POLITECNICO DI TORINO

Corso di Laurea in Ingengeria Matematica

Tesi di Laurea Magistrale

Analysis of learning dynamics in heterogeneous routing games

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Anno accademico 2020 – 2021

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Abstract

The main focus of this thesis is the analysis of the asymptotic behavior of deterministic logit dynamics in heterogeneous routing games on two-terminal directed multigraphs. We provide three results. The first one states that this dynamics admits a globally asymptotically stable fixed point when noise is sufficiently high. The second result, instead, shows that the fixed points of the dynamics always approach a subset of the Wardrop equilibria of the game, called the limit set, as the noise vanishes and that pure strategy Wardrop equilibria, i. e., equilibria where every population uses a single path and the other paths are strictly suboptimal, always belong to this limit set. Finally, the last result is that every pure strategy Wardrop equilibrium of the game is locally asymptotically stable under the logit dynamics in the limit of vanishing noise. In other words, the fixed points of the dynamics that converge to that equilibrium are locally asymptotically stable when noise is sufficiently low.

Acknowledgements

I thank prof. Giacomo Como and Leonardo Cianfanelli for their collaboration and their support in the development of the thesis and in its writing.

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Chapter 1

Introduction

In recent years, the optimal use of transportation networks has attracted significant interest as one of the main aspects of smart mobility. Routing games provide a powerful mathematical tool to model real-world scenarios whereby a set of users compete for shared resources. This class of games finds numerous applications within socio-technical systems, in particular in transportation networks. In this setting, the topology of the transportation network is modeled as a directed multigraph, whose links and nodes represent roads and junctions, respectively. Congestion effects are then captured by cost functions giving the travel time on each link as a function of the total flow along it. Users are modeled as fully rational decision makers choosing minimal cost paths from their origin to their destination. The users' perceived costs typically correspond to a combination of the total travel time along the route and possibly a monetary price associated to it (e. g., fuel consumption, tolls). Links are to be considered as resources and players using them create negative externalities because of congestion effects. We focus on two-terminal networks, in which users share the same origin and destination.

In non-atomic routing games, the users' population is modeled as a continuum, so that the change in the costs induced by a single individual's route choice is infinitesimal. In this case, variational inequalities and convex optimization lead to an elegant and insightful theory. Most of the interest is devoted to the analysis of the steady configurations reached by the system. A Wardrop equilibrium is a users' flow distribution in which every user cannot unilaterally decrease his cost by changing path [18], i. e., it is on the path with minimum cost. Assuming homogeneity of players, meaning that the cost functions over paths are the same for all of them, implies some significant results: the game always admits at least a Wardrop equilibrium and if cost functions are strictly increasing, then the equilibrium is unique. However, single population routing games are not always the most suitable models to use and, indeed, they can often prove to be limiting.

Heterogeneous routing games represent a much broader class of models that allows to describe more realistic scenarios, e. g., users might have different information on traffic [1], [3], [6], or users might perceive different route costs or have different priorities [11], [12]. This assumption divides users into multiple populations: the population they belong to determines the link costs. In this setting, existence of equilibria holds [16], while uniqueness does not, in general [14].

Routing games are usually equipped with learning dynamics. In this thesis, we choose to focus our attention on deterministic logit dynamics. This dynamics aims to describe the evolution of the system, given that users are characterized by a level of knowledge of the game, represented by a "noise level" parameter. In the single population case, the fixed points of the dynamics converge to the set of Wardrop equilibria of the game, as the noise level vanishes [15]. Because of its wide use and effectiveness, knowing if the logit dynamics might converge to a specific Wardrop equilibrium tells us whether this point represents a realistic situation or not. The same result holds in the heterogeneous case if we limit ourselves to consider specific types of multigraphs, namely parallel graphs and series of parallel graphs [10]. However, the literature provides some examples of more complex settings exhibiting bifurcation phenomena [10]. With the term "bifurcation" we identify all those phenomena in which changing the value of one or more parameters characterizing the dynamics influences the asymptotic behavior of the system.

To the best of our knowledge, the asympotic behavior of deterministic logit dynamics for heteogeneous routing games was studied only in [10] for games defined on series of simple graphs, so far. Given these motivations, this thesis aims to extend what has been done so far by analyzing the case of non-parallel graphs.

This thesis is organized as follows. In Chapter 2 we provide the background notions required to fully understand this thesis. Specifically, we give the definitions of the type of network we exploit and network flow and we introduce some basic notions on dynamical systems and their stability and some crucial results about this, as well, that we will use in the following chapters to study the logit dynamics. Then, in Chapter 3, after an overview on routing games, we provide the definition of routing game and we introduce the notion of Wardrop equilibrium. We distinct between single population and heterogenous routing games and we show the different implications brought by these two different assumptions, also by providing two significant examples. We also highlight how the network topology affects routing games. Finally, in Chapter 4 we introduce the deterministic logit dynamics. After looking at some significant numerical simulations, we provide our original theoretical results, which characterize the fixed points of the logit dynamics and their stability, both in the vanishing and large noise regimes.

Chapter 2

Preliminaries

In this chapter, we are going to provide the reader with the basic notions from graph theory we resort to in this thesis. Specificaly, we give the definitions of *two-terminal networks* and *network flow*. Two-terminal networks are the mathematical tool used to model transportation networks, while network flows aim to represent users travelling along it.

2.1 Two-terminal networks

Definition 2.1. We define a **directed multigraph** as a quadruple

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \theta, \kappa), \tag{2.1}$$

where:

- $\mathcal{V}$ is the finite set of **nodes**;
- $\mathcal{E}$ is the set of **links**;
- $\theta, \kappa : \mathcal{E} \to \mathcal{V}$ are the **tail** and **head functions**, respectively.

An important implication stems from Definition 2.1. There might exist two links sharing the same tail and head, hence two links $e_1, e_2 \in \mathcal{E}$ such that $\theta(e_1) = \theta(e_2)$ and $\kappa(e_1) = \kappa(e_2)$. These two links are then called *parallel*. Instead, if $\theta(e_1) = \kappa(e_2)$ and $\kappa(e_1) = \theta(e_2)$, then they are called *opposite*.

We now introduce the notion of reachability. Let us consider a directed multigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \theta, \kappa)$.

- A *walk* from i to j is a finite sequence of links $\gamma = (e_1, e_2, \ldots, e_l)$ such that $\kappa(e_i) = \theta(e_{i+1}), i = 1, \ldots, l-1$. Then, we say that the walk begins at $\theta(e_1)$, it ends in $\kappa(e_l)$ and its length is l.

- $j \in \mathcal{V}$ is *reachable* from $i \in \mathcal{V}$ if there exists a walk from i to j.
- A walk such that $\theta(e_h) \neq \theta(e_l)$, $\forall h \neq k$, $h, k \in \{1, \dots, l\}$ is called *path*. Basically, paths are walks with no repeating nodes.
- A path such that $\theta(e_0) = \kappa(e_l)$ is called *circuit*, while a path with this property is called *cycle*.

We now denote as $\Gamma_{(i,j)}$ the set of paths starting at i and ending at j, the *i-j paths*. Let us also denote as Δ the set of all closed paths in $\mathcal{G}$, which includes self-loops, length-2 circuits that are not repeating self-loops and cycles.

It is very useful to provide the following definitions.

Definition 2.2. Given a directed multigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \theta, \kappa)$, we define its **node-link incidence matrix** $B \in \mathbb{R}^{\mathcal{V} \times \mathcal{E}}$ in the following way:

$$B_{ie_j} = \begin{cases} +1, & \text{if} \quad \theta(e_j) = i \\ -1, & \text{if} \quad \kappa(e_j) = i \\ 0, & \text{otherwise} \end{cases} . \tag{2.2}$$

Then, we define its **link-path incidence matrix** $A^{(i,j)} \in \mathbb{R}^{\mathcal{E} \times \Gamma_{(i,j)}}$ in the following way:

$$A^{(i,j)}_{e\gamma} = \begin{cases} 1 & \text{if link } e \text{ is along path } \gamma \\ 0 & \text{if link } e \text{ is not along path } \gamma \end{cases} . \tag{2.3}$$

Finally, we define its **link-cycle incidence matrix** $C \in \mathbb{R}^{\mathcal{E} \times \Delta}$ in the following way:

$$C_{e\gamma} = \begin{cases} 1 & \text{if link } e \text{ is along cycle } \gamma \\ 0 & \text{if link } e \text{ is not along cycle } \gamma \end{cases} . \tag{2.4}$$

Basically, the columns of B are labelled by links. Each column has only two entries that differ from 0. The first one is associated to the tail of the edge which the column is labeled with and it is equal to $+1$, whereas the second entry is equal to -1 and it refers to the head of the edge. Notice also that each column of $A^{(i,j)}$ is a $\{0,1\}^{\mathcal{E}}$ vector with entries equal to one corresponding to the edges belonging to the path the column refers to. This same thing holds for the link-cycle incidence matrix C.

We now introduce the notion of network flow, which is crucial in order to obtain a full understanding of this thesis. Let us consider a directed multigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \theta, \kappa)$ and a vector of *exogenous flows* $\nu \in \mathbb{R}^{\mathcal{V}}$ satisfying the constraint

$$\mathbb{1}'\nu = 0. \tag{2.5}$$

Each component ν_i may be interpreted as an *exogenous inflow* at i if it is positive or as an *external outflow* at i if it is negative. In our work, we refer to nodes such that $\nu_i > 0$ as *origins* and the nodes such that $\nu_i < 0$ as *destinations*. At this point, it is possible to define the *throughput*, i. e., the total flow that runs across the network, as

$$\tau = \frac{1}{2} \sum_{i \in \mathcal{V}} |\nu_i|. \tag{2.6}$$

Definition 2.3. Given a directed multigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \theta, \kappa)$ and an exogenous flow $\nu \in \mathbb{R}^{\mathcal{V}}$ satisfying (2.5), a **network flow** is a vector $f \in \mathbb{R}_+^{\mathcal{E}}$ whose entries f_{e_j} (flow on link $e_j \in \mathcal{E}$) are such that

$$\nu_i + \sum_{e_j | \kappa(e_j) = i} f_{e_j} = \sum_{e_j | \theta(e_j) = i} f_{e_j}, \quad i \in \mathcal{V}. \tag{2.7}$$

(2.7) is basically a mass conservation law, since it prescribes that, for each node, the amount of flow entering it must be equal to the amount of flow leaving it. (2.7) can be rewritten in a more compact way by resorting to the node-link incidence matrix:

$$Bf = \nu. \tag{2.8}$$

The class of network flows on which we focus our attention in this thesis is that of *o-d* flows, which are network flows with a single origin-destination pair (o, d) and satisfy the following condition:

$$Bf = \tau \left(\delta^{(o)} - \delta^{(d)} \right), \tag{2.9}$$

Here, $\delta^{(u)} \in \mathbb{R}^{\mathcal{V}}$ indicates a one-hot vector such that the only non-zero component is the one associated to node $u \in \mathcal{V}$. Notice that in the case of *o-d* flows we can say that all the mass constituting the flow enters the network at o and leaves it at d. The notion of *o-d* flow is crucial, since it will be used to model flows of vehicules travelling across a network. We now state an important result from [2] showing that every assignment of flows to both *o-d* paths and cycles in the multigraph induces a unique network flow f on the links and that, conversely, for every network flow f on the links, there exists an assignment (possibly and tipicaly non unique) of flows to both *o-d* paths and cycles in the multigraph that induces f.

Theorem 2.1 (Flow Decomposition Theorem). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \theta, \kappa)$ be a directed multigraph with link-cycle incidence matrix C. Let $o \neq d$ be two nodes in $\mathcal{V}$ with d reachable from o and let $A^{(o,d)}$ be the associated link-path incidence matrix. Then, for every couple of vectors $z \in \mathbb{R}_+^{\Gamma^{(o,d)}}$ and $w \in \mathbb{R}_+^{\Delta}$,*

$$f = A^{(o,d)} z + Cw \tag{2.10}$$

is a o-d flow of throughput $\mathbb{1}' z = \tau$. Conversely, for every o-d flow with throughput τ, there exists vectors $z \in \mathbb{R}_+^{\Gamma^{(o,d)}}$ and $w \in \mathbb{R}_+^{\Delta}$ such that (2.10) holds true.

We conclude this section by introducing a narrower class of directed multigraphs, which is the one that we focused on in this thesis.

Definition 2.4. A **two-terminal network (TTN)** is a pair $\mathcal{G}_{(o,d)} = (\mathcal{G}, (o, d))$ of a directed multigraph $\mathcal{G}$ and two distinct nodes $o, d \in \mathcal{V}$ such that d is reachable from o in $\mathcal{G}$.

2.2 Stability of dynamical systems

In this section, we are going to talk about convergence phenomena of differential systems of equations of the form

$$\dot{x} = g(x), \tag{2.11}$$

where $g(x)$ represents a vector field from $\mathbb{R}^m$ to $\mathbb{R}^m$ of class $C^r, r \geq 1$. This chapter heavily relies on [4], [8].

A differentiable function $\gamma(t) : I \to \mathbb{R}^m$ is a solution of such that (2.11) if

$$\dot{\gamma}(t) = g(\gamma(t)), \quad \forall t \in I \tag{2.12}$$

The result we present in the following allows to relate (2.11) with the notion of *dynamical system* that we will introduce in the next section.

Theorem 2.2. *Let us consider system* (2.11) *and a point* $\bar{x} \in \mathbb{R}^m$*. Then, there exists a neighborhood* U *of* $\bar{x}$*,* $\epsilon > 0$ *and a map*

$$(t, x) \mapsto \varphi(t, x) : (-\epsilon, \epsilon) \times U \to \mathbb{R}^m \tag{2.13}$$

such that:

- $t \mapsto \varphi(t, x)$ *is the only solution of* (2.11) *in* $(-\epsilon, \epsilon)$*,* $\forall x \in U$*;*
- $x \mapsto \varphi(t, x)$ *is an invertible function of class* C^r *and whose inverse function is still of class* C^r *such that* $\varphi(0, x) = x$*,* $\forall t \in (-\epsilon, \epsilon)$*;*
- *it holds that*

$$\varphi(t, \varphi(s, x)) = \varphi(t + s, x), \quad \forall x \in U, \quad \forall\, t, s, t + s \in (-\epsilon, \epsilon). \tag{2.14}$$

We now provide the definition of dynamical system.

Definition 2.5. A **dynamical system** over $\mathbb{R}^m$ is a function

$$\varphi(t, x) : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m \tag{2.15}$$

satisfying the following properties:

1. φ is continuous;
2. $\varphi(0, x) = x,\ \forall x \in \mathbb{R}^m$;
3. $\varphi(t, \varphi(s, x)) = \varphi(t + s, x),\ \forall x \in \mathbb{R}^m,\ \forall t, s, t + s \in \mathbb{R}$.

The application $t \mapsto \varphi(t, x)$ is called **trajectory** of the dynamical system.

By looking at Theorem 2.2, it is easy to understand how system (2.11) defines a dynamical system as we just described it. Actually, it can be shown that g must be a *complete vector field* (all of its are gobally defined) in order for (2.11) to fully correspond to a dynamical system. This is the case, for example, when g is a Lipschitz function.

The analysis of the stability of a dynamical system consists in the research of its equilibrium points and in determining the nature and properties of these steady configurations.

Definition 2.6. $x \in \mathbb{R}^m$ is said to be a **fixed point** for the dynamical system (2.15) if $\varphi(t, x) = x,\ \forall t \in \mathbb{R}$.

Before proceeding, please notice that from now on we will use the notation $\mathcal{B}(x, r)$ to indicate the *ball in* $\mathbb{R}^m$ *of radius* r *and centered at* x. Please also notice that every notion of stability that we will provide in the following has to be intended *in the sense of Lyapunov*. Moreover, let M be a compact subset of $\mathbb{R}^m$.

Definition 2.7. M is said to be **stable** if

$$\forall \epsilon > 0, \exists \delta > 0 \mid \varphi(t, \mathcal{B}(M, \delta)) \subset \mathcal{B}(M, \epsilon), \quad \forall t \geq 0. \tag{2.16}$$

Basically, M is stable when trajectories starting inside it or next to it (the level of closeness is determined by the magnitude of δ) will stay next to it as the dynamical system evolves in time.

Definition 2.8. M is said to be **locally attractive** if

$$\exists \epsilon > 0 \mid \lim_{t \to +\infty} d(\varphi(t, x), M) = 0, \quad \forall x \in \mathcal{B}(M, \epsilon). \tag{2.17}$$

$\mathcal{B}(M, \epsilon)$ is said to be the **basin of attraction** of M. M is said to be **globally attractive** when (2.17) holds $\forall x \in \mathbb{R}^m$.

Definition 2.9. M is said to be **locally asymptotically stable** if it is stable and it is locally attractive. Analogously, M is said to be **globally asymtotically stable** if it stable and it is globally attractive.

Notice that points in $\mathbb{R}^m$ are compact sets, so the definitions provided above are valid also for them.

The following result turns out to be crucial in order to study non-linear differential systems, since it allows us to reduce the study of a non-linear system to the study of a linear one, simplifying things a lot.

Theorem 2.3. *Let us consider a system of the form*

$$\dot{x} = Ax + \tilde{h}(x), \quad x \in \mathbb{R}^m, \tag{2.18}$$

where $\tilde{h}$ is a function such that

$$\lim_{\|x\|\to 0} \frac{\|\tilde{h}(x)\|}{\|x\|} = 0.$$

Then, if all eigenvalues in the spectrum of A have negative real part, the origin is a locally asymptotically stable fixed point of the system. On the contrary, if there is an eigenvalue with positive real part, then the origin is an unstable fixed point for the system.

Theorem 2.3 can actually be applied on (2.11). It suffices to take $A = Dg(0)$, the jacobian matrix of $g(x)$ evaluated at 0. By resorting to Taylor's first order approximation with Peano's remainder, one can easily see that (2.11) satisfies to its conditions. The linear system

$$\dot{x} = Dg(0)x \tag{2.19}$$

is called *linear part* of (2.11) in $x = 0$. Finally, notice that one can apply Theorem 2.3 in every equilibrium point x_0 of a system just by performing a change of coordinates like $y = x - x_0$. In this case, the linear part of the system is determined by $Dg(x_0)$.

2.3 Contractivity of dynamical systems

We are now introducing the very useful notion of *contractivity*. Contractivity allows to prove the existence of globally asymptotically stable fixed point when a specific set of conditions is met. In the following, we are going to provide some new definitions and results based on contractivity that we exploit in our work.

We begin by defining the concept of matrix measure. Recall that, given a vector norm in an eucllidean space $\|\cdot\|_V$, the associated *induced matrix norm* is defined as

$$\|A\|_M = \sup_{\|x\|_V=1} \|Ax\|_V = \sup_{|x|\neq 0} \frac{\|Ax\|_V}{\|x\|_V}. \tag{2.20}$$

Notice that $\|\cdot\|_M$ corresponds to the norm of the linear map $F_A : x \mapsto Ax$ associated to A.

Definition 2.10. Given a matrix $A \in \mathbb{R}^{n\times n}$, we define its **matrix measure** $\mu(A)$ as the directional derivative of the matrix norm $\|\cdot\|_M$ in the direction of A an evaluated at I_n, that is:

$$\mu(A) := \lim_{h\to 0^+} \frac{\|I_n + hA\|_M - 1}{h}. \tag{2.21}$$

Matrix measure was introduced in [17]. The explicit form of this quantity depends on the euclidean norm used. For example, if $\|\cdot\|_M$ corresponds to the ℓ_1 norm, then:

$$\mu_1(A) = \max_{j\in\{1,\dots,n\}} \left(a_{jj} + \sum_{i=1, i\neq j}^{n} |a_{ij}| \right). \tag{2.22}$$

Now that we defined what a matrix measure is, we are ready to define a contractive system.

Definition 2.11. Let us consider the vector field $g : \mathbb{R}_{\geq 0} \times \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n$ and suppose that it is continuously differentiable (so that solutions exist and are locally unique) and denote its Jacobian by $Dg(t, x)$. Then, g is said to be

1. **infinitesimally contracting** on a set $C \subset \mathbb{R}^n$ if there exists a norm such that the associated matrix measure satisfies to the following condition, for some $c > 0$ (the contraction rate):

$$\mu(Dg(t, x)) \leq -c, \forall x \in C, \forall t \in \mathbb{R}_{\geq 0}; \tag{2.23}$$

2. **infinitesimally weakly contracting** (or **non-expansive**) if (2.23) holds when $c = 0$.

A very useful result from [8] exploiting the concept of contractive system is now reported.

Theorem 2.4. *Let us consider a continuously differentiable vector field g and a norm $\|\cdot\|_V$ with associated matrix measure $\mu(\cdot)$. Assume that:*

- *there exists a convex, closed and g-invariant set C;*
- *g is infinitesimally contracting with contraction rate $c > 0$ on the set C with respect to $\|\cdot\|_V$.*

Then:

- *there exists a unique equilibrium point x^* in C;*
- *x^* is exponentially stable with region of attraction containing C;*
- *$x \mapsto \|x - x^*\|$ e $x \mapsto \|g(x)\|$ are global Lyapunov functions.*

Chapter 3

Routing games

In this chapter, we provide an overview on *routing games*. Then, we introduce the notion of Wardrop equilibrium, a flow distribution in which none of the network users is interested in modifying its choice on the path, since he is already travelling on the one of minimum cost. We discuss existence and uniqueness of Wardrop equilibria, making a distinction between single population and heterogeneous games and highlight that in the latter case uniqueness does not hold, in general. Nevertheless, the network topology plays an important role and we show how, under certain assumptions, uniqueness is guaranteed also for heterogeneous games. We conclude the chapter with two significant examples.

3.1 Routing games

What routing games aim to do is modeling the congestion phenomena arising on a transportation network exploited by one or more population of users that want to move from one place to another. Players of a routing game correspond to the users of a transportation network. The set of adoptable strategies, instead, correspond to the set of paths and the associated cost functions represent the path travel times. In most applications, cost functions are non-decreasing with respect to the quantity of users resorting to the link they refer to: by choosing a path, every player increases the cost of all players choosing the same strategy. Notice that, in general, cost functions might also include other types of cost, such as monetary tolls or fuel consumption.

Routing games belong to the broader class of *population games* defined in [15]. Here, we do not formally define population games: rather, we just list the features we need to complete our introduction of routing games. Specifically, the aspects we are interested in are the following.

- *Populations*: each agents belongs to a specific population determining its cost functions, so the strategy it is going to adopt. An agent can belong to one and only one population.

- *Non-atomicity*: each population is modeled as a continuum, meaning that changes induced in the game by the actions of a single player are negligible. This assumption is equivalent to suppose that each population consistsin a very large number of agents.

- *Anonimity*: the congestion effects arising on the transportation network depend only on the empirical frequencies of users resorting to the available strategies and not on who uses them.

Before proceeding to define the model analyzed in this thesis, it should be noted that given the assumptions made up to now, it is also usual to suppose that the cost functions are continuous, coherently with the non-atomicity feature. We are now ready to describe the model we used in this thesis.

Let us consider a TTN $\mathcal{G}_{(o,d)}$. Let $\mathcal{P}$ be the set of populations, with $P = |\mathcal{P}|$ and let us assume that all populations share the same origin-destination pair (o, d). Let the p-th population have throughput τ^p, $p \in \mathcal{P}$ and let $\tau = \sum_{p=1}^{P} \tau^p$ be the aggregate throughput. Each of the P populations is characterized by a different set of continuous, non-decreasing cost functions over the set $\mathcal{E}$. Let $D_e^p(x) : \mathbb{R}_+ \to \mathbb{R}_+$ be the cost perceived by population p users when travelling along link $e \in \mathcal{E}$. This choice allows us to model a situation where populations are all affected by the congestion phenomenon, but possibly users from different populations react differently. Let D denote the vector containing all cost functions associated to a different pair $(e, p) \in \mathcal{E} \times \mathcal{P}$. It is crucial to keep in mind that we are assuming that individual users are negligible and they are playing the game anonymously. Also, observe that we are assuming *separability*, meaning that the cost on each edge is influenced by the amount of flow on that edge only. Finally, the set of possible actions corresponds to the set of o-d paths and does not depend on populations. Then, let $\mathcal{R} = \Gamma_{(o,d)}$ be the path set and let V, E and R be the cardinalities of $\mathcal{V}$, $\mathcal{E}$ and $\mathcal{R}$, respectively.

Now, for each population $p \in \mathcal{P}$ we can define an *admissible path distribution* as a vector $z^p \in \mathbb{R}_+^R$ satisfying the throughput constraint $\sum_{r \in \mathcal{R}} z_r^p = \tau^p$. We can then obtain the associated *edge flow distribution* vector

$$f^p = A^{(o,d)} z^p. \tag{3.1}$$

Unlike in (2.10) in Theorem 2.1, edge flow distribution vectors can be retrieved just by considering path flow distributions and neglecting flows on the cycles of the network. This is due to the fact that no rational player would pass through the same node more than once if its objective is to go from o to d. We can then retrieve the aggregate version of both these vectors in the following way:

$$f^{\text{agg}} = \sum_{p \in \mathcal{P}} f^p, \quad z^{\text{agg}} = \sum_{p \in \mathcal{P}} z^p. \tag{3.2}$$

Notice that the cost function on link $e \in \mathcal{E}$ will be determined by the same aggregate flow quantity f_e^{agg}, for every population $p \in \mathcal{P}$. This is due to the fact that every population suffers from the same congestion effects of the other populations on every link of the network. Finally, we also assume the cost to be *additive*, meaning that the cost of each path for population p is simply defined as the sum of the costs for that population of the edges that compose the path:

$$c_r^p(z) = \sum_{e \in \mathcal{E}} A_{er}^{(o,d)} D_e^p(f_e^{\text{agg}}). \tag{3.3}$$

We are now ready to provide the following definition.

Definition 3.1. A quadruple $(\mathcal{G}_{(o,d)}, \mathcal{P}, \{\tau^p\}_{p=1}^P, D)$ is called **routing game**.

The distinction between the single population case ($P = 1$) and the heterogeneous one, where each population is provided with a distinct set of cost functions,represents a crucial aspect that we will better investigate in the following. Now, in a routing game each player wants to minimize its perceived cost, accordingly to the cost functions.

Definition 3.2. Given a routing game $(\mathcal{G}_{(o,d)}, \mathcal{P}, \{\tau^p\}_{p=1}^P, D)$, a **Wardrop equilibrium** $z^* \in \mathbb{R}^{\mathcal{P} \cdot \mathcal{R}}$ is an admissible path flow distribution such that for every population $p \in \mathcal{P}$ and path $r \in \mathcal{R}$

$$(z^*)_r^p > 0 \;\Rightarrow\; c_r^p(z^*) \leq c_s^p(z^*), \;\forall s \in \mathcal{R}. \tag{3.4}$$

Wardrop equilibria actually correspond to Nash equilibria from game theory. The reason why they are named differently in the context of routing games is that they were named after J. G. Wardrop, the first who devoted his attention to this kind of problems in [18]. We are going to provide a classification of a Wardrop equilibrium based on the number of paths it involves and how populations distribute on them. Let $\mathcal{R}_{\text{opt}}^p(z^*)$ be the set of optimal paths for population $p \in \mathcal{P}$ at Wardrop equilibrium z^*.

Definition 3.3. Given a routing game $(\mathcal{G}_{(o,d)}, \mathcal{P}, \{\tau^p\}_{p=1}^P, D)$ and one of its Wardrop equilibria z^*, we say that z^* is **pure strategy** when $|\mathcal{R}_{\text{opt}}^p(z^*)| = 1, \;\forall p \in \mathcal{P}$, i. e., every population has a single optimal path on which it travels along.

Now, population games always admit at least one Nash equilibrium [15]. Then, since routing games are population games and Wardrop equilibria are Nash equilibria, it follows that routing games always admit at least one Wardrop equilibrium. Hence, existence of equilibrium configurations in these models is always guaranteed. Uniqueness, instead, turns out to be a much more delicate aspect. We want to anticipate that uniqueness, in general, does not hold and homogeneity and heterogeneity represent a major concern in this case.

3.2 Single population routing games

In the single population case, the following result guarantees uniqueness of the Wardrop equilibrium of the game by associating every single population routing games to a specific convex program with convex objective function.

Theorem 3.1 ([7])**.** *Let us consider the single population routing game. Then, a o-d flow f^* of throughput τ is a Wardrop equilibrium if and only if it is an optimal solution of the following minimization problem:*

$$\min_{\substack{f \geq 0 \\ Bf = \tau(\delta^{(o)} - \delta^{(d)})}} \sum_{e \in \mathcal{E}} \Delta_e(f_e), \tag{3.5}$$

where

$$\Delta_e(f_e) := \int_0^{f_e} D_e(s) ds. \tag{3.6}$$

Then, f^* is unique if the cost functions in D are strictly increasing.

First of all, notice that Theorem 3.1 is stated in terms of the link flow distribution f^*. When referring to f^* as Wardrop equilibrium, we are actually saying that all z^* such that $f^* = A^{(o,d)}z^*$ are Wardrop equilibria. This result might be restated in an equivalent form in terms of z by substituting f_e with $(A^{(o,d)}z)_e$ in (3.5). Nevertheless, the problem is only convex in z. Hence, it might be that the game admits a continuum of equilibrium route flow distributions that induce the same network flow f^*. Therefore, uniqueness of the Wardrop equilibrium with strictly increasing cost functions holds only from a link-perspective point of view.

Theorem 3.1 states that finding a Wardrop equilibrium in the single population case is equivalent to minimizing a convex potential function with linear constraints. Basically, what this result is saying is that single population routing games are actually *potential games*.

Definition 3.4. A routing game is said to be **potential** if there exists a function $V : \mathcal{Z} \to \mathbb{R}$ such that

$$c_r^p(z) - c_s^p(z) = \left(\frac{\partial}{\partial z_r^p} - \frac{\partial}{\partial z_s^p} \right) V(z), \quad \forall p \in \mathcal{P}, \ \forall r, s \in \mathcal{R}. \tag{3.7}$$

The function V is called the **potential** of the game.

The objective function in (3.5) is the potential of the single population routing game $(\mathcal{G}, \tau, D)$. Potential will turn out to be particularly useful in the next chapter when analyzing the asymptotic behavior of the deterministic logit dynamics in single population routing games. Please refer to [15] for more details on potential games and notice that Definition 3.4 is valid also for the broader class of population games.

In many cases, single population routing games do not describe real-word problems well, since homogeneity basically consists in the assumption that all users perceive link costs in the same way.

3.3 Heterogeneous routing games

The heterogeneity assumption allows us to model more realistic and complex systems. The main difficulty that arises when dealing with heterogeneous routing games is that they are not associated to a potential function. The lack of potential has several implications on the properties of the game, for instance on the uniqueness of the equilibria. In fact, we will see that the Wardrop equilibrium in heterogeneous routing games is in general not unique, in contrast with the single population case.

Definition 3.5. Let us consider a heterogeneous routing game $(\mathcal{G}_{(o,d)}, \mathcal{P}, \{\tau^p\}_{p=1}^P, D)$ and let $\mathcal{Z}^*$ be the set of its Wardrop equilibria. We say that the game admits an **essentially unique Wardrop equilibrium** if all its Wardrop equilibria have the same aggregate link flow distribution, i. e.,

$$A^{(o,d)} z^{\text{agg}} = A^{(o,d)} w^{\text{agg}}, \quad \forall z, w \in \mathcal{Z}^*. \tag{3.8}$$

In [14], the author shows that the largest class of TTN for which essential uniqueness holds is that of *series of nearly parallel graphs*. Notice that the definitions that will follow were originally provided in [14] for the more general case of undirected graphs. Nevertheless, the same article shows how they can be easily adapted to the special case of directed graphs.

Definition 3.6. A TTN is said to be **nearly parallel** when one of the following conditions is satisfied:

- it has a single *o*-*d* path;
- it has two parallel *o*-*d* paths;
- it derives from a TTN with two parallel paths added with one or more parallel paths with common end nodes.

A **series of nearly parallel graphs**, instead, is a TTN consisting in concatenation in series of two or more nearly parallel graphs, where two consecutive graphs are connected so that the first one has as destination node the origin node of the following. Notice that from now on, we will use the term *path* to denote an *o*-*d* path of a given TTN.

All typologies of nearly parallel graphs are shown in Figure 3.1. It is worth to mention the fact that nearly parallel graphs might also be defined as TTNs that do not embed none of the graphs reported in Figure 3.3, where the notion of embedding is provided in the following definition.

Definition 3.7 ([14])**.** A TTN $\mathcal{G}'_{(o',d')}$ is said to be **embedded in the wide sense** in another TTN $\mathcal{G}''_{(o'',d'')}$ if the latter can be derived from $\mathcal{G}'_{(o',d')}$ by applying one or more of the following operations (displayed in Fig 3.2) any number of times:

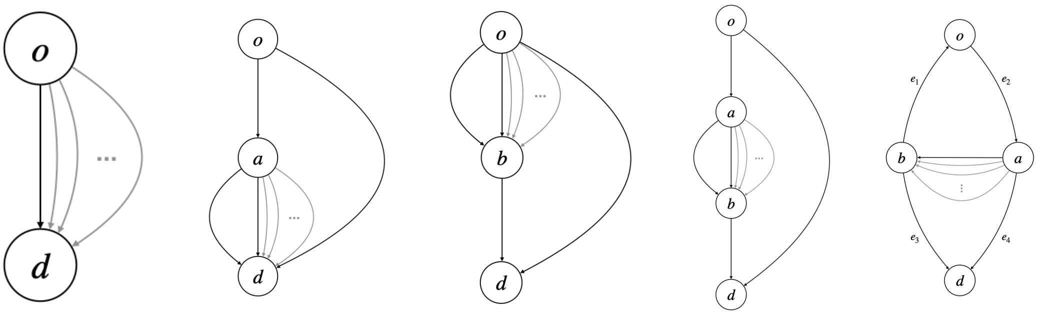

Figure 3.1: Sub-classes of nearly parallel graphs.

1. subdivision of an existing link;
2. addition of a new link;
3. subdivision of a terminal node (o or d).

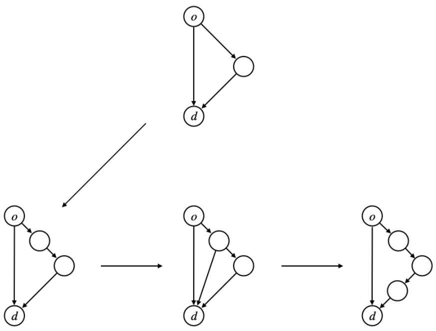

Figure 3.2: Embedding operation: *subdivision of an existing edge* (bottom left), *addition of an existing edge* (bottom center) and *subdivision of a terminal node* (bottom right).

Therefore, being a (series of) nearly parallel graph(s) and embedding one of the graphs in Figure 3.3 are mutually exclusive conditions. Let us call TTNs satisfying the second condition *non-parallel graphs*.

In [14], the author also proved that a TTN has the essential uniqueness property if and only if it is a (series of) nearly parallel graph(s). As a consequence, all TTNs that do not belong to this class of graphs admit a heterogeneous routing game on them with multiple Wardrop equilibria. We now report a significant example from [10]. Then, we dedicate a subsection to analyze a heterogeneous routing game defined on a non-parallel graph. This last example was originarly presented in [13]: here we show that there is an additional Wardrop equilibria that was not mentioned in the previous article.

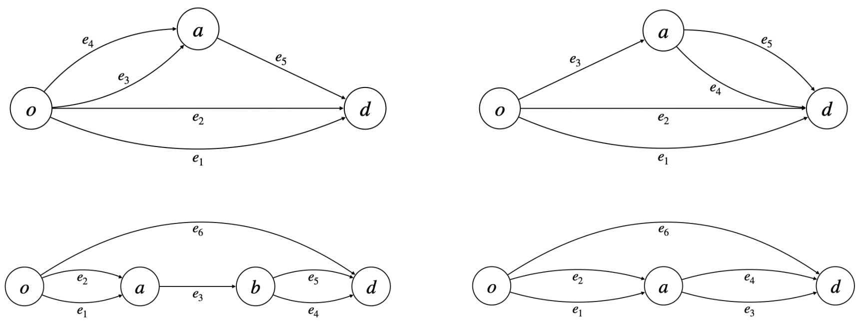

Figure 3.3: The *prohibited* graphs.

Example 1 Let us consider a TTN such that $\mathcal{E} = (e_1, e_2)$ and $\theta(e_i) = o$, $\kappa(e_i) = d$, $i = 1{,}2$. Let $|\mathcal{P}| = P = 2$ and τ^1, τ^2 throughputs of population 1 and 2, respectively. It is straightforward that if D is such that

$$D_1^1(f_1^*) = D_2^1(\tau - f_1^*), \quad D_1^2(f_1^*) = D_2^2(\tau - f_1^*), \tag{3.9}$$

for some aggregate link flow distribution $f^* = (f_1^*, \tau - f_1^*)$, then every link flow distribution satisfying

$$f_1^1 + f_1^2 = f_1^*, \quad f_2^1 + f_2^2 = \tau - f_1^*, \quad f_1^1 + f_2^1 = \tau^1, \quad f_1^2 + f_2^2 = \tau^2, \tag{3.10}$$

is a Wardrop equilibria for the heterogeneous routing game $(\mathcal{G}, \mathcal{P}, \{\tau^p\}_{p=1}^2, D)$. It is interesting to notice that the Wardrop equilibrium is essentially unique, since the aggregate link flow distribution is always the same, although population-wise the game admits a continuum of Wardrop equilibria.

3.3.1 An example of non essential uniqueness

Let us consider the TTN in Figure 3.4. We have that $\mathcal{V} = \{o, a, b, d\}$, where

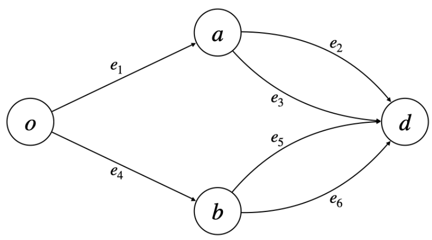

Figure 3.4: Example of non-parallel graph.

o and d are, respectively, the origin node and the destination node, and $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. Let also $\mathcal{R} = \{r_1, r_2, r_3, r_4\}$ be the set of paths from o to

d, where $r_1 = (e_1, e_2)$, $r_2 = (e_1, e_3)$, $r_3 = (e_4, e_5)$, $r_4 = (e_4, e_6)$. Finally, let $R = |\mathcal{R}|$. The throughputs of the populations are the following:

$$\tau^1 = 1.2, \quad \tau^2 = 1, \quad \tau^3 = 1. \tag{3.11}$$

Let us also assign the following cost functions to the edges of the TTN. Here, $D_e^p(x)$ quantifies the cost that an user of the p-th population experiences when it travels along the e-th edge, given that the flow that runs across it is equal to x:

$$\begin{aligned}
&D_1^1(x) = D_2^1(x) = D_4^1(x) = D_6^1(x) = 19 + x,\\
&D_1^2(x) = D_4^2(x) = D_1^3(x) = D_4^3(x) = 19 + x,\\
&D_3^1(x) = D_5^1(x) = D_3^2(x) = 100 + x,\\
&D_6^2(x) = D_2^3(x) = D_5^3(x) = 100 + x,\\
&D_2^2(x) = D_6^3(x) = 20x,\\
&D_5^2(x) = D_3^3(x) = 21 + x.
\end{aligned} \tag{3.12}$$

In this case, the link-path incidence matrix is

$$A^{(o,d)} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.13}$$

The path costs for each population involved in the game for the network taken into consideration are the following:

$$\begin{aligned}
c_1^1 &= 2z_1^{\text{agg}} + z_2^{\text{agg}} + 38, & c_2^1 &= z_1^{\text{agg}} + 2z_2^{\text{agg}} + 119,\\
c_3^1 &= \tau - z_1^{\text{agg}} - z_2^{\text{agg}} + z_3^{\text{agg}} + 119, & c_4^1 &= 2\tau - 2z_1^{\text{agg}} - 2z_2^{\text{agg}} - z_3^{\text{agg}} + 38,\\
c_1^2 &= 21z_1^{\text{agg}} + z_2^{\text{agg}} + 19, & c_2^2 &= z_1^{\text{agg}} + 2z_2^{\text{agg}} + 119,\\
c_3^2 &= \tau - z_1^{\text{agg}} - z_2^{\text{agg}} + z_3^{\text{agg}} + 40, & c_4^2 &= 2\tau - 2z_1^{\text{agg}} - 2z_2^{\text{agg}} - z_3^{\text{agg}} + 119,\\
c_1^3 &= 2z_1^{\text{agg}} + z_2^{\text{agg}} + 119, & c_2^3 &= z_1^{\text{agg}} + 2z_2^{\text{agg}} + 40,\\
c_3^3 &= \tau - z_1^{\text{agg}} - z_2^{\text{agg}} + z_3^{\text{agg}} + 119, & c_4^3 &= 21\tau - 21z_1^{\text{agg}} - 21z_2^{\text{agg}} - 20z_3^{\text{agg}} + 19.
\end{aligned} \tag{3.14}$$

They can be computed resorting to (3.3) and (3.12). Notice that they are expressed as functions depending on the components of the aggregate route flow distribution.

Proposition 3.1. *The only equilibria of the heterogeneous routing game on network in Fig. 3.4 with cost functions (3.12) and throughputs (3.11) are the flow vectors*

- $$f_1^{agg} = \left(\frac{6}{5}, \frac{6}{5}, 0, 2, 1, 1\right), \tag{3.15}$$

- $$f_2^{agg} = \left(2,1,1,\frac{6}{5},0,\frac{6}{5}\right), \tag{3.16}$$

- $$f_3^{agg} = \left(\frac{8}{5},\frac{113}{105},\frac{11}{21},\frac{8}{5},\frac{11}{21},\frac{113}{105}\right), \tag{3.17}$$

associated to the the following route distributions:

- $$\begin{aligned} z^1 &= \left(\frac{6}{5},0,0,0\right), \\ z^2 &= (0,0,1,0), \\ z^3 &= (0,0,0,1); \end{aligned} \tag{3.18}$$

- $$\begin{aligned} z^1 &= \left(0,0,0,\frac{6}{5}\right), \\ z^2 &= (1,0,0,0), \\ z^3 &= (0,1,0,0); \end{aligned} \tag{3.19}$$

- $$\begin{aligned} z^1 &= \left(\frac{3}{5},0,0,\frac{3}{5}\right), \\ z^2 &= \left(\frac{10}{21},0,\frac{11}{21},0\right), \\ z^3 &= \left(0,\frac{11}{21},0,\frac{10}{21}\right). \end{aligned} \tag{3.20}$$

Proof. The proof is contained in Appendix A. □

Therefore, the game admits three distinct Wardrop equilibria, each of them associated to a different aggregate link flow distribution. Hence, essential uniqueness does not hold in this case. Notice also that two of them are pure strategy Wardrop equilibria.

We showed that much more complex scenarios might arise when dealing with heterogeneous routing games defined on non-parallel graphs. This is the framework which we focus on in the next chapter and the results in it refer to.

Chapter 4

Logit learning in heterogeneous routing games

As outlined in [15], routing games provide a relatively simple framework to describe strategic multi-agent systems, making a certain number of assumptions in order to facilitate the construction of a model. Among all these assumptions, the *equilibrium knowledge assumption* is probably the most significant one. Under this hypothesis, we are basically saying that players have full knowledge of the game they are taking part to and they are able to choose the correct strategy to adopt by anticipating what other players will do, in fact. Many times it is more appropriate to build dynamical systems where, starting from a certain initial condition, players are free to change their choice over time and see where this evolution leads.

We are now going to describe a framework in which players may decide to change their strategy over time in order to minimize their perceived cost. By doing so, they may modify the current perceived costs of the other players who might also decide to change strategy and so on. The system will eventually converge to a configuration in which none of the players has a reason to make another change [15]. Specifically, in this chapter we devote our attention to the so called *deterministic logit dynamics* and we will equip heterogeneous routing games with this new tool in order to create a more realistic model. Actually, the main results provided in this thesis concern the asymptotic behavior of the aforementioned dynamical system.

4.1 Dynamics description

Deterministic logit dynamics belongs to the family of *evolutionary dynamics*. Evolutionary dynamics are continuous-time systems where players can adapt their decision depending on the current system configuration.

Definition 4.1. The **deterministic logit dynamics** reads as

$$\dot{z}_r^p = \tau^p \cdot \frac{\exp\left(-\eta \cdot c_r^p\left(z\right)\right)}{\sum_{s\in R} \exp\left(-\eta \cdot c_s^p\left(z\right)\right)} - z_r^p, \tag{4.1}$$

where η is the inverse of the **noise level**.

Basically, this dynamics aims to describe the evolution of the system given that users are characterized by a level of knowledge of the game, η. The higher is η, the lower is the noise level and the better are the choices based on the current state made by players. We also introduce the jabobian matrix of the logit dynamics $J_\eta(z) \in \mathbb{R}^{\mathcal{R}\cdot\mathcal{P}\times\mathcal{R}\cdot\mathcal{P}}$, whose entries have the following form:

$$\begin{aligned}\frac{\partial \dot{z}_r^p}{\partial z_v^q} &= \tau^p \eta \exp(-\eta \cdot c_r^p(z)) \frac{\sum_{s\in\mathcal{R}} \frac{\partial}{\partial z_v^q}\left(c_s^p(z) - c_r^p(z)\right)\exp(-\eta \cdot c_s^p(z))}{\left(\sum_{s\in\mathcal{R}} \exp(-\eta \cdot c_s^p(z))\right)^2} - \delta_{r,v}^{p,q} = \\ &= \frac{\eta}{\tau^p}\left(\dot{z}_r^p + z_r^p\right)\left(\sum_{s\in\mathcal{R}} \frac{\partial}{\partial z_v^q}\left(c_s^p(z) - c_r^p(z)\right)\left(\dot{z}_s^p + z_s^p\right)\right) - \delta_{r,v}^{p,q}.\end{aligned} \tag{4.2}$$

In the last equality, we used:

$$\frac{\dot{z}_r^p + z_r^p}{\tau^p} = \frac{\exp\left(-\eta \cdot c_r^p\left(z\right)\right)}{\sum_{s\in R} \exp\left(-\eta \cdot c_s^p\left(z\right)\right)}, \quad \forall r, p.$$

Our main goal is to find the fixed points of (4.1) and to study their stability. Stability of the equilibria is important, since it indicates if an equilibrium point will persist even if perturbations of the system will occur. Another important objective is to understand whether there is a connection between these fixed points and the Wardrop equilibria of the associated routing game. Specifically, we are interested in investigating if the system converges to one of the Wardrop equilibria when noise vanishes. This would tell us which of the Wardrop equilibria of the game represent a concrete possibility that the system might reach in a real-world application. Therefore, in this section we will show what is the asymptotic behavior of the logit dynamics both in the case of a single population and a heterogeneous routing game, highlighting the differences between the two.

4.2 Single population case

In Section 3.2 we said that single population routing games are potential games and, as such, they admit a unique Wardrop equilibrium. We now report a result determining the asymptotic behavior of (4.1) in single population routing games. Before stating this result, we need to introduce the so called *entropy function*:

$$H(z) := -\sum_{s\in\mathcal{R}} z_s \log\left(\frac{z_s}{\tau}\right) \tag{4.3}$$

Entropy is measure borrowed from Information Theory. In this particular case, it quantifies how varied the choices of the paths to be used by the users are. When all agents choose the same path, $H(z) = 0$. Then, as users start to travel along altenative paths, $H(z)$ increases: it will attain its maximum when z will coincide with the uniform distribution on $\mathcal{R}$. Thanks to the fact the single population games are potential games and result 7.1.4 in [15], it is possible to prove the following result and to illustrate how the logit dynamics admits a globally asymptotically stable fixed point which converges to the set of Wardrop equilibrium of the single population routing game, as noise vanishes.

Proposition 4.1. *Let us consider a single population routing game* $(\mathcal{G}, \tau, D)$ *with potential function*

$$V(z) = \sum_{e\in\mathcal{E}} \int_0^{(A^{(o,d)}z)_e} D_e(s)ds \tag{4.4}$$

and the associated logit dynamics (4.1)*. Let us define the following function:*

$$V_\eta(z) = V(z) - \frac{1}{\eta}H(z) \tag{4.5}$$

Then:

1. $V_\eta(z)$ *is a strictly convex function with respect to* z*;*
2. *for every initial condition* $z(0)$*, we have that*

$$\lim_{t\to+\infty} z(t) = z_\eta, \tag{4.6}$$

   where z_η *is the unique minimizer of* V_η*.*

3. $z_\eta \to \mathcal{Z}^*$*, where* $\mathcal{Z}^*$ *is the set of Wardrop equilibria of the single population routing game.*

In Section 3.2 we already pointed out that, since the potential is only convex in z, a single population routing game admits a continuum of Wardrop equilibria $\mathcal{Z}^*$. However, since the entropy V_η is a strictly convex function, it is possible to prove the uniqueness of a globally asymptotically stable fixed point, $\forall\eta \geq 0$. Finally, we highlight that Proposition 4.1 does not state that each $z^* \in \mathcal{Z}^*$ is approached by the fixed points of the dynamics, as noise vanishes.

4.3 Heterogeneous case

In [10], the authors show that on *(series of) simple graphs*, a sub-class of (series of) nearly parallel graphs, the logit dynamics (4.1) associated to a heterogeneous routing game admits a globally asymptotically stable fixed point which approaches the set of Wardrop equilibria, as the noise vanishes. Therefore, in this case both uniqueness and stability of the fixed point hold. In the following sections, we address a more general setting in which we do not make any assumption on the the TTN topology. We establish theoretical results that apply also to heterogeneous games for which the essential uniqueness of the equilibrium does not hold, as the one presented in Section 3.3.1.

4.3.1 Numerical experiments

We are going to discuss the results obtained from some of the numerical simulations we performed. The experimental phase of our work was very useful in obtaining guidelines to orient the theoretical research.

Our tests focus on the example previously analyzed in Section 3.3.1. In [10], the authors already highlighted that the logit dynamics associated to the game admits a bifurcation point. Our aim is to provide more insightful observations on numerical simulations. Finally, in the next sections, we will discuss theoretical results.

Experiment analysis

As already discussed, the noise level represents a measure of the knowledge of the game of the players. The larger the noise, the less users' strategy is influenced by the perceived path costs. On the contrary, the user will tend to make more rational choices, as noise vanishes. Figure 4.1 shows two trajectories corresponding to different initial conditions, with different values of η. The simulations suggest that the dynamics presents a bifurcation point: as η is small, the system converges to a unique asymptotically stable fixed point, while as η increases the two trajectories converge to different fixed points, which approach the two pure strict equilibria of the game as η increases. In particular, through various simulations it was possible to estimate that these two asymptotically stable branches originate starting from the approximate value $\eta^* = 3.225$.

The simulations show that this is a *pitchfork bifurcation*. For this reason, we looked for a third fixed points curve depending on η, specifically the one associated to the unstable fixed points characterizing this type of bifurcation. To this end, we performed again some numerical simulations. After a few attempts, we were able to identify this fixed points curve. The associated plots are reported in Figure 4.2.

The numerical simulations are based on the idea that both the asymptotically stable fixed points have an attraction basin. Therefore, the feasible path flow distribution space $\mathcal{Z}$ must divide into the two of them. The curve of unstable fixed

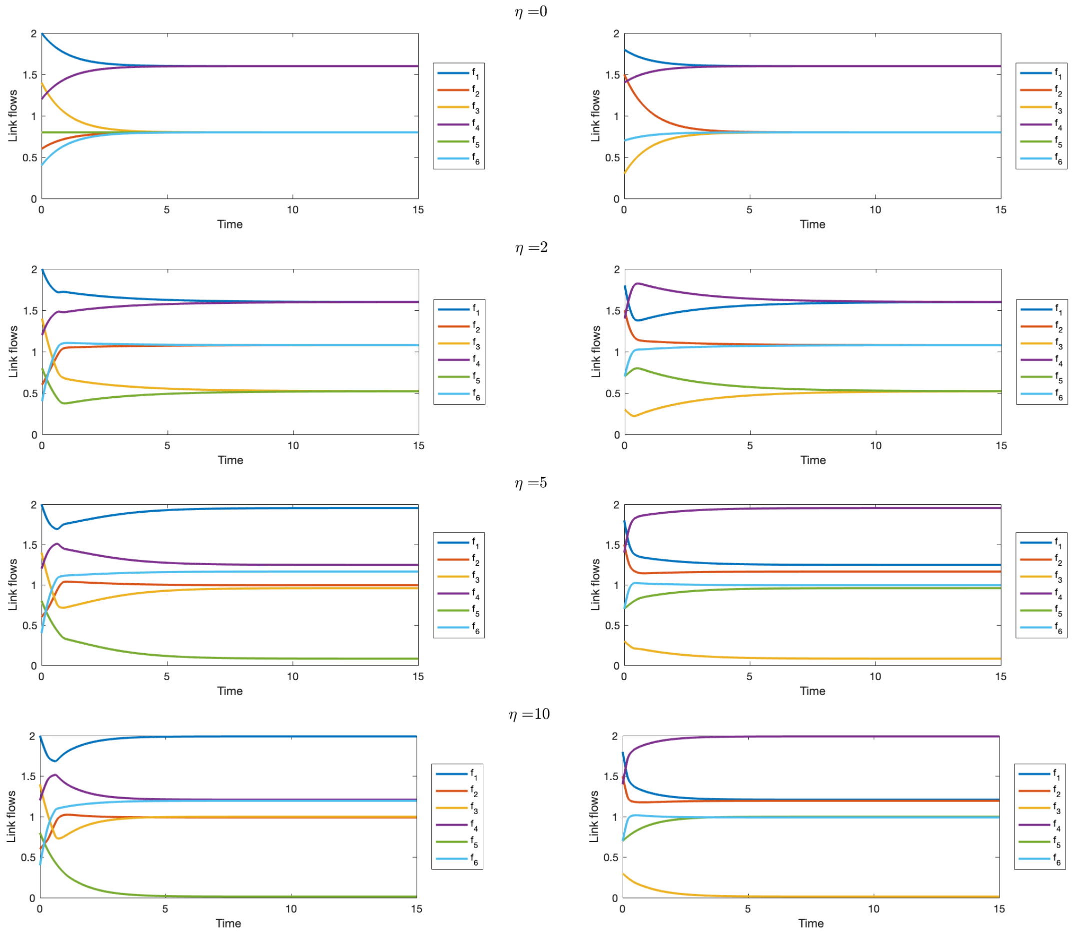

Figure 4.1: Two trajectories of the logit dynamics for different values of η. The trajectories are projected in the space of the aggregate flows.

points will therefore be placed, if exists, among them. Hence, we decided to test the asymptotic behavior of some convex combination of the two known fixed points:

$$z_\alpha^\eta = \alpha z_1^\eta + (1-\alpha)\, z_2^\eta, \qquad \alpha \in [0,1]\,,$$

where z_1^η e z_2^η are the two asymptotically stable fixed points of (4.1) when the noise level is $1/\eta$. Notice that for linearity it also holds that:

$$\begin{aligned} f_\alpha^\eta = A z_\alpha^\eta = A\left(\alpha z_1^\eta + (1-\alpha)\, z_2^\eta\right) = \alpha A z_1^\eta + (1-\alpha)\, A z_2^\eta = \\ = \alpha f_1^\eta + (1-\alpha)\, f_2^\eta, \qquad \alpha \in [0,1]\,, \end{aligned}$$

The rightmost plots in Figure 4.2 were obtained by fixing as initial condition the convex combination of the stable fixed points corresponding to $\alpha = 1/2$. The instability of this fixed point is suggested by some other simulations we performed,

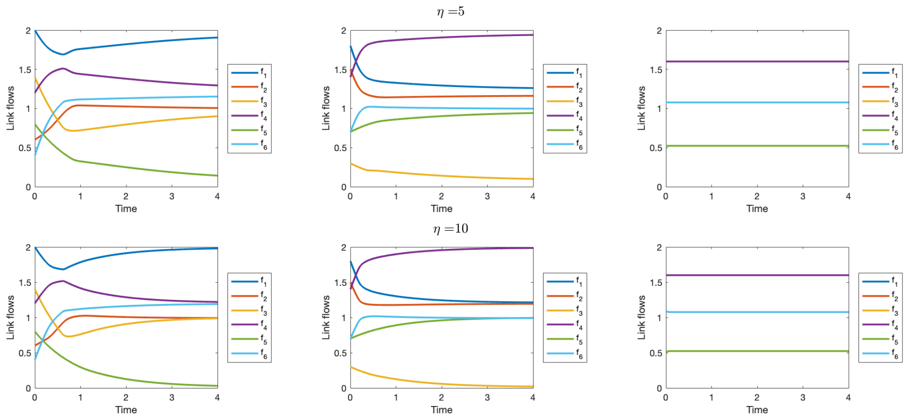

Figure 4.2: The rightmost panel shows the behavior of the dynamics corresponding to initial condition equivalent to the unstable equilibrium of the game.

whose results are reported in Figure 4.3. Here, one can see that by slightly perturbing the initial condition ($\delta\alpha = 0.001$), the system deviates towards one of the other two fixed points.

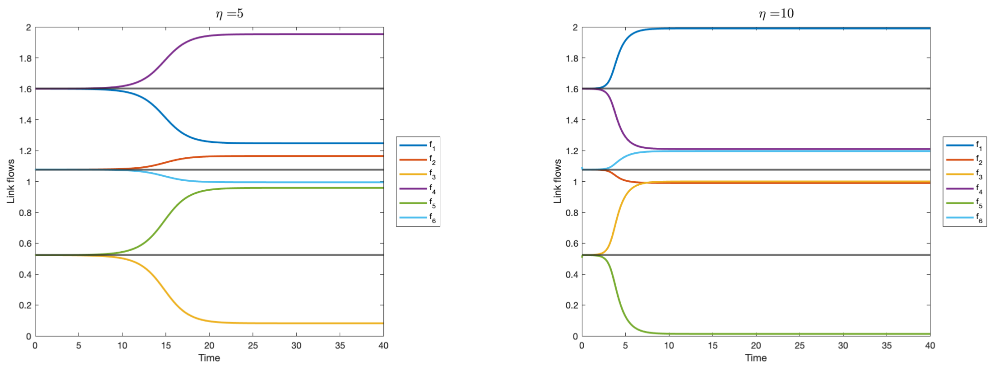

Figure 4.3: The plots show how small perturbations can deviate the system from the unstable fixed point (grey lines) towards one on the two asymptotically stable fixed points.

At this point, we determined the nature of these fixed points by studying the eigenvalues of the jacobian matrix (4.25) of the dynamics. From Theorem 2.3, we know that if one of the eigenvalues of $J_\eta(\bar{z})$ has positive real part, then $\bar{z}$ is an unstable fixed point. The tests we performed told us that this curve of fixed points has an eigenvalue which switches is sign when η overpasses the η^* threshold. This fact further confirms our hypothesis that the bifurcation phenomena originates in a neighborhood of η^*. Figure 4.4 shows the bifurcation phenomenon for each link of the network.

The aspects of main interest exhibited by this example are the following:

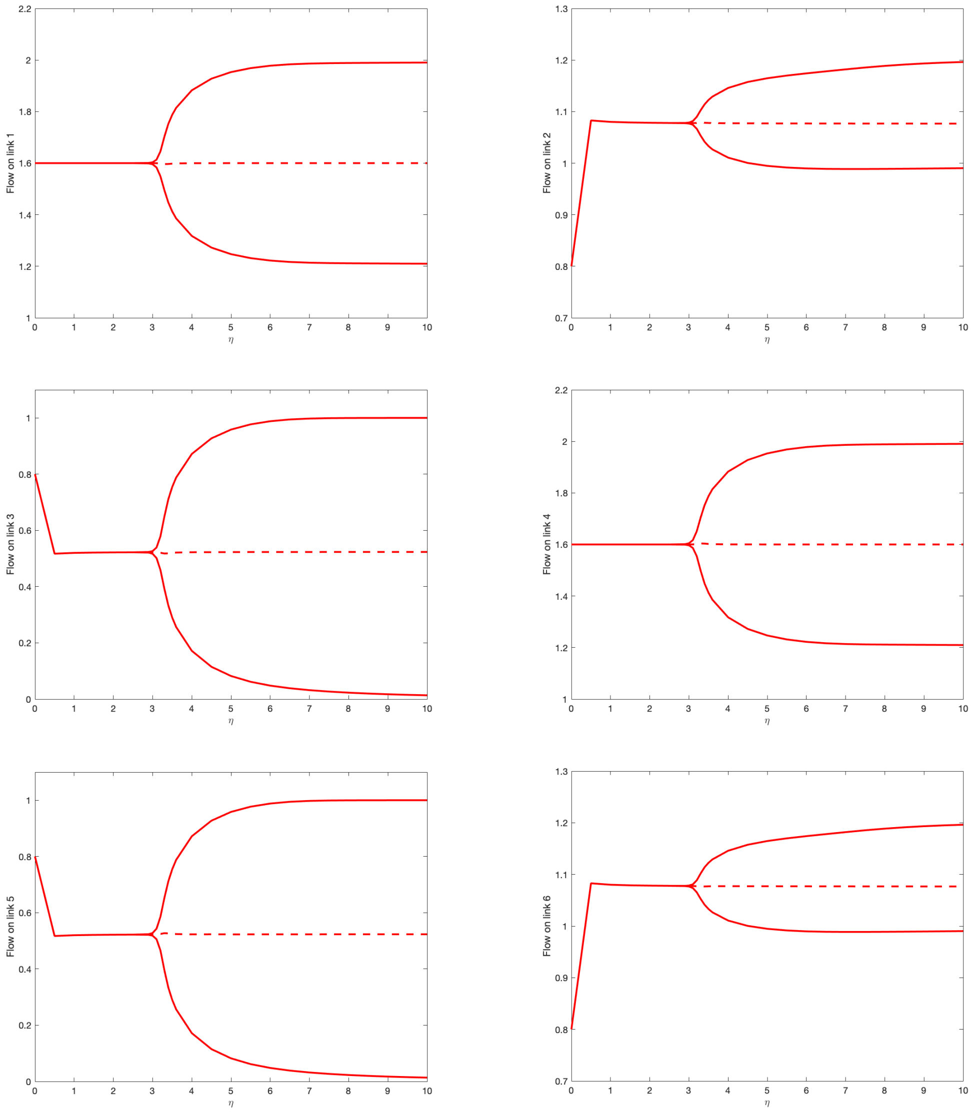

Figure 4.4: Evolution of the fixed points of the dynamics as noise decreases. These figures depict very well the bifurcation phenomenon and suggest that it might be the case of a pitchfork bifurcation.

- the system admits a unique globally asymptotically stable fixed point on the right neighborhood of $\eta = 0$;
- as noise vanishes, the fixed points curve converges towards one of the Wardrop equilibria of the heterogeneous routing game associated to the dynamics;
- asymptotical stability occurs when the fixed point converges to one of the pure strategy Wardrop equilibria.

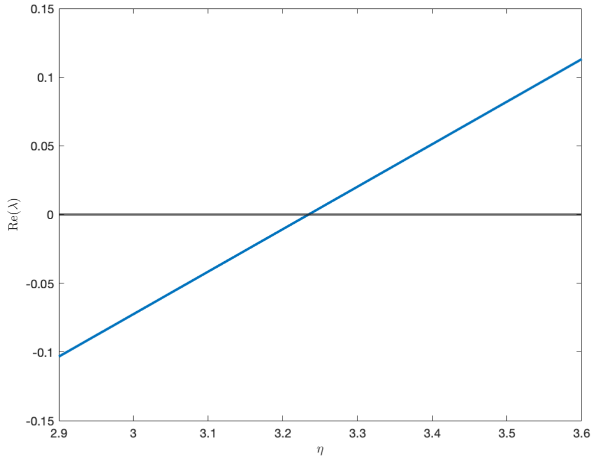

Figure 4.5: The evolution of the real part of the three eigenvalues with the largest magnitudes. The third plot shows how the real part of the third eigenvalue switches sign in the neighborhood of $\eta^* = 3.225$.

Inspired by the results obtained from these experiments, we channeled our efforts in their direction. As a result, in the following we are able to prove that:

- every routing game admits a unique globally asymptotically stable fixed point when the level of noise is very large ($\eta \to 0$);
- fixed points of logit dynamics converge to the set of Wardrop equilibria of the routing game when noise vanishes ($\eta \to +\infty$), even if the game is heterogeneous;
- fixed points of (4.1) are locally asymptotically stable when they converge to a pure strategy Wardrop equilibria, for a sufficiently high noise level.

4.3.2 Large noise regime

When noise tends to infinity ($\eta \to 0$), the system converges to an uniform distribution over all paths, i. e., each population randomizes among all routes. This is due to the fact that the high value of η makes costs irrelevant in (4.1). We report the following result.

Theorem 4.1. *There exist $\bar{\eta}$ such that the deterministic logit dynamics (4.1) admits a unique globally asymptotically stable fixed point, $\forall \eta \in [0, \bar{\eta})$.*

Proof. First of all, let us assume $\eta = 0$. In this case, (4.1) becomes the following linear system of differential equations:

$$\dot{z}_r^p = \frac{\tau^p}{R} - z_r^p, \qquad r = 1, \dots, R, \quad p = 1, \dots, P. \tag{4.7}$$

The general solution of (4.7) is given by

$$z_r^p = e^{-t}\left(\int \frac{\tau^p}{R} e^t\right) = Ce^{-t} + \frac{\tau^p}{R}, \qquad r = 1, \dots, R, \quad p = 1, \dots, P, \tag{4.8}$$

and by imposing the initial condition z_0^p, for every $p \in \mathcal{P}$ we get:

$$z^p = \left(z_0^p - \frac{\tau^p}{R}\mathbb{1}\right) e^{-t} + \frac{\tau^p}{R}\mathbb{1}, \qquad \mathbb{1} \in \mathbb{R}^P. \tag{4.9}$$

Hence, the dynamics converges to the unique asymtotically stable fixed point consisting in the uniform route flow distribution.

We now want to show that this holds true also in a right neighboorhood of $\eta = 0$. Matrix measures can be exploited in order to show existence and uniqueness of a globally asymptotically stable fixed point for this dynamical systems. The matrix measure associated to the ℓ_1 norm of our dynamics is given by:

$$\begin{aligned}
\mu(J_\eta(z)) &= \max_{r \in \mathcal{R}} \, (J_\eta(z))_{r,r} + \sum_{s \in \mathcal{R}\setminus\{r\}} |(J_\eta(z))_{s,r}| = \\
&= \max_{r \in \mathcal{R}} \; \eta \sum_{p=1}^{P} \tau^p \exp\left(-\eta \cdot c_r^p(z)\right) \frac{\sum_{h \in \mathcal{R}} \frac{\partial}{\partial z_r}\left(c_h^p(z) - c_r^p(z)\right)\left(\exp\left(-\eta \cdot c_h^p(z)\right)\right)}{\left(\sum_{h \in R} \exp\left(-\eta \cdot c_h^p(z)\right)\right)^2} - 1 \\
&\quad + \sum_{s \in \mathcal{R}\setminus\{r\}} \left| \eta \sum_{p=1}^{P} \tau^p \exp\left(-\eta \cdot c_s^p(z)\right) \frac{\sum_{h \in \mathcal{R}} \frac{\partial}{\partial z_r}\left(c_h^p(z) - c_s^p(z)\right)\left(\exp\left(-\eta \cdot c_h^p(z)\right)\right)}{\left(\sum_{h \in R} \exp\left(-\eta \cdot c_h^p(z)\right)\right)^2} \right|.
\end{aligned} \tag{4.10}$$

Notice that $\mu(J_\eta(z))$ is continuous, since all the functions at denominator are strictly positive and continuous functions and all the functions involved in sums or products are continuous, too. Moreover, the pointwise maximum of a set of continuous functions is a continuous function, as well. Now, it holds that $J_0(z) = -I_n$, for every z in the simplex, and it is easy to see that $\mu(-I_{\mathcal{R}\cdot\mathcal{P}}) = -1$. Now, by applying the Theorem of the permanence of sign [9], it follows that there must exists a right neighborhood of $\eta = 0$ in which $\mu(J_\eta(z))$ remains strictly negative, for every z in the simplex. Therefore, in this neighborhood it is still possible to apply Theorem 2.4. Hence, existence and uniqueness of the fixed point still holds and its region of attraction contains the whole set defined by the throughput constraints. □

4.3.3 Vanishing noise regime

We are now going to show that the fixed points of the dynamics approach the Wardrop equilibria of the associated heterogeneous routing game in the limit of zero noise ($\eta \to +\infty$).

Before continuing, we need to introduce the notion of *convergence of a set* and the famous result know as *Brouwer's fixed point Theorem*.

Definition 4.2. Let $\mathcal{X}$ be a compact set and let $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ be a sequence of compact sets. Then, we say that $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ **converges to** $\mathcal{X}$ and we indicate it as

$$\lim_{n\to+\infty} \mathcal{X}_n = \mathcal{X}$$

if:

- for every $x \in \mathcal{X}$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \in \mathcal{X}$, $\forall n$, and $\lim_{n\to+\infty} x_n = x \in \mathcal{X}$;
- for every converging sequence $\{x_n\}_{n\in\mathbb{N}}$, we have that $x_n \in \mathcal{X}_n$, $\forall n$, and $\lim_{n\to+\infty} x_n = x \in \mathcal{X}$.

Theorem 4.2 ([5])**.** *Let $f : K \to K$ a continuous map, where K is a convex and compact subset of a Euclidean space. Then f admits at least a fixed point, i. e., there exists $x \in K$ such that $f(x) = x$.*

We can now provide the main results of this section.

Theorem 4.3. *Consider a heterogeneous routing game $(\mathcal{G}_{(o,d)}, \mathcal{P}, \{\tau^p\}_{p=1}^P, D)$, where $\mathcal{G}$ is an arbitrary TTN. Let us assume that D is characterized by non-decreasing cost functions and let $\mathcal{Z}^* \subset \mathcal{Z}$ be the set of Wardrop equilibria of the routing game. Finally, let us associate the dynamics (4.1) to the routing game and let $\Omega_\eta \subset \mathcal{Z}$ be the set of its fixed points. Then:*

1. *Ω_η is a non-empty and compact set, $\forall \eta > 0$;*
2. *there exists a non-empty and compact set $\bar{\mathcal{Z}}^* \subset \mathcal{Z}^*$, called limit set, such that*
$$\lim_{\eta\to+\infty} \Omega_\eta = \bar{\mathcal{Z}}^*, \tag{4.11}$$
in the sense of Definition 4.2;
3. *every pure strategy Wardrop equilibrium belongs to $\bar{\mathcal{Z}}^*$.*

Proof. 1. Consider the function $F_\eta : \mathcal{Z} \to \mathcal{Z}$, defined by

$$(F_\eta)_r^p(z) = \tau^p \frac{\exp(-\eta \cdot c_r^p(z))}{\sum_{s\in\mathcal{R}} \exp(-\eta \cdot c_s^p(z))} \tag{4.12}$$

F_η is a continuous map and it maps $\mathcal{Z}$ to itself. Since $\mathcal{Z}$ is a convex and compact set, we can apply Theorem 4.2 that guarantees the existence of at least one fixed point for the dynamics (4.1) in $\mathcal{Z}$. Hence, Ω_η is non-empty, $\forall \eta > 0$. Notice also that Ω_η is compact for every η, since it is a level set of a continuous function on a compact set.

2. Theorem 4.2 ensures that the logit dynamics admits at least one fixed point, $\forall\eta \geq 0$. Then, it is possible to consider a map of fixed points $\tilde{z}(\eta)$ depending on η. Let us also consider a sequence η_n, such that $\lim_{n\to+\infty}\eta_n = +\infty$. Then, it follows that $\{\tilde{z}(\eta_n)\}_{n\in\mathbb{N}} \subset \mathcal{Z}$ admits a subsequence such that $\lim_{n_k\to+\infty}\tilde{z}(\eta_{n_k}) = z^* \in \mathcal{Z}$. Since all the elements of the sequence are fixed points, it holds that:

$$\dot{\tilde{z}}_r^p(\eta_{n_k}) = 0, \ \forall r \in \mathcal{R}, \ \forall p \in \mathcal{P}, \ \forall k \in \mathbb{N}. \tag{4.13}$$

It follows that $(z^*)_r^p$ also satisfies to this condition and it will be a fixed point of the dynamics, in the limit of $\eta \to \infty$. Let us now rewrite (4.1) in the following way:

$$\begin{gathered}\dot{\tilde{z}}_r^p(\eta_{n_k}) = (F_\eta)_r^p(\tilde{z}(\eta_{n_k})) - \tilde{z}_r^p(\eta_{n_k}),\\ (F_\eta)_r^p(\tilde{z}(\eta_{n_k})) = \tau^p \cdot \frac{\exp\left(-\eta \cdot c_r^p\left(\tilde{z}(\eta_{n_k})\right)\right)}{\sum_{s\in R}\exp\left(-\eta \cdot c_s^p\left(\tilde{z}(\eta_{n_k})\right)\right)}.\end{gathered} \tag{4.14}$$

From (4.13), it follows that all the fixed points satisfy to the following equation:

$$\tilde{z}_r^p(\eta_{n_k}) = (F_\eta)_r^p(\tilde{z}(\eta_{n_k})), \ \forall r \in \mathcal{R}, \ \forall p \in \mathcal{P}, \ \forall k \in \mathbb{N}. \tag{4.15}$$

By rewriting F_η as

$$(F_\eta)_r^p(\tilde{z}(\eta_{n_k})) = \tau^p \cdot \frac{1}{1 + \sum_{s\in R}\exp\left(-\eta(c_s^p(\tilde{z}_r^p(\eta_{n_k})) - c_r^p(\tilde{z}_r^p(\eta_{n_k}))\right)}, \tag{4.16}$$

it can be seen that the components of $\tilde{z}(\eta_{n_k})$, for each population p, that are not associated to the minimum cost will drop to 0, as $\eta \to \infty$. In fact, let us consider the limits attained by the cost functions as η diverges, namely $c_r^p(\tilde{z}^*), \ r \in \mathcal{R}, \ p \in \mathcal{P}$. Then, it holds that:

$$c_r^p(z^*) \geq \min_{v\in|\mathcal{R}|, q\in|\mathcal{P}|} c_v^q(z^*) = c_h^p(z^*). \tag{4.17}$$

This means that for all paths for which this inequallity holds strictly, there will be $\bar{k} \in \mathbb{N}$ such that:

$$c_r^p(\tilde{z}(\eta_{n_k})) > c_h^p(\tilde{z}(\eta_{n_k})), \quad \forall k > \bar{k}. \tag{4.18}$$

This implies that the exponential term in (4.16) will diverge for all those paths that are not associated to the minimum cost when there is no noise and the right-hand side of (4.16) will go to zero.

Finally, since

$$(z^*)_r^p = \lim_{\eta\to+\infty}(F_\eta)_r^p(z^*), \tag{4.19}$$

we find that:

$$(z^*)^p_r > 0 \Rightarrow c^p_r\left(z^*\right) \leq c^p_s\left(z^*\right), \quad \forall s \in |R|. \tag{4.20}$$

By definition, z^* is a Wardrop equilibrium for the heterogeneous routing game.

3. Let us consider a pure strategy Wardrop equilibrium z^* and denote by $r^*(p)$ the optimal path for population p under flow z^*. By definition of pure strategy equilibrium, all the throughput of population p flows through link $r^*(p)$. Then, for any $\epsilon \geq 0$, let us define

$$O_\epsilon := \{z \in \mathcal{Z} \mid z^p_{r^*(p)} \geq \tau^p(1-\epsilon), \forall p \in \mathcal{P}\}. \tag{4.21}$$

Basically, O_ϵ is the set containing all route flows distributions such that at least a fraction $1-\epsilon$ of every population travels along its optimal route $r^*(p)$ under z^*. Notice that $z^* \in O_\epsilon, \forall \epsilon \geq 0$. Now, let us define

$$\alpha := \min_{p\in\mathcal{P}} \min_{\substack{s\in\mathcal{R}\\ s\neq r^*(p)}} \left(c^p_s(z^*) - c^p_{r^*(p)}(z^*)\right). \tag{4.22}$$

Note that alpha is strictly positive. This is due to the fact that z^* is a pure strategy Wardrop equilibrium. Let us also define

$$\bar{\epsilon} = \max\left\{\epsilon \geq 0 \,\middle|\, \min_{z\in O_\epsilon} \min_{p\in\mathcal{P}} \min_{\substack{s\in\mathcal{R}\\ s\neq r^*(p)}} \left(c^p_s(z) - c^p_{r^*(p)}(z)\right) \geq \frac{\alpha}{2}\right\} \tag{4.23}$$

Similarly to α, $\bar{\epsilon} > 0$, since z^* is pure strategy and the cost functions are continuous. We now want to show that there exists $\bar{\eta}$ such that F_η maps O_ϵ to itself, for every $\eta \in [\bar{\eta}, +\infty)$ and $\epsilon \in [0, \bar{\epsilon}]$. First of all, notice that for every $\epsilon \in [0, \bar{\epsilon}]$ and population p, $r^*(p)$ is still the strictly optimal path and this holds true for every path flow distribution in O_ϵ. So, let us now pick an arbitrary $z \in O_\epsilon$. What we just said implies that by keeping the route flow configuration z fixed and by letting η go to infinity, we find that

$$\lim_{\eta\to+\infty} (F_\eta)^p_r(z) = \begin{cases} \tau^p, & \text{if } i = r^*(p), \\ 0, & \text{otherwise} \end{cases}, \quad \forall r \in \mathcal{R},\ \forall p \in \mathcal{P}. \tag{4.24}$$

Hence, $\lim_{\eta\to+\infty} F_\eta(z) = z^*$, for every $z \in O_\epsilon$, for every $\epsilon > 0$. Notice that z^* is contained in the internal part of O_ϵ, $\forall \epsilon > 0$. We conclude by pointing out that, by continuity of F_η with respect to η, there exists $\bar{\eta}$ such that, for $\eta \in [\bar{\eta}, +\infty)$, F_η maps O_ϵ to itself, $\forall \epsilon \in [0, \bar{\epsilon}]$. We can now show that there exists a sequence of fixed points of F_η converging to z^*. Firstly, observe that O_ϵ is a compact and convex set, so Theorem 4.2 guarantees that at least a fixed point of F_η in O_ϵ exists if $\eta \in [\bar{\eta}, +\infty]$, for all $\epsilon \in [0, \bar{\epsilon}]$. By noticing that O_ϵ collapse onto z^* as $\epsilon \to 0$, we can then find a sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ such that there exists a family of set $\{O_{\epsilon_n}\}_{n\in\mathbb{N}}$ such that $O_{\epsilon_{n+1}} \subseteq O_{\epsilon_n}$, each of them containing a fixed point z_{ϵ_n}. Then, this sequence of fixed points converges to z^*: $\lim_{n\to+\infty} z_{\epsilon_n} = z^*$. Finally, since there exists a sequence of fixed point of (4.1) converging to z^* , then $z^* \in \bar{\mathcal{Z}}^*$.

□

We now investigate the local stability of the fixed points of the dynamics when η is large.

Theorem 4.4. *Consider a heterogeneous routing game $(\mathcal{G}_{(o,d)}, \mathcal{P}, \{\tau^p\}_{p=1}^P, D)$, where $\mathcal{G}_{(o,d)}$ is an arbitrary TTN. Let $\{\eta_n\}_{n\in\mathbb{N}}$ be a sequence of inverse noise paramaters such that $\lim_{n\to+\infty} \eta_n = +\infty$, and for every n let $\tilde{z}_n \in \Omega_{\eta_n}$ be a fixed point of the corresponsing logit dynamics (4.1). If $\lim_{n\to+\infty} \tilde{z}_n = z^*$ and z^* is a strict Wardrop equilibrium, then, for large enough n, $\tilde{z}_n$ is locally asymptotically stable for (4.1).*

Proof. Note that for every pure strategy equilibrium z^*, a sequence of fixed points $\{\tilde{z}_n\}_{n\in\mathbb{N}}$ that converges to z^* always exists due to Theorem 4.3. Now, notice that at equilibrium, since $\dot{z}_r^p = 0,\ \forall r,p$, (4.2) becomes:

$$\frac{\partial \dot{z}_r^p}{\partial z_v^q} = \frac{\eta}{\tau^p} z_r^p \left(\sum_{s\in\mathcal{R}} \frac{\partial}{\partial z_v^q} \left(c_s^p(z) - c_r^p(z)\right) z_s^p \right) - \delta_{r,v}^{p,q}. \tag{4.25}$$

Notice that all the entries of $J_\eta(z)$ are continuous in η. Now, we can rewrite $J_\eta(z)$ in the following way:

$$J_\eta(z) = \eta M(z(\eta)) - I_{\mathcal{R}\cdot\mathcal{P}}, \quad M(z(\eta)) \in \mathbb{R}^{\mathcal{R}\cdot\mathcal{P}\times\mathcal{R}\cdot\mathcal{P}}, \tag{4.26}$$

where the entries of M at equilibrium are of the form:

$$M_{(p-1)\cdot R+r,(q-1)\cdot R+v}(z(\eta)) = \frac{z_r^p}{\tau^p} \left(\sum_{s\in\mathcal{R}} \frac{\partial}{\partial z_v^q} \left(c_s^p(z) - c_r^p(z)\right) z_s^p \right). \tag{4.27}$$

We now want to show that as $\eta \to +\infty$, $J_{\eta_n}(\tilde{z}_n)$ converges to $-I_{\mathcal{R}\cdot\mathcal{P}}$. This boils down to show that $\eta_n M(\tilde{z}_n) \to 0,\ \eta \to +\infty$, which is equivalent to show that the entries of $M(\tilde{z}_n)$ converges to 0 faster than linearly. (4.14) and (4.16) describe the evolution of the fixed point $\tilde{z}_n$ as a function of η and (4.17) and (4.18) tell us how a population moves on the minimum cost path. Using the same notation of (4.17), we can see that:

- when $r = h$, $(\tilde{z}_n)_r^p$ converges exponentially fast to τ^p;
- when $r \neq h$, $(\tilde{z}_n)_r^p$ converges exponentally fast to 0.

Now, if we look at (4.27), we can notice that:

- if $(\tilde{z}_n)_r^p \to 0$, then also $M_{(p-1)\cdot R+r,(q-1)\cdot R+v}(\tilde{z}_n)$ approaches 0 exponentially fast;

- if $(\tilde{z}_n)^p_r \to \tau^p$, then:
  - all the terms containing $(\tilde{z}_n)^p_s$, $r \neq s$, will go to 0 exponentially fast;
  - the term contaning $(\tilde{z}_n)^p_s$, $r = s$, is always zero.

Therefore, all the entries of $M(\tilde{z}_n)$ will approach zero exponentially fast, thus $\eta_n M(\tilde{z}_n) \to 0$, $\eta \to +\infty$. Finally, this means that $\lim_{\eta \to +\infty} J_{\eta_n}(\tilde{z}_n) \to -I_{\mathcal{R}\cdot\mathcal{P}}$. Hence, for a sufficiently low level of noise, all the eigenvalues of the jacobian matrix will have negative real part. The application of Theorem 2.3 concludes the proof. $\square$

Remark. It is worth to mention that the author in [15] shows a very similar result for a broader class of Wardrop equilibria. We are now going to restate in terms of our notation the definition of this class of equilibria and we will then highlight the difference between the two results. Before proceeding, let us indicate as $C(z)$ the vector containing all costs, as $DC(z) \in \mathbb{R}^{\mathcal{R}\cdot\mathcal{P}\times\mathcal{R}\cdot\mathcal{P}}$ the matrix containing all the cost derivatives with respect to the components of the path flow distribution vector z and let $\mathcal{Z}$ be the space of feasible path flow distribution vectors. Consider a routing game $(\mathcal{G}_{(o,d)}, \mathcal{P}, \{\tau^p\}_{p=1}^P, D)$. We say that z^* is a *regular Taylor evolutionarily stable state* (*ESS*) for the game if:

- z^* is a *quasi-strict equilibrium*:

$$z^p_r > 0, z^p_s = 0 \quad \Rightarrow \quad c^p_r(z^*) < c^p_s(z^*); \tag{4.28}$$

- the following condition is fulfilled:

$$(y - z^*)'C(z^*) = 0, y \in \mathcal{Z} \setminus \{z^*\} \quad \Rightarrow \quad (y - z^*)'DC(z^*)(y - z^*) < 0. \tag{4.29}$$

Notice that pure strategy Wardrop equilibria are regular Taylor ESS. The aforementioned result in [15] states that regular Taylor ESS admit a neighborhhod in which the logit dynamics (4.1) has a unique asymptotically stable fixed point and that this point actually converges to the ESS in the limit of zero noise. Although this result seems to be identical to Proposition 4.4, we want to underline that the latter does not require the the regular Taylor ESS to belong to int($\mathcal{Z}$) (internal region of $\mathcal{Z}$). Actually, pure strategy Wardrop equilibria lay on the border of $\mathcal{Z}$. This implies that our result is original.

Chapter 5

Conclusion

In this thesis, we provided three results about the asymptotic behavior of the deterministic logit dynamics associated to heterogeneous routing games. The interest behind this type of models is due to their ability in describing real-word scenarios in which users on a trasportation network may choose the route to travel by considering different criteria, such as travel time, fuel consumption, payment of tolls, etc.

The first result we proved guarantees the existence and uniqueness of a globally asymptotically stable fixed point for the dynamics, in the limit of infinite noise. While this result holds for every value of noise for single population routing games and for heterogeneous routing games on (series of) simple graph(s), we stress the fact that for heterogeneous routing games on arbitrary graphs the results holds only in the limit of vanishing noise.

The second theorem shows that the fixed points of the logit dynamics approach a subset of the set of Wardrop equilibria of the game and that pure strategy Wardrop equilbria are always included in it.

Finally, the third and last result provides additional information on the stability of the fixed points of the logit dynamics, for a sufficiently low noise level, when they are converging to a pure strategy Wardrop equilibria of the heterogeneous routing game.

There are several other paths that might be explored in order to continue this thesis and to extend our knowledge about heterogeneous routing games and the asymptotic behavior of the deterministic logit dynamics associated to it. The most natural follow-up to this thesis would be to understand if and how uniqueness of the Wardrop equilibria depends on the number of populations involved in the game and the network topology. Moreover, it would be interesting to understand whether it is possible to extend Theorem 4.4, in order to be able to deduce stability of fixed points of the dynamics also when they are converging to other types of Wardrop equilibria (non pure strategy).

Additional results about the logit dynamics might be provided for heterogeneous routing games. For example, an in-depth analysis of the behavior of the deterministic logit dynamics in the case of multiple origin-destination pairs would allow the description of even more realistic scenarios.

Finally, it would also be very interesting to associate heterogeneous routing games to other types of learning dynamics and to study the same problems we treated in this thesis in these new settings.

Appendix A

Proof of Proposition 3.1

First of all, we can find for each population the routes that it will never travel along. This routes are the whose cost is always higher than the one of another route, for every configuration of the aggregate route flow distribution. This can be done by resorting to the cost functions in (3.14). Let us consider population 1. This population will never run across r_2 and r_3 at a Wardrop equilibrium. In fact:

$$2z_1 + z_2 + 38 \leq z1 + 2z_2 + 119, \quad z_1 - z_2 \leq 81,$$

where for simplicity of notation we omit the index *agg* to indicate aggregate flows. By looking at the throughput constraints, one can see that this condition always stritly holds. Hence, $c_1^1 \leq c_2^1, \forall z$. Analogously, for r_3 holds that:

$$2z_1 + z_2 + 38 \leq \tau - z_1 - z_2 + z_3 + 119, \quad 3z_1 + 2z_2 - z_3 \leq \tau + 81.$$

This condition always strictly holds, hence $c_1^1 \leq c_3^1, \forall z$. Moreover, it turns out that r_2 and r_3 will always be more expensive of r_4, too:

$$\begin{gathered} z_1 + 2z_2 + 119 \leq 2\tau - 2z_1 - 2z_2 - z_3 + 38 \\ 3z_1 + 4z_2 + z_3 \leq 2\tau - 81 = -\tfrac{373}{5} < 0 \\ \Rightarrow c_4^1 < c_2^1, \ \forall z. \end{gathered}$$

$$\begin{gathered} \tau - z_1 - z_2 + z_3 + 119 \leq 2\tau - 2z_1 - 2z_2 - z_3 + 38 \\ z_1 + 2z_2 + 2z_3 \leq \tau - 81 = -\tfrac{389}{5} < 0 \\ \Rightarrow c_4^1 < c_3^1, \ \forall z. \end{gathered}$$

Let us now consider population 2. This population will never travel along r_2 and r_4 at a Wardrop equilibrium. In fact:

$$\begin{gathered} 21z_1 + z_2 + 19 \leq z_1 + 2z_2 + 119, \\ 20z_1 - z_2 \leq 100 \\ \Rightarrow \ c_1^2 < c_2^2, \ \forall z. \end{gathered}$$

$$\begin{gathered}21z_1 + z_2 + 19 \leq 2\tau - 2z_1 - 2z_2 - z_3 + 119,\\ 23z_1 + 3z_2 + z_3 \leq 2\tau + 100\\ \Rightarrow\ c_1^2 < c_4^2,\ \forall z.\end{gathered}$$

$$\begin{gathered}z_1 + 2z_2 + 119 \leq \tau - z_1 - z_2 + z_3 + 40,\\ 2z_1 + 3z_2 - z_3 \leq \tau - 79 = -\tfrac{379}{5} < 0\\ \Rightarrow\ c_3^2 < c_2^2,\ \forall z.\end{gathered}$$

$$\begin{gathered}2\tau - 2z_1 - 2z_2 - z_3 + 119 \leq \tau - z_1 - z_2 + z_3 + 40,\\ \tau + 79 \leq z_1 + z_2 + 2z_3\\ \Rightarrow\ c_3^2 < c_4^2,\ \forall z.\end{gathered}$$

Finally, population 3 will never run across r_1 and r_3 at a Wardrop equilibrium:

$$\begin{gathered}2z_1 + z_2 + 119 \leq z_1 + 2z_2 + 40,\\ z_1 - z_2 \leq -79,\\ \Rightarrow\ c_1^3 > c_2^3,\ \forall z.\end{gathered}$$

$$\begin{gathered}z_1 + 2z_2 + 40 \leq \tau - z_1 - z_2 + z_3 + 119,\\ 2z_1 + 3z_2 - z_3 \leq \tau + 79,\\ \Rightarrow\ c_3^3 > c_2^3, \forall z.\end{gathered}$$

$$\begin{gathered}2z_1 + z_2 + 119 \leq 21\tau - 21z_1 - 21z_2 - 20z_3 + 19,\\ 23z_1 + 23z_2 + 20z_3 \leq 21\tau - 100 = -\tfrac{164}{5} < 0,\\ \Rightarrow\ c_1^3 > c_4^3, \forall z.\end{gathered}$$

$$\begin{gathered}\tau - z_1 - z_2 + z_3 + 119 \leq 21\tau - 21z_1 - 21z_2 - 20z_3 + 19,\\ 20z_1 + 20z_2 + 21z_3 \leq 20\tau - 100 = -36 < 0,\\ \Rightarrow\ c_3^3 > c_4^3,\ \forall z.\end{gathered}$$

Therefore, we find out that a Wardrop equilibrium must satisfy to the following condition:

$$z_2^1 = 0 \wedge z_3^1 = 0,\ z_2^2 = 0 \wedge z_4^2 = 0,\ z_1^3 = 0 \wedge z_3^3 = 0, \tag{A.1}$$

Then, the cost functions may be written as follows thanks to (A.1) (for every population, we consider only the costs associated to the paths that might be travelled at a Wardrop equilibrium):

$$\begin{aligned}c_1^1 &= 2(z_1^1 + z_1^2) + z_2^3 + 38,\\ c_4^1 &= 2\tau - 2(z_1^1 + z_1^2) - 2z_2^3 - z_3^2 + 38,\\ c_1^2 &= 21(z_1^1 + z_1^2) + z_2^3 + 19,\\ c_3^2 &= \tau - (z_1^1 + z_1^2) - z_2^3 + z_3^2 + 40,\\ c_2^3 &= z_1^1 + z_1^2 + 2z_2^3 + 40,\\ c_4^3 &= 21\tau - 21(z_1^1 + z_1^2) - 21z_2^3 - 20z_3^2 + 19.\end{aligned} \tag{A.2}$$

Let us assume $z_4^1 = 0$, so that the entire population 1 travels along r_1. Then:

- population 2:

$$
\begin{gathered}
c_1^2 = 21(z_1^1 + z_1^2) + z_2^3 + 19 \leq \tau - (z_1^1 + z_1^2) - z_2^3 + z_3^2 + 40 = c_3^2, \\
22z_1^2 + 2z_2^3 - z_3^2 \leq \tau - 22\tau^1 + 21 = -\tfrac{11}{5} < 0.
\end{gathered}
$$

  Notice that this constraint is infeasible, since $0 \leq z_3^2 \leq 1$. This means that if population 1 entirely runs across r_1, then population 2 will use only r_3.

- population 3:

$$
\begin{gathered}
c_2^3 = z_1^1 + z_1^2 + 2z_2^3 + 40 \leq 21\tau - 21(z_1^1 + z_1^2) - 21z_2^3 - 20z_3^2 + 19 = c_4^3, \\
22z_1^2 + 23z_2^3 + 20z_3^2 \leq 21\tau - 22\tau^1 - 21, \\
23z_2^3 \leq 21\tau - 22\tau^1 - 41 = -\tfrac{1}{5} < 0.
\end{gathered}
$$

  Notice that this constraint is infeasible, since $z_2^3 \geq 0$. Hence, if populations 1 and 2 use only r_1 and r_3, respectively, then population 3 will entirely travel along r_4.

Therefore, we obtained the following Wardrop equilibrium:

$$
\begin{aligned}
z^1 &= \left(\frac{6}{5},0,0,0\right), \\
z^2 &= (0,0,1,0), \\
z^3 &= (0,0,0,1),
\end{aligned}
$$

which corresponds to (3.18).

Let us now assume $z_1^1 = 0$, so that the entire population 1 travels along r_4. Then:

- population 2:

$$
\begin{gathered}
c_1^2 = 21(z_1^1 + z_1^2) + z_2^3 + 19 \leq \tau - (z_1^1 + z_1^2) - z_2^3 + z_3^2 + 40 = c_3^2, \\
22z_1^2 + 2z_2^3 - z_3^2 \leq \tau + 21 = \tfrac{121}{5}.
\end{gathered}
$$

  Notice that this condition always holds, since $22z_1^2 + 2z_2^3 - z_3^2 \leq 24 \leq 121/5$. Then, if population 1 flows entirely along r_4, population 2 will use only r_1.

- population 3:

$$
\begin{gathered}
c_2^3 = z_1^1 + z_1^2 + 2z_2^3 + 40 \leq 21\tau - 21(z_1^1 + z_1^2) - 21z_2^3 - 20z_3^2 + 19 = c_4^3, \\
23z_2^3 \leq 21\tau - 22\tau^2 - 21 = \tfrac{121}{5}.
\end{gathered}
$$

  This constraint is always respected, since $z_2^3 \leq 1 < 121/115$. Then, if populations 1 and 2 entirely run across on r_4 and r_1, respectively, population 3 will flow entirely on r_2.

Therefore, we obtained the following Wardrop equilibrium:

$$\begin{aligned} z^1 &= \left(0,0,0,\frac{6}{5}\right), \\ z^2 &= (1,0,0,0), \\ z^3 &= (0,1,0,0), \end{aligned}$$

which corresponds to (3.19). Finally, by plugging z in the cost functions of population 1, we see that both configurations (3.18) and (3.19) are minimum cost solutions.

Now, we show that the game admits a third equilibrium in which all the populations uses both their allowed paths at Wardrop equilibrium, hence:

- r_1 and r_4 for population 1;
- r_1 and r_3 for population 2;
- r_2 and r_4 for population3.

First of all, let us write the condition such that both accessible paths share the same cost, for every population:

- population 1:

  $$c_1^1 = 2(z_1^1 + z_1^2) + z_2^3 + 38 = 2\tau - 2(z_1^1 + z_1^2) - 2z_2^3 - z_3^2 + 38 = c_4^1,$$
  $$4(z_1^1 + z_1^2) + 3z_2^3 + z_3^2 = \tfrac{32}{5}.$$

  Now, since $\tau^2 = 1$ and population 2 can only use r_1 and r_3, it holds that $z_1^2 = \tau^2 - z_3^2$. Therefore:

  $$4z_1^1 + 3z_1^2 + 3z_2^3 = \frac{27}{5}. \tag{A.3}$$

- population 2;

  $$21(z_1^1 + z_1^2) + z_2^3 + 19 = \tau - (z_1^1 + z_1^2) - z_2^3 + z_3^2 + 40,$$
  $$22(z_1^1 + z_1^2) + 2z_2^3 - z_3^2 = \tfrac{121}{5}.$$

  Since $z_1^2 = \tau^2 - z_3^2$, then:

  $$22z_1^1 + 23z_1^2 + 2z_2^3 = \frac{126}{5}. \tag{A.4}$$

- population 3:

  $$z_1^1 + z_1^2 + 2z_2^3 + 40 = 21\tau - 21(z_1^1 + z_1^2) - 21z_2^3 - 20z_3^2 + 19,$$
  $$22(z_1^1 + z_1^2) + 23z_2^3 + 20z_3^2 = \tfrac{231}{5}.$$

Thanks to $z_1^2 = \tau^2 - z_3^2$, we find that:

$$22z_1^1 + 2z_1^2 + 23z_2^3 = \frac{131}{5} \tag{A.5}$$

Now, (A.3), (A.4) and (A.5) define a linear system in three variables, whose unique solution is given by ($z_1^1 = 3/5$, $z_1^2 = 10/21$, $z_2^3 = 11/21$). Therefore, we find that the following route flow ditribution is a Wardrop equilibrium:

$$\begin{aligned} z^1 &= \left(\frac{3}{5}, 0, 0, \frac{3}{5}\right), \\ z^2 &= \left(\frac{10}{21}, 0, \frac{11}{21}, 0\right), \\ z^3 &= \left(0, \frac{11}{21}, 0, \frac{10}{21}\right), \end{aligned}$$

which corresponds to (3.20).

We now want to show that (3.18), (3.19) and (3.20) are the only equilibria of the game. To do this we first of all observe that population 1 dominates the others, in the sense that when this is distributed on one of the two roads available to it, it is automatically determined which are the roads that the other two populations will travel. Then, it suffices to prove that when population 1 uses both its available path, then the other populations do the same. Let us now assume that population 2 entirely run across r_1 ($z_1^2 = 1$). Let us verify whether populations 1 and 3 can flow on both the paths available for them. By resorting to (A.2), we obtain the following linear system:

$$\begin{cases} 4z_1^1 + 3z_2^3 = \frac{12}{5} \\ 22z_1^1 + 23z_2^3 = \frac{121}{5} \end{cases}$$

The solution of this linear system does not satisfy to the throughput constraints, hence it is infeasible. Let us now assume $z_3^2 = 1$. The associated linear system is the following:

$$\begin{cases} 4z_1^1 + 3z_2^3 = \frac{12}{5} \\ 22z_1^1 + 23z_2^3 = \frac{131}{5} \end{cases}$$

Also in this case we got an infeasible solution. Let us now consider population 3 and assume that it entirely flows across r_2 ($z_2^3 = 1$). In order for the populations 1 and 2 to be distributed on both available paths, the following system must be satisfied:

$$\begin{cases} 4z_1^1 + 3z_2^3 = \frac{12}{5} \\ 22z_1^1 + 23z_2^3 = \frac{116}{5} \end{cases}$$

As before, the solution of this linear system does not satisfy to the throughput constraints, hence it is infeasible. The same thing holds when we consider the linear system associated to the condtion $z_4^3 = 1$:

$$\begin{cases} 4z_1^1 + 3z_2^3 = \frac{27}{5} \\ 22z_1^1 + 23z_2^3 = \frac{126}{5} \end{cases}$$

Finally, we show that when populations 2 and 3 entirely run across one of the available paths we get (3.18) and (3.19).

- $z_1^2 = z_2^3 = 1$:

  $$c_1^1 = 2z_1^1 + 41$$
  $$c_4^1 = \frac{203}{5} - 2z_1^1$$

  It turns out that population 1 cannot use both paths in this configuration, since it does not exist z_1^1 satisfying to the throughput constraints such that $c_1^1 = c_4^1$. Moreover, it holds that $c_1^1 > c_4^1, \forall z_1^1 \in [0{,}6/5]$. Therefore, we obtain (3.19).

- $z_1^2 = z_4^3 = 1$:

  $$c_1^1 = 2z_1^1 + 40$$
  $$c_4^1 = \frac{232}{5} - 2z_1^1$$

  By equaling the two costs, we obtain that:

  $$4z_1^1 = \frac{32}{5}$$

  There are no feasible values of z_1^1 satisfying this equation. Nevertheless, it holds that $c_1^1 < c_4^1, \forall z_1^1 \in [0{,}6/5]$. Hence, $z_1^1 = 1, z_4^1 = 0$. This implies that $c_1^2 > c_3^2$, which is impossible, since we assumed $z_1^2 = 1, z_3^2 = 0$. Therefore, it cannot exist a route flow distribution such that $z_1^2 = z_4^3 = 1$.

- $z_3^2 = z_2^3 = 1$:

  $$c_1^1 = 2z_1^1 + 39$$
  $$c_4^1 = \frac{207}{5} - 2z_1^1$$

  By equaling the two costs, we obtain that:

  $$4z_1^1 = \frac{12}{5}$$

  This equation is satisfied when $z_1^1 = z_4^1 = 3/5$. Nevertheless, by plugging this values into the cost functions c_2^3 and c_4^3, we get that $c_2^3 > c_4^3$, which is in contrast with the initial assumptions. Moreover:

- if $z_1^1 = 1, z_4^1 = 0$, then$c_1^1 > c_4^1$, which is impossible;
- if $z_1^1 = 0, z_4^1 = 1$, then $c_1^1 < c_4^1$, which is impossible.

Hence, it cannot exist a configuration such that $z_3^2 = z_2^3 = 1$.

- $z_3^2 = z_4^3 = 1$:

$$c_1^1 = 2z_1^1 + 38$$
$$c_4^1 = \frac{217}{5} - 2z_1^1$$

By equaling the two costs, we obtain that:

$$4z_1^1 = \frac{27}{5}$$

There are no feasible values of z_1^1 satisfying this equation. Moreover, within such configuration it holds that $c_1^1 < c_4^1, \forall z_1^1 \in [0,6/5]$. Hence, we obtain (3.18).

Bibliography

[1] D. Acemoglu, A. Makhdoumi, A. Malekian, and A. Ozdaglar, *Informational Braess paradox: the effect of information on traffic congestion*, Operations Research, vol. 66, no. 4, pp. 893–917, 2018.

[2] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, *Network flows: Theory, algorithms, and applications*, Prentice Hall, 1993.

[3] S. Amin, J. Liu, M. Wu, *Informational aspects in a class of Bayesian congestion games*, American Control Conference (ACC). IEEE, 2017, pp. 3650–3657

[4] A. Bacciotti, *Analisi della stabilità*, Quaderni dell'UMI 49, Pitagora editrice, Bologna, 2006

[5] V. & F. Bayart, *Point fixe, et théorèmes du point fixe*, Bibmath.net. Archived, 2008.

[6] A. M. Bayen, N. Laurent-Brouty, J. Thai,*Negative externalities of GPS-enabled routing applications: a game theoretical approach*, IEEE 19th International Conference on Intelligent Transporta- tion Systems (ITSC). IEEE, 2016, pp. 595–601.

[7] M. Beckmann, C. B. McGuire, and C. B. Winsten, *Studies in the economics of transportation*, Tech. Rep., 1956.

[8] F. Bullo, P. Cisneros-Velarde, S. Jafarpour, *Weak and Semi-Contraction for Network Systems and Diffusively-Coupled Oscillators*, arXiv:2005.09774, 2020

[9] C. Canuto, A, Tabacco, *Analisi matematica I - Teoria ed esercizi*, 4^a edizione, Springer Verlag, 2014

[10] L.Cianfanelli, G. Como, *On stability of users equilibria in heterogeneous routing games*, IEEE 58th Conference on Decision and Control (CDC), 2019

[11] R. Cole, Y. Dodis, and T. Roughgarden, *Pricing network edges for heterogeneous selfish users*, Proceedings of the thirty-fifth annual ACM symposium on Theory of computing. ACM, 2003, pp. 521–530.

[12] L.Fleischer, K.Jain and M.Mahdian, *Tolls for heterogeneous selfish users in multicommodity networks and generalized congestion games*, 45th Annual IEEE Symposium on Foundations of Computer Science. IEEE, 2004, pp. 277–285.

[13] H. Konishi, *Uniqueness of User Equilibrium in Transportation Networks with Heterogeneous Commuters*, Transportation Science 38(3):315-330, 2004

[14] I. Milchtaich, *Topological conditions for uniqueness of equilibrium in networks*,

Mathematics of Operations Research, vol. 30, no. 1, pp. 225–244, 2005.

[15] W. H. Sandholm, *Population Games and Evolutionary Dynamics*, MIT Press, 2010

[16] D. Schmeidler, *Equilibrium points of nonatomic games*, Journal of statistical Physics, vol. 7, no. 4, pp. 295–300, 1973.

[17] G. Söderlind, *The logarithmic norm. History and modern theory*, BIT Numerical Mathematics, 46(3):631-652, 2006

[18] J. G. Wardrop, *Road paper: some theoretical aspects of road traffic research*, Proceedings of the institution of civil engineers, vol. 1, no. 3, pp. 325–362, 1952.