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Optimization and implementation of a software code for the automated analysis of fatigue data

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Abstract

Material fatigue is the most common cause of failures of components and must be properly considered during the design process. Fatigue is a highly structure-sensitive process, depending mainly on the material characteristics at sub microscopic and microscopic scale. If, in fact, samples of the same size and material are subjected to the same loading conditions, different results can be obtained. This induces a statistical variability of the material fatigue strength. Therefore, data obtained from fatigue tests should be analysed using statistical methodologies. The purpose of a statistical analysis of fatigue test results is to estimate the fatigue properties of the material, such as fatigue limit, or fatigue strength at a given number of cycles, S-N curve, etc., from the test data. The least squares method is generally used for assessing the parameters of the S-N curve, in the finite life range, while the Staircase method is used for assessing the fatigue strength in the infinite life range of the Wöhler curve. However, with the least-squares method run-outs data (i.e. tests that are interrupted before failure) are not considered. Furthermore, although the Staircase method considers both runout and failure data, it does not consider the actual number of cycles to failure but only whether the specimen fails or does not fail. Other methods for estimating the fatigue S-N curve permit to considers both failures and runout, e.g., methods based on the Maximum Likelihood Principle. The material parameters that maximize the Maximum Likelihood function are the best estimates for the considered S-N curves. This method permits the estimation of the parameters used in a unified model that is able to shape the complete SN curve with a single equation.

The goal of this thesis is to develop and optimise a user-friendly software implementing a complete model, whose parameters are estimated with the principle of maximum likelihood, for the statistical analysis of fatigue data. This software has been validated on several experimental datasets related to different types of materials and the results obtained were compared with those obtained with traditional approaches to highlight the differences between the methods. In addition, since the principle of maximum likelihood maximises the information contained in the experimental dataset, the possibility of reducing the number of tests to be carried out and thus reducing the time and costs of a characterisation campaign was also investigated. In order to validate the obtained results, virtual experimental datasets have been randomly simulated. In this way it was possible to verify how much the estimated parameters, computed with both methods, deviate from the real ones, and thus investigate which method was more accurate in estimating the fatigue parameters.

Keywords*: Fatigue testing; Statistical analysis; Staircase; Least-squares method; Maximum Likelihood; Numerical simulation*

Summary

In the first chapter I will briefly describe the experimental methods and the machines used in CRF, by which the fatigue tests were conducted and from which the datasets have been obtained and then analysed. In the second chapter, I will introduce the "traditional" method based on linear regression for the finite fatigue life regime and Staircase method for the infinite fatigue life regime. In the third chapter, the Maximum Likelihood Estimation (MLE) of the parameters for the construction of the Wöhler curve will be explained. Subsequently, in chapter 4, the methodology with which the two models were implemented in Matlab app Designer was presented. Then, in chapters 5, it has been made a comparison between the results of the analysis of experimental datasets obtained with the two models presented. Finally, chapter 6 presents the simulation study that was used to evaluate the accuracy of the two methods for a large number of virtual randomly datasets and for different configuration of test parameters.
In addition, as required by the instructions for thesis development for the Master of Science in Materials Engineering, an Italian language summary of the work is provided here. Some of the tables and figures considered relevant have been included in the summary.

Introduzione e obiettivi del lavoro di tesi

La fatica dei materiali è la causa più comune dei guasti dei componenti e deve essere adeguatamente considerata durante il processo di progettazione. La fatica è un processo altamente sensibile alle caratteristiche del materiale su scala sub microscopica e microscopica. Se, infatti, campioni della stessa dimensione e dello stesso materiale sono sottoposti alle stesse condizioni di carico, si possono ottenere risultati diversi. Questo induce una variabilità statistica della resistenza a fatica del materiale. Pertanto, i dati ottenuti dalle prove di fatica dovrebbero essere analizzati utilizzando metodologie statistiche. Lo scopo di un'analisi statistica dei risultati delle prove di fatica è quello di stimare le proprietà di fatica del materiale, come il limite di fatica, o la resistenza a fatica ad un dato numero di cicli, la curva S-N, ecc, dai dati della prova.

Il metodo dei minimi quadrati è generalmente usato per valutare i parametri della curva S-N, nel range di vita a termine, mentre il metodo Staircase è usato per valutare la resistenza a fatica nel range di vita infinita della curva di Wöhler. Tuttavia, con il metodo dei minimi quadrati le prove di run-out (cioè le prove interrotte prima della rottura) non vengono considerate. Inoltre, sebbene il metodo Staircase consideri sia i dati di runout che quelli di rottura, non considera l'effettivo numero di cicli a rottura, ma solo se il provino si rompe o non si rompe. Quindi, in entrambi i casi, si ha una perdita di informazioni importanti.

Altri metodi, come per esempio quelli basati sul principio della Massima Verosimiglianza, permettono di considerare sia le rotture che i runout per stimare la curva di fatica S-N dei materiali. Per questo tipo di metodi, i parametri del materiale che massimizzano la funzione di Massima Verosimiglianza, sono le migliori stime per le curve S-N considerate. Quest'approccio, sebbene sia più complesso dei metodi "tradizionali" perché prevede un processo di calcolo iterativo, permette la stima dei parametri di un modello che è in grado di descrivere la curva S-N completa con un'unica equazione.

Questa tesi fa parte di un più ampio progetto di ricerca svolto in collaborazione con il Centro Ricerche Fiat (CRF). L'obiettivo di questa tesi è lo sviluppo e l'ottimizzazione di un software user-friendly che implementi un modello completo, i cui parametri sono stimati con il principio della Massima Verosimiglianza, per l'analisi statistica dei dati di fatica. Questo software è stato validato su diversi dataset sperimentali relativi a diverse tipologie di materiali e i risultati ottenuti sono stati confrontati con quelli ottenuti con approcci tradizionali per evidenziare le differenze tra i metodi. Inoltre, poiché il principio della Massima Verosimiglianza massimizza l'informazione contenuta nel dataset sperimentale, è stata studiata anche la possibilità di ridurre il numero di prove da effettuare e quindi di ridurre i tempi e i costi di una campagna di caratterizzazione.

Al fine di convalidare i risultati ottenuti, è stata utilizzata anche la simulazione numerica. In particolare, è stato generato un gran numero di dataset virtuali, ed è stato possibile verificare quanto i parametri stimati, calcolati con entrambi i metodi, si discostino da quelli reali imposti e quindi indagare quale metodo fosse più accurato nella stima dei parametri di fatica.

Metodologia sperimentale

Il set-up delle prove di fatica ha seguito le linee guida della ISO 12107:2012 [1]. Lo scopo delle prove di fatica ad alto numero di cicli (HCF) è quello di determinare la resistenza a fatica nella regione a vita finita e nella regione a vita infinita ovvero costruire la curva di Wöhler per il materiale testato.
Le prove di fatica sono condotte ad una data sollecitazione, S, su una serie di provini accuratamente preparati per determinare i valori di vita a fatica per ciascuno di essi. Il provino può essere caricato fino a rottura o la prova viene terminata anche se non si verifica alcuna rottura al raggiungimento di un elevato numero di cicli, che viene scelto in base al tipo di materiale utilizzato e al campo di applicazione previsto. Le prove di fatica da cui sono stati ottenuti i datasets sperimentali, utilizzati per questo lavoro di tesi, sono state prove di fatica a flessione piana e prove di fatica assiale, a seconda del tipo di prova richiesta dal cliente. Anche le condizioni di prova (come frequenza, rapporto di carico R, temperatura) e il tipo di macchine di prova, sono state scelte in base al tipo di materiale e al tipo di caratterizzazione richiesta.

Approccio "tradizionale"

Una volta che i dati sperimentali relativi alla regione di vita a termine della curva S-N sono stati ottenuti, si effettua una regressione lineare dei dati per ottenere la curva S-N mediana. A tal proposito, viene utilizzato il metodo dei minimi quadrati. Lo standard ASTM E739-91 è uno standard comune nella costruzione delle curve S-N, che considera un comportamento lineare, su scala bi-logaritmica (equazione (1)), dell'ampiezza della sollecitazione rispetto al numero di cicli a rottura. Viene assunto che la vita a fatica risultante ad un dato livello di ampiezza della sollecitazione segua la distribuzione log-normale, il che significa che il $\log(N_f)$ segue una distribuzione normale. Si assume inoltre che la varianza del logaritmo della vita a fatica è costante nell'intervallo testato [3].

$$log_{10}(N) = log_{10}(A) + B \cdot \ log_{10}(S_a) \tag{1}$$

dove B è la pendenza e $\log_{10}(A)$ è l'intercetta. Includere campioni di runout in questo modello non è raccomandato come indicato da Barbosa et al. in [5]. La retta di regressione è quindi:

$$\hat{Y} = \hat{A} + \hat{B} \cdot X \tag{2}$$

L'Equazione (2) può essere riscritta in una forma che è comunemente usata per descrivere le curve mediane S-N:

$$S_a = S_f' \ (N)^b \tag{3}$$

dove b è l'esponente della resistenza a fatica e S_f' è il coefficiente di resistenza a fatica. Prendendo i logaritmi di entrambi i lati dell'equazione S-N (Equazione 3) e riorganizzando si ottiene l'equazione (4):

$$log_{10}(N) = -\frac{1}{b} \ log_{10}(S_f') + \frac{1}{b} log_{10}(S_a) \tag{4}$$

Confrontando l'equazione (4) con l'equazione (1), si può determinare che X=$log_{10}(S_a)$ e Y=$log_{10}(N)$ mentre le proprietà statistiche di fatica (b, S_f') sono determinate tramite i coefficienti dell'equazione dei minimi quadrati, $b = {}^{1}/_{\hat{B}}$, $S_f' = 10^{(-\hat{A} \cdot b)}$ [3]. Spesso però le curve mediane di vita a fatica S-N non sono sufficienti per l'analisi e la progettazione a fatica e quindi, per garantire maggiore affidabilità, sono richieste curve S-N relative a più alti percentili. Infatti, la curva mediana rappresenta la curva per la quale il 50% dei campioni fallisce al di sopra di questa curva e il 50% al di sotto di essa, ma si potrebbe ugualmente tracciare una curva dove solo il 10 % dei campioni si rompe al di sotto della curva e il 90% al di sopra di essa (curva R90). La curva S-N con una diversa probabilità di sopravvivenza può essere costruita da una retta con lo stesso esponente di resistenza a fatica *b*, che è spostata orizzontalmente a sinistra dalla linea mediana di una distanza pari a $z \cdot s$, dove *z* è il quantile *p-esimo* desiderato e s è la deviazione standard della vita alla fatica. *z* può essere ricavato dalle tabelle Z per una distribuzione normale oppure, come in questo caso, può essere calcolato tramite Matlab con il comando *"norminv (p)"*. Nel caso della curva R90, per esempio, *z* = *norminv* (0.9) = 1.28.

Per garantire un maggiore margine di sicurezza contro le rotture a fatica solitamente vengono però utilizzate le curve di design. Una curva di design è una curva per la quale la maggior parte dei dati si trova al di sopra di essa. In generale, un punto sulla curva di design, ad un livello di tensione specifico X_i è costruito usando l'equazione (5) [3]:

$$Y_L(X_i) = \hat{Y}(X_i) - K \cdot s \tag{5}$$

dove Y_L è definito come il limite inferiore di Y $(=log_{10}(N_f))$ ad un dato X_i $(=log_{10}(S_{a_i}))$ e K è un fattore dipendente dal numero di campioni adottati nelle prove sperimentali e dal livello di affidabilità e confidenza desiderato. Se si considera, per esempio, un'affidabilità del P % e un livello di confidenza del γ % questo significa che il (100 · γ) % delle volte Y_L definisce un punto sulla curva S-N che ha una probabilità di sopravvivenza maggiore del P %. Ci sono varie scelte per *K* ma l'approccio più comune, usato anche al CRF, è il fattore di tolleranza unilaterale di Owen. Questo metodo presuppone che la curva S-N di progetto abbia la stessa pendenza della curva S-N mediana. Owen [3] fornisce una formula per determinare il fattore K_{Owen}.

$$K_{Owen} = K_D \cdot R_{Owen} \tag{6}$$

dove K_D e R_{Owen} sono due fattori che dipendono da coefficienti sperimentali e dal livello di affidabilità e confidenza desiderato.

Per quanto riguarda invece l'analisi dei dati del tratto vita infinita, il metodo Staircase è il più diffuso, per la determinazione del comportamento statistico del limite di fatica [3]. Questo perché si basa su un protocollo semplice in cui un provino viene testato ad una data sollecitazione iniziale per un determinato numero di cicli o fino alla rottura, a seconda di quello che avviene prima. Se il primo provino è sopravvissuto fino al numero di cicli previsto (numero di cicli di runout), il provino successivo viene testato ad un livello di stress aumentato di Δ*S*. Se, invece, il provino fallisce prima del numero di cicli di runout, verrà successivamente testato ad un livello di stress ridotto di Δ*S* [9]. Pertanto, ogni prova dipende dal risultato della prova precedente e questa è una delle limitazioni di questa strategia perché rende impossibile effettuare le prove in parallelo. Questo metodo non tiene in considerazione il numero effettivo di cicli a rottura, ma l'unica informazione che considera è se il provino si è rotto o meno. Un tipico risultato di una prova Staircase è mostrato nella Figura 1 dove "O" è una prova di runout e "X" è una rottura. In questo esempio il numero di cicli oltre il quale il campione è considerato sopravvissuto è 1e7.

Level	**Stress Sa [Mpa]**	**Sequence number of specimens**														
i		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
6																
5																
4																
3	**90**					X		X		X		X				X
2	**85**				O		O		O		O		X		O	
1	**80**	X		O										O		
0	**75**		O													
Δs [MPa]	**5**															
Number of cycles		3.46E+06	1.00E+07	1.00E+07	1.00E+07	6.61E+06	1.00E+07	2.04E+06	1.00E+07	4.01E+06	1.00E+07	2.95E+06	6.28E+06	1.00E+07	1.00E+07	9.76E+06

Figura 1 Esempio di una prova Staircase

La prima prova viene condotta ad un livello di stress che può essere determinato o tramite dei dati S-N di materiali simili o dall'esperienza. Lo step size adottato per la prova, invece, deve essere nel range ($0.5 \cdot \sigma \div 2 \cdot \sigma$) dove σ è la deviazione standard del limite di fatica. Quindi, una conoscenza approssimativa della deviazione standard è necessaria per specificare lo step size prima di effettuare le prove. Una scelta non ottimale del punto di partenza e dell'ampiezza del passo può causare incertezza nella stima del limite di fatica e della deviazione standard e, eventualmente, richiedere un numero aggiuntivo di test, il che aumenterebbe i costi della campagna sperimentale.

Una volta completati i test, inizia l'elaborazione statistica dei risultati. Il metodo tradizionale che viene usato è il metodo proposto da Dixon e Mood [9] che permette di determinare la media e la deviazione standard dei dati delle prove di Staircase. Una delle condizioni necessarie per l'applicazione di questo metodo è che il limite di fatica segua una distribuzione normale [9]. Questa condizione sembra essere molto restrittiva, tuttavia, una trasformazione logaritmica o esponenziale dei valori di tensione può essere applicata per rendere la distribuzione normale. Denotando con n_i il numero dell'evento meno frequente (perché in questo metodo si considerano solo le rotture o solo le sopravvivenze) al livello di stress *i*, si possono calcolare tre quantità *A*, *B* e *C*:

$$A = \sum_{i=0}^{n_i} n_i\,, \quad B = \sum_{i=0}^{n_i} i \cdot n_i\,, \quad C = \sum_{i=0}^{n_i} i^2 \cdot n_i \tag{7}$$

La stima della media e della deviazione standard del limite di fatica è quindi:

$$\mu = S_0 + \Delta S \cdot \left(\frac{B}{A} \pm 0.5\right) \tag{8}$$

$$\sigma = 1.62 \cdot \Delta S \left(\frac{A \cdot C - B^2}{A^2} + 0.029\right) \tag{9}$$

se $\frac{A \cdot C - B^2}{A^2} \geq 0.3$ or,

$$\sigma = 0.53 \cdot \Delta S \tag{10}$$

se $\frac{A \cdot C - B^2}{A^2} < 0.3$

dove il segno "+" nell'Eq. (8) indica che l'evento meno frequente è la sopravvivenza del campione, mentre "-" indica che l'evento meno frequente è la rottura.

Il valore del limite di fatica associato ad un diverso livello di affidabilità può essere determinato sottraendo dalla media il prodotto $z \cdot \sigma$ dove z è il quantile *p*-esimo.

$$S_{e,R} = \mu - z \cdot \sigma \tag{11}$$

Allo stesso modo, il valore del limite di fatica associato ad una certa affidabilità e livello di fiducia può essere determinato tramite la formula (12).

$$S_{e,R,C} = \mu - K \cdot \sigma \tag{12}$$

dove il fattore *K* può essere determinato tramite le tabelle di Liebermann [12] o può essere calcolato con la formula proposta da Link et al. in [13].

Principio della Massima Verosimiglianza

L'idea alla base dell'approccio della Massima Verosimiglianza è di trovare i parametri della distribuzione che massimizzino la probabilità di ottenere i dati osservati sperimentalmente. Il metodo della Massima Verosimiglianza ha ottime proprietà asintotiche [15] convergendo al valore reale per un numero infinito di campioni, e questo lo rende adatto alla stima dei parametri di un modello statistico. Anche se la strategia Staircase è il metodo più diffuso per calcolare il limite di fatica, l'analisi dei dati è strettamente basata su un approccio rotto/non-rotto e, così facendo, un'informazione importante come il numero di cicli a rottura rimane inutilizzata. Il vantaggio del metodo basato sul principio della

Massima Verosimiglianza rispetto al metodo tradizionale è la capacità di poter stimare i parametri di un modello in grado di descrivere l'intera curva S-N con una singola equazione. Inoltre, a differenza dei metodi basati sui minimi quadrati, questo permette di considerare anche le prove interrotte (runout). Con questo modello, la rottura avviene se il provino è caricato ad un livello di sollecitazione S superiore al limite di fatica e se il numero di cicli N a S è maggiore di N_f. Quindi, la probabilità di rottura può essere espressa come il prodotto delle singole probabilità (Eq.13) come suggerito da Paolino et al. in [16-17]:

$$F_{Y|X=x} = \varphi\left(\frac{Log_{10}(S_a) - \mu_{X_l}}{\sigma_{X_l}}\right) \cdot \varphi\left(\frac{y - \left(a + b \cdot Log_{10}(S_a)\right)}{\sigma_Y}\right) \tag{13}$$

dove φ(·) è la funzione di distribuzione cumulativa normale standardizzata (cdf). Qui, è stato supposto che il numero di cicli a rottura segua una distribuzione log-normale, che il valore mediano di y, il logaritmo (in base 10) del numero di cicli, μ_y, segua la legge di Basquin [18] e che la deviazione standard della vita a fatica sia costante per qualsiasi valore di sollecitazione. μ_{X_l} e σ_{X_l} sono il valore medio e la deviazione standard di X_l cioè il logaritmo del limite di fatica (in questo modello si è assunta una distribuzione log-normale del limite di fatica quindi X_l può essere assunto come distribuito normalmente) mentre a e b sono coefficienti costanti relativi alla legge di Basquin. L'equazione (13) suppone quindi che la rottura può avvenire a seguito di una sollecitazione superiore al valore limite μ_{X_l} o dopo una vita a fatica superiore o uguale a quella definita dalla legge di Basquin. La cdf (13) dipende da cinque parametri: a, b, σ_Y, μ_{X_l}, σ_{X_l}. Il metodo della Massima Verosimiglianza viene usato per stimare questi parametri. Per utilizzare il principio della massima verosimiglianza, è necessario definire la funzione di verosimiglianza, L. La verosimiglianza di ottenere le misure (in questo caso, il numero di cicli a rottura) ottenute sperimentalmente è la probabilità di ottenere quelle misure dato il valore dei parametri della distribuzione $\boldsymbol{\theta}$. La funzione di verosimiglianza assume la forma [19]:

$$L[\boldsymbol{\theta}] = \prod_{i=1}^{n_f} f_{Y|X=x}\,[y_i, x_i, \boldsymbol{\theta}] \cdot \prod_{j=1}^{n_r} (1 - F_{Y|X=x}\,[y_j, x_j, \boldsymbol{\theta}]) \tag{14}$$

dove (x_i,y_i) e (x_j, y_j) sono l'insieme dei dati sperimentali relativi rispettivamente a rotture e runouts, n_f è il numero di rotture, n_r è il numero di runout, $\boldsymbol{\theta}$ denota l'insieme dei 5 parametri coinvolti nel modello statistico, $f_{Y|X=x}$ è la funzione di densità di probabilità (pdf) e $F_{Y|X=x}$ è la funzione di densità di probabilità cumulata (cdf). $f_{Y|X=x}[y_i, x_i, \boldsymbol{\theta}]$ rappresenta la probabilità di una rottura osservata a y_i mentre $\left(1 - F_{Y|X=x}\left[y_j, x_j, \boldsymbol{\theta}\right]\right)$ rappresenta la probabilità che la vita del campione sia superiore a y_j[19]. Come precedentemente anticipato, il principio della Massima Verosimiglianza afferma che i valori preferiti dei parametri di una funzione di verosimiglianza sono quelli che massimizzano la probabilità di ottenere i dati osservati sperimentalmente [20]. In pratica, per semplificare i calcoli, si usa generalmente una funzione di verosimiglianza logaritmica, $\mathcal{L} = \ln(L)$ perché il logaritmo di un prodotto è uguale alla somma dei logaritmi dei termini e la ricerca degli estremi della funzione somma è più facile rispetto a quella di una produttoria. Questo calcolo è stato fatto numericamente con il comando *"GlobalSearch"* su Matlab e

$$\widetilde{\boldsymbol{\theta}} = \left(\tilde{a}, \tilde{b}, \widetilde{\sigma_Y}, \widetilde{\mu_{X_l}}, \widetilde{\sigma_{X_l}}\right) \tag{15}$$

è il vettore dei parametri stimati.
I parametri così calcolati possono essere utilizzati per calcolare le curve S-N a diversi quantili. In particolare, se è di interesse la curva S-N del quantile α-esimo, cioè la curva S-N con la probabilità di rottura pari ad α, sostituendo $F_{Y|X=x} = \alpha$ nella (13), si può ottenere l'equazione (16):

$$\alpha = \varphi\left(\frac{Log_{10}(S_a) - \widetilde{\mu_{X_l}}}{\widetilde{\sigma_{X_l}}}\right) \cdot \varphi\left(\frac{y - \left(\tilde{a} + \tilde{b} \cdot Log_{10}(S_a)\right)}{\widetilde{\sigma_Y}}\right) \tag{16}$$

che, risolta rispetto a $Log_{10}(S_a)$ per diversi valori di y (=$Log_{10}(N_f)$), permette di costruire le diverse curve S-N desiderate.

Per calcolare la curva di design in questo caso è stato utilizzato il Profile Likelihood Ratio approach [17]. Gli intervalli di fiducia basati su quest'approccio sono più difficili da stimare rispetto al metodo usato da Owen e Dixon-Mood a causa del tempo di calcolo richiesto per l'applicazione del metodo, ma sono più accurati e sembrano funzionare meglio per un numero inferiore di campioni disponibili [21]. Se $\boldsymbol{\theta} = (\theta_1,\boldsymbol{\theta_2})$ è una partizione del vettore dei parametri da stimare, in cui θ_1 è il parametro di interesse, e $\boldsymbol{\theta_2}$ è il vettore degli altri parametri coinvolti nel modello, la funzione di Profile Likelihood per θ_1 è:

$$PL[\theta_1] = \frac{max_{\theta_2}\left[\mathcal{L}[\theta_1,\boldsymbol{\theta_2}]\right]}{\mathcal{L}\,[\widetilde{\boldsymbol{\theta}}]} \tag{17}$$

dove $\mathcal{L}\,[\widetilde{\boldsymbol{\theta}}]$ è il logaritmo della funzione di Verosimiglianza calcolata per il vettore $\widetilde{\boldsymbol{\theta}}$ ovvero il vettore dei parametri che massimizzano la funzione di Verosimiglianza. Si può dimostrare che il "Likelihood Ratio Statistics" definito come -2 ln($PL[\theta_1]$) segue una distribuzione Chi-quadro ad un grado di libertà [17]. Di conseguenza, gli intervalli di fiducia basati sull'approccio Profile Likelihood per il parametro θ_1 sono basati sull'equazione (18):

$$PL[\theta_1] \geq \mathrm{e}^{-\frac{\chi^2{}_{(1;\beta)}}{2}} \tag{18}$$

dove, $\chi^2{}_{(1;\beta)}$ è il quantile β della distribuzione χ^2 con 1 grado di libertà. Se δ è il livello di fiducia desiderato, cioè $\delta = C/100$, $\beta = (2\delta\text{-}1)$. Per stimare gli intervalli di fiducia per S_{a_α}, $PL[\theta_1]$ deve essere una funzione di S_{a_α} dove S_{a_α} è il quantile α-esimo della resistenza a fatica. Considerando $\theta_1 = \mu_{X_l}$ e tenendo conto dell'equazione (13) con $F_{Y|X=x} = \alpha$, si può ottenere l'eq. (19):

$$\mu_{X_l} = Log_{10}\left(S_{a_\alpha}\right) - \varphi^{-1}\left(\frac{\alpha}{\varphi\left(\frac{y_\alpha - \left(a + b\,\cdot\,Log_{10}\left(S_{a_\alpha}\right)\right)}{\sigma_Y}\right)}\right)\cdot\sigma_{X_l} \tag{19}$$

Ora, se si sostituisce l'equazione (19) nell'equazione (13), si ottiene:

$$F_{Y|X=x} = \varphi\left(\frac{y-\left(a+b\cdot Log_{10}(S_a)\right)}{\sigma_Y}\right)\cdot\,\varphi\left(\frac{Log_{10}(S_a)-\left(Log_{10}(S_{a_\alpha})-\varphi^{-1}\left(\frac{\alpha}{\varphi\left(\frac{y_\alpha-\left(a+b\cdot Log_{10}(S_{a_\alpha})\right)}{\sigma_Y}\right)}\right)\cdot\sigma_{X_l}\right)}{\sigma_{X_l}}\right) \tag{20}$$

L'equazione (20) permette di calcolare $max_{\theta_2}(\mathcal{L}[\theta_1,\boldsymbol{\theta_2}])$ e quindi $PL[\,\theta_1]$. Il range di valori di θ_1 che soddisfa l'eq. (18) rappresenta gli intervalli di fiducia del parametro θ_1 [23]. Questa procedura è stata implementata in un codice Matlab che calcola automaticamente gli intervalli di fiducia con questo metodo. Per costruire, per esempio, la curva R90C90 abbiamo bisogno di calcolare la funzione di Profile Likelihood per diversi valori di numero di cicli. La Fig. 2 si riferisce ad un esempio reale in cui è stato considerato un numero di cicli pari a N_f=2.48e7. È stata tracciata solo metà della curva *PL* perché siamo interessati solo al limite inferiore dell'intervallo di confidenza. L'intervallo di fiducia del 90% può essere letto dal grafico come l'insieme dei valori del parametro S_a con *PL* maggiore di $\exp\left(-\frac{\chi^2{}_{(1;0.8)}}{2}\right) = 0.439$ (questa linea critica è stata disegnata nel grafico).

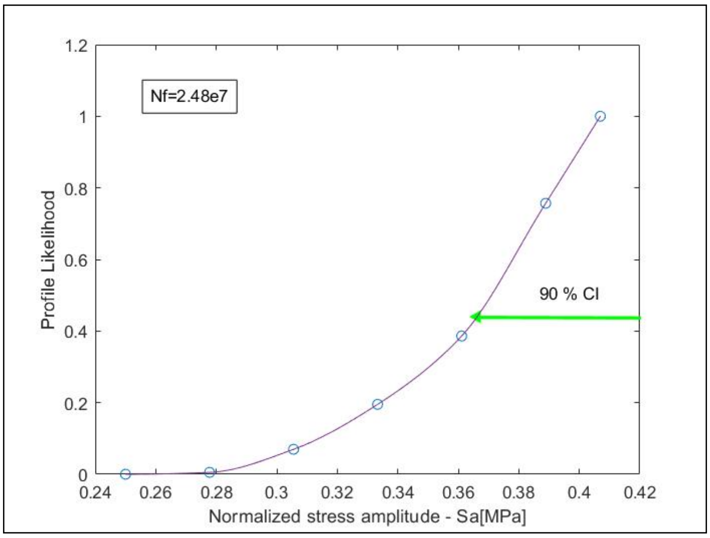


Figura 2 Esempio di curva di Profile Likelihood in funzione di S_a

Se iteriamo questo calcolo per diversi valori di numero di cicli, possiamo ottenere la curva R90C90 come mostrato in Figura 3. Le curve R50 e R90 in Fig. 3 si basano sull'equazione (16) in cui $\alpha = 0.5$ e $\alpha = 0.9$ rispettivamente. Il valore degli stress è stato normalizzato per riservatezza aziendale.

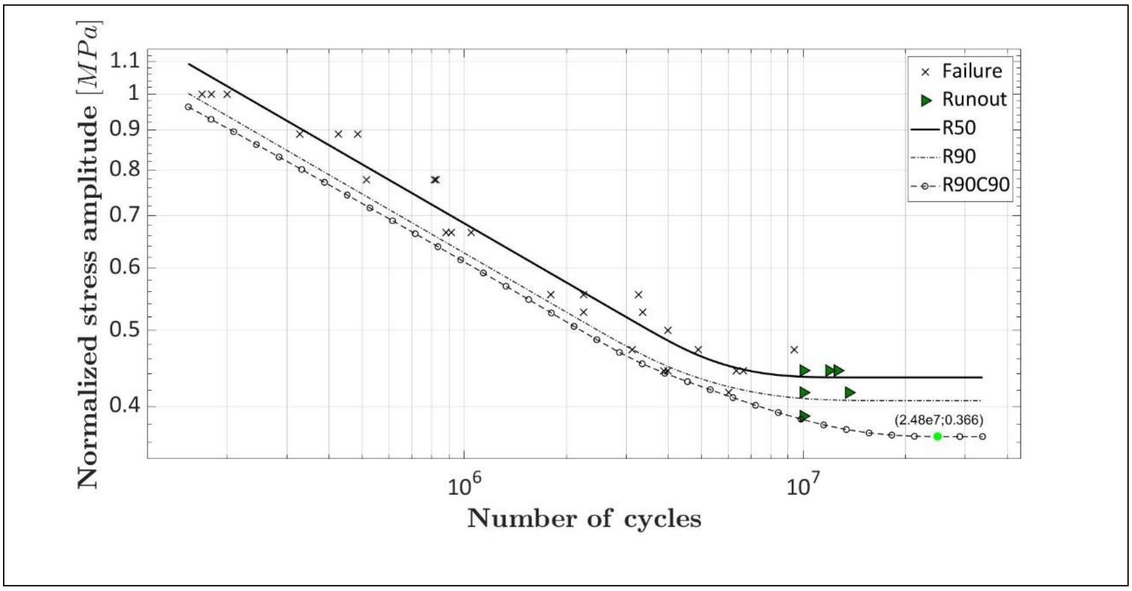


Figura 3 Curve R50, R90 e R90C90 calcolate con l'approccio di Massima Verosimiglianza

Sviluppo di un'applicazione per l'analisi automatica dei dati di fatica

L'analisi dei dati delle prove di fatica è stata effettuata per mezzo di un codice Matlab che è stato sviluppato e ottimizzato in modo tale da implementare entrambe le metodologie e permettere un confronto diretto dei loro risultati. Per facilitare l'utilizzo di questo codice, è stata sviluppata un'applicazione user-friendly utilizzando Matlab App Designer. L'applicazione permette di caricare il dataset sperimentale in formato .txt in cui la prima colonna corrisponde ai diversi livelli di stress, la seconda corrisponde al numero di cicli ad essi associato e la terza corrisponde allo stato di quel particolare test, cioè se si tratta di una rottura (= 1) o di un runout (= 0). Per presentare i seguenti risultati sono stati usati i dati di fatica dell'$AlSi_{10}Mg$.
Figura 4 e 5 mostrano il layout dell'app relativa allo Staircase e al metodo dei minimi quadrati.

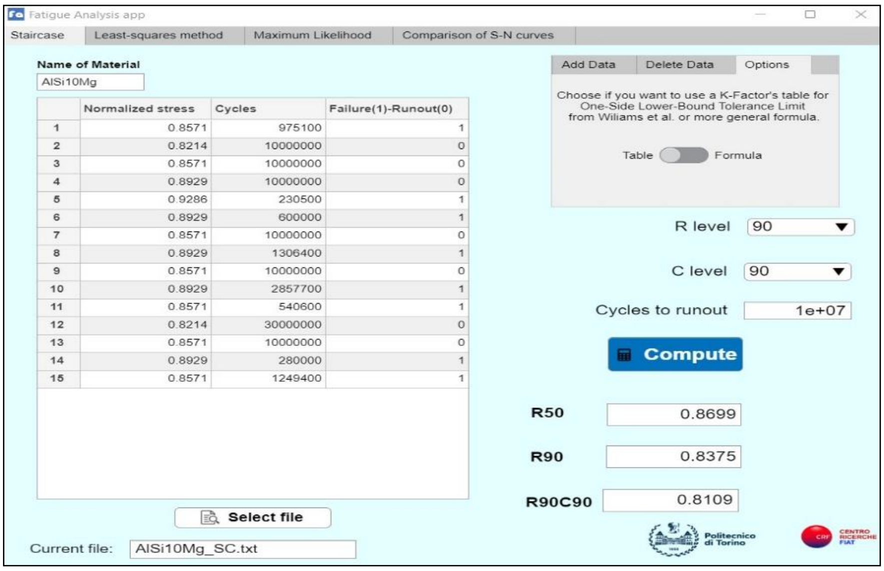


Figura 4 Staircase tab dell'app Fatigue Analysis

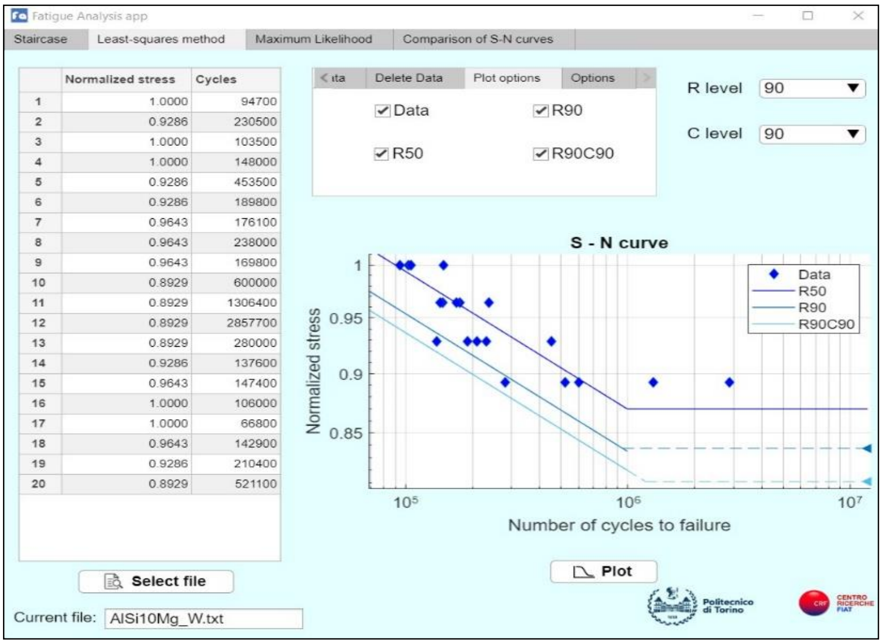


Figura 5 Metodo dei minimi quadrati implementato nell'app Fatigue Analysis

Figura 6 mostra invece il tab relativo all'implementazione del metodo basato sulla Massima Verosimiglianza. Siccome questo metodo permette la stima dei parametri di un modello capace di descrivere il comportamento a fatica di un materiale con una curva continua e non con due rette separate, come nel caso dell'approccio tradizionale, nell'implementazione del metodo va considerato il dataset completo consistente dei valori di stress e numero di cicli di tutte le prove sperimentali effettuate. In Figura 7 sono mostrate le curve S-N per l'$AlSi_{10}Mg$ calcolate con l'applicazione utilizzando il metodo della Massima Verosimiglianza.

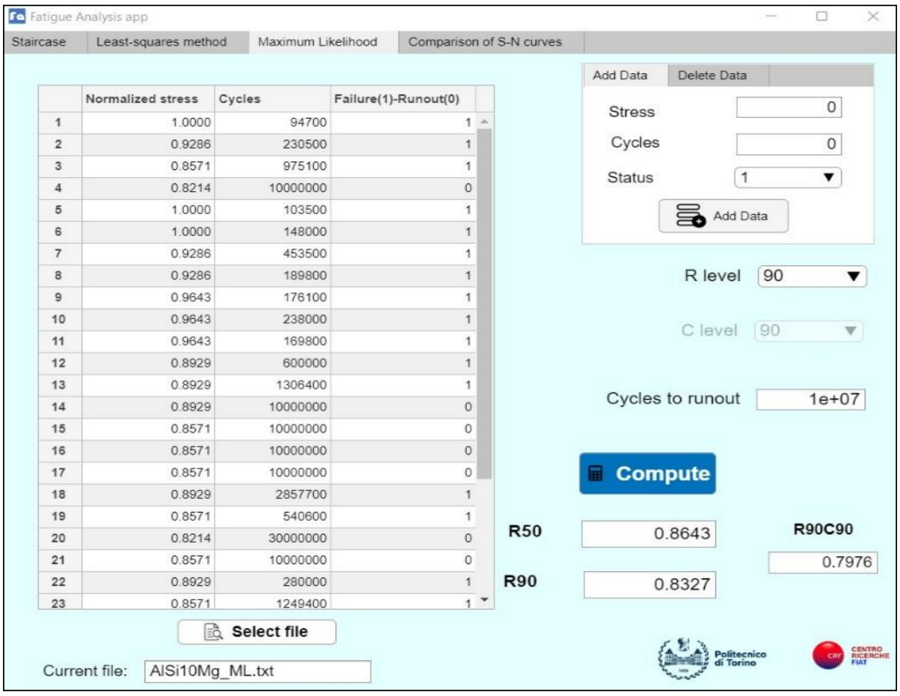


Figura 6 Tab per il metodo della Massima Verosimiglianza dell'app Fatigue Analysis

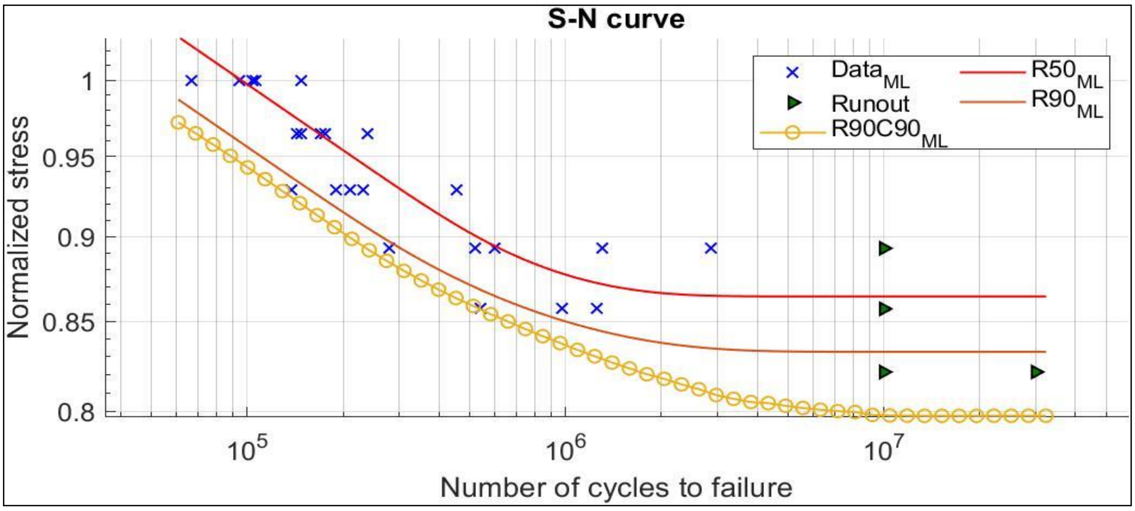


Figura 7 Esempio di curve S-N calcolate con l'app Fatigue Analysis applicando il principio della Massima Verosimiglianza

In Figura 4.3 - 4.6 e 4.9 del testo in inglese sono stati riportati i flowchart che schematizzano, in maniera semplificata, il codice dietro lo sviluppo di quest'applicazione. Uno degli aspetti importanti da sottolineare è la scelta dei valori iniziali per l'ottimizzazione dei parametri con il principio della Massima Verosimiglianza. Nell'applicazione è stato usato il comando "*GlobalSearch*", una funzione di Matlab che permette di calcolare il minimo globale della funzione di Massima Verosimiglianza (che corrisponde al massimo della funzione cambiata di segno) ma questa funzione richiede una stima iniziale dei parametri da prendere come punto di partenza per l'ottimizzazione. Come suggerito anche da Nelson in [19], una prima stima di a e b può essere derivata da un fitting ai minimi quadrati di una regressione lineare di $\log_{10}(N)$ rispetto a $\log_{10}(S)$ dove però sono stati considerati solo i dati relativi alle rotture perché sono quelli che generalmente sono allineati seguendo la relazione di Basquin. Questo è stato fatto con la funzione *"regress"* di Matlab che restituisce un vettore delle stime dei coefficienti di una regressione lineare, come è stato mostrato nel diagramma di flusso della Figura 4.9 (a). La funzione r*egress* calcola anche tutti i parametri statistici del modello di regressione lineare tra cui l'errore quadratico medio la cui radice quadrata è stata presa come stima preliminare di σ_Y.

Per quanto riguarda i parametri del tratto vita infinita, come stima iniziale di μ_{X_l} è stata considerata la media dei logaritmi dei valori degli stress derivanti dalle prove Staircase mentre come stima iniziale di σ_{X_l} è stata considerata la massima distanza dei logaritmi degli stress dello Staircase dal valore medio.

Come sottolineato nel capitolo precedente, nel calcolo degli intervalli di fiducia con il metodo del Likelihood Ratio, la funzione *PL* deve essere funzione di S_{a_α} e questo può essere ottenuto esplicitando, per esempio, μ_{X_l} come mostrato nell'equazione (19). Tuttavia, quando si è ad alti livelli di tensione, la probabilità che il limite di fatica sia al di sotto del livello di tensione applicato è uguale ad 1 e quindi se $\varphi\left(\frac{Log_{10}(S_{a_\alpha})-\mu_{X_l}}{\sigma_{X_l}}\right) = 1$ allora, per l'equazione (19) $\frac{\alpha}{\varphi\left(\frac{y_\alpha-\left(a+b\cdot Log_{10}(S_{a_\alpha})\right)}{\sigma_Y}\right)} = 1$ e quando viene calcolato

$$\mu_{X_l} = Log_{10}(S_{a_\alpha}) - \varphi^{-1}\left(\frac{\alpha}{\varphi\left(\frac{y_\alpha-\left(a+b\cdot Log_{10}(S_{a_\alpha})\right)}{\sigma_Y}\right)}\right)\cdot\sigma_{X_l}\text{ , si ha }\varphi^{-1}(1) = \infty.$$

Questo rendeva impossibile calcolare *PL* e interrompeva l'esecuzione del codice. In questi casi, invece di esplicitare μ_{X_l} dall'equazione (19), è stato esplicitato il parametro *a* dalla stessa equazione. Tuttavia, anche in questo caso, quando si è ad un alto numero di cicli, la probabilità che il provino si rompa ad un numero di cicli inferiore a quello effettivo è 1 quindi se $\varphi\left(\frac{y_\alpha-\left(a+b\cdot Log_{10}(S_{a_\alpha})\right)}{\sigma_Y}\right) = 1$, allora, dall'equazione (19), $\frac{\alpha}{\varphi\left(\frac{Log_{10}(S_{a_\alpha})-\mu_{X_l}}{\sigma_{X_l}}\right)} = 1$ e quando si calcola

$$a = log_{10}(y_\alpha) - b\cdot log_{10}(S_{a_\alpha}) - \varphi^{-1}\left(\frac{\alpha}{\varphi\left(\frac{Log_{10}(S_{a_\alpha})-\mu_{X_l}}{\sigma_{X_l}}\right)}\right)\cdot\sigma_Y\text{, si ha ancora }\varphi^{-1}(1) = \infty.$$

Per questa ragione, come schematizzato nel diagramma di flusso della figura 4. 9 (b), per il calcolo di PL_{θ_1} e quindi per la costruzione della curva di design, è stato implementato un controllo secondo il quale se $\frac{\alpha}{\varphi\left(\frac{Log_{10}(S_{a_\alpha})-\mu_{X_l}}{\sigma_{X_l}}\right)}$ è diverso da 1 allora viene esplicitato il parametro *a*, altrimenti viene esplicitato il parametro μ_{X_l}.

Risultati ottenuti

L'implementazione di entrambe le metodologie per l'analisi dei dati di fatica ha permesso il confronto dei due metodi e l'analisi delle somiglianze e delle differenze. In particolare, sono state confrontate le curve mediane di Wöhler, R50, e le curve R90C90 cioè le curve al di sotto delle quali il 90 % delle volte, il 90% dei campioni sopravvive. Inoltre, sono stati confrontati i parametri caratteristici del metodo della Massima Verosimiglianza, cioè i 5 parametri *a*, *b*, σ_Y per la regione di vita finita e μ_{X_l} e σ_{X_l} per la regione di vita infinita della curva, con gli stessi parametri derivati per il metodo tradizionale basato sulla regressione lineare per la regione di vita finita e sul metodo Dixon-Mood per l'analisi dello Staircase. In particolare, $A = a$, $B = b$, $s = \sigma_Y$, $\mu = \mu_{X_l}$ e $\sigma = \sigma_{X_l}$ dove *A*, *B*, *s*, μ e σ sono i parametri usati nel metodo tradizionale. È stato confrontato anche il valore del limite di fatica R90C90 per i due modelli, considerando per il metodo tradizionale il valore $S_{e,R,C}$ determinato dalla formula (12) mentre per il metodo basato sul Profile Likelihood, è stata considerata la media degli ultimi tre valori della curva R90C90, quando la curva mostra un asintoto orizzontale, oppure il valore della curva R90C90 in corrispondenza del numero di cicli di runout quando la curva non mostra un andamento asintotico con i dati disponibili. Per confrontare i due modelli sono stati usati i datasets, forniti dal CRF, provenienti da prove di fatica su 12 diversi tipi di materiali. Sono qui presentati 3 dei casi più esemplificativi analizzati. I valori degli stress per ogni dataset analizzato sono stati normalizzati per riservatezza aziendale.

Il primo dataset è relativo al TRIP1000 cioè un acciaio a plasticità indotta da trasformazione (Transformation Induced Plasticity steel) la cui resistenza a fatica è stata determinata tramite prove di fatica assiali condotte a temperatura ambiente, con un rapporto di carico R = -1, e una frequenza di 50 Hz. La Figura 8 mostra i risultati sperimentali delle prove effettuate e confronta le curve R50 e R90C90 calcolate con entrambi i metodi, mentre nella Tabella 1 è riportato il confronto dei parametri calcolati con i due metodi. Ci si riferisce all'approccio tradizionale come Model I e all'approccio basato sulla Massima Verosimiglianza come Model II.

Tabella 1 Confronto dei parametri della curva SN per il TRIP1000

a		b		σ_Y		μ_{X_l}normalizzato		σ_{X_l}		R90C90 normalizzato	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
74.2739	85.6732	-25.6623	-29.9005	0.2400	0.2489	0.9329	0.9325	7.0820	6.5221	0.899812	0.899626
Δ :	**13.31%**	**Δ :**	**14.17%**	**Δ :**	**3.57%**	**Δ :**	**0.04%**	**Δ :**	**8.59%**	**Δ :**	**0.02%**

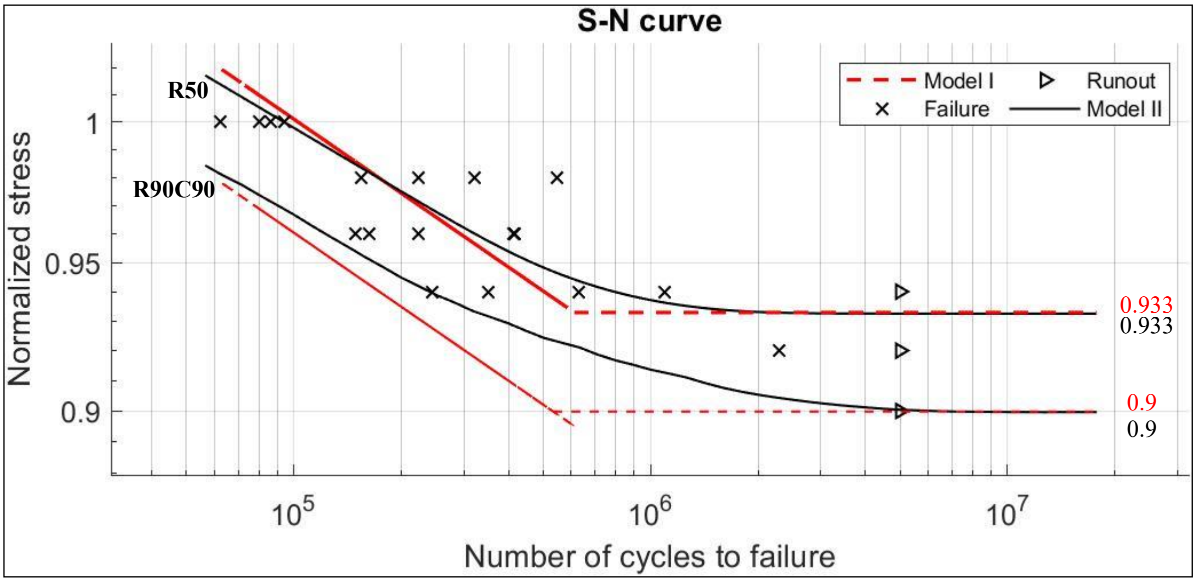


Figura 8 Risultati sperimentali, e curve R50, R90C90 per il TRIP1000 calcolate con entrambi i modelli

I valori del limite di fatica R50 e R90C90 sono praticamente coincidenti con una differenza inferiore allo 0,05 %, mentre i valori dei parametri della regione di vita finita sono leggermente diversi a causa del fatto che il Modello I, per convenzione interna del CRF, non considera la rottura a (0,92;2,29e6) che è al di sotto del valore mediano del limite di fatica 0,933. In Figura 9, invece, è stato riportato il grafico delle curve R50 e R90C90 in cui, per il Modello II è stato usato un dataset ridotto rispetto al Modello I, per cui invece è stato usato il dataset completo, al fine di dimostrare che nonostante limitate variazioni nei parametri, si ottengono risultati simili rispetto al caso precedente in cui è stato considerato l'intero dataset per entrambi i Modelli.

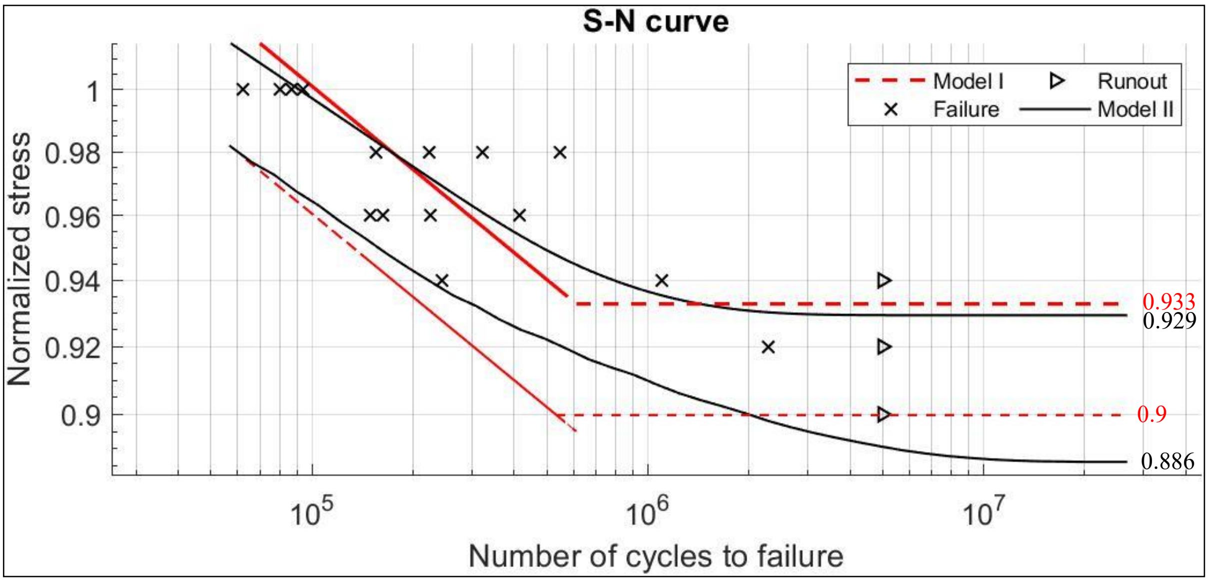


Figura 9 Curve R50 e R90C90 calcolate considerando un dataset ridotto per il Modello II e il dataset completo per il Modello I

Questo comporterebbe un risparmio di tempo di 3 giorni e 19 ore sulla caratterizzazione a fatica del TRIP1000. La tabella 2 mostra il confronto dei parametri in questa condizione. Dopo aver rimosso alcuni dati, è chiaro che le differenze nei valori dei parametri, ad eccezione di σ_{X_l}, aumentano e, soprattutto la curva R90C90 tracciata con il Modello II, mostra un asintoto a livelli di tensione inferiori. Questo perché, con un numero inferiore di prove, per avere una confidenza del 90 % che il 90 % dei provini sopravviva sotto questo livello di tensione, la curva si sposta inevitabilmente verso il basso.

Tabella 2 Confronto dei parametri per il TRIP1000 senza considerare 6 prove per il calcolo con il Modello II

a		b		σ_Y		μ_{X_l} normalizzato		σ_{X_l}		R90C90 normalizzato	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
74.2739	89.2218	-25.6623	-31.2193	0.2400	0.2661	0.9329	0.9294	7.0820	7.3182	0.899812	0.886231
Δ :	**16.75%**	**Δ :**	**17.80%**	**Δ :**	**9.82%**	**Δ :**	**0.37%**	**Δ :**	**3.23%**	**Δ :**	**1.53%**
		Saved time[day.h]:						**3.19**			

Il secondo dataset è relativo invece al 6060-T66 ovvero una lega di alluminio soggetta ad una tempra T66. Campioni a clesssidra con K_t di 1.06 di questo materiale sono stati sottoposti a prove di fatica assiali condotte a temperatura ambiente, con rapporto di carico R = -1 e frequenza di 100 Hz.

Come si può vedere dal confronto delle curve mediane in Figura 10, i due metodi danno risultati simili con un limite di fatica praticamente coincidente. Tuttavia, se si confrontano le curve R90C90 si nota una differenza significativa tra i due modelli. Infatti, in questo caso, con i dati disponibili, il Modello II, a ragion veduta, presenta il ginocchio della curva a un numero maggiore di cicli e trova un limite di fatica dopo un numero di cicli di 1e8, al contrario del Modello I per il quale si stabilisce una resistenza a fatica al numero prefissato di cicli, N_G = 1e7 che però non garantisce una condizione di sicurezza. La ragione di questa differenza può essere dovuta alla presenza della rottura, (0,64;9,76e6), molto vicino al numero di cicli di runout scelto. I valori dei parametri delle curve SN per i due modelli sono confrontati in Tabella 3.

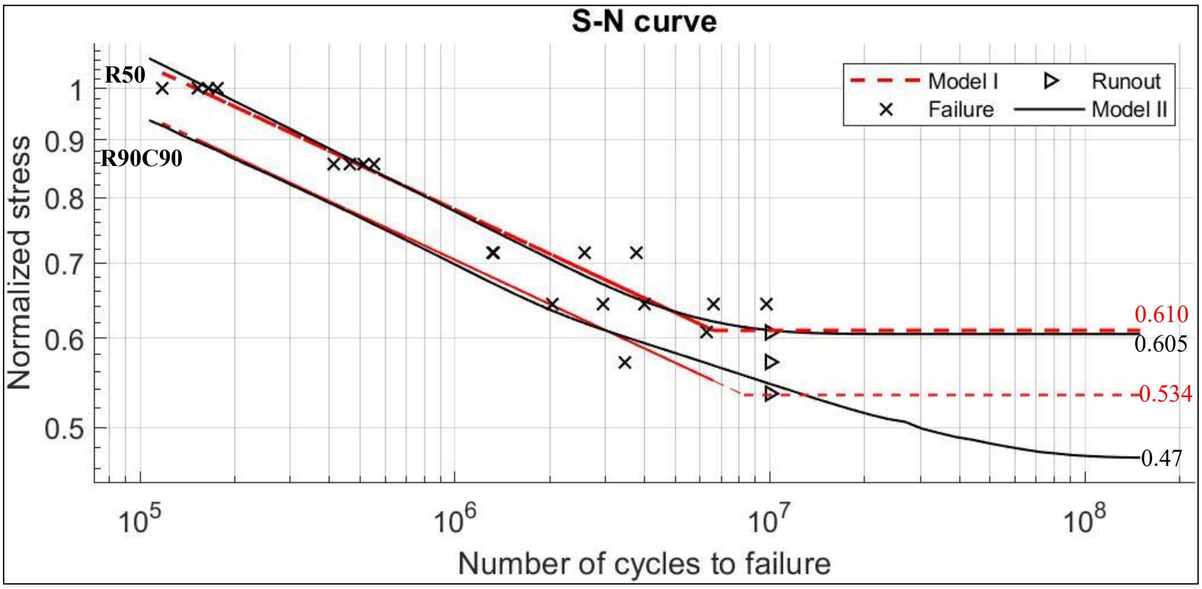


Figura 10 Risultati sperimentali, e curve R50, R90C90 per il 6060-T66 calcolate con entrambi i modelli

Tabella 3 Confronto dei parametri della curva SN per il 6060-T66

a		*b*		σ_Y		μ_{X_l} normalizzato		σ_{X_l}		R90C90 normalizzato	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
21.5802	20.5295	-7.6425	-7.1328	0.1777	0.1989	0.6097	0.6051	4.5329	4.6845	0.534157	0.46975
Δ :	**5.12%**	**Δ :**	**7.15%**	**Δ :**	**10.62%**	**Δ :**	**0.76%**	**Δ :**	**3.24%**	**Δ :**	**13.71%**

Il terzo dataset analizzato è invece relativo al G-AS7C3.5GM cioè una lega di alluminio usata dal CRF per testate motore. Questo materiale è stato sottoposto a prove di fatica assiali ad una temperatura di 150 °C, rapporto di carico -1 e frequenza di 150 Hz. Come riportato in Tabella 4 e in Figura 11, la differenza nei due modelli è piccola, ma in questo caso il metodo basato sul principio della Massima Verosimiglianza non riesce a trovare un limite di fatica con i dati a disposizione.

Tabella 4 Confronto dei parametri della curva SN per il G-AS7C3.5GM

a		*b*		σ_Y		μ_{X_l}normalizzato		σ_{X_l}	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
30.1326	29.6988	-12.1389	-11.9110	0.4156	0.4119	0.6227	0.6101	4.0349	4.4960
Δ :	**1.46%**	**Δ :**	**1.91%**	**Δ :**	**0.90%**	**Δ :**	**2.06%**	**Δ :**	**10.26%**

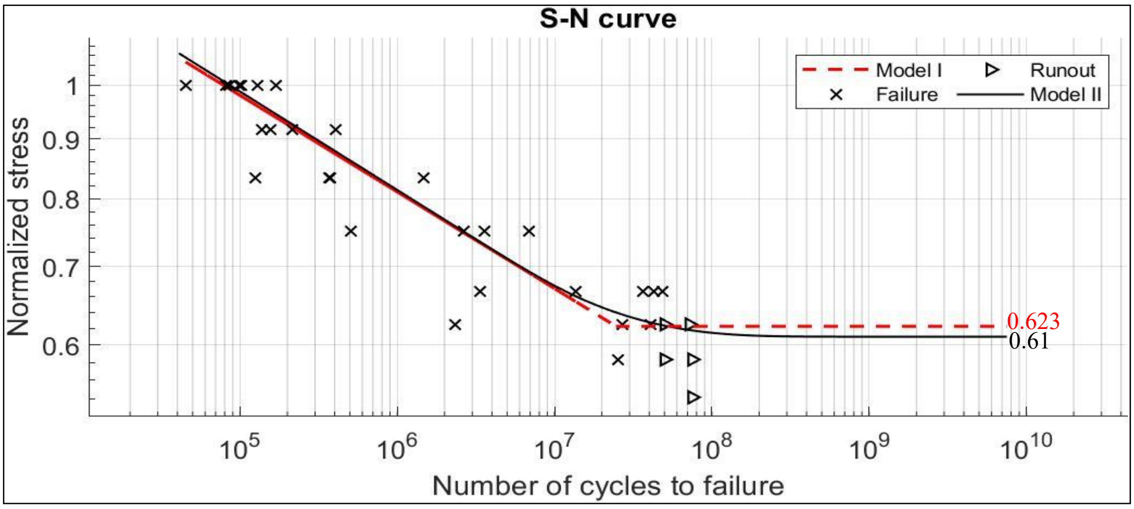


Figura 11 Risultati sperimentali e curve R50 calcolate con entrambi i metodi per il G-AS7C3,5GM @150°C

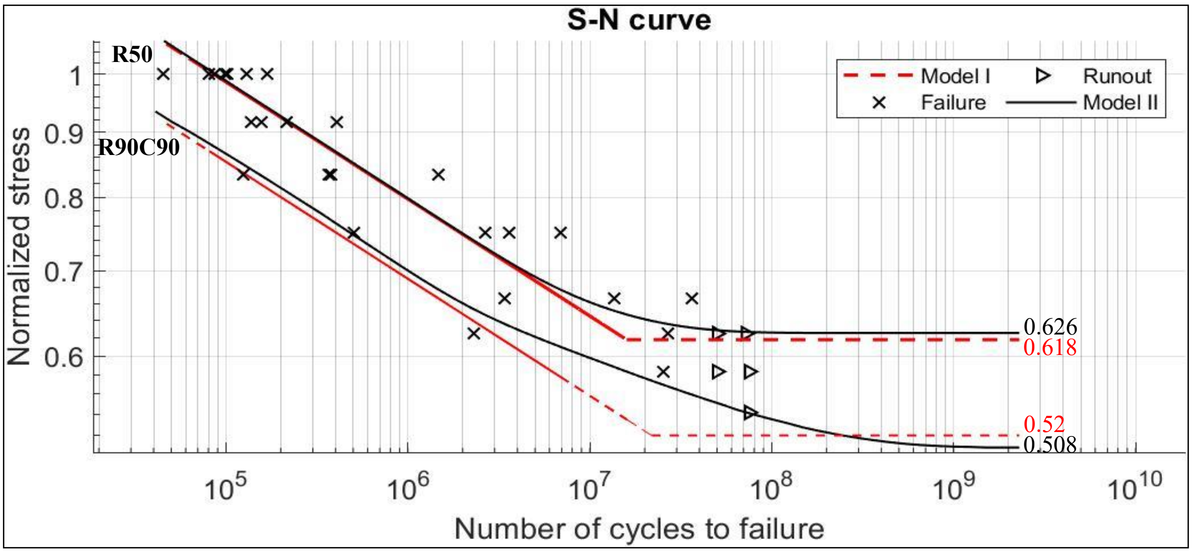


Figura 12 Curve R50 e R90C90 senza considerare le ultime 3 rotture in entrambi i modelli

Questo è probabilmente dovuto alla presenza di rotture molto vicine al numero scelto di cicli di runout, N_G = 5e7. Infatti, il metodo della Massima Verosimiglianza non riesce a dire con il 90% di confidenza che il 90% delle rotture si verifica sopra un certo valore di tensione, con questi dati a disposizione. Sarebbero quindi necessari più test effettuati ad un numero di cicli maggiore per avere più confidenza nei risultati. Infatti, se non considerassero le ultime tre rotture, cioè i dati (0,667;4,87e7), (0,667;4,3e7) e (0,625;4,05e7), avremmo le curve R50 e R90C90 come mostrato in Figura 12, in cui il Modello II è in questo caso in grado di calcolare la curva R90C90, ma comunque raggiunge un plateau a numeri di cicli superiori (maggiore di 1e9 cicli).

Dai risultati si può notare che quando le ultime rotture riscontrate durante la prova sono lontane dal numero di cicli di runout scelto e i runout sono lontani dal ginocchio della curva R50, i valori calcolati con entrambi i modelli sono confrontabili tra loro con una differenza che, in ogni caso analizzato, è inferiore al 2 %. Tuttavia, quando ci sono rotture vicino N_G, la stima del limite di fatica del Modello II è inferiore a quella del Modello I, e la differenza diventa quasi sempre ancora più significativa quando si considera il limite R90C90. Questo perché, nella regione di vita infinita della curva di Wöhler, mentre il Modello I usa solo l'informazione che un provino si è rotto o non si è rotto a un certo livello di tensione, il Modello II tiene conto anche del numero di cicli a rottura. Così, nel caso in cui le rotture sono sufficientemente lontane dal numero di cicli per il runout, allora la probabilità che il logaritmo della vita a fatica ad un certo livello di tensione S_a, sia inferiore a $\log_{10}(N_G)$ cioè il fattore $\varphi\left(\frac{Log_{10}(N_G)-(a+b\cdot Log_{10}(S_a))}{\sigma_Y}\right)$ è circa uguale a uno e la differenza tra i due modelli è trascurabile perché, in entrambi i casi, la probabilità a rottura può essere espressa come la probabilità che lo stress attuale sia superiore al limite di fatica ovvero con il termine $\varphi\left(\frac{Log_{10}(S_a)-\mu_{X_l}}{\sigma_{X_l}}\right)$. Al contrario, quando ci sono rotture vicino il numero di cicli di runout, il fattore $\varphi\left(\frac{Log_{10}(N_G)-(a+b\cdot Log_{10}(S_a))}{\sigma_Y}\right)$ è minore di uno e quindi il termine $\varphi\left(\frac{Log_{10}(S_a)-\mu_{X_l}}{\sigma_{X_l}}\right)$ nell'equazione (16) aumenta leggermente e di conseguenza μ_{X_l} diminuisce e questo porta ad una differenza nei due modelli.

Studio di simulazione

Per valutare quale dei due metodi fosse più accurato, si è effettuato uno studio di simulazione. In particolare, sono stati simulati dataset virtuali costruiti a partire da una distribuzione nota, cioè partendo da valori "veri" per i parametri a, b, σ_Y, μ_{X_l}, σ_{X_l}. Una volta calcolati i dataset per la regione di vita finita e di vita infinita della curva S-N, sono stati applicati i due metodi illustrati per stimare i parametri di interesse. Questa procedura viene poi ripetuta un gran numero di volte (di default 100 runs) al fine di avere una certa confidenza statistica dei risultati. Come output della simulazione, per ogni parametro e per ogni metodo di stima, è stato anche calcolato un errore relativo di stima (REE), espresso come la differenza tra il valore del parametro stimato e il valore vero rispetto al valore vero, cioè utilizzando il parametro dell'equazione (21).

$$Relative\ estimation\ error\ [\%] = \frac{Estimated\ value - True\ value}{True\ value} \cdot 100 \tag{21}$$

In Tabella 5 sono riassunti i risultati di una simulazione effettuata su 100 dataset in cui si è considerato, nella generazione dei dataset, un numero di campioni per il tratto vita infinita pari a 15 e un passo dello Staircase pari alla deviazione standard reale del limite di fatica. I risultati mostrano, per ogni parametro e per ciascuno dei due metodi, il valore massimo dell'errore relativo, il campo di variazione definito come la differenza tra il valore massimo e minimo del parametro stimato, la media campionaria di ogni parametro stimato e l'errore medio stimato (MEE) definito come:

$$Mean\ estimated\ error\ [\%] = \frac{Mean\ of\ estimated\ value - True\ value}{True\ value} \cdot 100 \tag{22}$$

Tabella 5 Risultati riassuntivi dell'analisi di 100 dataset simulati con $n = 15$, $\Delta S = \overline{\sigma_{X_l}}$

	a		***b***		σ_Y		μ_{X_l}		σ_{X_l}		***R90C90***	
Model	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**
Max REE (%)	30.8%	53.1%	55.52%	93.47%	37.91%	45.46%	8.95%	9.67%	81.48%	223.84%	35.67%	49.38%
Range of variation	7.3	12.7	3.796	6.483	0.190	0.208	11.626	11.457	7.456	14.955	28.070	37.313
Parameter mean	12.044	11.748	-3.455	-3.307	0.246	0.253	72.595	73.643	4.443	5.761	60.731	60.203
True value	12.271		-3.569		0.255		72.5		5.6		65.332	
MEE (%)	1.8%	4.3%	3.2%	7.4%	3.8%	0.8%	0.1%	1.6%	20.7%	2.9%	7.0%	7.9%

Per ogni parametro analizzato, il Modello I ha un valore di errore relativo massimo più alto rispetto al Modello II e un range di variazione più ampio in cui i valori sono distribuiti. Il Modello II ha anche un errore medio stimato (MEE) minore per i parametri *a*, *b*, μ_{X_l}e *R90C90* mentre il Modello I ha valori medi per le deviazioni standard che sono più vicini ai valori reali. Tuttavia, la dispersione nelle stime delle deviazioni standard è maggiore, soprattutto se si considera la deviazione standard del limite di fatica.

La stessa analisi è stata poi ripetuta variando lo step size e il numero di campioni utilizzati per lo Staircase e i risultati delle analisi sono riportate nelle Figure 6.2-6.10 del testo in cui sono stati confrontati i valori medi dei parametri stimati e la loro distribuzione rispetto al valore vero per ognuno dei due metodi. In Figura 13 e 14 sono riportati, a titolo d'esempio, i confronti dei risultati di simulazione per il parametro μ_{X_l}. In Figura 13 si può vedere il confronto delle stime di μ_{X_l} con i due modelli al variare dello step adottato nello Staircase.

Si vede come il Modello I presenta un valore medio dei parametri stimati che è più vicino al valore vero per passi più alti, mentre il Modello II è più accurato per passi uguali alla deviazione standard vera. Inoltre, il Modello II presenta quasi sempre meno dispersione ed è più accurato nella stima di μ_{X_l}rispetto al Modello I tranne quando ΔS =10 MPa, dove il Modello I è superiore.

Figura 14 invece riporta i risultati della simulazione per lo stesso parametro in funzione del numero di provini utilizzati nello Staircase. Si vede come, per entrambi i modelli, i valori medi di μ_{X_l}sono comparabili tra loro, e la dispersione si riduce aumentando il numero di provini per la prova. Inoltre, si può notare che il Modello II, indipendentemente dal numero di campioni, presenta risultati più simili tra loro e molto più vicini al valore vero rispetto al modello I.

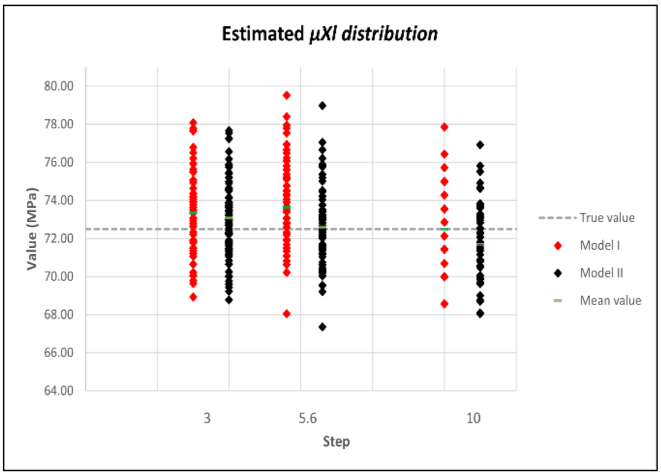


Figura 13 Risultati di simulazione per il parametro μ_{X_l} per ogni valore di step adottato

(a) (b)

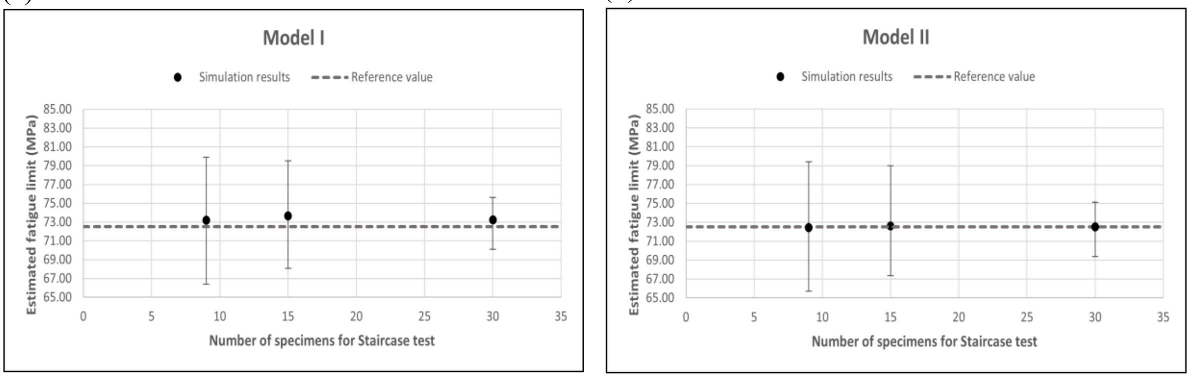


Figura 14 Stima del valore mediano del limite di fatica in funzione del numero di provini utilizzati nello Staircase per il Modello I (a) e II (b)

Conclusioni

Alla fine delle numerose analisi di datasets sperimentali e simulati effettuate in questo lavoro di tesi, è utile evidenziare i principali risultati ottenuti e gli obiettivi raggiunti. Come si è visto, la resistenza a fatica di un materiale può essere determinata attraverso la valutazione della curva S-N. Per la costruzione di tale curva sono stati esaminati due modelli: il primo prevede una semplice regressione lineare dei dati per la regione di vita finita utilizzando il metodo dei minimi quadrati e il metodo Staircase per la determinazione del limite di fatica, il secondo prevede l'utilizzo di un metodo basato sul principio della Massima Verosimiglianza per la determinazione della curva S-N completa. Il primo metodo è attualmente molto usato a livello industriale a causa della semplicità del protocollo di prova e dell'analisi dei risultati. Il secondo, pur richiedendo un'analisi dei dati leggermente più complessa, è in grado di massimizzare l'informazione contenuta in ogni dataset sperimentale. Infatti, se nel primo metodo l'unica informazione che viene utilizzata per la determinazione del limite di fatica (che in realtà è una resistenza a fatica ad un numero di cicli fissato) è la rottura o la non rottura dei campioni (a seconda di quale sia l'evento meno frequente), il secondo metodo utilizza informazioni aggiuntive tenendo conto del numero di cicli a rottura di ogni prova eseguita. Inoltre, nella progettazione dello Staircase, ci sono alcune limitazioni, cioè il valore di stress iniziale non deve essere troppo lontano dal limite di fatica, che si sta calcolando, e il passo deve essere nel range $0{,}5\cdot\sigma$ - $2\cdot\sigma$, il che richiede una conoscenza preliminare della deviazione standard del limite di fatica. D'altra parte, i metodi basati sul principio della Massima Verosimiglianza, non richiedono alcuna stima iniziale dei parametri né una successiva verifica e quindi consentono una maggiore libertà nella progettazione delle prove.

Poiché l'analisi dei risultati mediante il principio della Massima Verosimiglianza richiede un approccio iterativo, è stata sviluppata un'applicazione user-friendly che, una volta caricati i file relativi alle prove sperimentali effettuate in formato txt, consentisse di analizzare i dati con entrambi i metodi e di fornire in output le curve S-N corrispondenti a qualsiasi livello desiderato di affidabilità e confidenza e un file excel con un riepilogo di tutti i risultati ottenuti.

L'analisi dei dataset sperimentali ha mostrato che quando non ci sono rotture vicino il numero di cicli di runout e i runout sono lontani dal ginocchio della curva, allora i risultati, in termini di forma della curva S-N e di limite di fatica, sono molto simili tra loro. Quando, invece, ci sono rotture vicino il numero scelto di cicli di runout, allora i valori del limite di fatica calcolati con i due metodi differiscono tra loro, specialmente quando si considera la curva S-N ad un certo livello di confidenza (come la curva R90C90). Questo perché i due modelli non stimano lo stesso limite. Il metodo Staircase stima la resistenza a fatica al numero di cicli di runout, N_G, mentre il secondo modello permette di visualizzare il reale andamento suggerito dai dati sperimentali. Nei casi in cui sorge questa differenza nei modelli, vuol dire che la prova Staircase è stata effettuata ad un livello di run-out troppo basso. In particolare, si è notato che nei casi in cui i due modelli differiscono, il limite di fatica calcolato con il metodo basato sul principio della Massima Verosimiglianza è inferiore a quello calcolato con lo Staircase. La ragione di ciò è che, quando le rotture si verificano lontano dal numero di cicli di runout scelto, la probabilità che il numero effettivo di cicli a rottura sia inferiore al numero di cicli di runout, cioè il fattore $\varphi\left(\frac{Log_{10}(N_G)-(a+b\cdot Log_{10}(S_a))}{\sigma_Y}\right)$, nel modello considerato per il metodo della Massima Verosimiglianza, è approssimativamente uguale ad uno e quindi, per entrambi i modelli, la probabilità di rottura può essere espressa come $\varphi\left(\frac{Log_{10}(S_a)-\mu_{X_l}}{\sigma_{X_l}}\right)$ cioè come la probabilità che la sollecitazione applicata sia superiore al limite di fatica. D'altra parte, quando alcuni campioni si rompono vicino il numero di cicli di runout, il fattore $\varphi\left(\frac{Log_{10}(N_G)-(a+b\cdot Log_{10}(S_a))}{\sigma_Y}\right)$ è inferiore ad uno, e quindi il fattore $\varphi\left(\frac{Log_{10}(S_a)-\mu_{X_l}}{\sigma_{X_l}}\right)$ nel modello che prevede l'utilizzo della Massima Verosimiglianza per la stima dei parametri, aumenta e di conseguenza il valore mediano del limite di fatica, μ_{X_l}, diminuisce.

Per alcuni datasets, inoltre, si è visto che il metodo che utilizza il principio della Massima Verosimiglianza non è riuscito a trovare, a quel dato livello di confidenza, un limite di fatica. Questo non significa che il materiale non abbia un limite di fatica, ma che i dati a disposizione non permettono di dire che ci sia. In questi casi sarebbe infatti necessario effettuare prove fino a un numero di cicli superiore al numero di cicli di runout scelto. Infatti, anche se l'approccio basato sulla regressione lineare e lo Staircase permette di tracciare una retta orizzontale corrispondente alla resistenza a fatica a quel dato numero di cicli, questo può essere pericoloso perché i dati non mostrano un comportamento asintotico, come suggerito dal metodo basato sulla Massima Verosimiglianza, che invece descrive il reale andamento dei dati. È chiaro che anche con questo metodo, quando non è possibile trovare un asintoto per la curva S-N, sarebbe possibile ottenere la pendenza del tratto lineare, il ginocchio della curva e "forzare" una resistenza a fatica ad un dato numero di cicli, cioè determinare quelli che sono i 3 parametri che vengono solitamente utilizzati per la progettazione a fatica di un componente, ma questo non garantirebbe necessariamente una condizione di sicurezza perché i dati non suggeriscono la presenza di un plateau.

È stato anche investigato un altro vantaggio dei metodi basati sul principio della Massima Verosimiglianza, cioè quello di poter effettuare meno prove senza perdere troppe informazioni e quindi risparmiare tempo e costi per la caratterizzazione a fatica di un materiale. Per alcuni datasets sperimentali, sono stati infatti rimossi alcuni dei dati relativi allo Staircase e sono stati ottenuti risultati molto simili a quelli ottenuti considerando il dataset completo. Poiché i risultati ottenuti sono di particolare interesse, si suggeriscono verifiche simili per altri datasets al fine di validare questo risultato.

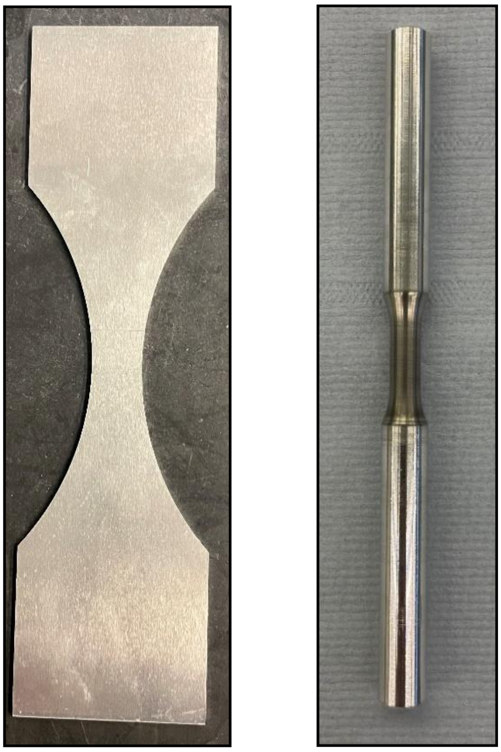
Una volta effettuate le analisi sui datasets sperimentali, per verificare quale dei due modelli fosse più preciso nel determinare i parametri della curva S-N, è stato effettuato uno studio di simulazione.

Datasets virtuali sono stati simulati a partire dai valori reali dei parametri da stimare, e sono stati poi analizzati con entrambi i metodi come se fossero dei datasets sperimentali, ma di cui si conoscevano già i valori veri dei parametri da stimare. I risultati delle simulazioni hanno mostrato che l'errore nella stima dei parametri con il metodo basato sulla Massima Verosimiglianza è quasi sempre inferiore a quello del metodo basato sulla regressione lineare e Staircase, e che anche nei casi in cui quest'ultimo metodo era superiore nella stima, presentava comunque più dispersione nei risultati. Questo è ragionevole perché nell'approccio tradizionale un'informazione importante come il numero di cicli alla rottura dei dati della regione di vita infinita non è presa in considerazione e quindi ci si aspettava che il campo di variazione dei parametri stimati fosse più ampio.

1 Experimental Methods

The fatigue test set-up followed the guidelines of ISO 12107:2012 [1]. The purpose of the high-cycle fatigue tests (HCF tests) is to determine the fatigue strength in the finite-life region and the infinite-life region i.e. to construct the Wöhler curve for the tested material. Fatigue tests are conducted at a given stress, S, on a series of well-prepared specimens to determine the fatigue life values for each of them. The number of specimens selected will depend on the purpose of the test and the availability of the test material. A set of 6-12 specimens is recommended for exploratory testing. For reliability purposes, however, at least 28 samples are recommended [2]. The test specimen may be loaded until failure occurs or the test is terminated even if no failure occurs when a high number of cycles, which is chosen according to the type of material and the field of employment, is reached.

The specimens used in CRF for fatigue testing were either flat or cylindrical, depending on the type of test required by the customer and the specimens available. Flat specimens are usually used when the material comes in sheets. The flat specimens are tapered so that all sections of the specimen are subjected to the same level of stress independently of the distance from the force application point, while the hourglass shape of the cylindrical specimens is necessary to provide a larger section in the clamping area where the stress concentration increases the local stress state, preventing fatigue failure from occurring there rather than in the controlled central section. The two types of specimens used are shown in Figure 1.1.



(a) (b)

Figure 1.1 Fatigue test specimen with flat (a) and cylindrical (b) section

Both types of specimens have a large fillet to reduce the notching effect ($k_t \cong 1$). The dimensions of the specimens used depend on various factors such as from which part of the workpiece the specimen was taken, the type of machine available for testing, the dispositions of the machine manufacturer itself etc. Normally, fatigue samples used to derive S-N curves have a mirror-polished surface to ensure that fatigue data is not affected by surface finish. In this sense, FCA's internal rules require the use of abrasive papers of different sizes until the machining witnesses, longitudinal to the applied stress, are no longer visible under the optical microscope. The fatigue tests carried out were flat bending fatigue tests and axial load fatigue tests. Figure 1.2 shows the resonant testing machine (CRACKTRONIC, Rumul) for the flat bending load tests (a), the servohydraulic equipment (MTS Landmark) for the axial load test (b) and the AMSLER vibrophorus, an economical alternative with extremely low energy consumption due to resonance, which was used for the axial load fatigue tests (c). The test conditions (such as frequency, stress ratio R, temperature) and the type of testing machines were chosen according to the type of material and the type of characterisation requirement.

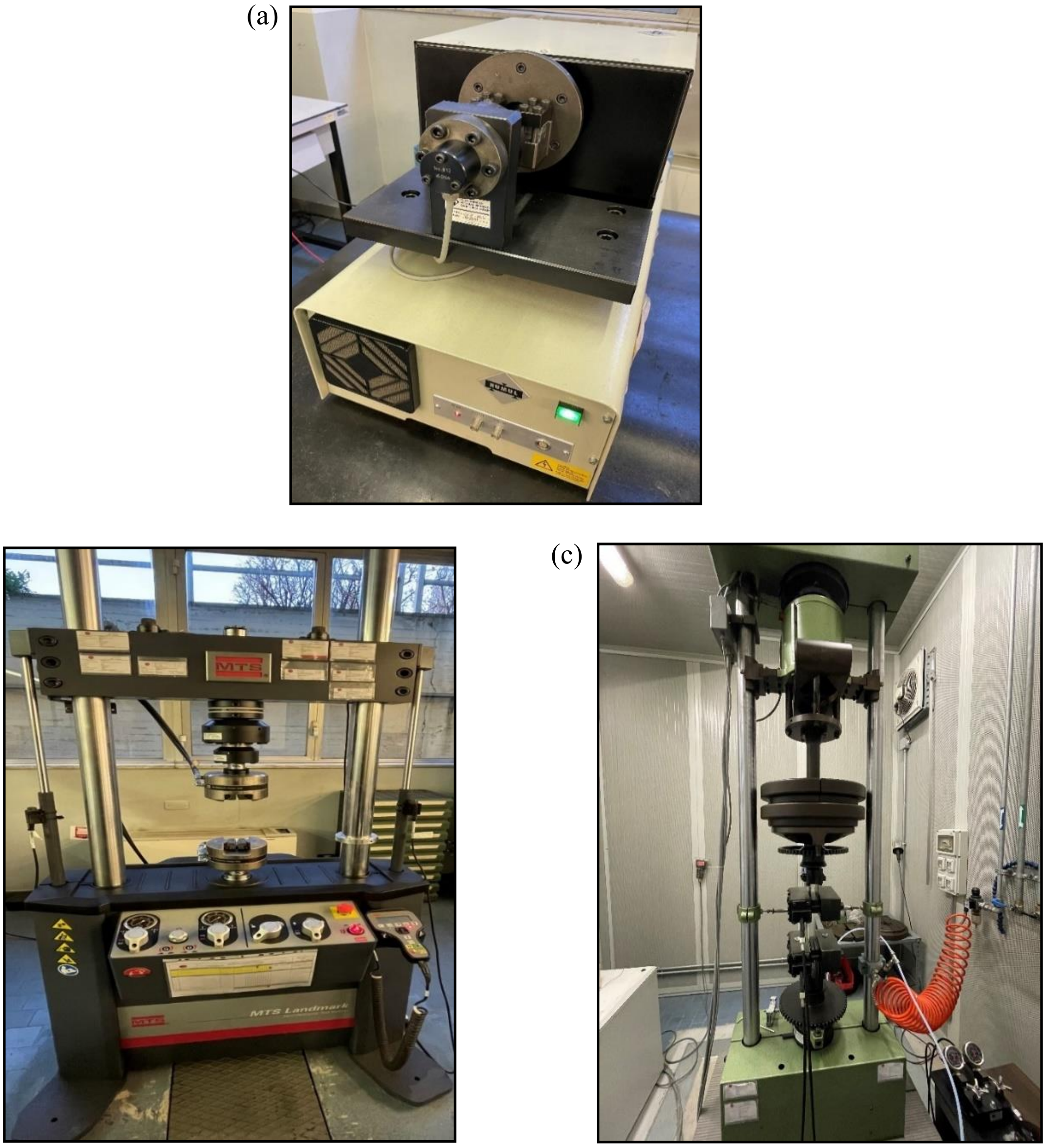


Figure 1.2 (a) Cracktronic resonant testing machine (b) MTS servohydraulic testing machine (c) AMSLER vibrophorus

2 Traditional approach

2.1 Analysis of fatigue data in finite life region

2.1.1 Median S-N curve

Once fatigue life data in the finite life region has been obtained from S–N tests, the generation of the finite life portion of the median S-N curves is done by a linear regression. The least-squares method for generating a line of best fit from the data is used. ASTM E739-91 standard is a common standard in S-N or ε-N curve construction, which considers the linear behavior, on log-log scale (equation (2.1)), of stress amplitude versus cycles to failure. It is assumed that the fatigue life follows a log-normal distribution, which means that its logarithm, log(Nf), follows a normal distribution, and that the variance of the logarithm of the fatigue life is constant over the tested range [3].

Fatigue test results are traditionally plotted with log(S_a) on the y-axis and log(N_f) on the x-axis. In curve fitting it is usually assumed that the parameter plotted on the x-axis is the independent variable and that plotted on the y-axis is the dependent variable, but in the case of fatigue data, the opposite is done by treating log(N_f) as the dependent variable [4].

The least-squares regression model is:

$$log_{10}(N) = log_{10}(A) + B \cdot \ log_{10}(S_a) \tag{2.1}$$

where B is the slope, $\log_{10}(A)$ is the intercept. In these equations, $log_{10}(S_a)$, is the stress amplitude, and can be replaced by strain if the aim is to find the ε-N counterpart relation while log(N) corresponds to fatigue life. Including runout specimens in this model is not recommended as mentioned by Barbosa et al. in [5]. The regression line can be expressed as:

$$\hat{Y} = \hat{A} + \hat{B} \cdot X \tag{2.2}$$

where the estimated values of $\hat{A}$ and $\hat{B}$ are obtained by minimizing the sum of the square of the distances between the observed and predicted Y data i.e.,

$$SSE = \Delta^2 = \sum_{i=1}^{n_s} \left(Y_i - \hat{Y}_i\right)^2 = \sum_{i=1}^{n_s} \left(Y_i - \hat{A} - \hat{B}X_i\right)^2 \tag{2.3}$$

where n_s is the number of test samples [3].

Equation (2.1) can be re-written in a form that is commonly used to describe median S-N curves in design rules:

$$S_a = S_f' \ (N)^b \tag{2.4}$$

where b is the fatigue strength exponent, and S_f' is the fatigue strength coefficient. By making the logarithms of equation (2.4) and rearranging, the following equation is obtained:

$$log_{10}(N) = -\frac{1}{b} \ log_{10}(S_f') + \frac{1}{b} log_{10}(S_a) \tag{2.5}$$

From the comparison of equation (2.5) and equation (2.2), it can be determined that X=$log_{10}(S_a)$ and Y=$log_{10}(N)$ while the statistical fatigue properties (b, S_f') are determined from the coefficients of least squares equation: b = ${}^{1}/_{\hat{B}}$, $S_f' = 10^{(-\hat{A} \cdot b)}$ [3].

2.1.2 Different reliability curve

Due to the dispersion of fatigue life, frequently median S-N curves are not sufficient for fatigue analysis and design, and so higher percentile S-N curves are required to ensure greater reliability. Indeed, the median curve, described in section 2.1.1, represents the curve for which 50% of the specimens fail above this curve and 50% below it, but one could equally draw a curve where only 10% of the specimens fail below the curve and 90% above it (R90 curve). The S-N curve with a different probability of survival can be constructed from a straight line with the same fatigue life exponent b, which is shifted horizontally to the left from the median line by a distance equal to z · s, where z is the desired *p*-th quantile and s is the standard deviation of fatigue life. z can be derived from the Z tables for a normal distribution or, as in this case, can be calculated using Matlab with the command '*norminv* (p)'.

If we consider the S-N curve with a reliability of 90 % (R90) i.e. the curve with 90 % of probability of survival and 10 % probability of failure, the logarithm of the fatigue life R90 can be determined by subtracting to the logarithm of the median fatigue life a value of $z \cdot s$ as shown in equation (2.6).

$$log_{10}\left(N_{f_{R90}}\right) = log_{10} - z \cdot s \tag{2.6}$$

where z in this case is equal to *norminv* (0.9) = 1.28. To calculate the S-N equation we need R90 fatigue strength coefficient $S'_{f,R90}$:

$$S'_{f,R90} = \frac{S_a}{\left(N_{f_{R90}}\right)^b} \tag{2.7}$$

where N_f in equation (2.6) and $S_a$in equation (2.7) are an arbitrarily chosen point on median S-N curve. Thus, the equation with 90% probability of survival is shown in equation (2.8):

$$S_a = S'_{f,R90}\left(N_f\right)^b \tag{2.8}$$

2.1.3 Design curve

In order to provide a greater safety margin against fatigue failure, however, design curves are usually used. A design curve is a curve for which most of the data lies above it. In general, a point on the design curve at a specific stress level X_i is constructed using equation (2.9) [3]:

$$Y_L(X_i) = \hat{Y}(X_i) - K \cdot s \tag{2.9}$$

where Y_L is defined as the lower limit of Y (=$log_{10}(N_f)$ at a given X_i (=$log_{10}(S_{a_i})$ and K is the so called one-sided tolerance limit factor. If we consider, for example, a reliability of P % and a confidence level of γ % this means that ($100 \cdot \gamma$) % of the time Y_L defines a point on the S-N curve which has a higher probability of survival than P %.

There are various choices for K, like considering $K = 2$ or $K = 3$ i.e. shifting the median S-N curve by a factor equal to two or three times the standard deviation associated with the experimental dataset [6], but the most common approach, also used at CRF, is the approximate Owen one-side tolerance limit. This method assumes that the design S-N curve has the same slope as the median S-N curve i.e. K_{Owen} it's not a function of stress level. This is not a particularly restrictive assumption because generally the variation of K_{Owen} factor within the stress range of the test is relatively small for sample sizes greater than six [7]. Owen [3] gives a formula for determining K_{Owen} factor.

$$K_{Owen} = K_D \cdot R_{Owen} \tag{2.10}$$

where,

$$K_D = c_1 K_R + K_C\sqrt{c_3 K_R^2 + c_2 a} \quad , \quad R_{Owen} = b_1 + \frac{b_2}{f^{b_3}} + b_4 \exp(-f) \tag{2.11}$$

in which

$$K_R = \varphi^{-1}(R)\,,\ K_C = \varphi^{-1}(C)\,, f = n_s - 2\,, a = \frac{1.85}{n_s} \tag{2.12}$$

R and C denote the reliability and confidence level, respectively, and $\varphi^{-1}(\cdot)$ is the inverse of the standard normal cumulative distribution function (cdf). b_i and c_i are empirical coefficients calculated by Shen et al.[7] and tabulated in Tables 2.1 and 2.2.

Table 2.1 Empirical coefficients b_i

Confidence Level (C)	b1	b2	b3	b4
0.95	0.9968	0.1596	0.60	-2.636
0.90	1.0030	-6.0160	3.00	1.099
0.85	1.0010	-0.7212	1.50	-1.486
0.80	1.0010	-0.6370	1.25	-1.554

Table 2.2 Empirical coefficients c_i

	c1	c2	c3
f<2	1	1	$\frac{1}{2f}$
f≥2	$1+\frac{3}{4(f-1.042)}$	$\frac{f}{f-2}$	$c_2 - c_1^2$

Based on the approximate Owen tolerance limits (i.e., Eqs. (2.10)),Williams et al. [8] derived a table for the K-factor which depends on the number of samples used for testing and the desired reliability and confidence level. This table is shown in Table 2.3.

Table 2.3 K-factor for the approximate Owen tolerance limits

ns	C=0.90			C=0.95		
	R=0.90	R=0.95	R=0.99	R=0.90	R=0.95	R=0.99
6	2.862	3.504	4.750	3.560	4.331	5.837
7	2.608	3.190	4.319	3.167	3.846	5.173
8	2.441	2.987	4.043	2.910	3.534	4.751
9	2.323	2.843	3.851	2.728	3.314	4.455
10	2.253	2.736	3.707	2.592	3.151	4.237
11	2.162	2.651	3.595	2.485	3.024	4.069
12	2.105	2.583	3.505	2.400	2.923	3.936
13	2.057	2.526	3.430	2.331	2.840	3.827
14	2.016	2.478	3.367	2.272	2.771	3.737
15	1.980	2.436	3.313	2.222	2.712	3.660
16	1.949	2.400	3.266	2.178	2.661	3.594
17	1.922	2.369	3.225	2.140	2.617	3.536
18	1.898	2.340	3.189	2.106	2.577	3.485
19	1.876	2.315	3.156	2.076	2.542	3.440
20	1.857	2.292	3.127	2.048	2.510	3.399
21	1.839	2.272	3.100	2.024	2.482	3.363
22	1.822	2.252	3.076	2.001	2.456	3.329
23	1.807	2.235	3.054	1.981	2.432	3.300
24	1.793	2.219	3.034	1.961	2.410	3.271
25	1.781	2.204	3.015	1.947	2.389	3.247
26	1.769	2.191	2.997	1.927	2.370	3.221
27	1.757	2.178	2.981	1.912	2.353	3.199
28	1.747	2.166	2.966	1.898	2.337	3.178
29	1.737	2.155	2.952	1.885	2.321	3.158
30	1.728	2.144	2.939	1.872	2.307	3.140

If we do fatigue tests on 20 samples (n_s=20) and we want to determine a lower-bound S–N curve with 90% probability of survival (R90) and 90% confidence level (C90) based on the approximate Owen tolerance limit method, we have to calculate the R90C90 fatigue life and fatigue strength coefficient as shown in equation (2.13) and (2.14).

$$log_{10}\left(N_{f_{R90C90}}\right) = log_{10}\left(N_f\right) - 1.857 \cdot s \tag{2.13}$$

$$S'_{f,R90C90} = \frac{S_a}{\left(N_{f_{R90C90}}\right)^b} \tag{2.14}$$

where N_f in equation (2.13) and $S_a$in equation (2.14) are an arbitrarily chosen point on median S-N curve. Thus, the equation of R90C90 curve is shown in equation (2.15):

$$S_a = S'_{f,R90C90}\left(N_f\right)^b \tag{2.15}$$

Figure 2.1 shows an example of the limited fatigue life region of S-N curves. Specifically, R50 curve is based on eqs. (2.4), R90 curve is based on eqs. (2.8) and R90C90 curve is based on eqs. (2.15). Stress values are normalized for corporate confidentiality.

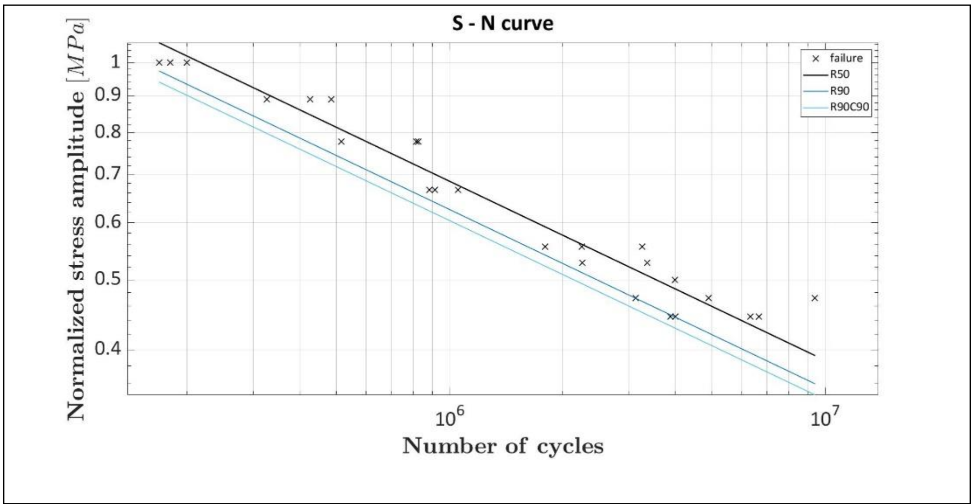


Figure 2.1 Example of the limited life region S-N curves based on traditional approach

2.2 Analysis of fatigue data in unlimited life region, Staircase testing approach

The aim of fatigue limit testing is to evaluate the statistical variability of the fatigue limit at a specific fatigue life. Endurance limit evaluation method that it's used in CRF is based on staircase testing method following ISO-12107/UNI 3964 std [1].
Staircase method is the most widely used for the analysis of infinite life data [3] because it is based on a simple protocol in which a specimen is tested at a given initial stress level for a given number of cycles or until failure, whichever comes first. If the first specimen has survived up to the expected number of cycles (number of runout cycles), the next specimen is tested at a stress level increased by ΔS. If, on the other hand, the specimen fails before the number of runout cycles, it will subsequently be tested at a stress level reduced by ΔS [9]. Therefore, each test depends on the result of the previous test and this is one of the limitations of this strategy because it makes it impossible to carry out the tests in parallel. This method does not take into account the actual number of cycles to failure, but the only information it considers is if the test specimen failed or not.

The staircase test, also known as up-an-down method, was first proposed by Dixon and Mood (1948) [9] for application to explosives testing. The tests were conducted at an initial height h_0, and if the weight detonated then the height for the next test was lowered otherwise it was raised by the same amount. Later this method was popularized by Little (1975) for application to fatigue testing. The original sample size proposed by Dixon-Mood for the application of this method was 40-50 specimens because this condition ensures that large sample theory, on which the analysis is based, can be applied [9]. However, Brownlee *et al* note that the distribution mean using Dixon-Mood analysis is "reasonably reliable even in samples as small as 5 to 10" [10]. In fact, the Japan Society of Mechanical Engineers (JSME) recommends a 6-specimen staircase for the determination of the fatigue limit [11].

The first test is conducted at a stress level that can be determined either by S-N data of similar materials or by experience. The step size adopted for the test, however, must be in the range ($0.5 \cdot \sigma \div 2 \cdot \sigma$) where σ is the standard deviation of the fatigue limit. Thus, an approximate knowledge of the standard deviation is necessary to specify the step size before testing. A poor choice of starting point and step size may cause uncertainty in the estimation of the fatigue limit and standard deviation and possibly require an additional number of tests, which would increase the costs of the experimental campaign.

A typical staircase test result is shown in Figure 2.2 where "O" is a runout specimen and "X" is a failure. In this example the number of cycles beyond which the sample is considered survival is 1e7.

Level	Stress Sa [Mpa]	Sequence number of specimens														
i		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
6																
5																
4																
3	**90**					X		X		X		X				X
2	**85**				O		O		O		O		X		O	
1	**80**	X		O										O		
0	**75**		O													
Δs [MPa]	**5**															
Number of cycles		3.46E+06	1.00E+07	1.00E+07	1.00E+07	6.61E+06	1.00E+07	2.04E+06	1.00E+07	4.01E+06	1.00E+07	2.95E+06	6.28E+06	1.00E+07	1.00E+07	9.76E+06

Figure 2.2 Example of the staircase fatigue data

When the tests have been completed, the statistical analysis of the results begins. The traditional method is the one proposed by Dixon and Mood [9] who used a simplified maximum likelihood estimation technique (described in Appendix A of [9]) to determine the mean and the standard deviation of the staircase test data. One of the necessary conditions for the application of this method is that the fatigue limit follows a normal distribution [9]. This condition seems to be very restrictive, however, a logarithmic or exponential transformation of the stress values can be applied to make the distribution normal. Denoting by n_i the number of the least frequent event (because in this method only failures or only survivals are considered) at stress level i, three quantities A, B and C can be calculated:

$$A = \sum_{i=0}^{n_i} n_i\,, \quad B = \sum_{i=0}^{n_i} i \cdot n_i\,, \quad C = \sum_{i=0}^{n_i} i^2 \cdot n_i \tag{2.16}$$

The estimate of the mean and standard deviation of the fatigue limit is then expressed in equation (2.17) - (2.29):

$$\mu = S_0 + \Delta S \cdot \left(\frac{B}{A} \pm 0.5\right) \tag{2.17}$$

$$\sigma = 1.62 \cdot \Delta S \left(\frac{A \cdot C - B^2}{A^2} + 0.029\right) \tag{2.18}$$

if $\frac{A \cdot C - B^2}{A^2} \geq 0.3$ or,

$$\sigma = 0.53 \cdot \Delta S \tag{2.19}$$

if $\frac{A \cdot C - B^2}{A^2} < 0.3.$

where the sign '+' in Eq. (2.17) indicates that the least frequent event is survival of the sample, while '-' indicates that the least frequent event is failure.

The lower-bound value associated with different reliability level can be determined by subtracting from the mean the $z \cdot \sigma$ factor in which *z* is the *p*-quantile corresponding to a probability of survival of $(p \cdot 100)$ % as shown in equation (2.20):

$$S_{e,R} = \mu - z \cdot \sigma \tag{2.20}$$

In the same way, the lower-bound value associated with a certain reliability and confidence levels can be determined by subtracting from the mean a factor *K* as follows:

$$S_{e,R,C} = \mu - K \cdot \sigma \tag{2.21}$$

where the K factor can be determined using Liebermann's tables [12] given in Tables 2.4.

Table 2.4 K-Factor for One-Side Lower-Bound Tolerance Limit for a normal distribution

	C=0.75					C=0.90				
n	R=0.75	R=0.90	R=0.95	R=0.99	R=0.999	R=0.75	R=0.90	R=0.95	R=0.99	R=0.999
3	1 464	2 501	3 152	4 396	5 805	2 602	4 258	5.31	7.34	9 651
4	1 256	2 134	2.68	3 726	4.91	1 972	3 187	3 967	5 437	7 128
5	1 152	1 961	2 463	3 421	4 507	1 698	2 742	3.40	4 666	6 112
6	1 087	1.86	2 336	3 243	4 273	1.54	2 494	3 091	4 242	5 556
7	1 043	1 791	2.25	3 126	4 118	1 435	2 333	2 894	3 972	5 201
8	1.01	1.74	2.19	3 042	4 008	1.36	2 219	2 755	3 783	4 955
9	0.984	1 702	2 141	2 977	3 924	1 302	2 133	2 649	3 641	4 772
10	0.964	1 671	2 103	2 927	3 858	1 257	2 065	2 586	3 532	4 629
11	0.947	1 646	2 073	2 885	3 804	1 219	2 012	2 503	3 444	4 515
12	0.933	1 624	2 048	2 851	3.76	1 188	1 966	2 448	3 371	4.42
13	0.919	1 606	2 026	2 822	3 722	1 162	1 928	2 403	3.31	4 341
14	0.909	1 591	2 007	2 796	3.69	1 139	1 895	2 363	3 257	4 274
15	0.899	1 577	1 991	2 776	3 661	1 119	1 866	2 329	3 212	4 215
16	0.891	1 566	1 977	2 756	3 637	1 101	1 842	2 299	3 172	4 164
17	0.883	1 554	1 964	2 739	3 615	1 085	1.82	2 272	3 136	4 118
18	0.876	1 544	1 951	2 723	3 595	1 071	1.8	2 249	3 106	4 078
19	0.87	1 536	1 942	2.71	3 577	1 058	1 781	2 228	3 078	4 041
20	0.865	1 528	1 933	2 697	3 561	1 046	1 765	2 208	3 052	4 009
21	0.859	1.52	1 923	2 686	3 545	1 035	1.75	2.19	3 028	3 979
22	0.854	1 514	1 916	2 675	3 532	1 025	1 736	2 174	3 007	3 952
23	0.849	1 508	1 907	2 665	3.52	1 016	1 724	2 159	2 987	3 927
24	0.845	1 502	1 901	2 656	3 509	1 007	1 712	2 145	2 969	3 904
25	0.842	1 496	1 895	2 647	3 497	0.999	1 702	2 132	2 952	3 882
30	0.825	1 475	1 869	2 613	3 545	0.966	1 657	2.08	2 884	3 794
35	0.812	1 458	1 849	2 588	3 421	0.942	1 623	2 041	2 833	3.73
40	0.803	1 445	1 834	2 568	3 395	0.923	1 598	2.01	2 793	3 679
45	0.795	1 435	1 821	2 552	3 375	0.908	1 577	1 986	2 762	3 638
50	0.788	1 426	1 811	2 538	3 358	0.894	1.56	1 965	2 735	3 604

	C=0.95					C=0.99				
n	R=0.75	R=0.90	R=0.95	R=0.99	R=0.999	R=0.75	R=0.90	R=0.95	R=0.99	R=0.999
3	3 804	6 158	7 655	10.55	13.86	0	0	0	0	0
4	2 619	4 163	5 145	7 042	9 215	0	0	0	0	0
5	2 149	3 407	4 202	5 741	7 501	0	0	0	0	0
6	1 895	3 006	3 707	5 062	6 612	2 849	4 408	5 409	7 334	9.55

7	1 732	2 755	3 399	4 641	6 061	2.49	3 856	4.73	6 411	8 348
8	1 617	2 582	3 188	4 353	5 686	2 252	3 496	4 287	5 811	7 566
9	1 532	2 454	3 031	4 143	5 414	2 085	3 242	3 971	5 389	7 014
10	1 465	2 355	2 911	3 981	5 203	1 954	3 048	3 739	5 075	6 603
11	1 411	2 275	2 815	3 852	5 036	1 854	2 897	3 557	4 828	6 284
12	1 366	2.21	2 736	3 747	4.9	1 771	2 773	3.41	4 633	6 032
13	1 329	2 155	2.67	3 659	4 787	1 702	2 677	3.29	4 472	5 826
14	1 296	2 108	2 614	3 585	4.69	1 645	2 592	3 189	4 336	5 561
15	1 268	2 068	2 566	3.52	4 607	1 596	2 521	3 102	4 224	5 507
16	1 242	2 032	2 523	3 463	4 534	1 553	2 458	3 028	4 124	5 374
17	1.22	2 001	2 486	3 415	4 471	1 514	2 405	2 962	4 038	5 268
18	1.2	1 974	2 453	3.37	4 415	1 481	2 357	2 906	3 961	5 167
19	1 183	1 949	2 423	3 331	4 364	1.45	2 315	2 855	3 893	5 078
20	1 167	1 926	2 396	3 295	4 319	1 424	2 275	2 807	3 832	5 003
21	1 152	1 905	2 371	3 262	4 276	1 397	2 241	2 768	3 776	4 932
22	1 138	1 887	2.35	3 233	4 238	1 376	2 208	2 729	3 727	4 866
23	1 126	1 869	2 329	3 206	4 204	1 355	2 179	2 693	3.68	4 806
24	1 114	1 853	2 309	3 181	4 171	1 336	2 154	2 663	3 638	4 755
25	1 103	1 838	2 292	3 158	4 143	1 319	2 129	2 632	3 601	4 706
30	1 059	1 778	2.22	3 064	4 022	1 249	2 029	2 516	3 446	4 508
35	1 025	1 732	2 166	2 994	3 934	1 195	1 957	2 431	3 334	4 364
40	0.999	1 697	2 126	2 941	3 866	1 154	1 902	2 365	3.25	4 255
45	0.978	1 669	2 092	2 897	3 811	1 122	1 857	2 313	3 181	4 168
50	0.961	1 646	2 065	2 863	3 766	1 096	1 821	2 269	3 124	4 096

It is also possible to develop an equation for determining tolerance limits that is not restricted to tabled values. In fact, sample sizes may vary and be larger than 50 (the extent of the table) and it may be desirable to vary the percentile being estimated or the desired level of confidence. In order to determine *K* without the use of tables, Link et al. [13] proposed an equation (Eq. 2.32) to calculate this factor. t, c_0, c_1, c_2, d_1, d_2, d_3, z_P and z_γ must be determined. An approximation for these quantities may be found in Abramowitz and Stegun [14].

$$t_p = \sqrt{\ln\left(\frac{1}{p^2}\right)} \tag{2.22}$$

$$t_\gamma = \sqrt{\ln\left(\frac{1}{\gamma^2}\right)} \tag{2.23}$$

where p = 1-R/100 and γ= 1-C/100.

$$c_0 = 2.515517 \tag{2.24}$$

$$c_1 = 0.802853 \tag{2.25}$$

$$c_2 = 0.010328 \tag{2.26}$$

$$d_1 = 1.432788 \tag{2.27}$$

$$d_2 = 0.189269 \tag{2.28}$$

$$d_3 = 0.001308 \tag{2.29}$$

$$z_p = t_p - \frac{c_0 + c_1 t_p + c_2 t_p^2}{1 + d_1 t_p + d_2 t_p^2 + d_3 t_p^3} \tag{2.30}$$

$$z_\gamma = t_\gamma - \frac{c_0 + c_1 t_\gamma + c_2 t_\gamma^2}{1 + d_1 t_\gamma + d_2 t_\gamma^2 + d_3 t_\gamma^3} \tag{2.31}$$

Finally, K-factor can be calculated as:

$$K = \frac{z_p(1-f) + \left\{ z_p^2(1-f)^2 - \left[(1-f)^2 - \frac{z_\gamma^2}{2(n_s-1)} \right] \left(z_p^2 - \frac{z_\gamma^2}{n_s} \right) \right\}^{\frac{1}{2}}}{(1-f)^2 - \frac{z_\gamma^2}{2(n_s-1)}} \tag{2.32}$$

where $f = 1/(4(n_s - 1))$ and n_s is the sample size.

3 Maximum Likelihood approach for data analysis

3.1 α-quantile of S-N curves

The idea behind the Maximum Likelihood approach is to find the parameters of the distribution that maximise the probability of obtaining the experimentally measured data. The Maximum Likelihood method has very good asymptotic properties [15] by converging to the true value for an infinite number of samples, which makes it suitable for estimating the parameters of a statistical model. Although the Staircase strategy is the most popular method for calculating the fatigue limit, the data analysis is strictly based on a broken/unbroken approach and, in this way, important information such as the number of cycles to failure remains unused. The advantage of the Maximum Likelihood method over traditional methods is the ability to estimate the parameters of a model that can describe the entire S-N curve with a single equation. In addition, unlike methods based on least squares, it also allows interrupted tests (runout) to be considered. With this model, failure occurs if the specimen is loaded to a stress level S above the fatigue limit and if the number of cycles N at S is greater than N_f. Thus, the probability of failure can be expressed as the product of the two independent probabilities (Eq.3.1) as suggested by Paolino et al. in [16].

$$F_{Y|X=x} = F_{X_l} \cdot F_{Y|surf} \tag{3.1}$$

where $F_{Y|X=x}$ is the cdf of the fatigue life *Y* for a given logarithm of the stress amplitude *x,* F_{X_l} is the cdf of the logarithm of the fatigue limit and $F_{Y|surf}$ is the cdf of the fatigue life if crack nucleates superficially (that is the only case considered here) [17].
F_{X_l} can be expressed as:

$$F_{X_l} = \varphi\left(\frac{x - \mu_{X_l}}{\sigma_{X_l}}\right) = \varphi\left(\frac{Log_{10}(S_a) - \mu_{X_l}}{\sigma_{X_l}}\right) \tag{3.2}$$

where φ(·) is the standardized Normal cumulative distribution function (cdf), while μ_{X_l} and σ_{X_l} are the mean value and standard deviation of X_l i.e. the logarithm of the fatigue limit (in this model a log-normal distribution of the fatigue limit was assumed so X_l can be assumed to be normally distributed).

$F_{Y|surf}$ can be expressed as in Eq. (3.3).

$$F_{Y|surf} = \varphi\left(\frac{y - (a + b \cdot x)}{\sigma_Y}\right) = \varphi\left(\frac{y - \left(a + b \cdot Log_{10}(S_a)\right)}{\sigma_Y}\right) \tag{3.3}$$

Here, it was assumed that the number of cycles to failure follows a log-normal distribution, that the median value of y, the logarithm (in base 10) of the number of cycles, $\mu_{Y|surf}$, follows Basquin's law [18] and that the standard deviation is constant for any stress value. *a* and *b* in Eq.(3.3) are constant coefficients related to the Basquin's law. In particular, *a* is related to the intercept and *b* to the slope of the linear section of the S-N curve while σ_Y denote the standard deviations of the fatigue life. With these assumptions, and by taking in account Eqs. (3.2) and (3.3), the probability of failure for limited and unlimited fatigue life regime can be expressed as:

$$F_{Y|X=x} = \varphi\left(\frac{Log_{10}(S_a) - \mu_{X_l}}{\sigma_{X_l}}\right) \cdot \varphi\left(\frac{y - \left(a + b \cdot Log_{10}(S_a)\right)}{\sigma_Y}\right) \tag{3.4}$$

Thus, in according to equation (3.4), the failure can occur as a result of a stress higher that the limit value μ_{X_l} or after the fatigue life higher than or equal to that defined by the Basquin's law. The cdf (3.4) depends on five parameters: *a*, *b*, σ_Y, μ_{X_l}, σ_{X_l} . The Maximum Likelihood method is used to determine these parameters.

In order to use Maximum Likelihood principle, it is necessary to define the Likelihood function, L. The Likelihood of obtaining the outcome measures (usually life) is the probability of obtaining those measures given the value of the distribution parameters $\boldsymbol{\theta}$. Likelihood function takes the form [19]:

$$L[\boldsymbol{\theta}] = \prod_{i=1}^{n_f} f_{Y|X=x}\,[y_i, x_i, \boldsymbol{\theta}] \cdot \prod_{j=1}^{n_r} (1 - F_{Y|X=x}\,[y_j, x_j, \boldsymbol{\theta}]) \tag{3.5}$$

where (x_i,y_i) and (x_j,y_j) are the experimental failure and runouts data respectively, n_f is the number of breaks, n_r is the number of runouts, $\boldsymbol{\theta}$ denotes the set of five parameters involved in the statistical model, $f_{Y|X=x}$ is the probability density function (pdf) and $F_{Y|X=x}$ is the cumulative density function (cdf). $f_{Y|X=x}[y_i, x_i, \boldsymbol{\theta}]$ represents the probability of an observed failure at y_i while $\left(1 - F_{Y|X=x}[y_j, x_j, \boldsymbol{\theta}]\right)$ represents the probability that the specimen's life is above y_j[19]. As mentioned before, the principle of Maximum Likelihood states that the preferred values of the parameters of a likelihood function are those which maximise the probability of obtaining the observed data [20]. In practice, to simplify calculations, a log-likelihood function is generally used, $\mathcal{L}$ = ln(L). In fact, since the logarithm is an increasing function, the calculation of the maximum (or minimum) of a function is equivalent to calculating the maximum (or minimum) of its logarithm. However, the logarithm of a product is equal to the sum of the logarithms of the terms, i.e.:

$$\begin{aligned} ln(L[\boldsymbol{\theta}]) &= ln\left(\prod_{i=1}^{n_f} f_{Y|X=x}\,[y_i, x_i, \boldsymbol{\theta}] \cdot \prod_{j=1}^{n_r} (1 - F_{Y|X=x}\,[y_j, x_j, \boldsymbol{\theta}])\right) \\ &= \sum_{i=1}^{n_f} \ln\,(f_{Y|X=x}[y_i, x_i, \boldsymbol{\theta}] + \sum_{j=1}^{n_r} \ln\left(\left(1 - F_{Y|X=x}[y_j, x_j, \boldsymbol{\theta}]\right)\right) \end{aligned} \tag{3.6}$$

The search for extremes of the summation function is easier than the productoria. Thus, the parameter estimates with the maximum likelihood method are the solutions of the following system:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \theta_1} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \theta_2} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \theta_3} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \theta_4} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \theta_5} = 0 \end{cases} \tag{3.7}$$

This calculation can be done numerically with the command *"GlobalSearch"* on Matlab as will be shown in chapter 5.

$$\widetilde{\boldsymbol{\theta}} = \left(\tilde{a}, \tilde{b}, \widetilde{\sigma_Y}, \widetilde{\mu_{X_l}}, \widetilde{\sigma_{X_l}}\right) \tag{3.8}$$

is the set of estimated parameters.

$\widetilde{\mu_{X_l}}$ is the ML estimation of the base-10 logarithm of the fatigue limit. Thus, the median value of fatigue limit can be calculated by equation (3.9):

$$\mu = \mathrm{e}^{\frac{\widetilde{\mu_{X_l}}}{log_{10}(\mathrm{e})}} \tag{3.9}$$

μ is the mean values of R50 fatigue limit. Standard deviation of the fatigue limit, that follows a Log-Normal distribution, can be calculated with Eqs. (3.10) and (3.11).

$$\mu^* = e^{\left(\frac{\widetilde{\mu_{X_l}}}{log_{10}(e)} + \frac{1}{2} \frac{\widetilde{\sigma_{X_l}}^2}{log_{10}(e)}\right)} \tag{3.10}$$

$$\sigma = \mu^* \sqrt{e^{\frac{\widetilde{\sigma_{X_l}}^2}{log_{10}(e)}} - 1} \tag{3.11}$$

where μ^* is the expected value of the Log-Normal distributed fatigue limit (that's difference from the median value for asymmetrical distribution) and σ is the standard deviation of fatigue limit.

Parameter estimates given in Eq. (3.8) can be used for computing quantile S–N curves. In particular, if the α-th quantile S–N curve, i.e., S-N curve with the probability of failure equals α, is of interest, by substituting $F_{Y|X=x} = \alpha$ in (3.4), equation (3.12) can be obtained:

$$\alpha = \varphi\left(\frac{Log_{10}(S_a) - \widetilde{\mu_{X_l}}}{\widetilde{\sigma_{X_l}}}\right) \cdot \varphi\left(\frac{y - \left(\tilde{a} + \tilde{b} \cdot Log_{10}(S_a)\right)}{\widetilde{\sigma_Y}}\right) \tag{3.12}$$

which, solved with respect to $Log_{10}(S_a)$ for different values of y (=$Log_{10}(N_f)$), allows the construction of the different probability of survival S-N curves.

3.2 Likelihood Ratio Confidence Intervals for the quantiles of S-N curve

In this case, the Profile Likelihood Ratio approach [17] was used to calculate the design curve. Confidence intervals based on this approach are more difficult to estimate than the method used by Owen and Dixon-Mood due to the computational time required to the application of the method, but they are more accurate and seem to work better for a smaller number of available samples [21].

If $\boldsymbol{\theta} = (\theta_1, \boldsymbol{\theta_2})$ is a subdivision of the vector of parameters to be estimated, where θ_1 is a parameter of interest, and $\boldsymbol{\theta_2}$ is a vector of other parameters involved in the model, the Profile likelihood for θ_1 is:

$$PL[\theta_1] = \frac{max_{\theta_2}\left[\mathcal{L}[\theta_1, \boldsymbol{\theta_2}]\right]}{\mathcal{L}\,[\widetilde{\boldsymbol{\theta}}]} \tag{3.13}$$

where $\mathcal{L}\,[\widetilde{\boldsymbol{\theta}}]$ is log-likelihood function calculated at the estimated vector $\widetilde{\boldsymbol{\theta}}$ i.e., the vector of parameters values that maximizes $\mathcal{L}[\boldsymbol{\theta}]$.

Thus, this Profile Likelihood method use a function of one parameter of interest, θ_1, by treating the other components of $\boldsymbol{\theta}$ as "disturbing" parameters and maximizing the likelihood over them [22]. It can be proved that the "Likelihood Ratio Statistics" defined as -2 ln($PL[\theta_1]$) is asymptotically Chi-squared with 1 degree of freedom [17]. As a result, likelihood ratio confidence intervals for θ_1 are based on the equation (3.14):

$$PL[\theta_1] \geq e^{-\frac{\chi^2_{(1;\beta)}}{2}} \tag{3.14}$$

where, $\chi^2_{(1;\beta)}$ is the β quantile of the χ^2 distribution with 1 degree of freedom. If δ is the confidence level, i.e., $\delta = C/100$, $\beta = (2\delta\text{-}1)$ for 1-sided confidence bounds.

In order to estimate the LRCIs for S_{a_α}, $PL[\theta_1]$ must be a function of S_{a_α} where S_{a_α} is the α-th quantile of fatigue strength. Considering, for example, $\theta_1 = \mu_{X_l}$ and by taking in account equation (3.4) with $F_{Y|X=x} = \alpha$, we can obtain eq. (3.15):

$$\mu_{X_l} = Log_{10}(S_{a_\alpha}) - \varphi^{-1}\left(\frac{\alpha}{\varphi\left(\frac{y_\alpha - \left(a + b \cdot Log_{10}(S_{a_\alpha})\right)}{\sigma_Y}\right)}\right) \cdot \sigma_{X_l} \tag{3.15}$$

Now, if we substitute eq. (3.15) in Eq. (3.4),

$$F_{Y|X=x} = \varphi\left(\frac{y-(a+b\cdot Log_{10}(S_a))}{\sigma_Y}\right)\cdot \varphi\left(\frac{Log_{10}(S_a) - \left(Log_{10}(S_{a_\alpha}) - \varphi^{-1}\left(\frac{\alpha}{\varphi\left(\frac{y_\alpha-(a+b\cdot Log_{10}(S_{a_\alpha}))}{\sigma_Y}\right)}\right)\cdot\sigma_{X_l}\right)}{\sigma_{X_l}}\right) \tag{3.16}$$

is obtained. Equation (3.16) allows to calculate $max_{\theta_2}(\mathcal{L}[\theta_1, \boldsymbol{\theta_2}])$ and then $PL[\ \theta_1]$. The range of θ_1 values which satisfies eq. (3.14) represents the LRCIs of parameter θ_1 [23]. This procedure has been implemented in a Matlab code that automatically compute the PL confidence intervals, as will be described in chapter 5.

Figure 3.1 shows an example of Profile Likelihood curve for fatigue strength. It is plotted only half of the curve because we are interested in the lower bound of confidence interval. Common choices for the confidence level C are 0.90, 0.95 and 0.99. To build, for example, the R90C90 curve we need to calculate the Profile Likelihood for each value of number of cycles. Fig. 3.1 refers to a real example where a number of cycles equal to N_f = 2.48e7 was considered and stress values have been normalised for corporate confidentiality. The 90% confidence interval can be read off the plot as the set of S_a values with PL greater than $\exp\left(-\frac{\chi^2_{(1;0.8)}}{2}\right)$ = 0.439 (this critical line is drawn in the plot).

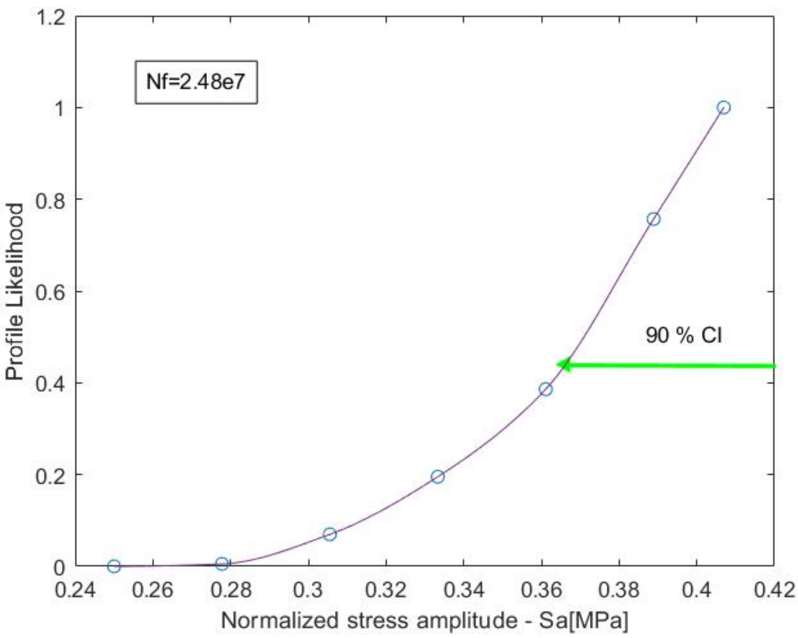


Figure 3.1 Example of Profile Likelihood curve for S_a

If we iterate this calculation for each number of cycles, we can obtain the R90C90 curve as shown in Figure 3.2. R50 and R90 curves in Fig. 3.2 is based on equation (3.12) in which $\alpha = 0.5$ and $\alpha = 0.9$ respectively. The value of the stresses was normalised for corporate confidentiality.

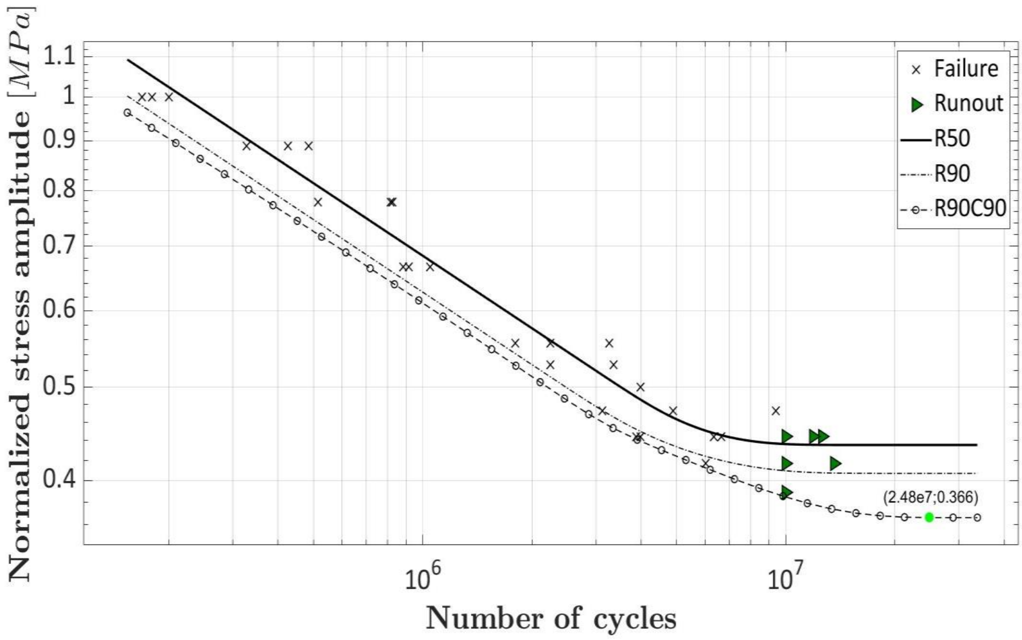


Figure 3.2 R50, R90 and R90C90 curves based on Maximum Likelihood approach

4 Implementation on MATLAB App Designer of a code for data analysis

The analysis of the fatigue test data was carried out by means of a Matlab code which was developed and optimised in such a way that it implemented both methodologies described in chapters 2 and 3 and allowed a direct comparison of their results. To facilitate the use of this code, a user-friendly application was developed using Matlab App Designer. Matlab App Designer is a software environment that facilitates the creation of apps in MATLAB providing a large set of interactive UI components. Permette inoltre di distribuire le app create tramite Matlab Drive o creando applicazioni standalone. It also allows apps to be shared via Matlab Drive or by creating standalone desktop applications.

Section 4.1 shows details regarding app features for the implementation of least squares and Owen method, for finite life region, and Staircase method for fatigue limit definition. Section 4.2 concerns the procedure by which the maximum likelihood method was implemented in App Designer.

4.1 Implementation of the "Traditional approach"

Fatigue Analysis App is the name of the app that was created for data analysis of experimental data. The app allows you to load the experimental dataset in .txt format in which the first column corresponds to the different stress levels, the second corresponds to the number of cycles associated with them and the third corresponds to the status of that particular test, i.e. whether it is a failure (= 1) or a runout (= 0).

As explained in Chapter 3, fatigue limit evaluation method that is used in CRF is based on Staircase approach. Figure 4.1 shows the layout of Staircase tab in Fatigue Analysis app. The values of stress in Figure 4.1 have been normalised for corporate confidentiality. Fatigue data for $AlSi_{10}Mg$ (polished), obtained in the axial load fatigue test, were used to present the results. Figure 4.2 shows tab group for adding and removing data directly from user interface. This is a useful function because it avoids the need to edit the txt file every time you want to add or remove an experimental data element.

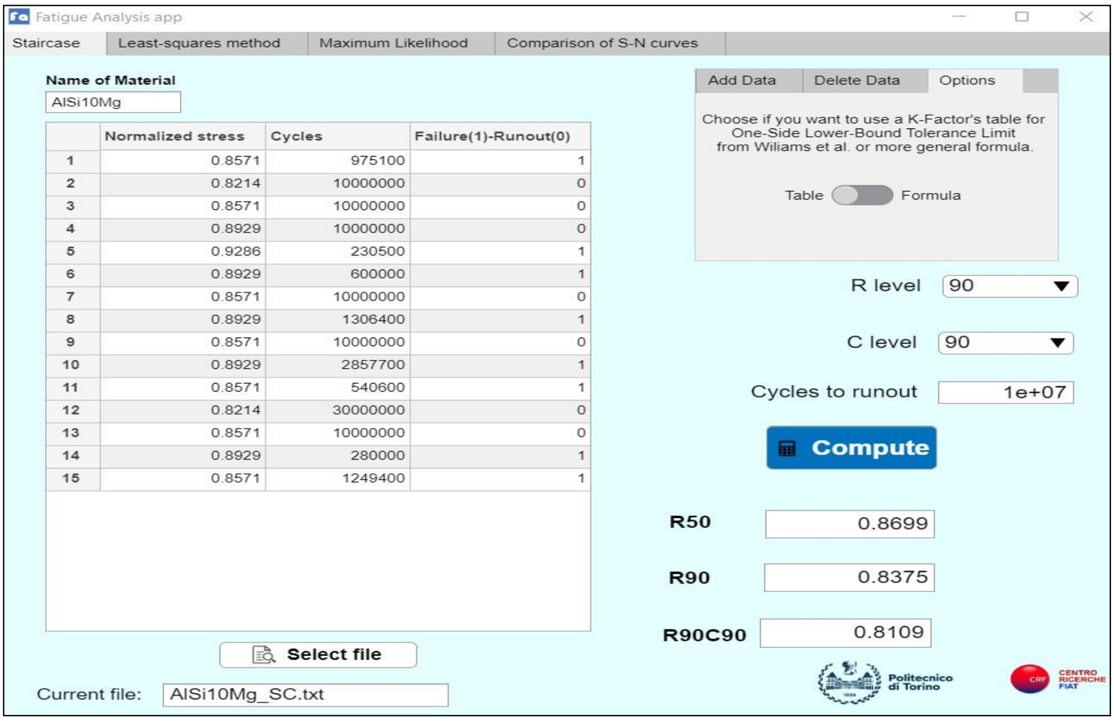


Figure 4.1 Staircase tab of Fatigue Analysis app

Once the staircase dataset has been loaded and the number of cycles to runout has been selected, the app automatically calculates the fatigue limit with different reliability and confidence level that can be chosen from a drop-down menu on the right.
In Figure 4.1, the Options tab allows you to choose whether to use the Williams et al. tables or the more general formula for calculating the K-factor of equation (2.25).

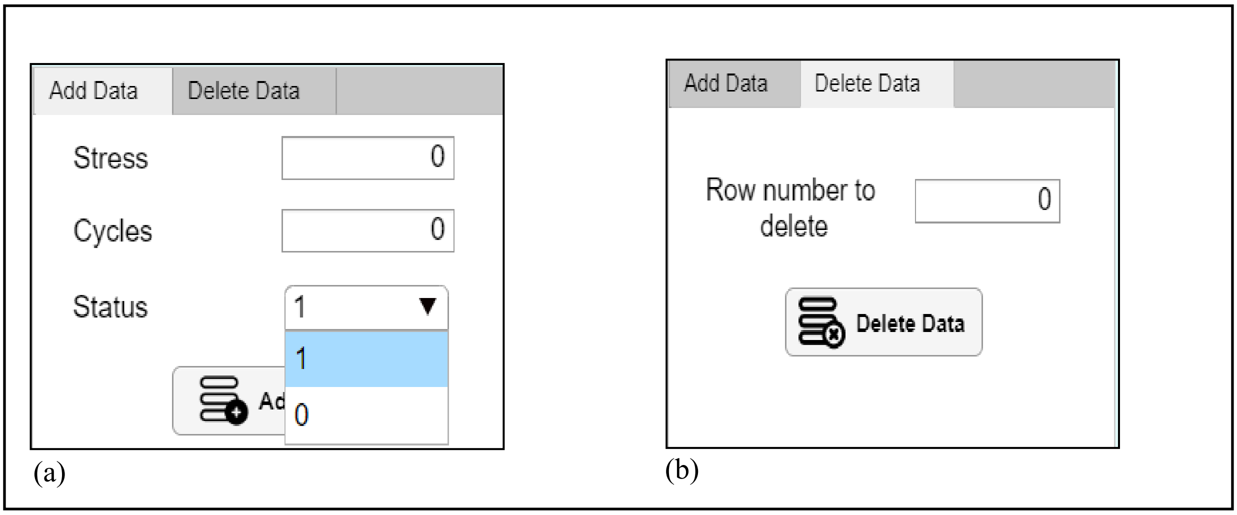


Figure 4.2 Add (a) and delete (b) data tab

Figure 4.3 shows the flowchart of the algorithm that was developed, in according to equations in section 2.2, to implement the Dixon-Mood method in the Fatigue Analysis app.

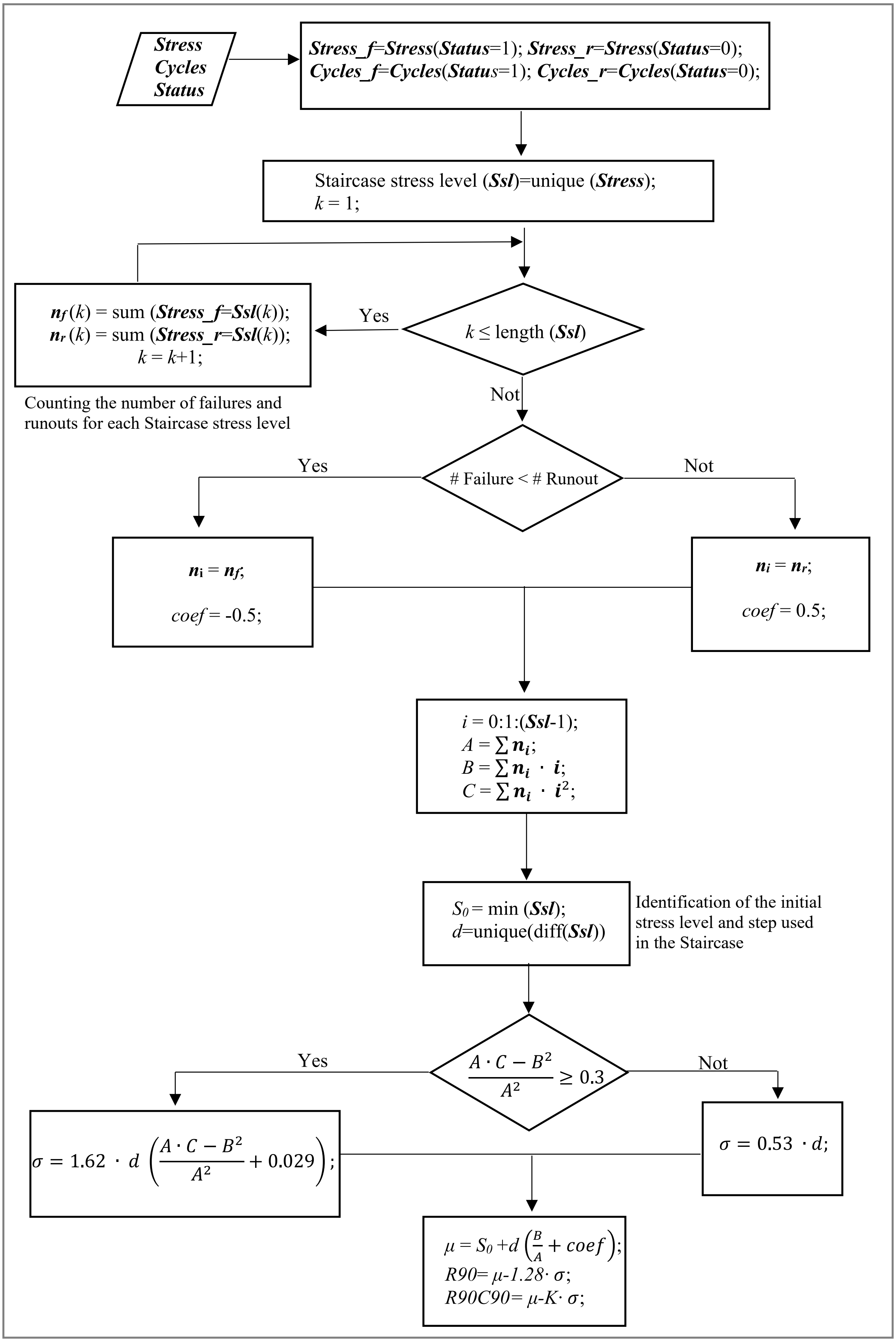


Figure 4.3 Flowchart of Dixon-Mood approach implemented in the app

On the other hand, with regards to the finite-life region of the S-N curve, the least-squares method was used to fit the experimental data. Figure 4.4 shows the app layout of Least-squares tab. The plot in Fig, 4.4 shows the median, R90 and R90C90 S-N curves of $AlSi_{10}Mg$ developed using the least-squares method combined with the Dixon-Mood method. Plot options allow to show a single S-N curve or all curves simultaneously.

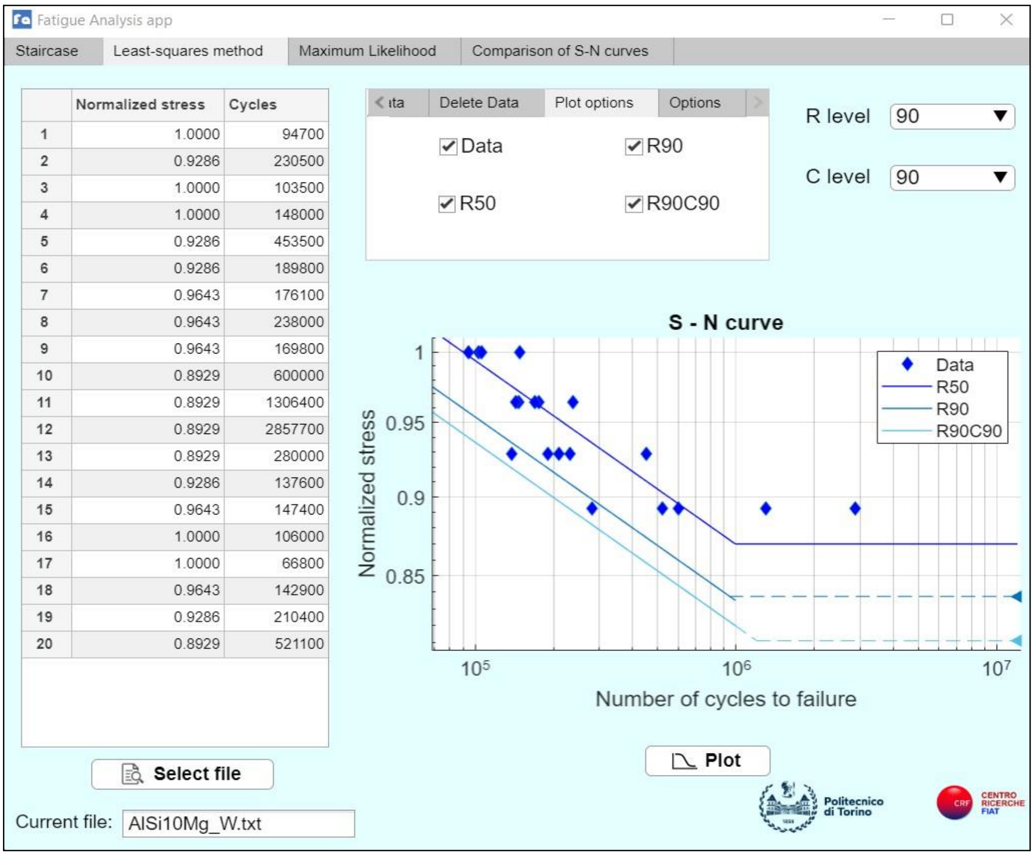


Figure 4.4 Least-squares tab of Fatigue Analysis app

In Figure 4.5, the Options tab allows you to choose whether to use the Williams et al. tables or the more general formula for calculating the K_{Owen} factor of equation (2.13).

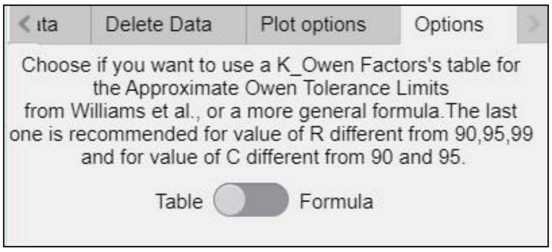


Figure 4.5 Options tab for K_{Owen} factor

Figure 4.6 shows the flowchart of the algorithm that was developed, in according to equations in section 2.1, to implement the least-squares method in the Fatigue Analysis app.

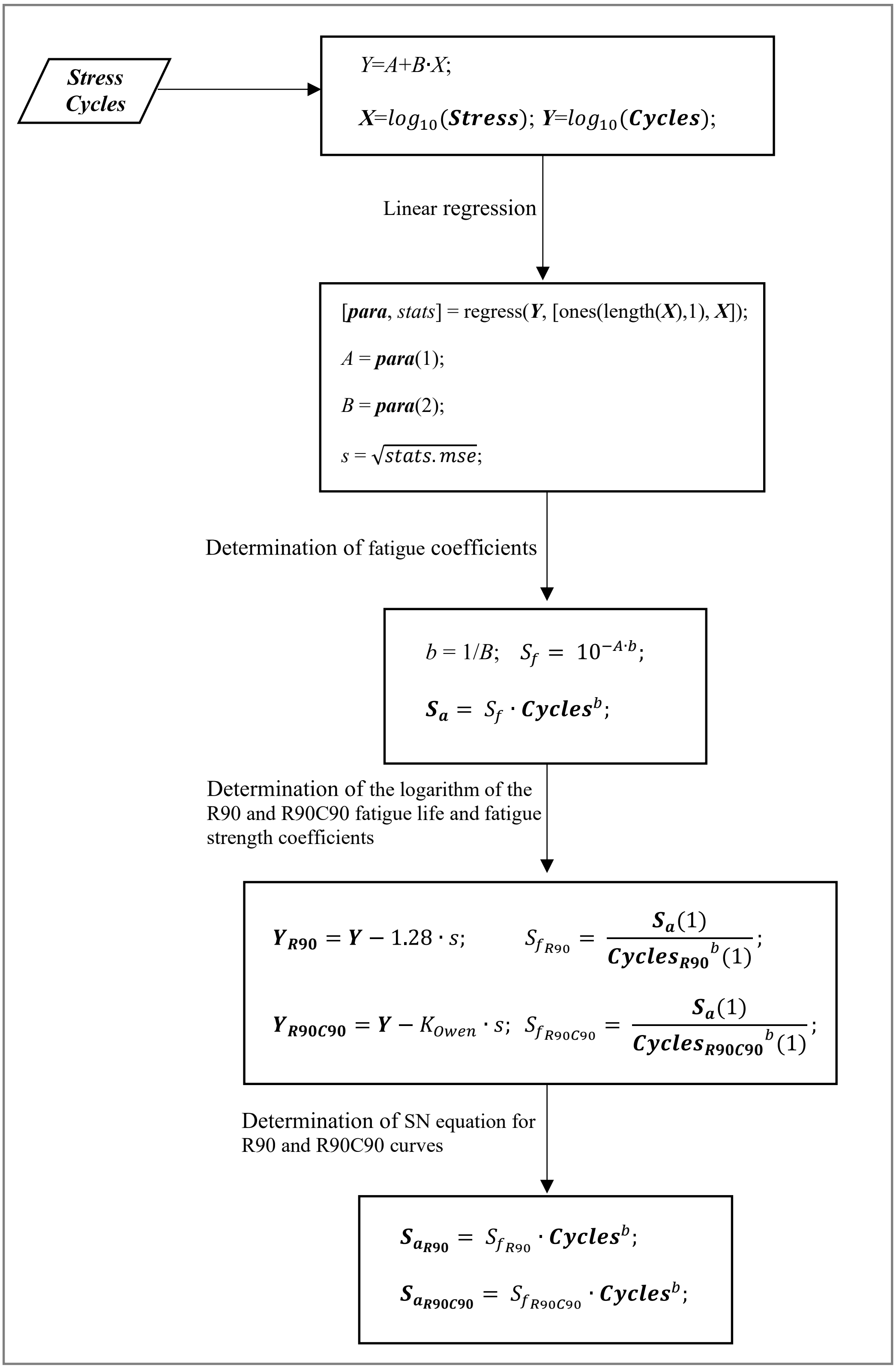


Figure 4.6 Flowchart of least-squares method implemented in the app

4.2 Implementation of Maximum Likelihood method

Since this method allows the estimation of the parameters of a model capable of describing the fatigue behaviour of a material with a continuous curve and not with two separate straight lines, as in the case of the traditional approach, in the implementation of the method the complete dataset consisting of the stress values and number of cycles of all the experimental tests carried out must be considered.
Figure 4.7 shows the Fatigue Analysis app tab like that shown in Figure 4.1 but using the maximum likelihood method for fatigue limit calculation instead of the Dixon-Mood method. Figure 4.8 shows the S-N curves of $AlSi_{10}Mg$ calculated with the app using the maximum likelihood method.

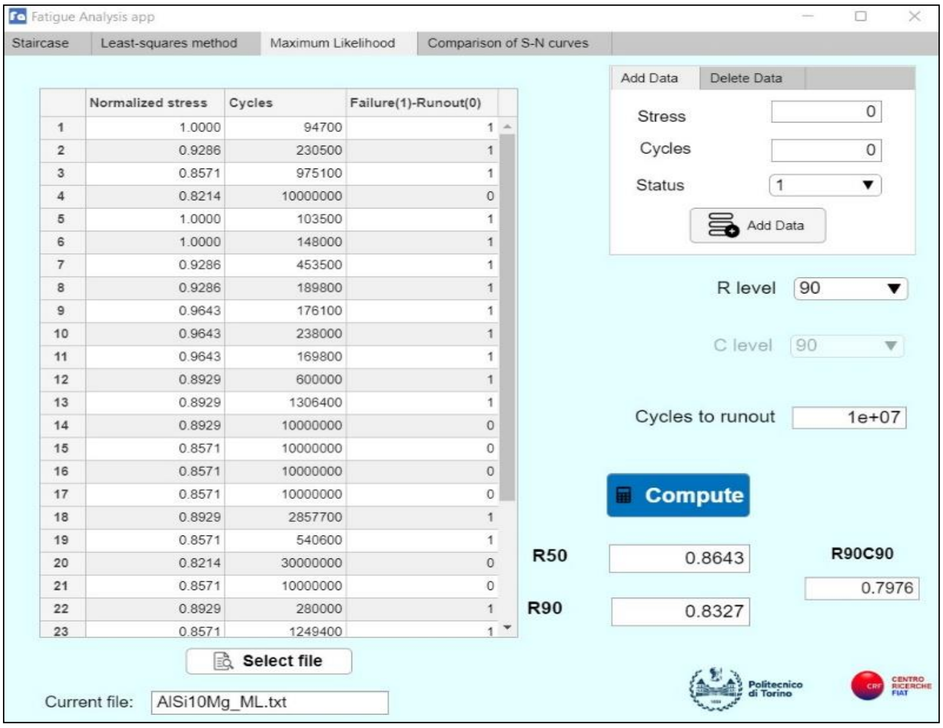


Figure 4.7 Maximum Likelihood tab of Fatigue Analysis app

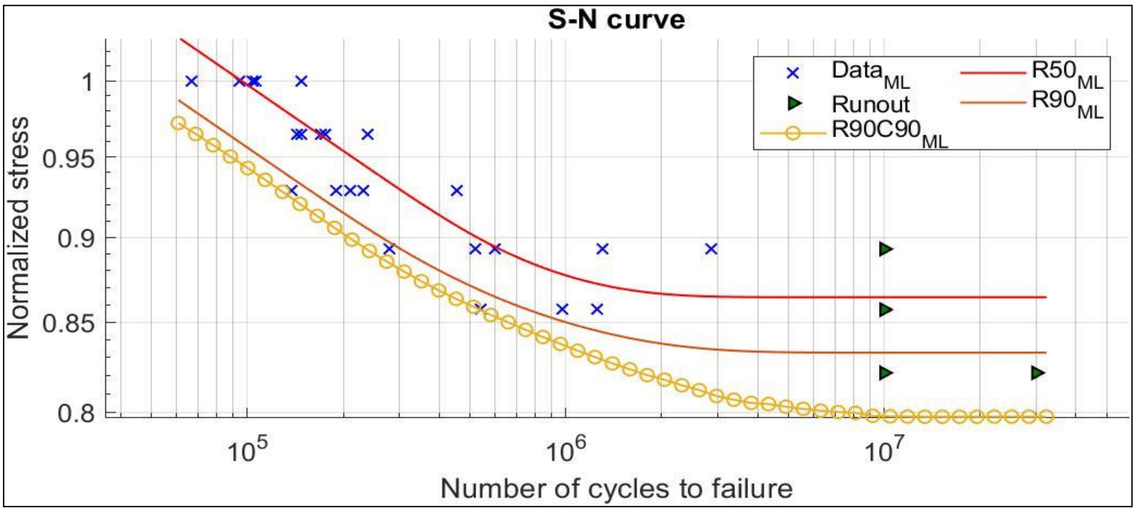


Figure 4.8 Example of S-N curves calculated with the app by applying the ML method

The algorithm behind this method, which is based on the equations presented in section 3, is schematised by the simplified flowchart in Figure 4.9 (a) and (b). The first flowchart explains the code used for the construction of the α-th quantile of the S-N curve while the second one explains the code developed for the construction of profile likelihood function and design curve.

To maximise the log-likelihood function and find the best estimate of five parameters involved in the statistical model two matlab functions were investigated*,"GlobalSearch"* and *"fminsearch".* The first one is the most focused on finding a global minimum (which coincides with the maximum of the opposite of the objective function) while the second one is one of the most common functions to search for a local minimum in the vicinity of the given starting point. Of course, the first function guarantees a higher probability of convergence while the second one is simple to use and less time consuming. For these reasons, in the construction of the Profile Likelihood function which requires that ML fitting has to be repeated multiple times, the function *fminsearch* was used starting from the values of the optimised parameters found with *GlobalSearch*. *GlobalSearch* function require an initial estimate of the parameters to be taken as a starting point for optimisation. As also suggested by Nelson in [19], a first estimate of *a* and *b* may be derived from a least-squares fit of a linear regression of $\log_{10}(N)$ with respect to $\log_{10}(S)$ where, however, only failures were considered because they are generally aligned following the Basquin relation. This was done with the *"regress"* function of Matlab that returns a vector *para* of coefficient estimates for a multiple linear regression as can be shown in the flowchart of Figure 4.9 (a). This function requires a column of one in the X matrix to compute coefficient estimates for a model with a constant term (intercept). *Regress* function also compute the statistics of linear regression model and in particular the mean squared error (*MSE*) was calculated and thus a preliminary estimate of σ_Y was the square root of this value.

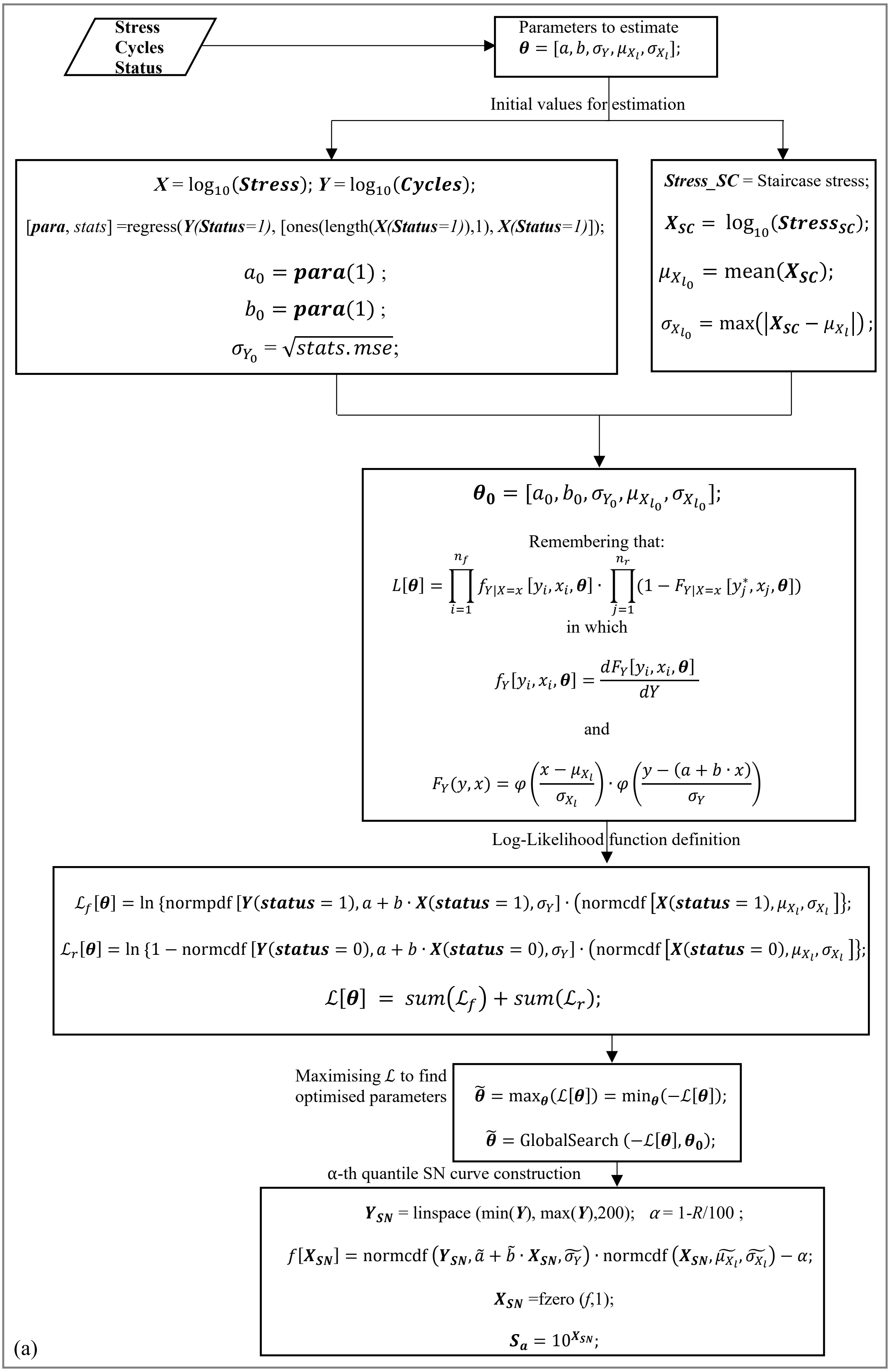
For the infinite life parameters, the average of the logarithms of the Staircase stresses was taken as the initial value for μ_{X_l} and the maximum distance from the average value of the logarithms of the Staircase stresses was taken as the initial value for σ_{X_l}.

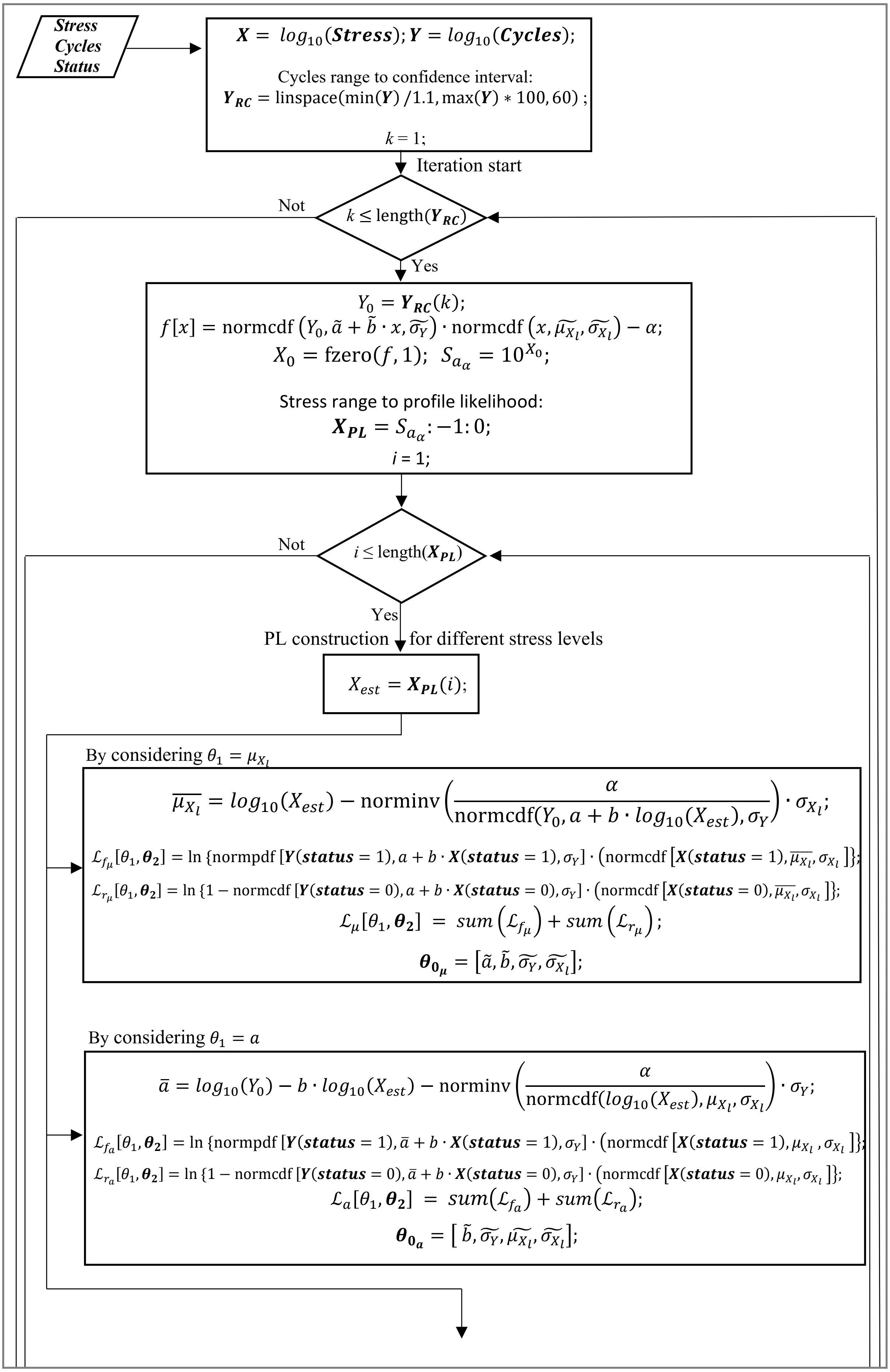
As regards the computation of LRCIs of S_{a_α}, *PL* must be a function of S_{a_α} and this can be obtained, for example, by making μ_{X_l} explicit as shown in equation (3.15). However, at high stress levels, the probability that the fatigue limit is below the applied stress is equal to 1 so if $\varphi\left(\frac{Log_{10}(S_{a_\alpha})-\mu_{X_l}}{\sigma_{X_l}}\right) = 1$ then, by equation (3.12), $\frac{\alpha}{\varphi\left(\frac{y_\alpha-(a+b\cdot Log_{10}(S_{a_\alpha}))}{\sigma_Y}\right)} = 1$ and when calculating

$$\mu_{X_l} = Log_{10}(S_{a_\alpha}) - \varphi^{-1}\left(\frac{\alpha}{\varphi\left(\frac{y_\alpha-(a+b\cdot Log_{10}(S_{a_\alpha}))}{\sigma_Y}\right)}\right)\cdot\sigma_{X_l}$$, we have $\varphi^{-1}(1) = \infty$.

This made it impossible to calculate *PL* and interrupted code execution. In these cases, instead of making μ_{X_l} explicit from equation (3.12), the parameter *a* was made explicit from the same equation. However, even in this case, when we are at high number of cycles, the probability of the specimen breaking at a lower number of cycles than the actual number of cycles is 1 so if $\varphi\left(\frac{y_\alpha-(a+b\cdot Log_{10}(S_{a_\alpha}))}{\sigma_Y}\right) = 1$, then, by equation (3.12), $\frac{\alpha}{\varphi\left(\frac{Log_{10}(S_{a_\alpha})-\mu_{X_l}}{\sigma_{X_l}}\right)} = 1$ and when calculating

$$a = log_{10}(y_\alpha) - b\cdot log_{10}(S_{a_\alpha}) - \varphi^{-1}\left(\frac{\alpha}{\varphi\left(\frac{Log_{10}(S_{a_\alpha})-\mu_{X_l}}{\sigma_{X_l}}\right)}\right)\cdot\sigma_Y,$$ we have $\varphi^{-1}(1) = \infty$. For this reason, as schematised in the flowchart of figure 4.9 (b), a check has been implemented according to which if $\frac{\alpha}{\varphi\left(\frac{Log_{10}(S_{a_\alpha})-\mu_{X_l}}{\sigma_{X_l}}\right)}$ is different from 1 then the parameter *a* is used, otherwise the parameter μ_{X_l} is used for the calculation of PL_{θ_1} and then for the construction of the design curve.


Stress
Cycles
Status
Parameters to estimate
$\boldsymbol{\theta} = [a, b, \sigma_Y, \mu_{X_l}, \sigma_{X_l}];$
Initial values for estimation
$\boldsymbol{X} = \log_{10}(\boldsymbol{Stress})$; $\boldsymbol{Y} = \log_{10}(\boldsymbol{Cycles})$;
[*para*, *stats*] =regress(*Y(Status*=*1)*, [ones(length(*X(Status*=*1)*),1), *X(Status*=*1)*]);
$a_0 = \boldsymbol{para}(1)$;
$b_0 = \boldsymbol{para}(1)$;
$\sigma_{Y_0} = \sqrt{stats.mse}$;
Stress_SC = Staircase stress;
$\boldsymbol{X_{SC}} = \log_{10}(\boldsymbol{Stress_{SC}})$;
$\mu_{X_{l_0}} = \mathrm{mean}(\boldsymbol{X_{SC}})$;
$\sigma_{X_{l_0}} = \max(|\boldsymbol{X_{SC}} - \mu_{X_l}|)$;
$\boldsymbol{\theta_0} = [a_0, b_0, \sigma_{Y_0}, \mu_{X_{l_0}}, \sigma_{X_{l_0}}]$;
Remembering that:
$L[\boldsymbol{\theta}] = \prod_{i=1}^{n_f} f_{Y|X=x}[y_i, x_i, \boldsymbol{\theta}] \cdot \prod_{j=1}^{n_r} (1 - F_{Y|X=x}[y_j^*, x_j, \boldsymbol{\theta}])$
in which
$f_Y[y_i, x_i, \boldsymbol{\theta}] = \frac{dF_Y[y_i, x_i, \boldsymbol{\theta}]}{dY}$
and
$F_Y(y, x) = \varphi\left(\frac{x - \mu_{X_l}}{\sigma_{X_l}}\right) \cdot \varphi\left(\frac{y - (a + b \cdot x)}{\sigma_Y}\right)$
Log-Likelihood function definition
$\mathcal{L}_f[\boldsymbol{\theta}] = \ln\{\mathrm{normpdf}[\boldsymbol{Y}(\boldsymbol{status} = 1), a + b \cdot \boldsymbol{X}(\boldsymbol{status} = 1), \sigma_Y] \cdot (\mathrm{normcdf}[\boldsymbol{X}(\boldsymbol{status} = 1), \mu_{X_l}, \sigma_{X_l}]\}$;
$\mathcal{L}_r[\boldsymbol{\theta}] = \ln\{1 - \mathrm{normcdf}[\boldsymbol{Y}(\boldsymbol{status} = 0), a + b \cdot \boldsymbol{X}(\boldsymbol{status} = 0), \sigma_Y] \cdot (\mathrm{normcdf}[\boldsymbol{X}(\boldsymbol{status} = 0), \mu_{X_l}, \sigma_{X_l}]\}$;
$\mathcal{L}[\boldsymbol{\theta}] = sum(\mathcal{L}_f) + sum(\mathcal{L}_r)$;
Maximising $\mathcal{L}$ to find optimised parameters
$\tilde{\boldsymbol{\theta}} = \max_{\boldsymbol{\theta}}(\mathcal{L}[\boldsymbol{\theta}]) = \min_{\boldsymbol{\theta}}(-\mathcal{L}[\boldsymbol{\theta}])$;
$\tilde{\boldsymbol{\theta}} = \mathrm{GlobalSearch}\ (-\mathcal{L}[\boldsymbol{\theta}], \boldsymbol{\theta_0})$;
α-th quantile SN curve construction
$\boldsymbol{Y_{SN}}$ = linspace (min(*Y*), max(*Y*),200); α = 1-*R*/100 ;
$f[\boldsymbol{X_{SN}}] = \mathrm{normcdf}(\boldsymbol{Y_{SN}}, \tilde{a} + \tilde{b} \cdot \boldsymbol{X_{SN}}, \widetilde{\sigma_Y}) \cdot \mathrm{normcdf}(\boldsymbol{X_{SN}}, \widetilde{\mu_{X_l}}, \widetilde{\sigma_{X_l}}) - \alpha$;
$\boldsymbol{X_{SN}}$ =fzero (*f*,1);
$\boldsymbol{S_a} = 10^{X_{SN}}$;
(a)


Stress
Cycles
Status
$\boldsymbol{X} = log_{10}(\boldsymbol{Stress}); \boldsymbol{Y} = log_{10}(\boldsymbol{Cycles});$
Cycles range to confidence interval:
$\boldsymbol{Y_{RC}} = \text{linspace}(\min(\boldsymbol{Y})/1.1, \max(\boldsymbol{Y}) * 100, 60);$
$k = 1;$
Iteration start
Not
$k \leq \text{length}(\boldsymbol{Y_{RC}})$
Yes
$Y_0 = \boldsymbol{Y_{RC}}(k);$
$f[x] = \text{normcdf}\left(Y_0, \tilde{a} + \tilde{b} \cdot x, \widetilde{\sigma_Y}\right) \cdot \text{normcdf}\left(x, \widetilde{\mu_{X_l}}, \widetilde{\sigma_{X_l}}\right) - \alpha;$
$X_0 = \text{fzero}(f, 1); \ S_{a_\alpha} = 10^{X_0};$
Stress range to profile likelihood:
$\boldsymbol{X_{PL}} = S_{a_\alpha}: -1: 0;$
$i = 1;$
Not
$i \leq \text{length}(\boldsymbol{X_{PL}})$
Yes
PL construction for different stress levels
$X_{est} = \boldsymbol{X_{PL}}(i);$
By considering $\theta_1 = \mu_{X_l}$
$\overline{\mu_{X_l}} = log_{10}(X_{est}) - \text{norminv}\left(\frac{\alpha}{\text{normcdf}(Y_0, a + b \cdot log_{10}(X_{est}), \sigma_Y)}\right) \cdot \sigma_{X_l};$
$\mathcal{L}_{f_\mu}[\theta_1, \boldsymbol{\theta_2}] = \ln\{\text{normpdf}[\boldsymbol{Y}(\boldsymbol{status} = 1), a + b \cdot \boldsymbol{X}(\boldsymbol{status} = 1), \sigma_Y] \cdot (\text{normcdf}[\boldsymbol{X}(\boldsymbol{status} = 1), \overline{\mu_{X_l}}, \sigma_{X_l}]\};$
$\mathcal{L}_{r_\mu}[\theta_1, \boldsymbol{\theta_2}] = \ln\{1 - \text{normcdf}[\boldsymbol{Y}(\boldsymbol{status} = 0), a + b \cdot \boldsymbol{X}(\boldsymbol{status} = 0), \sigma_Y] \cdot (\text{normcdf}[\boldsymbol{X}(\boldsymbol{status} = 0), \overline{\mu_{X_l}}, \sigma_{X_l}]\};$
$\mathcal{L}_\mu[\theta_1, \boldsymbol{\theta_2}] = sum\left(\mathcal{L}_{f_\mu}\right) + sum\left(\mathcal{L}_{r_\mu}\right);$
$\boldsymbol{\theta_{0_\mu}} = [\tilde{a}, \tilde{b}, \widetilde{\sigma_Y}, \widetilde{\sigma_{X_l}}];$
By considering $\theta_1 = a$
$\bar{a} = log_{10}(Y_0) - b \cdot log_{10}(X_{est}) - \text{norminv}\left(\frac{\alpha}{\text{normcdf}(log_{10}(X_{est}), \mu_{X_l}, \sigma_{X_l})}\right) \cdot \sigma_Y;$
$\mathcal{L}_{f_a}[\theta_1, \boldsymbol{\theta_2}] = \ln\{\text{normpdf}[\boldsymbol{Y}(\boldsymbol{status} = 1), \bar{a} + b \cdot \boldsymbol{X}(\boldsymbol{status} = 1), \sigma_Y] \cdot (\text{normcdf}[\boldsymbol{X}(\boldsymbol{status} = 1), \mu_{X_l}, \sigma_{X_l}]\};$
$\mathcal{L}_{r_a}[\theta_1, \boldsymbol{\theta_2}] = \ln\{1 - \text{normcdf}[\boldsymbol{Y}(\boldsymbol{status} = 0), \bar{a} + b \cdot \boldsymbol{X}(\boldsymbol{status} = 0), \sigma_Y] \cdot (\text{normcdf}[\boldsymbol{X}(\boldsymbol{status} = 0), \mu_{X_l}, \sigma_{X_l}]\};$
$\mathcal{L}_a[\theta_1, \boldsymbol{\theta_2}] = sum(\mathcal{L}_{f_a}) + sum(\mathcal{L}_{r_a});$
$\boldsymbol{\theta_{0_a}} = [\tilde{b}, \widetilde{\sigma_Y}, \widetilde{\mu_{X_l}}, \widetilde{\sigma_{X_l}}];$

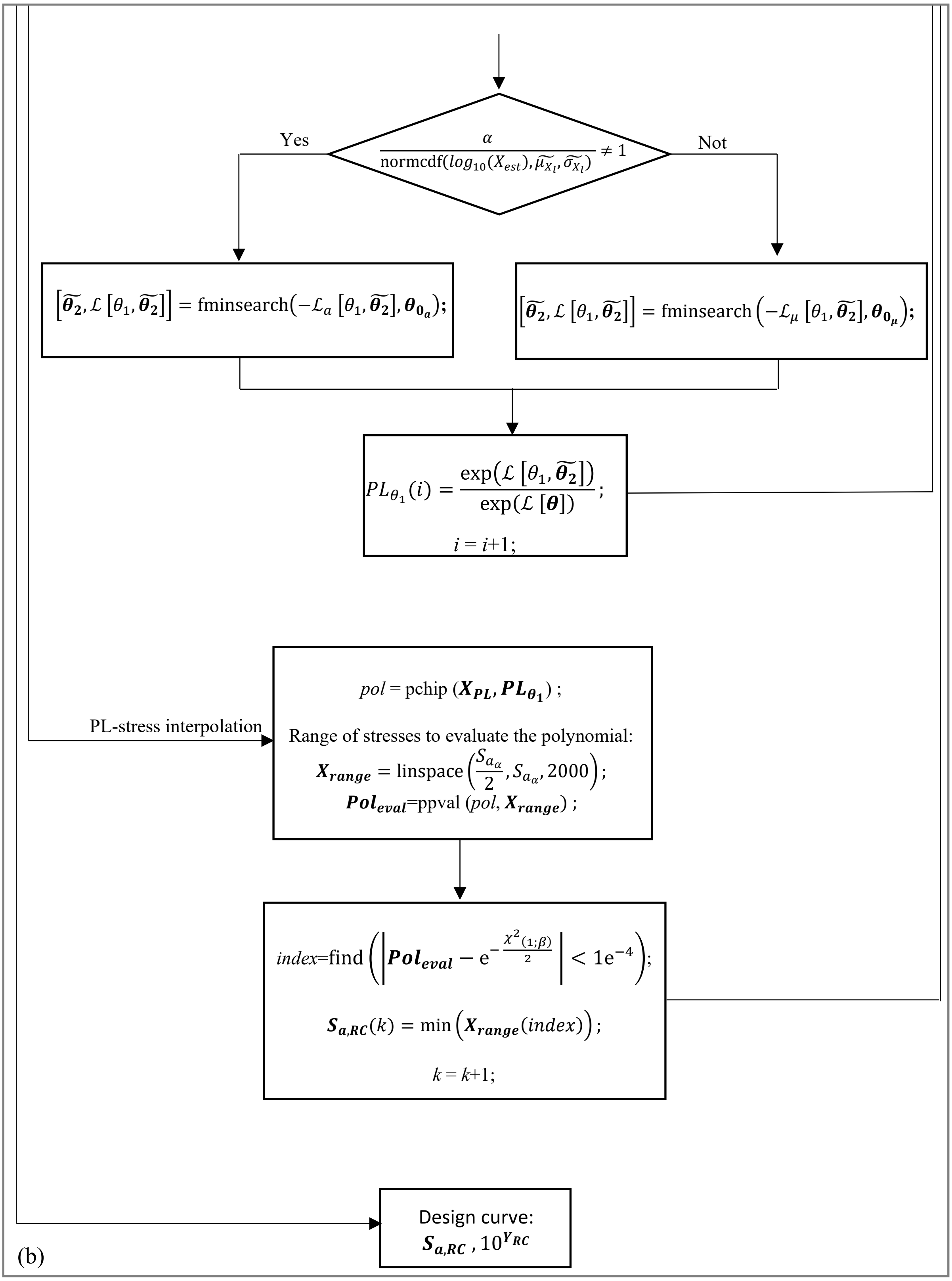


Figure 4.9 (a) Flowchart of ML method for the α-th quantile of SN curve implemented in the app. (b) Flowchart of ML method for profile likelihood and design curve construction.

5 Comparison of the two methods

The implementation of both methodologies for the analysis of fatigue data allowed the comparison of the two methods and the analysis of similarities and differences. In particular, the median Wöhler curves, R50, and the R90C90 curves i.e. the curves below which 90% of the time, 90% of the samples survive, will be compared. In the traditional model used at CRF the R90C90 curves are plotted up to the number of cycles corresponding to the knee of the R50 curve, N_A, however, the following comparison plots also provide the intersection with the R90C90 fatigue limit value calculated by Dixon-Mood, which may occur at a number of cycles above or below N_A. Furthermore, the characteristic parameters of the Maximum Likelihood method, i.e. the 5 parameters a, b, σ_Y for the finite life region and μ_{X_l} e σ_{X_l} for the infinite life region of the curve, introduced in Chapter 3, will be compared with the same parameters derived for the traditional method based on linear regression for the finite life region and on the Dixon-Mood method for the Staircase analysis. In particular, $A = a$, $B = b$, $s = \sigma_Y$, $\mu = \mu_{X_l}$ and $\sigma = \sigma_{X_l}$ where A, B, s, μ and σ are the parameters introduced in Chapter 2. The value of the fatigue limit R90C90 was also compared for the two models, considering for the traditional method the value $S_{e,R,C}$ determined by the formula (2.21) while for the method based on the Profile Likelihood the average of the last three values of the R90C90 curve was considered when the curve shows an horizontal asymptote or the value of the R90C90 curve in correspondence of the number of runout cycles when the curve does not show an asymptotic trend with the available data. In addition, for some materials, the S-N curves and the values of the parameters of the curve have been evaluated, considering, for the method based on Maximum Likelihood, a reduced dataset obtained by removing some data of the infinite life region, and compared with the respective results obtained with the traditional model but considering the starting dataset. This is because one of the advantages of Maximum Likelihood is that, compared to the Dixon-Mood method, it takes additional information into account, namely the lives of the failed and runout specimens. Therefore, the possibility of reducing the number of specimens, and thus saving time and costs for a characterisation campaign, was explored, and the extent to which these results differed from those obtained using the full dataset was verified. Data from the fatigue tests of 12 different materials, provided by CRF, were used to compare the models. In this chapter the traditional model will be identified as Model I and the Maximum Likelihood model will be identified as Model II. Stress values, in the following results, have been normalised for corporate confidentiality.

5.1 Experimental cases

5.1.1 $AlSi_7Mg_{0.3}$ -T6

The first material analysed is a cast aluminium alloy, $AlSi_7Mg_{0.3}$ subjected to a T6 temper. The fatigue behaviour of this material was investigated by means of flat bending fatigue tests. The tests were carried out on hourglass specimens with a load concentration coefficient (K_t) equal to 1.06, at room temperature, with a stress ratio (R) equal to -1 and a frequency of 80 Hz. Fig. 5.1 shows experimental test results for this material with median (R50) S-N curves plotted by using Model I and Model II. The tests were performed on 33 specimens recording 26 fractures and 7 runouts. 15 specimens were used for Staircase (8 specimens failed and 7 specimens reached the limit number of cycles N_G = 1e7). The coefficients obtained by using Model I and Model II were compared in Table 5.1. From this analysis it emerged that the parameters for the finite life region, a, b and σ_Y calculated with the two methods are very similar to each other with a difference of no more than 1.1 % while in the unlimited fatigue life region, the calculated normalized fatigue limit at 50 % probability of failure, is practically the same with a difference of only 0.7 %, while the standard deviations of the fatigue limit is slightly lower for Model II.

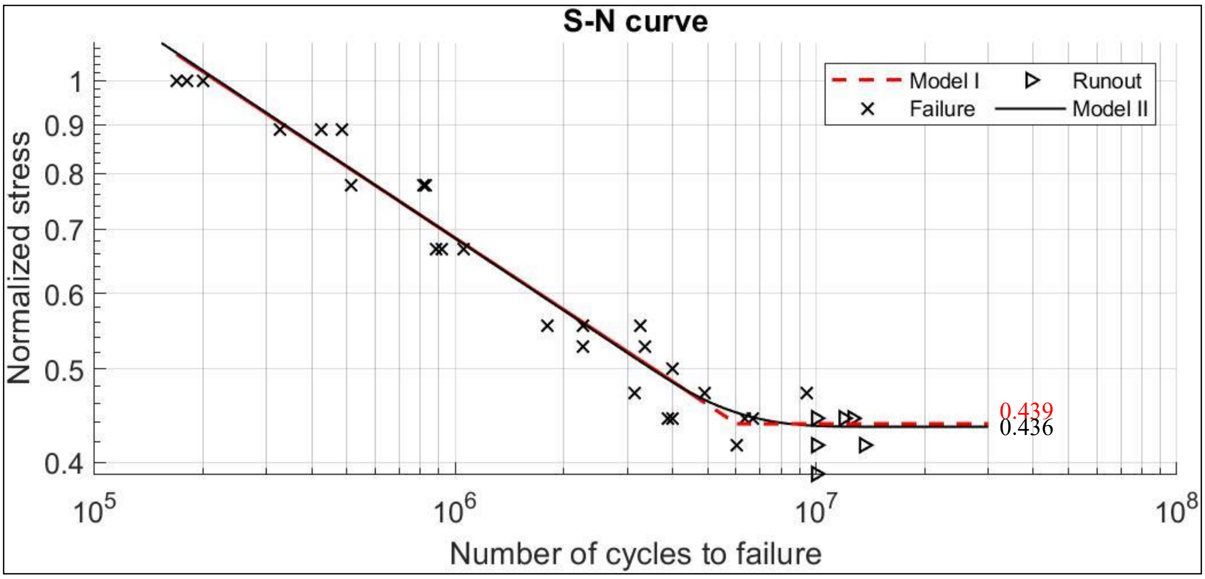


Figure 5.1 Fatigue data of $AlSi_7Mg_{0.3}$ and median fitted fatigue characteristics by using Model I and Model II

Thus, for this material, the parameter estimate for Model II is practically the same as for Model I, as can be seen in the median S-N curves shown in Figure 5.1.

Table 5.1 Comparison of SN curve parameters for $AlSi_7Mg_{0.3}$

a		*b*		σ_Y		Normalized μ_{X_l}		σ_{X_l}		Normalized R90C90	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
14.4244	14.3521	-4.0291	-3.9951	0.1232	0.1218	0.4385	0.4355	4.2022	4.1523	0.384026	0.366131
Δ :	**0.50%**	**Δ :**	**0.85%**	**Δ :**	**1.13%**	**Δ :**	**0.69%**	**Δ :**	**1.20%**	**Δ :**	**4.89%**

Even if the curve with 90% of reliability and 90% of confidence level (R90C90 curve) is considered, the difference between the two models increases slightly, but is still less than 5 %, as shown in Figure 5.2 and Table 5.1. From the analysis of the results, it would appear that the characteristic parameters of S-N curve, calculated with both methods, do not differ much from each other even if the Maximum Likelihood principle seems to be, with good reason, slightly more conservative resulting in a slightly lower R90C90 fatigue limit and in a knee of the curve shifted towards a higher number of cycles, as shown in Figure 5.2. This slight lowering of the fatigue limit is probably due to the presence of a failure very close to the number of runout cycles, N_G. Model II, unlike Model I which must follow the Staircase protocol in the infinite life section, allows to remove some data, especially the runouts which are more time consuming, without losing too much information and maximizing the available data.

Table 5.2 shows the results of fatigue analysis carried out by removing two runouts for the calculation with Model II and comparing it with the results obtained with Model I without removing any data. The first experimental data removed was the runout at the lowest stress level, (0.389;1e7). Since there are runouts at higher stress levels, removing this data does not significantly influence the curve shape. The second data removed was (0.417;1e7) because there were two tests at the same stress level and number of cycles which therefore did not give different information for the shape of the S-N curve with the Model II.

Clearly, although the characteristic parameters of the Wöhler curve remain almost unchanged, decreasing the number of data points in the infinite life section, increases the standard deviation of the fatigue limit and the curve at 90% confidence is shifted down, as shown in Figure 5.3. However, considering that the tests were carried out at a frequency of 80 Hz, and that the number of cycles at which they were interrupted is 1e7, removing two runouts results in a saving of 2 days and 21 hours in test time.

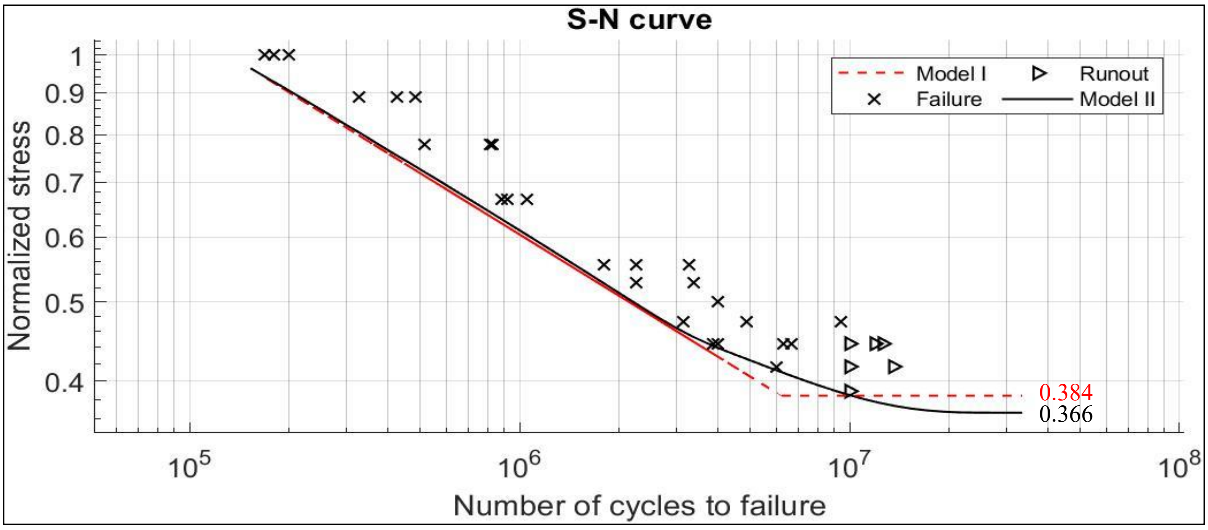


Figure 5.2 R90C90 curve of $AlSi_7Mg_{0.3}$ with Model I and Model II

Table 5.2 Comparison of fatigue parameters without two data for the Model II

a		*b*		σ_Y		Normalized μ_{X_l}		σ_{X_l}		Normalized R90C90	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
14.4244	14.3522	-4.0291	-3.9951	0.1232	0.1218	0.4385	0.4335	4.2022	4.6518	0.384026	0.339728
Δ :	**0.50%**	**Δ :**	**0.85%**	**Δ :**	**1.13%**	**Δ :**	**1.14%**	**Δ :**	**9.66%**	**Δ :**	**13.04%**
		Saved time [day.h]:						**2.21**			

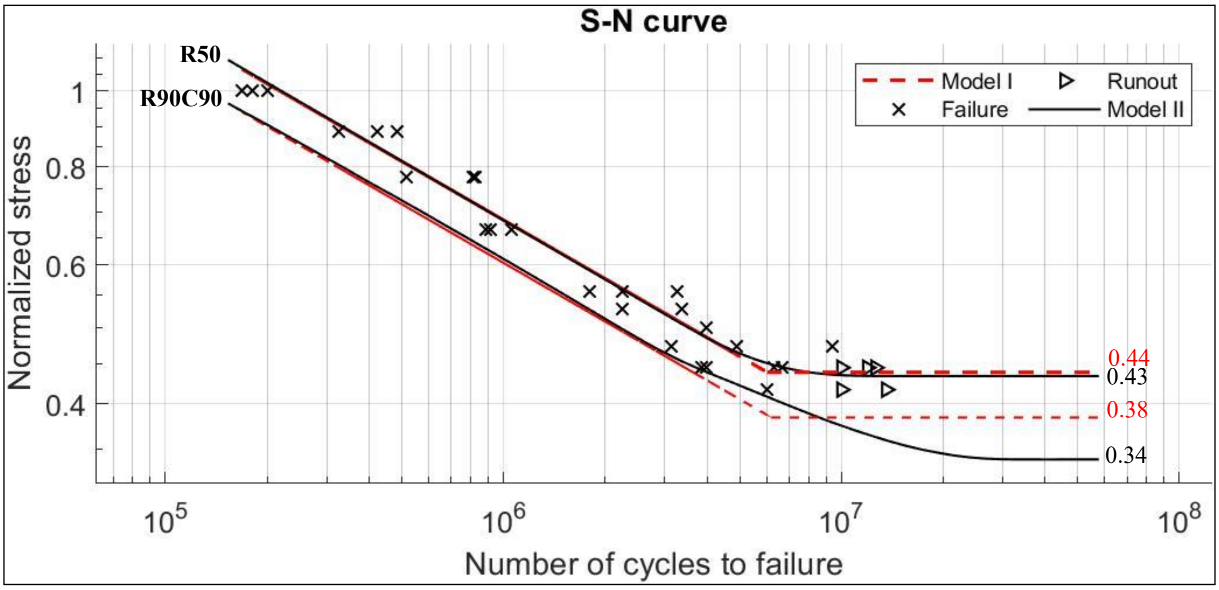


Figure 5.3 R50 and R90C90 curves of $AlSi_7Mg_{0.3}$ without considering two runout data for the Model II

5.1.2 $AlSi_{10}Mg$-beam shaping

The second material analysed was the aluminium alloy $AlSi_{10}Mg$ produced by selective laser melting and tested with axial load fatigue tests. The tests were carried out on hourglass specimens (K_t = 1.06) in a fully reversed cycle (*R* = -1), at room temperature and with a frequency of 100 Hz. Figure 5.4 shows the results of the experimental tests with the R50 S-N curves plotted by using Model I and Model II.

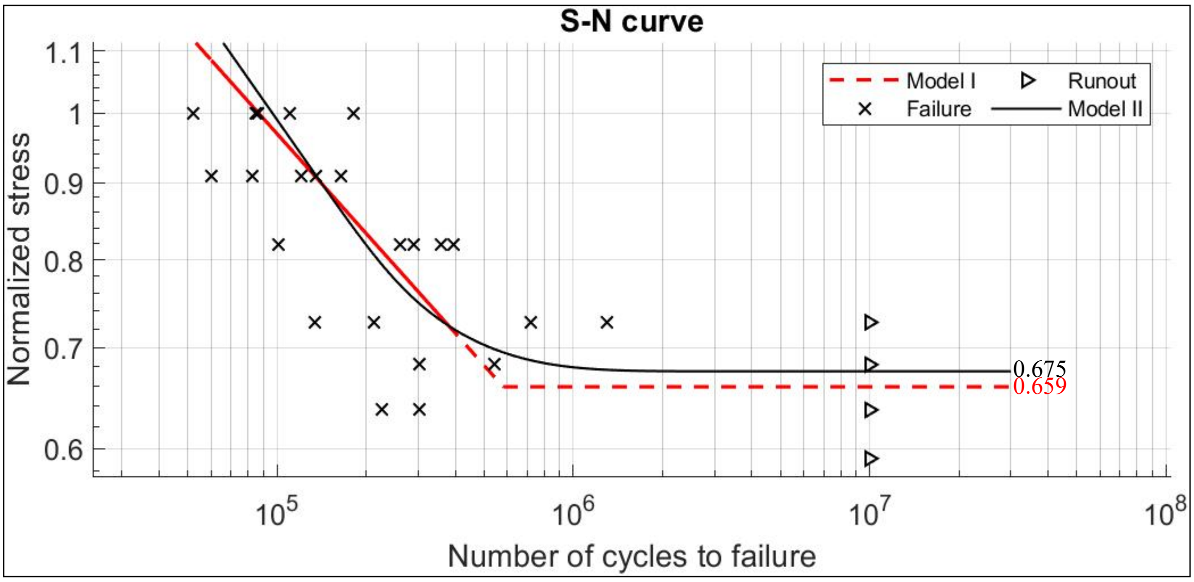


Figure 5.4 Fatigue data of $AlSi_{10}Mg$ and median fitted fatigue characteristics by using Model I and Model II

The tests were conducted on 32 specimens of which 23 specimens failed and 9 specimens survived. For the Staircase method, 15 specimens were used (6 specimens failed and 9 specimens reached the limit number of cycles N_G = 1e7). The characteristic parameters of the S-N curve were compared in Table 5.3.

Table 5.3 Comparison of SN curve parameters for $AlSi_{10}Mg$

a		*b*		σ_Y		Normalized μ_{X_l}		σ_{X_l}		Normalized R90C90	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
14.2438	12.2711	-4.5582	-3.5691	0.2583	0.2672	0.6591	0.6749	5.6349	6.7270	0.531332	0.537513
Δ :	**16.08%**	**Δ :**	**27.71%**	**Δ :**	**3.33%**	**Δ :**	**2.34%**	**Δ :**	**16.23%**	**Δ :**	**1.15%**

The median curve constructed with Model I has a greater slope and intercept, *a* and *b* respectively (in absolute value) than that constructed with Model II. This is due to the fact that the method using Maximum Likelihood considers not only all the data of the finite life section but also all the data of the infinite life section to determine the slope and intercept of the linear section. In this case, in fact, Model II also uses the failures at (0.636;3.01e5) and (0.636;2.26e5) for the calculation of slope and intercept, unlike Model I in which, by internal convention of the CRF, since they are below the mean value of the fatigue limit, 0.659, are used only in a pass/fail manner for the determination of the latter with the Staircase method, but losing the information relating to these two failures. In addition, Model II has a higher standard deviation in the unlimited fatigue life regime than Model I probably due to the higher scatter of specimen fatigue life near the fatigue limit. The difference in the value of the fatigue limit at 50 % probability of failure is less than 3 %, with a higher standard deviation in the case of the second model. If the curve below which 90 % of the specimens survive is considered, with 90 % confidence, i.e. the R90C90 curve, the difference in fatigue limits is reduced to 1 %, as shown in Figure 5.5.

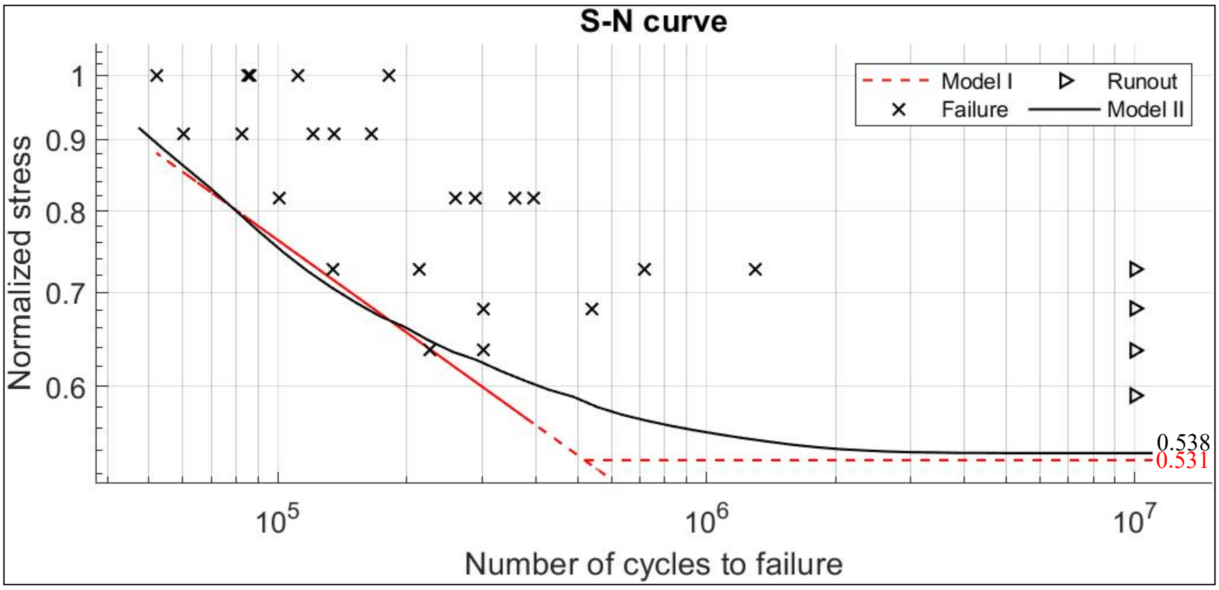


Figure 5.5 R90C90 curve of $AlSi_{10}Mg$ with Model I and Model II

Again, Model II allows greater freedom in the design of the test campaign, without the need to follow the Staircase protocol, and therefore the possibility of carrying out less tests, especially time consuming ones, such as runouts. In this case we tried to compare the results considering, for Model II, a reduced dataset obtained taking into consideration, for the infinite life section, only the first 9 experimental results obtained with the Staircase, as shown in Figure 5.6. In fact, if the other 6 tests were not considered, a sufficient number of runouts (from 9 to 5) and failure (from 6 to 4), distributed over the different stress levels, would still be available for the determination of the S-N curves and the fatigue limit, without significant differences on the median and R90C90 values of the fatigue limit, as shown in Figure 5.7 and Table 5.4.

Levels	Stress Sa [MPa]	Sequence number of specimens														
i		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	**0.727**									X				X		O
2	**0.682**		X						O		X		O		O	
1	**0.636**	O		X		X		O				O				
0	**0.591**				O		O									
ΔS [MPa]	**0.045**															
N° of cycles		**1.00E+07**	**5.41E+05**	**3.01E+05**	**1.00E+07**	**2.26E+05**	**1.00E+07**	**1.00E+07**	**1.00E+07**	**1.30E+06**	**3.03E+05**	**1.00E+07**	**1.00E+07**	**7.19E+05**	**1.00E+07**	**1.00E+07**

Figure 5.6 Original and reduce Staircase for $AlSi_{10}Mg$

By removing some data, the difference in the slope and intercept values calculated by the two methods inevitably increases, from a difference of 16% to 19.5% for parameter *a* and from 27.7% to 34% for parameter *b*. But if one accepts this level of accuracy on the results, one can save time, and therefore costs, for the fatigue characterisation of the material by 4 days and 17 hours, going from a time of 10 days and 19 hours to a time of 6 days and 3 hours for an entire characterisation campaign.

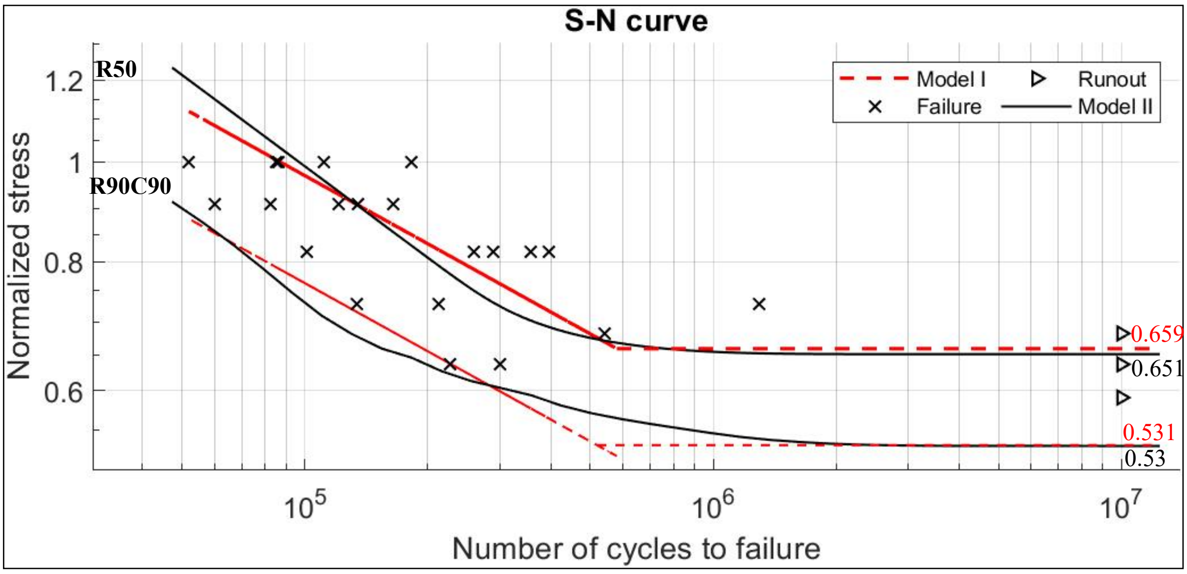


Figure 5.7 R50 and R90C90 curves of $AlSi_{10}Mg$ without considering 6 data for the Model II

Table 6.4 Comparison of fatigue parameters for $AlSi_{10}Mg$ without 6 data for the Model II

a		*b*		σ_Y		Normalized μ_{X_l}		σ_{X_l}		Normalized R90C90	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
14.2438	11.9217	-4.5582	-3.3980	0.2583	0.2656	0.6591	0.6509	5.6349	4.6843	0.531332	0.530383
Δ :	**19.48%**	**Δ :**	**34.14%**	**Δ :**	**2.77%**	**Δ :**	**1.26%**	**Δ :**	**20.29%**	**Δ :**	**0.18%**

Saved time[day.h]: **4.17**

5.1.3 6060-T66

6060-T66 is an extrusion aluminium alloy in the T66 temper. Hourglass specimens with a K_t of 1.06 of this material were subjected to axial fatigue tests, conducted at room temperature, with a stress ratio R = -1 and a frequency of 100 Hz. Twenty-seven specimens were used for the fatigue characterisation, of which 19 failures and 8 runouts. The Staircase was performed on 15 specimens (7 failures, 8 runouts). Figure 5.8 show experimental test results with R50 S-N curves plotted by using Model I and Model II.

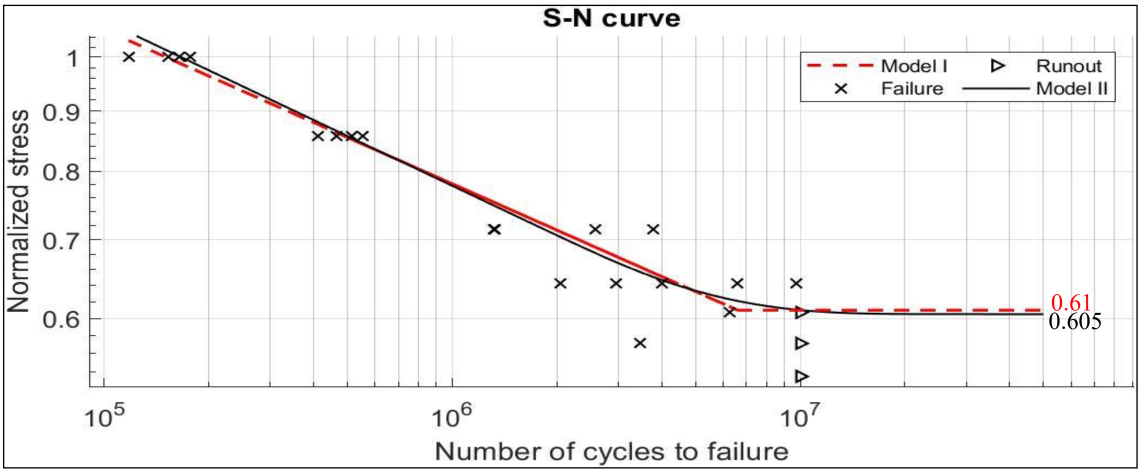


Figure 5.8 Fatigue data of 6060-T66 and median fitted fatigue characteristics by using Model I and Model II

Table 5.5 compares the results obtained using the two methods. As can also be seen from the comparison of the median curves in Figure 5.8, both methods give similar results with a practically coincident fatigue limit even though the standard deviations for limited and unlimited fatigue life regimes are slightly higher for the Model II.

Table 5.5 Comparison of SN curve parameters for 6060-T66

a		b		σ_Y		Normalized μ_{X_l}		σ_{X_l}		Normalized R90C90	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
21.5802	20.5295	-7.6425	-7.1328	0.1777	0.1989	0.6097	0.6051	4.5329	4.6845	0.534157	0.46975
Δ :	**5.12%**	**Δ :**	**7.15%**	**Δ :**	**10.62%**	**Δ :**	**0.76%**	**Δ :**	**3.24%**	**Δ :**	**13.71%**

However, if we compare the R90C90 curves we can see a significant difference between the two models, as shown in Figure 5.9. In fact, in this case, with the available data, the Model II, with good reasons, presents the knee of the curve at higher number of cycles and finds a fatigue limit after 1e8 number of cycles, on the contrary of the Model I for which a fatigue resistance is established at the prefixed number of cycles, N_G = 1e7 which however does not guarantee a safe situation. The reason for this difference may be due to the presence of the failure, (0.64;9.76e6), very close to the number of runout cycles, as can be seen in Figure 5.9.

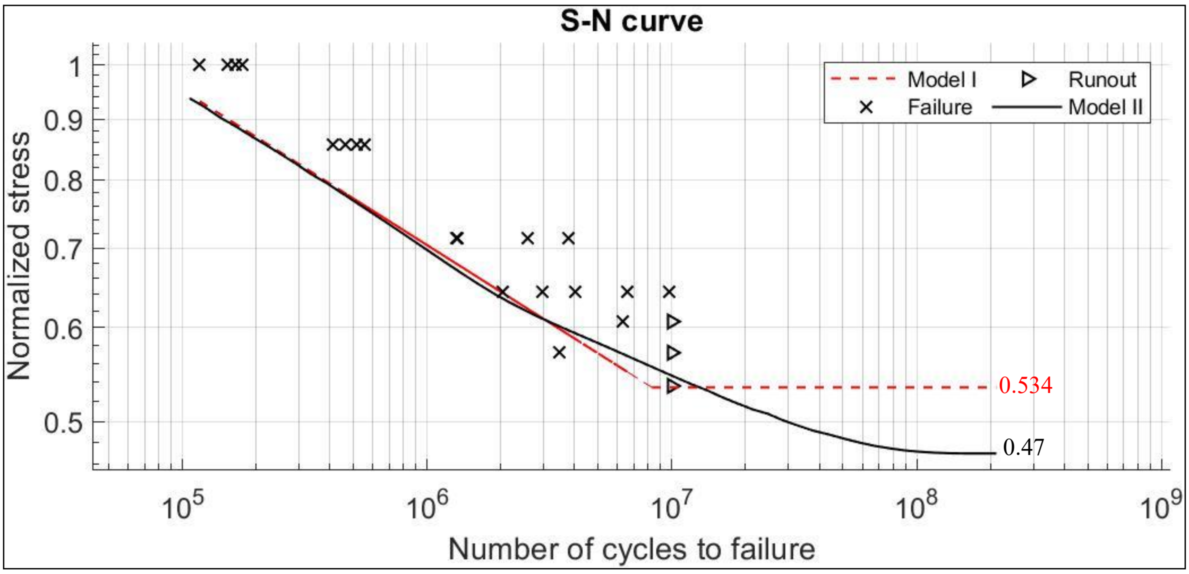


Figure 5.9 R90C90 curve of 6060-T66 with Model I and Model II

In fact, if the last failure was removed, the R50 curves would be even closer, with the difference in the median values of the fatigue limit reduced to 0.47 %, but especially the difference in the R90C90 curves would also decrease, with a difference in the values of the fatigue limit reduced to 2.25 %. In addition, in this case, Model II would find the horizontal asymptote for fewer number of cycles than in the previous case, as shown in Figure 5.10.

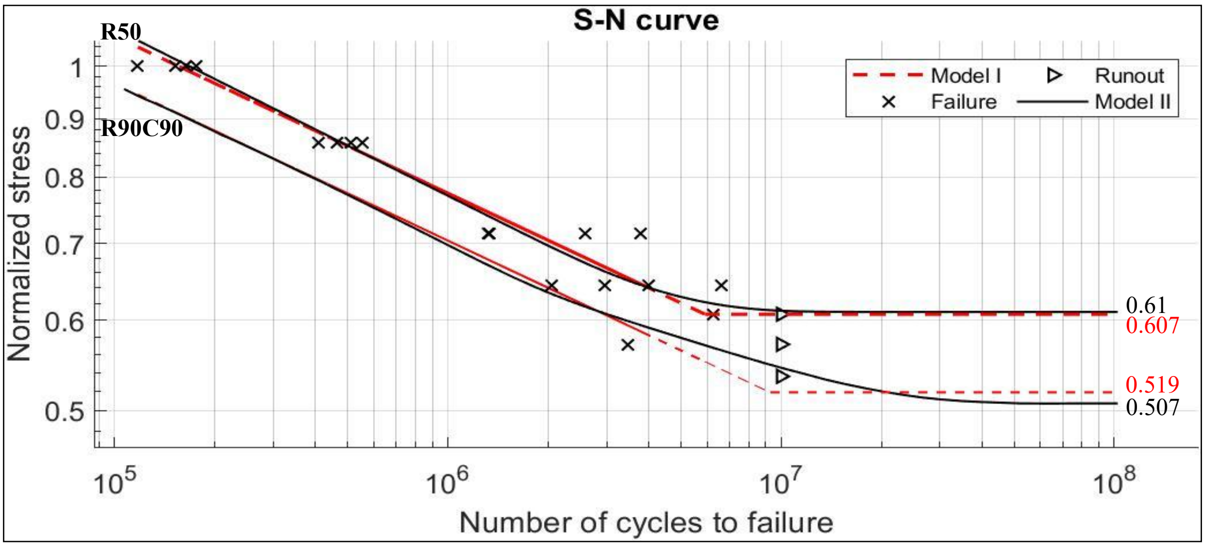


Figure 5.10 R50 and R90C90 curves for 6060-T66 without last failure

5.1.4 17-4 PH

Alloy 17-4 PH is a chromium-nickel-copper precipitation-hardening martensitic stainless steel with an addition of niobium, that combines high strength and hardness with good corrosion resistance. The fatigue characterisation of this material involved flat bending tests at room temperature with stress ratio of 0, K_t of 1.068 and frequency of 100 Hz. The tests were carried out on 32 hourglass specimens, of which 25 were broken and 7 run-outs. The Staircase was performed on 15 specimens (8 failures, 7 runouts). Figure 5.11 shows experimental test results and compares the R50 and R90C90 curves obtained with the two methods.

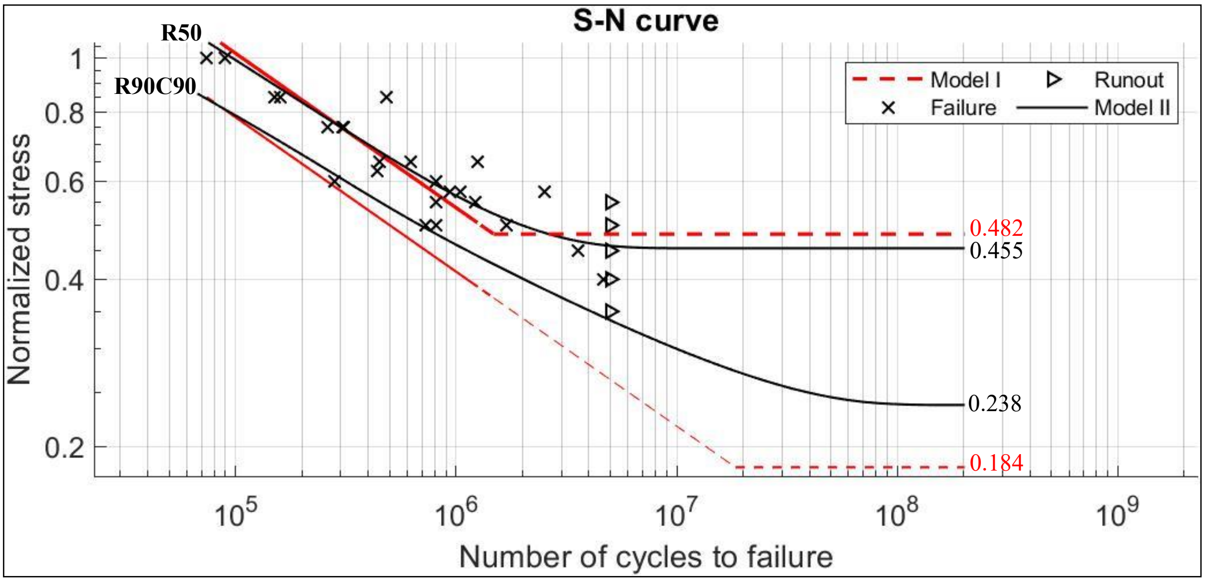


Figure 5.11 Experimental test results, R50 and R90C90 curves plotted by using Model I and II for 17-4 PH

Table 5.6 compares the characteristic parameters of the S-N curve calculated with the two methods. Both models present a high standard deviation of the fatigue limit probably due to the higher dispersion of specimen fatigue life near the fatigue limit. In addition, Model II shows a knee of the curve shifted towards higher number of cycles and a lower median value of the fatigue limit. In addition, the value of the R90C90 fatigue limit for Model II is higher than that of Model I since the latter has a higher standard deviation for the fatigue limit and therefore a higher value of the $K_{Owen} \cdot \sigma_{X_l}$ factor which lowers the curve.

Table 5.6 Comparison of SN curve parameters for 17-4 PH

a		*b*		σ_Y		Normalized μ_{X_l}		σ_{X_l}		Normalized R90C90	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
14.3989	15.2522	-3.6007	-3.9458	0.2290	0.2267	0.4821	0.4546	51.1927	34.4130	0.183562	0.237584
Δ :	**5.59%**	**Δ :**	**8.75%**	**Δ :**	**1.05%**	**Δ :**	**6.07%**	**Δ :**	**48.76%**	**Δ :**	**22.74%**

The difference in the median values of the fatigue limit is probably due to the presence of a failure very close to the number of runout cycles, N_G = 5e6. If, in fact, this last failure, (0.4;4.64e6), is not considered in the calculation of the S-N curves, the fatigue limit with Model I remains unchanged because in the Staircase protocol the least frequent event is still the runout (in case of an equal number of failures and runouts, the runouts are considered), while with Model II there is an increase in the median value of the fatigue limit, with a difference, compared to Model I, that is reduced to 0.52 %, as shown in Figure 5.12.

However, the R90C90 curves are even more different from each other because the standard deviation for Model I remains unchanged, since runouts and not failures are used in its calculation, while the already lower standard deviation for Model II decreases further because it takes all data into account for the curve analysis.

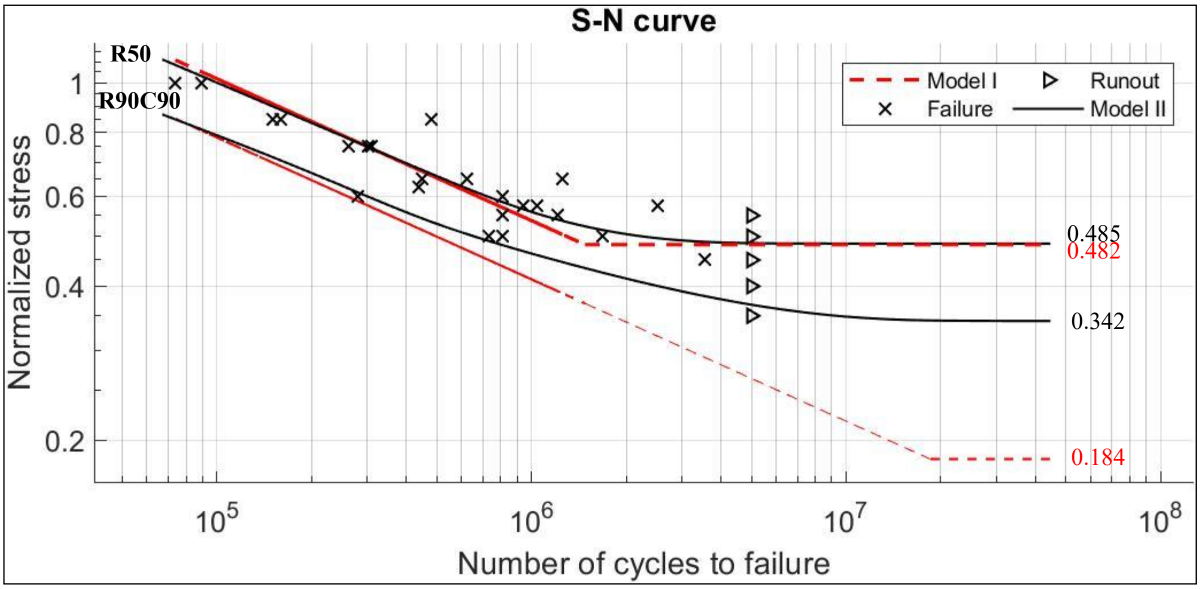


Figure 5.12 R50 and R90C90 curves of 17-4 PH without last failure

5.1.5 CP800HR

CP800HR is a Hot-Rolled Complex Phase steels with a tensile strength of 800 MPa and extremely fine grain size and microstructure containing small amounts of martensite, pearlite and retained austenite embedded in a ferrite-bainite matrix. High grain refinement is achieved by precipitation of micro alloying elements such as Nb, Ti or V. Hourglass specimens of this material were subjected to axial fatigue tests at room temperature, with a stress ratio of -1, specimen K_t of 1.06 and a frequency of 100 Hz. The fatigue characterisation was performed on 37 specimens of which 30 were broken and 7 were runouts. The specimens used for the Staircase are 15 of which 8 are broken and 7 runouts. Figure 5.13 shows the results of the experimental tests and the plotted R50 curves using both methods while Table

5.7 shows the values of the SN curve parameters calculated with the two models. The fatigue finite life parameters, a, b, σ_Y calculated with Model I and Model II are very similar to each other although they are not coincident because Model I does not use the data (0.652;3.21e6), (0.652;3.62e6) and (0.63;4.18e6) which are below the median value of the fatigue limit, 0.657, in the least squares regression for the calculation of the linear section of the fatigue life region.

Table 5.7 Comparison of SN curve parameters for CP800HR

a		b		σ_Y		Normalized μ_{X_l}		σ_{X_l}	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
29.0083	27.7664	-9.0018	-8.5212	0.1324	0.1353	0.6568	0.6129	17.6616	24.6795
Δ :	**4.47%**	**Δ :**	**5.64%**	**Δ :**	**2.15%**	**Δ :**	**7.17%**	**Δ :**	**28.44%**

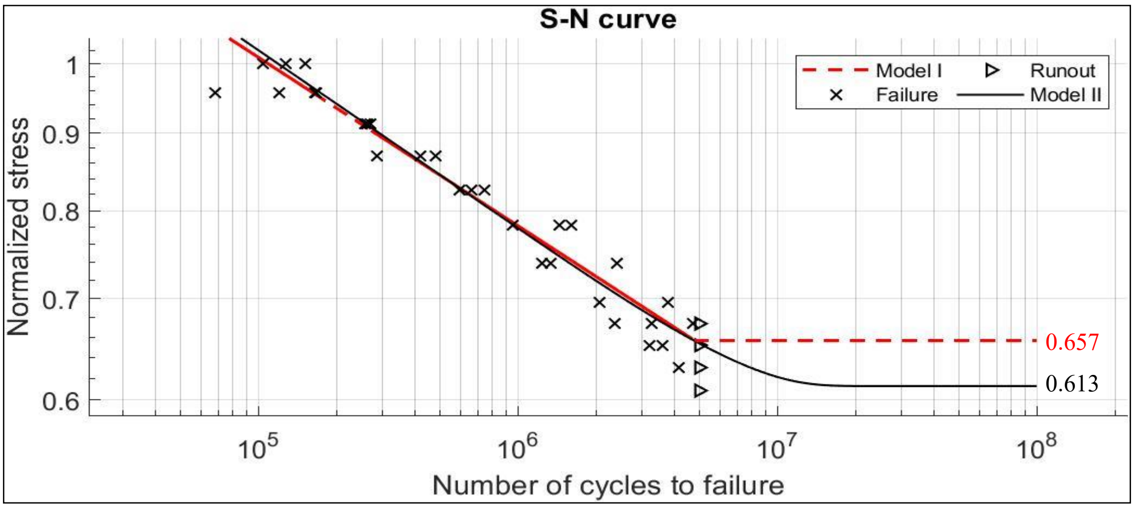


Figure 5.13 Experimental test results and R50 curves plotted by using Model I and II for CP800HR

The median value of the fatigue limit is lower in Model II which is more conservative since there are failures, such as those corresponding to the data (0.674;4.71e6), (0.63;4.18e6), (0.696;3.79e6), which are very close to the number of runout cycles, N_G = 5e6. In fact, if these last three failures were not considered, the two curves would be practically the same with the difference of the fatigue limit reduced to 0.7%, as shown in Figure 5.14.

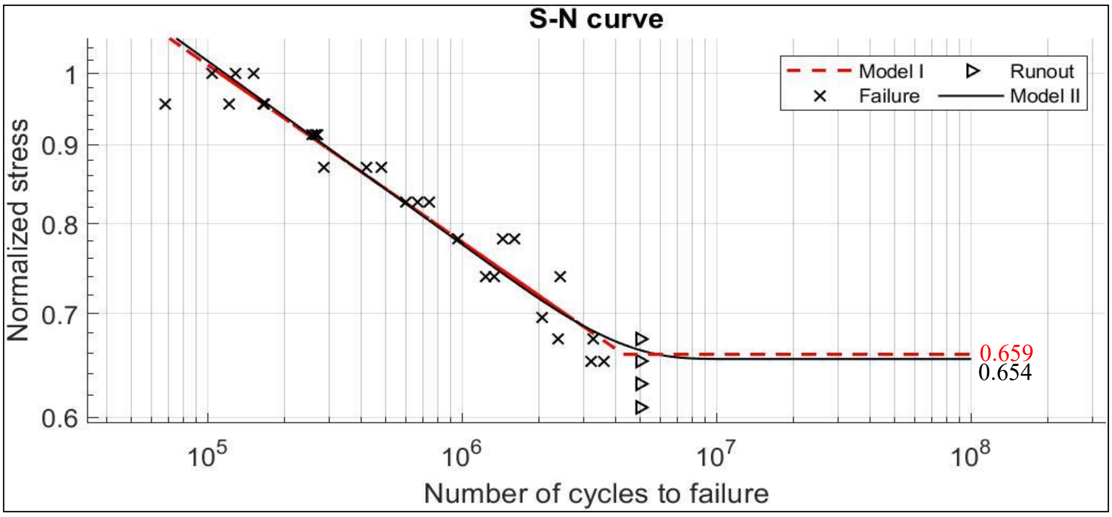


Figure 5.14 R50 curves for CP800HR without considering last three failures

When considering R90C90 curves, the Maximum Likelihood method fails to find a fatigue limit at this confidence level and with this data available. Furthermore, apart from the fact that there are failures very close to the number of runout cycles, the runout specimens are located even before the knee of the R50 curve plotted using Model II. This means that higher number of cycles to runout would be required to obtain results at a 90 % confidence level. Model I, on the other hand, only shifts the R50 curve by a factor of $K_{Owen} \cdot \sigma_{X_l}$ and therefore finds a fatigue strength of R90C90 at the number of runout cycles which, however, does not ensure a safe condition.

5.1.6 DP1000HF

DP1000HF is a dual phase steel with a calculated tensile strength of 980 MPa that has a ferritic–martensitic microstructure. Ferrite that imparts High Formability properties, and martensite that accounts for the strength. This material was cold rolled and the specimens were taken in the direction parallel to the rolling one. Hourglass specimens of this material were subjected to axial fatigue tests at room temperature, with a stress ratio equal to -1, a load concentration coefficient of 1.06 and a frequency of 50 Hz. The tests were carried out on 28 specimens of which 20 were broken and 8 were runouts. For the Staircase 15 specimens were used, 7 of which were broken and 8 runouts. Figure 5.15 shows the experimental results and the plotted R50 curves using Model I and Model II.

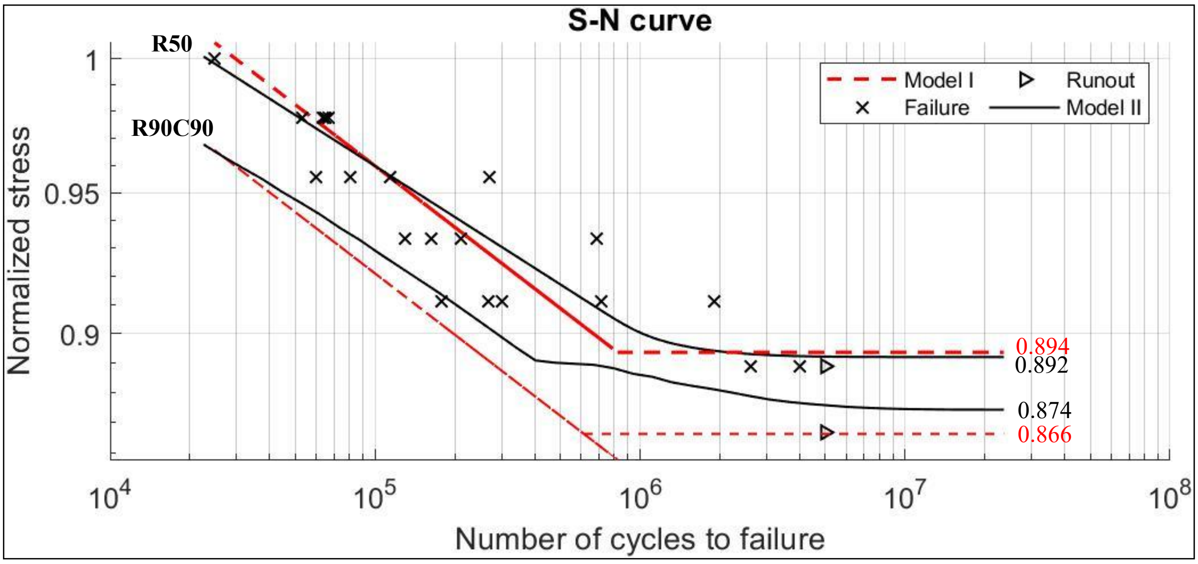


Figure 5.15 Experimental testing results, R50 and R90C90 curves of DP1000 with Model I and Model II

Table 5.8 shows the characteristic parameters of the SN curves computed with both methods. The parameters of the finite life regione are slightly different since Model I does not use all the data but only those above the median value of the fatigue limit, 0.894. The fatigue limit values are practically the same with a difference of less than 1 %.

Table 5.8 Comparison of SN curve parameters for DP1000HF

a		b		σ_Y		Normalized μ_{X_l}		σ_{X_l}		Normalized R90C90	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
82.8275	98.6135	-29.5319	-35.5222	0.2781	0.2984	0.8937	0.8920	5.3000	2.1924	0.866173	0.874228
Δ :	**16.01%**	**Δ :**	**16.86%**	**Δ :**	**6.81%**	**Δ :**	**0.18%**	**Δ :**	**141.75%**	**Δ :**	**0.92%**

The values of the standard deviation of the fatigue limit are instead very different because Model I uses the simplified formula (2.19), i.e. $\sigma_{X_l} = 0.53 \cdot \Delta S$, in which the step for the Staircase, ΔS, is equal to 10 MPa, otherwise the Staircase would not have been valid because the step would have been lower than $0.5 \cdot \sigma_{X_l}$ and one of the necessary conditions for the validity of the Staircase is that the step is in the range $0.5 \cdot \sigma_{X_l} \div 2 \cdot \sigma_{X_l}$. For Model II, no correction is provided. If the R90C90 curves are considered, the values of the fatigue limits are similar to each other with a difference of less than 1%, but the curve constructed with the Profile Likelihood seems to have some irregularities in its shape probably due to the presence of failures very close to the runouts but also due to the low value of the standard deviation of the fatigue limit caused by a bad design of the test with a too large step and with only 3 stress levels in the infinite life region. Figure 5.16 shows the R50 and R90C90 curves in which the last two failures have not been considered. In this case the R90C90 curve plotted using Model II is regular with the difference, compared to Model I, increasing to 2.3%.

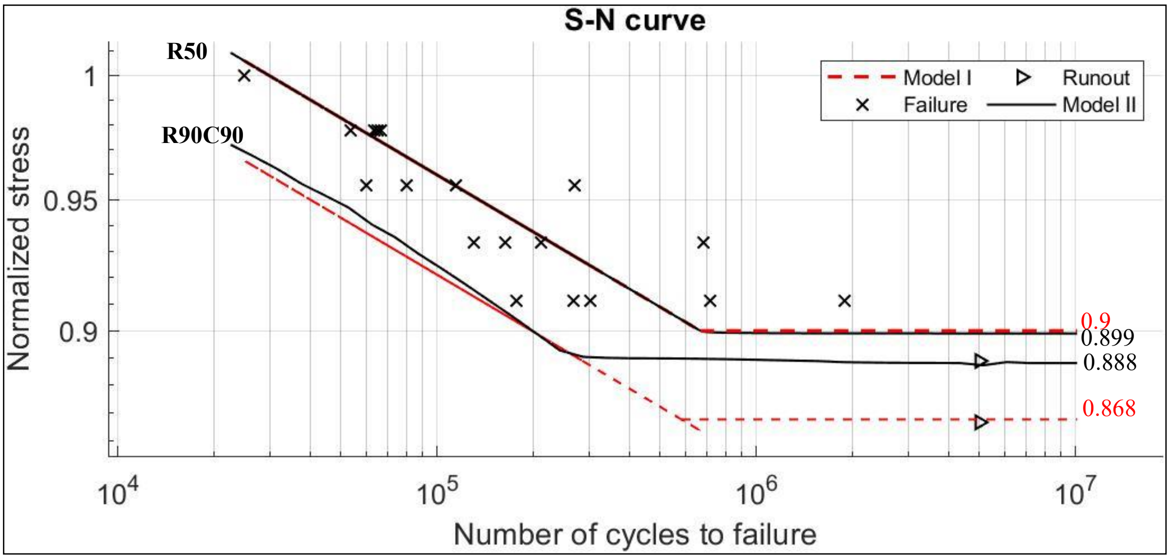


Figure 5.16 R50 and R90C90 curves for DP1000 without considering last two failure

5.1.7 Trip1000

TRIP1000 is a Transformation Induced Plasticity steel that offer an outstanding combination of strength (tensile strength is equal to 980 MPa) and ductility as a result of their microstructure. The microstructure of these steels consists of islands of hard residual austenite and carbide-free bainite dispersed in a ductile ferritic matrix. Austenite is transformed into martensite during plastic deformation. This material is cold rolled and hourglass specimens are taken in the direction parallel to the rolling direction and subjected to axial fatigue tests at room temperature with R = -1, K_t = 1.06 and frequency = 50 Hz. Tests were carried out on 26 specimens of which 18 failures and 8 runouts. The staircase was carried out on 15 specimens of which 7 failures and 8 runouts. Figure 5.17 shows the experimental results of the tests carried out and compares the R50 and R90C90 curves calculated with both methods while in Table 5.9 there is a comparison of the parameters calculated with the two methods.

Table 5.9 Comparison of SN curve parameters for TRIP1000

a		*b*		σ_Y		Normalized μ_{X_l}		σ_{X_l}		Normalized R90C90	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
74.2739	85.6732	-25.6623	-29.9005	0.2400	0.2489	0.9329	0.9325	7.0820	6.5221	0.899812	0.899626
Δ :	**13.31%**	**Δ :**	**14.17%**	**Δ :**	**3.57%**	**Δ :**	**0.04%**	**Δ :**	**8.59%**	**Δ :**	**0.02%**

The values of the fatigue limit R50 and R90C90 are practically coincident with a difference of less than 0.05 %, while the values of the parameters of the finite life section are slightly different due to the fact that Model I does not consider failure at (0.92;2.29e6) which is below the median value of the fatigue limit 0.933.

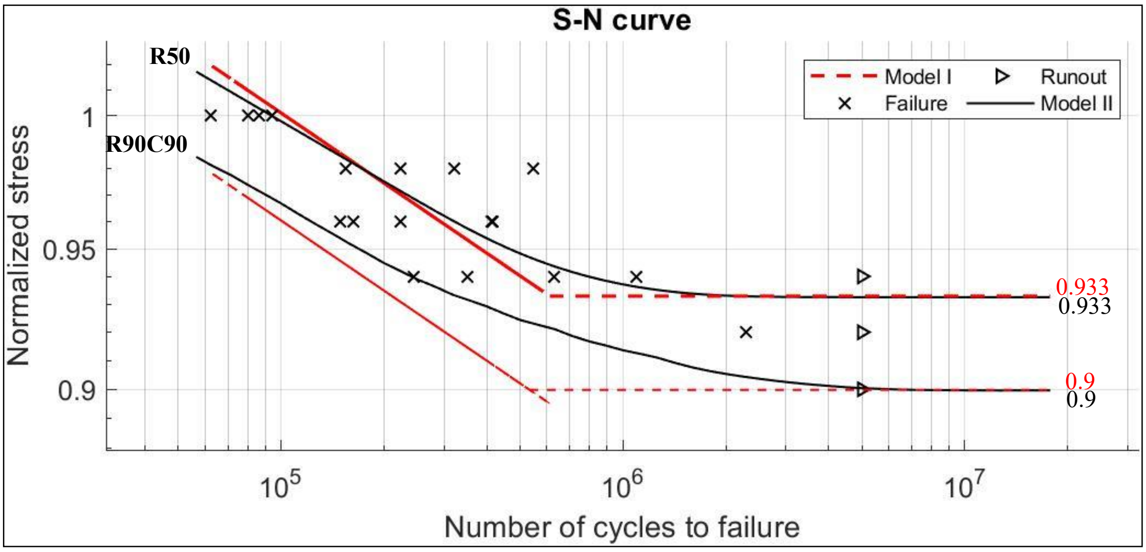


Figure 5.17 Experimental results, R50 and R90C90 curves for TRIP1000 plotted by using Model I and Model II

Figure 5.19 shows the plot of the curves R50 and R90C90 in which data were removed from the dataset used for the calculation of the curves with the method based on Maximum Likelihood principle (Model II), in order to demonstrate that despite small changes in the parameters, results with similar accuracy are obtained compared to the previous case in which the whole dataset was considered. In particular, the 6 experimental data, marked with a darker colour in Figure 5.18, relative to the infinite life region have been removed.

Levels	Stress Sa [MPa]	Sequence number of specimens														
i		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	0.960								X				X			
2	0.940					X		O		X		O		X		X
1	0.920		X		O		O				O				O	
0	0.900	O		O												
ΔS [MPa]	0.020															
N° of cycles		5.0E+06	2.3E+06	5.0E+06	5.0E+06	1.1E+06	5.0E+06	5.0E+06	1.5E+05	2.4E+05	5.0E+06	5.0E+06	4.1E+05	6.3E+05	5.0E+06	3.5E+05

Figure 5.18 Original and reduce Staircase for TRIP1000

This would result in a time saving of 3 days and 19 hours on the fatigue characterisation of the TRIP1000. Table 5.10 shows the comparison of the parameters in this condition. Having removed some of the data, it is clear that the differences in parameter values, with the exception of σ_{X_l}, increase and, especially the R90C90 curve plotted with Model II, shows an asymptotic trend at lower stress levels. This is because, with a lower number of tests, in order to have a 90% confidence that 90% of the specimens will survive below this stress level, the curve inevitably shifts downwards.

Table 5.10 Comparison of parameters for TRIP1000 without considering 6 data in the Model II

a		b		σ_Y		Normalized μ_{X_l}		σ_{X_l}		Normalized R90C90	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
74.2739	89.2218	-25.6623	-31.2193	0.2400	0.2661	0.9329	0.9294	7.0820	7.3182	0.899812	0.886231
Δ :	**16.75%**	**Δ :**	**17.80%**	**Δ :**	**9.82%**	**Δ :**	**0.37%**	**Δ :**	**3.23%**	**Δ :**	**1.53%**
		Saved time[day.h]:						**3.19**			

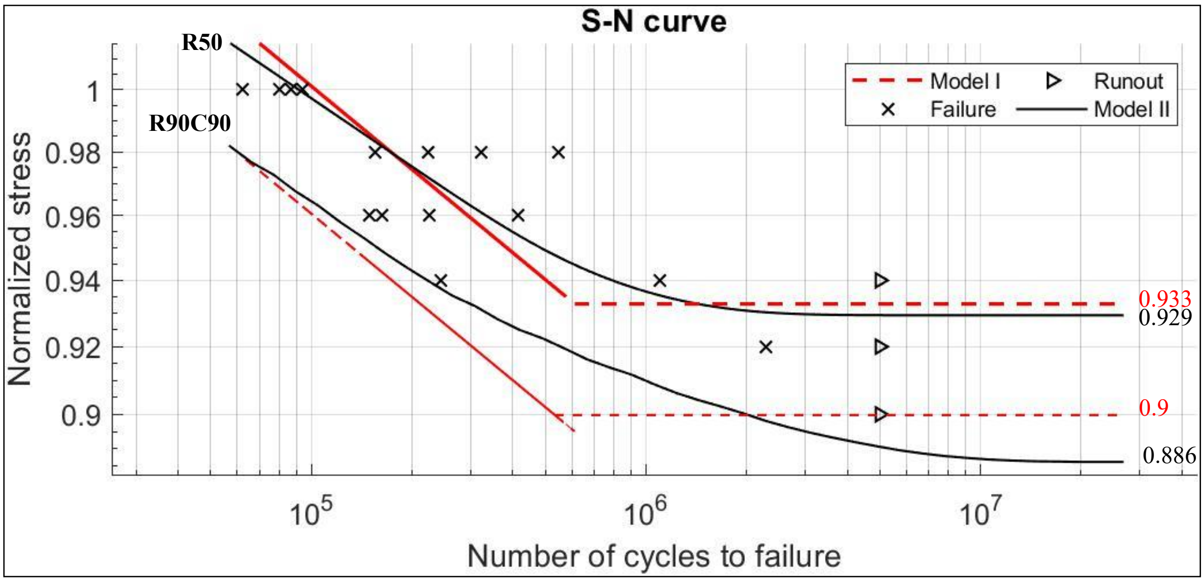


Figure 5.19 R50 and R90C90 curves without considering 6 data in the Model II

5.1.8 Cast Iron GH 60-38-10

GH 60-38-10 is a spheroidal cast iron with ferritic-pearlitic matrix that is characterised by the spherical shape of the graphite nodules offering the material enhanced mechanical strength and toughness. The first number indicates the minimum tensile strength expressed in ksi (kilopound per square inch, 1ksi = 6.895 MPa), the second the yield stress $R_{p0.2}$ in ksi and the third the minimum percentage elongation. Fatigue tests were carried out on 30 specimens, of which 21 failures and 9 runouts, which were subjected to flat bending fatigue tests at room temperature, and with $K_t = 1.068$, frequency = 100 Hz and R = 0. The Staircase was performed on 15 samples resulting in 6 failures and 9 runouts. The experimental results and the R50 and R90C90 curves for Model I and Model II are compared in Figure 5.20. The slope and intercept of the finite life region are clearly different because Model I does not use the failures corresponding to (0.783;2.96e5), (0.783;3.46e5), which are below the fatigue limit, 0.794. The values of the fatigue limit at 50 % probability of failure and at 90 % probability of failure with 90 % confidence differ by no more than 2 %. Both models show a large standard deviation for the fatigue life in the finite fatigue life region of the curves caused by the wide dispersion of the data. The parameters calculated with the two models are summarised in Table 5.11.

Table 5.11 Comparison of SN curve parameters for GH 60-38-10

a		b		σ_Y		Normalized μ_{X_l}		σ_{X_l}		Normalized R90C90	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
23.3187	19.4212	-7.3494	-5.7542	0.4301	0.4202	0.7944	0.8014	9.4599	8.7729	0.715801	0.727883
Δ :	**20.07%**	**Δ :**	**27.72%**	**Δ :**	**2.35%**	**Δ :**	**0.87%**	**Δ :**	**7.83%**	**Δ :**	**1.66%**

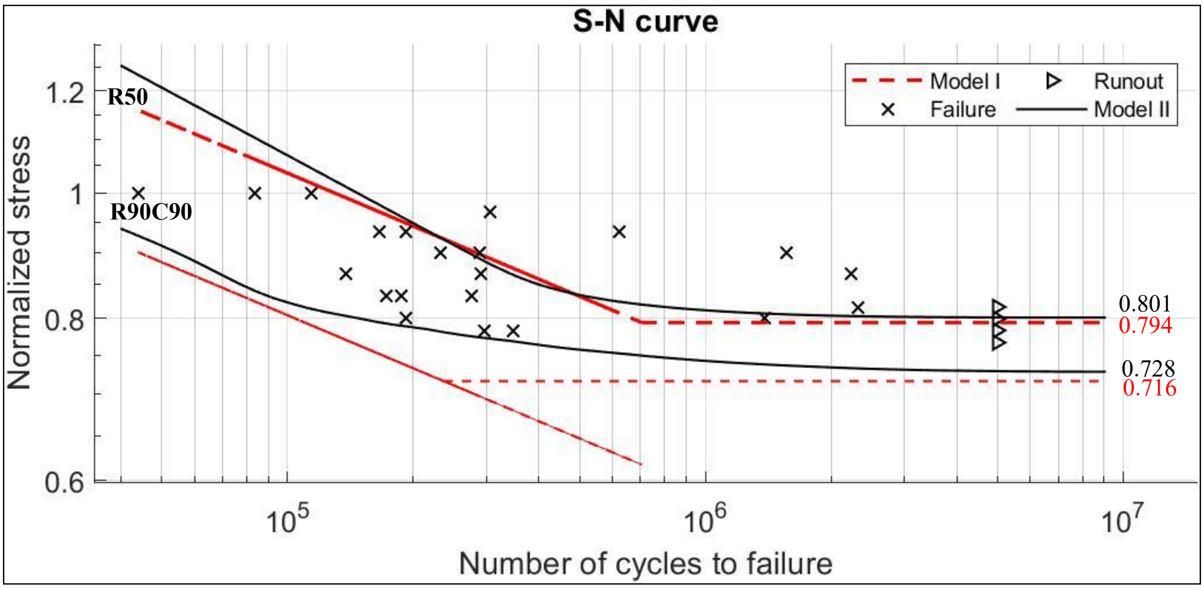


Figure 5.20 Experimental results, R50 and R90C90 curves for GH 60-38-10 plotted by using Model I and Model II

Levels	Stress Sa [MPa]	Sequence number of specimens														
i		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
4	0.833														X	
3	0.817									X				O		O
2	0.800						X		O		X		O			
1	0.783	X		X		O		O				O				
0	0.767		O		O											
ΔS [MPa]	0.017															
N° of cycles		3.46E+05	5.00E+06	2.96E+05	5.00E+06	5.00E+06	1.39E+06	5.00E+06	5.00E+06	2.32E+06	1.92E+05	5.00E+06	5.00E+06	5.00E+06	1.72E+05	5.00E+06

Figure 5.21 Original and reduced Staircase for GH 60-38-10

Even in this case we tried to decrease the data used in Model II and see how much the results differed from those of Model I and II calculated using the entire dataset. In particular, the last 6 data element of Staircase test were removed, as shown in Figure 5.21. In fact, in this way we still have a sufficient number of runouts and failures distributed over different stress levels from which we can draw important information for the construction of the curves. Figure 5.22 shows the R50 and R90C90 curves for GH 60-38-10 constructed considering for Model II a reduced dataset of 24 samples and for Model I the entire dataset of 30 samples. Table 5.12 shows, instead, the comparison of the parameters in this new condition.

Table 5.12 Comparison of parameters for GH 60-38-10 without considering 6 data in the Model II

a		b		σ_Y		Normalized μ_{X_l}		σ_{X_l}		Normalized R90C90	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
23.3187	22.8039	-7.3494	-7.1308	0.4301	0.4156	0.7944	0.7892	9.4599	5.1844	0.715801	0.731561
Δ :	**2.26%**	**Δ :**	**3.07%**	**Δ :**	**3.47%**	**Δ :**	**0.66%**	**Δ :**	**82.47%**	**Δ :**	**2.15%**
		Saved time[day.h]:						**2.08**			

The comparison of the parameters in Table 5.11 and Table 5.12 shows that by removing some data in Model II, there is a decrease in the difference between the parameters of the infinite life region and a slight increase in the difference between the values of the R90C90 fatigue limits which, however, is still around 2 %. However, there is an important decrease in the standard deviation of the fatigue limit calculated with Model II due to the removal of some data from the infinite life section. But if this level of accuracy in the results were accepted, a saving of 2 days could be made on the fatigue characterisation of this material.

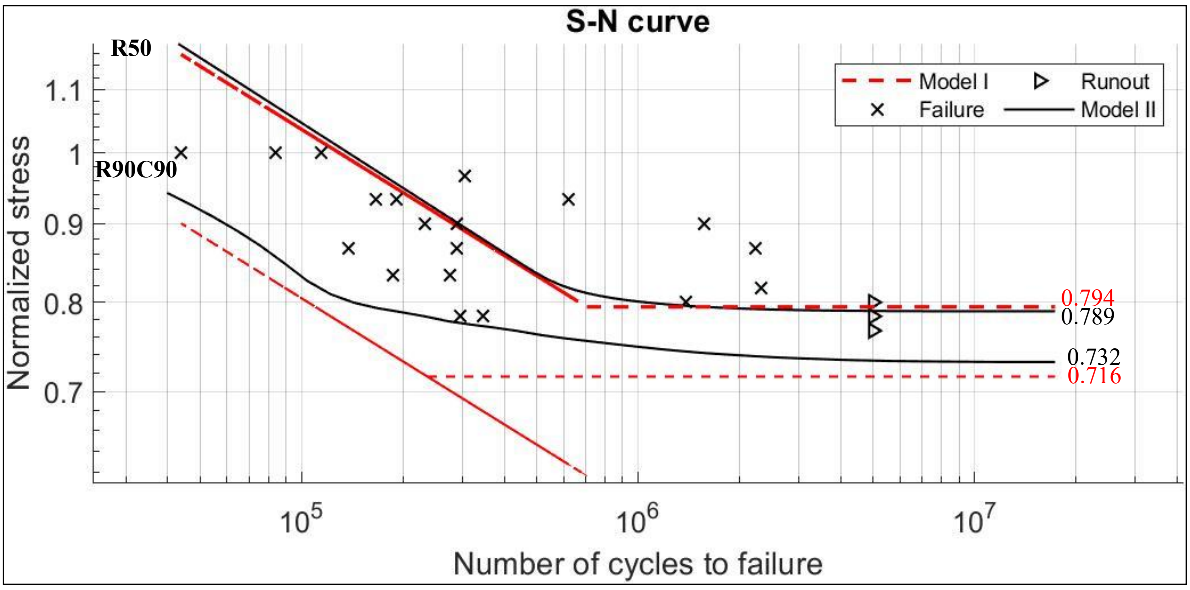


Figure 5.22 R50 and R90C90 curves for GH 60-38-10 without considering 6 data in the Model II

5.1.9 G-AS7C3,5GM

G-AS7C3.5GM is an aluminium alloy used for CRF engine heads. The mechanical properties of this material are summarised in Table 5.13.

Table 5.13 Mechanical properties of G-AS7C3,5GM. Information provided by FCA

Mechanical properties	**G-AS7C3,5GM**
Yield strength [MPa]	280
Tensile strength [MPa]	310
Elastic modulus [Gpa]	72.6
Poisson coef.	0.32
Coef. of thermal expansion [°C $^{-1}$]	1.8

Thirty-four hourglass specimens of this material, with a K_t of 1, were subjected to axial fatigue tests at room temperature, with a stress ratio of -1 and a frequency of 150 Hz. Twenty-five cracks and nine runouts were recorded. The staircase was performed on 19 specimens of which 10 failed and 9 survived. Fatigue data from G-AS7C3,5GM and fitted R50 and R90C90 curves plotted by using Model I and Model II are shown in Figure 5.23. Table 5.14 compares the coefficient values used for Model I and Model II. The coefficients of the finite life region and hence the slope and intercept of the curves plotted using the two models are slightly different because Model I, does not consider the failures below the median fatigue limit value, 0.686. The fatigue limit values are very similar with a difference of 0.14 % for the median values and 1.8 % for the R90C90 values.

Table 5.14 Comparison of fatigue coefficients of Model I and Model II for G-AS7C3,5GM

a		b		σ_Y		Normalized μ_{X_l}		σ_{X_l}		Normalized R90C90	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
18.5280	17.4134	-6.3699	-5.8297	0.2881	0.2990	0.6859	0.6849	5.6349	5.2994	0.593442	0.583005
Δ :	**6.40%**	**Δ :**	**9.27%**	**Δ :**	**3.65%**	**Δ :**	**0.14%**	**Δ :**	**6.33%**	**Δ :**	**1.79%**

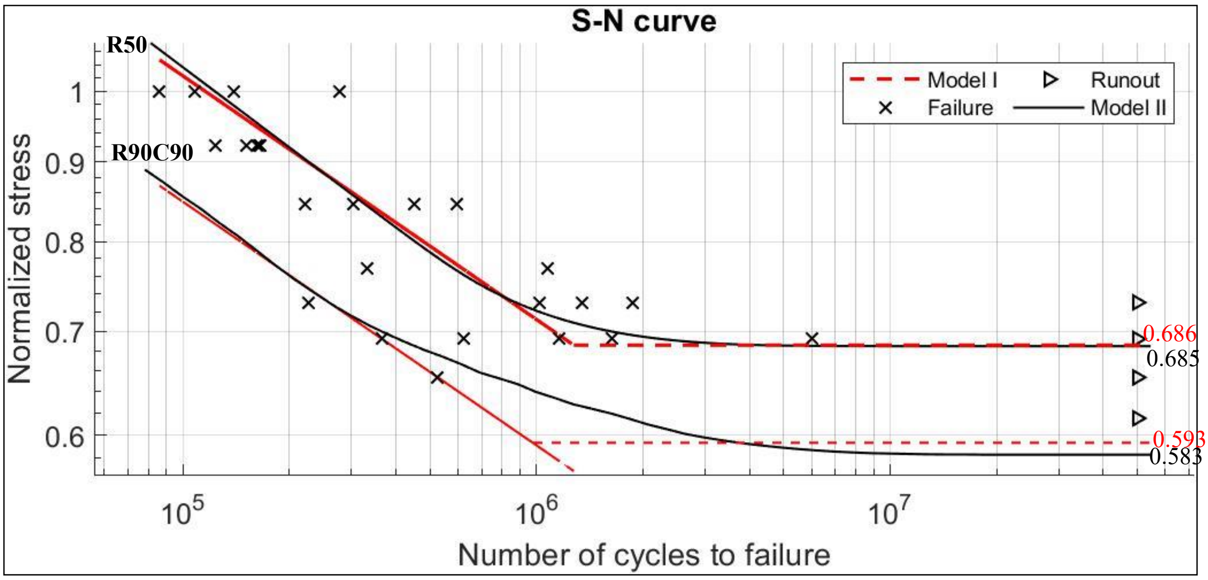


Figure 5.23 Experimental fatigue data for G-AS7C3,5GM and fitted R50 and R90C90 curves with Model I and Model II

The statistical analysis of the data with Model II on a reduced Staircase was also attempted for this material to see how much the results differed from those obtained with Model I considering all the data available and how much time could be saved in carrying out a smaller number of tests without losing too much information. Figure 5.24 shows the original Staircase carried out using all the 19 data element and the reduced Staircase carried out considering only the first 9 samples tested.

Levels	**Stress Sa [MPa]**	**Sequence number of specimens**																		
i		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
4	**0.769**															X				
3	**0.731**										X		X		O		X			
2	**0.692**	X		X				X		O		O		O				X		X
1	**0.654**		O		X		O		O										O	
0	**0.615**					O														
ΔS [MPa]	**0.038**																			
N° of cycles		**3.7E+05**	**5.0E+07**	**6.2E+05**	**5.3E+05**	**5.0E+07**	**5.0E+07**	**1.6E+06**	**5.0E+07**	**5.0E+07**	**1.3E+06**	**5.0E+07**	**2.3E+05**	**5.0E+07**	**5.0E+07**	**3.3E+05**	**1.0E+06**	**6.0E+06**	**5.0E+07**	**1.2E+06**

Figure 5.24 Original and reduce Staircase for G-AS7C3,5GM

From the analysis of the R50, R90C90 curves and the parameters obtained with the two models, it was found that the difference in the values of the fatigue limits calculated with the two methods remains very low, less than 2 %, as shown in Figure 5.25 and Table 5.15.

Clearly, by decreasing the available data, the difference between the values of the slope and intercept parameters and the standard deviations of the fatigue life and fatigue limit, increases. In this way, however, 16 days could be saved on the characterisation of this material, which has a very high number of runout cycles, N_G = 5e7, without having very different results from those obtained by considering 19 samples for the infinite life region.

Table 5.15 Comparison of parameters considering only 9 specimens for unlimited fatigue life region for Model II

a		b		σ_Y		Normalized μ_{X_l}		σ_{X_l}		Normalized R90C90	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
18.5280	15.1130	-6.3699	-4.7241	0.2881	0.2597	0.6859	0.6790	5.6349	4.0426	0.593442	0.586598
Δ :	**22.60%**	**Δ :**	**34.84%**	**Δ :**	**10.96%**	**Δ :**	**1.01%**	**Δ :**	**39.39%**	**Δ :**	**1.17%**
Saved time[day.h]:						**16.05**					

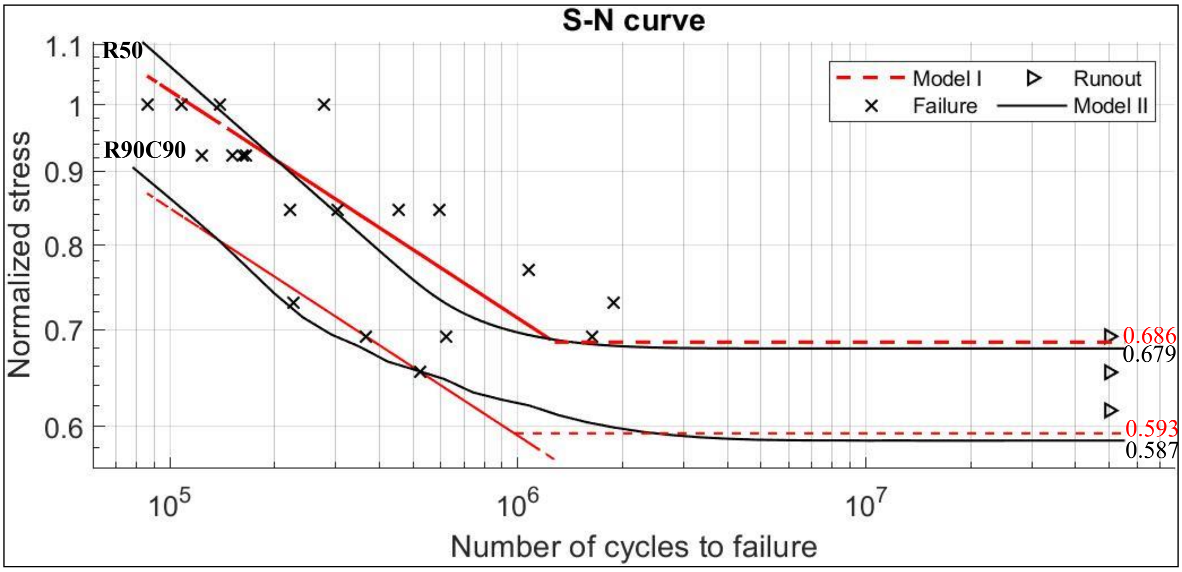


Figure 5.25 R50 and R90C90 curves for G-AS7C3,5GM plotted by using 19 and 9 specimens for unlimited fatigue life region for Model I and II respectively

5.1.10 G-AS7C3,5GM at a temperature of 150°C

The same material analysed in the previous paragraph was tested in the same way but at a temperature of 150°C. An additional 36 specimens were used for this purpose, recording 27 failures and 9 runouts. For the Staircase, in this case, 18 samples were used of which 9 were broken and 9 survived. The experimental results and R50 curves plotted with both methods are compared in Figure 5.26. Table 5.16 compares the values of the parameters calculated with the two methods.

Table 5.16 Comparison of parameters for G-AS7C3,5GM at 150°C

a		b		σ_Y		Normalized μ_{X_l}		σ_{X_l}	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
30.1326	29.6988	-12.1389	-11.9110	0.4156	0.4119	0.6227	0.6101	4.0349	4.4960
Δ :	**1.46%**	**Δ :**	**1.91%**	**Δ :**	**0.90%**	**Δ :**	**2.06%**	**Δ :**	**10.26%**

Even in this case the difference between the two models is small but the Model II does not find, with the available data, an R90C90 fatigue limit. This is probably due to the presence of failures very close to the chosen number of runout cycles, $N_G = 5e7$, as can be seen in Figure 5.26. In fact, the Maximum Likelihood method fails to say with 90 % confidence that 90 % of failures occur above a certain stress value with this data available. More tests at a higher number of cycles would therefore be needed to have more confidence in the results. In fact, if we did not consider the last three failures, i.e. the data (0.667;4.87e7), (0.667;4.3e7) and (0.625;4.05e7), we would have the R50 and R90C90 curves as shown in Figure 5.27, in which the Model II is able to calculate the R90C90 curve, but it reaches a plateau at higher numbers of cycles (higher than 1e9 cycles).

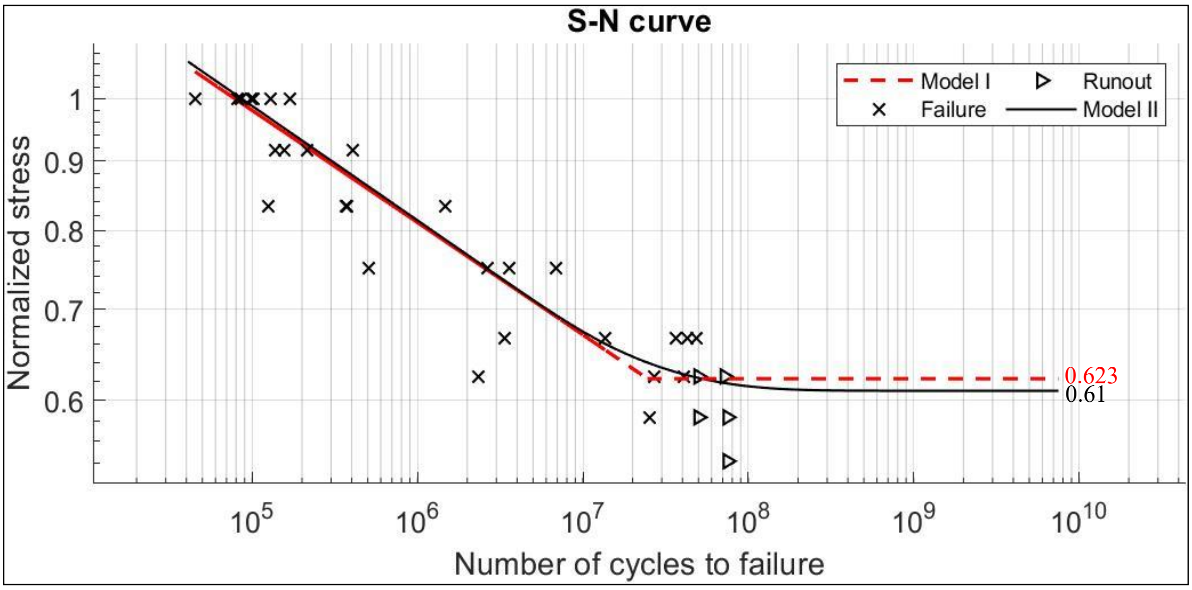


Figure 5.26 Experimental fatigue data for G-AS7C3,5GM @150°C and fitted R50 curves with Model I and Model II

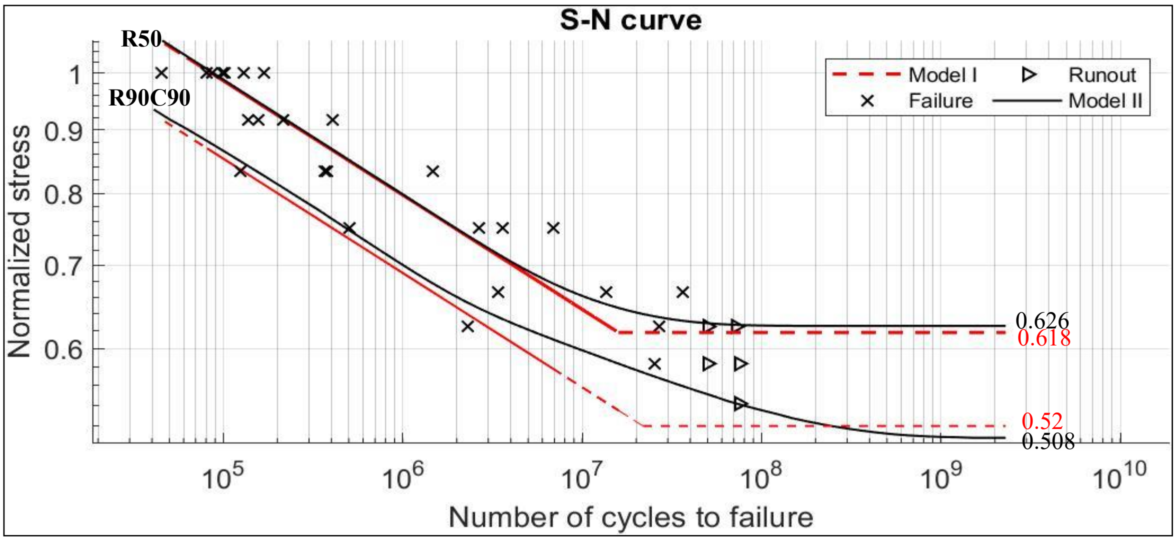


Figure 5.27 R50 and R90C90 curves without considering last three failure in Model I and Model II

5.1.11 Hifax TYC 462P

Hifax TYC 462P is a 30% talc reinforced, high flow, impact modified and UV stabilized polypropylene copolymer for injection molding. It combines an excellent flowability with good processability and durability and offers excellent esthetic properties for exterior automotive applications. Dog-bone specimens, with $K_t = 1$, of this material were subjected to asymmetric ($R = 0.1$) axial fatigue tests conducted at room temperature and a frequency of 3 Hz. Thirty-two specimens were tested and 24 failures and 8 runouts were recorded. Staircase was performed on 15 specimens with 7 failures and 8 runouts. Figure 5.28 shows the experimental results and the comparison of the plotted median curves using the two models.

The curves present a very similar trend, with a slightly lower median value of the fatigue limit for Model II than for Model I, as can also be seen from the data shown in Table 5.17. There is, however, a difference of 170 % for the standard deviation of the fatigue limit. In fact, the value calculated with Model II is much lower than that calculated with Model I, but this is due to the fact that in Model I the simplified formula (2.19) was used, which causes the standard deviation to be higher, while in Model II no correction is applied, but simply reflects the trend of the available data.

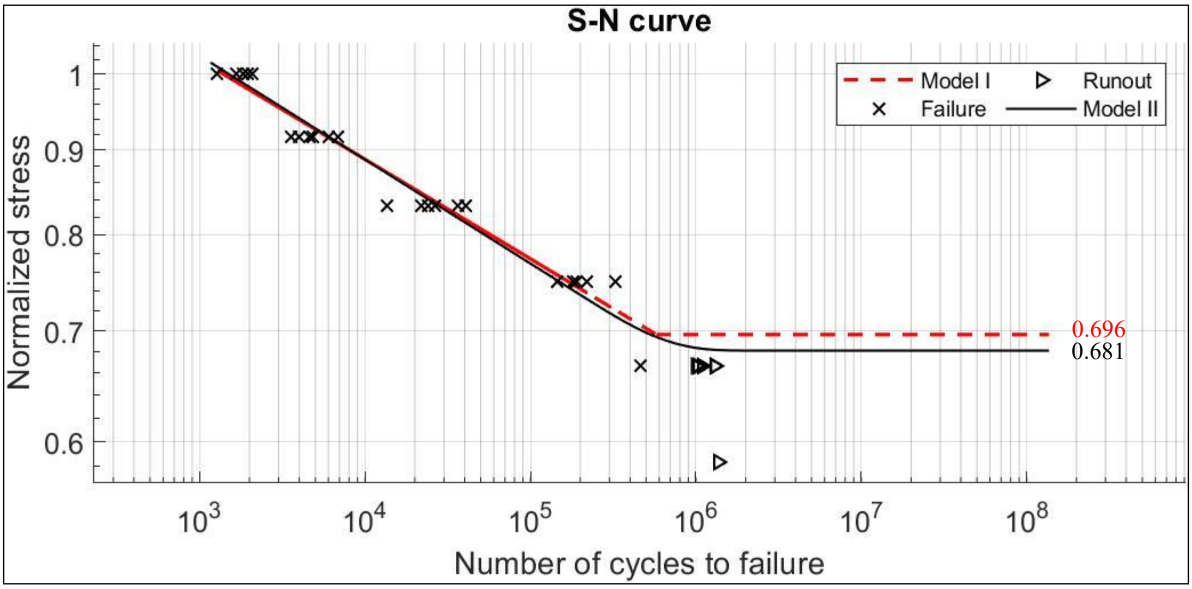


Figure 5.28 Experimental testing results and R50 curves for PP 30 % talc

Table 5.17 Comparison of parameters for PP 30% talc

a		b		σ_Y		Normalized μ_{X_l}		σ_{X_l}		Normalized R90C90	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
16.1028	15.5826	-16.6529	-15.9408	0.1456	0.1619	0.6964	0.6811	0.2650	0.0984	0.593388	0.614612
Δ :	**3.34%**	**Δ :**	**4.47%**	**Δ :**	**10.06%**	**Δ :**	**2.25%**	**Δ :**	**169.30%**	**Δ :**	**3.45%**

Figure 5.29 shows the comparison between the R90C90 curves. The R90C90 curve plotted using Model II deviates from the expected curvature. This may be caused by poor test design due to the presence of runouts near the knee of the curve, but it is not considered to be the "wrong" curve but rather the curve suggested by the full dataset for this material. From this curve, however, it is possible to extrapolate a slope of the linear section and an asymptote for the infinite life region which can be used in the design phase. The value of the R90C90 fatigue limit is given in Table 5.17.

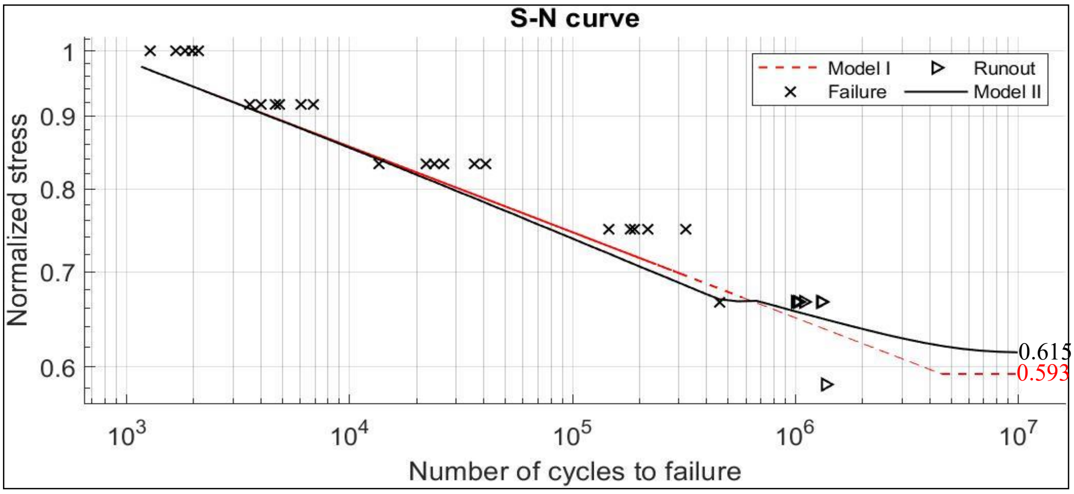


Figure 5.29 R90C90 curves plotted by using Model I and II for PP 30 % talc

5.1.12 Polynt SMC LP 2512 R33

Polyint SMC LP 2512 R33 is a sheet moulding thermosetting compound (SMC) based on an unsaturated polyester resin, with 33 % of glass fibres especially suitable for compression moulding of medium to large flat geometries. This material has been developed for structural parts and joint good mechanical properties and good varnish adhesion. The fatigue properties of this material were examined by means of asymmetric (R = 0,1) axial fatigue tests on dog-bone specimens with a K_t = 1. The tests were conducted at room temperature, and at a frequency varying between 2 and 8 Hz, on 30 specimens, recording 22 failures and 8 runouts. The Staircase was performed on 15 samples of which 7 were broken and 8 survived. The experimental results and the R50 and R90C90 curves are compared in Figure 5.30.

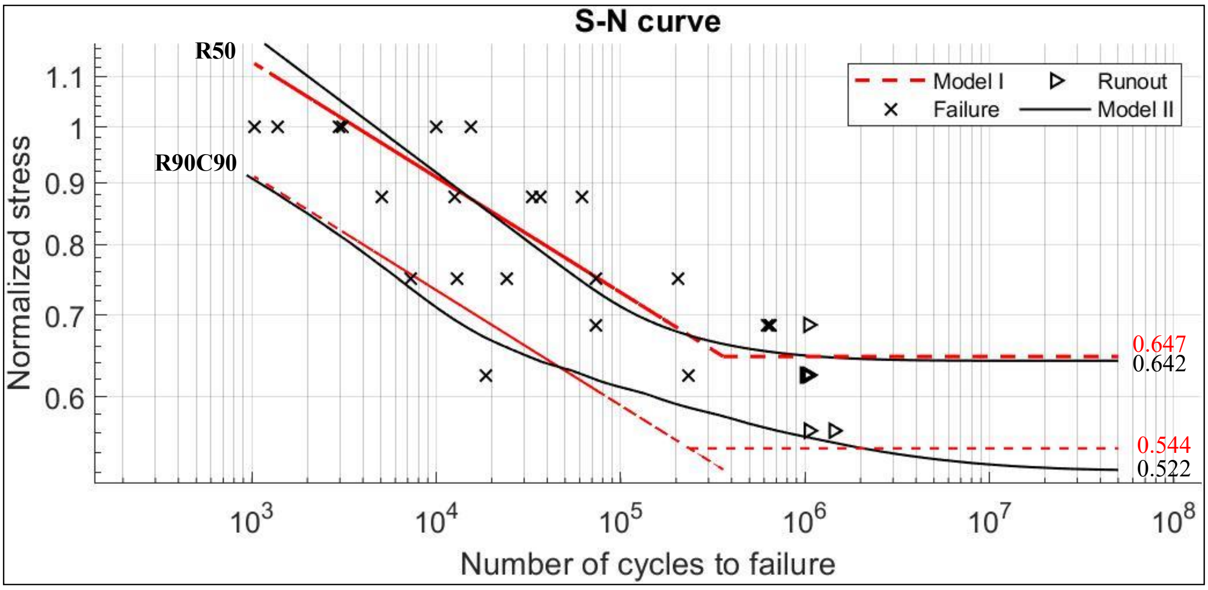


Figure 5.30 Experimental fatigue data for Polynt SMC LP 2512 and fitted R50 and R90C90 curves with Model I and Model II

The differences between the parameters of the linear section of the SN curves calculated by the two methods, which cause a different slope and intercept of the same curves, are due to the fact that Model I does not consider the breaks (0.625;2.33e5) and (0.625;1.87e4), which occur below the median value of the fatigue limit, 0.647, thus losing important information.
The difference in the median fatigue limit values is less than 1 %, while for the R90C90 values the difference increases to 4 %, as also shown in Table 5.18. The lower value of Model II is justified by the presence of failures in the vicinity of the number of cycles for the runout, N_G = 1e6.
In fact, if one did not consider the failure at (0.688;6.54e5) i.e. the failure closest to the runout, in both models, the difference would decrease to 2.8%, as shown in Figure 5.31. This proves that Model II, taking into account all the available data, is, with good reason, more cautelative than Model I, given the presence of failure and runout for numbers of cycles close to each other.

Table 5.18 Comparison of parameters for Polynt SMC LP 2512

a		b		σ_Y		Normalized μ_{X_l}		σ_{X_l}		Normalized R90C90	
Model I	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**
20.4653	17.7455	-10.5496	-8.7858	0.5282	0.5785	0.6473	0.6421	1.7705	1.5405	0.544056	0.522538
Δ :	**15.33%**	**Δ :**	**20.08%**	**Δ :**	**8.70%**	**Δ :**	**0.81%**	**Δ :**	**14.93%**	**Δ :**	**4.12%**

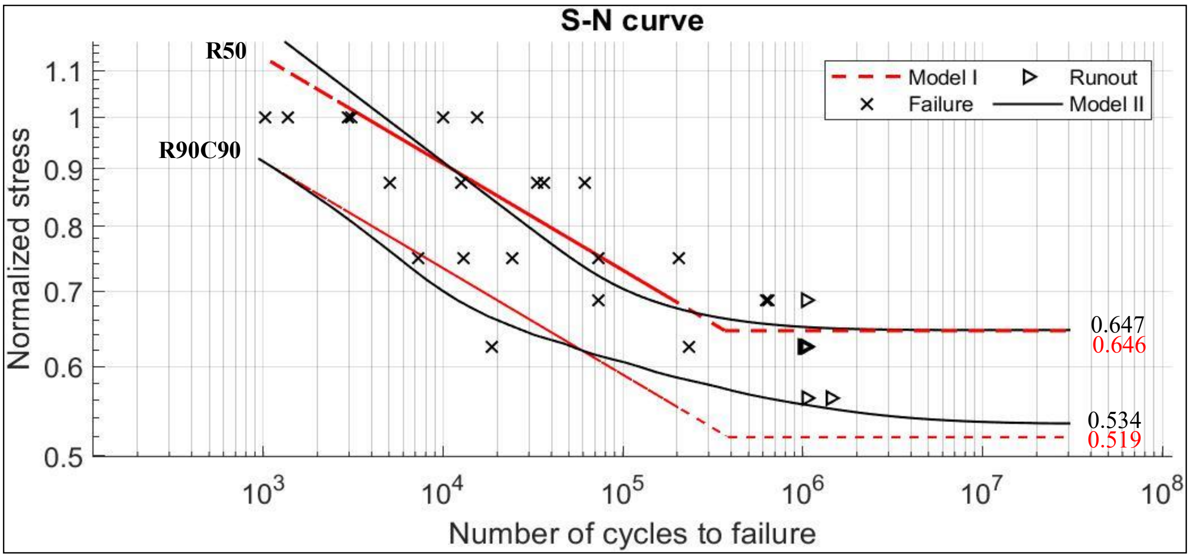


Figure 5.31 R50 and R90C90 curves for Polynt SMC LP 2512 without considering last failure

5.2 Summary results

Table 5.19 summarises the results of the analyses presented in terms of the median value of the fatigue limit and the value of the fatigue limit corresponding to the R90C90 curve, comparing, for each material, the difference of the values calculated with both models. In particular, the difference between the values is relative to the value calculated with Model II, i.e., for example $\Delta R50 = \left|\frac{R50_{Model\,II} - R50_{Model\,I}}{R50_{Model\,II}}\right| \cdot 100$. Furthermore, in Table 5.19 the distance of the last failure from the runout and the distance of the first runout from the knee of the R50 curve were also quantified with the parameters defined in equation (5.1) and (5.2).

$$\Delta(Runout - last\ failure) = \frac{N_G - N_{f,max}}{N_G} \cdot 100 \tag{5.1}$$

$$\Delta(Runout - Knee\ of\ curve) = \frac{N_G - N_{A,50}}{N_G} \cdot 100 \tag{5.2}$$

where$N_{f,max}$ is the number of cycles corresponding to the failure closest to the number of runout cycles and $N_{A,50}$ is the fatigue life at the knee of R50 S-N curve plotted by using Model II.

From the results it can be seen that when the last failure found during the test are far from the number of cycles for the chosen runout, i.e. when $\Delta_{R,L\text{-}F} > 50$ %, and the runouts are far from the knee of the curve, i.e when $\Delta_{R,K} > 60$ %, the values calculated with both models are comparable with each other with a difference that in each case is less than 2 %. However, when there are failures at N_G, the fatigue limit estimate of Model II is lower than the Model I estimate and the difference becomes almost always even more significant when considering the R90C90 limit. This is because, in the infinite life region of the Wöhler curve, while Model I use only the information that a specimen has broken or has not broken at a certain stress level, Model II use also the information about finite life of some specimens breaks near the number of runout cycles, N_G. Thus, in the case where there are failures sufficiently far from the number of cycles to runout, then the probability that the logarithm of the lives, at a certain stress level S_a, is below $\log_{10}(N_G)$ i.e. the factor $\varphi\left(\frac{Log_{10}(N_G)-(a+b\cdot Log_{10}(S_a))}{\sigma_Y}\right)$ is approximately equal to one and the difference between the two models is negligible because, in both cases, the probability of failure can be expressed as the probability that the current stress is higher than the fatigue limit, i.e. with the $\varphi\left(\frac{Log_{10}(S_a)-\mu_{X_l}}{\sigma_{X_l}}\right)$ factor of Eq.(3.4). On the contrary, when some of the finite lives are close to N_G, the factor $\varphi\left(\frac{Log_{10}(N_G)-(a+b\cdot Log_{10}(S_a))}{\sigma_Y}\right)$ is less than one and therefore the term $\varphi\left(\frac{Log_{10}(S_a)-\mu_{X_l}}{\sigma_{X_l}}\right)$ in equation (3.4) slightly increases and consequently μ_{X_l} decreases and there is a difference between the two models.

Table 5.19 Summary results for experimental datasets

Material	R50 Model I	R50 Model II	ΔR50	R90C90 Model I	R90C90 Model II	ΔR90C90	Δ(Runout-Last failure)	Δ(Runout-Knee of curve)	NOTES
AlSi7Mg03	0.438	0.436	0.69%	0.384	0.366	4.89%	6.00%	36.15%	R90C90 fatigue limit for Model II is slightly lower probably because there are failures quite close to the number of cycles to runout.
AlSi10Mg	0.659	0.675	2.34%	0.531	0.538	1.15%	87.00%	95.87%	There are runouts very distant from the last failure and knee of the curve.
6060-T66	0.610	0.605	0.76%	0.534	0.470	13.71%	2.39%	35.69%	R90C90 fatigue limit for Model II is lower because there is a failure close to the number of cycles to runout. There are runouts quite close to the knee of SN curve.
17-4PH	0.482	0.455	6.07%	0.184	0.238	22.74%	7.23%	51.77%	There is a failure close to the number of cycles to runout. There are runouts quite close to the knee of the SN curve.
CP800HR	0.657	0.613	7.17%	0.567	0.612	7.34%	5.78%	-71.61%	ML does not find a R90C90 fatigue limit with the available data. There are runouts close to the number of cycles to runout and there are runouts before the knee of SN curve.
DP1000	0.894	0.892	0.18%	0.866	0.874	0.92%	19.77%	71.84%	Data suggest an irregular pattern of the R90C90 curve which is probably caused by the presence of failures near the number of cycles to runout. Runouts are distant from the knee of the curve.
Trip1000	0.933	0.932	0.04%	0.900	0.900	0.02%	54.24%	83.76%	There are runouts far from the knee of the curve and quite distant to the last failure.
GH 60-38-10	0.794	0.801	0.87%	0.716	0.728	1.66%	53.62%	89.04%	There are runouts far from the knee of the curve and quite distant to the last failure.
GAS7C3,5GM @RT	0.686	0.685	0.14%	0.593	0.583	1.79%	87.93%	97.63%	There are runouts very distant from the last failure and knee of the curve.
GAS7C3,5GM @150 °C	0.623	0.610	2.06%	0.551	0.550	0.12%	2.54%	5.67%	ML does not find a R90C90 fatigue limit with the available data. This is probably due to the presence of failure very close to runout, and runouts very close to the knee of the curve.
PP 30%Talco	0.696	0.681	2.25%	0.593	0.615	3.45%	54.07%	25.93%	Data suggest an irregular pattern of the R90C90 curve which is probably caused by the presence of runouts close to the knee of SN curve.
Polynt LP-2512	0.647	0.642	0.81%	0.544	0.523	4.12%	34.62%	75.35%	R90C90 for Model II is slightly lower than that of Model I probably due to the fact that there is a failure close to runout. Runouts are quite distant to the knee of SN curve.

6 Simulation experiments

In order to validate the obtained results, simulation was used. In particular, virtual datasets were simulated, constructed from a known distribution i.e. by setting "true" values for the parameters a, b, σ_Y, μ_{X_l}, σ_{X_l}. Once the datasets for the finite life and infinite life region of SN curve had been calculated, the two methods illustrated in the previous chapters were applied to estimate the parameters of interest. In this way, knowing the real values of the parameters, it was possible to verify which of the two methods was more accurate in estimating the parameters. This procedure is then repeated a large number of times (100 runs) in order to have some statistical reliability of the results.

6.1 Simulation of dataset

The 'true' parameters from which we started are given in Table 6.1. The chosen values were taken as equal to the real parameter values calculated for an experimental dataset in order to have a realistic simulation, but another other choice of them would have been possible.

Table 6.1 True parameters values

$\bar{a}$	$\bar{b}$	$\overline{\sigma_Y}$	$\overline{\mu_{X_l}}$	$\overline{\sigma_{X_l}}$
12.271	- 3.569	0.255	72.5	5.6

A Staircase was simulated for the simulation of the infinite life data. The staircase starting stresses are specified for each simulation run. The traditional approach provides for $S_{init} = \mu$, however, use of offset starting stresses does have some beneficial effect for datasets with a limited number of samples, reducing standard deviation bias [23]. Thus, for each simulation run, a random starting stress between the range $\overline{\mu_{X_l}} - 0.5 \cdot \overline{\sigma_{X_l}} \div \overline{\mu_{X_l}} + 0.5 \cdot \overline{\sigma_{X_l}}$ was chosen. However, it has been verified that the effect of starting stress is rather small.

The step size for the staircase was varied in the range $0.5 \cdot \overline{\sigma_{X_l}} \div 2 \cdot \overline{\sigma_{X_l}}$, which is the range recommended by Dixon and Mood for successful application of their method, to analyse the effect of step size on parameter estimation. In particular, simulations were performed for step values ΔS = 3-5.6-10 MPa.

Typically, the Staircase is performed on 15 specimens, but simulations were also carried out for a smaller number of specimens in order to assess the possible benefits of using Maximum Likelihood principle for a small number of specimens compare to Dixon-Mood analysis. Simulations were also carried out for a larger number of specimens in order to assess the possible benefits of carrying out a larger number of tests. Thus, the number of specimens used was n = 9-15-30.

For each specimen, starting from the first stress level S_{init}, the number of cycles was calculated using equation (6.1). In particular, a model taking into account the entire S-N curve was considered, imposing a log-normal distribution for the fatigue life and a normal distribution for the fatigue limit. For each specimen, the value of α, i.e. the probability of failure, was randomly generated in the range [0;1]. Randomness has been achieved by using the Mersenne-Twister algorithm on Matlab.

$$N_{f,i} = 10^{\left(\bar{a}+\bar{b}\cdot Log_{10}\left(S_{a,i}\right)\right)+\overline{\sigma_Y}\cdot \varphi^{-1}\left(\frac{\alpha}{\varphi\left(\frac{S_{a,i}-\overline{\mu_{X_l}}}{\overline{\sigma_{X_l}}}\right)}\right)} \tag{6.1}$$

Once the number of cycles for the runout, N_G, is established, the current number of cycles $N_{f,i}$ is compared with N_G and if $N_{f,i} \geq N_G$ or $N_{f,i}$ the specimen can be considered a runout and the next stress is increased by a factor equal to the step, otherwise it is considered a failure and the next stress is decreased by the same amount, as represented schematically in Table 6.2. There are cases where $N_{f,i}$ is a Not a Number; this occurs when the probability that the applied stress level $S_{a,i}$ is higher than the median value of the fatigue limit $\overline{\mu_{X_l}}$, is very low (lower than α). In fact, in that case, $\varphi^{-1}(\cdot)$ of a number greater than one results in a Not a Number. Thus, according to the given model, a sample can be considered as a runout either if it has survived for a number of cycles higher than the established number of cycles N_G or if the stress level is below the fatigue limit.

Table 6.2 Staircase simulation protocol

if $N_{f,i} \geq N_G$ | $N_{f,i}$ =NaN
$S_{a,i+1} = S_{a,i} + step;$
$N_{f,i} = N_G;$
$Status_i = runout;$
else
$S_{a,i+1} = S_{a,i} - step;$
$N_{f,i} = N_{f,i};$
$Status_i = failure;$

In the results we excluded those datasets in which the first two specimens were two failures or two runouts to take into account the fact that generally a correct staircase starts from the first inversion, i.e. when the result changes from survival to failure, or vice versa. Furthermore, those datasets in which there was not a correct balance between the number of failures and runouts, were excluded.

As regard the dataset for the finite life section of the S-N curve, 20 specimens were used, subdividing the stress values, generated randomly, over 5 different stress levels, so that at least two specimens corresponded to each stress level and the value of the lowest stress level was greater than or equal to the highest stress level generated in the Staircase simulation. Also in this case, the number of cycles corresponding to each stress level was calculated using equation (6.1). All those datasets where $N_{f,i}$ is a NaN were excluded.

6.2 Simulation results

As output of simulation, for each of the 100 datasets, the value of each parameter, estimated with both models, was compared with the initially set true value. For each parameter and for each estimation method, a relative estimation error (REE) was also calculated, expressed as the difference between the estimated parameter value and the true value with respect to the true value, i.e. using the parameter in equation (6.2). This is the same parameter used by Snyder et al. [26] to compare Dixon-Mood method and probit analysis method. Tab 6.3 shows the estimation of the 5 parameters of interest and the value of the R90C90 fatigue limit calculated, with both methods, for 100 datasets where the number of samples used for the Staircase is 15 and the step size was set equal to the true standard deviation. The value corresponding to the R90 fatigue limit, calculated from the true value of the median fatigue limit as $\overline{\mu_{X_l}} - 1.28 \cdot \overline{\sigma_{X_l}} = 65.332$, was taken as the "true" value of the R90C90 fatigue limit.

$$Relative\ estimation\ error\ [\%] = \frac{Estimated\ value - True\ value}{True\ value} \cdot 100 \tag{6.2}$$

Table 6.3 Estimated parameter and relative estimation error (REE) for 100 datasets with a step size equal to standard deviation and 15 specimens used for the Staircase

N° Dataset	*a*	12.271			*b*	-3.569			σ_Y :	0.255		
	Model II		Model I		Model II		Model I		Model II		Model I	
	Value	REE	Value	REE	Value	REE	Value	REE	Value	REE	Value	REE
1	10.36	-16%	10.18	-17%	-2.59	-27%	-2.50	-30%	0.24	-5%	0.22	-12%
2	10.23	-17%	7.40	-40%	-2.48	-30%	-1.07	-70%	0.24	-6%	0.24	-6%
3	11.77	-4%	14.07	15%	-3.32	-7%	-4.49	26%	0.20	-23%	0.19	-24%
4	12.59	3%	9.75	-21%	-3.75	5%	-2.34	-35%	0.23	-10%	0.23	-9%
5	12.28	0%	13.70	12%	-3.58	0%	-4.30	20%	0.24	-6%	0.27	4%
6	13.18	7%	15.18	24%	-4.08	14%	-5.08	42%	0.31	22%	0.26	1%
7	11.09	-10%	11.91	-3%	-2.93	-18%	-3.34	-6%	0.24	-8%	0.24	-6%
8	13.16	7%	12.76	4%	-4.03	13%	-3.83	7%	0.27	7%	0.25	-2%
9	13.47	10%	12.39	1%	-4.18	17%	-3.65	2%	0.23	-12%	0.25	-1%
10	9.23	-25%	9.43	-23%	-1.99	-44%	-2.09	-42%	0.23	-11%	0.27	7%
11	12.29	0%	10.55	-14%	-3.60	1%	-2.72	-24%	0.25	-1%	0.23	-10%
12	11.18	-9%	12.42	1%	-3.01	-16%	-3.63	2%	0.23	-9%	0.24	-4%
13	10.37	-16%	9.90	-19%	-2.60	-27%	-2.36	-34%	0.27	5%	0.24	-5%
14	12.24	0%	10.42	-15%	-3.55	0%	-2.64	-26%	0.23	-12%	0.21	-17%
15	10.46	-15%	6.08	-50%	-2.62	-27%	-0.42	-88%	0.24	-7%	0.22	-15%
16	12.72	4%	11.81	-4%	-3.81	7%	-3.35	-6%	0.26	2%	0.23	-9%
17	10.00	-18%	11.31	-8%	-2.44	-32%	-3.09	-13%	0.27	4%	0.29	14%
18	10.28	-16%	9.42	-23%	-2.57	-28%	-2.14	-40%	0.25	-2%	0.29	13%
19	11.48	-6%	11.15	-9%	-3.15	-12%	-2.98	-16%	0.25	-4%	0.27	7%
20	10.99	-10%	9.80	-20%	-2.92	-18%	-2.32	-35%	0.24	-5%	0.27	4%
21	10.23	-17%	10.35	-16%	-2.52	-30%	-2.59	-28%	0.24	-8%	0.21	-19%
22	12.55	2%	9.97	-19%	-3.73	4%	-2.44	-32%	0.26	4%	0.27	5%
23	12.39	1%	9.23	-25%	-3.64	2%	-2.05	-42%	0.25	-1%	0.25	-1%
24	12.92	5%	9.45	-23%	-3.90	9%	-2.16	-39%	0.29	14%	0.30	19%
25	11.90	-3%	11.85	-3%	-3.31	-7%	-3.30	-8%	0.28	10%	0.31	22%
26	12.11	-1%	9.96	-19%	-3.49	-2%	-2.42	-32%	0.23	-11%	0.23	-10%
27	10.21	-17%	11.97	-2%	-2.49	-30%	-3.37	-6%	0.27	7%	0.28	11%
28	12.28	0%	13.03	6%	-3.59	0%	-3.97	11%	0.22	-15%	0.24	-8%
29	10.18	-17%	8.64	-30%	-2.51	-30%	-1.73	-52%	0.23	-9%	0.28	9%
30	14.54	18%	17.40	42%	-4.73	33%	-6.16	73%	0.27	5%	0.26	3%
31	12.66	3%	11.73	-4%	-3.78	6%	-3.32	-7%	0.25	0%	0.27	7%
32	12.31	0%	11.25	-8%	-3.61	1%	-3.07	-14%	0.22	-14%	0.20	-24%
33	10.47	-15%	11.79	-4%	-2.62	-26%	-3.28	-8%	0.30	16%	0.29	12%
34	13.98	14%	10.66	-13%	-4.42	24%	-2.76	-23%	0.20	-22%	0.19	-26%
35	12.60	3%	13.86	13%	-3.78	6%	-4.42	24%	0.24	-5%	0.25	-1%
36	13.48	10%	14.17	15%	-4.17	17%	-4.51	26%	0.23	-12%	0.25	-2%
37	12.99	6%	14.52	18%	-3.95	11%	-4.71	32%	0.23	-10%	0.26	1%
38	9.97	-19%	9.76	-20%	-2.37	-33%	-2.26	-37%	0.33	28%	0.35	37%
39	13.03	6%	14.32	17%	-3.93	10%	-4.57	28%	0.25	-3%	0.22	-12%
40	11.86	-3%	14.37	17%	-3.37	-6%	-4.62	29%	0.28	9%	0.30	16%
41	9.61	-22%	9.20	-25%	-2.19	-39%	-1.99	-44%	0.21	-19%	0.21	-19%
42	10.31	-16%	7.28	-41%	-2.59	-27%	-1.08	-70%	0.33	29%	0.35	37%
43	14.24	16%	15.50	26%	-4.56	28%	-5.19	45%	0.22	-14%	0.24	-5%
44	14.88	21%	15.50	26%	-4.95	39%	-5.27	48%	0.26	2%	0.29	14%
45	11.64	-5%	11.08	-10%	-3.24	-9%	-2.96	-17%	0.19	-25%	0.21	-17%
46	10.93	-11%	12.22	0%	-2.90	-19%	-3.54	-1%	0.33	28%	0.37	45%
47	10.71	-13%	9.96	-19%	-2.78	-22%	-2.40	-33%	0.24	-5%	0.26	0%
48	11.52	-6%	9.22	-25%	-3.17	-11%	-2.02	-43%	0.26	2%	0.29	13%
49	13.68	11%	13.77	12%	-4.31	21%	-4.35	22%	0.16	-38%	0.17	-33%

50	15.01	22%	14.28	16%	-4.97	39%	-4.61	29%	0.22	-16%	0.23	-11%
51	12.85	5%	12.72	4%	-3.85	8%	-3.79	6%	0.28	11%	0.31	21%
52	11.35	-8%	10.96	-11%	-3.09	-13%	-2.90	-19%	0.20	-21%	0.20	-20%
53	12.42	1%	9.91	-19%	-3.64	2%	-2.37	-34%	0.27	5%	0.26	2%
54	13.63	11%	14.56	19%	-4.27	20%	-4.74	33%	0.22	-15%	0.25	-4%
55	12.15	-1%	11.20	-9%	-3.50	-2%	-3.03	-15%	0.26	1%	0.30	17%
56	13.30	8%	11.15	-9%	-4.10	15%	-3.03	-15%	0.19	-25%	0.21	-19%
57	11.84	-4%	10.63	-13%	-3.32	-7%	-2.72	-24%	0.22	-13%	0.25	-3%
58	11.32	-8%	10.34	-16%	-3.09	-13%	-2.60	-27%	0.26	3%	0.26	0%
59	8.96	-27%	8.38	-32%	-1.88	-47%	-1.58	-56%	0.24	-7%	0.24	-4%
60	10.70	-13%	14.09	15%	-2.74	-23%	-4.44	24%	0.32	26%	0.30	17%
61	14.89	21%	17.41	42%	-4.89	37%	-6.17	73%	0.25	-2%	0.28	9%
62	14.63	19%	14.88	21%	-4.77	34%	-4.89	37%	0.21	-17%	0.22	-14%
63	9.70	-21%	9.80	-20%	-2.28	-36%	-2.34	-34%	0.21	-19%	0.20	-24%
64	8.49	-31%	11.22	-9%	-1.59	-56%	-2.98	-16%	0.23	-10%	0.22	-15%
65	10.80	-12%	11.48	-6%	-2.86	-20%	-3.19	-11%	0.22	-15%	0.20	-21%
66	9.10	-26%	8.41	-31%	-1.95	-45%	-1.60	-55%	0.24	-7%	0.27	6%
67	13.19	8%	8.85	-28%	-4.07	14%	-1.90	-47%	0.18	-31%	0.16	-36%
68	15.37	25%	18.79	53%	-5.19	45%	-6.90	93%	0.28	8%	0.30	17%
69	13.11	7%	8.27	-33%	-3.99	12%	-1.57	-56%	0.27	7%	0.25	-1%
70	12.11	-1%	11.88	-3%	-3.52	-1%	-3.40	-5%	0.23	-11%	0.23	-8%
71	12.93	5%	10.74	-12%	-3.90	9%	-2.80	-21%	0.26	0%	0.26	2%
72	11.25	-8%	12.44	1%	-3.03	-15%	-3.62	1%	0.28	11%	0.30	17%
73	9.07	-26%	8.30	-32%	-1.90	-47%	-1.51	-58%	0.24	-7%	0.26	1%
74	12.71	4%	14.77	20%	-3.78	6%	-4.80	35%	0.24	-4%	0.25	-1%
75	12.73	4%	12.97	6%	-3.82	7%	-3.94	10%	0.29	13%	0.32	24%
76	10.84	-12%	10.00	-18%	-2.84	-20%	-2.42	-32%	0.19	-24%	0.20	-21%
77	10.68	-13%	10.26	-16%	-2.74	-23%	-2.53	-29%	0.22	-13%	0.26	2%
78	14.20	16%	14.18	16%	-4.55	28%	-4.54	27%	0.26	3%	0.28	11%
79	15.77	28%	17.50	43%	-5.38	51%	-6.25	75%	0.19	-25%	0.21	-19%
80	14.15	15%	17.18	40%	-4.50	26%	-6.01	68%	0.24	-5%	0.22	-14%
81	12.27	0%	10.90	-11%	-3.53	-1%	-2.84	-21%	0.21	-19%	0.21	-19%
82	15.24	24%	9.65	-21%	-5.10	43%	-2.31	-35%	0.28	9%	0.26	1%
83	11.28	-8%	11.79	-4%	-3.07	-14%	-3.32	-7%	0.27	4%	0.31	20%
84	12.09	-2%	10.80	-12%	-3.46	-3%	-2.80	-22%	0.27	6%	0.28	8%
85	13.10	7%	14.05	15%	-4.02	13%	-4.50	26%	0.29	12%	0.33	29%
86	11.92	-3%	10.88	-11%	-3.45	-3%	-2.92	-18%	0.35	36%	0.34	34%
87	13.33	9%	9.45	-23%	-4.10	15%	-2.14	-40%	0.22	-14%	0.20	-21%
88	12.33	0%	9.97	-19%	-3.58	0%	-2.39	-33%	0.28	8%	0.32	24%
89	12.64	3%	15.20	24%	-3.77	6%	-5.04	41%	0.24	-7%	0.22	-13%
90	12.12	-1%	11.75	-4%	-3.50	-2%	-3.31	-7%	0.20	-24%	0.20	-20%
91	9.12	-26%	8.37	-32%	-1.97	-45%	-1.59	-55%	0.26	3%	0.28	8%
92	12.84	5%	13.25	8%	-3.87	9%	-4.07	14%	0.24	-8%	0.23	-9%
93	12.29	0%	11.88	-3%	-3.60	1%	-3.40	-5%	0.26	1%	0.26	1%
94	10.78	-12%	10.37	-15%	-2.84	-20%	-2.64	-26%	0.26	0%	0.22	-13%
95	11.11	-10%	13.44	9%	-2.99	-16%	-4.16	17%	0.21	-17%	0.20	-22%
96	11.83	-4%	8.53	-31%	-3.33	-7%	-1.68	-53%	0.25	-3%	0.26	2%
97	14.88	21%	13.95	14%	-4.96	39%	-4.49	26%	0.25	-1%	0.27	7%
98	12.77	4%	16.97	38%	-3.79	6%	-5.90	65%	0.22	-13%	0.25	-1%
99	12.09	-2%	15.77	29%	-3.47	-3%	-5.31	49%	0.23	-11%	0.25	-2%
100	11.62	-5%	8.66	-29%	-3.28	-8%	-1.81	-49%	0.23	-12%	0.23	-11%

N° Dataset	μ_{XI} :	72.5			σ_{XI} :	5.6			*R90C90* :	65.332		
	Model II		Model I		Model II		Model I		Model II		Model I	
	Value	REE	Value	REE	Value	REE	Value	REE	Value	REE	Value	REE
1	71.48	-1%	71.09	-2%	3.41	-39%	3.18	-43%	62.58	-4%	63.67	-3%
2	71.72	-1%	72.80	0%	4.60	-18%	5.84	4%	59.34	-9%	59.18	-9%
3	70.34	-3%	71.09	-2%	4.13	-26%	5.04	-10%	59.58	-9%	59.32	-9%
4	71.49	-1%	72.20	0%	5.10	-9%	5.84	4%	57.67	-12%	58.58	-10%
5	73.78	2%	75.37	4%	4.04	-28%	5.44	-3%	63.02	-4%	62.68	-4%
6	72.66	0%	73.66	2%	3.50	-37%	4.25	-24%	63.13	-3%	63.74	-2%
7	74.22	2%	75.63	4%	6.00	7%	8.22	47%	58.34	-11%	56.46	-14%
8	75.37	4%	76.09	5%	4.26	-24%	5.04	-10%	63.90	-2%	64.32	-2%
9	71.22	-2%	71.94	-1%	3.62	-35%	4.25	-24%	61.23	-6%	62.03	-5%
10	70.35	-3%	72.34	0%	6.40	14%	9.80	75%	53.48	-18%	49.47	-24%
11	72.65	0%	73.66	2%	3.45	-38%	4.25	-24%	63.39	-3%	63.74	-2%
12	71.36	-2%	74.06	2%	8.51	52%	12.58	125%	46.84	-28%	44.71	-32%
13	72.23	0%	73.40	1%	4.30	-23%	5.84	4%	60.69	-7%	59.78	-8%
14	70.48	-3%	70.83	-2%	4.36	-22%	5.44	-3%	58.91	-10%	58.14	-11%
15	71.60	-1%	72.20	0%	2.87	-49%	3.18	-43%	63.89	-2%	64.78	-1%
16	77.05	6%	77.80	7%	2.71	-52%	3.18	-43%	70.09	7%	70.38	8%
17	72.64	0%	73.66	2%	3.41	-39%	4.25	-24%	63.62	-3%	63.74	-2%
18	70.61	-3%	71.34	-2%	5.98	7%	7.03	25%	54.64	-16%	54.95	-16%
19	71.04	-2%	73.66	2%	7.86	40%	12.58	125%	49.33	-24%	44.31	-32%
20	77.05	6%	77.80	7%	2.71	-52%	3.18	-43%	70.10	7%	70.38	8%
21	72.02	-1%	71.94	-1%	4.28	-24%	4.25	-24%	60.95	-7%	62.03	-5%
22	73.10	1%	73.91	2%	4.36	-22%	5.04	-10%	61.57	-6%	62.15	-5%
23	72.04	-1%	71.94	-1%	4.32	-23%	4.25	-24%	61.15	-6%	62.03	-5%
24	69.20	-5%	70.63	-3%	6.01	7%	8.22	47%	52.97	-19%	51.46	-21%
25	73.40	1%	73.66	2%	3.72	-33%	4.25	-24%	63.85	-2%	63.74	-2%
26	76.21	5%	76.94	6%	3.58	-36%	4.25	-24%	66.63	2%	67.03	3%
27	71.68	-1%	72.80	0%	4.43	-21%	5.84	4%	60.05	-8%	59.18	-9%
28	74.03	2%	74.51	3%	4.01	-28%	5.04	-10%	63.97	-2%	62.75	-4%
29	73.25	1%	74.51	3%	3.99	-29%	5.04	-10%	62.62	-4%	62.75	-4%
30	72.45	0%	73.40	1%	4.47	-20%	5.84	4%	60.62	-7%	59.78	-8%
31	74.47	3%	76.23	5%	4.23	-24%	5.44	-3%	62.54	-4%	63.54	-3%
32	70.17	-3%	72.60	0%	9.76	74%	16.94	203%	44.46	-32%	33.07	-49%
33	72.68	0%	73.66	2%	3.55	-37%	4.25	-24%	63.12	-3%	63.74	-2%
34	72.65	0%	73.66	2%	3.45	-38%	4.25	-24%	63.54	-3%	63.74	-2%
35	70.24	-3%	71.69	-1%	4.20	-25%	5.04	-10%	58.94	-10%	59.92	-8%
36	72.06	-1%	72.80	0%	2.73	-51%	3.18	-43%	64.89	-1%	65.38	0%
37	72.06	-1%	72.80	0%	2.71	-52%	3.18	-43%	65.03	0%	65.38	0%
38	73.43	1%	73.66	2%	3.81	-32%	4.25	-24%	63.54	-3%	63.74	-2%
39	73.20	1%	75.11	4%	3.52	-37%	5.04	-10%	63.92	-2%	63.35	-3%
40	72.64	0%	73.66	2%	3.41	-39%	4.25	-24%	63.61	-3%	63.74	-2%
41	73.51	1%	73.66	2%	3.96	-29%	4.25	-24%	63.35	-3%	63.74	-2%
42	70.03	-3%	71.49	-1%	5.82	4%	7.82	40%	53.94	-17%	53.24	-19%
43	71.48	-1%	72.20	0%	5.05	-10%	5.84	4%	57.83	-11%	58.58	-10%
44	72.05	-1%	73.40	1%	4.15	-26%	5.84	4%	60.89	-7%	59.78	-8%
45	73.06	1%	75.11	4%	3.78	-33%	5.04	-10%	62.40	-4%	63.35	-3%
46	71.21	-2%	71.94	-1%	3.59	-36%	4.25	-24%	61.47	-6%	62.03	-5%
47	75.19	4%	75.83	5%	3.81	-32%	5.44	-3%	65.54	0%	63.14	-3%
48	71.20	-2%	71.94	-1%	3.53	-37%	4.25	-24%	61.85	-5%	62.03	-5%
49	70.06	-3%	71.49	-1%	5.85	4%	7.82	40%	54.34	-17%	53.24	-19%
50	72.65	0%	73.66	2%	3.45	-38%	4.25	-24%	63.57	-3%	63.74	-2%
51	72.98	1%	74.26	2%	5.27	-6%	7.03	25%	59.11	-10%	57.86	-11%
52	69.54	-4%	70.23	-3%	4.55	-19%	5.44	-3%	57.42	-12%	57.54	-12%

53	70.75	-2%	71.94	-1%	5.17	-8%	7.03	25%	57.10	-13%	55.55	-15%
54	72.81	0%	72.80	0%	3.27	-42%	3.18	-43%	64.67	-1%	65.38	0%
55	75.89	5%	76.69	6%	3.44	-39%	5.04	-10%	67.14	3%	64.92	-1%
56	71.20	-2%	71.94	-1%	3.53	-37%	4.25	-24%	61.86	-5%	62.03	-5%
57	72.83	0%	72.80	0%	3.31	-41%	3.18	-43%	64.40	-1%	65.38	0%
58	71.47	-1%	71.09	-2%	3.37	-40%	3.18	-43%	63.00	-4%	63.67	-3%
59	75.85	5%	78.40	8%	4.65	-17%	5.84	4%	62.25	-5%	64.78	-1%
60	67.36	-7%	68.06	-6%	3.82	-32%	4.25	-24%	56.90	-13%	58.14	-11%
61	71.20	-2%	71.94	-1%	3.53	-37%	4.25	-24%	61.96	-5%	62.03	-5%
62	71.20	-2%	71.94	-1%	3.56	-36%	4.25	-24%	61.78	-5%	62.03	-5%
63	71.20	-2%	71.94	-1%	3.53	-37%	4.25	-24%	61.91	-5%	62.03	-5%
64	72.45	0%	73.40	1%	4.47	-20%	5.84	4%	60.79	-7%	59.78	-8%
65	73.20	1%	74.51	3%	3.84	-31%	5.04	-10%	63.07	-3%	62.75	-4%
66	72.06	-1%	72.80	0%	2.71	-52%	3.18	-43%	65.03	0%	65.38	0%
67	71.60	-1%	72.20	0%	2.87	-49%	3.18	-43%	63.86	-2%	64.78	-1%
68	78.99	9%	79.51	10%	3.91	-30%	5.04	-10%	69.21	6%	67.75	4%
69	72.65	0%	73.66	2%	3.45	-38%	4.25	-24%	63.39	-3%	63.74	-2%
70	73.78	2%	75.37	4%	4.09	-27%	5.44	-3%	62.42	-4%	62.68	-4%
71	71.68	-1%	72.80	0%	4.46	-20%	5.84	4%	59.59	-9%	59.18	-9%
72	76.68	6%	77.80	7%	4.43	-21%	5.84	4%	65.02	0%	64.18	-2%
73	71.60	-1%	72.20	0%	2.87	-49%	3.18	-43%	63.90	-2%	64.78	-1%
74	71.66	-1%	74.26	2%	6.52	16%	9.80	75%	53.24	-19%	51.39	-21%
75	72.50	0%	75.37	4%	6.80	21%	10.99	96%	53.22	-19%	49.72	-24%
76	71.20	-2%	71.94	-1%	3.53	-37%	4.25	-24%	61.85	-5%	62.03	-5%
77	73.78	2%	75.37	4%	4.09	-27%	5.44	-3%	62.48	-4%	62.68	-4%
78	73.10	1%	73.91	2%	4.36	-22%	5.04	-10%	61.56	-6%	62.15	-5%
79	74.22	2%	75.63	4%	6.00	7%	8.22	47%	58.33	-11%	56.46	-14%
80	74.47	3%	74.77	3%	7.00	25%	8.22	47%	56.91	-13%	55.60	-15%
81	74.53	3%	75.23	4%	4.51	-19%	5.44	-3%	62.59	-4%	62.54	-4%
82	72.16	0%	73.06	1%	5.75	3%	7.03	25%	56.86	-13%	56.66	-13%
83	72.05	-1%	72.80	0%	2.71	-52%	3.18	-43%	65.10	0%	65.38	0%
84	70.26	-3%	70.83	-2%	3.90	-30%	5.44	-3%	60.38	-8%	58.14	-11%
85	72.53	0%	72.80	0%	4.82	-14%	5.84	4%	60.26	-8%	59.18	-9%
86	72.66	0%	73.66	2%	3.49	-38%	4.25	-24%	63.37	-3%	63.74	-2%
87	73.20	1%	74.51	3%	3.83	-32%	5.04	-10%	63.13	-3%	62.75	-4%
88	71.64	-1%	74.31	3%	9.02	61%	16.15	188%	47.75	-27%	36.63	-44%
89	73.03	1%	73.51	1%	4.39	-22%	5.04	-10%	61.31	-6%	61.75	-5%
90	69.52	-4%	70.23	-3%	4.47	-20%	5.44	-3%	57.89	-11%	57.54	-12%
91	73.23	1%	74.51	3%	6.55	17%	10.60	89%	56.38	-14%	49.79	-24%
92	72.65	0%	73.66	2%	3.45	-38%	4.25	-24%	63.39	-3%	63.74	-2%
93	75.85	5%	77.54	7%	3.62	-35%	4.25	-24%	65.51	0%	67.63	4%
94	71.48	-1%	72.20	0%	5.04	-10%	5.84	4%	58.12	-11%	58.58	-10%
95	75.75	4%	76.94	6%	5.17	-8%	7.03	25%	62.20	-5%	60.55	-7%
96	73.10	1%	73.91	2%	4.38	-22%	5.04	-10%	61.39	-6%	62.15	-5%
97	72.48	0%	77.94	8%	10.16	81%	18.13	224%	42.03	-36%	35.63	-45%
98	72.36	0%	73.06	1%	3.82	-32%	4.25	-24%	61.93	-5%	63.14	-3%
99	75.36	4%	76.09	5%	4.21	-25%	5.04	-10%	64.16	-2%	64.32	-2%
100	75.03	3%	76.49	5%	5.82	4%	7.82	40%	59.04	-10%	58.24	-11%

Table 6.4 summarises the results of Tab 6.3, showing, for each parameter and for each of the two methods, the maximum relative error, the range of variation defined as the difference between the maximum and minimum value of the estimated parameter, the sample mean of each estimated parameter and the mean estimated error (MEE) defined as:

$$Mean\ estimated\ error\ [\%] = \frac{Mean\ of\ estimated\ value - True\ value}{True\ value} \cdot 100 \quad (6.3)$$

Table 6.4 Summary results of 100 simulation with n = 15, $\Delta S = \overline{\sigma_{X_l}}$

	a		b		σ_Y		μ_{X_l}		σ_{X_l}		R90C90	
Model	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**	**Model II**	**Model I**
Max REE (%)	30.8%	53.1%	55.52%	93.47%	37.91%	45.46%	8.95%	9.67%	81.48%	223.84%	35.67%	49.38%
Range of variation	7.3	12.7	3.796	6.483	0.190	0.208	11.626	11.457	7.456	14.955	28.070	37.313
Parameter mean	12.044	11.748	-3.455	-3.307	0.246	0.253	72.595	73.643	4.443	5.761	60.731	60.203
True value	12.271		-3.569		0.255		72.5		5.6		65.332	
MEE (%)	1.8%	4.3%	3.2%	7.4%	3.8%	0.8%	0.1%	1.6%	20.7%	2.9%	7.0%	7.9%

For each analysed parameter, Model I has a higher maximum relative error than Model II and a wider range of variation in which the values are distributed. Model II also has a smaller mean estimated error for parameters a, b, μ_{X_l} and *R90C90* while Model I has mean values for standard deviations that are closer to the true values. However, the scatter in the standard deviations estimates is larger, especially in the estimation of the standard deviation of the fatigue limit. The same analyses were repeated by varying the step size, and the results are presented in Figures 6.1-6.6 which show the dispersion of the estimated results from the true value, for each step, for each parameter and for each method.

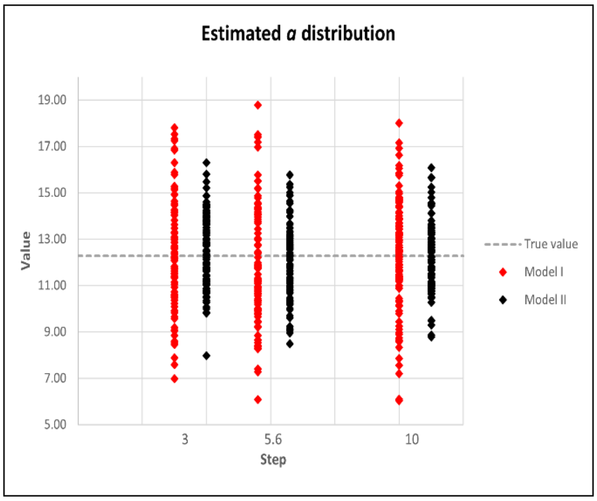


Figure 6.1 Simulation results for parameter a

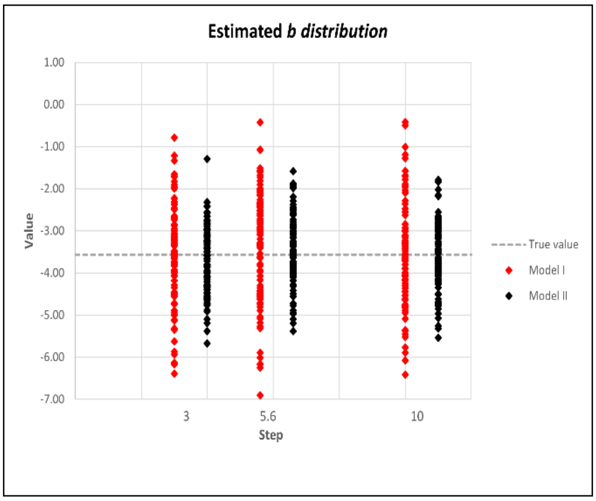


Figure 6.2 Simulation results for parameter b

Figures 6.1 and 6.2 show that Model I exhibits a greater dispersion of the estimates of parameters a and b than Model II even for step values below and above the true standard deviation. Figure 6.3 shows a similar pattern for the standard deviation of fatigue life, with the distribution of the estimated parameters that are similar for the two methods but with Model II presenting, in each case, a smaller dispersion in the results than Model I. Figure 6.4 relates to the median value of the fatigue limit; Model I presents an average value of the estimated parameters which is closer to the true value for higher steps while Model II is more accurate for steps equal to the true standard deviation. In particular, Model II presents less scatter and is more accurate in estimating μ_{X_l} than Model I except when ΔS =10 MPa, where Model I is superior, as confirmed also by Figure 6.7 which reports the mean estimated error for each step size and for each model. Figure 6.5 shows that the distribution of estimated σ_{X_l} for Model I is always much more dispersed than for Model II, especially for $\Delta S = \overline{\sigma_{X_l}}$.

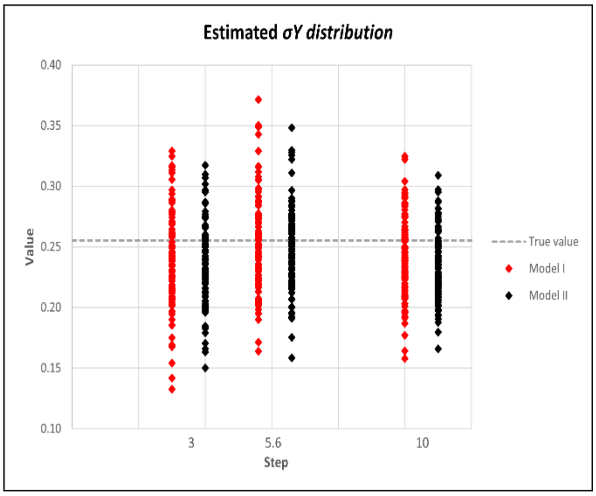


Figure 6.3 Simulation results for parameter σ_Y

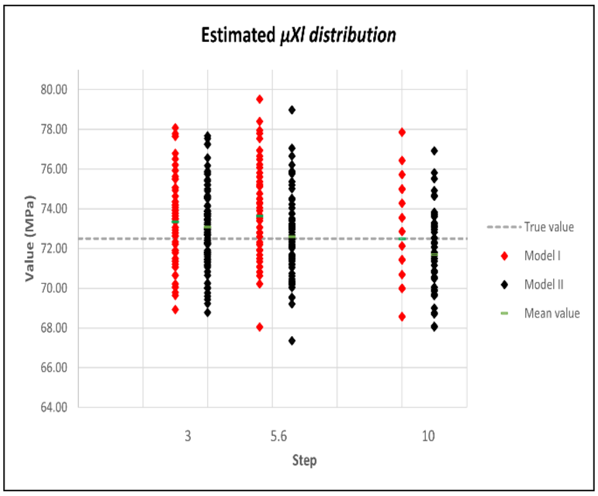


Figure 6.4 Simulation results for parameter μ_{X_l}

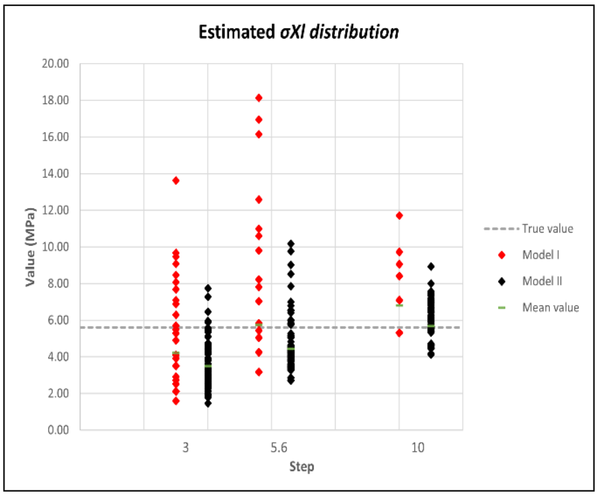


Figure 6.5 Simulation results for parameter σ_{X_l}

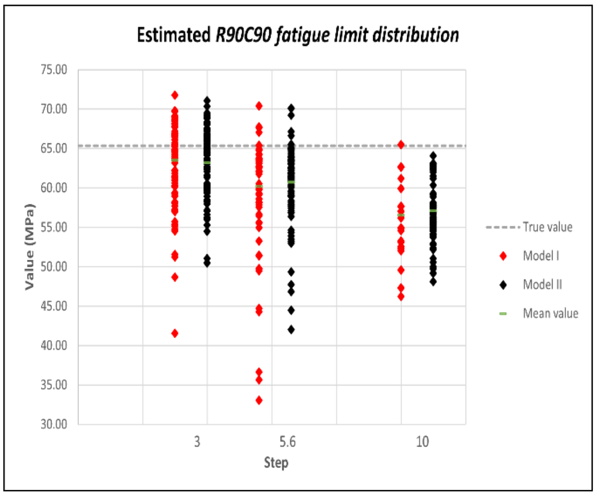


Figure 6.6 Simulation results for R90C90 fatigue limit

Model II has an average value of the estimated σ_{X_l} that is lower than the true value except when the step height is at 10 MPa where the estimate seems to be very accurate and the data poorly dispersed. Figure 6.6 refers instead to the estimation of the R90C90 fatigue limit with both methods. The mean value of the estimates with Model II is slightly closer to the true value for each step size even if the difference with Model I is quite small. Model I, however, presents a greater dispersion of the results especially when $\Delta S = \overline{\sigma_{X_l}}$, as shown also in Table 6.5 where the range of variation of the results for both models and for each step size is reported.

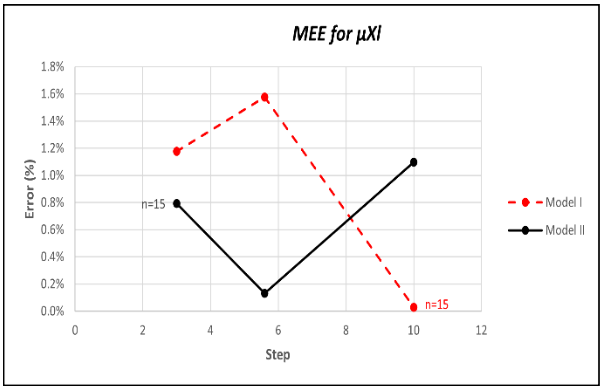


Figure 6.7 Mean estimated error for μ_{X_l}

Table 6.5 Range of variation for estimated R90C90 fatigue limit

	Range of variation	
Step	***Model I***	***Model II***
3	30.202	20.582
5.6	37.313	28.070
10	19.242	15.977

To determine the sensitivity of both models to the sample size, virtual tests were performed by varying the number of specimens on Staircase test with a constant step size equal to the true standard deviation. Figure 6.8 shows that, for both models, the mean values of μ_{X_l} are comparable, and the scatter is reduced by increasing the number of specimens on the test. Furthermore, it can be verify that Model II, independently of the number of samples, presents results that are more similar to each other and much closer to the true value than Model I.

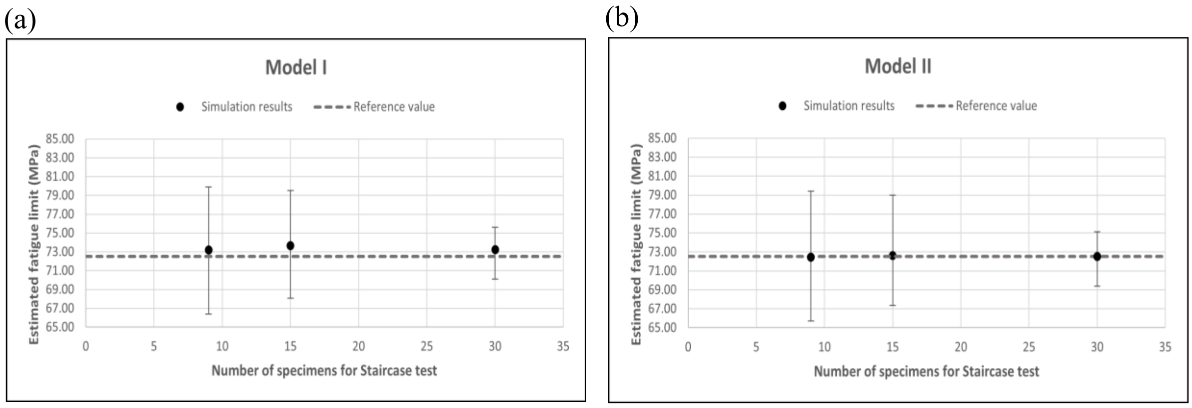


Figure 6.8 Calculated fatigue limit as function of the number of specimens for Model I (a) and II (b)

Figure 6.9 shows the same pattern for R90C90 fatigue limit. For both methods, the mean values are comparable with each other and as the number of samples increases, the mean value is closer to the true value. Also in this case, Model I presents more scatter in the results than Model II, especially for a smaller sample size.

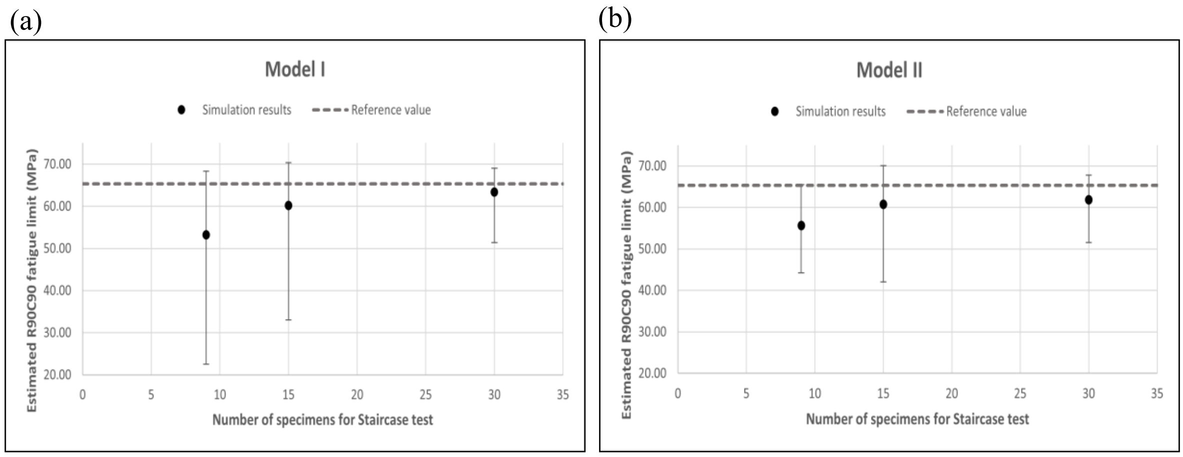


Figure 6.9 Calculated R90C90 fatigue limit as function of the number of specimens for Model I (a) and II (b)

The number of times the confidence interval contains the true value of R90 was also evaluated. Indeed, by definition of R90C90, 90 % of the times the R90C90 lower bound of the curve should be lower than the R90 S-N curve. However, as both models are only estimates, the number of times the confidence interval does not contain the true value of R90 is greater than 10 %. This was quantified by counting the number of times the lower bound of confidence interval is below the true R90 curve, and indicated with N_e in Table 6.6.

Table 6.6 Control of confidence interval estimate with both models

		$N_e/100$	
Step size	**Number of specimens**	***Model I***	***Model II***
3	9	82.0%	89.0%
5.6	9	97.0%	99.0%
10	9	100.0%	100.0%
3	15	57.0%	66.0%
5.6	15	89.0%	93.0%
10	15	98.0%	100.0%
3	30	49.0%	72.0%
5.6	30	65.0%	77.0%
10	30	99.0%	99.0%

It can be seen that for each value of step size and number of specimens, the number of times the estimate of the R90C90 curve with Model II is below the true R90 curve is always greater than or equal to number of times the estimate of the confidence interval with Model I contains the true value of R90.

7 Conclusion

At the end of the numerous analyses of experimental and simulated datasets carried out in this thesis work, it is useful to highlight the main results obtained and the objectives achieved. As we have seen, fatigue strength of a material can be determined by the evaluation of S-N curve. For the construction of that curve, two models have been examined: the first involves a simple linear regression of the data for the fatigue finite life region using the method of least squares and the Staircase method for the determination of the fatigue limit, the second one involves the use of a method based on the principle of Maximum Likelihood for the determination of the complete S-N curve. The first method is currently widely used in industry because of its simplicity in test protocol and analysis of results. The second one, although requiring a slightly more complex analysis of the data, is able to maximise the information of each experimental result. In fact, if in the first method the only information that is used for the determination of the fatigue limit is the failure or non-failure of the samples (depending on which is the less frequent event), the second method uses additional information taking into account the number of cycles to failure of each test performed. Moreover, in the design of the Staircase, there are some limitations, i.e. the initial stress value must not be too far from the fatigue limit, which is being calculated, and the step must be included in the range $0.5\cdot\sigma$ - $2\cdot\sigma$, which requires prior knowledge of the standard deviation of the fatigue limit. On the other hand, methods based on the principle of Maximum Likelihood, do not require any initial parameter estimation and subsequent verification and thus allow greater freedom in test design.

Since the analysis of the results by means of the Maximum Likelihood principle requires an iterative approach, especially for the calculation of the Profile Likelihood function necessary for the calculation of the confidence intervals of the S-N curve, a user-friendly application was developed which, once the files relating to the experimental tests carried out had been loaded in txt format, would allow the data to be analysed using both methods and to provide in output the S-N curves corresponding to any desired reliability and confidence level and an excel file with all the results obtained.

The analysis of the experimental datasets has shown that when there are no failures near the number of runout cycles and the runouts are far from the knee of the curve, then the results, in terms of curve shape and fatigue limit, are very similar to each other. When, however, there are failures near the chosen number of runout cycles, then the fatigue limit values computed with the two methods differ, especially when considering the S-N curve at a certain confidence level (such as the R90C90 curve). This is because the two models do not estimate the same limit. Staircase method estimates the fatigue strength at the number of runout cycles, N_G, while the second model allows to visualize the real trend suggested by the experimental data, whatever it is. In these cases, the staircase test was carried out at too low run-out level. In particular, it was found that in these cases the fatigue limit calculated with the method based on the principle of Maximum Likelihood is lower than that calculated with the Staircase. The reason for this is that, when the failures are sufficiently far from the number of cycles to runout, then the probability that the logarithm of the lives is below $\log_{10}(N_G)$, i.e. $\varphi\left(\frac{Log_{10}(N_G)-\left(a+b\cdot Log_{10}(S_{a_\alpha})\right)}{\sigma_Y}\right)$ factor, in the model considered for Maximum Likelihood method, is approximately equal to one and thus, for both models, the probability of failure can be expressed as $\varphi\left(\frac{Log_{10}(S_a)-\mu_{X_l}}{\sigma_{X_l}}\right)$ i.e. as the probability that the actual stress is higher than the fatigue limit. On the other hand, when some specimens break near the endurance life N_G, $\varphi\left(\frac{Log_{10}(N_G)-\left(a+b\cdot Log_{10}(S_{a_\alpha})\right)}{\sigma_Y}\right)$ factor is slightly lower than one, and thus the $\varphi\left(\frac{Log_{10}(S_a)-\mu_{X_l}}{\sigma_{X_l}}\right)$ factor in ML model, increases and consequently median value of fatigue limit, μ_{X_l}, decreases.

For some datasets, moreover, it was seen that the method based on ML failed to find, at that given confidence level, a fatigue limit. This does not mean that the material does not have a fatigue limit but that the data available does not allow me to say that it does. In such cases it would in fact be necessary to carry out tests up to a higher number of cycles than the number of cycles for the chosen runout. In fact, even if the approach involving linear regression and staircase makes it possible to draw a horizontal line corresponding to the fatigue strength at that given number of cycles, this can be dangerous because the data do not show asymptotic behaviour, as suggested by the method based on Maximum Likelihood, which instead describes the real trend of the data. It is clear that even with this method, when it is not possible to find an asymptote for the S-N curve, it would be possible to obtain the slope of the linear region, the knee of the curve and "force" a fatigue strength at a given number of cycles, i.e. to determine the 3 parameters that are usually used for the fatigue design of a component, but this does not necessarily guarantee a safe condition because the data do not suggest the presence of a plateau.

Another advantage of Maximum Likelihood methods was also investigated, namely that fewer tests can be carried out without losing too much information and thus saving time and costs for the fatigue characterisation of a material. For some of the experimental datasets, some of the Staircase data were removed and the results obtained were very similar to those obtained from the complete dataset. Since the obtained results are of particular interest, similar verifications are recommended for other datasets in order to validate this result.

Once the experimental datasets were analysed, a simulation study was carried out to determine which of the two methods was more accurate in determining the parameters of the S-N curve. Virtual datasets were simulated from the true values of the parameters to be estimated and were then analysed with both methods as if they were experimental datasets, but for which the real values of the parameters to be estimated were already known. The results of simulations show that the error in the estimation of the parameters with the method based on maximum likelihood was almost always lower than with the method based on linear regression and Staircase, and that even in cases where the latter method was superior in estimation, it still presented more scatter in the results. This is reasonable because in the traditional approach an important information such as the number of cycles to failure of the infinite life region data is not taken into account and therefore the range of variation of the estimated parameters was expected to be wider.

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