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THE VALUE AT RISK MEASURE FOR FINANCIAL PORTFOLIO RISK MANAGEMENT

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INTRODUCTION

The problem of measurement and management of risk has ancient origin in finance which can trace back to the end of the Middle Ages and the beginning of Renaissance when the first banks were born in order to finance commercial operations by investing their own and depositors' money. However, similar notions to the one of Value at Risk did not emerge until the late 1980s. A first track of such a concept was found in 1990 when the president of J.P. Morgan, Dennis Weatherstone, disappointed by the length of the report he received every day, wanted a simple summary of the bank's total exposure across its entire trading portfolio before the closing of the markets. From this need, the "4:15 report" was born, representing the time by which it was placed on Weatherstone's desk, and it constituted the first time in which the concept of Value at Risk was formally introduced. Moreover, in 1994 J.P. Morgan posted on the Internet a simplified version of their financial risk calculation system, called RiskMetrics™ in which Value at Risk was set as an industry-wide standard. From this point on, the V.a.R. application in the financial world has encountered a rapid development especially thanks to the Amendment to Basel I published in 1996 in which it was introduced the calculation of the minimum capital requirement for financial institutions to deal with the risk related to their trading book. In this paper it was specified that banks could estimate the capital requirement for market risk using two different approaches. The first one was the standardized method which was relatively simple and often used by small banks, while the second was the internal model which involves the use of validated internal models based on Value at Risk. Subsequently, with the development of Basel 2 and Basel 3 significant changes have occurred in the calculation of the capital requirement since it was refined the estimation of Value at Risk with a more conservative approach in order to improve the capability of financial institutions to cope with unexpected losses on their trading portfolio.

Hence, the scope of this thesis is to analyze in depth the main methods present in the academic literature to estimate the Value at Risk measure for a portfolio made up by financial assets in order to take a snapshot of the risk contained in such a portfolio. For each of them it is illustrated the underlying assumptions, the steps required for V.a.R. calculation and the main advantages and drawbacks. The thesis is divided into five chapters and for each of them it is provided a brief summary as follows.

In the first chapter it is given an overview about the risk in finance, especially focusing on market risk. Then, it is debated about the importance to quantify and manage risk for financial institutions and society in general, listing all the implications of a poor risk management. Subsequently, it is explained the concept of risk measures clarifying which properties they should respect to be defined as "coherent".

In the second chapter it is introduced the Value at Risk measure, which is one of the most popular measures used by regulators and financial firms to gauge the amount of capital needed to cover possible losses. In addition, it is defined a financial portfolio from a mathematical point of view and then it is presented the notion of the Loss distribution for a given time horizon. Moreover, it is given some comments about Value at Risk highlighting its drawbacks and mentioning alternate options such as Expected Shortfall.

In the third chapter it is presented the parametric approach which builds a model for the joint distribution of changes in market variables in order to estimate the model parameters of the Loss distribution. To start with, it is considered that daily returns of the market variables are normally distributed. Then, it is assumed that the change in portfolio's value is linearly dependent on changes the underlying market variables returns so that the Loss distribution can be considered multivariate normal. Hence, the Value at Risk measure is estimated as a quantile of such a distribution. Successively, it is enhanced the previous approximation considering a quadratic model which takes into account also the curvature of the relationship between the portfolio's value and the underlying market variables returns. Thus, the Loss distribution can no longer supposed to be normal and its quantile is estimated with a numerical procedure basing on its expected value, variance and skewness. Finally, it is shown how to build the Loss distribution using Monte Carlo simulation which is a procedure for sampling random outcomes for a given stochastic process.

In the fourth chapter it is talked about the historical simulation method, which has become the most popular method for calculating the Value at Risk measure since it does not require any assumption about the shape of the Loss distribution. In fact, it uses only past returns of the assets included in the financial portfolio in order to build an empirical distribution for losses. Furthermore, it is introduced two extensions of this method which aim to overcome the main drawbacks and improve the results obtained with the traditional historical simulation.

In the last chapter of the thesis it is presented a practical example showing how to estimate the V.a.R. measure of a given financial portfolio using all the methods presented so far. In addition, it is illustrated the benefits of diversification comparing the Value at Risk computed at the portfolio level with the one calculated for each security individually. Lastly, it is explained the concept of back testing and then it is implemented in the previous example in order to draw some conclusions about the adequacy of the chosen methods.

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1. FINANCIAL RISK

Financial risk can be defined as any financial event that may adversely affect the organization's ability to execute its strategies and achieve its objectives. This concept is strongly related to uncertainty and hence to the notion of probability.

An axiomatic definition of probability was given in Kolmogorov (1933) in which a probabilistic model is described by a triplet $(\Omega, \mathcal{F}, P)$ called the measure space. The term Ω is the sample space, $\mathcal{F}$ represents the set of all events and P denotes the probability measure. To model situations in which there is randomness, it is defined a one-period risky position X to be a function on the probability space $(\Omega, \mathcal{F}, P)$ and it is called a random variable. Most of the modelling of a risky position X is related to its distribution function $F_X(x) = P(X \leq x)$, which is the probability that the value of the risk X is less than or equal to a given number x. If time is introduced, several risky positions would then be denoted by a random vector $\boldsymbol{X} = (X_1, X_2, \dots, X_d)$ leading to the notion of stochastic processes which is termed X_t.

In the financial industry risk can be categorized in three type of risks which are market risk, credit risk, and operational risk. Market risk is the risk of incurring in portfolio's losses due to changes in the value of the underlying market variables and it will be further debated in this chapter. Credit risk deals with the possibility of not receiving the promised repayments on outstanding investments such as loans and bonds, due to the financial default of the borrower. Operational risk is the risk of losses resulting from inadequate internal processes, people, and systems or from external events. However, even if they have their own explanation, the boundaries of these three risk categories are not always clearly defined.

1.1 Market risk

Market risk is the risk of incurring in portfolio's losses due to changes in the value of the underlying market variables, which can be stocks, bonds, exchange rates, stock indices or commodities. More specifically, market risk can be classified in four categories, which are:

- **Price risk**: Basically, financial assets' prices depend on market trends and on expectations of the investors. More in detail, for the Law of One Price, the price of a security should equal the present value of the expected cash flows an investor will receive from owning it. For example, cash flows from a stock can be dividend payments or capital gains, while those from bonds can be interest payments or principal repayment. However, future cash flows cannot be precisely known and prices are affected by the perceptions of the investors which are driven by the flow of publicly available information. In this framework, price risk is defined as the risk which reflects the adverse fluctuations of price of the underlying financial assets. For instance, in *Figure 1* it is shown the pattern followed by Netflix stock price (NASDAQ: NFLX) between 20th and 23rd April 2021.

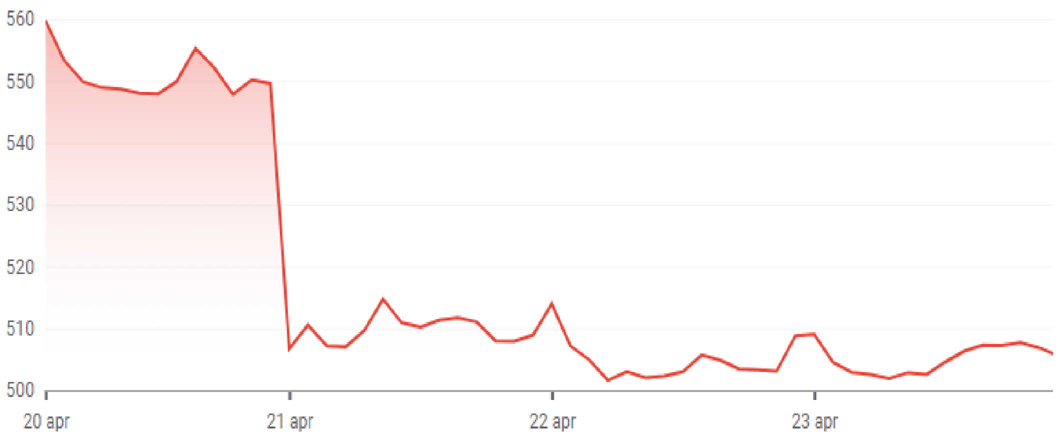

Figure 1: NFLX Stock Price (Source: Google Finance)

As it can be noticed from the graph, between the 4:00 PM of 20th April 2021 and the 9:30 AM of 21st April 2021, Netflix stock price decreased from

$549,57 to $506,68 experiencing a loss of $42,89 (-7,8%). Just before that stock price started to drop, Netflix published the financial statements concerning the first quarter of 2021, which showed an increase in the subscribers far below the expectations of the analysts. Therefore, many investors were scared by a possible setback in the growth of the company and they sold their stocks causing a so fast fall in the price.

- **Exchange rate risk**: The exchange rate is the ratio with which one currency can be traded for another one and it depends on many macroeconomic factors such as the differences between the interest rates of different countries and their monetary policies. Usually, the exchange rate is computed as the current spot price in the domestic currency of one unit of the foreign currency. Hence, exchange rate risk is the risk which depends on fluctuations in the exchange rate and it concerns those financial instruments denominated in foreign currencies. If there is a depreciation of a foreign currency with respect to the domestic one, the value of a financial instrument denominated in that foreign currency decreases accordingly. Basically, it is possible that although an asset denominated in a foreign currency is achieving excellent performance, an unfavorable movement in the exchange rate may cause a reduction in the value of the financial instrument for the investor. For instance, *Figure 2* shows the pattern followed by the exchange rate between Great Britain Pound (GBP) and United States Dollars (USD) during the second half of June and the beginning of July in 2016.

Figure 2: GBP/USD Exchange Rate (Source: Yahoo! Finance)

As it can be seen from the graph, between 23rd and 28th June the exchange rate GBP-USD dropped from 1,4789 to 1,3235 experiencing a loss of 0,1554 (-10,5%) in only three trading days. Such a fast devaluation of the GBP with respect to USD was caused by the publication of the outcome of the referendum made on 23rd June 2016 concerning the exit of the Great Britain from the European Union (Brexit). The uncertainty surrounding Brexit and the risks of a "no deal" (i.e. a British exit from the European Union without a deal) pushed the GBP down on the currency markets, with strong repercussions on the UK economy. Therefore, an US investors who held one million GBP in a British bank account on 23rd June 2016, saw the value of this deposit, calculated in the domestic currency (USD), dropped by around 10,5% within three trading days.

- **Interest rate risk**: Interest rate is defined as the percentage of the amount of money a borrower promises to pay to the lender in addition to the principal. For any given currency, many different types of interest rates are regularly quoted on the market using the convention of the effective annual yield (EAY), which indicates the actual amount of interest that will be earned at the end of one year. Basically, interest rates depend on the equilibrium between demand and supply of money. Demand of money is affected by the level of the investments of the firms and the confidence level of the investors, while its supply is driven by the monetary policies of central banks. In this context, interest rate risk is the risk which derives from the impact of movements in interest rates on financial assets. The most exposed financial instruments on interest rate movements are fixed-rate coupon bonds. For instance, if there is an increase in interest rates, an investor who holds a fixed-rate coupon bond in his portfolio will see its price decrease. In fact, if the interest rate changes, fixed-rate securities, conversely to floating-rate coupon bonds, cannot change their coupons and therefore the price changes in order to adjust the yield to the new rate levels. In *Figure 3* it is shown the pattern followed by the 12-month USD Libor in 2008.

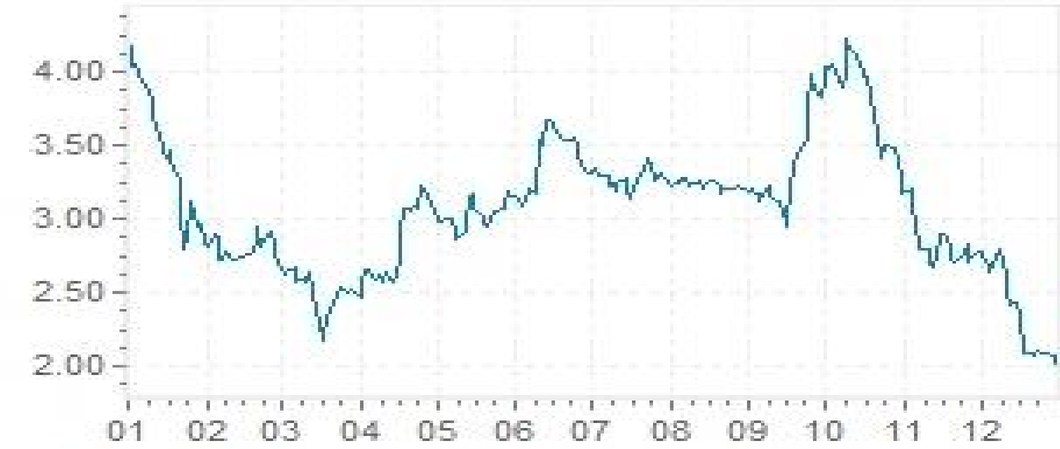

Figure 3: 12-month USD Libor (Source: Global-rates.com)

As it can be noticed from the graph, the 12-month USD Libor increased by more than 100 basis points (1 basis point=0,01%) from the mid of September until the beginning of October, when it started to decrease rapidly. Such a high volatility in the interest rate was caused by the crisis of subprime mortgages in USA which affected the global financial industry in 2008. In fact, in a first moment the interest rate rose due to the lack of liquidity in the market, while in a second moment it started a constant decline phase because a lot of money was injected in the market by the Federal Reserve (FED) in order to support the economy.

- **Spread risk**: Credit spread is given by the difference between the yields of bonds emitted by issuers with different creditworthiness but with the same maturity and liquidity. Generally, in order to compute the spread related to a specific issuer, it is utilized as reference the yield of bonds with the maximum creditworthiness such as German and U.S. government bonds, which are considered "safe". For instance, if the yields on 10-year German bonds and 10-year Italian BTP are respectively 1% and 5%, the spread between the bonds of the two countries is 400 basis points. Spread risk depends on the creditworthiness of the issuer of a bond and it occurs when changes in its price cannot be explained by movements of other market variables such as exchange rates or interest rates. More specifically, if the creditworthiness of an issuer of a bond gets worse it will increase the bond yield requested by the market and then its price will decrease. Spread risk can also occur when the creditworthiness of the issuer remains the same but there is an increase of the investors' risk aversion. In *Figure 4* it is plotted the evolution of the spread between 10-year Italian BTP and 10-year German Bund in 2018.

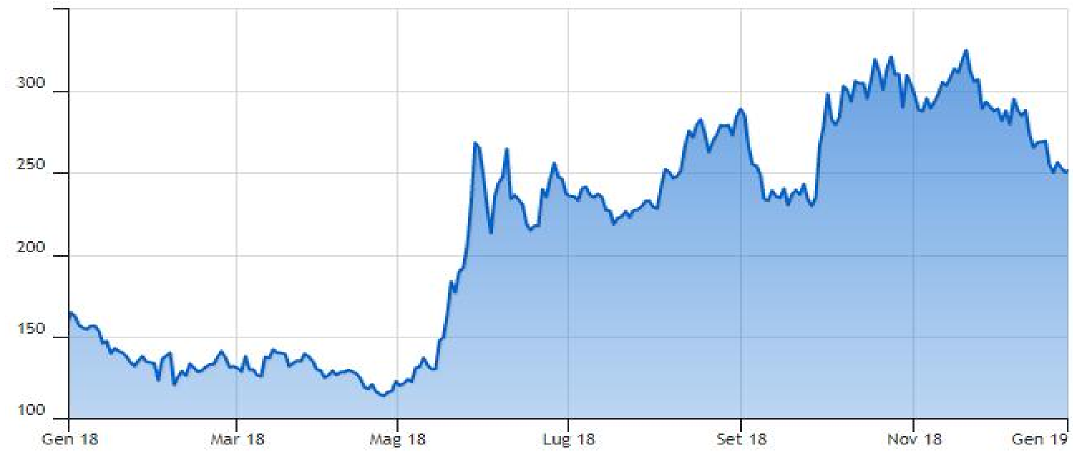

Figure 4: Spread 10-year BTP-Bund (Source: Teleborsa)

As it can be seen from the graph, the 10-year BTP-Bund spread doubled its value in May, going from 130 to 260 basis points in only one month. Such a high increase in the spread was caused by the institutional crisis which affected the Italian government and the threat of the impeachment charge made to the Italian president by political parties. Those events caused a fall in the confidence of the investors about the financial health of Italy and the chances of the country to cope with its obligations.

In the next chapters of the thesis, it will be examined in depth statistical models in order to quantify market risk, focusing mostly on price and exchange rate risk.

1.2 Quantitative risk management

Risk management is defined in Kloman (1990) as a discipline for living with the possibility that future events may cause adverse effects. The reasons for which a market player, as a financial institution or a fund manager, has to invest in quantitative risk management are various and they depend on which stakeholder is considered.

For shareholders, managing the risk is related to preserving the flow of profit and to techniques which aim to earn an adequate return on funds invested and to maintain a proper surplus of assets beyond liabilities. Companies must take risks if they want to survive and prosper. Thus, the primary responsibility of quantitative risk management is to understand the portfolio of risks that the company is currently taking and the risks it plans to take in the future. It must decide whether the risks are acceptable and, if they are not acceptable, what action should be taken in order to reduce it. On the other hand, society, governments, and regulators have a collective interest on the smooth functioning of the financial system and its stability. For society, quantitative risk management is viewed positively because it enhances this stability and safeguards its interests. Instead, government and regulators are strongly motivated by the fear of systemic risk, which means the danger that problems in a single financial institution may spill over and, in extreme situations, disrupt the normal functioning of the entire financial system. When a bank or other large financial institution does get into financial difficulties, governments have a difficult decision to make. If they allow the financial institution to fail, they are putting the financial system at risk. If they bail out the financial institution, they are sending wrong signals to the market. In fact, there is a danger that large financial institutions will be less vigilant in controlling risks if they know that they are "too big to fail" and the government will always bail them out. Therefore, it can be easily understood the crucial role played by quantitative risk management in the financial industry.

Existing approach to quantify the risk of a financial portfolio can be grouped into four different categories, which are:

- **Notional-amount approach:** The risk of a portfolio is determined as the sum of the notional values of the assets in the portfolio, where each notional value is weighted by a factor representing an assessment of the riskiness of the broad asset class to which the security belongs. This is the oldest approach to quantifying the risk of a portfolio consisting of financial assets and the advantage of the notional-amount approach concerns its apparent simplicity. However, it also presents a lot of drawbacks. First of all, the approach does not differentiate between long and short positions and there is no netting. In addition, this approach does not take into account the benefits of diversification on the overall risk of a portfolio. Finally, the notional-amount approach has problems when the portfolio contains derivative instruments, because the notional amount of the underlying asset and the value of the derivative position can differ widely.
- **Factor-sensitivity measures:** This approach gives the change in the value of a portfolio with respect to a given predetermined change in one of the underlying risk factors. The most important factor-sensitivity measures are the Greeks for portfolios of derivatives and the duration for bond portfolios. However, while these measures provide useful information about the robustness of the portfolio value with respect to certain factors, they are not capable to measure the overall riskiness of a position. Moreover, factor-sensitivity measures do not allow the aggregation of risks. In fact, for a given portfolio it is not possible to aggregate the sensitivity with respect to changes in different risk factors.
- **Risk measures based on a probability distribution:** This approach is based on statistical quantities describing the distribution of the change in the portfolio's value over a predetermined period. Risk measures based on a probability distribution are for instance the Value at Risk and the Expected

Shortfall. It is of course problematic to rely on any one particular statistic to summarize the risk contained in a portfolio. However, the advantages of this approach concern the diversification effect thanks to the aggregation property of these measures and the possibility to make comparisons among different portfolios. Instead, there are two major problems when working with probability distributions. The first one regards the fact that every statistical distribution is based on past data and if the laws governing financial markets change, these past data are of limited use in predicting future risk. The second problem is more practical and concerns the difficulty to estimate distributions accurately.

- **Scenario-based risk measures:** This approach considers a number of possible future risk-factors changes (scenarios) and risk of a portfolio is measured as the maximum loss of the portfolio under all scenarios, where certain extreme scenarios can be downweighed to mitigate their effect on the result. Scenario-based risk measures are a very useful risk-management tool for portfolios exposed to a relatively small set of risk factors. Moreover, they provide useful complementary information to measures based on statistics of the distribution. The main problem is of course to determine an appropriate set of scenarios and weighting factors. Moreover, it is difficult to make comparisons across portfolios which are affected by different risk factors.

This thesis debates about Value at Risk, which is one of the most used risk measures based on probability distributions in the financial industry. In the next paragraph, it will be explained what is a risk measure, while in the next chapters it will deepen the knowledge about the Value at Risk.

1.3 Risk measures

Fixing a probability space $(\Omega, \mathcal{F}, P)$ and a time horizon Δt, denote by $L^0(\Omega, \mathcal{F}, P)$ the set of all random variables on $(\Omega, \mathcal{F})$ which are almost finite. Financial risk is represented by a set $M \subset L^0(\Omega, \mathcal{F}, P)$ of random variables, which it is interpreted as portfolio losses over the period Δt. Furthermore, it is assumed that M is a convex cone which implies that for every $L_1 \in M, L_2 \in M, \lambda > 0$ it is verified that $L_1 + L_2 \in M$ and $\lambda \cdot L_1 \in M$. Risk measures are real-valued functions $g: M \rightarrow \mathbb{R}$ defined on such cone of random variables, satisfying certain properties. Therefore, $g(L)$ is interpreted as the amount of capital that should be added to a position with loss given by L. In Artzner et al. (1998) it is presented and justified a set of four desirable properties for measures of risk, and call "coherent" the measures satisfying these properties.

The axioms that a risk measure should satisfy in order to be called "coherent" are:

- **Translation invariance:** For all $L \in M$ and every $l \in \mathbb{R}$ the relationship:

  $$g(L + l) = g(L) + l$$

  is always satisfied. Translation invariance axiom states that by adding or subtracting a deterministic quantity l to a position leading to the loss L, the capital requirements is altered by exactly that amount l.

- **Subadditivity:** For all $L_1, L_2 \in M$ it is valid the inequality:

  $$g(L_1 + L_2) \leq g(L_1) + g(L_2)$$

  Subadditivity means that if there are two loss distributions for two portfolios, the overall loss distribution of the merged portfolio is bounded above by the sum of the losses of the individual portfolios. So, subadditivity reflects the idea that risk can be reduced by diversification. In addition, it makes decentralization of risk-management systems possible.

- **Positive homogeneity:** For all $L \in M$ and every $\lambda > 0$ it is true that:

$$g(\lambda \cdot L) = \lambda \cdot g(L)$$

Positive homogeneity implies the risk of a position is proportional to its size. Loosely speaking, if the size position is doubled then the risk capital will be doubled.

- **Monotonicity:** For $L_1, L_2 \in M$ such that $L_1 \leq L_2$, the relationship:

$$g(L_1) \leq g(L_2)$$

is always valid. Monotonicity implies that positions that lead to higher losses require more risk capital.

These axioms are not restrictive enough to specify a unique risk measure. So, the choice of a particular risk measure should presumably be made on the basis of the specific objectives of quantitative risk management. The most important purposes for which risk measures are used are:

- **Determination of risk capital and capital adequacy:** One of the principal functions of risk management in the financial sector is to determine the amount of capital a financial institution needs to hold as a buffer against unexpected future losses on its portfolio in order to satisfy a regulator, who is concerned with the solvency of the institution. A related problem is the determination by the clearing house of appropriate margin requirements for investors trading at an organized exchange.
- **Management tool:** Risk measures are often used by management as a tool for limiting the amount of risk that a unit may take within a firm. For instance, traders in a bank are often constrained by that the risk of their position should not exceed a given bound.
- **Insurance premiums:** Insurance premiums compensate an insurance company for bearing the risk of the insured claims. The size of this compensation can be viewed as a measure of the risk of these claims.

2. VALUE AT RISK MEASURE

The Value at Risk measure represents in a monetary form the level of risk at which an owner of a portfolio is subject. It has become widely used by many market players such as financial institutions, corporate treasurers, and fund managers. Under certain conditions, V.a.R. represents the maximum loss which might occur in a financial portfolio for a given period and probability. For instance, if the time horizon is 5 days and the confidence level is 99%, a V.a.R. of ten million dollars states that the maximum loss which may occur in the following 5 days will be not greater than ten million dollars with a probability of 99%.

Firstly, in this chapter it will be formally defined what is a portfolio in finance, then it will be described the Value at Risk measure of such a portfolio and finally it will be given some comments about the limitations of this measure.

2.1 V.a.R. of a financial portfolio

In finance, a portfolio is defined as a vector $\boldsymbol{h} = (q_1, q_2, \dots, q_n)$ of the quantities of the financial assets from which it is made up. It is important to notice that each q_i, with $i = 1,2, \dots, n$, can be either positive or negative in order to distinguish a long position from a short position. In the financial jargon, a long position involves owning a security or contract and there is a gain when the price of the asset increases. Instead, a short position involves selling a security or a contract and it is realized a gain when there is a decrease in the asset's price.

Denoting the current time as t, the value process of the portfolio $\boldsymbol{h}$ at time t is defined as:

$$V_{\boldsymbol{h},t} = \sum_{i=1}^{n} P_{i,t} \cdot q_i$$

where $P_{i,t}$ is the current price of the asset i which it is assumed to be observable at time t. As a consequence, also the value of the portfolio $V_{\boldsymbol{h},t}$ is known at time t. Then, considering a specific moment in the future which is denoted by $t + \Delta t$, the change in the value of the portfolio $\boldsymbol{h}$ from t to $t + \Delta t$ is represented by:

$$\Delta V_{\boldsymbol{h},t+\Delta t} = V_{\boldsymbol{h},t+\Delta t} - V_{\boldsymbol{h},t}$$

where $V_{\boldsymbol{h},t+\Delta t}$ is the value of the portfolio at time $t + \Delta t$. Unfortunately, $\Delta V_{\boldsymbol{h},t+\Delta t}$ is a random variable at time t, because it can be observed only at time $t + \Delta t$.

The distribution of $\Delta V_{\boldsymbol{h},t+\Delta t}$ is often termed profit and loss (P&L) distribution and it is used by many practitioners in finance. However, risk managers are mainly concerned with the probability of large losses and hence they often use the loss distribution instead of P&L distribution. For this purpose, it makes sense to introduce the loss of the portfolio $\boldsymbol{h}$ from t to $t + \Delta t$, which is defined as:

$$L_{\boldsymbol{h},t+\Delta t} = -\Delta V_{\boldsymbol{h},t+\Delta t}$$

As it is stated for $\Delta V_{\boldsymbol{h},t+\Delta t}$, also $L_{\boldsymbol{h},t+\Delta t}$ is a random variable at time t since it is observable only at time $t+\Delta t$. Thus, V.a.R. can be calculated as the quantile corresponding to a specific confidence level of the loss distribution.

In statistics, a point $x_0 \in \mathbb{R}$ is the α-quantile of a cumulative distribution function F if and only if the following two conditions are satisfied: $F(x_0) = \alpha$ and $F(x) < \alpha$ for all $x < x_0$. Therefore, as it is shown in *Figure 5*, using the distribution of the portfolio's loss $L_{\boldsymbol{h},t+\Delta t}$, the Value at Risk at the confidence level $\alpha \in (0,1)$ for a period of length Δt is defined as the number q for which:

$$VAR_{\boldsymbol{h},\alpha,t+\Delta t} = inf\{q \in \mathbb{R}^+ : P(L_{\boldsymbol{h},t+\Delta t} \geq q) \leq 1-\alpha\}$$

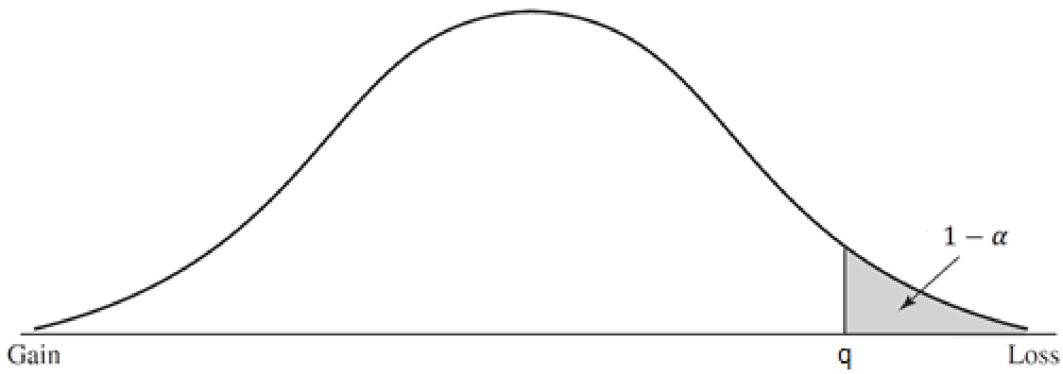

Figure 5: V.a.R. of the Loss Distribution

As it can be noticed, V.a.R. depends on two parameters: the time horizon and the confidence level. These parameters are chosen accordingly the purpose for which V.a.R. is computed, but there is not a single optimal value for them.

The time interval Δt should reflect the period over which a market player is committed to hold its portfolio. When the financial assets included in the portfolio are very liquid and actively traded, it makes sense to use a short time horizon like few days. Thus, if the Value at Risk calculated turns out to be unacceptable, the portfolio can be adjusted quickly. Instead, when V.a.R. is being calculated by the manager of a pension fund, a longer time horizon is likely to be used. This is because the portfolio is traded less actively and some of the instruments in the portfolio are less liquid. However, when the liquidity of a portfolio varies from one instrument to another, it makes sense to compute more than one V.a.R. measure, each with a different time horizon which takes into account the liquidity of a specific group of assets.

Instead, the confidence level α is chosen accordingly the risk aversion of a specific market player because holding all other conditions equal, the higher the confidence level the greater the Value at Risk. Thus, a relatively high confidence level leads to a greater protection against risk. However, the confidence level that is actually used for the V.a.R. calculation is sometimes much less than the one that is required by regulators in the financial world. In fact, as it will be shown in one of the following chapters, the standard error of estimate for the Value at Risk measure is greater when the confidence level is very high (typically greater than 99%). A general approach for increasing the confidence level is the use of the extreme value theory. This theory is a way of smoothing the tails of the probability distribution of daily changes in the portfolio's value and it leads to estimates of V.a.R. which reflect the whole shape of the tail distribution.

In addition to the choice of the time horizon and the confidence level, there is the necessity to also choose the method to calculate the Value at Risk measure. The most popular methods are the parametric approach and historical simulation, which will be extensively debated in the next chapters of the thesis.

2.2 Criticism on V.a.R.

One of the drawbacks of the Value at Risk measure regards the fact that it does not give any information about the severity of losses which occur with a probability less than $1 - \alpha$. By the way, in Taleb (2008) it is stated that:

"What I care about, with standard Value at Risk, is not the number that falls within the 99 percent probability. I care about what happens in the other 1 percent, at the extreme edge of the curve. The fact that you are not likely to lose more than a certain amount 99 percent of the time tells you absolutely nothing about what could happen the other 1 percent of the time. You could lose $51 million instead of $50 million — no big deal. That happens two or three times a year, and no one blinks an eye. You could also lose billions and go out of business. Value at Risk has no way of measuring which it will be. Something rare, something you have never considered a possibility. I call these events fat tails or black swans."

A measure that deals with this problem is Expected Shortfall (ES). It represents the expected loss during a period of length Δt conditional on the loss being worse than the V.a.R. measure. More formally, the Expected Shortfall at the confidence level α of a portfolio $\boldsymbol{h}$ calculated at time t for a period of length $t + \Delta t$ is defined as:

$$ES_{\boldsymbol{h},\alpha,t+\Delta t} = \frac{1}{1-\alpha} \cdot \int_{\alpha}^{1} VAR_{\boldsymbol{h},u,t+\Delta t} \cdot du$$

Then, instead of fixing a particular confidence level α, V.a.R. is averaged over all levels $u \geq \alpha$ and obviously Expected Shortfall is always greater than V.a.R. at the same confidence level.

If it is assumed, as it will be shown in the next chapter, that the daily loss distribution is the multivariate normal distribution with expected value 0 and variance $\sigma_{\boldsymbol{h}}^2$, the daily Expected Shortfall at the confidence level α can be written as follows:

$$ES_{\boldsymbol{h},\alpha,t+1} = \sigma_{\boldsymbol{h}} \cdot \frac{e^{-z_\alpha^2/2}}{\sqrt{2} \cdot \pi \cdot (1-\alpha)}$$

where z_α is the quantile of the standard normal distribution corresponding to the probability α. The methods used to estimate Expected Shortfall are the same as those used for the V.a.R. measure and it is shown that ES, like Value at Risk, is proportional to $\sigma_{\boldsymbol{h}}$ which is the daily volatility of the portfolio.

As it will be done with V.a.R., also the Expected Shortfall is usually estimated for a time horizon equal to one day and it is extended to a longer period Δt using the formula:

$$ES_{\boldsymbol{h},\alpha,\Delta t} = ES_{\boldsymbol{h},\alpha,t+1} \cdot \sqrt{\Delta t}$$

However, it represents an approximation unless the portfolio's losses on successive days are independent and identically distributed.

Furthermore, Value at Risk is fundamentally criticized because, in the most general case, it is not a coherent risk measure. In fact, although it satisfies the properties of translation invariance, positive homogeneity, and monotonicity, it does not meet the subadditivity property. However, it can be demonstrated that the V.a.R. measure does not satisfy this property if the assets included in the portfolio have very skewed loss distributions or they are independent but very heavy-tailed, such as defaultable bonds or options. On the other hand, V.a.R. is subadditive in the situation where the portfolio can be represented as a linear combination of the same set of underlying distributed risk factors. As it will be shown, the linear model may be a reasonable approximate model for various kinds of market variables, such as equity prices, commodity prices, currencies, and linear derivative instruments while it represents a rough approximation in case the portfolio contains non-linear derivative instruments such as options. Instead, in every case Expected Shortfall is a coherent risk measure because it satisfies all the four axioms of coherence.

Another criticism concerns the fact that V.a.R. neglects any problems related to market liquidity. By definition, a market is termed liquid if investors can buy or sell large amounts of an asset in a short time without affecting too much its price. Conversely, a market in which an attempt to trade has a large impact on price, or where trading is impossible since there is no counterparty willing to take the other side of the trade, is termed illiquid. However, it is difficult to implement in the model the effects of market illiquidity because they are hard to measure, and they depends on elusive factors such as market mood or the distribution of economic information among investors. Moreover, in illiquid markets traders are forced to close their position gradually over time to minimize the price impact of their transactions and it makes the aggregation of risk measures across portfolios impossible. Hence, in many practical situations, risk managers can therefore do not better than ignore the effect of market liquidity in computing risk measures.

3. PARAMETRIC APPROACH

So far it was not made any assumption about the probability distribution of the losses in the value of a portfolio. Then, the scope of the parametric approach is to build a mathematical model which describes the distribution function of the portfolio's loss over a specific period in order to estimate the Value at Risk measure. For this purpose, there is the need to understand the processes of the market variables included in the portfolio and analyze the relations among them. Following the market's usual terminology, it will be referred to market variables as equity prices, commodity prices, or exchange rates. Instead, derivative instruments such as forward contracts or options will be related to the underlying market variables.

In the first place, it will be supposed that, to a good approximation, the change in the value of the portfolio is linearly related to proportional changes in the market variables. Hence, it will be used a linear model to link the portfolio's loss to the returns of the underlying market variables in order to calculate the Value at Risk measure.

Successively, it will be used a quadratic model to shape the loss distribution from the returns of the market variables included in the portfolio in order to improve the estimate of the V.a.R. measure. In fact, if the portfolio contains non-linear derivative instruments, its loss cannot reasonably be considered to be linearly dependent on changes in the underlying market variables. Therefore, it is necessary to use a model for building the loss distribution which is more accurate than the linear one.

Furthermore, it will be illustrated how V.a.R. is calculated with the Monte Carlo simulation which is a more practical approach used to shape the loss distribution of a financial portfolio starting from the returns of the underlying market variables.

Finally, at the end of this chapter, it will be debated how volatilities and correlations of the market variables' returns are calculated because they play a crucial role in the calculation of the Value at Risk measure of a financial portfolio.

3.1 Linear model

Recalling the definition of the change in the value of a financial portfolio, it makes sense to relate it with the changes in the market variables which compose the portfolio. Therefore, if it is supposed that in a period of length Δt the quantity of the financial assets which compose the portfolio remains the same, the change in the portfolio's value over this period can be written as:

$$\Delta V_{\boldsymbol{h},t+\Delta t} = \sum_{i=1}^{n} (P_{i,t+\Delta t} - P_{i.t}) \cdot q_i = \sum_{i=1}^{n} \Delta P_{i,t+\Delta t} \cdot q_i$$

Therefore, in order to know the probability distribution of $\Delta V_{\boldsymbol{h},t+\Delta t}$, it is necessary to understand the value process of each market variable included in the portfolio. For this purpose, the following sections explain the models used to describe the process of the main financial instruments and what are the assumptions underlying these models.

3.1.1 Equities, Commodities and Currencies

In finance, it is a common practice to assume that the process followed by market variables such as equity prices, commodities prices, and currencies is the geometric Brownian motion, which is defined as:

$$\mathrm{d}P_t = (\mu - q) \cdot P_t \cdot \mathrm{d}t + \sigma \cdot P_t \cdot \mathrm{d}W_t$$

where:

- μ is the expected return of the asset;
- v is a term which summarizes the income ($v > 0$) or the cost ($v < 0$), expressed as a percentage of the price, provided by the asset in a unitary period of time. For instance, in case of stocks or stock indices q represents

the dividend yield, for commodities it is defined as the storage cost, while for currencies it represents the foreign risk-free interest rate;

- σ represents the volatility, which is a measure of the uncertainty about the returns provided by the asset. It is defined as the standard deviation of its return in a unitary period of time;
- W_t is the standard Weiner process which is a particular type of stochastic process with a mean change of zero and a variance rate equal to one.

An important result known as Ito's lemma, shows that if a variable x follows the Ito's process $dx_t = a(t,x) \cdot dt + b(t,x) \cdot dW_t$, in which the terms $a(t,x)$ and $b(t,x)$ depend also on x and t, a function f of the variable x and time t follows the process:

$$\mathrm{d}f = \left(\frac{\partial f}{\partial t} + a(t,x) \cdot \frac{\partial f}{\partial x} + \frac{1}{2} \cdot b(t,x)^2 \cdot \frac{\partial^2 f}{\partial x^2}\right) \cdot \mathrm{d}t + b \cdot \frac{\partial f}{\partial x} \cdot \mathrm{d}W$$

Thus, considering $f = ln(P)$, the process followed by f is:

$$\mathrm{d}ln(P_t) = \left(\mu - v - \frac{1}{2} \cdot \sigma^2\right) \cdot \mathrm{d}t + \sigma \cdot \mathrm{d}W_t$$

which is named drifted Brownian motion and it is a particular case of the Ito's process. So, the change in $ln(P)$ in a discrete time period, defined from t to $t + \Delta\mathrm{t}$, can be computed as:

$$ln(P_{t+\Delta t}) - ln(P_t) = \int_t^{t+\Delta t} \left(\mu - v - \frac{1}{2} \cdot \sigma^2\right) \cdot ds + \int_t^{t+\Delta t} \sigma \cdot dW_s$$

Firstly, it is important to notice that the terms in the left side of the above equation represents the logarithmic return of a specific market variable over a period of length Δt. Secondly, there is the need to make a distinction between the two integrals in the right side of the previous equation. In fact, the former is a Riemann-Stieltjes integral which can be easily calculated, while the second is a stochastic integral, also known as Ito's integral, which is not easy to manage. Nevertheless, since μ, v and σ are assumed to be constant it is valid that:

$$\int_t^{t+\Delta t} \left(\mu - v - \frac{1}{2}\cdot\sigma^2\right)\cdot ds = \left(\mu - v - \frac{1}{2}\cdot\sigma^2\right)\cdot \int_t^{t+\Delta t} ds = \left(\mu - v - \frac{1}{2}\cdot\sigma^2\right)\cdot \Delta t$$

$$\int_t^{t+\Delta t} \sigma \cdot dW_s = \sigma \cdot \int_t^{t+\Delta t} dW_s = \sigma \cdot (W_{t+\Delta t} - W_t)$$

and the previous equation can be rewritten as:

$$ln\left(\frac{P_{t+\Delta t}}{P_t}\right) = \left(\mu - v - \frac{1}{2}\cdot\sigma^2\right)\cdot \Delta t + \sigma \cdot (W_{t+\Delta t} - W_t)$$

From the definition of the Weiner process, the random variable $W_{t+\Delta t} - W_t$ is normally distributed with mean zero and variance rate equal to the difference between $t + \Delta t$ and t. So, it can be written:

$$W_{t+\Delta t} - W_t \sim N(0, \Delta t)$$

Therefore, it can be stated that the logarithmic return of a specific market variable over a period of length Δt is normally distributed with expected value and variance which are respectively:

$$E\left[ln\left(\frac{P_{t+\Delta t}}{P_t}\right)\right] = \left(\mu - v - \frac{1}{2}\cdot\sigma^2\right)\Delta t$$

$$var\left[ln\left(\frac{P_{t+\Delta t}}{P_t}\right)\right] = \sigma^2 \cdot \Delta t$$

because the term $\left(\mu - v - \frac{1}{2}\cdot\sigma\right)\cdot \Delta t$ is deterministic, while the term $\sigma \cdot (W_{t+\Delta t} - W_t) \sim N(0, \sigma^2 \cdot \Delta t)$ is the only random variable present in the equation of the return.

Logarithmic returns are widely used in finance because of their useful property of additivity over subsequent and non-overlapping intervals of time. However, in the framework to computing the Value at Risk of a portfolio, it is more convenient to use percentage returns because the return of the portfolio can be calculated as the weighted sum of the linear returns of the assets from which it is made up. For this purpose, by using the first-order Taylor expansion for $\frac{P_{t+\Delta t}}{P_t} \rightarrow 1$ it is valid that

$ln\left(\frac{P_{t+\Delta t}}{P_t}\right) \simeq \frac{P_{t+\Delta t}-P_t}{P_t}$. It means that if the returns of a portfolio are close to zero then the linear returns and the logarithmic returns of a portfolio are similar to each other. Moreover, in Miskolczi (2017) it is conducted an empirical study in which it is illustrated that the Value at Risk of a considered sample of financial portfolios is not significantly affected by the type of returns used.

Hence, from this point on, it will be used the normal distribution also to model linear returns and moreover it is supposed that their expected value is equal to zero. This assumption seems reasonable if it is considered a relatively short period of time (e.g. days), while it is not realistic for longer periods (e.g. years). In fact, if the annual expected return of a financial asset is supposed to be, let's say, 20% of its price, the daily return, considering 252 trading days in a year, is approximately 0,12% and it can be assumed to be equal to zero without making a huge approximation.

So, for the purpose of calculation of the Value at Risk, it is considered that time is measured in days and it is chosen a period of length equal to one day ($\Delta t = 1$) as time horizon. Thus, the daily change in the portfolio's value can be written as:

$$\Delta V_{\boldsymbol{h},t+1} = \sum_{i=1}^{n} q_i \cdot P_{i,t} \cdot \Delta x_i = \sum_{i=1}^{n} V_{i,t} \cdot \Delta x_i$$

where $\Delta x_i = \frac{P_{i,t+1}-P_{i.t}}{P_{i,t}}$ is the percentage return in the price of asset i incurred between t and $t+1$, while $V_{i,t} = q_i \cdot P_{i,t}$ represents the monetary position in the asset i.

3.1.2 Derivative instruments

So far, it is considered that the portfolio is made up only by equity prices, commodity prices and currencies. However, in order to include the possibility that the portfolio also contains derivatives instruments such as forward contracts or financial options, there is the need to make some additional considerations.

Therefore, without defining an additional market variable for every derivative instrument present in the portfolio, it makes sense to link the return of each derivative with the one of the underlying asset. For this purpose, it is introduced the concept of delta δ of a derivative instrument, which is defined as $\delta = \frac{\partial f}{\partial P}$ and it represents the rate of change in the value of the derivative f with respect to a unitary change in the price of the underlying asset P. For small changes in the value of the underlying asset and for small periods of time (e.g. a day), it is valid the relation $\delta = \frac{\Delta f}{\Delta P}$ which allow to obtain $\Delta f = \delta \cdot \Delta P_{t+1}$. So, having defined the daily return of the underlying asset as $\Delta x = \frac{\Delta P_{t+1}}{P_t}$, it can be written $\Delta f = \delta \cdot P_t \cdot \Delta x$.

Thus, for a portfolio $\boldsymbol{h}$ containing *n* financial assets, which can be also derivative instruments, the daily change in the value of the portfolio can be defined as:

$$\Delta V_{\boldsymbol{h},t+1} = \sum_{i=1}^{n} q_i \cdot P_{i,t} \cdot \delta_i \cdot \Delta x_i = \sum_{i=1}^{n} V_{i,t} \cdot \Delta x_i$$

where $V_{i,t} = q_i \cdot P_{i,t} \cdot \delta_i$ and it represents the monetary delta position in the asset i. It is important to remark that each Δx_i represents the return of the market variable i underlying the derivative instrument, which can be considered to be normally distributed. Fundamentally, when considering also derivative instruments, what changes in the distribution of $\Delta V_{\boldsymbol{h},t+1}$ is only the terms $V_{i,t}$, with $i = 1,2,\dots,n$, which contains the monetary delta position in every asset included in the portfolio.

It is interesting to notice that the last definition of the daily change in the value of the portfolio can be considered as a general case of the former. In fact, by definition the delta of a stock, commodity, exchange rate or stock index is equal to one and if the portfolio includes only these assets, the above-mentioned equation of $\Delta V_{\boldsymbol{h},t+1}$ becomes equal to the first one.

3.1.3 Estimating V.a.R.

In order to compute the Value at Risk of the portfolio, it is also necessary to consider that market variables are not independent each other, but there is a correlation of $\rho_{i,j}$ between the returns of every pair of assets i and j. Starting from the correlation coefficient, the covariance between each couple of variables is computed as:

$$cov_{i,j} = \rho_{i,j} \cdot \sigma_i \cdot \sigma_j$$

where σ_i and σ_j are the daily volatilities of asset i and j, respectively.

Hence, since the returns of the market variables are modeled with the univariate normal distribution $\Delta x_i \sim N(0, \sigma_i^2)$ and the change in the portfolio's value is a convex linear combination of these returns, it follows that $\Delta V_{\boldsymbol{h},t+1}$ is multivariate normally distributed. Thus, the expected value of $\Delta V_{\boldsymbol{h},t+1}$ is simply:

$$\mu_{\boldsymbol{h}} = 0$$

Instead, the variance is computed as:

$$\sigma_{\boldsymbol{h}}^2 = \boldsymbol{V}_t' \Sigma \boldsymbol{V}_t$$

where $\boldsymbol{V}_t$ is the column vector of size *n* containing the monetary delta position in each asset at time t, and Σ is the covariance matrix defined as:

$$\Sigma = \begin{pmatrix} \sigma_1^2 & cov_{1,2} & \dots & cov_{1,n} \\ cov_{1,2} & \sigma_2^2 & \cdots & cov_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ cov_{1,n} & cov_{2,n} & \cdots & \sigma_n^2 \end{pmatrix}$$

In the covariance matrix, which is a symmetric matrix, the entry in the i^{th} row and j^{th} column is the covariance between variable i and variable j. Instead, the diagonal entries in the matrix are the variances since the covariance of a variable and itself is its variance.

Generally, before proceeding in calculating the standard deviation of $\Delta V_{\boldsymbol{h},t+1}$, it is important to check whether the covariance matrix Σ is positive semidefinite, which means that it should satisfy the condition $\boldsymbol{\omega}^{'}\Sigma\boldsymbol{\omega} \geq 0$ for all $n \times 1$ vectors $\boldsymbol{\omega}$. To understand the intuition behind this condition, suppose that $\boldsymbol{\omega}_t$ is the vector of the values of the financial assets included in the portfolio at time t. The variance of the portfolio is defined as $\boldsymbol{\omega}_t^{'}\Sigma\boldsymbol{\omega}_t$ and it cannot be negative. To ensure that Σ is positive semidefinite, the covariance matrix has to be internally consistent, which means that variances and covariances should be estimated using the same models and parameters which will be further debated at the end of this chapter.

In conclusion, having shown that $\Delta V_{\boldsymbol{h},t+1} \sim N(0, \sigma_{\boldsymbol{h}}^2)$ and recalling the definition of the portfolio's loss which is $L_{\boldsymbol{h},t+\Delta t} = -\Delta V_{\boldsymbol{h},t+\Delta t}$, it can be stated that also the daily loss of the portfolio $\boldsymbol{h}$ is normally distributed with the following parameters:

$$L_{\boldsymbol{h},t+1} \sim N(0, \sigma_{\boldsymbol{h}}^2)$$

The previous expression can be written as $L_{\boldsymbol{h},t+1} \sim \sigma_{\boldsymbol{h}} \cdot \Phi$, where Φ represents the standard normal distribution. Hence, the daily Value at Risk of the portfolio $\boldsymbol{h}$ at the confidence level α, which is defined as a quantile of the loss distribution, is given by:

$$VAR_{\boldsymbol{h},\alpha,t+1} = z_\alpha \cdot \sigma_{\boldsymbol{h}}$$

where z_α is the quantile of the standard normal distribution corresponding to the chosen probability level α. This result is proved as follows:

$$P\left(L_{\boldsymbol{h},t+1} \leq VAR_{\boldsymbol{h},\alpha,t+1}\right) = P\left(L_{\boldsymbol{h},t+1} \leq z_\alpha \cdot \sigma_{\boldsymbol{h}}\right) = P\left(\frac{L_{\boldsymbol{h},t+1}}{\sigma_{\boldsymbol{h}}} \leq z_\alpha\right) = \Phi(z_\alpha) = \alpha$$

Furthermore, if the aim is to compute the Value at Risk of the same portfolio $\boldsymbol{h}$ for a longer time horizon than a single day, it can be used the formula:

$$VAR_{\boldsymbol{h},\alpha,t+\Delta t} = VAR_{\boldsymbol{h},\alpha,t+1} \cdot \sqrt{\Delta t}$$

This formula is exactly true when the changes in the value of the portfolio on successive days have independent identical normal distributions with mean zero, while in other cases it represents an approximation.

3.2 Quadratic model

The linear model has some assumptions that, on the one hand, make the model easy to apply, but on the other hand, ensure that the model strays too far from reality, effectively constituting a limit in case the portfolio contains non-linear derivatives. In fact, when a portfolio includes options and other non-linear derivatives, the probability distribution of the portfolio's loss over a short period of time can no longer reasonably assumed to be normal because the gamma exposures of the portfolio cause the probability distribution of change in its value to exhibit skewness. It is related to the third moment of the distribution and a positive skewness indicates that the right tail is heavier than the left tail, while a negative skewness indicates the reverse.

The gamma γ of a derivative is defined as $\gamma = \frac{\partial^2 f}{\partial P^2} = \frac{\partial \delta}{\partial P}$ and it represents the rate of change of the derivative's delta with respect to a unitary change in the price of the underlying asset. So, gamma γ measures the curvature of the relationship between the derivative value and the underlying market variable. Using the second-order Taylor expansion, the change in the value of the derivative is represented by:

$$\partial f = \frac{\partial f}{\partial t} \cdot \Delta t + \delta \cdot \Delta P + \frac{1}{2} \cdot \gamma \cdot (\Delta P)^2$$

Considering a small period of time, like a day, the term $\frac{\partial f}{\partial t} \cdot \Delta t$ can be neglected and the previous equation can be written as follows:

$$\Delta f_{t+1} = \delta \cdot \Delta P_{t+1} + \frac{1}{2} \cdot \gamma \cdot (\Delta P_{t+1})^2$$

It represents an improvement in the accuracy of the approximation with respect the previous case in which it is considered only the first-order relationship. Since $\Delta P_{t+1} = P_t \cdot \Delta x$ as it is shown before, the previous formula becomes:

$$\Delta f_{t+1} = \delta \cdot P_t \cdot \Delta x + \frac{1}{2} \cdot \gamma \cdot P_t^2 \cdot (\Delta x)^2$$

More generally, for a portfolio $\boldsymbol{h}$ with n underlying financial assets and with each instrument in the portfolio being dependent on only one of the market variables, the daily change in the portfolio's value can be expressed as:

$$\Delta V_{\boldsymbol{h},t+1} = \sum_{i=1}^{n} q_i \cdot P_{i,t} \cdot \delta_i \cdot \Delta x_i + \sum_{i=1}^{n} \frac{1}{2} \cdot q_i^2 \cdot P_{i,t}^2 \cdot \gamma_i \cdot (\Delta x_i)^2$$

Instead, when individual instruments in the portfolio depend on more than one market variable, the previous formula becomes:

$$\Delta V_{\boldsymbol{h},t+1} = \sum_{i=1}^{n} q_i \cdot P_{i,t} \cdot \delta_i \cdot \Delta x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} \cdot q_i \cdot q_j \cdot P_{i,t} \cdot P_{j,t} \cdot \gamma_{ij} \cdot \Delta x_i \cdot \Delta x_j$$

where γ_{ij} is the "cross gamma" defined as $\gamma_{ij} = \frac{\partial^2 f}{\partial P_i \cdot \partial P_j}$ which measures the rate of change of delta of the derivative instrument i with respect to a unitary change in its underlying asset i and another asset j. The last equation represents a generalization of the previous one because if each instrument in the portfolio depends on only one risk factor, there are no cross gammas so that $\gamma_{ij} = 0$ except when $i = j$.

It is interesting to note that if the portfolio does not include any non-linear derivative instrument, this model will become equivalent to the linear one and the V.a.R. measure is calculated as illustrated in the previous paragraph. Conversely, considering the most general case in which there are also non-linear derivative instruments in the portfolio, $\Delta V_{\boldsymbol{h},t+1}$ results not to be normally distributed and it is more difficult to compute the Value at Risk measure. However, if there are only a small number of variables, the previous equation can be used to calculate moments for the distribution of $\Delta V_{\boldsymbol{h},t+1}$.

Denoting $V_{i,t} = q_i \cdot P_{i,t} \cdot \delta_i$ and $\beta_{ij} = \frac{1}{2} \cdot q_i \cdot q_j \cdot P_{i,t} \cdot P_{j,t} \cdot \gamma_{ij}$, the change in the portfolio's value can be written as:

$$\Delta V_{\boldsymbol{h},t+1} = \sum_{i=1}^{n} V_{i,t} \cdot \Delta x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} \cdot \Delta x_i \cdot \Delta x_j$$

So, the daily loss of the portfolio $\boldsymbol{h}$ is defined as:

$$L_{\boldsymbol{h},t+1} = -\left(\sum_{i=1}^{n} V_{i,t} \cdot \Delta x_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_{ij} \cdot \Delta x_i \cdot \Delta x_j\right)$$

Then, assuming that each Δx_i is normally distributed, it can be shown that the first, second, and third moment of the probability distribution of $L_{\boldsymbol{h},t+1}$ are respectively:

$$E\left[L_{\boldsymbol{h},t+1}\right] = -\sum_{i,j} \beta_{ij} \cdot \sigma_{ij}$$

$$E[(L_{\boldsymbol{h},t+1})^2] = \sum_{i,j} V_{i,t} \cdot V_{j,t} + \sum_{i,j,k,l} \beta_{ij} \cdot \beta_{kl} \cdot (\sigma_{ij} \cdot \sigma_{kl} + \sigma_{ik} \cdot \sigma_{jl} + \sigma_{il} \cdot \sigma_{jk})$$

$$E[(L_{\boldsymbol{h},t+1})^3 = -3 \cdot \sum_{i,j,k,l} V_{i,t} \cdot V_{j,t} \cdot \beta_{kl} \cdot (\sigma_{ij} \cdot \sigma_{kl} + \sigma_{ik} \cdot \sigma_{jl} + \sigma_{il} \cdot \sigma_{jk})$$
$$+ \sum_{i_1,i_2,i_3,i_4,i_5,i_6} \beta_{i_1 i_2} \cdot \beta_{i_3 i_4} \cdot \beta_{i_5 i_6} \cdot Q$$

where the variable Q consists of fifteen terms of the form $\sigma_{p_1 p_2} \cdot \sigma_{p_3 p_4} \cdot \sigma_{p_5 p_6}$ where $p_1, p_2, p_3, p_4, p_5, p_6$ are permutations of $i_1, i_2, i_3, i_4, i_5, i_6$.

From the moments of the distribution of $L_{\boldsymbol{h},t+1}$, it can be estimated the mean, variance, and skewness of this distribution as:

$$\mu_{\boldsymbol{h}} = E[L_{\boldsymbol{h},t+1}]$$

$$\sigma_{\boldsymbol{h}}^2 = E[(L_{\boldsymbol{h},t+1})^2] - E[L_{\boldsymbol{h},t+1}]^2$$

$$\xi_{\boldsymbol{h}} = \frac{E[(L_{\boldsymbol{h},t+1} - \mu_{\boldsymbol{h}})^3]}{\sigma_{\boldsymbol{h}}^3}$$

Hence, using a procedure named Cornish-Fisher expansion which approximates the quantile of a random variable based only on these parameters, it can be shown that for the portfolio $\boldsymbol{h}$ the V.a.R. measure at the confidence level α is determined as:

$$VAR_{\boldsymbol{h},\alpha,t+1} = \mu_{\boldsymbol{h}} + w_\alpha \cdot \sigma_{\boldsymbol{h}}$$

where $w_\alpha = z_\alpha + \frac{1}{6} \cdot (z_\alpha^2 - 1) \cdot \xi_{\boldsymbol{h}}$ and z_α is the percentile corresponding to the probability α of the standard normal distribution.

Finally, if there is the need to calculate the Value at Risk for a time horizon longer than one day, it can be used the formula shown in the linear model, which is:

$$VAR_{\boldsymbol{h},\alpha,t+\Delta t} = VAR_{\boldsymbol{h},\alpha,t+1} \cdot \sqrt{\Delta t}$$

However, using the quadratic model, this formula always represents an approximation and it should be aware about its accuracy in the V.a.R. computations for longer periods of time.

3.3 Monte Carlo simulation

The parametric approach can be also implemented using Monte Carlo simulation, especially when there is the necessity to deal with non-linear derivative instruments. By definition, the Monte Carlo simulation is a procedure for sampling random outcomes for a given stochastic process. This procedure consists of some steps, which allow to obtain a single outcome of the process and they have to be repeated, generally thousands of times, in order to build the probability distribution for $L_{\boldsymbol{h},t+1}$ from which the Value at Risk measure is calculated.

Firstly, it is necessary to value the portfolio using the current values of market variables and for this purpose it is recalled the equation:

$$V_{\boldsymbol{h},t} = \sum_{i=1}^{n} P_{i,t} \cdot q_i$$

Secondly, since it is assumed that $\Delta x_i \sim N(0, \sigma_i^2)$, it has to sample the value of the daily return Δx_i of each market variable from the normal distribution:

$$\Delta x_i = \sigma_{i,t+1} \cdot \varepsilon_i$$

where ε_i is a random sample obtained from a standard normal distribution and $\sigma_{i,t+1}$ is the daily volatility of the market variable i estimated for the next day.

After that, it must be obtained a simulation trial of the daily change in the portfolio's value. In case of a portfolio consisting of n financial assets, a simulation trial involves getting n samples of the Δx_i, where $1 \leq i \leq n$, from a multivariate standard normal distribution. However, since between each pair of variables i and j there is a correlation $\rho_{i,j}$, the samples of Δx_i and Δx_j cannot be independent and there must be a relationship between the random samples ε_i and ε_j obtained from the standard normal distribution. Thus, denoting z_i the independent sample obtained from a univariate normal distribution, the required samples ε_i are then obtained using a procedure known as Cholesky decomposition. For instance, if there are three market variables, the relation of the three samples obtained from the standard normal distribution is the following:

$$\varepsilon_1 = a_{11} \cdot z_1$$

where $a_{11} = 1$,

$$\varepsilon_2 = a_{21} \cdot z_1 + a_{22} \cdot z_2$$

where $a_{21} \cdot a_{11} = \rho_{1,2}$ and $a_{21}^2 + a_{22}^2 = 1$,

$$\varepsilon_3 = a_{31} \cdot z_1 + a_{32} \cdot z_2 + a_{33} \cdot z_3$$

where $a_{31} \cdot a_{11} = \rho_{1,3}$, $a_{31} \cdot a_{21} + a_{32} \cdot a_{22} = \rho_{2,3}$ and $a_{31}^2 + a_{32}^2 + a_{33}^2 = 1$.

In general, for a given market variable i, it is valid the relation:

$$\varepsilon_i = \sum_{k=1}^{i} a_{ik} \cdot z_k$$

where $\sum_{j=1}^{k} a_{ij} \cdot a_{kj} = \rho_{i,k}$ with $k = 1, \dots, i-1$ and $\sum_{j=1}^{i} a_{ij}^2 = 1$ with $i = 1, \dots, n$.

If the equations for the $a's$ do not have real solutions, it means that the assumed correlation structure is internally inconsistent.

Therefore, one simulation outcome of the change in the portfolio's value can be defined as:

$$\Delta V_{\boldsymbol{h},t+1} = \sum_{i=1}^{n} q_i \cdot P_{i,t} \cdot \delta_i \cdot \sigma_{i,t+1} \cdot \varepsilon_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} \cdot q_i \cdot q_j \cdot P_{i,t} \cdot P_{j,t} \cdot \gamma_{ij} \cdot \sigma_{i,t+1} \cdot \sigma_{j,t+1} \cdot \varepsilon_i \cdot \varepsilon_j$$

Finally, the portfolio's loss of one simulation trial is calculated as usual: $L_{\boldsymbol{h},t+1} = -\Delta V_{\boldsymbol{h},t+1}$.

The above explained procedure must be repeated several times until it is generated a sufficient number of trials (typically thousands of simulations) to approximate the real distribution of the daily portfolio's loss $L_{\boldsymbol{h},t+1}$. It is then constructed a vector where each sample of $L_{\boldsymbol{h},t+1}$ is sorted from the biggest to the smallest value. At this point, the Value at Risk corresponding to the confidence level α, can be estimated as the value present in the position $(1-\alpha)\cdot m$, where m is the number of trials performed. Again, if the chosen time horizon is longer than one day, the V.a.R. measure can be computed multiplying the daily V.a.R. with the square root of the period's length, even if this calculation often represents an approximation.

The accuracy of the result given by Monte Carlo simulation depends on the number of trials chosen. Usually, the accuracy of this method is calculated as the standard deviation of the distribution of $L_{\boldsymbol{h},t+1}$ given by the simulation trials. Denoting the standard deviation of this distribution as $s_{\boldsymbol{L}}$, the standard error of the estimate is $\frac{s_{\boldsymbol{L}}}{\sqrt{m}}$ where m is the number of trials. This shows that uncertainty is inversely proportional to the square root of the number of trials. Hence, the greater the number of simulations performed the better the accuracy of this method.

On the other hand, one of the main drawbacks of the Monte Carlo simulation concerns the computational cost for large portfolios. In fact, the number of computations can be considerable, as every simulation requires the revaluation of the portfolio. This is particularly problematic if the portfolio contains many derivatives which cannot be priced in closed form.

3.4 Estimating volatilities and correlations

Whatever the model used, in order to compute the Value at Risk measure in the parametric approach, it is fundamental to analyze the current level of volatilities and correlations of the underlying market variables. The most common mathematical models used to estimate volatilities and correlations are the exponentially weighted moving average (EWMA) and the generalized autoregressive conditional heteroscedasticity (GARCH). The distinctive feature of these models is that they recognize that volatilities and correlations are not constant. So, they try to keep track of the variations in the volatility or correlation through time.

First of all, it is important to remark that the process followed by volatilities and correlations is considered to be discrete in time, so there is the need to define a fixed time bucket, usually a day, in which these variables are observed and estimated. Secondly, before explaining how the models are constructed, it would be useful to introduce the variables included in these models, which are:

- $x_t = \frac{P_t - P_{t-1}}{P_{t-1}}$ is the linear return of a particular financial asset from $t-1$ to t, which is observed at time t;
- σ_t^2 is the variance of the return at time t, which is estimated at $t-1$;
- $cov_{i,j,t}$ is the covariance between the returns of two generic market variables at time t, which is estimated at $t-1$.

An unbiased estimator of the variance of the return of a specific financial asset, computed using the last m observations, is:

$$\sigma_t^2 = \frac{1}{m-1} \cdot \sum_{k=1}^{m} (x_{t-k} - \bar{x})^2$$

Since from one day to the next one, the expected return $\bar{x}$ is considered to be approximately 0 and replacing $m-1$ with m, the estimator of the variance becomes:

$$\sigma_t^2 = \frac{1}{m} \cdot \sum_{k=1}^{m} x_{t-k}^2$$

What is obtained is a maximum likelihood estimator of the variance per day using the most recent m observations. However, it makes sense to give more weight to recent data and less to the past ones, so it can be assigned a weight for each return observation. Hence, the estimate of the variance weighting the observation becomes:

$$\sigma_t^2 = \frac{1}{m} \cdot \sum_{k=1}^{m} \alpha_k \cdot x_{t-k}^2$$

where α_k represents the weight of the observation k and it decreases as it is moved back in time. A constraint is that the weights of the m returns must sum to unity.

Making the same considerations, an estimator of the covariance between the returns of two generic market variables, computed weighting each observation is defined as:

$$cov_{i,j,t} = \frac{1}{m} \sum_{k=1}^{m} \alpha_k \cdot (x_{i,t-k} \cdot x_{j,t-k})$$

3.4.1 EWMA (Exponentially Weighted Moving Average)

The exponentially weighted moving average (EWMA) is a model where the weights α_k decrease exponentially as it is gone back in time. More in detail, $\alpha_k = \lambda \cdot \alpha_{k-1}$ where λ is a constant between 0 and 1. So, the model used to calculate the variance estimate is given by the formula:

$$\sigma_t^2 = \lambda \cdot \sigma_{t-1}^2 + (1-\lambda) \cdot x_{t-1}^2$$

To understand why in the previous equation the weights decrease exponentially in time, it can be substituted σ_{t-1} with the formula of its estimate, obtaining:

$$\sigma_t^2 = \lambda \cdot [\lambda \cdot \sigma_{t-2}^2 + (1-\lambda) \cdot x_{t-2}^2] + (1-\lambda) \cdot x_{t-1}^2$$
$$= \lambda^2 \cdot \sigma_{t-2}^2 + (1-\lambda) \cdot (x_{t-1}^2 + \lambda \cdot x_{t-2}^2)$$

So, in general it is valid the following relation:

$$\sigma_t^2 = \lambda^m \cdot \sigma_{t-m}^2 + (1-\lambda) \cdot \left(\sum_{k=1}^{m} \lambda^{k-1} \cdot x_{t-k}^2\right)$$

It is easy to notice that the weight for each x_{t-k} declines at a rate λ as it is moved back in time.

Instead, the model used to estimate the covariance between the returns of two market variables i and j is given by the formula:

$$cov_{i,j,t} = \lambda \cdot cov_{i,j,t-1} + (1-\lambda) \cdot x_{i,t-1} \cdot x_{j,t-1}$$

One of the advantages of the EWMA model is that relatively little data need to be stored. In fact, the only data needed are the most recent variance or covariance rate and the last return over the considered time bucket. In addition, choosing the value of the constant λ, it can be decided the responsiveness of the model to the last observation. A relatively low value of λ (close to 0) gives more weight to the last return observation x_{t-1} and less to the older ones, which are included in the term σ_{t-1}^2 or cov_{t-1}. So, the estimates of the volatilities or covariances tend to respond rapidly to new information provided by the last return. Instead, a relatively high value of λ (close to 1) gives less weight to the last percentage change in the considered market variable and it makes the estimates more stable through time and less responsive to potential noises.

3.4.2 GARCH (Generalized Autoregressive Conditional Heteroscedasticity)

The generalized autoregressive conditional heteroscedasticity (GARCH) computes the variance from a long-run average variance rate V_L, the previous variance rate

σ_{t-1}^2 and the last return observation x_{t-1}, giving a weight to each parameter. So, it is defined as:

$$\sigma_t^2 = \alpha \cdot x_{t-1}^2 + \beta \cdot \sigma_{t-1}^2 + \gamma \cdot V_L$$

Also, the GARCH model is used to update covariance estimates. The covariance between two financial assets i and j is computed assigning appropriate weights to the long-run covariance rate $COV_{L,i,j}$, the most recent covariance rate $cov_{i,j,t-1}$ and the last return observations $x_{i,t-1}$ and $x_{j,t-1}$. Thus, the formula is represented by:

$$cov_{i,j,t} = \alpha \cdot x_{i,t-1} \cdot x_{j,t-1} + \beta \cdot cov_{i,j,t-1} + \gamma \cdot COV_{L,i,j}$$

The weights α, β, γ must sum to unity. GARCH model is very similar to EWMA, but the former gives a predetermined importance γ to the long-run variance or covariance rate. Also, it can be noticed that in GARCH model the weights of the past data on variance or covariance decline exponentially at rate β which is called "decay rate" and it plays the same role of λ in EWMA. However, GARCH model recognizes that over time variance and covariance tends to get pulled back to a long-run average level of V_L and $COV_{L,i,j}$ at a rate γ. It is called mean reversion and empirical studies show that variance and covariance rates tend to be mean reverting. Thus, the presence of this feature makes GARCH model be more appealing than EWMA.

4. HISTORICAL SIMULATION METHOD

The historical simulation has become the most popular method for calculating the Value at Risk measure of a financial portfolio because the parametric approach has some assumptions that ensure that the model strays too far from reality. In fact, the empirical evidence indicates that assets' returns, especially the daily ones, are rather non-normal. Excess kurtosis will cause losses greater than V.a.R. to occur more frequently and be more extreme than those predicted by the normal distribution. In addition, a large number of financial markets crash together and the correlation forecasts used to calculate V.a.R. in the parametric approach failed to predict such a synchronous crash. Instead, the historical simulation method overcomes the previous problems since it involves using the day-to-day changes in the values of market variables that have been observed in the past in a direct way, in order to estimate the empirical probability distribution of the daily change in the value of the current portfolio.

In the first place, it will be described the mechanics of the historical simulation explaining how this method is developed starting from the empirical probability distribution of the portfolio's loss. Successively, it will be debated a couple of extensions which can improve the accuracy of the V.a.R. estimate obtained from the traditional historical simulation method.

4.1 Empirical distribution and V.a.R. estimate

The empirical distribution is the cumulative distribution function which can be used to describe a sample of observations of a given variable. Its value at any specified value of the measured variable is the fraction of observations of the measured variable which are less than or equal to the specified value.

Assuming that m is the sample size, denote with x a generic observation of the variable of interest and sort all the observations in an ascending order, so that $x_j > x_{j-1}$ for $j = 2,3, \dots, m$. Then, the empirical distribution $F(x)$ of the variable x, can be written as:

$$F(x) = \begin{cases} 0 & if\ x < x_1 \\ \frac{1}{m} & if\ x_1 \leq x < x_2 \\ \frac{2}{m} & if\ x_2 \leq x < x_3 \\ \vdots & \vdots \\ \frac{m-1}{m} & if\ x_{m-1} \leq x < x_m \\ 1 & if\ x \geq x_m \end{cases}$$

Hence, $F(x)$ is a step function which is everywhere flat except at each of the m data points, where it jumps up by $1/m$. The main feature of the empirical distribution is that it can be considered as an estimate of the cumulative distribution function which generates the points in the sample. In fact, according to the Glivenko-Cantelli theorem, the empirical distribution converges with probability 1 to the underlying distribution as the sample size tends to infinity ($m \to \infty$).

In the framework of calculating the Value at Risk measure, it is considered the empirical distribution of the daily portfolio's loss $L_{h,t+1}$ which depends on the daily returns in the underlying market variables. In order to build such a distribution, the first step is to identify the market variables affecting the portfolio. These market variables are sometimes referred to as risk factors which can be stocks, exchange rates, commodities, stock indices, interest rates and so on. It is important to notice that, conversely of what it is done in the parametric approach, in the historical simulation method derivative instruments as options or forward contracts are considered as distinct market variables from their underlying assets. Then, data are collected on movements in the market variables over the most recent $m+1$ days. This provides m alternative scenarios for what can happen between today and tomorrow. Denote the first day for which data are available as Day 0, the second day as Day 1, and so on. Scenario 1 is where the percentage changes in the values of all variables are the same as they were between Day 0 and Day 1, Scenario 2 is where they are the same as between Day 1 and Day 2, and so on.

For each scenario j, the loss of the portfolio between today and tomorrow is calculated and it is denoted as $L_{\boldsymbol{h},j}$. Thus, it can be constructed a m-dimensional vector which includes all the sample observations $L_{\boldsymbol{h},i}$ with $j=1,2,\dots,m$ sorted in an ascending order. This vector defines the empirical probability distribution for daily loss of the portfolio.

Hence, the daily Value at Risk measure, which is the quantile of the distribution of $L_{\boldsymbol{h},t+1}$ corresponding to the confidence level α, can be estimated as the $\alpha \cdot m$ value of the sample. For example, if $m=500$, the 99% daily V.a.R. corresponds to the 495th value of the sample, which is the fifth highest loss. Instead, if the result of $\alpha \cdot m$ does not correspond to an integer number the V.a.R. measure will be calculated using linear interpolation. For instance, if $m=250$, the 99% daily V.a.R. corresponds the mean between the second and the third highest loss of the sample.

Again, if the aim is to compute the V.a.R. of the portfolio for a longer period than one day, it can be used the formula shown in the previous chapter, which is:

$$VAR_{\boldsymbol{h},\alpha,t+\Delta t} = VAR_{\boldsymbol{h},\alpha,t+1} \cdot \sqrt{\Delta t}$$

The historical-simulation method has obvious attractions: it is easy to implement, reduces the risk-measure estimation problem to a one-dimensional problem and there is no necessity to make any particular assumption about the distribution of the change in the portfolio’s value. However, the main disadvantages of historical simulation are that it is computationally slow and does not easily allow volatility updating schemes to be used. In fact, an implicit assumption of this method is that the volatility of the portfolio remains stable in time and therefore the future values of the losses in the portfolio’s value are well estimated by the past ones. Moreover, the success of the approach is highly dependent on the ability to collect sufficient quantities of relevant, synchronized data for all market variables. In the next paragraphs it will be debated some extensions to the traditional model which allow the historical simulation method to overcome these drawbacks.

4.2 Accuracy of V.a.R.

The historical simulation method estimates the distribution of the portfolio's loss from a finite number of observations. As a result, the estimates of quantiles of the distribution are subject to error.

In Kendall and Stuart (1961) it is described how to calculate a confidence interval for the percentile of a probability distribution when it is estimated from sample data. Suppose that the α-percentile of the distribution is estimated as x. The standard error of the estimate, denoted with SE is:

$$SE = \frac{1}{f(x)}\sqrt{\frac{(1-\alpha)\cdot\alpha}{m}}$$

where m is the number of observations and $f(x)$ is an estimate of the probability density function of the change in the value evaluated at x. The probability density $f(x)$, can be estimated approximately by fitting the empirical data to an appropriate distribution whose properties are known.

From the above formula it can be observed that the standard error declines as the density function $f(x)$ increases. Assuming that the density function is considered to be approximately normal, it means that the confidence level α used to estimating V.a.R. should be reduced in order to obtain results which are more accurate. Moreover, it can be noticed that SE decreases as the sample size is increased, but only as its square root. For instance, in order to halve the standard error of the estimate it is needed to quadruple the sample size which involves increasing the computational cost of implementing the historical simulation method.

4.3 Hybrid approach

When the Value at Risk is calculated with the historical simulation method, the assumption is that recent history is in some sense a good guide to the future. More precisely, it is that the empirical probability distribution estimated for market variables over the immediately preceding period is a good guide to the behavior of the market variables over the next day. Unfortunately, the behavior of market variables is non-stationary. Sometimes the volatility of a market variable is high, while sometimes it is low. For this purpose, in Boudoukh et al. (1997) it is suggested a method which combines the historical simulation with the parametric approach by estimating V.a.R. from the empirical distribution of $L_{\boldsymbol{h},t+1}$, using declining weights on past data. Thus, it makes sense that more recent observations should be given more weight because they are more reflective of current volatilities and current macroeconomic conditions.

The natural weighting scheme to use is one where weights decline exponentially, which is already used in the previous chapter when developing the EWMA (Exponentially Weighted Moving Average) model for monitoring volatility. The weight assigned to Scenario 1 (which is the one calculated from the most distant data) is λ times that assigned to Scenario 2. This in turn is λ times that given to Scenario 3, and so on. In general, the weight given to Scenario k is:

$$\frac{\lambda^{m-k} \cdot (1-\lambda)}{1-\lambda^m}$$

where m is the number of scenarios. It is important to notice that the previous definition of the different weights for the m observations is consistent since they sum to unity:

$$\sum_{j=1}^{m} \frac{\lambda^{m-j} \cdot (1-\lambda)}{1-\lambda^m} = 1$$

The parameter $\lambda \in (0,1]$ can be chosen by trying different values and seeing which one is the best according to back testing. Furthermore, it is interesting to observe that as λ approaches 1, this method approaches the basic historical simulation where all observations are given a weight of $1/m$:

$$\lim_{\lambda \to 1} \frac{\lambda^{m-j} \cdot (1-\lambda)}{1-\lambda^m} = \frac{1}{m}$$

Then, having defined all the weights, the Value at Risk measure is calculated by ranking the observations from the worst outcome to the best. Starting from the worst outcome, weights are summed until the required quantile of the distribution is reached. For instance, if it is calculating V.a.R. with a 95% confidence level, the weights are continued to be summed until the sum just exceed $0{,}05$. Thus, in the hybrid approach the Value at Risk measure depends on how recent the worst scenarios occurred. In fact, using the basic historical simulation method over a period of 100 days, the daily V.a.R. of the portfolio at the 95% confidence level is always estimated as the fifth highest loss. Instead, using the hybrid approach, if the greatest losses occur in a relatively recent time the V.a.R. measure is represented by the value of a worse loss than the fifth one, while if they occur in the most distant scenarios V.a.R. corresponds to a better outcome than those calculated with the basic historical simulation.

Although it is empirically shown that the hybrid approach represents an improvement in the accuracy of V.a.R. estimates compared to the ones obtained with the traditional historical simulation and parametric approach, this method is criticized on the grounds that it represents an indirect and somewhat inefficient way of considering stochastic volatility in the empirical distribution of $L_{\boldsymbol{h},t+1}$. In fact, in the hybrid approach a short run sequence of abnormally large positive (or negative) returns of the market variables will markedly skew the predicted distribution to the right (or the left). For instance, when $\lambda = 0{,}98$, the most recent observation is assigned a probability of about 2% so that a single large outcome is enough to generate this sort of skew. As a consequence, the hybrid method shortens the effective sampling period to capture the behavior of stochastic volatility. Unfortunately, in doing so it captures the stochastic behavior of all other sample moments of the distribution.

4.4 Volatility updating schemes

A different method used in quantitative risk management to overcome the main drawback of the hybrid approach is described in Hull and White (1998). In this paper it is proposed to incorporate volatility updating schemes into historical simulation in order to bridge the gap between this method and the parametric approach. In fact, it is observed that the probability distribution of a market variable's return, when scaled by an estimate of its volatility, is found to be approximately stationary. So, historical simulation can be improved by considering the volatility changes experienced during the period covered by the historical data. For instance, if the current volatility of a market variable is 2% per day and three months ago the volatility was only 1% per day, the data observed three months ago understates the changes it is expected to see now. On the other hand, if the volatility was 3% per day three months ago the reverse is true.

For the purpose of V.a.R. calculation, it is monitored the daily volatility of each market variable included in the portfolio using either a GARCH or EWMA model. Thus, it is defined:

- $x_{i,k}$ the historical linear return in the variable i on day k of the period covered by the historical sample ($k = 1,2, \dots, m$);
- $\sigma_{i,k}^2$ the historical GARCH/EWMA estimate of the daily variance of the linear return in the variable i made for day k at the end of day $k - 1$.

The most recent estimate of the daily variance in the return of the variable i is $\sigma_{i,m+1}^2$ made at the end of day m for day $m + 1$. Since it is assumed that the probability distribution of $x_{i,k}/\sigma_{i,k}$ is stationary, it is replaced $x_{i,k}$ by $x_{i,k}^*$ where:

$$x_{i,k}^* = x_{i,k} \cdot \frac{\sigma_{i,m+1}}{\sigma_{i,k}}$$

and set the sample return at day k for variable i to $x_{i,k}^*$ instead of $x_{i,k}$.

Therefore, instead of using the actual historical percentage returns in market variables for the purposes of calculating Value at Risk, this approach builds the empirical distribution of $L_{\boldsymbol{h},t+1}$ using historical returns that have been adjusted to reflect the ratio of the current daily volatility with respect to the one at the time of the observation. So, from this point on, the steps required to compute the V.a.R. measure of the portfolio are the same to the ones explained for the traditional historical simulation. In addition, it is interesting to notice that the V.a.R. estimates can be greater than any of the historical losses that would have occurred for the current portfolio during the historical period considered, especially if the current volatility of the underlying market variables is relatively high.

To conclude, in their scientific article Hull and White produce evidence using exchange rates and stock indices to show that this approach is superior to traditional historical simulation and to the hybrid method described earlier because it produces V.a.R. estimates which are demonstrated to be more accurate in quantifying the portfolio losses at a given confidence level.

5. EMPIRICAL ANALYSIS

In this chapter it will be calculated the Value at Risk measure of a financial portfolio made up by four of the most important stocks included in the index NASDAQ 100, which are:

- Google (GOOGL)
- Microsoft (MSFT)
- Apple (AAPL)
- Intel (INTC)

Successively, it will be shown the benefits of diversification related to this portfolio comparing the V.a.R. calculated at a portfolio level with the one computed for each asset individually.

In conclusion, it will be explained the concept of back testing and then it is used to monitor the accuracy of the estimation methods in quantifying the portfolio losses at a given confidence level.

5.1 Example of V.a.R. calculation

To illustrate how the Value at Risk measure is estimated with all the methods explained so far, it is assumed that an U.S. investor on 1st May 2021 holds a portfolio consisting of:

- 1.000 shares of Google (GOOGL)
- 10.000 shares of Microsoft (MSFT)
- 20.000 shares of Apple (AAPL)
- 50.000 shares of Intel (INTC)

which are four stocks belonging to the technological sector of the NASDAQ 100. The numbers of shares are chosen to balance the current value of each asset in the portfolio and they represent the vector of the quantities $\boldsymbol{h}$ in the given models.

Before starting the description of the steps required to estimate the Value at Risk measure of this portfolio, it is necessary to choose the confidence level and the time horizon. Hence, for all the methods that will be used, it is decided to set the confidence level equal to 99% and the time horizon equal to one day.

Then, in order to build the distribution of the daily portfolio's loss, it is collected the closing prices of these market variables on the most recent 501 trading days (from 07/05/2019 to 30/04/2021), which are summarized in *Table 1*:

DATE	DAY $(\boldsymbol{k})$	GOOGL $(\boldsymbol{i=1})$	MSFT $(\boldsymbol{i=2})$	AAPL $(\boldsymbol{i=3})$	INTC $(\boldsymbol{i=4})$
07/05/2019	0	\$1.178,86	\$125,52	\$50,72	\$50,48
08/05/2019	1	\$1.170,78	\$125,51	\$50,73	\$49,24
09/05/2019	2	\$1.167,97	\$125,50	\$50,18	\$46,62
10/05/2019	3	\$1.167,64	\$127,13	\$49,29	\$46,20
13/05/2019	4	\$1.136,59	\$123,35	\$46,43	\$44,76
⋮	⋮	⋮	⋮	⋮	⋮
23/03/2020	221	\$1.054,13	\$135,98	\$56,09	\$49,58
24/03/2020	222	\$1.130,01	\$148,34	\$61,72	\$52,40
25/03/2020	223	\$1.101,62	\$146,92	\$61,38	\$51,26
26/03/2020	224	\$1.162,92	\$156,11	\$64,61	\$55,54
27/03/2020	225	\$1.110,26	\$149,70	\$61,94	\$52,37

⋮	⋮	⋮	⋮	⋮	⋮
26/04/2021	496	$2.309,93	$261,55	$134,72	$58,76
27/04/2021	497	$2.290,98	$261,97	$134,39	$57,97
28/04/2021	498	$2.359,04	$254,56	$133,58	$57,62
29/04/2021	499	$2.392,76	$252,51	$133,48	$58,28
30/04/2021	500	$2.353,50	$252,18	$131,46	$57,53

Table 1: Market variables' prices over the last 501 trading days (Source: Investing)

Successively, from the market variables' closing prices, it is calculated the daily percentage returns using the following formula:

$$x_{i,k} = \frac{P_{i,k} - P_{i,k-1}}{P_{i,k-1}}$$

where $i = 1,2,3,4$ and $k = 1,2, \dots ,500$. So, in *Table 2* are shown the results of these computations:

DAY	GOOGL	MSFT	AAPL	INTC
1	-6,85E-03	-7,97E-05	1,97E-04	-2,46E-02
2	-2,40E-03	-7,97E-05	-1,08E-02	-5,32E-02
3	-2,83E-04	1,30E-02	-1,77E-02	-9,01E-03
4	-2,66E-02	-2,97E-02	-5,80E-02	-3,12E-02
5	-1,03E-02	1,12E-02	1,59E-02	9,16E-03
⋮	⋮	⋮	⋮	⋮
221	-1,32E-02	-9,97E-03	-2,13E-02	8,18E-02
222	7,20E-02	9,09E-02	1,00E-01	5,69E-02
223	-2,51E-02	-9,57E-03	-5,51E-03	-2,18E-02
224	5,56E-02	6,26E-02	5,26E-02	8,35E-02
225	-4,53E-02	-4,11E-02	-4,13E-02	-5,71E-02
⋮	⋮	⋮	⋮	⋮
496	4,35E-03	1,53E-03	2,98E-03	-8,10E-03
497	-8,20E-03	1,61E-03	-2,45E-03	-1,34E-02
498	2,97E-02	-2,83E-02	-6,03E-03	-6,04E-03
499	1,43E-02	-8,05E-03	-7,49E-04	1,15E-02
500	-1,64E-02	-1,31E-03	-1,51E-02	-1,29E-02

Table 2: Market variables' daily percentage returns

In addition, for each day of the considered period it is calculated the portfolio's value as:

$$V_{\boldsymbol{h},k} = q_1 \cdot P_{1,k} + q_2 \cdot P_{2,k} + q_3 \cdot P_{3,k} + q_4 \cdot P_{4,k}$$

where $k = 0,1, \dots ,500$ and the resulting values are summarized in *Table 3*:

DATE	PORTFOLIO'S VALUE
07/05/2019	$5.972.460,00
08/05/2019	$5.902.480,00
09/05/2019	$5.757.570,00
10/05/2019	$5.734.740,00
13/05/2019	$5.536.690,00
⋮	⋮
23/03/2020	$6.014.730,00
24/03/2020	$6.467.810,00
25/03/2020	$6.361.420,00
26/03/2020	$6.793.220,00
27/03/2020	$6.464.560,00
⋮	⋮
26/04/2021	$10.557.830,00
27/04/2021	$10.496.980,00
28/04/2021	$10.457.240,00
29/04/2021	$10.501.460,00
30/04/2021	$10.381.000,00

Table 3: Portfolio's value over the last 501 trading days

As it can be easily noticed, the portfolio has almost doubled its value with respect to the beginning of the given period.

From now on the steps followed in the calculation of the Value at Risk measure depend on the specific approach used. Hence, in order to be as clear as possible, every method is treated separately and for each of them it is dedicated a section within this paragraph.

5.1.1 Linear model example

In the linear model it is supposed that the market variables' returns follow a normal distribution with expected value equal to zero. Hence, first of all it is necessary to verify if the mean of the daily asset's returns can be assumed zero. For this purpose, it is performed a bilateral hypothesis test at 95% confidence level in which it is tested the null hypothesis $H_0: \mu_i = 0$ against the alternative hypothesis $H_1: \mu_i \neq 0$ for all the considered market variables ($i = 1,2,3,4$).

Thus, using the results represented in *Table 2* it is calculated for each asset's return the sample mean and the standard deviation. Then, it is performed the above-mentioned hypothesis test and the outcomes are shown in *Table 4*:

	SAMPLE MEAN	SAMPLE STANDARD DEVIATION	P-VALUE ($\mu_i = 0$)
GOOGL	1,60E-03	2,06E-02	46,92%
MSFT	1,63E-03	2,17E-02	47,00%
AAPL	2,19E-03	2,39E-02	46,35%
INTC	6,48E-04	2,77E-02	49,07%

Table 4: Hypothesis test

Since the p-value is far greater than the pre-defined threshold ($\alpha/2 = 2{,}5\%$) for all the market variables, the null hypothesis cannot be rejected at the significance level of 95%.

Having shown that the mean of the market variables returns' can be assumed zero, the next step consists in the calculation of their variance. For this purpose, it is used the Exponentially Weighted Moving Average (EWMA) with $\lambda = 0{,}94$. However, it is necessary to decide how to initialize the model, which means to choose for each market variable a value of the variance for the first day. In the scientific literature there are two main approaches used for the initialization of the model, which are:

- Set the first variance of each market variable equal to the sample variance of the returns in the selected period;
- Set the first variance of each asset equal to its first squared return.

It is decided to go with the first option because it produces consistent values of variance from the beginning of the given period even if this choice does not affect the final values in a significative way. Then, from Day 2 variances are calculated using the following formula:

$$\sigma_{i,k}^2 = 0{,}94 \cdot \sigma_{i,k-1}^2 + (1 - 0{,}94) \cdot x_{i,k-1}^2$$

where $i = 1{,}2{,}3{,}4$ and $k = 2{,}3, \ldots 501$. The results of these calculations are reported in *Table 5*:

DAY	GOOGL	MSFT	AAPL	INTC
1	4,26E-04	4,71E-04	5,73E-04	7,68E-04
2	4,04E-04	4,42E-04	5,38E-04	7,58E-04
3	3,80E-04	4,16E-04	5,13E-04	8,82E-04
4	3,57E-04	4,01E-04	5,01E-04	8,34E-04
5	3,78E-04	4,30E-04	6,73E-04	8,42E-04
⋮	⋮	⋮	⋮	⋮
221	2,17E-03	3,72E-03	3,22E-03	5,67E-03
222	2,05E-03	3,50E-03	3,06E-03	5,73E-03
223	2,24E-03	3,79E-03	3,48E-03	5,58E-03
224	2,15E-03	3,57E-03	3,27E-03	5,28E-03
225	2,20E-03	3,59E-03	3,24E-03	5,38E-03
⋮	⋮	⋮	⋮	⋮
496	2,34E-04	1,85E-04	2,77E-04	5,73E-04
497	2,21E-04	1,74E-04	2,61E-04	5,42E-04
498	2,12E-04	1,64E-04	2,45E-04	5,21E-04
499	2,52E-04	2,02E-04	2,33E-04	4,92E-04
500	2,49E-04	1,94E-04	2,19E-04	4,70E-04
501	2,50E-04	1,82E-04	2,19E-04	4,52E-04

Table 5: Variances of the market variables' daily returns

Thereafter, it is necessary also to compute the covariances between the returns of each pair of market variables. For this purpose, it is used again the EWMA model with $\lambda = 0{,}94$ and are still valid the considerations made for the variance calculation. However, the only difference is related to the initialization of the model since the first values of covariances are set equal to the sample covariance of each pair of market variables' returns. Instead, for all the other days the covariances are calculated as:

$$cov_{i,j,k} = 0{,}94 \cdot cov_{i,j,k-1} + (1 - 0{,}94) \cdot x_{i,k-1} \cdot x_{j,k-1}$$

where $i = 1{,}2{,}3{,}4$, $j > i$ and $k = 2{,}3, \ldots 501$. The results of these calculations are shown in *Table 6*:

DAY	GOOGL-MSFT	GOOGL-AAPL	GOOGL-INTC	MSFT-AAPL	MSFT-INTC	AAPL-INTC
1	3,61E-04	3,42E-04	3,32E-04	4,15E-04	4,01E-04	4,00E-04
2	3,39E-04	3,22E-04	3,23E-04	3,90E-04	3,77E-04	3,76E-04
3	3,19E-04	3,04E-04	3,11E-04	3,67E-04	3,55E-04	3,88E-04
4	2,99E-04	2,86E-04	2,92E-04	3,31E-04	3,27E-04	3,74E-04
5	3,29E-04	3,62E-04	3,25E-04	4,15E-04	3,63E-04	4,60E-04
⋮	⋮	⋮	⋮	⋮	⋮	⋮

221	2,75E-03	2,52E-03	3,16E-03	3,31E-03	4,34E-03	3,87E-03
222	2,59E-03	2,39E-03	2,91E-03	3,12E-03	4,03E-03	3,53E-03
223	2,83E-03	2,68E-03	2,98E-03	3,48E-03	4,10E-03	3,66E-03
224	2,68E-03	2,52E-03	2,83E-03	3,27E-03	3,87E-03	3,45E-03
225	2,72E-03	2,55E-03	2,94E-03	3,28E-03	3,95E-03	3,51E-03
⋮	⋮	⋮	⋮	⋮	⋮	⋮
496	1,63E-04	1,77E-04	8,82E-05	1,76E-04	1,24E-04	1,53E-04
497	1,54E-04	1,68E-04	8,08E-05	1,66E-04	1,16E-04	1,43E-04
498	1,44E-04	1,59E-04	8,25E-05	1,56E-04	1,07E-04	1,36E-04
499	8,48E-05	1,38E-04	6,68E-05	1,56E-04	1,11E-04	1,30E-04
500	7,28E-05	1,29E-04	7,26E-05	1,47E-04	9,89E-05	1,22E-04
501	6,97E-05	1,37E-04	8,09E-05	1,40E-04	9,40E-05	1,26E-04

Table 6: Covariances of the market variables' daily returns

Thus, using the most recent values of variances and covariances which are present in the last row of the previous two tables, it can be built the covariance matrix Σ, which is represented in *Table 7*:

	GOOGL	MSFT	AAPL	INTC
GOOGL	2,50E-04	6,97E-05	1,37E-04	8,09E-05
MSFT	6,97E-05	1,82E-04	1,40E-04	9,40E-05
AAPL	1,37E-04	1,40E-04	2,19E-04	1,26E-04
INTC	8,09E-05	9,40E-05	1,26E-04	4,52E-04

Table 7: Covariance Matrix

Successively, it is constructed the vector $\boldsymbol{V}_t$ multiplying the number of shares of a given market variable held by the investor with the respective price observed on the last trading day (Day 500):

$$\boldsymbol{V}_t = \begin{pmatrix} V_{1,t} \\ V_{2,t} \\ V_{3,t} \\ V_{4,t} \end{pmatrix} = \begin{pmatrix} \$2.353.500,00 \\ \$2.521.800,00 \\ \$2.629.200,00 \\ \$2.876.500,00 \end{pmatrix}$$

The values in this vector represent the monetary positions on the most recent trading day in Google stock, Microsoft stock, Apple stock and Intel stock, respectively.

Then, it can be estimated the daily portfolio's volatility (measured in \$) for the next trading day with the following formula:

$$\sigma_{\boldsymbol{h}} = \sqrt{\boldsymbol{V}_t' \Sigma \boldsymbol{V}_t} = \$128.603,36$$

At this point all the required computations have been performed and it can be stated that the daily portfolio's loss has the following distribution:

$$L_{\boldsymbol{h},t+1} \sim N(0, \sigma_{\boldsymbol{h}}^2) \rightarrow L_{\boldsymbol{h},t+1} \sim N(\$0, \$128.603,36^2)$$

Finally, the estimate of the Value at Risk measure for the next trading day using the linear model is obtained as:

$$VAR_{\boldsymbol{h},\alpha,t+1} = z_\alpha \cdot \sigma_{\boldsymbol{h}} = \$299.176,15$$

where $z_\alpha = 2{,}33$ and it represents the quantile of the standard normal distribution corresponding to the 99% probability level.

Since the assets included in the portfolio are only stocks, the quadratic model has no differences compared to the linear one and it is redundant to show what are the calculations underlying this model.

5.1.2 Monte Carlo simulation example

The steps to be followed in the Monte Carlo simulation are the same as those explained in the linear model until the calculation of the portfolio's covariance matrix for the next trading day. However, using this method it is more convenient to work with the correlation matrix instead of the covariance matrix. Therefore, starting from the latter, it can be obtained the correlation matrix which is summarized in *Table 8*:

	GOOGL	MSFT	AAPL	INTC
GOOGL	1,00	0,32	0,58	0,24
MSFT	0,32	1,00	0,68	0,31
AAPL	0,58	0,68	1,00	0,39
INTC	0,24	0,31	0,39	1,00

Table 8: Correlation matrix

Before proceeding further, in the Monte Carlo simulation it must be chosen the number of trials to be performed in order to build the probability distribution of the daily portfolio's losses. Hence, it is decided to perform one thousand simulations because this number is considered to be a good tradeoff between the computational complexity of this approach and the resulting error of the estimate.

In order to build one trial of the given sample, the starting point is to obtain four random values (one for each market variable) from the standard normal distribution which are extracted using the Microsoft Excel's built-in function NORM.S.INV (RANDOM ()) and termed z_1, z_2, z_3 and z_4. Then, this procedure is repeated one thousand of times and the outcomes are shown in *Table 9*:

TRIAL	GOOGL (z_1)	MSFT (z_2)	AAPL (z_3)	INTC (z_4)
1	0,9877	-0,3130	0,9403	-0,4478
2	-1,2366	0,9847	0,1221	0,6370
3	0,7286	1,0220	0,7515	-0,6467
4	-1,5245	1,7060	0,9422	1,1943
5	-1,0397	0,6304	-0,6525	-1,1886
⋮	⋮	⋮	⋮	⋮
498	0,7120	-1,2703	0,8472	-0,2894
499	-1,5800	-0,2197	1,1657	0,6060
500	-1,3919	0,0683	-0,7844	1,0345
501	0,3356	1,0797	0,6570	1,5872
502	-1,7357	-0,7123	-1,5094	-1,4475
⋮	⋮	⋮	⋮	⋮
996	0,7217	0,4950	0,3709	3,1010
997	0,1511	-0,4946	-0,7326	-1,2760
998	-0,5327	0,6243	1,6299	-1,5468
999	-0,4659	0,6563	0,6682	1,2934
1000	-0,3706	-1,2090	-0,7653	-0,6144

Table 9: Random values obtained from the standard normal distribution

After that, in order to find out the random values of the standard normal distribution according to the correlation scheme among the market variables included in the portfolio, it must be calculated the value of the coefficients a_{ij} where $i = 1,2,3,4$ and $j \leq i$. Hence, basing on the formulas presented in the paragraph in which it is explained the Monte Carlo simulation, every coefficient is calculated as:

$$a_{11} = 1$$

$$a_{21} = \frac{\rho_{1,2}}{a_{11}} = 0{,}3262$$

$$a_{22} = \sqrt{1 - a_{21}^2} = 0{,}9453$$

$$a_{31} = \frac{\rho_{1,3}}{a_{11}} = 0{,}5826$$

$$a_{32} = \frac{\rho_{2,3} - a_{31} \cdot a_{21}}{a_{22}} = 0{,}5382$$

$$a_{33} = \sqrt{1 - a_{31}^2 - a_{32}^2} = 0{,}6090$$

$$a_{41} = \frac{\rho_{1,4}}{a_{11}} = 0{,}2406$$

$$a_{42} = \frac{\rho_{2,4} - a_{41} \cdot a_{21}}{a_{22}} = 0{,}2635$$

$$a_{43} = \frac{\rho_{3,4} - a_{41} \cdot a_{31} - a_{42} \cdot a_{32}}{a_{33}} = 0{,}1947$$

$$a_{44} = \sqrt{1 - a_{41}^2 - a_{42}^2 - a_{43}^2} = 0{,}9137$$

So, for a given trial it is obtained the random values of the standard normal distribution which are based on the correlation scheme among the market variables according to the following relations:

$$\varepsilon_1 = a_{11} \cdot z_1$$

$$\varepsilon_2 = a_{21} \cdot z_1 + a_{22} \cdot z_2$$

$$\varepsilon_3 = a_{31} \cdot z_1 + a_{32} \cdot z_2 + a_{33} \cdot z_3$$

$$\varepsilon_4 = a_{41} \cdot z_1 + a_{42} \cdot z_2 + a_{43} \cdot z_3 + a_{44} \cdot z_4$$

Repeating these calculations for all the considered simulations, it is obtained the outcomes shown in *Table 10*:

TRIAL	GOOGL (ε_1)	MSFT (ε_2)	AAPL (ε_3)	INTC (ε_4)
1	0,9877	0,0263	0,9797	-0,0709
2	-1,2366	0,5274	-0,1162	0,5677

3	0,7286	1,2038	1,4322	0,0001
4	-1,5245	1,1153	0,6037	1,3572
5	-1,0397	0,2567	-0,6639	-1,2971
⋮	⋮	⋮	⋮	⋮
498	0,7120	-0,9685	0,2471	-0,2628
499	-1,5800	-0,7231	-0,3289	0,3426
500	-1,3919	-0,3895	-1,2520	0,4755
501	0,3356	1,1301	1,1767	1,9433
502	-1,7357	-1,2395	-2,3139	-2,2217
⋮	⋮	⋮	⋮	⋮
996	0,7217	0,7033	0,9128	3,2095
997	0,1511	-0,4183	-0,6243	-1,4024
998	-0,5327	0,4164	1,0183	-1,0596
999	-0,4659	0,4684	0,4886	1,3726
1000	-0,3706	-1,2638	-1,3327	-1,1180

Table 10: Random values obtained according to the correlation scheme

Successively, it is computed for each trial the daily return of each market variable as:

$$\Delta x_{i,k} = \sigma_{i,t+1} \cdot \varepsilon_{i,k}$$

where $i = 1,2,3,4$ and $k = 1,2,\dots,1000$. Then, for every simulation it can be calculated the resulting portfolio's loss with the following equation:

$$L_{\boldsymbol{h},t+1,k} = -\sum_{i=1}^{4} q_i \cdot P_{i,t} \cdot \sigma_{i,t+1} \cdot \varepsilon_{i,k}$$

where $k = 1,2,\dots,1000$. Finally, daily losses are ranked from the biggest to the smallest one and the results are summarized in *Table 11*:

TRIAL	LOSS
667	$524.835,30
759	$402.340,12
487	$381.240,90
61	$373.982,37
381	$365.928,69
34	$357.694,38
611	$345.333,88
502	$332.790,83
508	$322.380,42
390	$321.792,07
700	$321.591,89

578	$312.587,82
914	$308.612,60
144	$307.550,77
41	$306.398,45
863	$304.223,12
563	$298.155,33
⋮	⋮

Table 11: Ranked simulated portfolio's losses

So, the Value at Risk measure corresponds to the loss present in the tenth position, which is $321.792,07.

Using the Monte Carlo simulation method, it is also interesting to see the density distribution of the losses resulting from the performed trials. Therefore, in *Figure 6* it is shown the frequency with which the losses belong to a given bin:

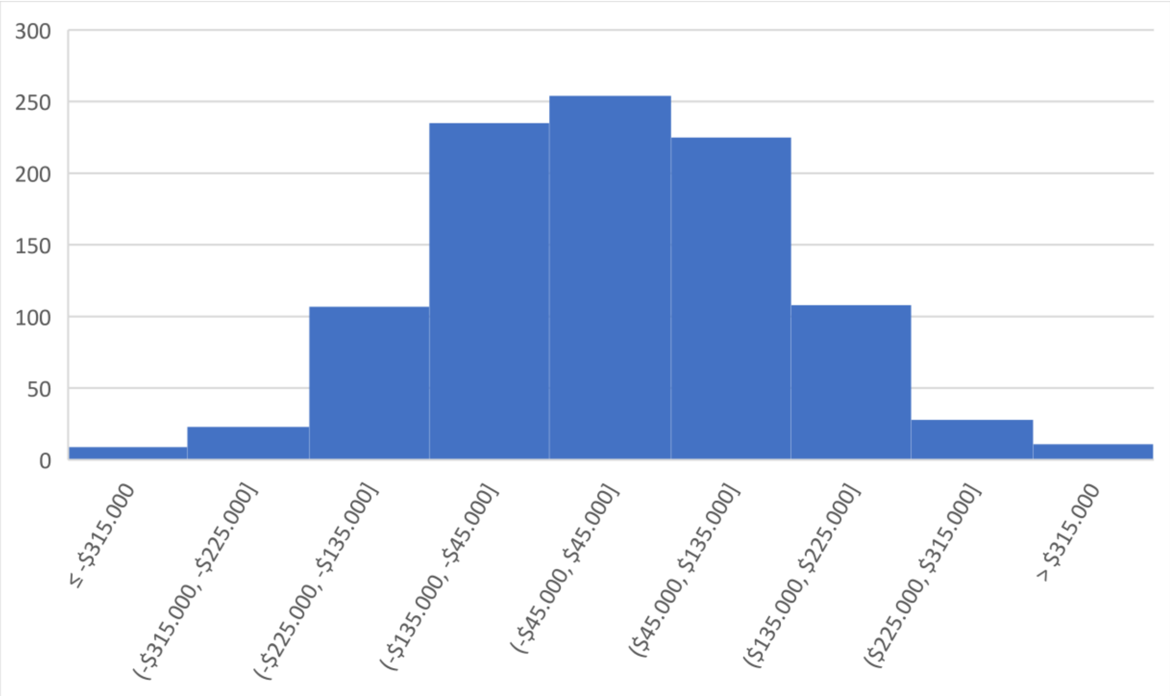

Figure 6: Loss Distribution obtained from Monte Carlo simulation

From the previous figure, it can be observed that the frequency distribution is a bell-shaped function and the majority of the losses (714) belong to the range [-$135.000,$135.000].

In addition, calculating the sample standard deviation of the loss distribution, which is $s_L = \$129.570,67$, it can be noticed that it can be well approximated by the normal distribution also because about two-thirds of the observations (684) fall within one standard deviation from the average value, which is assumed to be zero.

Finally, it can be calculated the standard error of the estimate as:

$$SE = \frac{s_L}{\sqrt{m}} = \frac{\$129.570{,}67}{\sqrt{1000}} = \$4.097{,}38$$

This result confirms that, as it was said before, using 1000 trials the error of the estimate is very small (around 1,5%) in relation to the value of the Value at Risk measure calculated with the Monte Carlo simulation.

5.1.3 Historical simulation example

Using the traditional historical simulation it is not necessary to calculate variances and covariances as it is done in the previous two methods, but from the market variables' returns it can be computed directly the portfolio's loss for each day of the given period using the following formula:

$$L_{\boldsymbol{h},k} = -\sum_{i=1}^{4} q_i \cdot P_{i,k-1} \cdot \Delta x_{i,k}$$

where $i = 1,2,3,4$ and $k = 1,2, \dots 500$. Then, all the losses are sorted in a descending order and the results are summarized in *Table 12*:

DAY	LOSS
216	\$1.045.170,00
214	\$687.050,00
211	\$549.850,00
336	\$534.780,00
307	\$516.240,00
204	\$482.020,00
277	\$461.800,00
338	\$394.810,00
374	\$371.280,00
455	\$350.160,00
201	\$344.000,00

225	$328.660,00
470	$311.180,00
241	$305.190,00
213	$302.570,00
228	$294.650,00
452	$291.550,00
⋮	⋮

Table 12: Ranked portfolio's losses

The daily Value at Risk measure corresponds to the fifth highest loss, which is $516.240,00.

In addition, it is interesting to see the empirical density function of the portfolio's loss over the last 500 trading days which is shown in *Figure 7*:

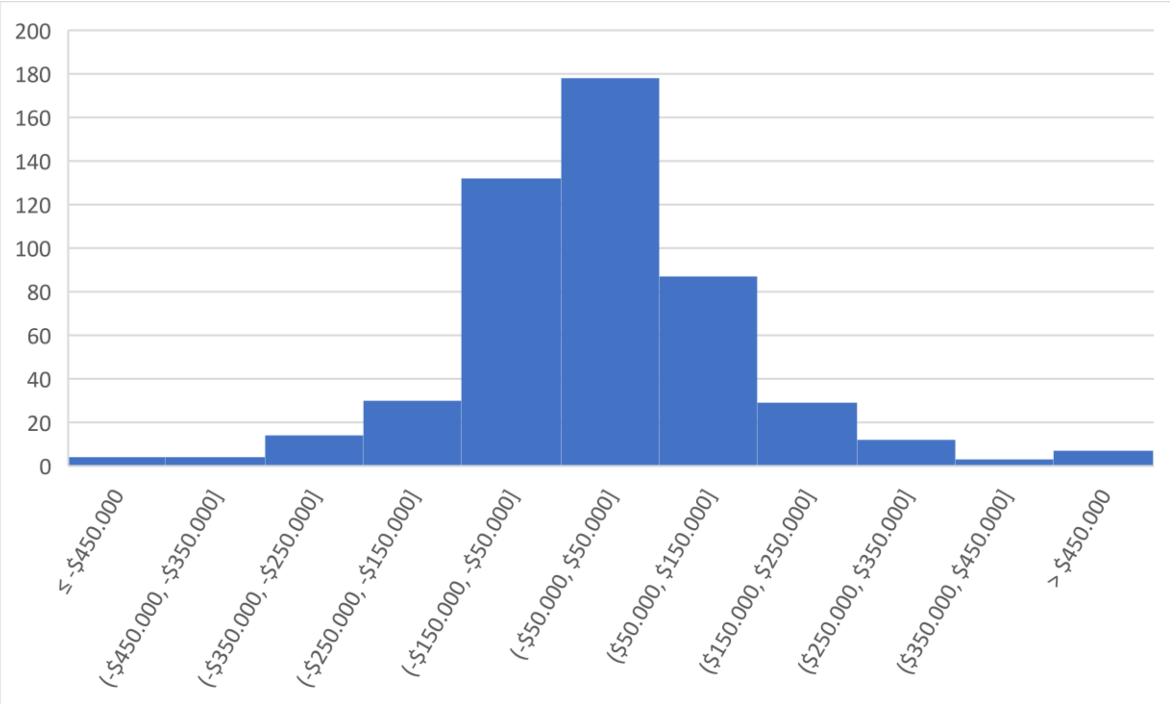

Figure 7: Historical Portfolio's Loss Distribution

From the previous graph, it can be observed that the distribution is a bell-shaped function and the majority of the losses (397) belong to the range [-$150.000,$150.000]. Then, it can be also estimated the average value (-$8.817,08) and the sample deviation ($157.090,25) of the historical losses.

Finally, assuming that empirical losses can be fitted using the normal distribution with the previous parameters, it can be calculated the probability density function

$f(x)$ of the quantile corresponding to a confidence level equal to 99%, which is $1{,}93 \cdot 10^{-7}$. So, the standard error of the estimate can be calculated as:

$$SE = \frac{1}{f(x)}\sqrt{\frac{(1-\alpha)\cdot\alpha}{m}} = \frac{1}{1{,}93\cdot 10^{-7}}\sqrt{\frac{(1-0{,}99)\cdot 0{,}99}{500}} = \$23.165{,}64$$

Comparing the standard error of the estimate with the Value at Risk measure calculated in the historical simulation, it can be noticed that it is relatively small since it is about 4,5% of the V.a.R. measure.

5.1.4 Hybrid approach example

The steps required to compute the V.a.R. measure in the hybrid approach are the same followed in the traditional historical simulation. The only difference concerns the fact that each loss has a different weight basing on the day in which it occurs. The chosen weighting scheme is the one presented in the related paragraph with $\lambda = 0{,}94$ and the loss observed on Day k is given a weight equal to:

$$\frac{0{,}94^{500-k}\cdot(1-0{,}94)}{1-0{,}94^{500}}$$

For instance, the loss occurs on Day 1 is given a weight equal to $2{,}34 \cdot 10^{-15}$ while the one observed on Day 500 has a weight equal to $0{,}06$. Then, as it was done in the traditional historical simulation, losses are ranked from the highest to lowest one but for each of them it is also reported the corresponding weight as it can be noticed in *Table 13*:

DAY	LOSS	WEIGHT
216	$1.045.170,00	1,40E-09
214	$687.050,00	1,24E-09
211	$549.850,00	1,03E-09
336	$534.780,00	2,35E-06
307	$516.240,00	3,91E-07
204	$482.020,00	6,67E-10
277	$461.800,00	6,10E-08

338	\$394.810,00	2,66E-06
374	\$371.280,00	2,47E-05
455	\$350.160,00	3,71E-03
201	\$344.000,00	5,54E-10
225	\$328.660,00	2,45E-09
470	\$311.180,00	9,38E-03
241	\$305.190,00	6,58E-09
213	\$302.570,00	1,16E-09
228	\$294.650,00	2,94E-09
452	\$291.550,00	3,08E-03
⋮	⋮	⋮

Table 13: Ranked portfolio's losses with weights

The daily Value at Risk measure of the portfolio corresponds to the highest loss for which the cumulative weight overcomes 1% and it occurs on Day 470, when the loss is \$311.180,00. It is interesting to notice that this approach produces a V.a.R. measure which is lower than the one calculated with traditional historical simulation since the greatest losses occurred not so recently.

5.1.5 Volatility updating scheme example

Using the volatility updating scheme it is necessary to compute variances for the market variables' returns. Again, it is used the EWMA model with $\lambda = 0{,}94$ and the results are already calculated in *Table 5*.

Successively, for each market variable i and day k, it is calculated the volatility ratio using the following formula:

$$\frac{\sigma_{i,501}}{\sigma_{i,k}}$$

where $i = 1{,}2{,}3{,}4$ and $k = 1{,}2{,}\dots{,}500$. The outcomes of these computations are summarized in *Table 14*:

DAY	GOOGL (ε_1)	MSFT (ε_2)	AAPL (ε_3)	INTC (ε_4)
1	0,77	0,62	0,62	0,77
2	0,79	0,64	0,64	0,77
3	0,81	0,66	0,65	0,72

4	0,84	0,67	0,66	0,74
5	0,81	0,65	0,57	0,73
⋮	⋮	⋮	⋮	⋮
221	0,34	0,22	0,26	0,28
222	0,35	0,23	0,27	0,28
223	0,33	0,22	0,25	0,28
224	0,34	0,23	0,26	0,29
225	0,34	0,23	0,26	0,29
⋮	⋮	⋮	⋮	⋮
496	1,03	0,99	0,89	0,89
497	1,06	1,02	0,92	0,91
498	1,09	1,05	0,95	0,93
499	1,00	0,95	0,97	0,96
500	1,00	0,97	1,00	0,98

Table 14: Volatility ratio

In addition, in *Figure 8* it is plotted the evolution of the volatility ratio of each market variable over the considered period:

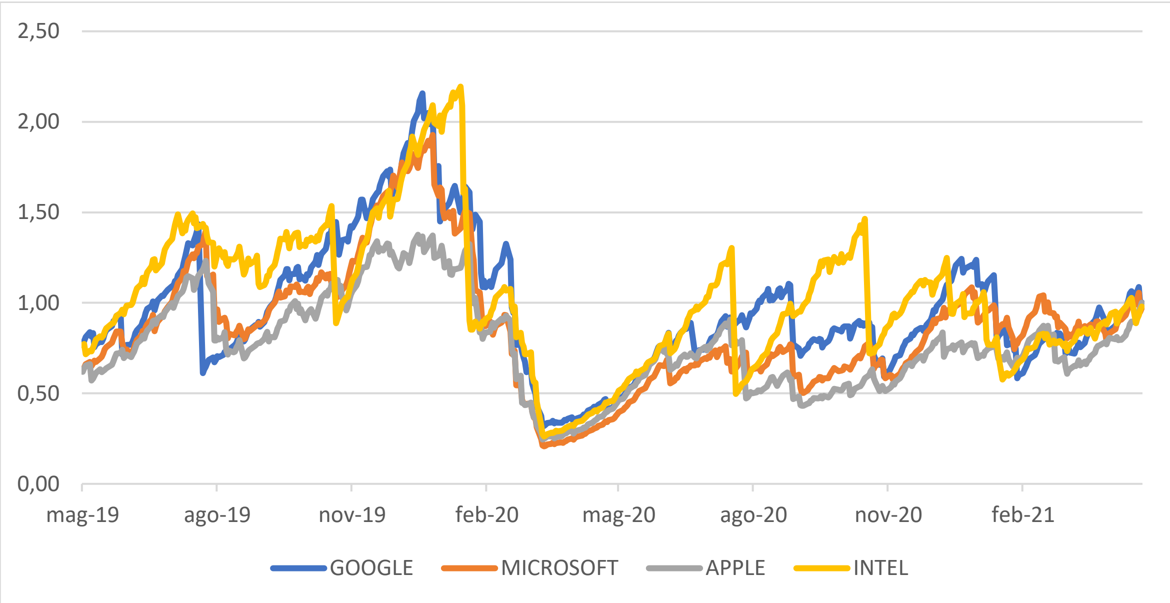

Figure 8: Evolution of volatility ratios

As it can be seen from the previous figure, at the beginning of the considered period the market variables' volatilities are relatively low and their volatility ratios reach the peak at the end of 2019. Instead, during the first months of 2020 it can be noticed a fast increase in the level of volatilities and consequently a decrease in the assets' volatility ratios caused by the Covid-19 pandemic.

After obtaining the volatility ratios for each trading day, it is calculated the adjusted returns as:

$$x_{i,k}^{*} = x_{i,k} \cdot \frac{\sigma_{i,501}}{\sigma_{i,k}}$$

where $i = 1,2,3,4$ and $k = 1,2,\dots,500$.

Then, portfolio's losses are calculated from the adjusted returns of the market variables and they are ranked in a descending order, as it can be seen in *Table 15*:

DAY	ADJUSTED LOSS
307	\$657.578,83
204	\$417.591,99
336	\$404.959,37
371	\$391.826,18
277	\$353.086,78
201	\$349.454,26
211	\$329.172,77
214	\$323.314,04
216	\$318.954,80
455	\$302.198,03
182	\$283.906,98
374	\$274.479,50
452	\$257.548,23
62	\$253.995,35
410	\$253.270,32
470	\$248.307,68
338	\$247.231,79
⋮	⋮

Table 15: Losses calculated from adjusted returns

Therefore, the daily Value at Risk measure is estimated as the fifth highest loss, which corresponds to \$353.086,78.

5.2 Benefits of diversification

Diversification is a risk management strategy that aims to reduce the riskiness of a financial portfolio simply investing in a wide variety of assets. Harry Markowitz was one of the first researchers to study the benefits of diversification for a portfolio manager and he was awarded with the Nobel prize in 1990 for his publications about this theme. The rationale behind this technique is that a portfolio constructed of different kinds of assets will, on average, yield higher long-term returns and lower the risk of any individual holding or security. Diversification strives to smooth out unsystematic risk, which is unique to a specific company, so that the positive performance of some investments neutralizes the negative performance of others. The benefits of diversification hold only if the securities in the portfolio are not perfectly correlated, so they respond differently to market influences.

In this paragraph it will be quantified the benefits of diversification for the financial portfolio analyzed so far, choosing the daily Value at Risk at the 99% confidence level as a measure of the portfolio's risk. More specifically, for each method it will be made a comparison between the V.a.R. calculated at a portfolio level and the sum of V.a.R. measures computed for each asset individually.

Concerning the V.a.R. calculated at a portfolio level, it was already obtained the value of this measure for each of the methods explained in this thesis and the results are summarized in *Table 16:*

METHOD	DAILY 99% V.a.R.
Linear Model	$299.176,15
Monte Carlo Simulation	$321.792,07
Historical Simulation	$516.240,00
Hybrid Approach	$311.180,00
Volatility Updating Scheme	$353.086,78

Table 16: Daily 99% V.a.R. of the portfolio

Instead, concerning the Value at Risk measure calculated for each asset individually it is necessary to repeat all the computations made in the previous paragraph neglecting the correlations among the market variables. However, without showing all the steps necessary to obtain this measure for each market variable individually, in *Table 17* are reported for each method all the final values of Value at Risk:

METHOD	GOOGL	MSFT	AAPL	INTC	TOTAL
Linear Model	$86.654,62	$79.213,37	$90.594,33	$142.220,16	$398.682,48
Monte Carlo Simulation	$91.709,90	$84.766,30	$85.362,62	$148.222,08	$410.060,90
Historical Simulation	$88.080,00	$115.900,00	$129.200,00	$285.000,00	$618.180,00
Hybrid Approach	$67.860,00	$74.100,00	$87.200,00	$166.500,00	$395.660,00
Volatility Updating Scheme	$73.992,84	$62.015,43	$75.960,05	$187.067,97	$399.036,30

Table 17: Daily 99% V.a.R. computed for each asset individually

The first consideration that stands out is that historical simulation produces V.a.R. measures which are always higher with respect to any other method. This can be explained by the fact that the dataset takes into account the pandemic period in which stock prices have suffered sharp drops and the historical simulation is the method which suffers most.

In addition, it can be easily noticed from *Table 17* that for every method the sum of all the V.a.R. measures at the security level is always greater than the one computed for the portfolio on its whole. Hence, the benefits of diversification for each approach can be quantified by the difference between these two values and the results are shown in *Table 18*:

METHOD	BENEFITS OF DIVERSIFICATION	PERCENTAGE ON PORTFOLIO'S V.a.R. MEASURE
Linear Model	$99.506,33	33,26%
Monte Carlo Simulation	$88.268,84	27,43%
Historical Simulation	$101.940,00	19,75%
Hybrid Approach	$84.480,00	27,15%
Volatility Updating Scheme	$45.949,52	13,01%

Table 18: Benefits of diversification

From the results reported in the previous table it can be stated that on average risk, measured by the portfolio’s daily 99% Value at Risk, is reduced by approximately one fourth (25%) of its value thanks to the diversification effect.

5.3 Back testing

Whatever the method used for calculating the Value at Risk, an important reality check is back testing. It involves testing how well the V.a.R. estimates would have performed in the past.

Denoting as $VAR_{\boldsymbol{h},t+\Delta t}$ the Value at Risk measure of a portfolio $\boldsymbol{h}$ which is estimated at time t for a period of length equal to Δt, once it is arrived at time $t + \Delta t$ there is the opportunity to compare the daily estimate with what actually happened.

By definition of Value at Risk:

$$P(L_{\boldsymbol{h},t+\Delta t} \geq VAR_{\boldsymbol{h},t+\Delta t}) = 1 - \alpha$$

so that the probability of a so-called violation of V.a.R. is $1 - \alpha$. Hence, it is introduced an indicator notation for violations of the Value at Risk estimates which is defined as:

$$\hat{I}_{t+\Delta t} := I_{\{L_{\boldsymbol{h},t+\Delta t} \geq VAR_{\boldsymbol{h},t+\Delta t}\}}$$

If the estimation method is reasonable then this indicator should behave like a Bernoulli random variable with probability of success close to $1 - \alpha$. More in detail, if it is considered m periods, the number of violations over this m periods should follow a binomial distribution with expected value equal to $m \cdot (1 - \alpha)$. For example, suppose that it is computed a daily Value at Risk at a confidence level of 99%. Back testing would involve looking at how often the loss in a day exceeded the V.a.R. measure that would have been calculated for that day. If this happened on about 1% of the days, the methodology for calculating V.a.R. can be considered appropriate. If it happened on, say, 10% of days, the methodology must be revised.

5.4 Monitor the V.a.R. performance

In this paragraph it will be applied back testing to show the performance that the daily 99% Value at Risk measure would have had in the 500 previous trading days for the chosen financial portfolio.

It means that for all the considered period, it is compared the V.a.R. estimated on Day k for Day $k+1$ with the loss occurred on Day $k+1$. Thus, if the loss exceeds the V.a.R. calculated for a given day it will be registered a violation. So, it is monitored the number of the violations occurred over the considered period in order to draw some conclusions about the adequacy of the estimation methods. However, given the huge computational effort due to the calculation of 500 daily V.a.R. measures for each method, back testing is applied only for linear model and historical simulation.

Before showing the implementation of this technique for the above-mentioned methods, it is important to mention that, in order to have consistent values of V.a.R. from the first day of the given period, it is collected the closing prices of the chosen market variables on the 500 trading days (from 10/05/2017 to 06/05/2019) ahead of the considered period.

5.4.1 Back testing and linear model

Concerning the linear model, it is calculated the assets' returns from the market variables' closing prices and then, according to the EWMA model with $\lambda = 0{,}94$, it is computed their variances and covariances. Successively, for each day it is calculated the covariance matrix Σ_{k+1} from the values of variances and covariances estimated on Day k for Day $k+1$ and also the vector of the monetary position in the market variables $\boldsymbol{V}_k$.

After that, it is computed the daily portfolio's volatility using the following formula:

$$\sigma_{\boldsymbol{h},k+1} = \sqrt{\boldsymbol{V}_k^{'} \Sigma_{k+1} \boldsymbol{V}_k}$$

and successively it is calculated the corresponding daily 99% V.a.R. measure as:

$$VAR_{\boldsymbol{h},\alpha,k+1} = z_\alpha \cdot \sigma_{\boldsymbol{h},k+1}$$

where $k = 0,1, \dots ,499$ and $z_\alpha = 2{,}33$.

Then, the Value at Risk estimated on Day k for Day $k+1$ is compared to the loss observed on Day $k+1$. If the loss on a given day is higher than the respective V.a.R. it is registered a violation and the occurrence of these violations during the considered period are reported in *Table 19:*

DAY	DATE	VALUE AT RISK	LOSS
4	13/05/2019	\$172.583,16	\$198.050,00
62	05/08/2019	\$154.243,61	\$227.570,00
76	23/08/2019	\$203.481,56	\$222.040,00
182	27/01/2020	\$211.792,62	\$247.840,00
186	31/01/2020	\$245.607,81	\$245.770,00
201	24/02/2020	\$244.942,31	\$344.000,00
204	27/02/2020	\$295.288,26	\$482.020,00
211	09/03/2020	\$461.453,33	\$549.850,00
214	12/03/2020	\$560.351,55	\$687.050,00
216	16/03/2020	\$892.936,72	\$1.045.170,00
277	11/06/2020	\$392.394,85	\$461.800,00
307	24/07/2020	\$326.468,16	\$516.240,00
336	03/09/2020	\$302.443,44	\$534.780,00
338	08/09/2020	\$384.953,90	\$394.810,00
374	28/10/2020	\$322.527,63	\$371.280,00
455	25/02/2021	\$324.057,86	\$350.160,00

Table 19: Violations for daily 99% V.a.R. computed with linear model

So, it can be noticed that using the linear model the number of violations in the previous 500 trading days is 16 which corresponds to 3,2% of the total. Since the number of violations over this period should behave like a binomial random variable with expected value $m \cdot (1-\alpha)$, corresponding to 5, it can be computed the variance of this random variable as:

$$m \cdot \alpha \cdot (1-\alpha) = 500 \cdot 0{,}99 \cdot (1 - 0{,}99) = 4{,}95$$

Then, it can be verified if the actual number of violations resulting from the linear model is significantly greater than the expected value from a statistical point of view. For this purpose, it is performed a unilateral hypothesis test at 99% significance level in which it is tested the null hypothesis $H_0: \mu = 5$ against the alternative hypothesis $H_1: \mu > 5$. Since it is considered a number of days relatively high ($m = 500$), the binomial distribution can be well approximated by the normal one. So, with the normal distribution the resulting p-value is 0,00004% and then the null hypothesis should be rejected for the chosen significance level because this value is far below the pre-defined threshold (1%). Thus, it means that during the considered period the linear model systematically underestimated the portfolio's Value at Risk and it can be related to the fact that the normal distribution is not the most suitable distribution to model daily assets' returns.

5.4.2 Back testing and historical simulation

Concerning the historical simulation, for each day of the considered period it is calculated the portfolio's losses over the preceding 500 trading days. Then, it is constructed a vector of losses in which they are ranked in a descending order and the Value at Risk measure for Day $k + 1$ corresponds to the fifth highest loss of the vector. This procedure is repeated for $k = 0,1, \dots ,499$ and for each day it is compared the resulting V.a.R. with the corresponding loss, recording a violation for each time the loss exceeds the respective V.a.R. measure.

So, in *Table 20* it is reported the occurrence of these violations during the considered period:

DAY	DATE	VALUE AT RISK	LOSS	EXCEEDING AMOUNT
201	24/02/2020	$249.190,00	$344.000,00	$94.810,00
204	27/02/2020	$254.110,00	$482.020,00	$227.910,00
207	03/03/2020	$257.790,00	$289.300,00	$31.510,00
211	09/03/2020	$274.510,00	$549.850,00	$275.340,00

213	11/03/2020	$289.300,00	$302.570,00	$13.270,00
214	12/03/2020	$292.210,00	$687.050,00	$394.840,00
216	16/03/2020	$302.570,00	$1.045.170,00	$742.600,00
277	11/06/2020	$344.000,00	$461.800,00	$117.800,00
307	24/07/2020	$461.800,00	$516.240,00	$54.440,00
336	03/09/2020	$482.020,00	$534.780,00	$52.760,00

Table 20: Violations for daily 99% V.a.R. computed with historical simulation

It can be noticed that using the historical simulation method the number of violations in the previous 500 trading days is 10 which corresponds to the 2% of the total. Then, it can be performed the same hypothesis test as the one made for the linear model in order to verify if the actual number of violations is significantly higher than its expected value. So, the resulting p-value is 1,23% and even if this value is near to the pre-defined threshold (1%), the null hypothesis cannot be rejected at the 99% significance level. Hence, the difference between the actual number of violations and the expected value can be considered not to be statistically significant.

5.4.3 Conclusions

From the previous considerations it seems that it is better to use the historical simulation method rather than the linear model for the Value at Risk calculation of the given financial portfolio.

However, taking into account only the number of violations as the indicator of the adequacy of a specific method, it is achieved a partial view of the reality. In fact, the number of violations does not consider the magnitude of each violation and hence it does not distinguish a relatively small violation from a high one. Thus, in order to consider this difference, for both methods it is computed the excess amount of each violation (EA) as:

$$EA_{\boldsymbol{h},k+1} = \max\left(L_{\boldsymbol{h},k+1} - VAR_{\boldsymbol{h},k+1}; 0\right)$$

where $k = 0,1, \dots ,499$.

Then, in *Table 21* it is shown the results of these calculations only for those days in which at least one of the two considered methods has a violation:

			EXCESS AMOUNT	
DAY	**DATE**	**LOSS**	**LINEAR MODEL**	**HISTORICAL SIMULATION**
4	13/05/2019	$198.050,00	$25.466,84	-
62	05/08/2019	$227.570,00	$73.326,39	-
76	23/08/2019	$222.040,00	$18.558,44	-
182	27/01/2020	$247.840,00	$36.047,38	-
186	31/01/2020	$245.770,00	$162,19	-
201	24/02/2020	$344.000,00	$99.057,69	$94.810,00
204	27/02/2020	$482.020,00	$186.731,74	$227.910,00
207	03/03/2020	$289.300,00	-	$31.510,00
211	09/03/2020	$549.850,00	$88.396,67	$275.340,00
213	11/03/2020	$302.570,00	-	$13.270,00
214	12/03/2020	$687.050,00	$126.698,45	$394.840,00
216	16/03/2020	$1.045.170,00	$152.233,28	$742.600,00
277	11/06/2020	$461.800,00	$69.405,15	$117.800,00
307	24/07/2020	$516.240,00	$189.771,84	$54.440,00
336	03/09/2020	$534.780,00	$232.336,56	$52.760,00
338	08/09/2020	$394.810,00	$9.856,10	-
374	28/10/2020	$371.280,00	$48.752,37	-
455	25/02/2021	$350.160,00	$26.102,14	-
CUMULATIVE EXCESS AMOUNT			$1.382.903,23	$2.005.280,00
AVERAGE EXCESS AMOUNT			$86.431,45	$200.528,00

Table 21: Excess amount of violations

From the previous table it can be easily noticed that both the cumulative and the average excess amount computed for the linear model is lower than the one calculated for the historical simulation method. So, considering the average excess amount of the violations as the indicator of the adequacy of a specific method to the V.a.R. calculation of the given financial portfolio, it is preferred to use the linear model instead of the historical simulation.

In addition, it is interesting to visualize the evolution of the V.a.R. values and the losses over the considered period for both the methods. For this purpose, *in Figure 9* it is plotted the V.a.R. measures and the corresponding loss for every day of the considered period:

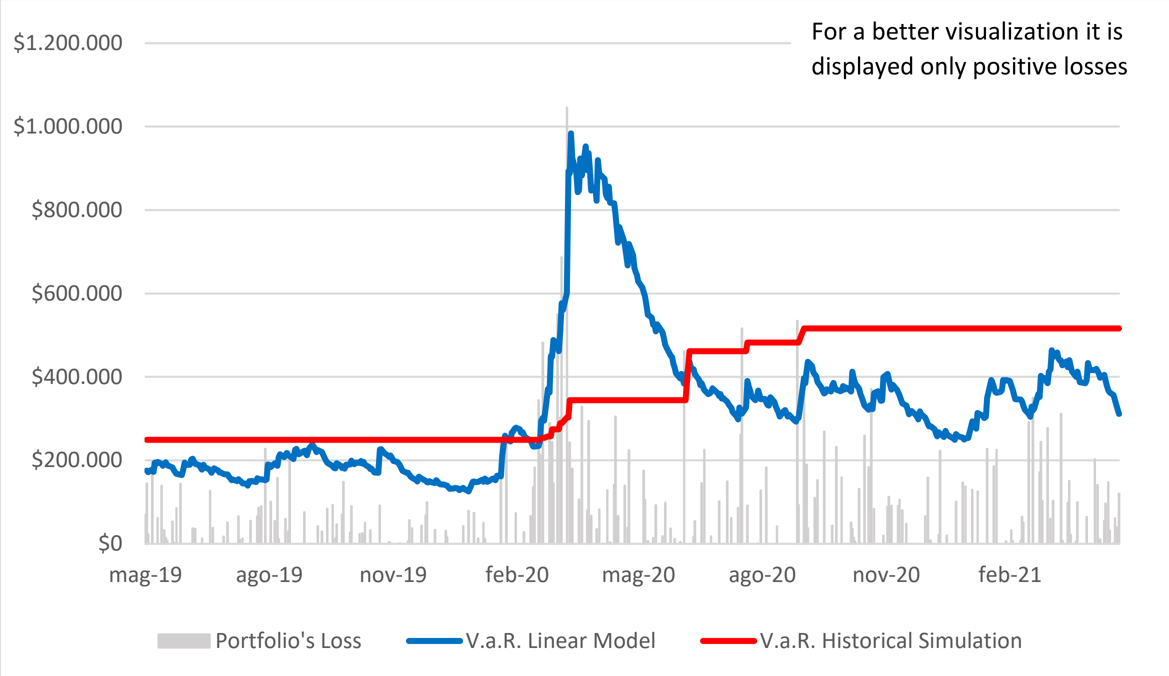

Figure 9: Evolution of V.a.R. measures and losses

As it can be seen from this picture, the V.a.R. measure estimated with the linear model has the capability to adapt well to the different levels of losses. Instead, the Value at Risk computed with the historical simulation method has a trend which is quite different with respect to the evolution of the past portfolio's losses. A possible reason could be related to the fact that the portfolio's V.a.R. measure estimated with the historical simulation is based on the implicit assumption that the volatility of the portfolio remains stable in time, while it can be easily noticed that it is not valid in this specific case.

In conclusion, it cannot be stated a priori that a method can be considered to be better than the other one. However, it can be chosen the most suitable method basing on the purpose for which the Value at Risk of the portfolio is calculated. In fact, in this example the linear model would be preferred if V.a.R. is calculated as a management tool for limiting the amount of risk that a unit may take within a firm because it has been proved that the resulting measure well represents the portfolio's volatility. Instead, if Value at Risk of the given portfolio is calculated for the determination of the capital amount the investor needs to hold as a buffer against unexpected future losses, it would be preferred the historical simulation method since it has been shown with back testing that the resulting measure is not systematically underestimated.

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