Master's Degree in Engineering and Management

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Asian Option pricing methods Construction and Realization

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Contents

Introduction

Options as one of the most important financial derivatives that help investors to hedge investment risk or avoid market risks. And for pricing, the options researchers have worked for decades, some main pricing methods are usually adept into the real market.

In this thesis, we will focus on the Asian option which underlying the asset without dividend.

In Section 1, We have introduced options and stock prices. Based on the detailed knowledge, the BSM model that have been carefully introduced as well as the foundation theorem Itô's formula. The last part of Section 1 is mainly focused on the derivation of the BSM model for the European option that provides basic equations for the Asian one.

In Section 2, we firstly clarified the Asian option and the difference with the European option. After that, there are two types of Asian options as well as pricing methods has been provided as the BSM model gives the basic solution of Asian option pricing.

However, with the development of computer science, the Monte Carlo method is widely used in pricing options now. So that, Section 3 is mainly presenting the relative content of the Monte Carlo method. It contains the introduction of the Monte Carlo method for option pricing, and for easy understanding, a small part of given a slight introduction about how the Monte Carlo simulation works have been created as well.

In the last section, we introduce a set of real data from the trading market for the modified BSM model and the Monte Carlo method to pricing, the aim is to test the feasibility of the above methods.

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Section 1 Introduction of options and BSM model for European option pricing

1.1 Introduction of Options and stocks price process

This thesis is mainly concerned with the pricing of Asian options underlying the stock without dividends. The option is a type of derivative underlying by stock, it can be named differently according to the different types of derivatives with the different contract contents.

Definition 1.1 A derivative is a contract between no less than two parties whose value is based on one agreed-upon underlying financial asset or a set of assets.

Derivatives, as one of the most historical investment instruments that help investors hedge risks or even make profits, have greatly influenced the financial market.

Based on the above situation, the proper and correct way to value the option is a topic worthy of research.

Definition 1.2 An option is a contract between two parties that gives the buyer the right to buy or sell the agreed-upon asset.

Stock options, for example, are a kind of derivatives. Their value depends on the price of the stock. We will focus on stock options in this thesis. The options trading has two positions. The position of the buy-side of the option contract is called the long position, and the opposite side of the sell side of the option contract is called the short position.

Options can be divided into two categories in terms of trading direction. A call option means that the buyer of the option contract has the right to buy the amount of stock agreed upon in the contract from the seller at the agreed upon price. A put option means that the buyer of the option has the right to sell the quantity of shares agreed in the contract at the agreed price. The agreed price is called the strike price, which is set with the contract.

There are different types of options in the financial market. The chart below shows the main types of options, but not all options are traded in every market. For better explanation, their comparative descriptions are also listed on the chart.

Options Name	Description	Long position Payoff	Notes
European Options	A type of option that only can be exercised on maturity time T.	Call:$\max(S_T - K, 0)$ Put:$\max(K - S_T, 0)$	S_T : stock price at the maturity date K: strike price
American Options	A type of option that can be exercised on any trading day that before expiration.	Call:$\max(S_t - K, 0)$ Put:$\max(K - S_t, 0)$	S_t: the stock price at the exercise date K: strike price
Asian/ Average Options	A type of option which is determined by the average underlying asset price over the predetermined period.	Call:$\max(AT - K, 0)$ Put:$\max(K - AT, 0)$	AT: Average stock price during the contract period K: strike price
Basket Options	A type of option which is underlying a group of a weighted sum or average of different assets	$\max(S_{basket} - X_{basket}, 0)$	S_{basket}: the weighted sum of average assets price at the maturity date X_{basket}: the strike price of the group assets

Table 1.1 Different type of options

From the above descriptions, it is clear that the strike price is a very important factor in the composition of options. For pricing the Asian options, studying the European option as prior knowledge is very important as they are similar, but the European option is much easier to price.

The European option is a traditional option that has a long history. The payoff of a European option is determined by the stock price S_T at maturity time T and the strike price K. For example, the payoff of the long position of a call option is $\max(S_T - K, 0)$ at the maturity time T.

Hence, the stock price S_T at the maturity time T is a key parameter in a European option, because it can directly affect the option holder's payoff. Thus, it is important to understand what the stock price is at maturity time T or how the stock price changes during the period from time 0 to time T.

It is common knowledge that the stock price can rise or fall in any time interval, which means that the stock can reach a different price in the future with different probability. To measure this kind of probability, we have two different ways: probability measure P and probability measure Q.

Definition 1.2 The probability measure Q and the probability measure P are equivalent probability measure on what has probability 0 to happen.

$$P(A) = 0 \ if \ and \ only \ if \ Q(A) = 0$$

Definition 1.3 The probability measure that based on the market historical data is called probability measure P, which can be called real-world measure as well.

Definition 1.4 The probability measure Q is called martingale measure or risk-neutral measure if the following the below conditions:

$$\boldsymbol{S_0 = \frac{1}{1 + r_s} E^Q(S_1)}$$

where the $\boldsymbol{r_s}$ is the spot interest rate with simple compounding for the period which providing from a risk-free asset, for example treasury bond.

Definition 1.5 The return of an asset in unit time is defined by

$$\boldsymbol{\frac{S_1 - S_0}{S_0} = \mu}$$

Stock price can be affected by the attitude of investors, which means that it can be affected by the expected return of different investors. According to the above theory, at time 0, investors can have an expected return rate which is the expected return for doing this investment.

In the real financial market, the expected return rate satisfies the below equation:

$$\mu_t = r + \sigma u_t \qquad (1\text{-}1)$$

where the u_t is the market price of risk in time t, with a more risk-averse attitude, the higher expected return rate μ is required in investment (Kroese, Taimre & Botev, 2011). However, it is almost impossible to find out the risk attitude of all investors in the market and the discount rate in the market, and a risk-neutral method must be found for pricing.

Therefore, the risk-neutral is necessary and helpful in the pricing of derivatives, so the following procedure is needed based on the risk neutral world. The so-called risk-neutral world means that all investors are risk neutral, i.e., the expected return rate μ for each investor is the risk-free interest rate r.

Stock price is affected not only by risk attitude but also by arbitrage activity. Arbitrage opportunities have a great impact on the stock price in the investment market. Performing arbitrage in the financial market is very convenient and fast. Although this kind of arbitrage is convenient, the existence of arbitrage opportunities in the financial market is always temporary, because once there is an arbitrage opportunity, investors will quickly implement the arbitrage and bring the market back to the arbitrage-free equilibrium. Therefore, arbitrage-free equilibrium is often used to price financial products.

Definition 1.6 A financial market is called Arbitrage-free if there is no possibility to create positive wealth from zero or negative initial wealth without incurring any risk.

$$\boldsymbol{V_0 = 0}$$
$$\boldsymbol{V_1 > 0, with\ the\ probability\ 0}$$

where the $\boldsymbol{V}$ presents investors wealth.

In this condition we could say if and only if the market exists probability measure Q is equivalent to probability measure P.

Hence, by no arbitrage opportunity and the risk-neutral world condition, the future asset price is the discounted expected value of the future payoff, we call this method is under the risk neutral measure, or $Q-$ measure.

However, even under the probability measure Q, the stock price changes stochastically. Thus, the stock price process can be modeled by a stochastic process. To be more precise, researchers in finance usually use the Geometric Brownian motion to model the stock price process in the financial and investment market.

Definition 1.6 A stochastic process is a collection of random variables indexed by time.

For continuous time:

$$\boldsymbol{z = \{z(t), t \in T\} = (z(t,\omega), t \in T, \omega \in \Omega)}$$

where $\boldsymbol{T}$ is a real interval, and the $\boldsymbol{\omega \in \Omega}$ is a random variable

For discrete time:

$$\boldsymbol{z = \{z(k), k \in \mathbb{Z}_+\}}$$

where $\boldsymbol{\mathbb{Z}_+ = \{0, 1, 2, \dots\}}$

The Geometric Brownian motion is a special type of stochastic process generated from Brownian motion. Therefore, we must first introduce Brownian motion before the Geometric Brownian motion.

The so-called Brownian motion is a kind of stochastic process first discovered in 1828 by the Scottish botanist Robert Brown when he makes the observed suspension under the microscope for pollen particles and dust.

The first use of the Brownian motion as a possible model for exploration in the financial industry is by Bachelier in 1900 for the study of stock market prices. By

1923, however, the Brownian motion was strictly defined by Norbert Wiener, in honor of which this type of stochastic process was called the Wiener process (Schoutens, 2003).

Definition 1.7 A stochastic process $z = \{z(t), t \geq 0\}$ is called standard Brownian motion or Wiener process on some probability space $(\Omega, \mathcal{F}, P)$ if the following conditions hold

1, $z(0) = 0$

2, The process z has independent increments

3, The process z has normally distributed increments:

$$z(s+t) - z(s) \sim N(0, t)$$

4, The process w has continuous sample path

Based on the above definition, the change of Δz during a small time period Δt is

$$\Delta z = \varepsilon\sqrt{\Delta t} \tag{1-2}$$

where the ε has a standard normal distribution: $\varepsilon \sim N(0,1)$

Below is a figure of the sample path of a standard Brownian motion.

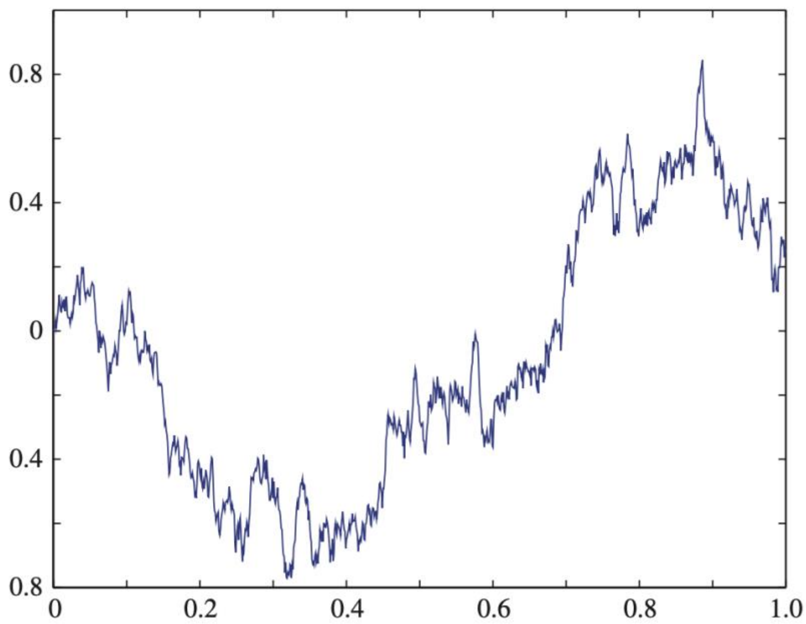

Figure 1: A sample path of a standard Brownian motion

The standard Brownian motion definition shows the standard Brownian motion has a 0 value at $t = 0$. And in any limited time-interval Δt, the standard Brownian motion satisfy the normal distribution with the mean 0 and the variance Δt, moreover, the variance increases linearly with the length of the time interval.

Independent increment means that the changes in Brownian motion in any time interval have nothing to do with the changes in other time intervals that do not overlap with it.

The Brownian motion has some properties that are very important in describing the stock price. The first property is the path property.

Since the function z_t is a continuous function of t, the Brownian motion has continuous paths. But these continuous paths are too erratic to be differentiable everywhere since the trajectory of Brownian motion is completely different from any continuous and smooth equation trajectory we are familiar with. Moreover, based on this kind of erratic, the paths of Brownian motion can be seen have infinite variation. And for a Brownian motion we have that

$$P\left(\sup_{t\geq 0} z(t) = +\infty \text{ and } \inf_{t\geq 0} z(t) = -\infty\right) = 1 \qquad (1\text{-}3)$$

The equation (1-2) shows that the Brownian motion path keeping oscillating between positive and negative values.

Assuming we use Brownian motion to describe the stock prices, these properties mean that there is a high probability that the stock price will fluctuate above or below the opening price, rather than staying above or below the opening price, this shows that it is very close to the reality of the stock market.

Now we can consider adding a drift part associated only with time t and a diffusion rate σ to the standard Brownian motion, then we get the drifted Brownian motion.

Definition 1.8 A stochastic process follows a drifted Brownian motion if the following condition holds

$$\boldsymbol{dX(t) = \mu dt + \sigma dz(t)}$$

The above equation is called stochastic differential equation or SED that is an extension of the ordinary differential equation. The difference is that at least one stochastic process is included in the stochastic differential equation.

The $dz(t)$ has a clear meaning, which represents the change of Brownian motion in an infinitely small-time interval.

Since the drifted Brownian motion is generated from the Brownian motion, the drifted Brownian motion clearly has an independent and stationary increment, then the distribution of drifted Brownian motion within an arbitrary time length t can be obtained

$$z(t) \sim N(\mu t, \sigma^2 t) \qquad (1\text{-}4)$$

Although the drifted Brownian motion has the drift part and the diffusion rate part, it is not the best model to describe the stock price process. This is because X(t) or z(t) have the probability of having a negative value, but the stock price in the real world clearly cannot be negative. Clearly, the stock price cannot be a negative number, but the return on the stock can be positive or negative, so the $X(t)$ can be used to describe the return.

We suppose $S(t)$ is the stock price process, the $\mathrm{d}S(t)$ is the amount of change in the stock price in an infinitely small-time interval, and then, the $\mathrm{d}S(t)/\ S(t)$ is the return rate during this infinitely small-time interval.

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dz(t) \tag{1-5}$$

Definition 1.8 A stochastic process $X(t)$ follows a geometric Brownian motion if the following stochastic differential equation holds

$$\boldsymbol{dS(t) = \mu S(t)dt + \sigma S(t)dz(t)}$$

The stock price that satisfies the above stochastic differential equation is a geometric Brownian motion that provides a basis for people to study stock prices.

1.2 Introduction of the Itô's formula and BSM differential equation

The above section has introduced the stock price in the real world. However, an option is a derivative contract with the underlying assets in a fixed period of time, so time is another important parameter to consider.

However, we must first address how to differentiate the function of a random process. The Japanese mathematician Itô Kiyoshi provided us with the Itô's formula to solve this problem (K. Itô, 1951).

Regardless of the geometric Brownian motion or Brownian motion, the drift and the variance are real numbers, which means that this format cannot show the situation of the stochastic process where the drift and variance change with time, then the Itô process must be introduced to describe the drift and variance change with time.

Definition 1.9 A stochastic process follows the Itô process if the following condition holds

$$\boldsymbol{dx = a(x,t)dt + b(x,t)dz}$$

where the a and b are functions of time t and variable x and z is the Brownian motion. The drift for Itô process is a and the variance is then b^2. Then, we could introduce Itô's formula to show the stochastic differential of derivatives.

Theorem 1.1 Itô's formula: Assume the process $X(t)$ follows Itô process, and f is the second-order continuous differentiable function, and the first-order derivative of t, then the process $G(t)$ is defined by $G(t) = f(X(t), t)$ that has a stochastic differential given by

$$\boldsymbol{dG = \left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial X}a + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2\right)dt + \frac{\partial G}{\partial X}bdz}$$

where the dz is the Brownian motion, therefore the function G follows the Itô process and then follows the same Brownian motion. For the same principle, the drift for the above function is

$$\left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial X}a + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2\right) \tag{1-6}$$

by the same principle, the variance for the function is

$$\left(\frac{\partial G}{\partial X}\right)^2 b^2 \tag{1-7}$$

With Itô's formula, we could find which stochastic process what lnS follows, where the S follows the process showed in the Definition 1.8. By this setting, $G(t) =$

$lnS(t)$ where $S(t)$ follows $dS(t) = \mu S(t)dt + \sigma S(t)dz(t)$, therefore the below elements can be generated by Theorem 1.1:

$$\frac{\partial G}{\partial S} = \frac{1}{S}, \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}, \frac{\partial G}{\partial t} = 0 \tag{1-8}$$

Then, dG therefore is

$$dG = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dz \tag{1-9}$$

From the equation (1-9), it is showed that the drift for the above function is $\left(\mu - \frac{\sigma^2}{2}\right)$, and the variance is σ^2 times random part dz, where $\sigma^2$is constant.

Hence for $G = lnS$ in time t, it follows the below normal distribution:

$$lnS_t \sim \emptyset \left[lnS_0 + \left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right] \tag{1-10}$$

Based on the above equation, we could say that the lnS_t follows normal distribution, hence, the stock price S_t follows logarithmic normal distribution.

Then the stock price dynamic can be present as below

$$S_t = S_0 \cdot e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma z_t} \tag{1-11}$$

Based on the above equation, the below equation can present the stock price changes between the small-time interval Δt

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \Delta z \tag{1-12}$$

Now, assuming a derivative f which is a call option of a stock S, then the value change in small period dt can be got by the equation in the Definition 1.8 and Itô's formula, it is

$$df = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S dz \tag{1-13}$$

As the stock price S_t follows a Wiener process z, based on Theorem 1.1, the derivative f follow the same Wiener process.

The equation (1-13) can be re-written as the below format to present the changes between the small-time interval Δt

$$\Delta f = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S \Delta z \tag{1-14}$$

From the above equation, which has a random part containing the Wiener process. However, we could create an appropriate investment portfolio to eliminate the Wiener process. To be precise, we can construct a portfolio that shorts one unit of derivative f and buys and $\frac{\partial f}{\partial S}$ units of stock like:

$$\begin{gathered} -1: Options \\ +\frac{\partial f}{\partial S}: stock \end{gathered} \tag{1-15}$$

The value of the portfolio can be represented by π as below

$$\pi = -f + \frac{\partial f}{\partial S} S \tag{1-16}$$

The value change of the portfolio $\Delta\pi$ in the time interval Δt is

$$\Delta\pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S \tag{1-17}$$

By substituting the equation (1-14) and the equation (1-12) in to the equation (1-17), the $\Delta\pi$ can be got as below.

$$\Delta\pi = -\left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t \tag{1-18}$$

In this way, the right-side random part can be eliminated, as the same reason, the portfolio π in the time interval Δt is risk-free. This portfolio must have the expected return rate as the same as the risk-free interest rate r, otherwise, there will be arbitrage opportunities, then

$$\Delta G = G r \Delta t \tag{1-19}$$

where r is risk-free interest rate with continuous compounding, by sum up the above equations, the Black-Scholes-Merton differential equation:

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 = rf \tag{1-20}$$

1.3 Introduction and derivation of the BSM model for European option pricing

The Black Scholes Merton model is also known as the BSM model. It was first proposed by two American economists Myron Scholes and Fischer Sheffey Black in the last century. The advent of this model brought a boom in the options market. This model is widely used, although in many cases users have made certain modifications and corrections. Many empirical tests have shown that this formula is close enough to the market price, but there are also times when there are differences. The BSM model is mainly used for pricing European options, and through modification it can also price Asian options.

In order to implement the BSM model, a number of assumptions must be made. Firstly, we are in a frictionless market and there are two assets in this market which is riskless assets like bonds and the other is the risky assets like stocks.
Secondly, the world in BSM model is risk neutral which means the probability measure is under the $Q - measure$, and the market is arbitrage-free.

Now we can focus on how to price a European option. A European option is the option to which the holder has the right, but can only be exercised at maturity time T with the strike price K. So, that a European can provide a maximum payoff P is $S_T - K$ if the holder exercises the option and a minimum payoff P is 0 if the holder does not exercise the option at maturity time T. Hence, the payoff from a European call option at maturity time T is

$$P = (S_T - K)^+ \tag{1-21}$$

By the risk natural pricing method, the price of the call option C can be seen as the expected value of the relative option

$$C = e^{-rT} E^Q(P) \tag{1-22}$$

In order to facilitate the calculation, it is need to define two new variables. By setting the $\mu_1 = lnS_0 + \left(r - \frac{\sigma^2}{2}\right)T$, and $\sigma_1 = \sigma\sqrt{T}$, the equation (1-10) changes to

$$lnS_T \sim \emptyset[\mu_1, \sigma_1] \tag{1-23}$$

which means S_T follows logarithmic normal distribution as we mentioned in the above section.

Now, by setting $y = S_T$, and $g(y)$ is the probability density function of S_T under the probability measure Q, the below equation can be got:

$$g(y) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{(lny-\mu_1)^2}{2\sigma_1^2}} \tag{1-24}$$

then the expected payoff P in time maturity time T is

$$P = \int_k^{\infty} (y - K) g(y) dy \tag{1-25}$$

Introduce the equation (1-24) to the equation (1-25), we get

$$P = \int_k^{\infty} (y - K) \frac{1}{\sigma_1 \sqrt{2\pi}} e^{-\frac{(lny - \mu_1)^2}{2\sigma_1^2}} dy \tag{1-26}$$

And introduce $lny = x$, the expected payoff in time maturity time T is

$$\int_{lnK}^{\infty} \frac{e^x}{\sigma_1 \sqrt{2\pi}} e^{-\frac{(x - \mu_1)^2}{2\sigma_1^2}} dx - \int_{lnK}^{\infty} \frac{K}{\sigma_1 \sqrt{2\pi}} e^{-\frac{(x - \mu_1)^2}{2\sigma_1^2}} dx \tag{1-27}$$

We set the right part be the function F_1 and the right part be the function F_2. By define $N(z)$ is the probability of the variable smaller than z with 0 mean and standard deviation is 1, the function F_1 is

$$F_1 = \frac{1}{\sigma_1 \sqrt{2\pi}} \int_{lnK}^{\infty} e^t e^{-\frac{(x - \mu_1)^2}{2\sigma_1^2}} dx$$

$$= e^{\mu_1 + \frac{\sigma_1^2}{2}} \left\{ 1 - N \left[\frac{lnK - (r + \sigma_1^2)}{\sigma_1} \right] \right\}$$

By expand the μ_1 *and* σ_1

$$= S_0 e^{rT} N \left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} \right)$$

$$= S_0 e^{rT} N(d_1) \tag{1-28}$$

Where the d_1 is

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} \tag{1-29}$$

By the same computing method, the function F_2 can be easily get

$$F_2 = \frac{1}{\sigma_1 \sqrt{2\pi}} \int_{lnK}^{\infty} K e^{-\frac{(x - \mu_1)^2}{2\sigma_1^2}} dx$$

$$= K * N[-\frac{lnK - \mu_1}{\sigma_1}]\}$$

By expand the μ_1 *and* σ_1

$$= KN \left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} \right)$$

$$= KN(d_2) \tag{1-30}$$

Where the d_2 is

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \tag{1-31}$$

By introduce the equation (1-28) and the equation (1-30) to the equation (1-26), the expected value in time maturity time T is

$$\mathrm{E}\,[\max(S_T - K, 0)] = S_0 e^{rT} N(d_1) - KN(d_2) \tag{1-32}$$

Combining the equation (1-32) and the equation (1-22), we get the price of the call option C is

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2)] \tag{1-33}$$

where d_1 and d_2 are respectively showed in the equation (1-29) and the equation (1-31).

Now we have derived the BSM model for pricing European options, where the underlying stock paid no dividend.

But the original BSM model cannot be used directly for the Asian options we are studying. Therefore, we will discuss in detail how to modify the model to price Asian options and the BSM.

Section 2 Introduction of Asian options and modifying BSM model to pricing Asian options

2.1 Introduction of Asian options

The Asian option is one of the European type options that give the holder the right but can only be exercised at maturity time T with the strike price K with the underlying asset prices calculated by the average method in the constant period set at the beginning of the contract. Therefore, the Asian options can provide a payoff is

$$P = (AT - K)^{+} \tag{2-1}$$

where the AT is determined by the average base price over the specified time interval. We will introduce how the needed AT can be computed in the following section parts.

In the Asian option, the point-to-point calculation method is not appropriate. And usually there are three algorithms of averaging in Asian option contract trading, namely arithmetic average, geometric average and weighted average. For weighted average, the most important thing is to determine the weighting vector, and the vector is decided by two parties before two parties sign the contract, it needs to be discussed case by case, so it is not discussed in this thesis.

2.2 Geometric average Asian option pricing based on BSM model framework

The above BSM model is useful for pricing the European option, but it cannot price the geometric average Asian option. To price the geometric average Asian option, one must introduce how the price is averaged.

Firstly, the mean of geometric average stock price during the period from time 0 to time T is AT that can be got by geometric average formula:

$$AT = \sqrt[T]{\prod_{t=1}^{T} S_t} \tag{2-2}$$

where t is positive integer.

Because the S_t can be considered as a continuous process as S_t contains a Wiener process, so that AT can be applied with the calculus method:

$$AT = \exp\,[\frac{1}{T}\int_1^T \log S_t dt] \tag{2-3}$$

By the risk-neutral pricing method, the Asian call option has an expected value C is

$$C = e^{-rT}E^Q(AT - K)^+ \tag{2-4}$$

Then it comes down to finding the payoff percentage. In 1983, Robert Jarrow and Andrew Rudd solved the way to obtain the value of $(AT - K, 0)^+$. According to the work of Robert Jarrow and Andrew Rudd (Jarrow & Rudd, 1983), they define the expected value P for an Asian option with the strike price K and maturity time T is

$$P = S_0 N(d)e^{\tilde{d}} - KN(d - \sigma\sqrt{\frac{1}{3}T}) \tag{2-5}$$

where the N is as in the above sections representing the standard cumulative normal distribution function, and the $\tilde{d}$ is

$$\tilde{d} = \frac{1}{2}(r - \sigma^2/6\,)T \tag{2-6}$$

and the d is

$$d = \frac{\ln\left(\frac{S_0}{K}\right) + \frac{1}{2}(r + \sigma^2/6)T}{\sigma\sqrt{\frac{1}{3}T}} \tag{2-7}$$

From the work of Robert Jarrow and Andrew Rudd, we can summarize their work as putting the growth rate of the risky asset at $(r - \frac{\sigma^2}{6})/2$ instead of r, and volatility at $\sigma/\sqrt{3}$ instead of σ.

Under the above set, the Asian option can be seen as a European option with volatility $\sigma' = \sqrt{3}$ and expected return $q = d_* = \frac{1}{2}(r - \sigma^2/6\,)T$, then, rearranging the equation (2-6) and the equation (2-7), we can get the model for pricing the Asian call option:

$$C_a = e^q e^{-rT} S_0 N(d_1) - e^{-rT} K N(d_2) \tag{2-8}$$

where the d_1 and d_2 are

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \frac{1}{2}(r + \sigma^2/6)T}{\sigma\sqrt{\frac{1}{3}T}} \tag{2-9}$$

$$d_2 = d_1 - \sigma\sqrt{\frac{1}{3}T} \tag{2-10}$$

Equation (2-9) can be obtained for pricing the Asian option with the geometric average algorithm. The table below shows how Algorithm 1: Geometric Average Algorithm works for pricing an Asian call option by modified BSM model.

Algorithm 1: Geometric average algorithm to pricing an Asian call option by modified BSM model
Input: Initial stock price: S_0 Strike price K Maturity time$T$ Risk-free interest rate r Stock price volatility σ
Output: the Asian call option price
0: input all the required related Asian option date by correct way 1: to compute q by $q = \frac{1}{2}(r - \sigma^2/6\,)T$ 2: to compute d_1 by the equation (2-9) which is $d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \frac{1}{2}(r + \sigma^2/6)T}{\sigma\sqrt{\frac{1}{3}T}}$ 3: to compute d_2 by the equation (2-10) which is

$$d_2 = d_1 - \sigma\sqrt{\frac{1}{3}T}$$

4: to compute the related Asian option price by the equation (2-8), which is

$$C_a = e^q e^{-rT} S_0 N(d_1) - e^{-rT} K N(d_2)$$

5: output the related Asian option price as result

The following code is the algorithm of the geometric average Asian call option for the Matlab, since in the Matlab has the inner function of the pricing of an option, so the function is used.

```
function V=BSprice(S0,K,T,r,sigma)
q=0.5*(r-(sigma^2)/6)*T;
a=log(S0/K);
b=sigma/sqrt(3);
d1=(a+0.5*(r+(sigma^2)/6)*T)/b*sqrt(T);
d2=d1-b*T;
[call]=exp(q)*exp(-r*T)*S0*cdf('normal',d1,0,1)-exp(-
r*T)*K*cdf('normal',d2,0,1);
price=call
```

The above is just a study of Asian options pricing with geometric average algorithm. In fact, geometric average has no practical significance at all. At present, the arithmetic average method is used in the industry. For example, if a farmer wants to buy feed, he will buy it once a month. He has to think more about the average cost of purchase per month. The geometric average is meaningless.

Therefore, we focus on the pricing of the arithmetic average algorithm of the Asian options and do not compare the pricing of the geometric average method in the following realistic data comparison.

2.3 Arithmetic average Asian option pricing based on BSM model framework

In general, if the averaging algorithm is arithmetic averaging, there is no analytical pricing formula for Asian options. This is because there is no standard lognormal distribution for arithmetic average. But the distribution of the arithmetic average is close to the lognormal distribution, so we can assume that the underlying asset obeys the lognormal distribution.

This is more reasonable, because the stock price obeys geometric Brownian motion, so the geometric average of the stock price obeys the log-normal distribution, and the arithmetic average of its price also approximately obeys the log-normal distribution.

So that we assume, the stock arithmetic average price from time 0 to time T is AT, and AT follows log-normal distribution, where AT in the arithmetic averaging is

$$AT = \frac{1}{T}\int_0^T S_t dt \tag{2-11}$$

where t is positive integer.

Then, a popular method to pricing the stock arithmetic average option price is to fit the lognormal distribution to the first two moments of AT, and then use the future pricing model which found by Black (Turnbull & Wakeman, 1991).

Suppose the parameters M_1 and M_2 are the first two moments of AT, the arithmetic average Asian options can be regarded as futures, with,

$$F_0 = M_1 \tag{2-12}$$

$$\sigma^2 = \frac{1}{T}\ln\left(\frac{M_2}{M_1^2}\right) \tag{2-13}$$

Where F_0 is the future price and σ is the volatility of the future price, and M_1 and M_2 are respectively (Hull, 2015):

$$M_1 = \frac{e^{(r-q)T} - 1}{(r-q)T} S_0 \tag{2-14}$$

$$M_2 = \frac{2e^{(2r-2q+\sigma^2)T} S_0^2}{(r-q+\sigma^2)(2r-2q+\sigma^2)T^2} + \frac{2S_0^2}{(r-q)T^2}\left(\frac{1}{2r-2q+\sigma^2} - \frac{e^{(r-q)T}}{r-q+\sigma^2}\right) \tag{2-15}$$

where r is the risk-less interest rate and q is the dividends payout ratio of the underlying stock, in this thesis we do not consider the stock that paying dividends, hence the q is 0.

Based on the above, then, according to the future pricing model of Black, we can get the Asian option price below.

$$c = e^{-rT}[F_0N(d_1) - KN(d_2)] \tag{2-16}$$

where the d_1 and d_2 are

$$d_1, d_2 = \frac{\ln(F_0/K) \pm \sigma^2 T/2}{\sigma\sqrt{T}} \tag{2-17}$$

The below table shows Algorithm 2: Arithmetic average algorithm to pricing an Asian call option by modified BSM model.

Algorithm 2: Arithmetic average algorithm to pricing an Asian call option by modified BSM model
Input: Initial stock price: S_0 Strike price K Maturity time$T$ Risk-free interest rate r Stock price volatility σ Payout ratio q
Output: the Asian call option price
0: input all the required date by correct way 1: compute M_1 by $M_1 = \frac{e^{(r-q)T}-1}{(r-q)T}S_0$ 2: compute M_2 by $M_2 = \frac{2e^{(2r-2q+\sigma^2)T}S_0^2}{(r-q+\sigma^2)(2r-2q+\sigma^2)T^2} + \frac{2S_0^2}{(r-q)T^2}\left(\frac{1}{2r-2q+\sigma^2} - \frac{e^{(r-q)T}}{r-q+\sigma^2}\right)$ 3: give the value of M_1 to F_0 4: compute σ^2 by $\sigma^2 = \frac{1}{T}\ln\left(\frac{M_2}{M_1^2}\right)$ 5: compute d_1 and d_2 by $d_1, d_2 = \frac{\ln(F_0/K)\pm\sigma^2 T/2}{\sigma\sqrt{T}}$ 6: compute the price by $c = e^{-rT}[F_0N(d_1) - KN(d_2)]$

The following code is the algorithm of the arithmetic average Asian call option for the Matlab:

```
function c=AsianBS(s0,k,t,r,sigma,q)
m1=s0*(exp(r*t-q*t)-1)/(r*t-q*t);
m21=2*(s0)^2*exp((2*r-2*q+sigma^2)*t)/((r-q+sigma^2)*(2*r-
2*q+sigma^2)*t^2);
m22=2*(s0)^2*(1/(2*r-2*q+sigma^2)-exp((r-q)*t)/(r-q+sigma^2))/((r-q)*t^2);
m2=m21+m22;
f0=m1;
sigmaa=(log(m2/m1^2))/t;
```

```
sigmaf=sqrt(sigmaa);
d1=(log(f0/k)+sigmaa*t/2)/sigmaf*sqrt(t);
d2=(log(f0/k)-sigmaa*t/2)/sigmaf*sqrt(t);
a=exp(-r*t)*(f0*cdf('normal',d1,0,1)-k*cdf('normal',d2,0,1))
c=max(a,0)
```

The BSM model has provided a classic way to pricing the options, but with the development of computer science, people found another way to pricing the derivatives by simulating the path of underlying asset value changes during the constant period.

Section 3 The Monte Carlo method and the application in option pricing

3.1 Introduction of the Monte Carlo method

The above algorithms gave some ways to price the option, those methods can provide the analytical solution or approaching boundary solution. In addition to these algorithms, to simulate the stock price process and simulate the Asian option payoff is widely used in the modern financial industry.

The Monte Carlo method is based on two theorems in probability theory: the law of large numbers and the central limit theorem. The law of large numbers shows that when the number of trials or the number of test samples is larger, the arithmetic mean has a higher probability of being close to its expected value. The central limit theorem shows that the distribution of the mean values of many mutually independent variables is close to the normal distribution after standardization.

So that, we can say the principle of the Monte Carlo method is to use repeated random sampling to obtain results. The Monte Carlo method is to repeat lots of times of simulations to find the required mean data in the confidence interval. Hence, by this method, it is easy to simulate the path of the required data.

Based on the principle, the Monte Carlo method can be applied by the below main steps for the pricing European style options：

Firstly, we need to assume that the given stochastic differential equation under the $Q - measure$ which we have mentioned before. By this assumption, the stochastic differential equation describes the price of underlying assets by a risk-neutral behavior model. Then, the derivatives can get values from the model.

And, the first step needs to simulate N sample paths of the underlying asset prices in the relevant time period $[0, T]$ under the $Q - measure$. This step usually needs to be applied by numerical approaches to approximate the solution of stochastic differential equations.

Then, by the risk-neutral pricing method, we can evaluate the discounted payoff of the asset prices in each sample path.

Lastly, the theoretical derivative value can be computed by Monte Carlo estimate way in applying the average result among the N sample pat
hs.

The above steps show a basic way to apply Monte Carlo method into derivative, especially for the European style option pricing. However, for the Asian option, the most difference is the Asian option rely on the stock price among the whole time period, therefore, the first step is most important and basic way we need to understand and simulate.

Hence, based on the Monte Carlo method principle and the first step showed above, we can use the Monte Carlo method to easily get the path of the stock price by simulating the price at fixed time points, to achieve the purpose of pricing it.

3.2 Simulation of stock price by the Monte Carlo method

From Section 1, we can know that the stock price follows a geometric Brownian motion, then the stock process in the risk-neutral world can be got based on the equation (1-11), which is

$$S_t = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma Z_t} \tag{3-1}$$

where the time t belongs to the time interval $[0,T]$.

Then we can use Monte Carlo method to simulate the stock price process. The below table shows how Algorithm 3: stock price process simulation

Algorithm 3: stock price process simulation by Monte Carlo method
Input: Initial stock price: S_0 Risk-free interest rate r Stock price volatility σ Number of sample-path simulations N Number of simulation year T
Output: the diagram of stock process and the mean stock price
0: input all the required date by correct way 1: Set each time step dt is 1/252 which is one trading day 2: Compute the total trading days and start a cycle, when i bigger than the value of T, stop the cycle 3: During the cycle doing the compute of $path = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma z_t$, and $z_t \sim N(0,t)$, and repeat this for "N" times. 4: Compute $path = S_0 * e^{path}$ 5: Get the mean of path 6: Let $i = i + 1\ step$, till the end of the 1. 7: Drawing the diagram of stock price and time point 8: Show the diagram of the stock process path.

The following code is the stock price process algorithm by Monte Carlo method in the Matlab.

```
function [P,CI]=PriceProcess(s0,r,sigma,N,T)
K = 0:1/252:T;
n=252*T;
i = T;
X=nan(N,1);
while i<=length(K)
    dt = K(i);
```

```
        for k=1:N
                path=(r-sigma^2/2)*dt+sigma*sqrt(dt)*randn(1,n);
                path=s0*exp(path);
        end
        y(i)=mean(path);
        i = i+1;
end
figure(1)
plot(K,y)
```

Below is a diagram for the stock with a $\$50$ initial price and volatility is 0.01, in the financial market with a 0.5% risk-free interest rate, and simulate 10000 times, each time-discrete 252 times.

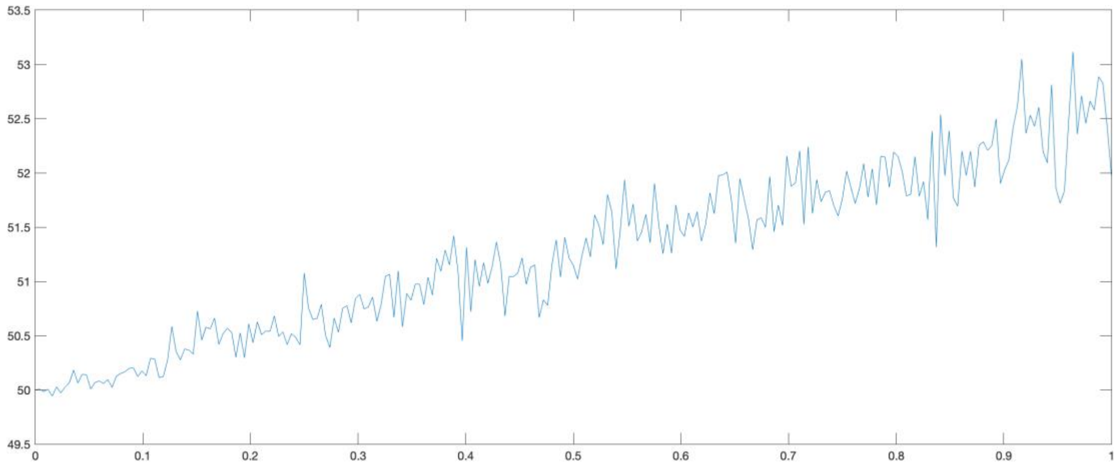

Figure 2: Stock process path

The above figure shows that the stock price after one year reaches approximately \$52, and the broken line shows the stock price path from initial time 0 to the end time 1.

This shows that the Monte Carlo method is efficient to simulate the path of the underlying asset, then we could try to use the Monte Carlo method to price the related Asian option.

3.3 Pricing Asian options by the Monte Carlo method

From the above knowledge, the stock price process is a stochastic process that is very clear. And by the Monte Carlo method, the stock pricing process can be simulated as above picture. Next is to focus on the payoff. The payoff of an Asian call option is $(AT-K)^+$, it is easy to get AT when trading days n during the contract period $[0,T]$ are relatively large by the following formula:

$$AT=\frac{1}{n+1}\sum_{t=0}^{n}S_t \tag{3-2}$$

where the S_t shows as the above equation (3-1).

Then the discount payoff X_n of the Asian option at the initial time $t=0$ is

$$X_n=e^{-rT}(AT-K)^+ \tag{3-3}$$

where the n belongs to the interval $[0,N]$, the N is the simulation times that need to be set before simulation, normally 10000 can be chosen.

According to the expected payoff of N independent realizations, we can get the Monte Carlo estimator of the price of the Asian option $\mathbb{C}$ is

$$\mathbb{C}=\frac{1}{N}\sum_{n=1}^{N}X_n \tag{3-4}$$

Then the Monte Carlo Method can be used to pricing the Asian call option. The below table shows how Algorithm 4: Monte Carlo Method to pricing an Asian call option.

Algorithm 4: Monte Carlo Method to pricing an Asian call option.
Input: Initial stock price: S_0 Strike price: K Risk-free interest rate r Maturity time T Stock price volatility σ Number of discrete times per time (trading days during the period from time0 to time T) n Number of sample-path simulations N
Output: the price of the relative Asian call option price
0: input all the required date by correct way 1: Set each time step 1/252 2: Compute dt by $dt=T/n$ 3: Start a cycle from 1 to the N to repeat N paths simulation 4: Compute $path=\left(r-\frac{\sigma^2}{2}\right)*dt+\sigma\sqrt{t}*random\ number$, the random number satisfy the normal distribution of $(1,n)$

5: Get single path value by $path = S_0 * e^{path}$
6: Get the mean value $[P]$,of all the paths, this value is the relative Asian call option price

The following code is the Monte Carlo method of Asian call option pricing algorithm in the Matlab.

```
function [P,CI]=AsianMC(s0,k,r,T,sigma,n,N)
dt=T/n;
X=nan(N,1);
for i=1:N
    path=(r-sigma^2/2)*dt+sigma*sqrt(dt)*randn(1,n);
    path=cumprod([s0,exp(path)]);
    path1(i)=exp(-r*T)*max(mean(path)-k,0);
end
[P]=mean(path1);
```

The above sections have introduced and constructed the different ways to pricing an Asian option, in the following section we will focus on the realization of the above algorithms with the realistic stock data.

Section 4 Realization of the different Asian option pricing methods

4.1 Introduction of the realistic data of the stock and the Asian option that will be priced

Most of the Asian options trade over-the-counter, which means the data of real Asian options may not reliable and not objective. This is because over-the-counter transactions are carried out directly between the two parties without exchange supervision, hence, the price which the two parties traded with may be influenced by some private considerations, hence it is not objective. So that for the examination of the algorithms, we need to create an Asian option by the principle of Asian options from the realistic data of the stock market.

In this thesis, we will take the stock called Apple, which is a famous American multinational technology company in the world, and it is trading on the Nasdaq Stock Market. The related data of Apple's stock in the period from July 1 2020 to September 30 2020 is in Appendix 1.

For the Asian option, we define the Asian option with 3 months from October 1 2020 to December 31 2020 and the strike price is 117$.

4.2 Related stock volatility and riskless interest rate in the financial market

To comparison different algorithms, we need to find out the volatility during the period before the pricing period we mentioned.

To estimate the volatility of a stock, we take every day as the observed intervals of time, and there 64 trading days in the half year which we mentioned above. Then, we need to first find out the return rete u_i in every observed interval by the below formula.

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)\ for\ i = 1,2 \cdots n \tag{4-1}$$

Then, the standard deviation of the u_i is

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u-\bar{u})^2} \tag{4-2}$$

From the equation (1-20), we can get

$$ln\frac{S_T}{S_0} \sim \varphi[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T] \tag{4-3}$$

This means the standard deviation of the u_i is $\sigma\sqrt{\gamma}$, hence s can be an estimate of $\sigma\sqrt{\gamma}$. Hence, the estimate volatility $\tilde{\sigma}$ is

$$\tilde{\sigma} = \frac{s}{\sqrt{\gamma}} \tag{4-4}$$

where the γ is the length of time intervals in years, in our computation is $\frac{1}{252}$. We take one day as the observed interval, and by the equations above, the volatility of the stock Apple is 0.02803.

In the financial market, the riskless interest rate is determined by those assets which providing a risk-less return. In the financial market, the interest rate of US Treasury notes is generally recognized as the risk-free interest rate in the market. This is because the credibility of the US government is recognized by the market and there will be no default behavior. Hence, in this thesis, we will take the US 10-year treasury interest rate as the riskless interest rate.

According to the data from US Department of the 10-year treasury from May 15 2020 to May 15 2030 is r_s equal to 0.625% with simple compounding, the document is in the Appendix 2.

For the later computation, we need the continuous compounding interest rate. The treasury notes pay interest on a semi-annual basis. The continuous compounding interest rate can be got by below equation

$$r = mln(1 + \frac{r_s}{m}) \tag{4-5}$$

Where the m is the interest payment times in a year, hence, the continuous compounding interest rate is 0.624%.

4.3 The Asian option price by different Algorithms

Now we have all the data we need. Hence, we can price the Asian option which is the derivative of the stock price of Apple.

The price of the stock in the time period is starting at 116.79, and with 0 dividends paying. As mentioned above, the strike of the Asian option which we will price is 117, and the maturity time is 3 months. The volatility of the stock Apple is 0.02803, and the riskless interest is 0.624% with continuous compounding.

For the Algorithm 2: Arithmetic average algorithm to pricing an Asian call option by modified BSM model, we have the needed data in the below table.

s0	116.79
k	117
t	0.25
r	0.00624
sigma	0.02803
q	0

Then, we got the price from Algorithm 2 is 0.0363$.

For the Algorithm 4: Monte Carlo Method to pricing an Asian call option, we have we have the needed data in the below table.

s0	116.79
k	117
r	0.00624
T	0.25
sigma	0.02803
n	64
N	10000

Then, we got the price from Algorithm 4 is 0.3139$.

Above tables are the Asian option prices which be got by two algorithms.
From the result, we can find the price between the modified BSM model algorithm to other algorithms has a big difference, this proves the analytical solution or approaching boundary solution is different from the simulated solutions caused by the different realization ways.

Summary

In this thesis, we have constructed and realized the modified BSM model for pricing the Asian options algorithm, and the Monte Carlo method for pricing the Asian options algorithm.

And in the last section, we did the pricing for the Asian option that we set upon the real data from the APPLE stock in the Nasdaq Stock Market, and obtained the price by each method and find the prices are different caused by the different realization ways for the different algorithms.

Although options are currently one of the most popular and important financial instruments, there is still a lack of related Asian options products in some countries in the world. However, with the development of financial markets and the gradual increase of transactions in financial products by countries around the world, Asian options will usher in better development.

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Appendix 1

Apple’s stock from July 1 2020 to September 30 2020

Date	Open	High	Low	Close
2020/7/1	91.279999	91.839996	90.977501	91.027496
2020/7/2	91.962502	92.6175	90.910004	91.027496
2020/7/6	92.5	93.945	92.467499	93.462502
2020/7/7	93.852501	94.654999	93.057503	93.172501
2020/7/8	94.18	95.375	94.089996	95.342499
2020/7/9	96.262497	96.317497	94.672501	95.752502
2020/7/10	95.334999	95.980003	94.705002	95.919998
2020/7/13	97.264999	99.955002	95.2575	95.477501
2020/7/14	94.839996	97.254997	93.877502	97.057503
2020/7/15	98.989998	99.247498	96.489998	97.724998
2020/7/16	96.5625	97.404999	95.904999	96.522499
2020/7/17	96.987503	97.147499	95.839996	96.327499
2020/7/20	96.417503	98.5	96.0625	98.357498
2020/7/21	99.172501	99.25	96.7425	97
2020/7/22	96.692497	97.974998	96.602501	97.272499
2020/7/23	96.997498	97.077499	92.010002	92.845001
2020/7/24	90.987503	92.970001	89.144997	92.614998
2020/7/27	93.709999	94.904999	93.480003	94.809998
2020/7/28	94.3675	94.550003	93.247498	93.252502
2020/7/29	93.75	95.230003	93.712502	95.040001
2020/7/30	94.1875	96.297501	93.767502	96.190002
2020/7/31	102.885002	106.415001	100.824997	106.260002
2020/8/3	108.199997	111.637497	107.892502	108.9375
2020/8/4	109.1325	110.790001	108.387497	109.665001
2020/8/5	109.377502	110.392502	108.897499	110.0625
2020/8/6	110.404999	114.412498	109.797501	113.902496
2020/8/7	113.205002	113.675003	110.292503	111.112503
2020/8/10	112.599998	113.775002	110	112.727501
2020/8/11	111.970001	112.482498	109.107498	109.375
2020/8/12	110.497498	113.275002	110.297501	113.010002
2020/8/13	114.43	116.042503	113.927498	115.010002
2020/8/14	114.830002	115	113.044998	114.907501
2020/8/17	116.0625	116.087502	113.962502	114.607498
2020/8/18	114.352501	116	114.0075	115.5625
2020/8/19	115.982498	117.162498	115.610001	115.707497

2020/8/20	115.75	118.392502	115.732498	118.275002
2020/8/21	119.262497	124.8675	119.25	124.370003
2020/8/24	128.697495	128.785004	123.9375	125.857498
2020/8/25	124.697502	125.18	123.052498	124.824997
2020/8/26	126.18	126.9925	125.082497	126.522499
2020/8/27	127.142502	127.485001	123.832497	125.010002
2020/8/28	126.012497	126.442497	124.577499	124.807503
2020/8/31	127.580002	131	126	129.039993
2020/9/1	132.759995	134.800003	130.529999	134.179993
2020/9/2	137.589996	137.979996	127	131.399994
2020/9/3	126.910004	128.839996	120.5	120.879997
2020/9/4	120.07	123.699997	110.889999	120.959999
2020/9/8	113.949997	118.989998	112.68	112.82
2020/9/9	117.260002	119.139999	115.260002	117.32
2020/9/10	120.360001	120.5	112.5	113.489998
2020/9/11	114.57	115.230003	110	112
2020/9/14	114.720001	115.93	112.800003	115.360001
2020/9/15	118.330002	118.830002	113.610001	115.540001
2020/9/16	115.230003	116	112.040001	112.129997
2020/9/17	109.720001	112.199997	108.709999	110.339996
2020/9/18	110.400002	110.879997	106.089996	106.839996
2020/9/21	104.540001	110.190002	103.099998	110.080002
2020/9/22	112.68	112.860001	109.160004	111.809998
2020/9/23	111.620003	112.110001	106.769997	107.120003
2020/9/24	105.169998	110.25	105	108.220001
2020/9/25	108.43	112.440002	107.669998	112.279999
2020/9/28	115.010002	115.32	112.779999	114.959999
2020/9/29	114.550003	115.309998	113.57	114.089996
2020/9/30	113.790001	117.260002	113.620003	115.809998

Appendix 2

TREASURY NEWS

Department of the Treasury • Bureau of the Fiscal Service

For Immediate Release
May 12, 2020

CONTACT: Treasury Auctions
202-504-3550

TREASURY AUCTION RESULTS

Term and Type of Security	10-Year Note
CUSIP Number	912828ZQ6
Series	C-2030
Interest Rate	0-5/8%
High Yield [1]	0.700%
Allotted at High	28.91%
Price	99.276869
Accrued Interest per $1,000	None
Median Yield [2]	0.650%
Low Yield [3]	0.590%
Issue Date	May 15, 2020
Maturity Date	May 15, 2030
Original Issue Date	May 15, 2020
Dated Date	May 15, 2020

	Tendered	Accepted
Competitive	$85,952,460,000	$31,988,685,800
Noncompetitive	$11,349,600	$11,349,600
FIMA (Noncompetitive)	$0	$0
Subtotal [4]	**$85,963,809,600**	**$32,000,035,400[5]**
SOMA	$16,923,161,000	$16,923,161,000
Total	**$102,886,970,600**	**$48,923,196,400**

	Tendered	Accepted
Primary Dealer [6]	$47,845,500,000	$6,563,637,500
Direct Bidder [7]	$8,604,000,000	$4,266,000,000
Indirect Bidder [8]	$29,502,960,000	$21,159,048,300
Total Competitive	**$85,952,460,000**	**$31,988,685,800**

[1]All tenders at lower yields were accepted in full.
[2]50% of the amount of accepted competitive tenders was tendered at or below that yield.
[3]5% of the amount of accepted competitive tenders was tendered at or below that yield.
[4]Bid-to-Cover Ratio: $85,963,809,600/$32,000,035,400 = 2.69
[5]Awards to TreasuryDirect = $8,677,600.
[6]Primary dealers as submitters bidding for their own house accounts.
[7]Non-Primary dealer submitters bidding for their own house accounts.
[8]Customers placing competitive bids through a direct submitter, including Foreign and International Monetary Authorities placing bids through the Federal Reserve Bank of New York.