\vec{\sigma}| = Tr|\vec{r}\cdot\vec{\sigma}| = 2|\vec{r}| \tag{C.9}$$

as the eigenvalues of the operator $\vec{r}\cdot\vec{\sigma}$ are given by $\lambda_\pm = \pm|\vec{r}|$[1]. We apply now the partial SWAP map T:

$$D(T[\rho], T[v]) = Tr|T[\rho] - T[v]| = Tr|\vec{r}'\cdot\vec{\sigma}| = 2|\vec{r}'| \tag{C.10}$$

where, looking at Eq.C.5, $\vec{r}' = \vec{\omega}' - \vec{v}' = s^2\vec{t} + \mathbf{T}\vec{\omega} - s^2\vec{t} - \mathbf{T}\vec{v} = \mathbf{T}(\vec{\omega}-\vec{v}) = \mathbf{T}\vec{r} = c^2\vec{r} - 2cs\vec{t}\times\vec{r}$.
Now, we observe that $|\vec{r}'|^2 = c^4|\vec{r}|^2 + 4c^2s^2|\vec{t}\times\vec{r}|^2 = |\vec{r}|^2c^2(c^2 + 4s^2|\vec{t}|^2\sin\beta^2$, where $\beta \leq \pi$ is the angle between the vectors $\vec{t}$ and $\vec{r}$, $|\vec{t}|^2 \leq \frac{1}{4}$. Thus, applying the condition for a map to be contractive $D(T[\rho], T[v]) \leq kD(\rho, v)$, which translates in $|\vec{r}'| \leq k|\vec{r}|$, we can identify the contraction coefficient $k = c$. Thus, if $c = \cos\eta < 1$, then the map T is contractive and the convergence to the fixed point ξ is guaranteed by Banach Theorem. This concludes our formal proof.

[1] This because we set $\hbar = 1$ and we collected the factor $\frac{1}{2}$ in the basis of the space $\mathcal{T}(\mathcal{H})$.

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